Generalized entropic criterion for separability

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We discuss the entropic criterion for separability of compound quantum systems for general nonadditive entropic forms based on arbitrary concave functions *f*. For any separable state, the generalized entropy of the whole system is shown to be not smaller than that of the subsystems, for *any* choice of *f*, providing thus a necessary criterion for separability. Nevertheless, the criterion is not sufficient and examples of entangled states with the same property are provided. This entails, in particular, that the conjecture about the positivity of the conditional Tsallis entropy for all *q*, a more stringent requirement than the positivity of the conditional von Neumann entropy, is actually a necessary but not sufficient condition for separability in general. The direct relation between the entropic criterion and the largest eigenvalues of the full and reduced density operators of the system is also discussed.

The concept of quantum entanglement [1] has aroused great interest in recent years, due to its deep implications for quantum computation [2], quantum cryptography [3], and quantum teleportation [4]. The relation between entropy and quantum entanglement has also attracted attention from several authors [5–15]. It is well known, for instance, that the von Neumann entropy of a compound quantum system may be larger or smaller than that of a subsystem [16,17]. However, if the system is in a separable (i.e., unentangled) state, the von Neumann entropy of the whole system is not smaller than that of a subsystem [5,6]. Unfortunately, the converse is not true, i.e., the same may occur when the system is in an inseparable (i.e., entangled) state, so that this entropy provides only a necessary test for separability. The von Neumann based criterion is actually rather weak, being less stringent than other equally simple necessary conditions [5,18,19]. As discussed in [7,14,15], the von Neumann entropy is in fact not a good entanglement indicator even in those cases where entanglement is fully determined by the eigenvalues of the density operator ρ.

These facts suggest consideration of other information measures which could capture more effectively the effects associated with the separability or inseparability of a compound quantum system. In particular, it has been shown that nonadditive information measures like that of Tsallis [20] do provide more stringent conditions for separability [11,12]. Moreover, this entropy depends on a parameter *q* which can be optimized. In fact, for $q \to \infty$, necessary and sufficient conditions for separability were obtained with this entropy for some important classes of states, like Werner states for *n* qubits and also *n* qudits [11,12]. In other situations [13], entanglement was detected, however, at *finite* values of *q*, rather than in the $q \to \infty$ limit. Hence, the questions arise of whether this entropy could provide a necessary and/or sufficient test in general and whether other information measures could lead to the same result.

In this article we will examine more general entropic forms based on arbitrary concave functions, which include as particular cases the von Neumann and Tsallis entropies. We will show that any of these forms provide *necessary* conditions for separability, which are not sufficient in general. It

will also become clear why the Tsallis form provides necessary and sufficient conditions for Werner states in the *q* $\to \infty$ limit, and why it is not so in other situations. Finally, other entropic forms providing similar results are given.

Let us consider a quantum system described by a density operator ρ. We will examine the general entropic forms [15]

$$S_f(\rho) = \mathrm{Tr}\, f(\rho) = \sum_i f(p_i), \qquad (1)$$

where *f* is a smooth concave function [$f'(p)$ decreasing for $p \in (0,1)$] satisfying $f(0) = f(1) = 0$, and p_i, $i = 1, \ldots, n$, are the eigenvalues of ρ ($\Sigma_i p_i = 1$). We assume a finite dimension *n*. The von Neumann entropy is recovered for

$$f(p) = -kp \ln p, \qquad (2)$$

with $k > 0$, while the Tsallis entropy corresponds to [20]

$$f(p) = (p - p^q)/(q-1), \quad q > 0, \qquad (3)$$

which approaches $-p \ln p$ for $q \to 1$. The generalized entropies (1) satisfy most basic properties of the conventional entropy, except those related to additivity. In particular, $S_f(\rho) \geq 0$, with $S_f(\rho) = 0$ if and only if the system is in a pure state ($\rho^2 = \rho$), while its maximum is attained for the fully mixed state $\rho = I/n$ [21]. Concavity of *f* ensures concavity of $S_f(\rho)$ [17] [$S_f(\Sigma_j q_j \rho_j) \geq \Sigma_j q_j S_f(\rho_j)$ for $0 \leq q_j \leq 1$, $\Sigma_j q_j = 1$]. It can be shown [21,15] that if $[pf''(p)]' \leq 0$ (≥ 0), then S_f is sub-(super)additive, i.e., $S_f(\rho_A \otimes \rho_B) - S_f(\rho_A) - S_f(\rho_B) \leq 0$ (≥ 0). The condition $[pf''(p)]' = 0$ determines in fact Eq. (2). The Tsallis entropy is, accordingly, sub-(super)additive for $q > 1$ ($q < 1$).

