

Characterization of correlations in two-fermion systems based on measurement induced disturbances

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Abstract. We introduce an approach for the characterization of quantum correlations in two-fermion systems based upon the state disturbances generated by the measurement of “local” observables (that is, quantum observables represented by one-body operators). This approach leads to a concept of quantum correlations in systems of identical fermions different from entanglement.

1 Introduction

Considerable attention has, recently, been devoted to the applications of tools and concepts from quantum information theory to the study of correlations in systems of identical fermions [1–16]. A key role in these developments was played by the work of Ghirardi and collaborators [1,2], who advanced a clear and physically motivated formulation of the concept of separability for systems of identical particles. Most of the research conducted on quantum correlations in fermion systems has focused on the analysis of quantum entanglement. However, it is well-known that the concept of entanglement does not capture all the relevant, information-theoretical aspects of the quantum correlations exhibited by composite systems. Indeed, as was established in a pioneering work by Ollivier and Zurek [17], even separable mixed states can be endowed with correlations exhibiting non-trivial quantum features. In the case of systems consisting of distinguishable parts, various measures have been advanced to characterize quantitatively the different ways (besides entanglement) in which quantum correlations can manifest themselves [17–31]. Prominent among these are quantum discord and the measures of correlations based upon the disturbances of quantum states due to local measurements proposed by Luo [21,22], Li and Luo [23], and by SaiToh and collaborators [28,29]. In the case of pure states, these measures reduce to quantum entanglement. In the case of mixed states, however, these measures describe physical properties of quantum states that are different from entanglement. A thermodynamical approach of quantum correlations has been extensively studied by

Oppenheim et al. [32] and correlations related to non-locality have been explored in references [33].

Quantum discord, introduced by Ollivier and Zurek [17], is based on the difference between two quantum versions of classically equivalent expressions of the mutual information. An alternative but closely related quantity has been derived by Henderson and Vedral [18]. Quantum discord characterizes in a bipartite system the quantumness of correlations, quantifying the minimum change in the state and in the information of one of the parts of the system induced by a measurement on the other part. This measure has been calculated for different families of quantum states and compared with entanglement [20,22,23]. Several modified versions and generalizations of quantum discord have been advanced [24,25,27].

The measurement induced disturbance notion of quantum correlations introduced by Luo in [21] exhibits two attractive features. First, it admits an intuitive straightforward interpretation in terms of the basic idea that in classical scenarios one can perform a measurement upon a system without disturbing it. On the contrary, in the quantum domain measurements usually produce disturbances on the systems being measured. Luo applies these concepts to the analysis of correlations in bipartite systems. According to this approach, a bipartite system is endowed only with classical correlations if there exist local measurements on both subsystems that can be conducted without disturbing the global state of the composite system. If this is not the case, the (minimum) magnitude of the disturbance due to local measurements can be regarded as a quantitative measure of the quantumness of the correlations exhibited by the system. The second advantage of Luo’s proposal is that this measure of the quantum character of correlations is sometimes easier to

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compute than alternative measures, such as quantum discord. It is important to emphasize that both quantum discord and the notion of quantum correlations based upon measurement induced disturbances determine the same family of classical states of a quantum bipartite system. These states are those described by density matrices that are diagonal in a product basis $\{|i\rangle|j\rangle, i = 1, \dots, N_1; j = 1, \dots, N_2\}$, where $\{|i\rangle, i = 1, \dots, N_1\}$ and $\{|j\rangle, j = 1, \dots, N_2\}$ are orthonormal bases associated with the two subsystems, $N_{1,2}$ being the dimensions of the concomitant Hilbert spaces. Indeed, it is shown in reference [21] that a quantum state ρ of a bipartite system is undisturbed by appropriate (un-read) local measurements if and only if ρ is diagonal in a product basis. This suggests a natural way of assessing the “amount of quantumness” exhibited by the correlations present in a quantum state ρ , by recourse to the minimum possible “distance” between ρ and the disturbed state $\Pi(\rho)$ resulting from a local measurement [21].

There exist possible implementations of quantum computation that could take advantage of quantum correlations different from entanglement. Indeed, these correlations seem to play a role in the exponential speedup exhibited by the scheme of deterministic quantum computation with one qubit (DQC1) introduced in reference [34], as a model of mixed state quantum computation [35] (see, however, Ref. [25]). Discord is also relevant in connection with other quantum information protocols such as, for example, assisted optimal state discrimination [26].

The purpose of the present work is to investigate manifestations of the quantum correlations in fermion systems that do not correspond to quantum entanglement, focusing on the measurement induced disturbance approach. Quantum discord does not seem to admit a counterpart in the case of systems of identical fermions, because its definition involves a strong asymmetry between the constituting parts of the composite system under consideration. On the other hand, as we shall see, the measures of correlations based upon measurement induced disturbances do admit a natural generalization to the fermion systems.

2 Preliminaries

2.1 Entanglement in systems of identical fermions

A pure state of a composite system consisting of two identical fermions is regarded as separable (that is, non-entangled) if and only if it can be described as a single Slater determinant. Pure states like this are said to have Slater rank 1. Here, by “entanglement” in fermion systems we mean entanglement between particles (as opposed to entanglement between modes). Mixed separable states are those that can be expressed as a statistical mixture of pure states of Slater rank 1. A separable pure state of two identical fermions can be obtained by antisymmetrizing a product state $|\alpha_1\rangle \otimes |\alpha_2\rangle$,

$$|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} [|\alpha_1\rangle \otimes |\alpha_2\rangle - |\alpha_2\rangle \otimes |\alpha_1\rangle], \quad (1)$$

where $|\alpha_1\rangle, |\alpha_2\rangle$ are two orthogonal and normalized single-particle states.

It is useful to regard a system constituted by identical fermions with a single-particle Hilbert space of dimension $2k$ (with $k \geq 2$) as a system consisting of spin- s particles, with $s = (2k - 1)/2$ [10,12,13]. An orthonormal basis $\{|i\rangle, i = 1, \dots, 2k\}$ of the single-particle Hilbert space can then be identified with the states $|s, m_s\rangle$, with $m_s = s - i + 1, i = 1, \dots, 2k$. These states can be denoted by the shorthand notation $\{|m_s\rangle, m_s = -s, \dots, s\}$, because each of the examples discussed here corresponds to a given value of k (and, therefore, s). Within this angular momentum representation, the antisymmetric joint eigenstates $\{|j, m\rangle, -j \leq m \leq j, 0 \leq j \leq 2s\}$ of the total angular momentum operators J_z and J^2 constitute a natural basis for the Hilbert space associated with a system of two identical fermions. The antisymmetric states $|j, m\rangle$ are those characterized by an even value of the quantum number j [36,37]. In what follows the notation $|j, m\rangle$ is always meant to refer to the angular momentum representation.

The following is a list of the antisymmetric total angular momentum eigenstates for two fermions of spin- $\frac{3}{2}$ with the value for the concurrence (see Eq. (5)) indicated on the right:

	C
$ 2, 2\rangle = -\frac{1}{\sqrt{2}} \frac{1}{2}\frac{3}{2}\rangle + \frac{1}{\sqrt{2}} \frac{3}{2}\frac{1}{2}\rangle$	0
$ 2, 1\rangle = -\frac{1}{\sqrt{2}} -\frac{1}{2}\frac{3}{2}\rangle + \frac{1}{\sqrt{2}} \frac{3}{2}-\frac{1}{2}\rangle$	0
$ 2, 0\rangle = -\frac{1}{2} -\frac{3}{2}\frac{3}{2}\rangle - \frac{1}{2} -\frac{1}{2}\frac{1}{2}\rangle + \frac{1}{2} \frac{1}{2}-\frac{1}{2}\rangle + \frac{1}{2} \frac{3}{2}-\frac{3}{2}\rangle$	1
$ 2, -1\rangle = -\frac{1}{\sqrt{2}} -\frac{3}{2}\frac{1}{2}\rangle + \frac{1}{\sqrt{2}} \frac{1}{2}-\frac{3}{2}\rangle$	0
$ 2, -2\rangle = -\frac{1}{\sqrt{2}} -\frac{3}{2}-\frac{1}{2}\rangle + \frac{1}{\sqrt{2}} -\frac{1}{2}-\frac{3}{2}\rangle$	0
$ 0, 0\rangle = -\frac{1}{2} -\frac{3}{2}\frac{3}{2}\rangle + \frac{1}{2} -\frac{1}{2}\frac{1}{2}\rangle - \frac{1}{2} \frac{1}{2}-\frac{1}{2}\rangle + \frac{1}{2} \frac{3}{2}-\frac{3}{2}\rangle$	1

Notice that the states $|0, 0\rangle$ and $|2, 0\rangle$ are maximally entangled, while all the other states of two spin- $\frac{3}{2}$ fermions listed in the above table correspond to single Slater determinants and, therefore, have zero entanglement.

