Charged Scalar-Tensor Boson Stars: Equilibrium, Stability and Evolution

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We study charged boson stars in scalar-tensor (ST) gravitational theories. We analyse the weak field limit of the solutions and analytically show that there is a maximum charge to mass ratio for the bosons above which the weak field solutions are not stable. This charge limit can be greater than the GR limit for a wide class of ST theories. We numerically investigate strong field solutions in both the Brans Dicke and power law ST theories. We find that the charge limit decreases with increasing central boson density. We discuss the gravitational evolution of charged and uncharged boson stars in a cosmological setting and show how, at any point in its evolution, the physical properties of the star may be calculated by a rescaling of a solution whose asymptotic value of the scalar field is equal to its initial asymptotic value. We focus on evolution in which the particle number of the star is conserved and we find that the energy and central density of the star decreases as the cosmological time increases. We also analyse the appearance of the scalarization phenomenon recently discovered for neutron stars configurations and, finally, we give a short discussion on how making the correct choice of mass influences the argument over which conformal frame, the Einstein frame or the Jordan frame, is physical.

I. INTRODUCTION

Boson stars are localised, asymptotically flat configurations of gravitationally bound zero temperature bosons. Mathematically, the boson field is described by a complex wave function ψ whose Lagrangian possesses an internal $U(1)$ symmetry that gives rise to a conserved charge N, interpreted as the total number of bosons. The time dependence of ψ does not appear in the field equations, and the solutions describe bound eigenstates of ψ with a number of nodes that increases with the energy. The first boson star solutions were found by Kaup [1] and independently by Ruffini & Bonazzola [2], who studied spherically symmetric stars in General Relativity (GR). They found that the solutions were qualitatively similar to those describing neutron stars and white dwarfs, although they were of much smaller mass. Colpi, Shapiro & Wasserman [3] extended their work by examining GR boson stars whose matter Lagrangian includes a quartic self-interaction term. This additional term contributes to the pressure of the star, increasing both its mass and particle number. More general self-interaction terms can be considered. The first research on this subject was undertaken by Lee and co-workers [4] who, in a series of papers, studied the properties of non-topological solitons. These are boson stars whose self-interaction term allows localised, non-singular solutions to exist even in the absence of gravity. To accomplish this, the self-interaction must have attractive terms and must be at least sixth order in the boson field amplitude. The work of Lee et al. was performed independently of the main body of boson star research. For the case of non-soliton stars, it is not immediately obvious how the inclusion of a non-quartic self-interaction affects the properties of the star. This problem will be addressed elsewhere [5]; for our purposes, it will be sufficient to only consider the usual quartic self-interaction term. For reviews see Ref. [6].

Charged boson stars in GR were first studied by Jetzer & van der Bij [7] who found that the inclusion of the charge increases the star mass and particle number, in much the same way as the inclusion of the quartic self-interaction. In addition, they found that there was an upper limit on the charge to mass ratio of the bosons above which no non-singular solutions could be found: stars made up of bosons whose charge is beyond this limit generate a repulsive Coulomb field that overcomes the gravitational attraction, regardless of the number of bosons that make up the star. The stability properties of charged boson stars in GR were studied by Jetzer [8] before the advent of the use of catastrophe theory in stellar equilibrium. Also, Jetzer and coworkers [9] have discussed the influence of charged boson stars on the stability of the vacuum and have considered their contribution to cosmological dark matter.

Recently, there has been renewed interest in alternative theories of gravity, in particular scalar-tensor (ST) theories, in which the usual metric gravitational field is augmented by a scalar field ϕ which couples to the curvature via a

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parameter $\omega(\phi)$. The strength of this coupling increases with $1/\omega$. The simplest of these is Brans Dicke (BD) theory [10] in which ω is constant. These present studies of ST theories are motivated by the fact that they appear as the low energy limit of string theory [11] (the simplest string effective action being the $\omega = -1$ BD action) and that scalar gravitational fields arise from dimensional reduction of higher dimensional theories (see Ref. [12] and citations therein). These theories may better describe gravity in the early Universe and, in addition, several models of inflation are driven by the same scalar field of ST gravity [13]. Finally, ST theories provide a self-consistent framework in which one can study the possible variation of the gravitational coupling G.

The strongest constraints on ST theories are usually assumed to come from the solar system weak field tests [14]. As is well known, observations constrain the BD parameter to have a value of $\omega > 500$ to within 1σ. However, more general ST theories can satisfy the rather stringent constraints determined at the current epoch, while still differing considerably from GR in the past [15]. In addition, it has been shown that strong non-perturbative ST gravitational effects may arise in ST neutron star configurations at the present epoch, even when the parameters of the theory accord well with observational tests [16].

The current popularity of alternative models of gravity has prompted several authors to study boson stars in the framework of ST theory. The first of these were Gunderson & Jensen [17], who considered BD boson stars with $\omega = 6$. More general ST boson stars were considered by Torres [18] and later by Comer & Shinkai [19]. These studies showed that the inclusion of the ϕ field introduces no qualitative change in the solutions, although in all cases, boson stars have a slightly smaller mass than their GR counterparts.

Spherically symmetric charge distributions in ST theory have also been investigated before. Singh & Usham [20] studied particular solutions of the BD field equations with a charged perfect fluid source in Dicke's conformally transformed units. van der Bergh [21] analysed the case of a ST spacetime containing an electrostatic field, with traceless energy-momentum tensor, and found both naked time-like singularities and non-singular solutions. Reddy [22] and Reddy & Rao [23] also obtained spherically symmetric, static, conformally flat solutions of the BD electro-vacuum field equations. Neither of these authors, however, have considered the case of massive particles bounded to form a star, and the existing literature does not provide an analysis of the charged boson star system as we do here.

While charged stars, either in GR or in a ST theory, are interesting objects from a theoretical standpoint, until last year it was thought unlikely that they would play a significant astrophysical role. The argument to support this view was that of selective accretion: any charged object will naturally accrete matter of opposite sign and ultimately become neutral, as would, for example, a Kerr-Newmann black hole. However, a recent study by Punsly [24] established that a charged rotating black hole will be able not only to preserve its charge, but will also generate observable gamma rays. In an extremely simplified view of his model, he considered the black hole to be surrounded by a ring of opposite charge, but otherwise isolated in interstellar space. The ring, while neutralizing the whole system as seen by an observer at spatial infinity, would allow the black hole to keep its charge. Magnetic flux lines would enter the hole through its axis of symmetry and pair creation would produce gamma rays. This process may be important in understanding the nature of some of the unidentified galactic gamma ray sources recently discovered by the NASA Compton satellite, in its EGRET experiment [25].

The already initiated program of searching for observational signals of boson stars has lead us to some significant analogies with black holes [26]. Therefore, it is natural to expect that research into the subject of charged boson stars would extend this analogy. For instance, it may be possible that a rotating charged boson star may replace the central engine of Punsly's mechanism, although a careful analysis should be made to prove or disprove this statement (something which is far from the purpose of the present work). It is worth noting, however, that the more realistic models of (uncharged) rotating boson stars have been investigated and solutions were shown to exist [27]. There is no reason to suppose that rotating charged boson star solutions do not also exist. All in all, it is encouraging that charged objects are found to produce testable effects from which their astrophysical signatures may be extracted: the presence of charge allows a wider range of phenomena to be observable. In the context of our present work, boson stars, either charged or uncharged, are relativistic constructs that may help us understand the influence of a different gravitational theory and its cosmology on the astrophysical world.

In a ST theory, at the level of the action, Newton's gravitational constant G is replaced by a field $G^\star = G/\phi$. In a cosmological setting $G^\star$ will evolve with time, giving rise to the twin phenomena of gravitational memory and gravitational evolution [28]. Consider a compact object in equilibrium that forms at some time t_1 at which the gravitational coupling strength is $G^\star_1$. The value of $G^\star_1$ will determine to some extent the structure of the object. At some later cosmological time, $G^\star$ will have evolved to some new value $G^\star_2$ and the object may either "remember" its local value of $G^\star_1$ or evolve to some new configuration in which its local value of $G^\star$ matches the cosmological value $G^\star_2$. In actual fact, one would expect the behaviour of the object to lie somewhere between these two extremes. Boson stars seem to be ideal candidates for the study of these phenomena: they are relatively simple objects whose equilibrium solutions are easy to find, they are non-singular and they are fully relativistic. Gravitational memory and gravitational evolution in boson stars were first analysed by Torres, Liddle & Schunck [29], who constructed sequences of static configurations at different cosmological times. One can also consider the evolution of other compact objects, such as the white dwarfs studied by García-Berro et al. [30] and Benvenuto et al. [31], who found that the variation of $G^\star$ produced observable changes in the stars' luminosities.

The bosons that make up a boson star are assumed to have identical mass m and, as we shall show in detail in the following Section, m and the Planck mass M_{pl} serve as a scale against which all physical properties of the star may be measured. Here we shall briefly outline the range of physical magnitudes possible for given choices of m. For a mini-soliton star (made of uncharged non self-interacting bosons), the choice $m = 30$ GeV implies that the mass of the star is of order 10^{10} kg, which is 20 orders of magnitude smaller that the solar mass $M_\odot$, while its radius is of order 10^{-17} m. This gives a density of around 10^{48} times that of a neutron star [6] and these objects are obviously highly relativistic. Only when the boson mass is reduced to around 10^{-10}eV does the star resemble a more conventional stellar object. As we have mentioned above, including a self-interaction potential $V(\psi)$ can substantially alter these figures. The importance of the self-interaction term is measured by the ratio $V(\psi)/(m^2|\psi|^2)$, which for the quartic self-interaction considered by Colpi et al. [3] is approximately $\lambda M_{pl}^2/m^2$, where λ is a constant and one assumes that $|\psi| \sim M_{pl}$. In this case, the self-interaction can only be neglected if λ is exceedingly small: $\lambda \ll m^2/M_{pl}^2 = 6.7\times10^{-39}$ GeV^{-2}m^2. Rewriting the potential as the dimensionless number $\Lambda := \lambda M_{pl}^2/(4\pi m^2)$, the equilibrium solutions can be parameterised by Λ and for $\Lambda \gg 1$, the boson star masses scale as $\Lambda^{1/2}M_{pl}^2/m \sim \lambda^{1/2}M_{pl}^3/m^2 \sim \lambda^{1/2}M_{ch}$, where M_{ch} is the Chandrasekhar mass (approximately $1M_\odot$). For instance, if $m = 1$ GeV, the star has a mass of $1.6\sqrt{\lambda}M_\odot$ and a radius of $6\sqrt{\lambda}$km. If, instead, we take $m = 1$ MeV, then the mass becomes $10^6\sqrt{\lambda}M_\odot$ and the radius becomes $10^6\sqrt{\lambda}$ km, which is similar to the Sun's radius but encompasses one million solar masses. For non-topological soliton stars, the mass was found to scale as M_{pl}^4/m^3 and one can construct soliton stars of very large mass and radius (the latter quantity being of the order of a few light years). The fact that both m and λ are free parameters allows one to construct boson star solutions with physical properties whose values range over all astrophysically interesting magnitudes. With the inclusion of the gauge charge, the range of physical magnitudes will depend upon a third independent parameter. We shall discuss this in more detail when we analyse numerical solutions in Section VI.

The rest of this work is organized as follows. Section II introduces the basic formalism, both for gravity and the stellar model. An analysis of the charge, the mass definition and the asymptotic limit is made in Section III. In Section IV we discuss the boundary values for all fields and provide scaling relations valid for our field equations, which we shall later use. We then analytically prove, in Section V, that there exist a maximum charge for weak field configurations (characterised by small central densities) which differs from the GR result (we shall also explicitly derive this last result as the correct limit of the ST case). In Section VI we numerically analyse strong field solutions for both BD gravity and a representative power law ST theory. These latter solutions demonstrate how ST boson stars may produce observable ST effects at the current epoch. Section VII is devoted to the discussion of gravitational memory and evolution of boson stars, both charged and uncharged. We give our final comments in Section VIII.