A fundamental property of the forms (1) which will be employed in this work, and which justifies their use as information measures, is that if ρ is more mixed than a density operator ρ', then

$$S_f(\rho) \geq S_f(\rho'), \qquad (4)$$

for any f of the previous form [17]. Labeling the eigenvalues of ρ and ρ' in decreasing order, i.e., $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$, ρ is said to be more mixed (or disordered) than ρ' if

$$\mathcal{S}_i = \sum_{j=1}^{i} p_j \leqslant \mathcal{S}_i' = \sum_{j=1}^{i} p_j', \quad i=1,\ldots,n-1, \quad (5)$$

i.e., if $p_1 \leqslant p_1'$, $p_1 + p_2 \leqslant p_1' + p_2'$, etc. (for $i=n$, $\mathcal{S}_n = \mathcal{S}_n' = 1$). Mathematically, this states that the set of probabilities $(p_1,\ldots,p_n)$ is majorized by $(p_1',\ldots,p_n')$. Equation (4) can be immediately derived by writing $p_i = \mathcal{S}_i - \mathcal{S}_{i-1}$ in Eq. (1), with $\mathcal{S}_0 = 0$. $S_f(\rho)$ is then a decreasing function of $\mathcal{S}_i$ for $1 \leqslant i \leqslant n-1$, as $\partial S_f / \partial \mathcal{S}_i = f'(p_i) - f'(p_{i+1}) \leqslant 0$ if $p_i \geqslant p_{i+1}$ and f is concave [Eq. (4) follows then from the mean value theorem; note that the allowed values of $\mathcal{S}_i$ form a convex set defined by $\mathcal{S}_i \leqslant \mathcal{S}_{i+1}$, $\mathcal{S}_i \geqslant (\mathcal{S}_{i-1} + \mathcal{S}_{i+1})/2$, with $\mathcal{S}_0 = 0$, $\mathcal{S}_n = 1$].

Moreover, it can be shown [17] that ρ is more mixed than ρ' if and only if $\mathrm{Tr} f(\rho) \geqslant \mathrm{Tr} f(\rho')$ for any concave f, i.e., if and only if Eq. (4) holds $\forall f$ of the previous form [the conditions $f(0)=f(1)=0$ fix just an arbitrary linear term $ap+b$ that can be added to f without affecting concavity or Eq. (4)]. If the dimensions of ρ and ρ' differ, we may apply the same definition of "more mixed" by adding zero eigenvalues to the density with the smallest dimension, which leaves S_f unchanged.

Let us consider now a system composed of two subsystems A and B. The quantity

$$S_f^A(\rho) \equiv S_f(\rho) - S_f(\rho_A) = \mathrm{Tr} f(\rho) - \mathrm{Tr}_A f(\rho_A), \quad (6)$$

where $\rho_A = \mathrm{Tr}_B \rho$ is the reduced density matrix of system A and $\mathrm{Tr} = \mathrm{Tr}_A \mathrm{Tr}_B$, plays the role of a *conditional* entropy. In the von Neumann case, Eq. (6) becomes the usual conditional entropy [17],

$$S_f^A(\rho) = S(B|A) = -\mathrm{Tr}\,\rho[\ln\rho - \ln\rho_A \otimes I_B],$$

whereas, in the Tsallis case, it is proportional to the q-conditional entropy defined in [11,12], $S_q(B|A) = S_f^A(\rho)/\mathrm{Tr}\,\rho_A^q$.

