Brans-Dicke wormholes in nonvacuum spacetime

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Abstract

Analytical wormhole solutions in Brans-Dicke theory in the presence of matter are presented. It is shown that the wormhole throat must not be necessarily threaded with exotic matter.

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The field equations of general relativity, being local in character, admit solutions with nontrivial topology. Among these, wormholes have been extensively studied [1]. Their most salient feature is that an embedding of one of their spacelike sections in Euclidean space displays two asymptotically flat regions joined by a throat.

The interest on wormholes is twofold. From the point of view of the Euclidean path integral formulation of quantum gravity, Coleman [2] and Giddings and Strominger [3], among others, have shown that the effect of wormholes is to modify low energy coupling constants and to provide probability distributions for them. In particular, Coleman [4] showed that, in the dilute wormhole approximation, the probability distribution for universes is infinitely peaked at $\Lambda = 0$, rendering all other values of the cosmological constant improbable.

On the purely gravitational side, the interest has been recently focused on traversable wormhole [1,5–8]. Most of the efforts are directed to study static configurations [9] that must have a number of specific properties in order to be traversable. The most striking of these properties is the violation of the energy conditions [10]. It implies that the matter that generates the wormhole is exotic [1], *viz.* its energy density is negative, as seen by static observers. Geometrically, this is a direct consequence of the singularity theorems of Hawking and Penrose [11]. Although we do not know of any such exotic material to date, quantum field theory might come to the rescue [12].

Finally, we should mention yet another proposal related to wormholes. It has been shown [5,13] that a nonstatic wormhole's throat can be transformed into a time tunnel. Physical effects in this type of spacetimes have been studied in [14].

Wormhole solutions have also been discussed in alternative theories of gravity, such as $R + R^2$ theories [15], Moffat's nonsymmetric theory [16], Einstein-Gauss-Bonnet theory [17], and Brans-Dicke (BD) theory [18]. In the last case, static wormhole solutions were found in vacuum, the source of gravity being the scalar field. Dynamical solutions are discussed in [19]. The aim of this paper is to look for static wormhole solutions of Brans-Dicke theory in a general setting, *i.e.* in the presence of matter that obeys a generic equation of state [20]. We shall also discuss whether the BD scalar can be the "carrier" of exoticity, as was shown in [18] for the vacuum case.

Following the conventions of [21], the field equations of Brans-Dicke theory are

$$R_{\mu\nu} = \frac{8\pi}{\Phi}\left(T_{\mu\nu} - \frac{\omega+1}{2\omega+3}\,T\,g_{\mu\nu}\right) + \omega\,\frac{\Phi_{;\mu}\Phi_{;\nu}}{\Phi^2} + \frac{\Phi_{;\mu;\nu}}{\Phi} \tag{1}$$

$$\Phi^{;\mu}_{\ ;\mu} = \frac{8\pi}{2\omega+3}\,T \tag{2}$$

The assumption of a static spacetime entails that it is possible to choose a metric and a scalar field such that

$$g_{\mu\nu,t} = 0 \qquad \Phi_{,t} = 0 \qquad g_{ti} = 0 \tag{3}$$

$(i = r, \theta, \phi)$. We further require spherical symmetry, so that the line element can be written in Schwarzschild form:

$$ds^2 = -e^{2\psi}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4}$$

For the stress-energy tensor of matter we choose

$$T^t_{\ t} = -\rho(r) \qquad T^r_{\ r} = -\tau(r) \qquad T^\theta_{\ \theta} = T^\phi_{\ \phi} = p(r) \tag{5}$$

and zero otherwise. Finally, we adopt the following equation of state for matter:

$$-\tau + 2p = \epsilon\rho \tag{6}$$

where ϵ is a constant. Now, the trace of the stress-energy tensor can be written as $T = -\tau + 2p - \rho = \rho(\epsilon - 1)$. The field equations take the form

$$-\psi'' - (\psi')^2 + \lambda'\psi' + 2\frac{\lambda'}{r} = -\frac{8\pi}{\Phi}\left[\tau + \frac{\omega+1}{2\omega+3}T\right]e^{2\lambda} + (\omega+1)(\ln\Phi)'^2 + (\ln\Phi)'' - \lambda'(\ln\Phi)' \tag{7a}$$

$$1 - re^{-2\lambda}\left[\psi' - \lambda' + \frac{1}{r}\right] = \frac{8\pi}{\Phi}\left[p - \frac{\omega+1}{2\omega+3}T\right]r^2 + re^{-2\lambda}(\ln\Phi)' \tag{7b}$$

$$e^{2(\psi-\lambda)}\left[\psi'' + (\psi')^2 - \lambda'\psi' + 2\frac{\psi'}{r}\right] = \frac{8\pi}{\Phi}\left[\rho + \frac{\omega+1}{2\omega+3}T\right]e^{2\psi} - \psi' e^{2(\psi-\lambda)}(\ln\Phi)' \tag{7c}$$

$$\Phi'' - \Phi' \left(\lambda' - \psi' - \frac{2}{r} \right) = \frac{8\pi}{2\omega + 3} \, T \, e^{2\lambda} \tag{7d}$$