II. FORMALISM

The system we study is that of a complex scalar field ψ with a $U(1)$ charge in the framework of ST gravity. Its action is given by

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 16\pi L_m\right], \tag{1}$$

where $g = \mathrm{Det}(g_{\mu\nu})$, R is the scalar curvature, ω is the coupling function and L_m is the matter Lagrangian. We take L_m to be

$$L_m = -\frac{1}{2}g^{\mu\nu}(\overline{D}_\mu\overline{\psi})(D_\nu\psi) - \frac{1}{2}m^2\psi\overline{\psi} - \frac{1}{4}\lambda(\psi\overline{\psi})^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} \tag{2}$$

where

$$F_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu, \tag{3}$$

$$D_\mu = \partial_\mu + ie\mathcal{A}_\mu, \tag{4}$$

$\mathcal{A}_\mu$ is the gauge field and the over-bar denotes complex conjugation. This Lagrangian is invariant under a gauge transformation of the $U(1)$ group. This implies the existence of a conserved current

$$\mathcal{J}^\mu = \sqrt{-g}g^{\mu\nu}\left[ie\left(\overline{\psi}\partial_\nu\psi - \psi\partial_\nu\overline{\psi}\right) - 2e^2\mathcal{A}_\nu\psi\overline{\psi}\right], \qquad \partial_\mu\mathcal{J}^\mu = 0. \tag{5}$$

Varying the action with respect to $g^{\mu\nu}$ and ϕ we obtain the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\left[\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi^{,\alpha}\phi_{,\alpha}\right] + \frac{1}{\phi}\left[\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi\right] \tag{6}$$

and

$$\Box\phi = \frac{1}{2\omega+3}\left[8\pi T - \frac{d\omega}{d\phi}\phi^{,\alpha}\phi_{,\alpha}\right], \tag{7}$$

where

$$T_{\mu\nu} = (\overline{D}_\mu\overline{\psi})(D_\nu\psi) + \frac{1}{4\pi}F_\mu{}^\sigma F_{\nu\sigma} - \tfrac{1}{2}g_{\mu\nu}\left[g^{\alpha\beta}(\overline{D}_\alpha\overline{\psi})(D_\beta\psi) + m^2\psi\overline{\psi} + \tfrac{1}{2}\lambda(\psi\overline{\psi})^2 + \frac{1}{8\pi}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{8}$$

is the energy-momentum tensor for the matter fields and T is its trace.

Varying the action with respect to the mater fields ψ, $\overline{\psi}$ and $\mathcal{A}_\mu$ we obtain the boson wave equations

$$g^{\alpha\beta}\nabla_\alpha D_\beta\psi = (m^2 + \lambda\overline{\psi}\psi)\psi - ieg^{\alpha\beta}\mathcal{A}_\alpha D_\beta\psi, \quad g^{\alpha\beta}\nabla_\alpha\overline{D}_\beta\overline{\psi} = (m^2 + \lambda\overline{\psi}\psi)\overline{\psi} + ieg^{\alpha\beta}\mathcal{A}_\alpha\overline{D}_\beta\overline{\psi} \tag{9}$$

and the Maxwell equation

$$\nabla_\nu F^{\mu\nu} = 2\pi J^\mu, \tag{10}$$

where

$$J^\mu := \frac{1}{\sqrt{-g}}\mathcal{J}^\mu. \tag{11}$$

We consider static, spherically symmetric solutions and write the line element in the Schwarzschild form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\Omega^2, \tag{12}$$

where $d\Omega^2$ is the line element of the unit 2-sphere. We look for solutions of minimum energy. It has been shown [4] that for spherically symmetric systems of uncharged bosons, since L_m is universally coupled to the curvature, the form of ψ compatible with this requirement is

$$\psi(r,t) = \chi(r)\exp\left[i\varpi t\right], \tag{13}$$

where ϖ is a real positive constant. To prove this, one minimises the total energy of the boson star subject to the constraint that the particle number (which we shall define below) be conserved. The parameter ϖ plays the role of a Lagrange multiplier and this result is independent of the form of the boson self-interaction $V(\phi)$. This result also holds when one includes a $U(1)$ charge [32], and so we take Eq. 13 to be valid in the present work.

Equation (13) is also consistent with the assumption that $g_{\mu\nu}$ is static. The scalar field ϕ inherits the symmetries of the line element and is a function of r, while the gauge field may be chosen such that we have only electric charges and no magnetic ones, so we set

$$\mathcal{A}_\mu = (\mathcal{A}_0(r), 0, 0, 0). \tag{14}$$

Before we explicitly write down the full set of equations of structure of the star, we shall introduce some dimensionless quantities which will simplify the comparison with the previous work cited in Section I. Using the boson mass m and the present value of the Planck mass $M_{pl} := \sqrt{1/G}$ to define length and energy scales, the new quantities are:

$$\Omega(r) = \frac{\varpi + e\mathcal{A}_0}{m}, \tag{15}$$

$$\sigma = \sqrt{4\pi}\frac{\chi(r)}{M_{pl}} \tag{16}$$

and

$$\Lambda = \frac{\lambda}{4\pi}\left(\frac{M_{pl}}{m}\right)^2 \tag{17}$$

for the boson sector,

$$C = \frac{\mathcal{A}_0}{M_{pl}} \tag{18}$$

and

$$q = \frac{eM_{pl}}{m} \tag{19}$$

for the gauge field, and

$$\Phi = \frac{\phi}{M_{pl}^2} \tag{20}$$

for the Brans-Dicke field. We also introduce a rescaled radial coordinate x defined by

$$x := mr. \tag{21}$$

In terms of these quantities, the independent components of the field equations (6), (7), (9) and (10) are

- The generalized Einstein equations:

$$A' = Ax\left[\frac{\sigma'^2(2\omega+1)}{\Phi(2\omega+3)} + \frac{\sigma^2 A}{\Phi(2\omega+3)}\left(\frac{\Omega^2}{B}(2\omega+5)+2\omega-1\right) + \frac{\sigma^4 A\Lambda(2\omega-1)}{2\Phi(2\omega+3)} \right.$$
$$\left. -\frac{B'\Phi'}{2B\Phi} + \frac{\omega\Phi'^2}{2\Phi^2} - \frac{\Phi'^2}{\Phi(2\omega+3)}\left(\frac{d\omega}{d\Phi}\right) + \frac{C'^2}{B\Phi} + \frac{1}{x^2}(1-A)\right] \tag{22}$$

$$B' = \frac{x}{2\Phi + x\Phi'}\left[2B\sigma'^2 + 2\sigma^2 A(\Omega^2 - B) - \sigma^4 AB\Lambda - \frac{4B}{x}\Phi' + \frac{B\omega}{\Phi}\Phi'^2 - 2C'^2 + \frac{2B\Phi}{x^2}(A-1)\right] \tag{23}$$

- The scalar field equation:

$$\Phi'' = \Phi'\left(\frac{A'}{2A} - \frac{B'}{2B} - \frac{2}{x}\right) + \frac{2A}{2\omega+3}\left[\left(\frac{\Omega^2}{B}-2\right)\sigma^2 - \frac{\sigma'^2}{A} - \Lambda\sigma^4\right] - \frac{1}{2\omega+3}\frac{d\omega}{d\Phi}\Phi'^2 \tag{24}$$

- The boson field equation:

$$\sigma'' = \sigma'\left(\frac{A'}{2A} - \frac{B'}{2B} - \frac{2}{x}\right) - A\left[\left(\frac{\Omega^2}{B} - 1\right)\sigma - \Lambda\sigma^3\right] \tag{25}$$

- The Maxwell equation:

$$C'' = C'\left(\frac{B'}{2B} + \frac{A'}{2A} - \frac{2}{x}\right) + qA\Omega\sigma^2 \tag{26}$$

Here a prime denotes d/dx. We mainly consider two forms of ω: the Brans Dicke coupling in which ω is constant and a power law coupling, where $2\omega + 3 = 4n/3\Phi^n$.

Equations (22) to (26) reduce to the general relativistic ones [7] when $\omega \to \infty$, $\Phi \to 1$ and $d\omega/d\Phi \to 0$. Note, however, that this is not a simple issue whenever the trace of the matter energy-momentum tensor vanishes [33]. This is not the case here, although we still verify the general relativistic limit for large values of ω.

III. CHARGE, MASS AND THE ASYMPTOTIC LIMIT

We shall first look for solutions that are asymptotically flat in the sense that, in the limit $x \to \infty$,

$$A \sim 1 + \mathcal{O}\left(\frac{1}{x^n}\right), \quad B \sim B_\infty + \mathcal{O}\left(\frac{1}{x^n}\right), \quad \Phi \sim \Phi_\infty + \mathcal{O}\left(\frac{1}{x^n}\right), \quad C \sim C_\infty + \mathcal{O}\left(\frac{1}{x^n}\right), \tag{27}$$

where $n \geq 1$ and the subscript '∞' denotes values at space-like infinity. We also require σ to vanish at least as fast as x^{-1} so that the boson matter is localised and the solutions describe bound eigenstates of ψ. In fact, as we shall show below, the conditions above imply that σ falls off exponentially as $x \to \infty$. In addition, the field equations (22) to (26) are invariant under the rescaling $C \to C + c$, $\varpi \to \varpi - qc$ where c is a constant, and we use this remaining gauge freedom to set

$$C_\infty = 0. \tag{28}$$

This implies that $\Omega_\infty = \varpi/m$.

Given the asymptotic flatness conditions, we may define expressions for the total charge and mass (or energy) of the boson stars. From the conserved current equation (5) one may derive a conserved (time independent) charge

$$Q = q \int_0^\infty \Omega \sqrt{\frac{A}{B}}\, \sigma^2 x^2 \, dx. \tag{29}$$

This quantity may be interpreted as the star's total electrical charge per unit boson mass. Since all of the bosons are assumed to carry identical charge per unit mass q, we may also define a conserved particle number

$$N := \frac{Q}{q} = \int_0^\infty \Omega \sqrt{\frac{A}{B}}\, \sigma^2 x^2 \, dx. \tag{30}$$

Due to the fact that we are using the boson mass m as the mass scale, the numerical value of N is also equal to the value of the 'rest mass' of the star, a term we use to describe the sum of the masses of the bosons measured in some non-gravitational way. Then the particle number is measured in units of M_{pl}^2/m^2 while the rest mass is measured in units of M_{pl}^2/m. We also define a Newtonian mass

$$M_N := \frac{N}{\Phi_\infty} \tag{31}$$

which measures the active gravitational mass of the star whose bosons are dispersed to infinity.

Integrating the Maxwell equation (26) we find an alternative expression for the total charge:

$$Q = \int_0^\infty \frac{d}{dx}\left(\frac{x^2}{\sqrt{AB}} \frac{dC}{dx}\right) dx = \lim_{x \to \infty} \left(x^2 \frac{dC}{dx}\right), \tag{32}$$

where we have use the asymptotic flatness conditions to derive the second equation. In an analogous way, we define the 'scalar charge'

$$S := \lim_{x \to \infty} \left(x^2 \frac{d\Phi}{dx}\right). \tag{33}$$

Note that this is not in general a conserved quantity as there is no internal symmetry associated with Φ in the action (1). We introduce it here since it is included in the expression for the mass of the star which we shall give below. Integrating the wave equation (24) gives an alternative expression for the scalar charge:

$$S = \int_0^\infty x^2 \left[\Phi' \left(\frac{A'}{2A} - \frac{B'}{2B}\right) + \frac{2A}{2\omega + 3}\left[\left(\frac{\Omega^2}{B} - 2\right)\sigma^2 - \frac{\sigma'^2}{A} - \Lambda\sigma^4\right] - \frac{1}{2\omega + 3}\frac{d\omega}{d\Phi}\Phi'^2\right] dx. \tag{34}$$

The generalisation of the Schwarzschild mass is defined implicitly by the familiar relation

$$A := \left(1 - \frac{2M(x)}{x}\right)^{-1}. \tag{35}$$

Note that this equation does not explicitly include the scalar field Φ: here the gravitational coupling strength $G^* = G/\Phi$ has been factored into M. Using Eq. (22) the Schwarzschild mass may be written as the integral

$$M(x) = \int_0^x d\tilde{x}\, \frac{\tilde{x}^2}{2}\left[\frac{\omega\Phi'^2}{2A\Phi^2} - \frac{\Phi'^2}{A\Phi(2\omega+3)}\left(\frac{d\omega}{d\Phi}\right) + \frac{\sigma^2}{\Phi(2\omega+3)}\left(\frac{\Omega^2}{B}(2\omega+5)+2\omega-1\right)\right.$$
$$\left. + \frac{\sigma'^2(2\omega+1)}{A\Phi(2\omega+3)} + \frac{\sigma^4\Lambda(2\omega-1)}{2\Phi(2\omega+3)} - \frac{B'\Phi'}{2AB\Phi} + \frac{C'^2}{AB\Phi}\right]. \tag{36}$$

One can show that $M_\infty := \lim_{x\to\infty} M = M_{ADM}$, where M_{ADM} is the ADM mass of the star. Note that the integrand in Eq. (36) contains non-positive definite terms, so that M' and M themselves may be negative for weak field solutions. This is because the scalar field terms on the right hand side of Eq. (6) may cause $G_{\mu\nu}$ to violate the dominant energy condition. This fact has been exploited in the construction of ST wormhole solutions for which the matter energy-momentum tensor satisfies all of the energy conditions [34] and even when there is no matter tensor at all [35].