For a *discrete classical system* described by a joint probability distribution p_{ij}, Eq. (6) is always non-negative, i.e.,

$$\sum_{i,j} f(p_{ij}) - \sum_i f(p_i) \geqslant 0, \quad p_i = \sum_j p_{ij}, \quad (7)$$

since for any concave f satisfying $f(0)=0$ we have $f(p+q) \leqslant f(p)+f(q)$ if $p \geqslant 0$, $q \geqslant 0$ (it may also be seen that the set of probabilities $\{p_{ij}\}$ is more mixed than $\{p_i\}$). This implies that $S_f^A(\rho) \geqslant 0$ for any uncorrelated density $\rho = \rho_A \otimes \rho_B$ (i.e., $p_{ij} = p_i^A p_j^B$) as well as for any density diagonal in a basis of product states ($\rho = \Sigma_{i,j} p_{ij} |i_A j_B\rangle\langle i_A j_B|$). Nevertheless, in the general quantum case, $S_f^A(\rho)$ may of course be negative. In particular, for a pure state $\rho = |\Psi\rangle\langle\Psi|$, $S_f(\rho) = 0$ and the positive eigenvalues of ρ_A and ρ_B are identical [17], whence

$$S_f^A(\rho) = -S_f(\rho_A) = -S_f(\rho_B) \leqslant 0. \quad (8)$$

For $f(p) = -p \log_2 p$, this is just the usual definition of the entanglement of a pure state $|\Psi\rangle$ [22,23].

Negative values of $S_f^A(\rho)$ are then indicative of distinctive quantum correlations. In particular, for the case (3) it has been conjectured [11–13] that the sign of the difference (6) may provide a criterion for determining the *separability* of ρ [13]. Let us recall that a mixed state ρ is separable (or classically correlated) if and only if it can be written as a convex combination of uncorrelated densities [24],

$$\rho = \sum_\alpha \omega_\alpha \rho_A^\alpha \otimes \rho_B^\alpha, \quad 0 \leqslant \omega_\alpha \leqslant 1, \quad (9)$$

with $\Sigma_\alpha \omega_\alpha = 1$. Otherwise it is called *entangled* or *inseparable*. For the Tsallis case, it has been shown [11,12] that the criterion $S_f^A(\rho) \geqslant 0$ leads, for $q \to \infty$, to the necessary and sufficient condition for separability for some important classes of states, like Werner states. Nevertheless, we will show here that this does not hold in general. In particular, for an entangled state $S_f^A(\rho)$ and $S_f^B(\rho)$ may in fact both be positive *for* any concave f (including the $q \to \infty$ limit in the Tsallis case), indicating that entanglement cannot always be detected by such entropic criteria (or, in general, by information based on the eigenvalues of ρ and $\rho_{A,B}$ alone). This may occur already for a two-qubit system, where the Peres necessary criterion for separability [18] is known to be sufficient [19], so that the entropic criterion is here weaker than the Peres criterion.

Let us first show that Eq. (6) is indeed positive for any separable ρ. A fundamental theorem demonstrated in [25] states that if ρ is separable, then ρ is more mixed than ρ_A and ρ_B (disorder criterion for separability). Hence, Eq. (4) implies that if ρ is separable, then

$$S_f^A(\rho) \geqslant 0, \quad (10)$$

and similarly $S_f^B(\rho) \geqslant 0$, for any concave f [satisfying $f(0) = 0$]. This is in fact an equivalent entropic formulation of the disorder criterion. For a separable state, Eq. (10) will therefore hold $\forall q > 0$ in the case (3), implying $\mathrm{Tr}\,\rho^q - \mathrm{Tr}_A \rho_A^q \leqslant 0$ ($\geqslant 0$) if $q > 1$ ($0 < q < 1$). Note that this entails $S_\alpha(\rho) \geqslant S_\alpha(\rho_A) \forall \alpha > 0$, where $S_\alpha(\rho) = (1-\alpha)^{-1} \ln \mathrm{Tr}\,\rho^\alpha$ is the *Rényi* entropy [26,5] [which is additive but not of the form (1), and approaches the von Neumann entropy for $\alpha \to 1$]. The disorder criterion is, however, not sufficient [25], so that Eq. (10) provides in general only a necessary test for separability, as will be explicitly seen below.

