A comment on bosonization in $d \geq 2$ dimensions

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Abstract

We discuss recent results on bosonization in $d \geq 2$ space-time dimensions by giving a very simple derivation for the bosonic representation of the original free fermionic model both in the abelian and non-abelian cases. We carefully analyse the issue of symmetries in the resulting bosonic model as well as the recipes for bosonization of fermion currents.

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It is only very recently that the powerful techniques of bosonization of two-dimensional fermion systems [1] have been extended with some success to $d > 2$ space-time dimensional models [2]-[5] (see refs.[6] for previous efforts in this direction). In particular, the well known 2-dimensional abelian bosonization recipe for the fermionic current in terms of a bosonic field ϕ,

$$\bar{\psi}\gamma^\mu\psi \to (1/\sqrt{\pi})\epsilon^{\mu\nu}\partial_\nu\phi \tag{1}$$

has been shown to be generalizable at least in the case of $d = 3$ abelian fermionic models to the formula [4]

$$\bar{\psi}\gamma^\mu\psi \to \pm i\sqrt{\frac{1}{4\pi}}\epsilon^{\mu\nu\alpha}\partial_\nu A_\alpha \tag{2}$$

where A_α is a gauge field with Chern-Simons dynamics and the recipe is only valid to order $1/m$ (m being the fermion mass). This result, already implicit in ref.[2], has been extended to the case of free non-Abelian fermions in $d = 3$ dimensions (again to order $1/m$) [5],

$$\bar{\psi}\gamma^\mu t^a\psi \to \pm i\sqrt{\frac{1}{4\pi}}\epsilon^{\mu\nu\alpha}F^a_{\nu\alpha} \tag{3}$$

where $F_{\nu\alpha}$ is the field strength of a non-abelian gauge field with Chern-Simons dynamics. As explained in [5], formula (3) is, in some sense, the 3-dimensional analogue of the 2-dimensional non-abelian fermion-boson mapping

$$j_+ \to -\frac{i}{4\pi}h^{-1}\partial_+ h \tag{4}$$

$$j_- \to -\frac{i}{4\pi}h\partial_- h^{-1}. \tag{5}$$

Although the approaches of refs.[2]-[5] have many points in common and lead to equivalent results, the road they follow to attain bosonization is somehow diverse. For example, the results in refs.[4]-[5] are based on the construction of the master or interpolating bosonic Lagrangian introduced in refs.[7]-[9] for studying self-dual systems and seems to be the most appropriate for generalizations to non-abelian systems (at least in $d = 3$). The approach of ref.[2] exploits the connection between bosonization and duality transformations discovered in [10] and leads in a very elegant way to the abelian boson-fermion mapping; it seems to be the most adequate for studying $d > 3$ models.

It is the purpose of this note to comment on the connection between these different proposals, showing their common origin through the analysis of an alternative way of deriving the fermion-boson mapping. As we shall see, this alternative approach is related with a priori disconnected ideas, as those in Faddeev-Shatashvili proposal of introducing new gauge degrees of freedom for the consistent quantization of anomalous gauge theories [11]-[13] or in the "smooth" bosonization of certain 2 dimensional models [14]-[15].

We start from the (Euclidean) Lagrangian for free massive Dirac fermions in d dimensions

$$\mathcal{L}_F = \bar{\psi}(i\partial\!\!\!/ + m)\psi \tag{6}$$

where fermions are in the fundamental representation of some group G. In most cases, we shall consider $G = U(N)$ so that the conserved fermion current associated with Lagrangian (6) reads

$$j^{a\mu} = \bar{\psi}^i t^a_{ij} \gamma^\mu \psi^j \tag{7}$$

with t^a the $U(N)$ generators.

This current stems from the <u>global</u> $U(N)$ invariance of (6) under the transformation

$$\begin{aligned} \psi &\to g\psi \\ \bar{\psi} &\to \bar{\psi} g^{-1} \end{aligned} \tag{8}$$

with g an element of $U(N)$. At the quantum level, current conservation can be derived using Noether method by starting from the partition function

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int \bar{\psi}(i\partial\!\!\!/ + m)\psi d^d x], \tag{9}$$

promoting g to a local transformation,

$$\begin{aligned} \psi &\to g(x)\psi \\ \bar{\psi} &\to \bar{\psi} g^{-1}(x) \end{aligned} \tag{10}$$

and then taking (10) as a change of the fermionic variables in Z_F. After considering $g(x)$ infinitesimally close to the identity one straightforwardly obtains

$$< \partial_\mu j^a_\mu >= 0. \tag{11}$$

Now, the local transformation (10) plays a central role in our route to bosonization. Indeed, our procedure starts by taking (10) as a (finite) change of variables in (9),

$$\begin{aligned} \psi &= g(x)\psi' \\ \bar{\psi} &= \bar{\psi}' g^{-1}(x). \end{aligned} \tag{12}$$

One can always define for Dirac fermions a path integral measure invariant under transformation (12). Then, after the change of variables the partition function becomes

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int \bar{\psi}(i\not{\partial} + m + ig^{-1}\not{\partial}g)\psi d^d x]. \tag{13}$$