As shown by Lee in a usually uncited paper [36], the correct definition of mass for an isolated source in BD gravity is the Tensor mass

$$M_T := M_{ADM} - \frac{S}{2\Phi_\infty}. \tag{37}$$

This definition is also appropriate for more general ST theories. From the Tensor and Newtonian masses, we define the binding energy

$$\mathcal{E} := M_T - M_N \tag{38}$$

and the fractional binding energy

$$\mathcal{B} := \frac{M_T - M_N}{M_N} \tag{39}$$

which measures the binding energy per unit boson mass per boson. A necessary condition for dynamical stability is that $\mathcal{E} < 0$, which implies that $\mathcal{B} < 0$.

There are several reasons for preferring M_T over M_{ADM} as the correct definition of the mass of the star in ST gravity. Lee [36] has shown that for a localised source that emits gravitational radiation, the time rate of change of M_T evaluated at future null infinity is non-positive definite (in other words, gravitational wave emission can only reduce the Tensor mass of the source). This is not true for M_{ADM}. Creighton and Mann [37] have given a quasi-local definition of mass in ST gravity based on a rigorous consideration of the boundary terms both in the variation of the Lagrangian and in the formulation of the Hamiltonian. It turns out in either case that the expression for the mass is given by the usual integral of the extrinsic curvature on the boundary of the region under consideration, plus an additional term analogous to the scalar charge defined above. For an asymptotically flat spacetime, and in the appropriate limit, one can show that their mass tends to the Tensor mass given above. In addition, one can show that, at least for uncharged boson stars in BD theory, any solution with extremal Tensor mass is also one of extremal Newtonian mass (and hence extremal particle number) [38]. This result is easily extended to more general ST theories and is important when one uses catastrophe theory to determine the stability of a boson star [39]: a plot of mass against particle number for a set of solutions should show "cusps" at the extremal values of the particle number. The first of these cusps marks the point at which the solutions become dynamically unstable.

Expanding the metric and matter field equations in powers of $1/x$ about $x = \infty$ one can write the field equations in a linearised form, similar to that given in Section V for the weak field limit. Solving these equations and using the boundary conditions (27) and (28), we find that the scalar field and vector potential have the asymptotic form

$$\Phi = \Phi_\infty - \frac{S}{x} + \mathcal{O}\left(\frac{1}{x^2}\right), \qquad C = -\frac{Q}{x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{40}$$

while the line element in this limit is

$$ds^2 = -\left[1 - \frac{2M_K}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right]dt^2 + \left[1 + \frac{2M_{ADM}}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right]dx^2 + x^2 d\Omega^2. \tag{41}$$

Here

$$M_K := M_{ADM} - \frac{S}{\Phi_\infty} \tag{42}$$

is the Keplerian mass of the star. Note that Eq. (40) can be obtained directly from Eqs. (32) and (33).

Substituting the metric components appearing in the asymptotic form of the line element into the boson wave equation (25) gives

$$\sigma'' = -\frac{2\sigma'}{x} - \sigma\left(1 + \frac{2M_{ADM}}{x}\right)\left[\left(\Omega_\infty - \frac{qQ}{x}\right)^2\left(1 + \frac{2M_K}{x}\right) - 1\right] + \mathcal{O}\left(\frac{1}{x^2}\right). \tag{43}$$

This equation has the solution

$$\sigma = x^b e^{-\kappa x}\left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right], \tag{44}$$

where the constants κ and b are given by

$$\kappa = \sqrt{1 - \Omega_\infty^2}, \qquad b = -1 + \frac{1}{\kappa}\left[(2\Omega_\infty^2 - 1)M_T - \frac{S}{2\Phi_\infty} - qQ\Omega_\infty\right]. \tag{45}$$

Hence the boson field falls off exponentially with x in the asymptotic region. The quantity κ may be interpreted as the reciprocal radius of the star and must be real. This implies that $\Omega_\infty < 1$.

We conclude this section with a note on the physical interpretation of M_K and M_T and the choice of the physical frame. The Keplerian mass is the active gravitational mass measured by a non self-gravitating test particle in a circular orbit at space-like infinity about the star, while the Tensor mass is the active gravitational mass measured by a 'test black hole' in a similar orbit [40]. By test black hole we mean an object whose mass is negligible compared to the mass of the boson star and is made up entirely of gravitational binding energy. Its orbit is geodesic in the Einstein frame, whose metric $\widetilde{g}_{\mu\nu}$ is related to the physical metric by the conformal transformation $\widetilde{g}_{\mu\nu} = \Phi g_{\mu\nu}$. Calculating the angular velocity of a circular orbit in the Einstein frame, using Kepler's third law and transforming the result back into the Jordan frame, we find that M_T is given by

$$M_T = \lim_{x\to\infty}\left[\frac{r^2(\Phi B' + B\Phi')}{2\Phi + r\phi'}\right]. \tag{46}$$

The expression for M_T in the Einstein frame shows that this quantity is non-negative.

The ADM mass in the Jordan frame cannot be interpreted in terms of particle orbits. In the limit of large ω, Φ is only weakly coupled to the curvature and S is small compared with M_{ADM}. Hence, in this limit the differences between M_{ADM}, M_K and M_T are negligible and in the exact GR theory all three definitions are the same.

Recently, Faraoni, Gunzig & Nardone [41] have argued that the Einstein frame should be considered to be the physical frame, primarily because the Jordan frame ADM mass is non-positive definite, even when the matter energy-momentum tensor $T_{\mu\nu}$ satisfies all of the energy conditions. However, we believe that this reasoning is based upon the wrong choice of mass definition: the physical mass in the Jordan frame is M_T and one can show that M_T is positive in the following way. After a conformal transformation into the Einstein frame, one can show that the new Einstein tensor will obey the dominant energy condition if the Jordan frame matter energy-momentum tensor $T_{\mu\nu}$ obeys the dominant energy condition. Then the Einstein frame ADM mass will be non-negative, by the usual positive energy theorems. However, by analysing the effect of a conformal transformation on the definition of M_{ADM}, one can show that the Einstein frame ADM mass is identical to the Jordan frame Tensor mass, up to a (positive) factor of Φ_∞ [42]. Recently Santiago & Silbergleit [43] have rewritten the Jordan frame field equations of ST gravity by defining a new connection that isolates the dynamical degrees of freedom of the metric gravitational field. Their formalism is algebraically equivalent to making a conformal transformation into the Einstein frame, so it not surprising that their result corroborates with the proof outlined above. The scalar field contribution to the total energy-momentum of the spacetime can now be made non-negative, contrary to the previous claim of Faraoni et al. Here we simply use the conformal relationship between the two frames as a computational aid, and give our results in the physical Jordan frame.

IV. BOUNDARY VALUES AND SCALING RELATIONS

In addition to requiring asymptotic flatness, we impose the boundary conditions

$$A_0 = 1, \quad \sigma_0' = \Phi_0' = C_0' = 0, \tag{47}$$

where the subscript '0' denotes values at $x = 0$. These conditions ensure that the solutions are regular at the origin and, combined with the asymptotic flatness conditions (27), they make the field equations eigenvalue equations for ϖ which automatically lead to the asymptotic conditions $\sigma_\infty' = B_\infty' = \Phi_\infty' = C_\infty' = 0$, $A_\infty = 1$. The values of Φ_∞ and B_∞ are not determined by the field equations, and we next show how they may each be rescaled to take on any value.

The field equations (22) to (26) are invariant under the global rescaling

$$B \to \gamma B, \quad \varpi \to \sqrt{\gamma}\varpi, \quad C \to \sqrt{\gamma} C, \quad \Omega \to \sqrt{\gamma}\Omega, \tag{48}$$

where γ is some constant. Equation (48) simply rescales the magnitude of the time-like Killing vector field, and leaves all solutions physically unchanged. With the gauge freedom allowed by Eq. (48), the field equations become eigenvalue equations for B_0 with Ω arbitrary and their solution in general leads to a value of $B_\infty \neq 1$. We use Eq. (48) to set $B_\infty = 1$ for each solution.

The equations of motion are also invariant under the global rescaling

$$\Phi \to k^2\Phi, \quad \sigma \to k\sigma, \quad C \to kC, \quad q \to \frac{q}{k}, \quad \Lambda \to \frac{\Lambda}{k^2}, \tag{49}$$

where k is some constant, provided $\omega(\Phi)$ is held invariant. For a general ST theory, this will require a change in the functional form of ω.

Equation (49) leaves the ratio S/Φ_∞ and the ADM, Keplerian, Tensor and Newtonian masses invariant and rescales the total charge and particle number as

$$Q \to kQ, \quad N \to k^2 N. \tag{50}$$

Physically, the rescaling exchanges gravitating matter (measured by the number of bosons) for gravitational field energy (which depends qualitatively upon the gravitational coupling strength $G^* = G/\Phi$) in such a way as to leave the total mass of the boson star invariant.

Consider now the set of solutions

$$\mathcal{S}(\sigma_0; \Phi_\infty, q, \Lambda) = \{M(\sigma_0; \Phi_\infty, q, \Lambda),\ N(\sigma_0; \Phi_\infty, q, \Lambda),\ Q(\sigma_0; \Phi_\infty, q, \Lambda)\} \tag{51}$$

parameterised by σ_0 for fixed Φ_∞, q and Λ, where M represents all of the mass values discussed in Section IV. Under the rescaling $\Phi_\infty \to k^2\Phi_\infty$ the mass curves are re-parameterised as

$$M(\sigma_0; \Phi_\infty, q, \Lambda) \to M(k\sigma_0; k^2\Phi_\infty, k^{-1}q, k^{-2}\Lambda) \tag{52}$$

while the particle number and charge curves are both re-parameterised and rescaled as

$$N(\sigma_0; \Phi_\infty, q, \Lambda) \to k^2 N(k\sigma_0; k^2\Phi_\infty, k^{-1}q, k^{-2}\Lambda) \tag{53}$$

$$Q(\sigma_0; \Phi_\infty, q, \Lambda) \to kQ(k\sigma_0; k^2\Phi_\infty, k^{-1}q, k^{-2}\Lambda). \tag{54}$$

Hence, by applying the rescaling (49), $\mathcal{S}$ may be used to generate a new physically distinct set of solutions

$$\begin{aligned} \widetilde{\mathcal{S}}(\sigma_0; k^2\Phi_\infty, q, \Lambda) &= \{\widetilde{M}(\sigma_0; k^2\Phi_\infty, q, \Lambda),\ \widetilde{N}(\sigma_0; k^2\Phi_\infty, q, \Lambda),\ \widetilde{Q}(\sigma_0; k^2\Phi_\infty, q, \Lambda)\} \\ &= \{M(k^{-1}\sigma_0; \Phi_\infty, kq, k^2\Lambda),\ k^2 N(k^{-1}\sigma_0; \Phi_\infty, kq, k^2\Lambda),\ kQ(k^{-1}\sigma_0; \Phi_\infty, kq, k^2\Lambda)\}. \end{aligned} \tag{55}$$

Thus, Eq. (49) may be used to compare boson stars in cosmological settings with different values of Φ_∞.

The existence of this scaling relation means that we need to vary at most two of the parameters Φ_∞, Λ and q in order to investigate completely the properties of a solution in any given theory. For example, by fixing Φ_∞ to some constant, the complete set of solutions may be generated by the subset for which only Λ and q are varied. In addition, if either of these two quantities is zero, we need only vary the other to build up a complete set of solutions. This will be of use when we discuss our numerical results in Section VI. There we shall be primarily concerned with stars for which $\Lambda = 0$ and we shall show complete sets of solutions for several theories by only considering different values of q.

Finally, we comment on another possible choice of mass and how it is affected by Eq. (49). Torres, Schunck & Liddle [44] *define* the Schwarzschild mass by the relation

$$A = \left(1 - \frac{2M^*}{x\Phi_\infty}\right)^{-1}. \tag{56}$$

Note that $A(x)$ is the same metric potential both here and in Ref. [44]. Comparing this expression with Eq. (35) we have $M^* = \Phi_\infty M$, which implies that M^* rescales by a factor of k^2 under Eq. (49) with $q = 0$. This will be important when we analyse the behaviour of M^* in a star undergoing gravitational evolution (see Section VII). They then define the binding energy by using $M^*_\infty = \lim_{x\to\infty} M^*$ together with the rest mass (as we have defined it) to give

$$\mathcal{E}^* = M^*_\infty - N, \tag{57}$$

which also rescales by a factor of k^2 under Eq. (49). Their choice is such that the binding energy of any solution has the same sign when compared with ours and the fractional binding energy (defined as $\mathcal{B} = \mathcal{E}^*/N$ in [44]) is identical for both papers. Thus, if

$$M^*_\infty - N < 0 \tag{58}$$

then

$$M_\infty - M_N \;=\; \frac{M^*_\infty}{\Phi_\infty} - M_N \;=\; \frac{M^*_\infty}{\Phi_\infty} - \frac{N}{\Phi_\infty} < 0 \tag{59}$$

and

$$\frac{M^*_\infty - N}{N} \;\equiv\; \frac{M_\infty - M_N}{M_N}. \tag{60}$$

Here we have used the ADM mass (as we have defined it) instead of the Tensor mass since Torres et al. use the former in [44]. As they point out, for the large values of ω they consider, the numerical difference between M_{ADM} and M_T is negligible.