For a system of two qubits, Eq. (10) is actually an immediate consequence of the more obvious fact that for any separable state,

$$p_1 \leqslant p_1^A, \quad (11)$$

where p_1 (p_1^A) denotes the largest eigenvalue of ρ (ρ_A). This is so because the difference

$$\rho_d = \rho_A \otimes I_B - \rho = \sum_\alpha \omega_\alpha \rho_A^\alpha \otimes (I_B - \rho_B^\alpha) \quad (12)$$

is a non-negative operator if all $\omega_\alpha \geq 0$ [27]. Hence, denoting by $|i\rangle$ any eigenstate of ρ, we have

$$0 \leq \langle i|\rho_d|i\rangle = \langle i|\rho_A \otimes I_B|i\rangle - p_i \leq p_1^A - p_i, \qquad (13)$$

since $\langle i|\rho_A \otimes I_B|i\rangle \leq \langle 1_A j_B|\rho_A \otimes I_B|1_A j_B\rangle = p_1^A$, where $\rho_A|1_A\rangle = p_1^A|1_A\rangle$ and $|j_B\rangle$ is any state of B. For a two-qubit system, Eq. (11) already implies that ρ is more mixed than ρ_A: $\Sigma_{j=1}^i p_j \leq p_1^A + p_2^A = 1$ for $i = 2,3,4$.

There are two important remarks to make here. First, if $p_1 > p_1^A$, the state is certainly entangled, but ρ_A is not necessarily more mixed than ρ, entailing that $S_f^A(\rho)$ is not necessarily negative for any f. Nevertheless, in the Tsallis case, as well as for any set of entropic functions

$$f(p) = k[p - g_q(p)], \qquad (14)$$

where $k > 0$ and $g_q(p)$ is a convex increasing function satisfying $g_q(0) = 0$, $g_q(1) = 1$, and

$$\lim_{q \to \infty} g_q(p')/g_q(p) = 0 \text{ if } p' < p, \qquad (15)$$

$S_f(\rho)$ will be a decreasing function of the largest eigenvalue p_1 for sufficiently large q and finite dimension [$S_f(\rho) \approx k(1 - d_1 g_q(p_1))$ in this limit, with d_1 the multiplicity of p_1]. Hence, if $p_1 > p_1^A$, $S_f^A(\rho)$ will become negative for sufficiently large q, and the entropic criterion will be able to detect entanglement. In other words, for $q \to \infty$, $S_f^A(\rho) < 0$ if and only if $p_1 > p_1^A$, which is a sufficient condition for inseparability. Note that Eq. (3) is of the form (14) for $q > 1$ and satisfies Eq. (15). Another example is [15]

$$f(p) = \frac{1}{q}\left[p - \frac{e^{qp} - 1}{e^q - 1}\right], \qquad (16)$$

which is concave $\forall\, q$, approaches $\frac{1}{2}p(1-p)$ for $q \to 0$ [$q = 2$ case in Eq. (3)] and is of the form (14) for $q > 0$.

Nonetheless, and this is the second important remark, there are entangled states for which $p_1 \leq p_1^A$ and p_1^B, i.e., for which the greatest eigenvalue of ρ remains smaller than that of ρ_A and ρ_B. This may occur already for a system of two qubits, in which case ρ will remain more mixed than ρ_A and ρ_B, and $S_f^A(\rho), S_f^B(\rho)$ will both be non-negative for any concave f. This type of entanglement will therefore not be detected by the previous entropic criterion.

An example is the state considered in [18],

$$\rho = x|\Psi_0\rangle\langle\Psi_0| + (1-x)|\uparrow\uparrow\rangle\langle\uparrow\uparrow|, \quad 0 \leq x \leq 1, \quad (17)$$