To solve the system made up of Eqs. (7) we shall follow the philosophy sketched in [22]. We shall look for a differential equation relating ψ and λ, starting from the equations of motion and the equation of state. The equation we shall obtain is second order and nonlinear in ψ but, after a change of variables, first order and linear in λ. We shall then make a specific choice for ψ consistent with asymptotic flatness and nonexistence of horizons and singularities. We shall finally substitute this ψ into the linear equation and solve for λ.

As explained in [21], from Eqs. (6), (7c), and (7d), it can be shown that $\Phi = \Phi_0 \, e^{c\,\psi}$ where $c = (\epsilon - 1)/[2\omega + 3 + (\omega + 1)(\epsilon - 1)]$, and Φ_0 is related to the value of the gravitational coupling constant when $r \to \infty$. In the case $\omega \to \infty$ or $\epsilon \to 1$, we get general relativity back (although in the latter case, other solutions different from $\Phi = \text{const}$ might exist).

After a bit of algebra, we get the equation:

$$A \, \psi'' + B \, (\psi')^2 + 2A \, \psi' - A \, \lambda' \psi' + \frac{2}{r^2} \, (e^{2\lambda} - 1) = 0 \tag{8}$$

where

$$A = -2 \, \frac{2 + \epsilon + 2\omega}{2 + \epsilon + \omega(1 + \epsilon)}$$

$$B = - \, \frac{8 + \epsilon^2(\omega + 2) + 4\omega^2(1 + \epsilon) + 8\epsilon + 11\omega + 12\omega\epsilon}{[2 + \epsilon + \omega(\epsilon + 1)]^2}$$

In the spirit of [22], we make the ansatz $\psi = -\alpha/r$, where α is a positive constant. With this election, which guarantees that the gravitational constant takes the correct value at $r \to \infty$, Eq. (8) takes the form

$$h(r) + f(r) \, e^{2\lambda} + g(r) \, \lambda' = 0 \tag{9}$$

where

$$h(r) = B \left(\frac{\alpha}{r^2} \right)^2 - \frac{2}{r^2} \qquad f(r) = \frac{2}{r^2} \qquad g(r) = -\frac{A\alpha}{r^2} + \frac{4}{r}$$

A suitable change of variables transforms Eq. (9) into a Bernoulli equation, and afterwards into a linear equation. Its general solution is given by

$$e^{-2\lambda} = \frac{e^{2s/\varphi}}{\varphi}\left(1+\frac{R}{\varphi}\right)^{-(8l+1)} \{I+\mathcal{K}\} \tag{10}$$

where

$$\varphi = \frac{r}{\alpha} \qquad s = \frac{B}{A} \qquad R = -\frac{A}{4} \qquad l = -\frac{B}{A^2}$$

$$I \equiv \int e^{-2s/\varphi}\left(1+\frac{R}{\varphi}\right)^{8l} d\varphi$$

and $\mathcal{K}$ is a constant. It is not valid when $A \to 0$, *i.e.* for $\omega = -1-\epsilon/2$. The binomial $(1+R/\varphi)^{8l}$ is related to the hypergeometric function ${}_2F_1$ [23]. Using the relation [23]

$$e^t \, {}_pF_q(\alpha_1,\dots\alpha_p;\beta_1\dots\beta_q;-xt) = \sum_{n=0}^{\infty} {}_{p+1}F_q(-n,\alpha_1,\dots\alpha_p;\beta_1,\dots\beta_q;x)\frac{t^n}{n!}\,, \tag{11}$$

the integral I can be written

$$I = 2s\sum_{n=0}^{\infty}\int {}_3F_1(-n,-8l,b;b;R/2s)\left(\frac{-2s}{\varphi}\right)^n \tag{12}$$

Integrating out the terms corresponding to $n=0$ and $n=1$, we finally get

$$I = \varphi - 8\,l\,R\,\ln\varphi + \varphi\sum_{n=2}^{\infty} {}_3F_1(-n,8l,b;b;R/2s)(-1)^n\left(\frac{2s}{\varphi}\right)^n \frac{1}{n!\,(n-1)} \tag{13}$$

It is easily seen that $e^{2\lambda} \to 1$ when $\varphi \to \infty$.