Necessary and sufficient separability criteria for pure states $|\Psi\rangle$ of two identical fermions can be formulated in terms of appropriate entropic measures evaluated on the single-particle reduced density matrix

$$\rho_r = \text{Tr}_2(|\Psi\rangle\langle\Psi|). \quad (2)$$

A pure state $|\Psi\rangle$ of two identical fermions is separable iff any of the following two conditions hold (see Ref. [11] and references therein),

$$\begin{aligned} S[\rho_r] &= 1, \\ \text{Tr}(\rho_r^2) &= \frac{1}{2}, \end{aligned} \quad (3)$$

where $S[\rho_r] = -\text{Tr}(\rho_r \log \rho_r)$ is the von Neumann entropy of ρ_r . We use \log to denote logarithm of base 2 throughout the paper. The above separability criteria naturally lead to the following quantitative measures of entanglement for

pure states of the fermion system,

$$\begin{aligned} E_{vN}[|\Psi\rangle] &= S[\rho_r] - 1, \\ E_L[|\Psi\rangle] &= \frac{1}{2} - \text{Tr}(\rho_r^2). \end{aligned} \quad (4)$$

The above two quantities are non-negative and vanish iff $|\Psi\rangle$ has Slater rank 1. They provide quantitative indicators of how strongly the separability conditions (3) are violated and, consequently, of how entangled is the state $|\Psi\rangle$ under consideration.

The development of entanglement criteria, or of practical entanglement measures or indicators for mixed states of systems of two fermions remains a largely unexplored field. A closed analytical expression for the amount of entanglement exhibited by a general (pure or mixed) state of a system of two fermions is known only for the case of fermions described by a single-particle Hilbert space of dimension four. This is, by the way, the fermion system of lowest dimensionality exhibiting the phenomenon of entanglement.

In order to compute the amount of entanglement, we have an analytical expression for the concurrence of general states of two fermions only for systems with a single-particle Hilbert space of dimension four [10],

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6\}, \quad (5)$$

where the λ_i 's are, in decreasing order, the square roots of the eigenvalues of $\rho\tilde{\rho}$ with $\tilde{\rho} = \mathbb{D}\rho\mathbb{D}^{-1}$, where \mathbb{D} is given by,

$$\mathbb{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \kappa, \quad (6)$$

κ is the complex conjugation operator and \mathbb{D} is expressed with respect to the total angular momentum basis in the following order $|2, 2\rangle, |2, 1\rangle, |2, 0\rangle, |2, -1\rangle, |2, -2\rangle$, and $i|0, 0\rangle$.

2.2 Majorization

Suppose $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ are two d -dimensional vectors with real, non-negative, and summing up to one components. $x^\downarrow = (x_1^\downarrow, \dots, x_d^\downarrow)$ denotes the vector with its components rearranged into decreasing order, $x_1^\downarrow \geq x_2^\downarrow \geq \dots \geq x_d^\downarrow$. Then, x is majorized by y , $x \prec y$, if

$$\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow, \quad (7)$$

for $k = 1, \dots, d-1$ and with equality when $k = d$. This relation is connected with disorder [38] and it can be shown that majorization is a notion of disorder stronger than entropy [39,40] in the sense that if $x \prec y$ then it follows that $H(x) \geq H(y)$.

3 Correlations in fermion systems and measurement induced disturbance

As already mentioned, a pure state of a system of two identical fermions is non-entangled if and only if it can be written as a single Slater determinant,

$$|\psi(1, 2)\rangle = \frac{1}{\sqrt{2}} [|\alpha_1\rangle \otimes |\alpha_2\rangle - |\alpha_2\rangle \otimes |\alpha_1\rangle], \quad (8)$$

where the single-particle states $|\alpha_1\rangle, |\alpha_2\rangle$ are two orthogonal and normalized states. A state like (8) exhibits the ‘‘classical-like’’ feature that both constituents of the composite system possess a complete set of properties [2]. That is, one can objectively say that one particle possesses the complete set of properties associated with the single-particle pure state $|\alpha_1\rangle$ and the other particle possesses the set of properties corresponding to $|\alpha_2\rangle$ (of course, it makes no sense to ask ‘‘which particle possesses which set of properties’’). States having the form (8) are the only pure states of two fermions exhibiting this classical property. Indeed, the possibility of assigning a definite set of properties to each of the two fermions constitutes one of the strong conceptual reasons for regarding the state (8) as non-entangled.

The above discussion naturally leads to the question of how to characterize the set of mixed states that share the ‘‘classical-like’’ features of (8). There are at least two possible ways of extending the above discussion to the case of mixed states of systems of two identical fermions. On the one hand, we can consider the set of mixed states that are expressible as a statistical mixture of a family of pure states, each one being of the form (8). That is, we may consider states of the form,

$$\begin{aligned} \rho_{sep} &= \sum_k \frac{p_k}{2} [|\phi_1^{(k)}\rangle \otimes |\phi_2^{(k)}\rangle - |\phi_2^{(k)}\rangle \otimes |\phi_1^{(k)}\rangle] \\ &\times \left[\langle \phi_1^{(k)} | \otimes \langle \phi_2^{(k)} | - \langle \phi_2^{(k)} | \otimes \langle \phi_1^{(k)} | \right], \end{aligned} \quad (9)$$

where $0 \leq p_k \leq 1$, $\sum_k p_k = 1$, and the single-particle pure states $|\phi_i^{(k)}\rangle$ verify,

$$\langle \phi_i^{(k)} | \phi_j^{(k)} \rangle = \delta_{ij}. \quad (10)$$

Equation (9) represents the standard definition of a non-entangled, or separable, mixed state of two identical fermions. Notice that in equation (9) no special relation between states $|\phi_i^{(k)}\rangle$ with different values of the label k is assumed. In particular, the overlap between two states with different labels k is not necessarily equal to 0 or 1. This, in turn, means that the overlap between two different members of the family of (separable) two-fermion pure states participating in the statistical mixture leading to equation (9) may be non-zero.

The above considerations suggest an alternative, and complementary, way of extending to mixed states the ‘‘classical-like’’ features exhibited by pure states of the form (8). One can consider statistical mixtures of states

like (8) such that for all these states the two (complete) sets of properties associated with the pair of particles belong to the same family \mathcal{F} of mutually exclusive sets of (complete) single-particle properties. This family \mathcal{F} corresponds to an orthonormal basis $\{|\alpha_i\rangle, i = 1, 2, 3, \dots\}$ of the single-particle Hilbert space. Such a state then has the form

$$\rho_{class} = \sum_{i < j} \frac{p_{ij}}{2} [|\alpha_i\rangle \otimes |\alpha_j\rangle - |\alpha_j\rangle \otimes |\alpha_i\rangle] \times [\langle \alpha_i | \otimes \langle \alpha_j | - \langle \alpha_j | \otimes \langle \alpha_i |], \quad (11)$$

with $0 \leq p_{ij} \leq 1$, $\sum_{i < j} p_{ij} = 1$. The density operator (11) is diagonal in an orthonormal basis of the two-fermion state space consisting of all the states of the Slater determinant form, $\frac{1}{\sqrt{2}}(|\alpha_i\rangle \otimes |\alpha_j\rangle - |\alpha_j\rangle \otimes |\alpha_i\rangle)$, $i < j$, that can be constructed with states belonging to the single-particle basis $\{|\alpha_i\rangle\}$. Such a basis of the two-fermion system will be called a ‘‘Slater basis’’. We shall say that this Slater basis is constructed from, or induced or generated by the single-particle orthonormal basis $\{|\alpha_i\rangle\}$. Let us now consider a single-particle non-degenerate observable A_{sp} with eigenbasis $\{|\alpha_i\rangle\}$ and corresponding eigenvalues $\{a_i\}$, $A_{sp} = \sum_i a_i |\alpha_i\rangle \langle \alpha_i|$, and also the two-fermion observable (which we also assume to be non-degenerate)

$$A = A_{sp}^{(1)} \otimes \mathbb{I}^{(2)} + \mathbb{I}^{(1)} \otimes A_{sp}^{(2)}. \quad (12)$$

The two-fermion observable A has as its eigenbasis the Slater basis constructed from the single-particle basis $\{|\alpha_i\rangle\}$, the eigenvalue corresponding to the eigenvector $\frac{1}{\sqrt{2}}(|\alpha_i\rangle \otimes |\alpha_j\rangle - |\alpha_j\rangle \otimes |\alpha_i\rangle)$ being $a_i + a_j$. We shall call the measurement of an observable of the form (12) a ‘‘local’’ measurement. In other words, a local measurement is a measurement in a Slater basis. To each possible outcome of the measurement of A we can associate the projector

$$P_{ij} = \frac{1}{2} (|\alpha_i\rangle \otimes |\alpha_j\rangle - |\alpha_j\rangle \otimes |\alpha_i\rangle) \times (\langle \alpha_i | \otimes \langle \alpha_j | - \langle \alpha_j | \otimes \langle \alpha_i |), \quad i < j. \quad (13)$$

These projectors satisfy,

$$P_{ij} P_{i'j'} = P_{ij} \delta_{ii'} \delta_{jj'}, \quad i < j, \quad i' < j' \\ \sum_{i < j} P_{ij} = \mathbb{I}. \quad (14)$$

The notion of locality for identical fermions considered here is associated with operations or processes that do not involve interaction between the particles constituting the system. An example of a local measurement is given by the measurement of the energy of a system of two non-interacting fermions. The Hamiltonian operator, associated with the energy observable, is then of the form,

$$H = H_{sp}^{(1)} \otimes \mathbb{I}^{(2)} + \mathbb{I}^{(1)} \otimes H_{sp}^{(2)}, \quad (15)$$

where H_{sp} is the single-particle Hamiltonian. Local unitary operations for a system of two identical fermions are

those corresponding to the time evolution operator determined by a Hamiltonian of the form (15),

$$U = U_{sp}^{(1)} \otimes U_{sp}^{(2)}, \quad (16)$$

where $U_{sp} = \exp[-iH_{sp}t/\hbar]$. An essential feature of the concept of separability for fermions advanced by Ghirardi and collaborators [1,2], is that separable states evolve into separable states under local unitary operations of the above form. Moreover, the relevant quantitative measures of entanglement between particles for systems of identical fermions are also invariant under local unitary operations [10]. It is also clear that the two-fermion state resulting from the application of a local unitary operation (16) upon a classically correlated state (11) yields another classically correlated fermionic state.