Since M^* and the masses we define in this paper are measured in the same units, it is apparent that they measure physically different quantities. As mentioned in the text after Eq. (35), the masses we define implicitly include a factor of $1/\Phi_\infty$. This means that M_T (or M_{ADM} when ω is very large) measures the total gravitational energy and M_K measures the active gravitational mass as seen by an orbiting test particle with negligible self-gravity. Our mass definitions correspond to the product $G\mathcal{M}$ of Newtonian theory, where $\mathcal{M}$ is the Newtonian rest mass. The Newtonian analogue of Eq. (48) is $\mathcal{M} \to k^2\mathcal{M}$, $G \to G/k^2$ which leaves $G\mathcal{M}$ invariant: the rescaling swaps energy between the matter and the gravitational field. In the weak field limit and when ω is large, M^*_∞ describes the rest mass of the star in which the gravitational coupling strength $1/\Phi_\infty$ has been factored out (see the discussion after Eq. (83) in the following Section). Thus M^*_∞ roughly corresponds to $\mathcal{M}$. However, for strong field solutions one cannot easily interpret M^*_∞, since one cannot decouple the gravitational field strength from the total energy.

V. WEAK FIELD LIMIT AND THE LIMITING CHARGE

In this Section we discuss the weak field limit of the solutions which we define as the limit in which σ_0 is small but non-zero. The analysis that follows is in part a generalisation of the approach followed by Kaup [45] when dealing with uncharged boson stars in GR and involves expanding the solutions in power series about flat spacetime. We start by defining a function $\Pi(x)$ which has the boundary value $\Pi_0 = 1$ and the functional form given implicitly by

$$\sigma = \varepsilon\Pi. \tag{61}$$

The constant parameter $\varepsilon := \sigma_0/\Pi_0$ measures the degree to which the solution differs from flat spacetime and we shall use it as the expansion parameter in the analysis. We also define a rescaled radial coordinate by $a := \sqrt{\varepsilon}x$. Hence, Π is now considered to be a function of a. Note that, by definition, $\Pi(a)$ has the same asymptotic form as σ and decreases exponentially as $a \to \infty$.

The field variables and energy parameter we expand in powers of ε about flat spacetime. We implicitly define new field variables α, β, Σ and Γ by the first order expansions

$$A = 1 + \varepsilon\alpha(a) + \mathcal{O}(\varepsilon^2), \tag{62}$$

$$B = 1 + \varepsilon\beta(a) + \mathcal{O}(\varepsilon^2), \tag{63}$$

$$\Phi = \Phi_\infty + \varepsilon\Sigma(a) + \mathcal{O}(\varepsilon^2), \tag{64}$$

$$C = \frac{\varepsilon}{2}\Gamma(a) + \mathcal{O}(\varepsilon^2) \tag{65}$$

and

$$\Omega = 1 - \frac{\varepsilon}{2}E + \frac{\varepsilon}{2}q\Gamma(a) + \mathcal{O}(\varepsilon^2), \tag{66}$$

where Φ_∞ is the value of the scalar field at space-like infinity and E is a constant for each solution. We assume that the scalar field coupling parameter has the form

$$\omega = \omega_\infty + \varepsilon W(a) + \mathcal{O}(\varepsilon^2), \quad \frac{d\omega}{d\Phi} = \Delta(a) + \mathcal{O}(\varepsilon), \tag{67}$$

where ω_∞ is a constant, $\Delta = \lim_{\varepsilon\to 0}\left(\frac{d\omega}{d\Phi}\right)$ and we have written the second term in the expansion for ω explicitly in terms of a. These assumptions restrict the applicability of the following analysis to a subset of all possible scalar-tensor theories. However, Eqs. (67) are satisfied for both BD theory and the power law theory. In the latter case we have

$$\omega = \frac{2n}{3\Phi_\infty^n}\left(\Phi_\infty + \varepsilon\Sigma + \mathcal{O}(\varepsilon^2)\right)^n - \frac{3}{2}. \tag{68}$$

Expanding this expression binomially, we see that the power law theory has $W = 2n^2\Sigma/(3\Phi_\infty)$ and $\Delta = 2n^2/(3\Phi_\infty)$.

In terms of the new variables the field equations (22) to (26) become

$$(a\alpha)' = \frac{4\Pi^2 a^2(\omega_\infty + 1)}{\Phi_\infty(2\omega_\infty + 3)} + \mathcal{O}(\varepsilon), \tag{69}$$

$$\beta' = \frac{\alpha}{a} - \frac{2}{\Phi_\infty}\frac{d\Sigma}{da} + \mathcal{O}(\varepsilon), \tag{70}$$

$$\left(a^2\beta'\right)' = \frac{4\Pi^2 a^2(\omega_\infty + 2)}{\Phi_\infty(2\omega_\infty + 3)} + \mathcal{O}(\varepsilon), \tag{71}$$

$$\left(a^2\Pi'\right)' = \Pi(E + \beta - q\Gamma) + \mathcal{O}(\varepsilon), \tag{72}$$

$$\left(a^2\Sigma'\right)' = \frac{-2\Pi^2 a^2}{2\omega_\infty + 3} + \mathcal{O}(\varepsilon) \tag{73}$$

and

$$\left(a^2\Gamma'\right)' = 2q\Pi^2 a^2 + \mathcal{O}(\varepsilon), \tag{74}$$

where in this Section a prime denotes d/da. We have derived Eq. (71) by differentiating Eq. (70) and substituting in Eqs. (69) and (73). Equations (62) to (65) taken together with the asymptotic conditions (27) and (28) imply that we must impose the boundary conditions

$$\alpha_\infty = \beta_\infty = \Gamma_\infty = \Sigma_\infty = 0. \tag{75}$$

Equations (71), (73) and (74) imply that both β and Γ are strictly increasing functions of a while Σ is strictly decreasing. Combining these properties with the boundary conditions above we have $\beta \leq 0$, $\Gamma \leq 0$ and $\Sigma \geq 0$, where the equalities are only true in the limit $a \to \infty$. From Eq. (66), the requirement that $\Omega_\infty = \varpi/m < 1$ and the vanishing of Γ_∞ imply that $E \geq 0$, where the equality is only true when $\Pi = 0$.

Combining Eqs. (32) and (74) we have

$$Q = \tfrac{1}{2}\sqrt{\varepsilon}\lim_{a\to\infty}\left(a^2\frac{d\Gamma}{da}\right) = \sqrt{\varepsilon}\int_0^\infty q\Pi^2 a^2\, da + \mathcal{O}\left(\varepsilon^{3/2}\right) := \sqrt{\varepsilon}q\mathcal{N} + \mathcal{O}\left(\varepsilon^{3/2}\right), \tag{76}$$

where we have defined

$$\mathcal{N} := \int_0^\infty a^2\Pi^2\, da. \tag{77}$$

The quantity $\sqrt{\varepsilon}\mathcal{N}$ is the rest mass (or particle number) to lowest order in ε. From the definition (31) the Newtonian mass in the weak field limit is given by

$$M_N = \frac{\sqrt{\varepsilon}\mathcal{N}}{\Phi_\infty} + \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{78}$$

From Eqs. (33) and (73), the scalar charge is given by

$$S = \sqrt{\varepsilon}\lim_{a\to\infty}\left(a^2\frac{d\Sigma}{da}\right) = -\sqrt{\varepsilon}\int_0^\infty \frac{2\Pi^2 a^2}{2\omega_\infty+3}\, da + \mathcal{O}\left(\varepsilon^{3/2}\right) = -\frac{2\sqrt{\varepsilon}\mathcal{N}}{2\omega_\infty+3} + \mathcal{O}\left(\varepsilon^{3/2}\right), \tag{79}$$

where we have used Eq. (77) to obtain the final equality. Integrating Eq. (69) and using the definition (35), we obtain the weak field ADM mass

$$M_{ADM} = \frac{2\sqrt{\varepsilon}(\omega_\infty+1)\mathcal{N}}{\Phi_\infty(2\omega_\infty+3)} + \mathcal{O}\left(\varepsilon^{3/2}\right), \tag{80}$$

where again we have used Eq. (77). Combining Eqs. (79) and (80) with the definitions (37) and (42) gives the weak field tensor mass

$$M_T = \frac{\sqrt{\varepsilon}\mathcal{N}}{\Phi_\infty} + \mathcal{O}(\varepsilon) \tag{81}$$

and the weak field Keplerian mass

$$M_K = \frac{2\sqrt{\varepsilon}(\omega_\infty+2)\mathcal{N}}{\Phi_\infty(2\omega_\infty+3)} + \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{82}$$

Equations (81) and (82) show that the Keplerian and Tensor masses may be written as the products $M_K = \sqrt{\varepsilon}G_K\mathcal{N}$ and $M_T = \sqrt{\varepsilon}G_T\mathcal{N}$, where the coupling strengths G_K and G_T are given by

$$G_K = \frac{2(\omega_\infty+2)}{\Phi_\infty(2\omega_\infty+3)}, \quad G_T = \frac{1}{\Phi_\infty}. \tag{83}$$

Hence, the weak field solutions are Newtonian in the sense that the Keplerian and Tensor masses, which are the active masses to which test particles respond, may be decomposed as products of the star's rest mass and a coupling strength. In addition we have $M_T = M_N$ to lowest order in ε, which motivates the labelling of the quantity defined in Eqn. (31) as the Newtonian mass.

To calculate the fractional binding energy, we need to write M_T and M_N up to order $\varepsilon^{3/2}$. Rewriting the integrals (34) and (36) in terms of the weak field variables, one can show that the ADM mass and scalar charge are given by

$$\begin{aligned} M_{ADM} = {} & \frac{2\sqrt{\varepsilon}(\omega_\infty+1)\mathcal{N}}{\Phi_\infty(2\omega_\infty+3)}\, da + \varepsilon^{3/2}\int_0^\infty \frac{a^2}{\Phi_\infty}\left[\frac{\Pi'^2(2\omega_\infty+1)}{2(2\omega_\infty+3)} + \frac{\Gamma'^2}{8} - \frac{\Sigma'\beta'}{4} + \frac{\Sigma'^2\omega_\infty}{4\Phi_\infty}\right. \\ & \left. - \frac{\Delta\Sigma'^2}{2(2\omega_\infty+3)} + \frac{\Pi^2}{(2\omega_\infty+3)}\left(\frac{2W}{2\omega_\infty+3} + (2\omega_\infty+5)(q\Gamma - E - \beta) - \frac{2\Sigma}{\Phi_\infty}(\omega_\infty+1)\right)\right] da + \mathcal{O}\left(\varepsilon^{5/2}\right) \end{aligned} \tag{84}$$

and

$$\begin{aligned} S = {} & \frac{-2\sqrt{\varepsilon}\mathcal{N}}{2\omega_\infty+3} + \varepsilon^{3/2}\int_0^\infty a^2\left[-\frac{2\Pi'^2}{2\omega_\infty+3} - \frac{\Delta\Sigma'^2}{2\omega_\infty+3} + \frac{\Sigma'}{2}(\alpha'-\beta')\right. \\ & \left. + \frac{2\Pi^2}{(2\omega_\infty+3)}\left(q\Gamma - E - \beta - \alpha + \frac{2W}{(2\omega_\infty+3)}\right)\right] da + \mathcal{O}\left(\varepsilon^{5/2}\right). \end{aligned} \tag{85}$$

Similarly, rewriting Eq. (29) and using the definition (30) we have

$$M_N = \frac{\sqrt{\varepsilon}\mathcal{N}}{\Phi_\infty} + \varepsilon^{3/2}\int_0^\infty \frac{a^2\Pi^2}{2\Phi_\infty}\left(q\Gamma - \beta - E + \alpha\right)\, da + \mathcal{O}\left(\varepsilon^{5/2}\right). \tag{86}$$

Combining Eqs. (37), (84), (85) and (86) with the expression (39) for the fractional binding energy we have

$$\mathcal{B} = \frac{\varepsilon}{\mathcal{N}}\int_0^\infty a^2\left[\frac{\Sigma'^2\omega_\infty}{4\Phi_\infty} - \frac{\Sigma'\alpha'}{4} + \frac{\Pi'^2}{2} + \frac{\Gamma'^2}{8} + \frac{\Pi^2}{2}\left(\frac{-\alpha(2\omega_\infty+1)}{2\omega_\infty+3} - \frac{4\Sigma(\omega_\infty+1)}{\Phi_\infty(2\omega_\infty+3)}\right)\right]\, da + \mathcal{O}\left(\varepsilon^2\right). \tag{87}$$

We need to simplify this expression considerably. To do this we integrate Eq. (87) by parts in order to write each term as some numerical factor times the same positive definite integral. We carry out each partial integration separately below, noting that all surface terms vanish due to the asymptotic conditions (75) and the fact that all of the fields are finite at $a = 0$.