In order to fix the constant $\mathcal{K}$, we must select a value for the dimensionless radius ($\varphi_{\rm th}$) such that the "flaring out" condition

$$\lim_{\varphi\to\varphi_{\rm th}^+} e^{-2\lambda} = 0^+ \tag{14}$$

is satisfied. In the case $R \leq 0$, $\varphi_{\rm th}$ must necessarily be greater than $|R|$, so that the flaring out condition holds for all values of ω and ϵ except, obviously, those where R diverges, which are given by $\omega = -(2+\epsilon)/(1+\epsilon)$. Nevertheless, the absolute size of the throat also depends on α [1]. The aforementioned properties of λ, together with the definition of ψ, bear out that the metric tensor describes two asymptotically flat spacetimes joined by a throat.

Let us now study the issue of weak energy condition (WEC) violation. Using the field equations and the expression for the trace, we easily obtain

$$\frac{2e^{2\lambda}}{r^2} - \frac{4\psi'}{r} - \frac{2}{r^2} = \frac{16\pi}{\Phi}\tau e^{2\lambda} + \frac{4}{r}\frac{\Phi'}{\Phi} - \omega\left(\frac{\Phi'}{\Phi}\right)^2 + 2\frac{\Phi'}{\Phi}\psi' \tag{15}$$

At the throat, $e^{2\lambda} \to \infty$, and then

$$\tau_{\rm th} \approx \frac{\Phi_{\rm th}}{8\pi r_{\rm th}^2} \tag{16}$$

To calculate $\rho_{\rm th}$, we use the nontrivial component of the equation $T^{\mu}_{\nu;\mu} = 0$:

$$\tau' = \psi'(\rho - \tau) - \frac{2\tau}{r} - \frac{\epsilon\rho + \tau}{r} \tag{17}$$

Using Eqs. (16) and (17), and the derivative of Eq. (15),

$$\rho_{\rm th} \approx \tau_{\rm th}\,\frac{c + 1 + \varphi_{\rm th}}{1 - \epsilon\varphi_{\rm th}} \tag{18}$$

And finally, from Eq. (6),

$$p_{\rm th} \approx \frac{\tau_{\rm th}}{2}\,\frac{\epsilon\,(c+1) + 1}{1 - \epsilon\,\varphi_{\rm th}} \tag{19}$$

We shall show now that WEC may be violated (at least near the throat) with nonexotic matter. This means that we shall present the parameters for which a wormhole solution exists whenever the matter content of the theory satisfying the inequalities

$$\rho_{\rm th} \geq 0 \qquad \rho_{\rm th} - \tau_{\rm th} \geq 0 \qquad \rho_{\rm th} + p_{\rm th} \geq 0 \tag{20}$$

or equivalently,

[1] This situation is analogous to what Kar and Sahdev have found for wormholes in general relativity [22].

$$\frac{c+1+\varphi_{\rm th}}{1-\epsilon\varphi_{\rm th}} \geq 1 \tag{21}$$

$$\frac{\epsilon(c+1)+3+2(c+\varphi_{\rm th})}{1-\epsilon\varphi_{\rm th}} \geq 0 \tag{22}$$

In addition, a necessary condition for the violation of the weak energy condition for matter plus Brans-Dicke field at the throat is given by

$$\frac{2(\omega+1)+\epsilon}{2\omega+3}\,\rho_{\rm th} \leq 0 \tag{23}$$

As an example, let us study the case $\epsilon = 2$. From Eqs. (16), (18), and (19), the inequalities (20) will be satisfied if

$$\left(\varphi_{\rm th} \geq -\frac{1}{9\,\omega+12} \quad \text{and} \quad \varphi_{\rm th} < \frac{1}{2}\right) \quad \text{or} \quad \left(\varphi_{\rm th} \leq -\frac{1}{9\,\omega+12} \quad \text{and} \quad \varphi_{\rm th} > \frac{1}{2}\right) \tag{24}$$

Inequality (23) will be satisfied for $\omega \in (-2, -3/2)$. Finally, we have to impose that $\varphi_{\rm th} \geq |A/4|$, which implies that

$$\varphi_{\rm th} \geq \left|\frac{2+\omega}{4+3\omega}\right| \tag{25}$$

These inequalities constrain $\varphi_{\rm th}$ to an interval in which a nonexotic wormhole can be constructed, for instance, in the case $\omega = -1.75$. We should recall that a definite interval for $\varphi_{\rm th}$ does not determine the radius of the throat, because of the dependence of φ on α.

Summing up, we showed that Brans-Dicke theory in the presence of matter with a fairly general equation of state admits analytical wormhole solutions. They generalize the vacuum ones presented by Agnese and La Camera [18]. It should be noted that there exists some regions of the parameter space in which the Brans-Dicke field may play the role of exotic matter, implying that it might be possible to build a *wormholelike* spacetime with the presence of ordinary matter at the throat.

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