The process of measurement in quantum mechanics is associated with an alteration of the state. If the two-fermion system is initially in the state ρ , the state immediately after the measurement (and before the observation) is given by

$$\Pi(\rho) = \sum_{i < j} P_{ij} \rho P_{ij}. \quad (17)$$

If the initial state ρ is of the form (11) then one has $\Pi(\rho) = \rho$. In other words, for a state of the form (11) there always exists a local measurement that leaves the state undisturbed. As a particular instance of two-fermion states with this property we have the pure, separable states (8). We then propose to adopt this property as the criterion characterizing two-fermion states (pure or mixed) with minimal quantum correlations, which we shall call ‘‘classically correlated states’’¹. In summary, *a two-fermion state has minimal quantum correlations if there exists a local measurement that leaves the state undisturbed (in the sense that $\Pi(\rho) = \rho$)*. This constitutes an extension to the case of systems of identical fermions of the approach to analyze quantum correlations for distinguishable systems advanced by Luo in [21].

It follows from the above definition of classically correlated two-fermion states that the following statements are equivalent (see Appendix A):

1. The state ρ is classically correlated.
2. There exists a local measurement, with associated projectors P_{ij} (of the form (13)) such that ρ commutes with each P_{ij} .

¹ It is important to mention that there is not a direct correspondence, between states of two fermions and states of two distinguishable systems, that preserves the classical (or not classical) character of the correlations. For instance, let us consider the mixed state $\rho = \frac{1}{3}|0\rangle|0\rangle\langle 0|\langle 0| + \frac{2}{3}|1\rangle|+\rangle\langle 1|\langle +|$ of a system constituted by two distinguishable subsystems, where $|0\rangle, |1\rangle$ are orthonormal states and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The state ρ is not classical. If one tries naively to construct a ‘‘fermionic correlate’’ of this state, by anti-symmetrizing (and normalizing) each of its eigenstates, one obtains the pure state $\frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle)$ which, as a fermionic state, is classically correlated. The concept of classical correlations for systems of identical fermions does not ‘‘boil down’’ to the corresponding concept for systems with distinguishable subsystems.

3. The state ρ can be represented as

$$\rho = \sum_{i < j} \frac{p_{ij}}{2} (|\alpha_i\rangle \otimes |\alpha_j\rangle - |\alpha_j\rangle \otimes |\alpha_i\rangle) \times (\langle \alpha_i| \otimes \langle \alpha_j| - \langle \alpha_j| \otimes \langle \alpha_i|) \quad (18)$$

for some single-particle orthonormal basis $\{|\alpha_i\rangle\}$ and some probability distribution $\{p_{ij}\}$ (normalized as $\sum_{i < j} p_{ij} = 1$).

The above three statements are similar to the ones obtained by Luo for distinguishable subsystems [21]. We see that the structure of these statements is preserved when going from the distinguishable subsystems scenario to the one involving identical fermions, in spite of the fact that the underlying formalism in the latter case (based on Slater determinants) is quite different from the one corresponding to distinguishable particles.

The single-particle reduced density matrix ρ_r (see Eq. (2)) associated with a two-fermion state of the form (18) is given by,

$$\rho_r = \sum_{i < j} \frac{p_{ij}}{2} (|\alpha_i\rangle \langle \alpha_i| + |\alpha_j\rangle \langle \alpha_j|). \quad (19)$$

4 Measure of quantum correlations for two-fermion systems

The above definition of classically correlated two-fermion states (states with minimal quantum correlations) suggests that one adopts as a quantitative measure of quantum correlations of a two-fermion state ρ the minimum “distance” between ρ and the disturbed state $\Pi(\rho)$ arising from an (un-read) local measurement. That is, a measure of the form

$$\xi_D(\rho) = \inf_{\Pi} D(\rho, \Pi(\rho)), \quad (20)$$

where the infimum is taken over all complete local projective measurements and D may be almost any distance or divergence measure for quantum states. As already mentioned in the introduction, a similar proposal was advanced by Luo for treating systems with distinguishable subsystems [21]. To calculate ξ_D from the above definition it is necessary to implement an optimization procedure to determine the local measurement leading to the minimal disturbance, which is in general a very difficult problem. A more tractable approach is given by the expression

$$\xi_D^{sp}(\rho) = D(\rho, \Pi_{sp}(\rho)), \quad (21)$$

where the measurement Π_{sp} is the one induced by the spectral resolution of the single-particle reduced state ρ_r . That is, in (21) we consider a local measurement in the Slater basis constructed from the (single-particle) eigenbasis of ρ_r . The main problem with the measure (21) is that it is not unique when ρ_r has degenerate eigenvalues. This problem obviously disappears if one introduces in (21) a minimization over all the Slater bases induced by an eigenbasis of ρ_r (a similar situation arises in the case

of distinguishable subsystems [30,31], see Appendix C). If we call these bases “local bases”, we can then adopt the measure

$$\xi_D^{\text{local}}(\rho) = \inf_{\text{local bases}} D(\rho, \Pi_{sp}(\rho)). \quad (22)$$

It is clear that a measurement associated with a local basis leaves the single-particle reduced density matrix ρ_r undisturbed.

A convenient way of implementing the above ideas is the one advanced by SaiToh et al. [28] in the case of distinguishable subsystems: we can define as a measure of correlations,

$$\xi(\rho) = \min_{\text{local bases}} S[\Pi(\rho)] - S[\rho]. \quad (23)$$

This is the measure we are going to use in order to characterize the quantum correlations in systems of two identical fermions. Notice that we always have $S[\Pi(\rho)] \geq S[\rho]$ and, consequently, the measure (23) is always a non-negative quantity. In fact, it vanishes if and only if ρ is a classically correlated state.

In order to evaluate (23), we have to determine the local measurement that minimizes $S[\Pi(\rho)]$ under the constraint that ρ_r remains undisturbed (from here on this constraint is always assumed when we discuss optimization processes over the set of local measures or, equivalently, over the set of Slater bases). As we are going to see in the following sections, in many cases this optimization problem can be conveniently tackled using the concept of majorization. Let us consider a local measurement associated with the Slater basis $\{|sl_1\rangle, |sl_2\rangle, \dots\}$. We denote by $\lambda^{(\Pi(\rho))} = \{\langle sl_1|\rho|sl_1\rangle, \dots\}$ the eigenvalues of $\Pi(\rho)$. If we now compare two local measurements, using the majorization technique introduced in Section 2.2, we have that

$$\lambda^{(\Pi(\rho))} \prec \lambda^{(\Pi^*(\rho))} \rightarrow S[\Pi^*(\rho)] \leq S[\Pi(\rho)]. \quad (24)$$

Consequently, if we find a local measurement associated with a Slater basis $\{|sl_i^*\rangle\}$ such that the eigenvalues $\lambda^{(\Pi^*(\rho))}$ satisfy $\lambda^{(\Pi(\rho))} \prec \lambda^{(\Pi^*(\rho))}$ for any other local measurement, then we have that

$$\xi(\rho) = S[\Pi^*(\rho)] - S[\rho]. \quad (25)$$

Summing up, the optimization problem is solved if one finds a local measurement such that the set of eigenvalues $\lambda^{(\Pi^*(\rho))}$ majorizes the set of eigenvalues $\lambda^{(\Pi(\rho))}$ associated with any other local measurement.