where $|\Psi_0\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ is the singlet (a maximally entangled state) and $|\uparrow\uparrow\rangle$ a maximally polarized separable state. As shown in [18], the Peres criterion determines that this state is entangled $\forall\, x > 0$: the partial transpose of ρ (defined as the transposition with respect to the indices of system A), which is still a density operator if ρ is separable, has always a negative eigenvalue for $x > 0$, namely, $\sigma_1 = \frac{1}{2}[1 - x - \sqrt{1 - 2x(1-x)}]$ [$\sigma_1 = -x^2/4 + O(x^3)$ for $x \to 0$].
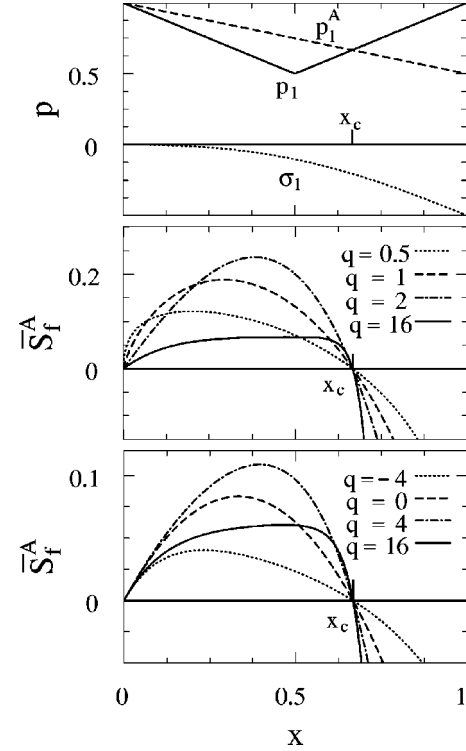

FIG. 1. Top: The largest eigenvalue p_1 of ρ and p_1^A of ρ_A, for the density (17), as a function of the parameter x. The dotted line depicts the lowest eigenvalue σ_1 of the partial transpose of ρ. Center: The normalized entropic difference (18) for the Tsallis case (3), at the indicated values of q. The curve for $q = 1$ corresponds to the von Neumann entropy, in which case $\overline{S}_f^A = S(B|A)$. Bottom: The same quantity for the entropic function (16). The curve for $q = 0$ depicts the limit $\overline{S}_f^A = (1/2)\mathrm{Tr}[\rho_A^2 - \rho^2]$. The point where $p_1 = p_1^A$ is indicated by x_c. Both x and the quantities plotted are dimensionless.

However, as the eigenvalues of ρ are $(x, 1-x, 0, 0)$, and those of ρ_A and ρ_B are $(1 - x/2, x/2)$, the greatest eigenvalue of ρ ($p_1 = x$ for $x > \frac{1}{2}$) is greater than that of ρ_A ($p_1^A = 1 - x/2$) only for $x > x_c = 2/3$ (Fig. 1). Hence, for $0 < x < 2/3$, entanglement will not be detected by $S_f^{A,B}(\rho)$, for any f. This can also be directly seen from the explicit expression

$$S_f^A(x) = f(x) + f(1-x) - [f(x/2) + f(1 - x/2)].$$

Since for a two state system, the entropy $f(p) + f(1-p)$ is a decreasing function of the largest eigenvalue [$f'(p) - f'(1-p) < 0$ for $p > 1/2$ and f concave], in this case $S_f^A(\rho) < 0$ if and only if $p_1 > p_1^A$, i.e., $S_f^A(x) < 0$ if and only if $x > 2/3$, for any f. The sign of $S_f^A(x)$ is independent of the choice of entropic function f in this example, i.e., independent of q in the Tsallis case or in Eq. (16), as shown in Fig. 1. For normalization purposes, we have plotted the quantity

$$\overline{S}_f^A(\rho) = S_f^A(\rho)/\mathrm{Tr}\, g_q(\rho_A), \qquad (18)$$

where $g_q(p) = p^q$ in the Tsallis case (3) [so that $\overline{S}_f^A(\rho) = S_q(B|A)$] and $g_q(p) = (e^{qp} - 1)/(e^q - 1)$ for Eq. (16).

This situation is actually not very special. Consider for instance the more general state

$$\rho = x|\Psi_0\rangle\langle\Psi_0| + (1-x)|uv\rangle\langle uv|, \quad (19)$$

where $|uv\rangle = |u\rangle_A|v\rangle_B$ is an arbitrary separable pure state of the two qubits. This state is again entangled $\forall \, x > 0$, since the partial transpose of ρ has a negative eigenvalue

$$\sigma_1 = \tfrac{1}{2}[1 - x - \sqrt{1 - 2x(1-x)r}], \quad r = |\langle u|v\rangle|^2,$$

with $\sigma_1 = -x(1-r)/2 + O(x^2)$ for $x \to 0$. On the other hand, the eigenvalues of ρ are

$$(\tfrac{1}{2}(1+z), \tfrac{1}{2}(1-z), 0, 0), \quad z = \sqrt{1 - 2x(1-x)(1+r)},$$