Using the wave equation (72) one can show that

$$\int_0^\infty a^2\Pi'^2\, da = -\int_0^\infty a^2\Pi^2(E + \beta - q\Gamma)\, da. \tag{88}$$

Integrating the product $a\Pi'(a^2\Pi')'$ by parts and using Eq. (88) one can also show that

$$\int_0^\infty a^2\Pi'^2\, da = \tfrac{1}{2}\int_0^\infty a^3\Pi^2(\beta' - q\Gamma')\, da. \tag{89}$$

Integrating $a^2\Sigma'\alpha'$ by parts and using Eqs. (70) and (73) we have

$$\int_0^\infty a^2\Sigma'\alpha'\, da = \int_0^\infty \frac{2\Pi^2a^3}{2\omega_\infty+3}\left(\beta' + \frac{2\Sigma'}{\Phi_\infty}\right)\, da. \tag{90}$$

Integrating both $a^2\Sigma'^2$ and the product $a\Sigma'(a^2\Sigma')'$ by parts, using the wave equation (73) and combining the results one can show that

$$\int_0^\infty a^2\Sigma'^2\, da = -\int_0^\infty \frac{4a^3\Pi^2\Sigma'}{2\omega_\infty+3}\, da = \int_0^\infty \frac{2a^2\Pi^2\Sigma}{2\omega_\infty+3}\, da. \tag{91}$$

Similarly, one can also show that

$$\int_0^\infty a^2\Gamma'^2\, da = \int_0^\infty 4a^3\Pi^2 q\Gamma'\, da = -\int_0^\infty 2a^2\Pi^2 q\Gamma\, da. \tag{92}$$

Using Eq. (69) we have

$$\int_0^\infty a^2\Pi^2\alpha\, da = \int_0^\infty a^3\Pi^2\left(\beta' + \frac{2\Sigma'}{\Phi_\infty}\right)\, da. \tag{93}$$

Substituting Eqs. (89) to (93) into our expression for the fractional binding energy (87) gives

$$\mathcal{B} = \frac{\varepsilon}{\mathcal{N}}\int_0^\infty a^3\Pi^2\left[\frac{q\Gamma'}{4} + \frac{1}{2\omega_\infty+3}\left(\frac{\Sigma'}{\Phi_\infty}(\omega_\infty+2) - \frac{\beta'}{4}(2\omega_\infty+1)\right)\right]\, da + \mathcal{O}\left(\varepsilon^2\right). \tag{94}$$

To simplify this expression further, we use Eq. (74) to obtain

$$\int_0^\infty a^3\Pi^2\Gamma' = \int_0^\infty a\Pi^2\left(a^2\Gamma'\right)\, da = \int_0^\infty 2qXa\Pi^2\, da, \tag{95}$$

where we have defined a new function

$$X(a) := \int_0^a \tilde{a}^2\Pi^2\, d\tilde{a}. \tag{96}$$

Similarly, from Eqs. (71) and (73) we have

$$\int_0^\infty a^3\Pi^2\beta'\, da = \int_0^\infty \frac{4Xa\Pi^2(\omega_\infty+2)}{\Phi_\infty(2\omega_\infty+3)}\, da \tag{97}$$

and

$$\int_0^\infty a^3\Pi^2\Sigma'\, da = -\int_0^\infty \frac{2Xa\Pi^2}{2\omega_\infty+3}\, da. \tag{98}$$

The quantity Π^2 is positive or zero over the interval of the integration which implies that X is an increasing positive function of a that vanishes at $a = 0$. The exponential decrease of Π in the limit $a \to \infty$ implies that X has a finite limit X_∞. Hence the integrals (95), (97) and (98) are finite and substituting them into Eq. (94) gives

$$\mathcal{B} = \frac{\varepsilon}{\mathcal{N}}\left(\frac{q^2}{2} - \frac{\omega_\infty+2}{\Phi_\infty(2\omega_\infty+3)}\right)\int_0^\infty aX\Pi^2\, da + \mathcal{O}\left(\varepsilon^2\right). \tag{99}$$

This result is independent of Λ and is true for any form of the coupling parameter $\omega(\Phi)$ that satisfies Eq. (67).

Equation (99) implies that the fractional binding energy is negative provided

$$q < q_{max}(\omega_\infty, \Phi_\infty) = \sqrt{\frac{2(\omega_\infty+2)}{\Phi_\infty(2\omega_\infty+3)}}\,. \tag{100}$$

For the star to be stable $\mathcal{B}$ must be negative. Hence Eq. (100) places an upper limit on the charge to mass ratio of the bosons that form a stable star in the weak field limit. For a star with $q > q_{max}$ the Coulomb repulsion dominates over the gravitational attraction and no stable solution exists. Note that, as one would intuitively expect, $q_{max} \propto \sqrt{1/\Phi_\infty}$ for fixed ω, so that the greater the gravitational coupling strength, the greater the amount of charge that can be bound into a stable object. In Section VI we shall numerically examine how q_{max} is modified for strong field solutions.

For finite ω we have $q_{max} > 1$ for $\Phi_\infty = 1$ which implies that in ST gravity it is possible to construct a stable star from bosons whose charge is greater than that allowed by GR. In the GR limit, where $\omega \to \infty$, Eq. (100) gives $q_{max} = 1$, which is the upper limit on q given by Jetzer and van der Bij by just comparing the electrostatic and electromagnetic forces [7] (note that our unit of charge differs from theirs by a factor of $\sqrt{2}$). Hence, the above derivation serves also as a detailed proof of the maximum charge in General Relativity.

VI. NUMERICAL SOLUTIONS

In this Section we discuss strong field solutions of the field equations (22) to (26) for both BD theory and the power law ST theory. In their full form, the equations are complicated enough to require numerical integration. To generate each solution in the set $\mathcal{S}$, the numerical routine that finds the eigenvalue B_0 must be nested inside a second iterative routine that searches for the the value of Φ_0 that gives a value of Φ_∞ common to all solutions in $\mathcal{S}$. When generating each solution, the numerical routine halts at the value of x at which σ becomes too small for the code to proceed further, typically at a value $\sigma \sim 10^{-10}$. Then the Tensor mass is calculated using Eq (46) with the limit replaced by values of x near to the termination point of the routine and a power series approximation about $x = \infty$ is used to estimate M_T for the solution. A similar power series method is used to calculate N and S, based on the behaviour of $\Phi(x)$ and the integral of dN/dx out to the termination point.

For the sake of ease of comparison, we have chosen $\Phi_\infty = 1$ for all of the solution sets we discuss here, and this choice implies that M_N and N are numerically identical for each solution. The sets of equilibrium solutions are shown in Figures 1 to 5. In each of these Figures we show Tensor mass and Newtonian mass curves as a function of σ_0 for several sets of solutions within the same theory, choosing a different value of q for each pair of curves. The results of the numerical integrations are then used to construct Figures 6 to 9. We shall discuss each figure separately below.

Figure 1 shows the mass curves for several sets of solutions in the $\omega = 500$, $\Lambda = 0$ BD theory, for which the weak field charge limit is $q_{max} = 1.001$. As one would expect from such a large value of ω, the curves in Figure 1 are virtually indistinguishable from the mass and particle number curves calculated for GR boson stars in [7]. As q approaches q_{max} from below, the locations of the maxima in the M_T and M_N curves shift to lower values of σ_0 and the binding energies $\mathcal{E}$ decrease in magnitude, while the gradients $dM_T/d\sigma_0$ and $dM_N/d\sigma_0$ of the curves near $\sigma_0 = 0$ increase. In addition, as q increases, the mass curves shift to successively higher values, since a star consisting of bosons with large q must generate a strong gravitational field to overcome its own Coulomb repulsion. This is achieved by an increase in N (hence an increase in M_N) which leads to an increase in M_T. Since ω is large for these solutions, Φ

remains approximately homogeneous throughout each star in each solution set shown in the Figure. For example, in the maximum mass $q = 0.99$ solution, Φ_0 and Φ_∞ differ by less that 1%. Hence the solutions in the Figure show no strong scalar-tensor gravitational effects.

In Figure 2 we show mass curves for several sets of solutions in the $\omega = -1$, $\Lambda = 0$ BD theory. As mentioned in Section I, this choice of the coupling constant may be relevant to the study of boson stars in the very early Universe, although in this case our value of $\Phi_\infty = 1$ is physically unrealistic: to approximate a boson star embedded in the early (possibly string theory dominated) Universe, we should choose a value $\Phi_\infty < 1$. However, from the scaling relation (49), rescaling the asymptotic value of Φ merely rescales the horizontal axis of Figure 2 and leaves the form of the curves invariant (although for $\Phi_\infty \neq 1$, M_N and N are no longer numerically equal and q must be rescaled). Hence the figure may be used as a template to generate $\omega = -1$ boson stars solutions with any value of Φ_∞.

For the $\omega = -1$ BD theory, the weak field charge limit is $q_{max} = \sqrt{2}$. As Figure 2 shows, this limit does not extend to the strong field solutions (in contrast with the GR charged boson stars for which the limit $q_{max} = 1$ holds for all values of σ_0). The mass curves of the solution set for which $q_{max} > q > q_c$, where q_c is some critical value of q, diverge at finite values of σ_0, and the value of σ_0 at which this happens decreases with increasing q. The mass curves of the solution set satisfying $q < q_c$ do not diverge, although for the values of q presented here the value of σ_0 at which the mass curves are maximal increases with q. We have numerically found that $q_c \approx 0.90$ for the $\omega = -1$ solutions. This value is approximate since the equations are very hard to integrate when q is close to q_c.

Physically, we may understand the reason for the divergent behaviour of the masses as follows. For each solution, the ratio $\Phi_0/\Phi_\infty > 1$ and this quantity increases with σ_0 since Φ is strongly coupled to the curvature. Hence, within any particular solution set satisfying $q > q_c$, the active gravitational mass of each boson (measured qualitatively by the product $m\Phi^{-1}$) decreases as σ_0 increases. At some finite σ_0, the number of bosons with charge q needed to generate a gravitational field sufficient to overcome their own Coulomb repulsion diverges. The results shown in Figure 2 imply that the limiting charge q_{max} is a decreasing function of σ_0 that has the weak field value in the limit $\sigma_0 \to 0$ and asymptotes towards q_c as $\sigma_0 \to \infty$. Solutions with $q > q_{max}(\sigma_0)$ do not exist, just as in the GR case.

Figure 3 shows sets of solutions for $\omega = 1$ BD gravity with $\Lambda = 0$. The weak field charge limit in this case is $\sqrt{1.2}$. The coupling between Φ and the curvature is weaker than it is in the $\omega = -1$ theory and the mass curves show behavior intermediate between those in Figures 1 and 2. For values of q up to $q \approx 0.9$, the location of the maxima in the M_T and M_N curves decreases with increasing q. As q increases further, the location of the maxima starts to shift towards higher values of σ_0 until the mass curves diverge at $q = q_c \approx 0.976$. This value is smaller than the $\omega = -1$ critical charge and the numerical solutions suggest that q_c is also a decreasing function of ω for fixed σ_0 that only approaches the GR charge limit as $\omega \to \infty$.

Figure 4 shows mass curves for the power law ST theory, where $2\omega(\Phi) + 3 = \frac{4}{3}n\Phi^n$. We have set $n = 4$ in these solutions; this parameter choice is in agreement with current observational constraints on this kind of ST theories [46]. These are based on primordial nucleosynthesis calculations, which for this theory are stronger than the constraints imposed by solar system tests. (In fact, weak field test do not impose any bound on the exponent of the power law. This is a consequence of the form of the matter era cosmological solutions [47], which are such that $\omega \to \infty$ and $\omega^{-3}d\omega/d\Phi \to 0$ when $t \to \infty$ for all n.) As with the BD solutions, the mass curves diverge for values of q greater than the critical charge, which for this theory is $q_c \approx 0.986$. The $n = 4$ power law boson stars could conceivably be observed today, and it is interesting that their mass curves are very different from their GR counterparts.

We next outline how one converts from the dimensionless charge unit we use here to the SI unit of charge. Rewriting the weak field relationship (100) as an equation that balances gravitational and electrostatic forces on a boson in the asymptotic region of the star, one finds that the charge $\widetilde{e}$ of each boson, measured in coulombs, is given by

$$\widetilde{e} = qm\sqrt{\frac{4\pi\epsilon G}{\Phi_\infty}} \tag{101}$$

where ϵ is the permitivity of the boson star. For a star of particle number N, the total charge $\widetilde{Q}$, measured in coulombs, is

$$\widetilde{Q} = qm\sqrt{\frac{4\pi\epsilon G}{\Phi_\infty}}N\left(\frac{M_{pl}}{m}\right)^2. \tag{102}$$

For example, if we take $m = 30GeV \approx 5.3 \times 10^{-26}$ kg, then $\widetilde{e} \approx (4.5 \times 10^{-36})q$ C. For a boson star with $N = q = 1$, one has $\widetilde{Q} \approx 1$ C. This charge is small. However, such a star will have a very small radius (approximately 10^{-17} m) so that its charge density is very large.