5 Pure states of two identical fermions

First, we are going to analyze the quantum correlations exhibited by pure states of a two-fermion system. Now, we are going to evaluate the measure $\xi(\rho)$ defined in (23) on a pure state $\rho = |\psi\rangle\langle\psi|$ of a two-fermion system with a single-particle Hilbert space of dimension $2k$, $k \geq 2$. In order to evaluate $\xi(\rho)$ in this case it will prove convenient to use the fermionic Schmidt decomposition of the

state $|\psi\rangle$. It is always possible to find an orthonormal basis $\{|1\rangle, |2\rangle, \dots, |2k\rangle\}$ of the single-particle Hilbert space (the ‘‘Schmidt basis’’) such that the state $|\psi\rangle$ can be cast as,

$$|\psi\rangle = \sum_{i=1}^k \sqrt{\frac{\lambda_i}{2}} (|2i-1\rangle|2i\rangle - |2i\rangle|2i-1\rangle), \quad (26)$$

with the Schmidt coefficients λ_i satisfying $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^k \lambda_i = 1$. The single-particle reduced density operator is,

$$\rho_r = \sum_{i=1}^k \frac{\lambda_i}{2} (|2i-1\rangle\langle 2i-1| + |2i\rangle\langle 2i|), \quad (27)$$

so that the Schmidt basis is an eigenbasis of ρ_r , and the halved Schmidt coefficients, $\lambda_i/2$, are the eigenvalues of ρ_r . Notice that each of these eigenvalues is (at least) two-fold degenerate. Since in this case, we have $S[\rho] = 0$, the correlations measure (23) reduces to the infimum of $S[\Pi(\rho)]$ over all the possible local measurements.

Let us first discuss the case where the k Schmidt coefficients are all different. Each eigenvalue of ρ_r is then two-fold degenerate: the eigenvectors $|2i-1\rangle$ and $|2i\rangle$ of ρ_r share the same eigenvalue $\lambda_i/2$. Consequently, we have to minimize $S[\Pi(\rho)]$ over all possible local bases consisting of Slater determinants constructed from single-particle bases of the form,

$$\begin{aligned} |\varepsilon_i\rangle &= c_{11}^{(i)}|2i-1\rangle + c_{12}^{(i)}|2i\rangle, \\ |\varepsilon_i^\perp\rangle &= c_{21}^{(i)}|2i-1\rangle + c_{22}^{(i)}|2i\rangle, \quad i = 1, \dots, k, \end{aligned} \quad (28)$$

with appropriate coefficients $c_{jl}^{(i)}$ such that $|\varepsilon_i\rangle$ and $|\varepsilon_i^\perp\rangle$ are normalized and orthogonal. However, it can be verified that, for any of these local bases we have

$$\begin{aligned} \Pi(\rho) &= \sum_{i=1}^k \frac{\lambda_i}{2} (|2i-1\rangle|2i\rangle - |2i\rangle|2i-1\rangle) \\ &\quad \times (\langle 2i-1| \langle 2i| - \langle 2i| \langle 2i-1|). \end{aligned} \quad (29)$$

That is, in this case the disturbed two-fermion density operator $\Pi(\rho)$ is the same for all the possible local bases. Consequently, $S[\Pi(\rho)]$ is constant over all the associated local measurements, and so we have that the quantum correlations measure is,

$$\xi(\rho) = - \sum_{i=1}^k \lambda_i \log \lambda_i. \quad (30)$$

Now, suppose that two or more λ_i 's are equal. Assume, for instance, that t Schmidt coefficients have the same value, $\lambda_{j_i} = \lambda$, $i = 1, 2, \dots, t$. In such a case we have within the Schmidt expansion of $|\psi\rangle$ a term of the form,

$$\sqrt{\frac{\lambda}{2}} \sum_{i=1}^t (|2j_i-1\rangle|2j_i\rangle - |2j_i\rangle|2j_i-1\rangle), \quad (31)$$

with $t \leq k$. The eigenvalue $\lambda/2$ of ρ_r is then $2t$ -fold degenerate. Consequently, within the single-particle orthonormal basis inducing the local (Slater) two-fermion basis we can substitute the set $\{|2j_i-1\rangle, |2j_i\rangle, i = 1, 2, \dots, t\}$ by any other set of $2t$ orthonormal linear combinations of these vectors. The corresponding two-fermion local basis (characterizing a local measurement) will then include the $t(2t-1)$ Slater determinants constructed with these new $2t$ single-particle vectors. Let us now compare the set of eigenvalues $\lambda^{(\Pi(\rho))}$ of the disturbed density matrix $\Pi(\rho)$ associated with this new local basis (resulting from the above substitution) with the set $\lambda^{(\text{Sch.})} = \{\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots\}$ of eigenvalues of the disturbed density matrix $\Pi^{(\text{Sch.})}(\rho)$ associated with the local (Slater) basis induced by the Schmidt basis $\{|1\rangle, |2\rangle, \dots, |2k\rangle\}$. Let $|\zeta\rangle$ be one of the Slater determinants constructed with two of the above-mentioned orthonormal linear combinations of the states $\{|2j_i-1\rangle, |2j_i\rangle, i = 1, 2, \dots, t\}$. It can be shown, after some algebra (see Appendix B), that

$$\left| \langle \zeta | \frac{1}{\sqrt{2}} \sum_{i=1}^t (|2j_i-1\rangle|2j_i\rangle - |2j_i\rangle|2j_i-1\rangle) \right| \leq 1. \quad (32)$$

This means that, as a result of the above-mentioned substitution, the eigenvalue λ , which appears t times in $\lambda^{(\text{Sch.})}$, is substituted in $\lambda^{(\Pi(\rho))}$ by a new set of eigenvalues, each one of them less or equal to λ , and adding up to $t\lambda$. This substitution leads to a $\lambda^{(\Pi(\rho))}$ that is majorized by $\lambda^{(\text{Sch.})}$. That is, we have

$$\lambda^{(\Pi(\rho))} \prec \lambda^{(\text{Sch.})}, \quad (33)$$

and thus

$$- \sum_{i=1}^k \lambda_i \log \lambda_i \leq S[\Pi(\rho)], \quad (34)$$

meaning that the quantum correlation measure for the pure two-fermion state is again given by equation (30).

The expression on the right hand side of equation (30) coincides with the amount of entanglement of the two-fermion pure state $|\psi\rangle$. This means that, in the case of pure states the concept of quantum correlation for two-fermion systems introduced here by us coincides with entanglement. In particular, our measure vanishes for a pure state if and only if this state has Slater rank equal to one (that is, if we have one Schmidt coefficient $\lambda_l = 1$ and $\lambda_i = 0 \forall i \neq l$).

6 Mixed states of two identical fermions

In this section, we shall analytically compute the above-introduced measure of quantum correlations for some relevant instances of mixed states of two-fermion systems. We shall consider fermionic analogues of important states, like the Werner states [41] and the Gisin states [42], that in the context of distinguishable subsystems constitute paradigmatic examples that proved to be useful in illuminating numerous aspects of entanglement and other types

of quantum correlations. Aside from their intrinsic interest, these states exhibit a high degree of symmetry and typically allow for the exact, analytical evaluation of relevant measures of quantum correlations. In point of fact, some of these states have been successfully employed by Luo to illustrate his measurement induced disturbance approach to quantum correlations.

Here, we shall use the angular momentum representation for two-fermion states as described in Section 2. Within this representation, as already explained, the states $|j, m\rangle$ with even j constitute a natural basis for the two-fermion state space. We use a compact notation according to which, for instance, the ket $|0, 0\rangle$ stands for $|j = 0, m = 0\rangle$.

6.1 Werner-like states

We first consider fermions with a single-particle Hilbert space of dimension four. We shall evaluate the correlation measure for the Werner-like state [41],

$$\rho = p|0, 0\rangle\langle 0, 0| + (1 - p)\rho_{mix}, \quad (35)$$

where

$$\rho_{mix} = \frac{1}{6}(|0, 0\rangle\langle 0, 0| + \sum_{m=-2}^2 |2, m\rangle\langle 2, m|) \quad (36)$$

is the totally mixed state of the two-fermion system. The state (35) is entangled if $p > 0.4$. The single-particle reduced density matrix ρ_r corresponding to this state is proportional to the identity matrix. Then, the choice of the local measurement (in a Slater basis constructed from an eigenbasis of ρ_r) is not uniquely defined. Using the majorization technique we can optimize this local measurement, finding the one leading to the disturbed matrix $\Pi^*(\rho)$ of minimum entropy.

When performing a local measurement on ρ the eigenvalues of the resulting $\Pi(\rho)$ are of the form,

$$\langle Sl|\rho|Sl\rangle = p|\langle Sl|0, 0\rangle|^2 + \frac{1-p}{6}, \quad (37)$$

where $|Sl\rangle$ is a two-fermion state of Slater rank 1. For these states one always has $|\langle Sl|0, 0\rangle|^2 \leq \frac{1}{2}$. Equality here can be achieved by $|Sl\rangle = \frac{1}{\sqrt{2}}[|\frac{3}{2}\rangle|\frac{3}{2}\rangle - |-\frac{3}{2}\rangle|-\frac{3}{2}\rangle]$ [12].