while those of ρ_A, ρ_B are again $(1 - x/2, x/2)$. Hence, $p_1 = (1+z)/2$, $p_1^A = (1 - x/2)$, and $p_1 > p_1^A$ only for

$$x > x_c = 2r/(1 + 2r) .$$

Thus, $S_f^A(\rho) < 0$ if and only if $x_c < x < 1$, for any concave f. Again, the entropic criterion fails to detect entanglement for $0 < x < x_c$. For $r = 1$, we recover the results of the previous example, whereas for $r = 0$, i.e., $|uv\rangle = |\uparrow\downarrow\rangle$, $\sigma_1 = -x/2$, and $x_c = 0$, so that $S_f^A(\rho) < 0 \, \forall \, x > 0$. This is the only case where the entropic criterion predicts the full interval of inseparability.

Let us still consider the example of Refs. [28,18],

$$\rho = x|\Psi\rangle\langle\Psi| + (1-x)(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)/2, \quad (20)$$

with $|\Psi\rangle = a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle$, $|a|^2 + |b|^2 = 1$. As shown in [18], this state is entangled just for

$$x > x_e = (1 + 2|ab|)^{-1},$$

since the lowest eigenvalue of the partial transpose is $\sigma_1 = [1 - x(1 + 2|ab|)]/2$ for $x > [2(1+|ab|) - \|a|^2 - |b|^2\|]^{-1}$. However, the eigenvalues of ρ are $(x, (1-x)/2, (1-x)/2, 0)$ while those of ρ_A, ρ_B are $[1 \pm x(|b|^2 - |a|^2)]/2$. The largest eigenvalue of ρ ($p_1 = x$ for $x > 1/3$) is greater than that of ρ_A $[p_1^A = (1 + x\|a|^2 - |b|^2\|)/2]$ only for

$$x > x_c = (2 - \|a|^2 - |b|^2\|)^{-1}.$$

But $x_c \geq x_e$, with $x_c = x_e$ just for $|a| = |b|$ or in the trivial separable cases $b = 0$ or $a = 0$. Hence, if $|a| \neq |b|$ and $ab \neq 0$, $S_f^A(\rho)$ will not detect entanglement for $x_e < x < x_c$. Note also that, for $x > x_c$, $S_f^A(\rho)$ is in this case not necessarily negative for any f, but will become negative for sufficiently large q in the Tsallis case or in Eq. (14) or Eq. (16), as shown in Fig. 2. The value of x where $S_f^A(\rho) = 0$ actually converges exponentially fast to x_c for $q \to \infty$ in Eq. (3) or Eq. (16). This will occur whenever the degeneracies of p_1 and p_1^A coincide.

The entropic criterion will provide, however, necessary and sufficient conditions for separability for any density ρ diagonal in the Bell basis [7], i.e., the basis of maximally
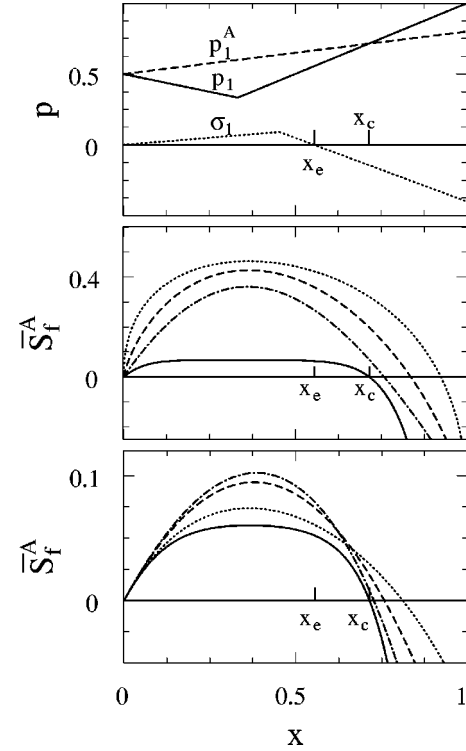

FIG. 2. Same details as Fig. 1 for the density (20) with $|a|^2 = 4/5$. The values of q for the different lines in the center and bottom panels are the same as those of Fig. 1.