In all of these solutions we have found that, for a given N and q, the magnitude of the scalar charge S increases while the mass M_T decreases, as the coupling strength $1/\omega$ is increased. This implies that for a given number of bosons of charge q, the magnitude of the binding energy $\mathcal{E}$ increases with the coupling strength. Thus the bosons become more tightly bound even though the mass is smaller in the strong coupling case. One can understand this apparent discrepancy as follows. For all of these solutions, the scalar charge is negative and, comparing Eqs. (37) and (42), this implies that $M_K > M_T$, where the difference between these two quantities increases with the coupling strength. It is even possible for M_K to exceed M_N for particularly strong couplings (see [38] for an example). One can show that the fractional binding energies $\mathcal{B}$ are reasonably small for all of these solutions (even for the rather extreme $\omega = -1$ solutions, for which $\mathcal{B}$ never exceeds 0.2), so that to a first order of approximation the bosons that make up the atmosphere of a given star may be treated as a cloud of non self-gravitating test particles. Here, the term atmosphere rather loosely describes the asymptotic region of the star which encloses most of the mass, electrical charge and scalar charge. However, as mentioned at the end of Section III, non self-gravitating test particles respond to the Keplerian mass. Hence, in Newtonian terms, the gravitational force experienced by the bosons in the atmosphere tends to increase as the coupling strength increases, even though the energy of the star decreases. For a star with $q < q_c$, the bosons in the central region of the star experience much the same conditions. The effects described here are entirely separate to the weakening of the gravitational field strength within the star caused by the increase of the ratio Φ_0/Φ_∞ with coupling strength. Although this reduces the Keplerian mass, in a star with $q < q_c$ this effect is swamped by the increase in the scalar charge. For stars with $q > q_c$, the weakening of the gravitational field strength dominates over the increase in scalar charge and the divergent behaviour discussed above occurs. The fact that the scalar charge is large for strongly coupled stars will be important when we discuss the scalarization phenomenon below.

Finally we consider the effect of introducing a quartic self-interaction term into the matter Lagrangian. For the GR solutions, the inclusion of the self-interaction increases the mass of the stars but does not alter the value of q_{max} [7]. We have found that for ST theories, the inclusion of the Λ term not only increases the mass but also slightly decreases the charge limit of the strong field solutions. To illustrate this, Figure 5 shows mass curves for the $\omega = 1$ BD theory in the limit $\Lambda \to \infty$. The field equations in this limit are obtained by making the change of variables $\sigma \to \sigma^* = \sigma\Lambda^{-1/2}$ and $x \to x^* = x\Lambda^{1/2}$, substituting these new variables into the field equations (22) to (26), and taking the limit $\Lambda \to \infty$ (see [3], [7] and [17] for details). The resulting field equations are identical to Eqs. (22) to (26) except that the σ'^2 terms vanish, $\sigma^4\Lambda$ is replaced by σ^{*4}, σ and x are replaced by σ^* and x^* respectively, and Eq. (25) is replaced by the algebraic equation

$$\sigma^{*2} = \frac{\Omega^2}{B} - 1. \tag{103}$$

The masses are now measured in units of $\Lambda^{1/2}M_{pl}^2/m$, while N is measured in units of $\Lambda^{1/2}M_{pl}^2/m^2$. This implies that one must insert a factor of $\Lambda^{1/2}$ in Eq. (102) to find the total charge on these stars in SI units.

The curves in Figure 5 are qualitatively similar to those in Figure 3 and the weak field value q_{max} is the same as in the $\Lambda = 0$ case, since the limit imposed by Eq. (100) is independent of Λ. However, the critical charge has decreased to a value $q_c \approx 0.905$. This is to be expected, since a factor of $\sigma^4\Lambda$ appears in the source term of the scalar wave equation (25) which implies that a large value of Λ leads to larger inhomogeneities in Φ. Nevertheless, this effect is still fairly minimal. Numerical calculations for other choices of ω show similar behaviour.

Figures 6 and 7 are derived from the numerical output used to generate Figure 4. Figure 6 is a bifurcation diagram for the stars. The first five curves are for solutions with $q < q_c$ and each shows a cusp at which the solutions become unstable. The remaining three curves, with $q > q_c$, show no cusps and all are stable. The other solution sets we have analysed have similar diagrams and all of the solutions with $q > q_c$ are stable. However, these objects have such large total charges and such small binding energies that their formation may be halted by their own Coulomb repulsion.

Figure 7 shows the behaviour of the coupling parameter $\alpha := -S/M_N$ for the power law boson stars. Note that α here is different from the weak field metric potential defined in Section V. This coupling parameter measures the strength of the coupling between the scalar field and the normal matter (in this case the bosonic matter). In the GR limit, $\alpha \to 0$. As the Figure shows, this quantity is non-negligible even though the theory satisfies the current observational constraints.

The coupling parameter α was first introduced by Damour and Esposito-Farese in the study of neutron star equilibrium solutions [16]. They found that a phenomenon known as spontaneous scalarization (SS) develops in the solutions: beyond a certain state of compactness, a ST neutron star develops a non-trivial scalar field configuration even when the coupling parameter ω of the ST theory is chosen so that $\omega \to \infty$ far from the star. Their results are given in Figure 2 of Ref. [16] and show that, when the baryonic mass M_B of the star is less that some critical value, α is negligible, while as M_B is increased beyond this value, α rapidly grows to a maximum before decreasing again as M_B is increased to its maximum value. M_B is analogous to M_N for bosons stars and Figure 7 shows that the power law boson stars exhibit similar behaviour to neutron stars for large value of M_N but differ considerably when M_N is small. In this case, α increases towards some limiting value as M_N decreases. This limit may be calculated as follows. As $M_N \to 0$, $\sigma_0 \to 0$ and from the weak field equations (78) and (79) we have

$$\lim_{\sigma_0 \to 0} \alpha = -\lim_{\sigma_0 \to 0} \left(\frac{S}{M_N} \right) = \frac{2\Phi_\infty}{2\omega_\infty + 3}. \tag{104}$$

The difference between these two behaviours may be due to the choice of the coupling parameter $\omega(\Phi)$. In Ref. [16], this coupling has the form $2\omega + 3 \sim 1/(\log \Phi)$ which implies that, in the spatially asymptotic region, $\omega \to \infty$ for the choice $\Phi_\infty = 1$. For the power law boson stars we have $\omega_\infty = 2/3\, n - 3/2$ for $\Phi_\infty = 1$, which is finite.

VII. GRAVITATIONAL EVOLUTION

When we consider a boson star (or any other compact object) in an evolving cosmological background, we have to take into account the influence of the time varying cosmological value of Φ on the structure of the star. To a first approximation, and in want of a more rigorous analytical analysis, we assume that Φ_∞ increases slowly with cosmological time and that the boson star evolves quasi-statically so that, at any point in its evolution, it is described by a static equilibrium solution with the appropriate value of Φ_∞. This approximation is a good one if the free fall timescale of the star is much smaller than the characteristic timescale over which $G^\star$ varies, and is often made when considering other compact objects such as white dwarfs. Stars with smaller values of Φ_∞ are assumed to exist at earlier cosmological times, since in a general ST cosmological model Φ tends to increase with time. The star is modelled as a sequence of asymptotically flat equilibrium solutions, where each star in the sequence has a value of Φ_∞ greater than the preceding (earlier time) star. This continues the approach adopted by Torres et al. [29,44], who showed how an increase in Φ_∞ affects the physical characteristics of the star. They particularly discuss the evolution of a star for which σ_0 remains constant, although they also consider some other cases of more general evolution.

In this Section we discuss the possible gravitational evolution of both charged and uncharged boson stars of fixed particle number N. The assumption that N is constant during the evolution of an asymptotically flat configuration is justified since in this case, Eq. (30) is valid and N is a conserved charge. However, in a cosmological setting (in which the spacetime is not asymptotically flat), we cannot in general write a global conservation law for any quantity, but instead must approximate a quasi-local conservation equation. This problem is not unique to ST boson stars in a cosmological setting: for any metric theory of gravity the formulation of global conservation laws in a general spacetime is very difficult. In this analysis we assume that the asymptotic region of the boson star matches smoothly to a homogeneous cosmological solution at some large value x_1 of the radial coordinate where the boson field amplitude σ is negligible. We assume that the quasi-local value of the particle number of the star in a cosmological setting (evaluated as an integral out to the matching surface $x = x_1$ over some space-like hypersurface) is approximately equal to N, the particle number the star would have if isolated in an asymptotically flat spacetime. Note that the Jordan frame representation of ST gravity that we use in this paper embodies the Einstein Equivalence Principle (defined in [14]) so that local charge conservation holds and q is constant during the star's evolution. Hence if N is constant, this implies that Q remains constant. However, as we shall show below, M_T is not conserved during the evolution. Physically, the assumption that N is constant is reasonable since classically, this quantity may be literally interpreted as a count of the number of bosons in the star, a procedure that is coordinate independent. The mass, on the other hand, is treated as a quasi-local quantity when the star is in a cosmological setting, and there are no conservation laws or physical arguments to constrain its behaviour.

We first examine the evolution of an uncharged boson star of particle number N with $\Lambda = 0$ in BD gravity. We make use of the notation of Section IV in which a set of solutions is denoted $\mathcal{S}(\sigma_0; \Phi_\infty, q, \Lambda)$, where Φ_∞, q and Λ are fixed and σ_0 is a parameter that labels the different elements of $\mathcal{S}$. In the discussion that follows, σ_0 is an independent variable while σ_1 and σ_2 are particular values of this variable that label particular solutions within the set.

Let the initial asymptotic value of the scalar field be $\Phi_1 = \Phi_\infty(t_1)$ at some initial time t_1 so that the star is a member of the set $\mathcal{S}_1(\sigma_0; \Phi_1, q = 0, \Lambda = 0)$. We assume that the star is initially stable, which implies that it lies to the left of the first maximum in the $N(\sigma_0)$ curve [39]. Then the pair (N, Φ_1) uniquely determines the initial central amplitude σ_1 of the boson field. The star will have a mass $M_1 = M_T(\sigma_1, \Phi_1, q = 0, \Lambda = 0)$. Here σ_1 is the particular value of σ_0 that gives the element of $\mathcal{S}_1$ with particle number N. At some later time t_2, the asymptotic value of the scalar field has evolved to $\Phi_2 = \Phi_\infty(t_2) = k^2\Phi_1$ where $k > 1$. At this time the star is a member of the set $\mathcal{S}_2(\sigma_0; \Phi_2, q = 0, \Lambda = 0)$ with particle number N. In general, this solution will have a central scalar field amplitude σ_2 different from σ_1. From Eqs. (51) and (55), the later time solution is generated by a rescaling of the member of $\mathcal{S}_1$ that has particle number $k^{-2}N$. Since $k > 1$, and since for stable stars in $\mathcal{S}_1$ the particle number curve $N(\sigma_0)$

obeys $dN/d\sigma_0 > 0$, this generating solution has $\sigma_2 < \sigma_1$. For stable solutions $dM_T/d\sigma_0 > 0$, which implies that the mass $M_2 = M_T(\sigma_2, \Phi_1, q = 0, \Lambda = 0)$ of the generating solution obeys the relation $M_2 < M_1$. To find the physical characteristics of the new solution at time t_2 we must use Eq. (49) to rescale the parameters of the generating solution using the rescaling factor k. However, M_T is invariant under this rescaling which implies that the later time boson star has a smaller mass than the earlier time solution. Repeating this process for successively higher values of Φ_∞ (or, equivalently, later cosmological times), one can see that the physical properties of the star during its evolution may be calculated from the single initial solution set $\mathcal{S}_1$: as the cosmological time increases, the generating solution is located at an ever decreasing value of σ_0 and its mass, as well as the mass of the physical solution it generates, decreases. In addition, the scalar field becomes increasingly homogeneous and its central value decreases, as viewed from the rest frame of the star. Physically, the mass loss may be accounted for by the generation of scalar gravitational radiation, which carries energy out from the star, and the weakening of the gravitational coupling strength. As pointed out in [44], within a single solution set the radius of a stable boson star is an increasing function of $1/\sigma_0$ (stars approach infinite radius in the weak field limit $\sigma_0 \to 0$). The radius, like the mass, is invariant under Eq. (49), so that this quantity also increases with cosmological time. Note that if the star starts out as being stable at time t_1, then it will remain stable throughout its evolution.