Let us first consider the local measurement performed in the Slater basis generated from the single-particle basis $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$. Let $\Pi^*(\rho)$ denote the density matrix resulting from this local measurement, and $\lambda^{(\Pi^*(\rho))} = \{\lambda_1^*, \dots, \lambda_6^*\}$ the corresponding set of eigenvalues. We now prove that this set majorizes the eigenvalues $\lambda^{(\Pi(\rho))} = \{\lambda_1, \dots, \lambda_6\}$ corresponding to any other local measurement. The members of $\lambda^{(\Pi^*(\rho))}$ are $\lambda_1^* = \lambda_2^* = \frac{p}{2} + \frac{1-p}{6}$ and $\lambda_3^* = \dots = \lambda_6^* = \frac{1-p}{6}$. The eigenvalues of the operator $\Pi(\rho)$ corresponding to a general local measurement are of the form,

$$\langle Sl_i|\rho|Sl_i\rangle = p\epsilon_i + \frac{1-p}{6}, \quad 0 \leq \epsilon_i \leq \frac{1}{2}; \quad \sum_{i=1}^6 \epsilon_i = 1. \quad (38)$$

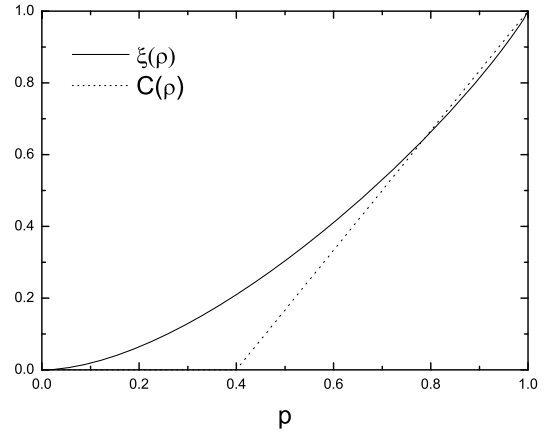


Fig. 1. $\xi(\rho)$ (solid line) and concurrence (dotted line) for the state given by equation (35).

The majorization inequalities $\sum_{i=1}^t \lambda_i^* \geq \sum_{i=1}^t \lambda_i, 1 \leq t \leq 6$ are then satisfied, and consequently we have that $\lambda^{(\Pi(\rho))} \prec \lambda^{(\Pi^*(\rho))}$, meaning that the quantum correlations measure is equal to $S[\Pi^*(\rho)] - S[\rho]$. Thus, for the state (35) we have,

$$\xi(\rho) = \frac{1-p}{6} \log \frac{1-p}{6} + \frac{1+5p}{6} \times \log \frac{1+5p}{6} - \frac{1+2p}{3} \log \frac{1+2p}{6}. \quad (39)$$

The concurrence of this state is given by

$$C(\rho) = \begin{cases} 0, & \text{if } 0 \leq p \leq 0.4 \\ \frac{5p-2}{3}, & \text{if } 0.4 < p \leq 1. \end{cases} \quad (40)$$

When $p = 1$, the state (35) is a pure, maximally entangled state of two fermions, and the quantum correlations measure adopts its maximum value, $\xi(\rho) = 1$. On the other hand, when $p = 0$ the state is equal to the maximally mixed one, ρ_{mix} and in this case $\xi(\rho) = 0$. However, we have non-vanishing quantum correlations, i.e. $\xi(\rho) \neq 0$, for non-entangled states. $\xi(\rho)$ can be larger than the concurrence for some states and it can be smaller for other states (see Fig. 1). In this respect, the behaviour of the fermionic quantum correlations measure exhibits some similarities with the behaviour of the quantum correlations measure corresponding to distinguishable systems [22].

The previous example admits a generalization to systems of two identical fermions with a d -dimensional single-particle Hilbert space, where $d = 2k, k \geq 2$. Let $d^* = \frac{d(d-1)}{2}$ denote the dimension of the corresponding two-fermion state space. We consider states consisting of a mixture of a maximally entangled state $|\psi\rangle$ and the maximally mixed one,

$$\rho = p|\psi\rangle\langle\psi| + (1-p)\frac{2}{d(d-1)}\mathbb{I}. \quad (41)$$

Here, \mathbb{I} is the $(d^* \times d^*)$ identity matrix, and $|\psi\rangle$ can be written as a superposition of non-overlapping Slater terms

$$|\psi\rangle = \frac{1}{\sqrt{d}} [|2\rangle|1\rangle - |1\rangle|2\rangle + |4\rangle|3\rangle - |3\rangle|4\rangle + \dots + |d\rangle|d-1\rangle - |d-1\rangle|d\rangle], \quad (42)$$

where $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$ is a single-particle orthonormal basis. Let

$$|Sl\rangle = \frac{1}{\sqrt{2}} [|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle], \quad (43)$$

be an arbitrary pure state of Slater rank one, constructed from the pair of orthonormalized single-particle states, $|\phi_1\rangle$ and $|\phi_2\rangle$. Then

$$|\langle\psi|Sl\rangle| \leq \sqrt{\frac{2}{d}}, \quad (44)$$

with equality obtained for states of the form $\frac{1}{\sqrt{2}}(|l+1\rangle|l\rangle - |l\rangle|l+1\rangle)$ (see Appendix B). The eigenvalues of ρ are $\frac{(1-p)}{d^*}$ with multiplicity $d^* - 1$ and $p + \frac{1-p}{d^*}$ with multiplicity 1, and the single-particle reduced density operator is $\rho_r = \mathbb{I}/d$.

Let $\Pi^*(\rho)$ denote the density matrix resulting from the local measurement associated with the Slater basis generated by the single-particle basis $\{|1\rangle, \dots, |d\rangle\}$, and $\lambda^{(\Pi^*(\rho))} = \{\lambda_1^*, \dots, \lambda_{d^*}^*\}$ the corresponding set of eigenvalues. Let $\lambda^{(\Pi(\rho))} = \{\lambda_1, \dots, \lambda_{d^*}\}$ be the eigenvalues of the $\Pi(\rho)$ corresponding to any other local measurement. The members of $\lambda^{(\Pi^*(\rho))}$ are $\frac{p(d-2)+1}{d^*}$ with multiplicity $\frac{d}{2}$ and $\frac{1-p}{d^*}$ with multiplicity $\frac{d(d-2)}{2}$. On the other hand, due to (44), the members of $\lambda^{(\Pi(\rho))}$ are of the form $p\epsilon_i + \frac{1-p}{d^*}$, with $\epsilon_i \leq \frac{2}{d}$ and $\sum_{i=1}^{d^*} \epsilon_i = 1$. It follows that $\lambda^{(\Pi(\rho))} \prec \lambda^{(\Pi^*(\rho))}$, and therefore we have,

$$\begin{aligned} \xi(\rho) &= S[\Pi^*(\rho)] - S[\rho] = (d^* - 1) \frac{(1-p)}{d^*} \log \frac{(1-p)}{d^*} \\ &+ \left(p + \frac{1-p}{d^*} \right) \log \left(p + \frac{1-p}{d^*} \right) - \frac{d(d-2)}{2} \frac{(1-p)}{d^*} \\ &\times \log \frac{(1-p)}{d^*} - \frac{d}{2} \frac{p(d-2)+1}{d^*} \log \frac{p(d-2)+1}{d^*}. \end{aligned} \quad (45)$$

6.2 Gisin-like states

We shall now compute the quantum correlations measure of the Gisin-like state [42]

$$\begin{aligned} \rho &= p|0,0\rangle\langle 0,0| + (1-p)[q|2,-2\rangle\langle 2,-2| \\ &+ (1-q)|2,2\rangle\langle 2,2|], \end{aligned} \quad (46)$$

with $0 \leq p, q \leq 1$. It will prove convenient to re-write this state under the guise $\rho = p\rho_1 + (1-p)\rho_2$, where $\rho_1 = |0,0\rangle\langle 0,0|$ and $\rho_2 = q|2,-2\rangle\langle 2,-2| + (1-q)|2,2\rangle\langle 2,2|$. Then, it is possible to prove that the set of eigenvalues

$\lambda^{(\Pi^*(\rho))}$ of the density matrix $\Pi^*(\rho)$ resulting from the local measurement in the Slater basis generated by the single-particle states $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$ is the one that majorizes the set of eigenvalues $\lambda^{(\Pi(\rho))}$ associated with any other local measurement. The single-particle reduced states corresponding to the three states ρ, ρ_1 and ρ_2 are all diagonal in the same single-particle basis. Consequently, these three states share the same family of admissible local measurements. Our strategy will be to show that the local measurement in the Slater basis associated with the single-particle basis $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$ is the optimal one both for ρ_1 and ρ_2 , and then conclude that it is optimal for ρ as well. To that effect first note that, if one has four probability distributions $\lambda^{(1)}, \lambda^{(1*)}, \lambda^{(2)}, \lambda^{(2*)}$, such that $\lambda^{(1)} \prec \lambda^{(1*)}$ and $\lambda^{(2)} \prec \lambda^{(2*)}$ then for any p ($0 \leq p \leq 1$) we have that

$$p\lambda^{(1)} + (1-p)\lambda^{(2)} \prec p\lambda^{(1*)} + (1-p)\lambda^{(2*)}. \quad (47)$$