entangled states $|\Psi_0\rangle, |\Psi_1\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}, |\Psi_{2,3}\rangle = (|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)/\sqrt{2}$. In such a case,

$$\rho = \sum_{i=0}^{3} q_i |\Psi_i\rangle\langle\Psi_i| \quad (21)$$

is known to be entangled if and only if $p_1 > 1/2$ [5], where $p_1 = \text{Max}[\{q_i\}]$ is the largest eigenvalue of ρ. This may be obtained directly with the Peres criterion, as the partial transpose of ρ has eigenvalues $\tfrac{1}{2} - q_i$. Now, for any pure Bell state $|\Psi_i\rangle\langle\Psi_i|$, the reduced density matrices are fully mixed, with eigenvalues $(\tfrac{1}{2}, \tfrac{1}{2})$, so that the same will occur for any state of the form (21). The condition $p_1 \leq p_1^A$ then becomes equivalent, for *any* state (21), to $p_1 \leq 1/2$, i.e., to the necessary and sufficient condition for separability. The entropic criterion will therefore always lead to this condition for $q \to \infty$ in Eq. (14).

This explains why the entropic criterion for $q \to \infty$ yields the necessary and sufficient condition for separability for Werner-Popescu states [24,29],

$$\rho = x|\Psi_0\rangle\langle\Psi_0| + (1-x)I/4, \quad (22)$$

where $I = \Sigma_{i=0}^{3}|\Psi_i\rangle\langle\Psi_i| = I_A \otimes I_B$ is the identity. The eigenvalues of ρ are $p_1 = (1 + 3x)/4$ and $(1-x)/4$ (threefold degenerate), and the equation $p_1 \leq \tfrac{1}{2}$ yields $x \leq \tfrac{1}{3}$, the necessary and sufficient condition [18,30]. Accordingly, for $x > x_c = \tfrac{1}{3}$, $S_f^A(\rho)$ will become negative for sufficiently large q. The root x_r where $S_f^A(\rho) = 0$ will approach x_c for $q \to \infty$, as
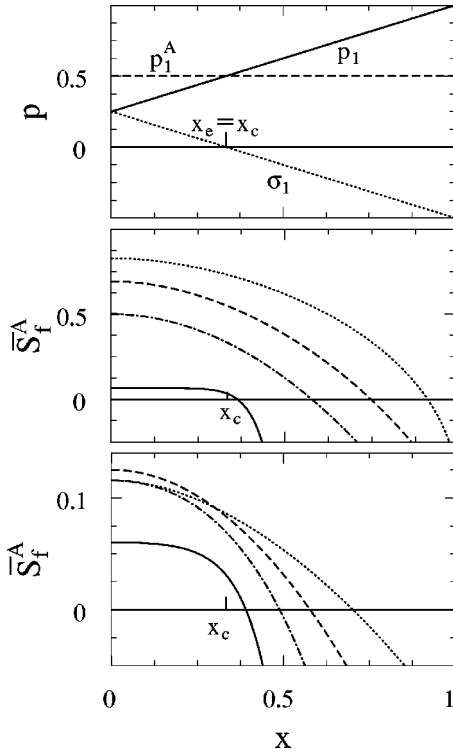
FIG. 3. Same details as Fig. 1 for the density (22). The values of q for the different lines in the center and bottom panels are the same as those of Fig. 1.

seen in Fig. 3, although the convergence is in this case less rapid due to the different degeneracies of p_1 and p_1^A. For large q,

$$x_r \approx \frac{1}{3} + \frac{2\gamma \ln 2}{3q}, \qquad (23)$$

where $\gamma = 1$ for the Tsallis case and $\gamma = 2$ for Eq. (16).

Note that the state (20) also becomes of the form (21) for $a = \pm b$ (where the entropic criterion works), as in this case $|\Psi\rangle = |\Psi_0\rangle$ or $|\Psi_1\rangle$ [the remaining term in Eq. (20) is proportional to $\Sigma_{i=2,3}|\Psi_i\rangle\langle\Psi_i|$].