Figure 8 shows evolution curves for four stable, uncharged boson stars in $\omega = -1$ BD gravity, where each star has a different particle number N. Each star evolves from an initial equilibrium solution with Φ_∞=1 to a configuration with $\Phi_\infty = 6$. The generating solutions all lie on the $q = 0$ curves of Figure 2, and all of the stars evolve towards lower values of M_T and σ_0 (in other words, the generating solutions approach the weak field limit as the cosmological time increases). For the stable part of each set $\mathcal{S}$, the quantity $\sqrt{1 - \Omega_\infty^2}$, which gives a measure of the inverse radius of the star, tends towards 0 as $\sigma_0 \to 0$ (this is true for all of the solutions investigated here). Hence the radius of each star increases as it evolves. If we were to continue the evolution beyond $\Phi_\infty = 6$, we would find that in the limit $\Phi_\infty \to \infty$, the stars tend towards a state of zero mass, zero boson field amplitude and infinite radius (i. e. they disperse). Furthermore, since the ratio Φ_0/Φ_∞ decreases towards unity as σ_0 decreases, the scalar field in the interior of an evolving star becomes increasingly homogeneous.

We now compare these results with the evolution curves shown in Figure 1 of Ref. [44]. There, it was found that the mass M_∞^* of the equilibrium solutions increased with time. This behaviour may be understood if we recall the discussion of Section IV. The definition of mass used in [44] differs from the one used here by a factor of Φ_∞. On using the definition in the former paper, one finds that the mass increases when we rescale from the generating solution (with $\Phi_\infty = 1$) to the physical solution (with $\Phi_\infty > 1$).[1] The mass we use here is invariant under this rescaling. A quantitative example of how these results differ for constant N evolution is given in Table 1. The conclusions in [44] are seen to be correct once one recognises that their mass is physically different to the definition we adopt here.

For the sake of completeness, we briefly outline how an uncharged star of constant σ_0 evolves. This kind of evolution is discussed in Ref. [44] and we reproduce and expand on their results. Since the rescaling factor k increases with cosmic time, the generating solutions must be those of ever decreasing values of σ_0 that approach the weak field limit as time increases. As is the case for the fixed N evolution, the mass M_T of the constant σ_0 stars decreases with time. Here N is no longer conserved and in the physical solution will in general increase with time. This can be seen as follows. Under a small change $\delta\sigma_0$ in the generating solution we have $\sigma_0 \to \widetilde{\sigma}_0 = \sigma_0 + \delta\sigma_0$. Since σ_0 is constant in the physical solution, we have $k = 1 - \delta\sigma_0/\sigma_0$ to first order in $\delta\sigma_0/\sigma_0$ and, since $k > 1$, this implies that $\delta\sigma_0 < 0$ to lowest order. The particle number N changes according to the relation $N \to \widetilde{N} = N + \delta\sigma_0(dN/d\sigma_0)$, where $\widetilde{N}$ is the particle number of the generating solution at $\widetilde{\sigma}_0$ and we have written this expression to lowest order in $\delta\sigma_0\sigma_0$. Rescaling to find the corresponding particle numbers of the physical solutions and taking their difference we have $k^2\widetilde{N} - N = \delta\sigma_0(dN/d\sigma_0 - 2N/\sigma_0)$, which is positive wherever the generating solution satisfies the inequality $(\sigma_0/N)\, dN/d\sigma_0 < 2$. One can show numerically that this is true for all stable solutions with $q < q_c$. For any particular set of solutions with $q > q_c$, the inequality is satisfied only up to some particular value of σ_0 that varies with the parameters of the set. Again, as discussed in Section IV, the mass M_∞^* used in Ref. [44] is different from the one used here, and it rescales by a factor of k^2 under Eq. (49). In a constant σ_0 evolution, one can show that M_∞^* will be increasing with time if the condition $(\sigma_0/M_\infty^*)\, dM_\infty^*/d\sigma_0 < 2$ holds for the generating solution. This is true for all solutions with $q < q_c$ and for solutions with $q > q_c$ whose central boson field amplitude is below some particular

[1] We note here that the choice $\Phi_\infty = 1$ for the generating solution is for convenience only and similar curves could be drawn for stars evolving from the distant past (with $\Phi_\infty < 1$) through to the present cosmic time (with $\Phi_\infty = 1$) by making a second rescaling of the parameters of the physical solutions.

value.

We next consider the evolution of a charged boson star with $\Lambda = 0$ in BD gravity. The situation here is more complex since we must also deal with the charge rescaling in Eq. (49). To start with, we consider a stable boson star of particle number N and mass M_1 made up of bosons with charge q, where the asymptotic scalar field has the value Φ_1. The solution is a member of the set $\mathcal{S}_1(\sigma_0; \Phi_1, q, \Lambda = 0)$. The choice of N and q uniquely determines the central boson field amplitude σ_1 of the solution. Again, keeping N fixed we evolve Φ_∞ to a new value $\Phi_2 = \Phi_\infty(t_2) = k^2\Phi_1$ where $k > 1$. The star is now a member of the set $\mathcal{S}_2(\sigma_0; \Phi_2, q, \Lambda = 0)$ with particle number N. However, in this case the new solution cannot be generated by rescaling a solution in $\mathcal{S}_1$, since we must also take into account the rescaling of q. Instead the generator of the new solution is the member of the set $\mathcal{S}_3(\sigma_0; \Phi_1, kq, \Lambda = 0)$ with particle number $k^{-2}N$. Repeating this process for successively higher values of Φ_∞ we see that the generating solutions at each cosmological time are those of successively higher values q.

Figure 9 shows the evolution of several charged boson stars in $\omega = -1$ BD theory. The stars have charges $q = 0.7$, $q = 0.8$, $q = 0.9$ and $q = 1.0$, and each has $N = 1$ particles. Each of the solutions evolves from an initial value of $\Phi_\infty = 1$ to a final value of $\Phi_\infty = 2.8$. The direction of increasing time in the Figure is from top right to bottom left, and each star evolves to a configuration of lower mass and lower central boson field amplitude. Some of the $\Phi_\infty = 1$ solutions used to generate the late time solutions are shown as points in Figure 2, where later time generating solutions are those of higher values of q. Their corresponding rescaled (physical) solutions are shown as points in Figure 6. Unlike in the uncharged case, charged boson stars cannot evolve indefinitely. At some point in a star's evolution, its generating solution has a value of q that exceeds q_c and the star will no longer be stable: as the cosmological value of $G^\star$ becomes weaker (ie. Φ_∞ increases), Φ in the interior of the star becomes increasingly more homogeneous while its central value also increases. Eventually, the gravitational field inside the star will become too weak for the star to remain gravitationally bound and it will be dispersed by its own Coulomb field after some finite time. This phenomenon is unique to ST gravity, since it is only in these theories that the gravitational field strength in the star may weaken with time.

The evolution of a star in a more general ST theory is a little more complex since ω will not be invariant under a rescaling of Φ unless its functional form is changed. For example, given the power law coupling considered above, one would also have to rescale the numerical coefficient of $4/3$ by a factor of k^{-2n}. Hence, to analyse the evolution of a ST star one would have to start with a sequence of generating solutions, each with a different choice of ω, much as in the case of a charged BD stars in which we needed a sequence of generating solutions, each of a different charge.

VIII. CONCLUSIONS

In this work, we have studied the equilibrium configurations of charged boson stars both in Brans-Dicke theory and scalar-tensor gravity with a power law coupling, and we have re-examined uncharged boson star solutions. We have found that, theoretically, these structures are stable and might exist in the real Universe, provided of course that a ST theory correctly describes the gravitational interaction. However, it is likely that only rotating stars will exist in an astrophysical setting. While we have made no attempt to construct solutions describing rotating charged stars, the existence of rotating stars in the uncharged case leads us to suppose that charged objects of this kind may also exist in GR and ST gravity. The use of charged objects to generate astrophysical effects was recently analysed for the case of black holes, and this constitutes an additional motivation for the consideration of other charged stellar structures. Additionally, the study of the range of possible stellar structures in ST gravity is far from being complete, and a careful understanding of the influence of an evolving $G^\star$ in a cosmological background is still missing. Without stating again each of the results we have obtained in the previous sections, we may say that the use of simple relativistic objects, such as the boson stars explored here, provide a useful setting in which to discuss gravitational issues and to compare ST theories with GR. For example, we have found that the properties of both static and evolving charged ST solutions in a cosmological background differ considerably from those of GR solutions. By clarifying the mass definitions and the properties of the rescaling, we have shown for the first time that gravitational evolution may drive a compact object towards a state of decreased energy and central density. The degree to which our conclusions are true for more complex and realistic astrophysical objects is unclear. However, it seems likely that these objects will yield observable astrophysical signals, like those reported in Refs. [30,31] for the case of white dwarfs. Much more work has yet to be done, particularly in the field of gravitational memory and gravitational evolution.

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TABLE I. Here we show some data from an $\omega = -1$, $q = 0$ BD boson star evolution, in which the particle number is conserved. We have chosen $N = 0.6$. Columns 2, 3 and 4 show the parameters of the generating solutions: central boson field amplitude σ_0, particle number N and mass (here we use the Jordan frame ADM mass for ease of comparison). Columns 5 and 6 show σ_0 and the mass M^*_∞ of the physical solution, using the mass definition adopted in the paper by Torres, Schunck and Liddle. Since this mass rescales by a factor k^2, it is larger than in the generating solution and increases with Φ_∞. In the present paper, the physical solution has equal ADM mass to the generating solution, since M_{ADM} is invariant under the rescaling. Thus this mass decreases with cosmic time.

Φ_∞	σ_0	N	M_{ADM}	σ_0	M^*_∞
1.00000	0.242902	0.600000	0.520288	0.242902	0.520288
1.20000	0.159362	0.500000	0.446675	0.174572	0.536010
1.40000	0.117526	0.428571	0.393866	0.139059	0.551412
1.60000	9.14216E-02	0.375000	0.352852	0.115640	0.564563
1.80000	7.16179E-02	0.333333	0.315893	9.60855E-02	0.568607
2.00000	5.77638E-02	0.300000	0.285910	8.16904E-02	0.571820

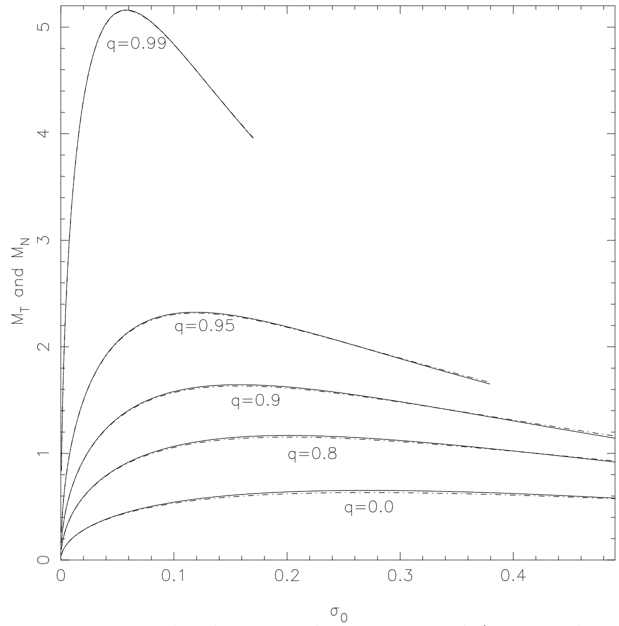

FIG. 1. Mass curves for the $\omega = 500$ Brans Dicke theory with $\Phi_\infty = 1$ and $\Lambda = 0$. The curves are parameterised by σ_0 and labelled by the boson charge-to-mass ratio q. For each value of q two curves are shown: the Tensor mass M_T (broken curves) and the Newtonian mass M_N (solid curves). Both masses are measured in units of M_{pl}^2/m. Our choice of Φ_∞ implies that M_N is numerically equal to the particle number N and M_∞^* is numerically equal to M_{ADM}.