Now, it is clear that for any local measurement we have $\lambda^{(\Pi(\rho_1))} \prec \lambda^{(\Pi^*(\rho_1))}$, since this is a particular instance of the previously considered case corresponding to the state (35). Now, the state ρ_2 is a convex linear combination of the states $\rho_{2a} = |2,-2\rangle\langle 2,-2|$ and $\rho_{2b} = |2,2\rangle\langle 2,2|$. It is plain that $\lambda^{(\Pi(\rho_{2a}))} \prec \lambda^{(\Pi^*(\rho_{2a}))}$ and $\lambda^{(\Pi(\rho_{2b}))} \prec \lambda^{(\Pi^*(\rho_{2b}))}$, since for both $\lambda^{(\Pi^*(\rho_{2a}))}$ and $\lambda^{(\Pi^*(\rho_{2b}))}$ we have one eigenvalue equal to 1 and the rest equal to zero (remember that the states $|2,-2\rangle$ and $|2,2\rangle$ are themselves members of the Slater basis induced by the single-particle basis $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$). Then, since $\lambda^{(\Pi(\rho_2))} = q\lambda^{(\Pi(\rho_{2a}))} + (1-q)\lambda^{(\Pi(\rho_{2b}))}$ and $\lambda^{(\Pi^*(\rho_2))} = q\lambda^{(\Pi^*(\rho_{2a}))} + (1-q)\lambda^{(\Pi^*(\rho_{2b}))}$ it follows from (47) that $\lambda^{(\Pi(\rho_2))} \prec \lambda^{(\Pi^*(\rho_2))}$. Then, taking into account that for any local measurement we have $\lambda^{(\Pi(\rho))} = p\lambda^{(\Pi(\rho_1))} + (1-p)\lambda^{(\Pi(\rho_2))}$, and applying once more the relation (47), we obtain that $\lambda^{(\Pi(\rho))} \prec \lambda^{(\Pi^*(\rho))}$.

So, finally, we find that $\xi(\rho) = p$. Thus, we see that for the family of states (46) the measure ξ depends only on the parameter p . On the other hand, the concurrence $C(\rho) = C(p, q)$ of these states depends on both parameters p and q . For the particular case $q = \frac{1}{2}$ we obtain

$$C(\rho) = \begin{cases} 0, & \text{if } 0 \leq p \leq 0.5 \\ 2p-1, & \text{if } 0.5 < p \leq 1. \end{cases} \quad (48)$$

Note that in this case ρ is entangled for $p > 0.5$. We plot the concurrence and $\xi(\rho)$ in Figure 2 for this state, with $q = \frac{1}{2}$.

6.3 Mixture of a pure and a maximally mixed state

We now consider the following state,

$$\rho = p|\Psi\rangle\langle\Psi| + (1-p)\rho_{mix}, \quad (49)$$

with ρ_{mix} as in equation (36) and

$$\begin{aligned} |\Psi\rangle &= \frac{\sin\theta}{\sqrt{2}} \left[\left| -\frac{3}{2} \right\rangle \left| \frac{3}{2} \right\rangle - \left| \frac{3}{2} \right\rangle \left| -\frac{3}{2} \right\rangle \right] \\ &+ \frac{\cos\theta}{\sqrt{2}} \left[\left| -\frac{1}{2} \right\rangle \left| \frac{1}{2} \right\rangle - \left| \frac{1}{2} \right\rangle \left| -\frac{1}{2} \right\rangle \right]. \end{aligned} \quad (50)$$

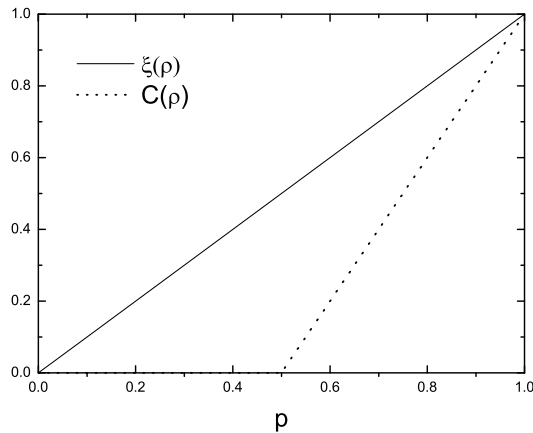


Fig. 2. $\xi(\rho)$ (solid line) and concurrence (dotted line) for the state given by equation (46). For this state $\xi(\rho)$ does not depend on q . The concurrence is evaluated by setting $q = \frac{1}{2}$.

The eigenvalues of ρ are $\{\frac{1-p}{6}, \dots, \frac{1-p}{6}, \frac{1+5p}{6}\}$ and the eigenvalues of the single-particle reduced density matrix ρ_r are $\frac{p \sin^2 \theta}{2} + \frac{(1-p)}{4}$ (corresponding to the eigenvectors $|-\frac{3}{2}\rangle$ and $|\frac{3}{2}\rangle$) and $\frac{p \cos^2 \theta}{2} + \frac{(1-p)}{4}$ (corresponding to the eigenvectors $|-\frac{1}{2}\rangle$ and $|\frac{1}{2}\rangle$). The admissible local measurements are thus those done in the Slater basis generated by a single-particle orthonormal basis $\{|\alpha_i\rangle\}$ consisting of four states of the form,

$$\begin{aligned} |\alpha_1\rangle &= d_{11} \left|-\frac{3}{2}\right\rangle + d_{12} \left|\frac{3}{2}\right\rangle, \\ |\alpha_2\rangle &= d_{21} \left|-\frac{3}{2}\right\rangle + d_{22} \left|\frac{3}{2}\right\rangle, \\ |\alpha_3\rangle &= d_{31} \left|-\frac{1}{2}\right\rangle + d_{32} \left|\frac{1}{2}\right\rangle, \\ |\alpha_4\rangle &= d_{41} \left|-\frac{1}{2}\right\rangle + d_{42} \left|\frac{1}{2}\right\rangle, \end{aligned} \quad (51)$$

with complex coefficients d_{ij} such that the vectors $\{|\alpha_i\rangle\}$ are orthonormal. Now, it can be verified after some algebra that the eigenvalues of the statistical operator $\Pi(\rho)$ resulting from any of these local measurements are always the same (that is, they do not depend on the particular values adopted by the coefficients d_{ij}). These eigenvalues are $\{\frac{1-p}{6}, \dots, \frac{1-p}{6}, \frac{1-p}{6} + p \cos^2 \theta, \frac{1-p}{6} + p \sin^2 \theta\}$. Consequently, we have $\xi(\rho) = S[\Pi(\rho)] - S[\rho]$, yielding

$$\begin{aligned} \xi(\rho) &= \frac{1-p}{6} \log \frac{1-p}{6} + \frac{1+5p}{6} \log \frac{1+5p}{6} \\ &\quad - \left[\frac{1-p}{6} + p \cos^2 \theta \right] \log \left[\frac{1-p}{6} + p \cos^2 \theta \right] \\ &\quad - \left[\frac{1-p}{6} + p \sin^2 \theta \right] \log \left[\frac{1-p}{6} + p \sin^2 \theta \right]. \end{aligned} \quad (52)$$

The concurrence of the state (49) is,

$$C(\rho) = \frac{1}{6} (\sqrt{c_1 + c_2} - \sqrt{c_1 - c_2}) - 4 \frac{1-p}{6}, \quad (53)$$

where c_1 and c_2 are given by the following expressions,

$$\begin{aligned} c_1 &= 1 + 4p + 4p^2 - 9p^2 \cos(4\theta) \\ c_2 &= 3p \sqrt{2[2 + 8p - p^2(1 + 9 \cos(4\theta))]} |\sin(2\theta)|. \end{aligned} \quad (54)$$

We plot the concurrence and ξ for this state in Figures 3a and 3b respectively. Setting $p = 1$ gives $\rho = |\Psi\rangle\langle\Psi|$ and so we obtain

$$\xi(\rho, p = 1) = -\sin^2 \theta \log(\sin^2 \theta) - \cos^2 \theta \log(\cos^2 \theta). \quad (55)$$

We plot the slice $p = 1$ in Figure 3c and the difference $C(\rho) - \xi(\rho)$ in Figure 3d.