Similar considerations hold for Werner-like states for n qubits [31],

$$\rho = x|\Psi\rangle\langle\Psi| + (1-x)I/d^n, \qquad (24)$$

where $d = 2$, $|\Psi\rangle = (|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle)/\sqrt{2}$ is a maximally entangled state (a Greenberger-Horne-Zeilinger state [32]) and I the identity. The eigenvalues of Eq. (24) are $p_1 = x + (1-x)/d^n$ and $(1-x)/d^n$ [(d^n-1)-fold degenerate]. Now, for a subsystem A_m with m qubits $(1\leq m\leq n-1)$, the reduced density matrix ρ_m can be easily shown to have eigenvalues $p_1^m = x/d + (1-x)/d^m$ (d-fold degenerate) and $(1-x)/d^m[(d^m-d)$-fold degenerate]. The necessary condition for separability between the m and $n-m$ subsystems, $p_1 \leq p_1^m$, leads to

$$x \leq x_c^m \equiv \left[1 + \frac{d^{n-1}(d-1)}{d^{n-m}-1}\right]^{-1}, \qquad (25)$$

which is a decreasing function of m. The most stringent condition is then obtained for $m = n-1$, i.e., $x \leq (1+d^{n-1})^{-1}$, which, according to Refs. [31,33], is just the necessary and sufficient condition for full separability. The entropic criterion $S_f^{A_m}(\rho) \geq 0$ will then lead to Eq. (25) for $q\to\infty$ (as shown in [11] for the Tsallis case). If d is an arbitrary integer (≥ 2), the previous discussion and expressions are actually also valid for n *qudits* (nd-dimensional systems), when $|\Psi\rangle$ is the fully entangled state $\Sigma_{k=0}^{d-1}|k\rangle_1\cdots|k\rangle_n/\sqrt{d}$ [33].

It should be stressed that for bipartite systems with subsystem dimension $d>2$, the first violation of the majorization relation between ρ and ρ_A in an entangled state may also occur for $i>1$ in Eq. (5). For instance, let us briefly discuss the example given in [13], dealing with a system of two identical harmonic oscillators. It was shown that for certain densities $S_f^A(\rho)$ becomes negative just in a finite interval of q values in the Tsallis case, remaining positive for arbitrary large q. This indicates that ρ is not more mixed than ρ_A, and hence entangled, but still has $p_1 < p_1^A$, which ensures that $S_f^A(\rho)$ remains positive for $q\to\infty$. The first violation of the inequalities (5) is therefore taking place for $i>1$ (we have verified that this occurred for $i=2$). Nevertheless, it should be remarked that in such situations, if $\mathcal{S}_i$ is only slightly larger than $\mathcal{S}_i^A$ and $i>1$, $S_f^A(\rho)$ may remain positive for all $q>0$ in the case (3), and is then unable to detect entanglement. The same happens with the entropy (16).

In summary, we have shown that the generalized entropic criterion $S_f^A(\rho) = S_f(\rho) - S_f(\rho_A) \geq 0$ constitutes, for *any* concave entropic function f, a necessary condition for separability. For $q\to\infty$ in Eq. (3), or in general Eq. (14), it becomes equivalent to the condition (11) between the largest eigenvalues of ρ and ρ_A. Nonetheless, the entropic criterion is not a sufficient one in general. We have provided examples of entangled densities of two qubits where $p_1 < p_1^A$, in which case ρ remains more mixed than ρ_A, implying $S_f^A(\rho) \geq 0$ for any choice of entropic function f. However, the condition $p_1 \leq p_1^A$ becomes sufficient in some important cases, which include any density diagonal in the Bell basis in a two-qubit system, and also Werner-like states in n-qubit (or qudit) systems. In these cases the inequality $S_f^A(\rho) \geq 0$ will lead, for $q\to\infty$ in Eqs. (3) or Eq. (14), to the necessary and sufficient condition for separability.

The condition $S_f^A(\rho) \geq 0$ for any concave entropic function f is equivalent to the requirement that ρ be more mixed than ρ_A, a general necessary condition for separability [25]. Let us remark that this requirement is stronger than the condition $S_f^A(\rho) \geq 0$ $\forall$ $q>0$ in Eq. (3) [or $\forall$ q in Eq. (16)]. Other families of concave entropic functions are required in general to detect that ρ is not more mixed than ρ_A when the first violation of Eqs. (5) occurs for $i>1$, although in many cases this can also be seen with the entropies (3) or (16). In such situations $S_f^A(\rho)$ will remain positive for $q\to\infty$ but may become negative at finite values of q.

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