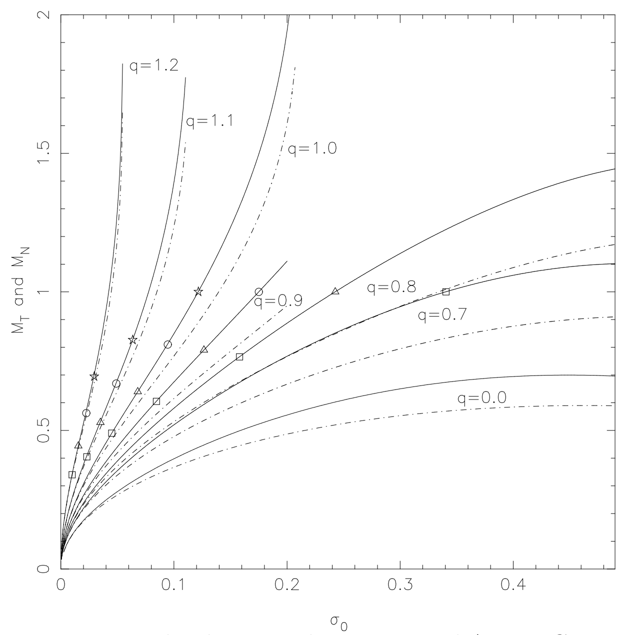

FIG. 2. Mass curves for the $\omega = -1$ Brans Dicke theory with $\Phi_\infty = 1$ and $\Lambda = 0$. Curve labelling and parameterisation are the same as in Figure 1. The weak field charge limit is $q_{max}^2 = 2$ and the critical charge is $q_c = 0.90$, above which the mass curves diverge. The data points show solutions used to generate the evolution sequences in Figure 9. Four sets of generating solutions are shown: $q = 1.0$ (marked by stars), $q = 0.9$ (marked by circles), $q = 0.8$ (marked by triangles) and $q = 0.7$ (marked by squares). The initial solution for each evolution sequence is at $M_N = 1$ and the later time generating solutions are those of higher values of q and lower values of M_N, M_T and σ_0. See the text in Section VII for further explanation.

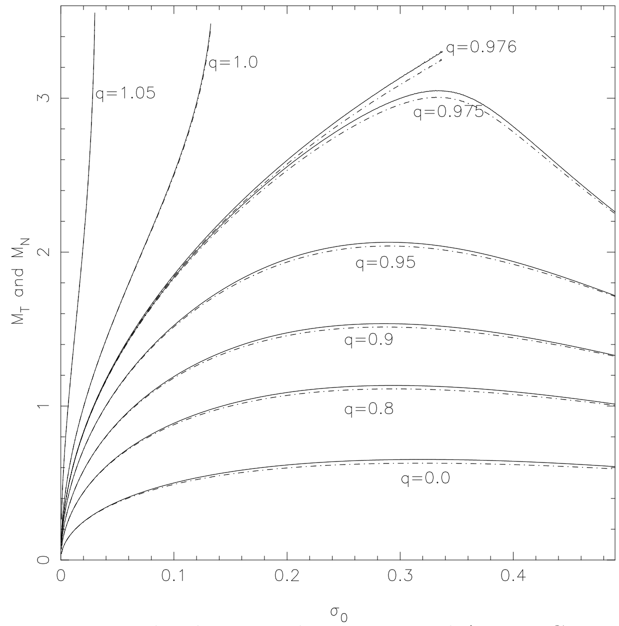

FIG. 3. Mass curves for the $\omega = 1$ Brans Dicke theory with $\Phi_\infty = 1$ and $\Lambda = 0$. Curve labelling and parameterisation are the same as in Figure 1. The weak field charge limit is $q^2_{max} = 1.2$ and the critical charge is $q_c = 0.976$.

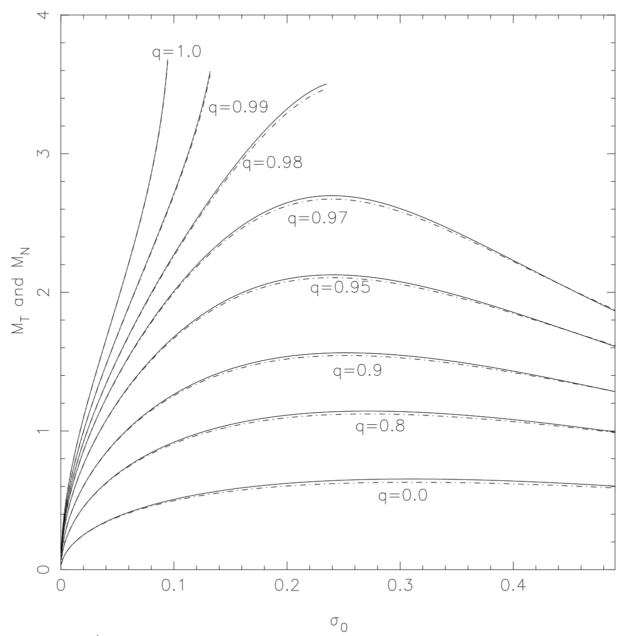

FIG. 4. Mass curves for the $2\omega + 3 = \frac{4}{3}n\Phi^n$ power law ST theory with $n = 4$, $\Phi_\infty = 1$ and $\Lambda = 0$. Curve labelling and parameterisation are the same as in Figure 1. The weak field charge limit is $q^2_{max} = 19/16$ and the critical charge is $q_c = 0.986$.

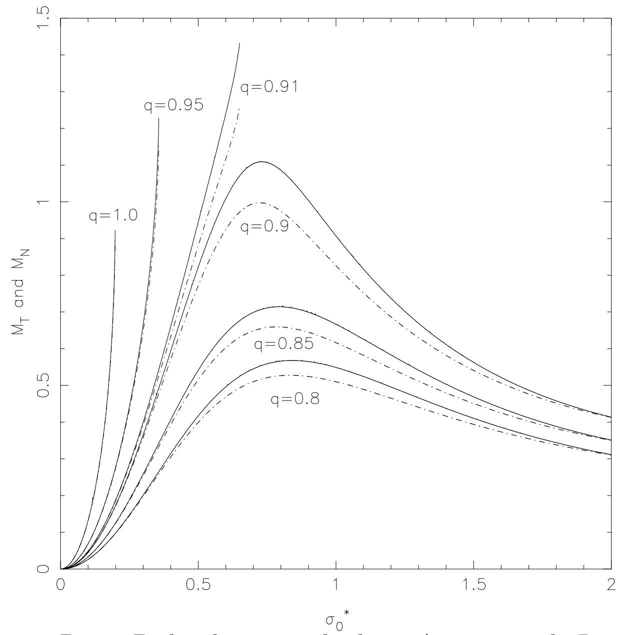

FIG. 5. Mass curves for the $\omega = 1$ Brans Dicke theory in the limit $\Lambda \to \infty$ with $\Phi_\infty = 1$. Curve labelling is the same as in Figure 1. The curves are parameterised by the rescaled central density σ^* and the masses are measured in units of $M_{pl}^2/(m\Lambda^{1/2})$. The weak field charge limit is $q_{max}^2 = 1.2$ (the same as in Figure 3) while the critical charge is $q_c = 0.905$.

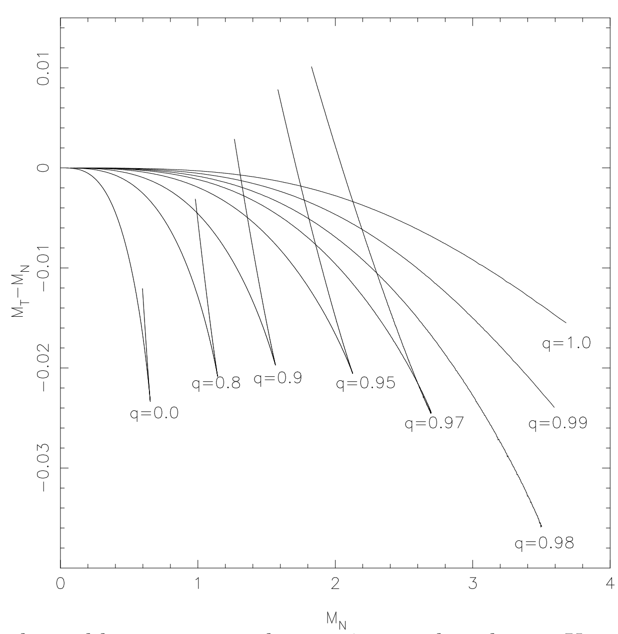

FIG. 6. Bifurcation diagram for charged boson stars in the $n = 4$ power law theory. Using catastrophe theory one may show that the appearance of cusps signals a change in the stability of the stellar object. Branches that start at the coordinate origin are stable.

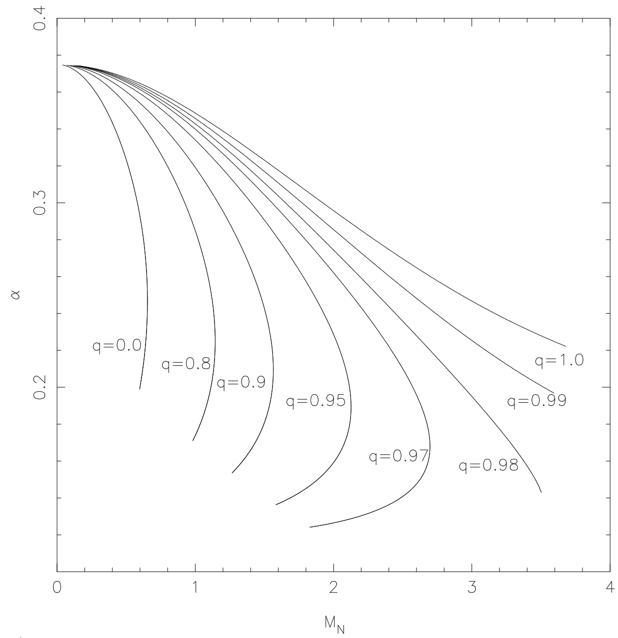

FIG. 7. Scalar coupling $\alpha := -S/M_N$ as a function of rest mass for charged boson stars in the $n = 4$ Power Law theory. The limiting value as $M_N \to 0$ may be obtained analytically using Eq. (104).

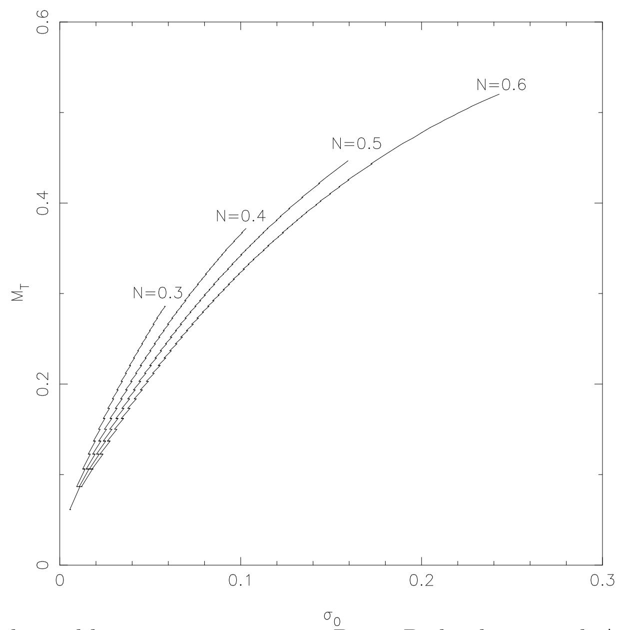

FIG. 8. Evolution curves for uncharged boson stars in $\omega = -1$ Brans Dicke theory with $\Lambda = 0$. The curves are labelled by N and evolve from configurations with $\Phi_\infty = 1$ to configurations with $\Phi_\infty = 6$. The direction of increasing time along the curves is towards the origin of the graph.

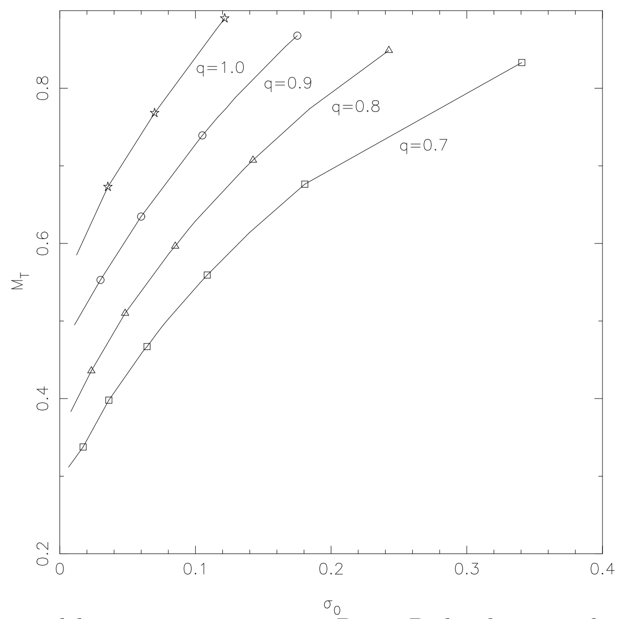

FIG. 9. Evolution curves for charged boson stars in $\omega = -1$ Brans Dicke theory with $\Lambda = 0$. All stars have $N = 1$ and evolve from configurations with $\Phi_\infty = 1$ to configurations with $\Phi_\infty = 2.8$. The direction of increasing time along the curves is towards the origin of the graph. The symbols (squares, triangles, circles and stars) on each curve are the rescaled $\Phi_\infty = 1$ generating solutions shown in Figure 2.