6.4 Mixture of two maximally entangled pure states

We consider now a mixture of two orthogonal, maximally entangled states,

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{\sqrt{2}} (|2, 2\rangle + |2, -2\rangle) \\ |\phi_2\rangle &= \frac{1}{\sqrt{2}} (|2, 2\rangle - |2, -2\rangle). \end{aligned} \quad (56)$$

That is, we shall now study the state

$$\rho = p|\phi_1\rangle\langle\phi_1| + (1-p)|\phi_2\rangle\langle\phi_2|. \quad (57)$$

The concurrence of (57) is given by $C = |2p-1|$. The eigenvalues of the state $\Pi^*(\rho)$ resulting from a local measurement in the Slater basis induced by the single-particle basis $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$, are $\lambda^{(\Pi^*(\rho))} = \{\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\}$. Now, for any two-fermion state $|Sl\rangle$ of Slater rank 1 we have

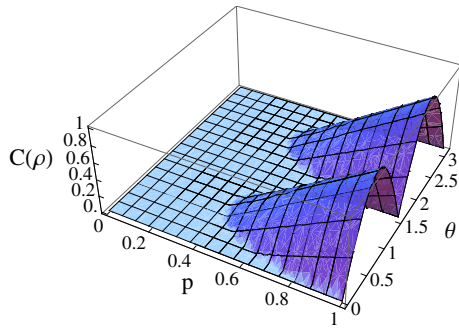
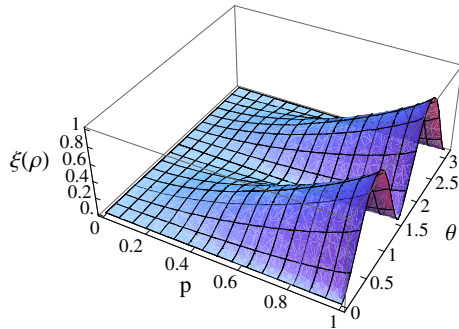
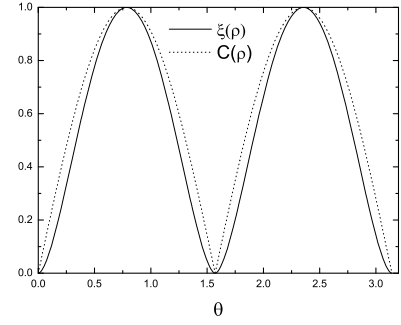
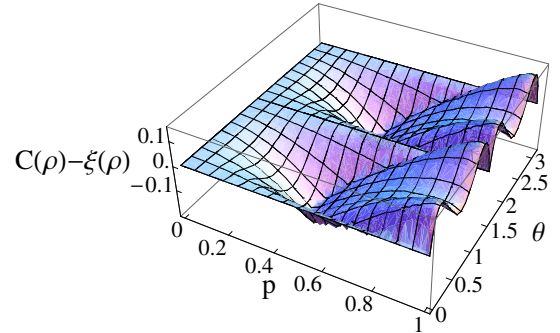
$$\begin{aligned} \langle Sl|\rho|Sl\rangle &= p|\langle\phi_1|Sl\rangle|^2 + (1-p)|\langle\phi_2|Sl\rangle|^2 \\ &\leq \frac{1}{2}. \end{aligned} \quad (58)$$

Thus, the two non-vanishing eigenvalues of $\Pi^*(\rho)$ adopt the maximum possible value, equal to $\frac{1}{2}$. It is plain then that $\lambda^{(\Pi^*(\rho))}$ majorizes the set of eigenvalues $\lambda^{(\Pi(\rho))}$ corresponding to any other possible local measurement. This leads to a quantum correlations measure for (57) equal to,

$$\xi(\rho) = 1 + p \log p + (1-p) \log(1-p). \quad (59)$$

7 Linear subspaces that admit non-classical states but have no entangled ones

Let us consider a system consisting of two identical fermions with a single-particle Hilbert space of dimension four. As usual, let $\{|\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$ denote a single-particle orthonormal basis. Let us focus on the linear subspace (of the two-fermion Hilbert space) spanned by the three Slater determinants that can be constructed with the three single-particle states $\{|\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$.

(a) Concurrence as a function of p and θ .(b) $\xi(\rho)$ as a function of p and θ .(c) $\xi(\rho)$ (solid line) and concurrence (dotted line) for $p = 1$.(d) Difference between concurrence and $\xi(\rho)$, $C(\rho)$ is not always larger than $\xi(\rho)$.**Fig. 3.** The graphs pertain to the state given by equation (49).

This subspace supports no entanglement. Any state belonging to this subspace is expressible as one single Slater determinant (see Appendix D) and, consequently, is non-entangled (and any statistical mixture of such states is non-entangled as well). However, *this subspace does involve non-classicality*. For instance, consider a mixed state of the form,

$$\rho = \alpha|\Phi_1\rangle\langle\Phi_1| + \beta|\Phi_2\rangle\langle\Phi_2| + \gamma|\Phi_3\rangle\langle\Phi_3|, \quad (60)$$

where,

$$\begin{aligned} |\Phi_1\rangle &= \frac{1}{\sqrt{2}} (|\Psi_1\rangle|\Psi_2\rangle - |\Psi_2\rangle|\Psi_1\rangle), \\ |\Phi_2\rangle &= \frac{1}{\sqrt{2}} (|\Psi_3\rangle|\Psi_4\rangle - |\Psi_4\rangle|\Psi_3\rangle), \\ |\Phi_3\rangle &= \frac{1}{\sqrt{2}} (|\Psi_3\rangle|\Psi_5\rangle - |\Psi_5\rangle|\Psi_3\rangle), \end{aligned} \quad (61)$$

with $0 < \alpha, \beta, \gamma < 1$, $\alpha \neq \beta \neq \gamma$, $\alpha + \beta + \gamma = 1$, and

$$\begin{aligned} |\Psi_1\rangle &= \left| -\frac{3}{2} \right\rangle, \quad |\Psi_2\rangle = \left| -\frac{1}{2} \right\rangle, \quad |\Psi_3\rangle = \left| \frac{1}{2} \right\rangle, \\ |\Psi_4\rangle &= \frac{1}{\sqrt{2}} \left(\left| -\frac{3}{2} \right\rangle + \left| -\frac{1}{2} \right\rangle \right), \\ |\Psi_5\rangle &= \frac{1}{\sqrt{2}} \left(\left| -\frac{3}{2} \right\rangle - \left| -\frac{1}{2} \right\rangle \right). \end{aligned} \quad (62)$$

The state (60) describes a statistical mixture of states belonging to the aforementioned subspace. It is clear that

this state is not diagonal in a Slater basis: the eigenstates of ρ are $|\Phi_1\rangle$, $|\Phi_2\rangle$, and $|\Phi_3\rangle$. Each of these eigenstates is itself a Slater determinant, but they can not be constructed from the members of one single-particle orthonormal basis. Consequently, the state ρ is non-entangled, but it is not classically correlated either. Summing up, this means that *systems of two identical fermions admit linear subspaces (of dimension larger than one) involving no entanglement but admitting non-classical states*. A similar situation is impossible in the case of bipartite systems consisting of two distinguishable subsystems.

8 Conclusions

We introduced an approach for the analysis of quantum correlations in fermion systems based upon the state disturbances generated by the measurement of “local” observables (that is, quantum observables represented by one-body operators). The concomitant concept of quantum correlations in systems of identical fermions differs from entanglement. According to this approach, the quantum states of two identical fermions exhibiting the minimum amount of quantum correlations, i.e. classically correlated states, are those that are diagonal in a Slater basis (induced by a single-particle basis). We proposed a quantitative measure for the quantum correlations of two-fermion systems, and computed it analytically for some relevant states. In the case of pure states of two identical

fermions, the present concept of quantum correlations coincides with entanglement, and the measure of quantum correlations reduces to the amount of entanglement exhibited by the fermionic state.

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Appendix A: Quantum states undisturbed by a projective measurement

Let $\{|k\rangle\}$ be an orthonormal basis of a quantum system's Hilbert space and $\{P_k = |k\rangle\langle k|\}$ the corresponding complete set of one-dimensional projectors, so that $P_k P_{k'} = \delta_{kk'} P_k$ and $\mathbb{I} = \sum_k P_k$ is the identity operator. Then, given a quantum state ρ , the following three statements are equivalent:

- (i) The state ρ is undisturbed by a measurement in the basis $\{|k\rangle\}$. That is, $\rho = \sum_k P_k \rho P_k$.
- (ii) The density operator ρ commutes with all the projectors: $P_k \rho = \rho P_k$.
- (iii) The state ρ is of the form $\rho = \sum_k \lambda_k P_k$, with $0 \leq \lambda_k \leq 1$ and $\sum_k \lambda_k = 1$.

It follows from (i) that $P_k \rho = P_k \sum_{k'} P_{k'} \rho P_{k'} = \rho P_k$. Therefore, (i) \rightarrow (ii). Now, if ρ verifies (ii) we have $\rho = \sum_{kk'} P_k \rho P_{k'} = \sum_{kk'} \rho P_k P_{k'} = \sum_k \rho P_k = \sum_k P_k \rho P_k = \sum_k \lambda_k P_k$, with $\lambda_k = \langle k | \rho | k \rangle$. Therefore, (ii) \rightarrow (iii). Finally, it is plain that (iii) \rightarrow (i).

The equivalence between the three statements concerning classically correlated states of two fermions, discussed in Section 3, follows immediately from the above considerations if we identify the projectors $\{P_k\}$ with the projectors associated with a local measurement of the system (that is, with the projectors corresponding to a Slater basis of the two-fermion system).

Appendix B: Upper bound for the overlap between a maximally entangled state and a state of Slater rank one

A maximally entangled state of two fermions with single-particle Hilbert space of dimension d ($d = 2k, k \geq 2$) can be written as a superposition of non-overlapping Slater determinants,

$$|\psi\rangle = \frac{1}{\sqrt{d}} [|2\rangle|1\rangle - |1\rangle|2\rangle + |4\rangle|3\rangle - |3\rangle|4\rangle + \dots + |d\rangle|d-1\rangle - |d-1\rangle|d\rangle], \quad (\text{B.1})$$

where $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$ is a single-particle orthonormal basis. Let

$$|Sl\rangle = \frac{1}{\sqrt{2}} [|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle], \quad (\text{B.2})$$

be an arbitrary pure state of Slater rank one, constructed from the pair of orthonormalized single-particle states, $|\phi_1\rangle = \sum_{i=1}^d \alpha_i |i\rangle$ and $|\phi_2\rangle = \sum_{i=1}^d \beta_i |i\rangle$. Then

$$\langle \psi | Sl \rangle = \sqrt{\frac{2}{d}} [\alpha_2 \beta_1 - \alpha_1 \beta_2 + \alpha_4 \beta_3 - \alpha_3 \beta_4 + \dots + \alpha_d \beta_{d-1} - \alpha_{d-1} \beta_d], \quad (\text{B.3})$$

$$\begin{aligned} |\langle \psi | Sl \rangle| &= \sqrt{\frac{2}{d}} | \alpha_2 \beta_1 - \alpha_1 \beta_2 + \alpha_4 \beta_3 \\ &\quad - \alpha_3 \beta_4 + \dots + \alpha_d \beta_{d-1} - \alpha_{d-1} \beta_d | \\ &\leq \sqrt{\frac{2}{d}} [|\alpha_2| |\beta_1| + |\alpha_1| |\beta_2| + |\alpha_4| |\beta_3| \\ &\quad + |\alpha_3| |\beta_4| + \dots + |\alpha_d| |\beta_{d-1}| + |\alpha_{d-1}| |\beta_d|], \end{aligned} \quad (\text{B.4})$$

and using the Schwartz inequality, we obtain

$$|\langle \psi | Sl \rangle| \leq \sqrt{\frac{2}{d}}. \quad (\text{B.5})$$

The equality is obtained for states of the form,

$$\frac{1}{\sqrt{2}} (|l+1\rangle|l\rangle - |l\rangle|l+1\rangle). \quad (\text{B.6})$$

Appendix C: Optimization of the measurement induced disturbance for systems of two distinguishable qubits

We consider as an example a Werner-like state of two distinguishable qubits

$$\rho = p |\psi\rangle\langle\psi| + \frac{(1-p)}{4} \mathbb{I}, \quad (\text{C.1})$$

where $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. The marginal density matrices corresponding to both qubits are equal to $\frac{1}{2} \mathbb{I}_2$, where \mathbb{I}_2 is the 2×2 identity matrix. Both marginal density matrices have a degenerate eigenvalue spectrum and, therefore, the local measurements leaving the marginal density matrices unchanged are not unique. Consequently, in this case one has to solve a nontrivial optimization problem in order to find the local measurement that generates the smallest possible disturbance upon the global state of the two qubits. Using the majorization argument, we can find the local basis that optimizes

$$\xi(\rho) = \min_{\text{local bases}} S[H(\rho)] - S[\rho]. \quad (\text{C.2})$$

This optimization problem was numerically solved using a random search of local bases in reference [28]. Lets define

$$|\psi_b\rangle = |\phi_1\rangle|\phi_2\rangle, \quad (\text{C.3})$$

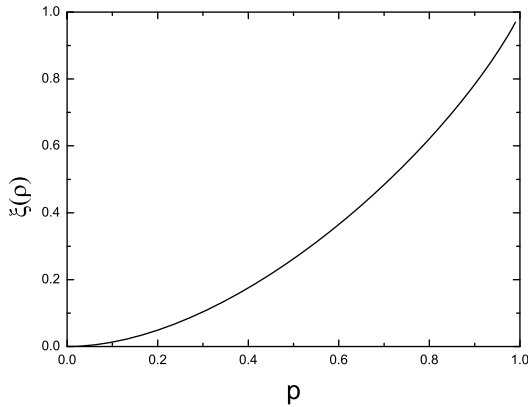


Fig. 4. $\xi(\rho)$ for the two-qubit Werner-like state (C.1).

with normalized, orthogonal single-particle states

$$\begin{aligned} |\phi_1\rangle &= a_0|0\rangle + a_1|1\rangle \\ |\phi_2\rangle &= b_0|0\rangle + b_1|1\rangle. \end{aligned} \quad (\text{C.4})$$

Then

$$\langle\psi|\psi_b\rangle = \frac{1}{\sqrt{2}}(a_0b_0 + a_1b_1), \quad (\text{C.5})$$

and

$$|\langle\psi|\psi_b\rangle| \leq \frac{1}{\sqrt{2}}(|a_0||b_0| + |a_1||b_1|). \quad (\text{C.6})$$

Using the Schwartz inequality

$$|\langle\psi|\psi_b\rangle| \leq \frac{1}{\sqrt{2}}, \quad (\text{C.7})$$

and, thus,

$$\max |\langle\psi|\psi_b\rangle| = \frac{1}{\sqrt{2}}. \quad (\text{C.8})$$

Choosing the single-particle basis $\{|0\rangle, |1\rangle\}$, we obtain for the eigenvalues of $\Pi^*(|\psi\rangle\langle\psi|)$: $\{\frac{1}{2}, \frac{1}{2}, 0, 0\}$. Then, we have

$$\lambda[\Pi^*(\rho)] = \left(\frac{1+p}{4}, \frac{1+p}{4}, \frac{1-p}{4}, \frac{1-p}{4} \right) \quad (\text{C.9})$$

and given the condition

$$\begin{aligned} \langle\psi_b|\rho|\psi_b\rangle &= p|\langle\psi|\psi_b\rangle|^2 + \frac{1-p}{4} \\ &\leq \frac{p}{2} + \frac{1-p}{4} \end{aligned} \quad (\text{C.10})$$

we have that $\lambda[\Pi(\rho)] \prec \lambda[\Pi^*(\rho)]$. Then we can analytically calculate $\xi(\rho) = \min_{\text{local bases}} S[\Pi(\rho)] - S[\rho]$, obtaining the exact result

$$\xi(\rho) = \frac{1+3p}{4} \log \frac{1+3p}{4} + \frac{1-p}{4} \log \frac{1-p}{4} - \frac{1+p}{4} \log \frac{1+p}{4}, \quad (\text{C.11})$$

which coincides with the numerical calculation of $\xi(\rho)$ reported in reference [28]. We plot $\xi(\rho)$ against the parameter p in Figure 4.

Appendix D: Linear combinations of the Slater determinants constructed with three orthonormal single-particle states

Any linear combination of the three Slater determinants that can be constructed with the single-particle states $\{|\frac{1}{2}\rangle, |-\frac{1}{2}\rangle, |-\frac{3}{2}\rangle\}$ is itself always expressible as one Slater determinant. This is the basic reason for the well-known fact that the system of two identical fermions of smallest dimension admitting entanglement is the one corresponding to a single-particle Hilbert space of dimension four. For the sake of completeness we provide, here, a brief discussion. Considering a normalized linear combination of the alluded Slater determinants, we have

$$\begin{aligned} &\frac{\alpha}{\sqrt{2}} \left(\left| -\frac{3}{2} \right\rangle \left| -\frac{1}{2} \right\rangle - \left| -\frac{1}{2} \right\rangle \left| -\frac{3}{2} \right\rangle \right) \\ &+ \frac{\beta}{\sqrt{2}} \left(\left| -\frac{3}{2} \right\rangle \left| \frac{1}{2} \right\rangle - \left| \frac{1}{2} \right\rangle \left| -\frac{3}{2} \right\rangle \right) \\ &+ \frac{\gamma}{\sqrt{2}} \left(\left| -\frac{1}{2} \right\rangle \left| \frac{1}{2} \right\rangle - \left| \frac{1}{2} \right\rangle \left| -\frac{1}{2} \right\rangle \right) \\ &= \frac{1}{\sqrt{2}} (|\xi_1\rangle|\xi_2\rangle - |\xi_2\rangle|\xi_1\rangle), \end{aligned} \quad (\text{D.1})$$

where

$$\begin{aligned} |\xi_1\rangle &= \sqrt{\alpha^2 + \beta^2} \left(\left| -\frac{3}{2} \right\rangle + \frac{\beta\gamma}{\alpha^2 + \beta^2} \left| -\frac{1}{2} \right\rangle - \frac{\alpha\gamma}{\alpha^2 + \beta^2} \left| \frac{1}{2} \right\rangle \right) \\ |\xi_2\rangle &= \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left(\alpha \left| -\frac{1}{2} \right\rangle + \beta \left| \frac{1}{2} \right\rangle \right). \end{aligned} \quad (\text{D.2})$$

The single-particle states $|\xi_1\rangle$ and $|\xi_2\rangle$ are orthonormal and, therefore, the right hand side of equation (D.1) is clearly a Slater determinant. We assumed that the coefficients α and β are not both equal to zero. If, on the contrary, $\alpha = \beta = 0$ it is obvious that the linear combination on the left hand side of equation (D.1) reduces to one Slater determinant.

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