(we have omitted primes in the new fermionic variables). Being Z_F g-independent, we can integrate both sides in eq.(13) over g using a Haar measure Dg, this amounting to a trivial change in the normalization of the path-integral. In performing this integration we include an arbitrary weight $\hat{F}[g]$ which will play an important role in what follows [1]. We then have

$$Z_F = \mathcal{N} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}g\hat{F}[g] \exp[-\int \bar{\psi}(i\not{\partial} + m + ig^{-1}\not{\partial}g)\psi d^d x]. \tag{14}$$

It is evident that $ig^{-1}\not{\partial}g$ in (14) can be thought as a flat connection and can be then replaced by a "true" gauge field connection provided a constraint is introduced to assure its flatness. Indeed, using the identity

$$\int \mathcal{D}g\mathcal{H}[ig^{-1}\not{\partial}g] = \int \mathcal{D}b_\mu \mathcal{H}[b]\delta[\epsilon_{\mu_1\mu_2\ldots\mu_d} f_{\mu_1\mu_2}] \tag{15}$$

with

$$f_{\mu\alpha} = \partial_\mu b_\alpha - \partial_\alpha b_\mu + i[b_\mu, b_\alpha] \tag{16}$$

we can rewrite (14) in the form

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}b_\mu F[b]\delta[\epsilon_{\mu_1\mu_2\ldots\mu_d} f_{\mu_1\mu_2}] \exp[-\int \bar{\psi}(i\not{\partial} + m + \not{b})\psi d^d x]. \tag{17}$$

The next step in our derivation is to integrate out fermions so that the partition function becomes:

[1] The suggestion of introducing this functional and its relevance concerning symmetries was done by N.Bralić [16].

$$Z_F = \int Db_\mu F[b] det(i\partial\!\!\!/ + m + b\!\!\!/)\delta[\epsilon_{\mu_1\mu_2\ldots\mu_d} f_{\mu_1\mu_2}]. \tag{18}$$

The central point in our bosonization route is now at sight: whether one would arrive to an exact bosonization formula or to an approximate recipe will depend on the possibility of computing the fermion determinant in a closed form. If this were possible, then the fermionic degrees of freedom, which disappeared from the partition function would be replaced by new bosonic degrees of freedom, in an exactly equivalent bosonic model. Otherwise, bosonization will be approximate.

As it is well-known, in the $d = 2$ massless case, the fermion determinant can be computed exactly both in the abelian and non-abelian cases and then exact bosonization rules can be trivially obtained from (18). In $d = 3$ the fermion determinant can be computed as a $1/m$ expansion this leading to a fermion-boson mapping valid at large-distances. This and other approximation methods would lead to (approximate) bosonization rules in higher dimensions.

In order to proceed, one has to exponentiate the delta function in (18) by means of a Lagrange multiplier field. As explained above, it is this bosonic field, a scalar in $d = 2$, a vector in $d = 3$, an antisymmetric (Kalb-Rammond) field in $d > 3$ dimensions, which will play the central role in the bosonized theory, once the auxiliary b-field is integrated out. This will become clear in the following simple examples:

The $d = 2, U(1)$, massless case

In this case we can use the well-known result:

$$\log det(i\partial\!\!\!/ + b\!\!\!/) = -\frac{1}{2\pi}\int d^2x b_\mu(\delta_{\mu\nu} - \partial_\mu \Box^{-1}\partial_\nu)b_\nu. \tag{19}$$

This, together with the representation

$$\delta[\epsilon_{\mu\nu} f_{\mu\nu}] = \int D\phi \exp(-\frac{1}{\sqrt{\pi}}\int d^2x \phi\epsilon_{\mu\nu} f_{\mu\nu}) \tag{20}$$

leads, after a trivial gaussian integration over b_μ, to the result:

$$Z_F = \int D\phi \exp(-\frac{1}{2}\int d^2x \partial_\mu\phi\partial_\mu\phi) \tag{21}$$

relating the free fermion partition function Z_F with the partition function of the corresponding free boson field. Notice that in arriving to eq.(21) we have fixed the arbitrary function $F[b] = 1$. As we shall see below, the choice of $F[b]$ is related to the question of gauge invariance of the resulting bosonic theory and it is in the non-abelian case where a non-trivial choice of $F[b]$ becomes important.

Concerning bosonization rules for fermion currents, let us note that the addition of a fermion source s_μ in Z_F amounts to the inclussion of this source in the fermion determinant

$$Z_F[s] = \int Db_\mu det(i\partial\!\!\!/ + b\!\!\!/ + s\!\!\!/)\delta[\epsilon_{\mu\nu} f_{\mu\nu}]. \tag{22}$$

Now, a trivial shift $b + s \to b$ in the integration variable b puts the source dependence into the constraint

$$Z_F[s] = \int Db_\mu det(i\partial\!\!\!/ + b\!\!\!/)\delta[\epsilon_{\mu\nu}(f_{\mu\nu} - 2\partial_\mu s_\nu)] \tag{23}$$

so that, instead of (21) one ends with

$$Z_F[s] = \int D\phi \exp(-\frac{1}{2}\int d^2x(\partial_\mu\phi\partial_\mu\phi + \frac{2}{\sqrt{\pi}} s_\mu \epsilon_{\mu\nu}\partial_\nu\phi)). \tag{24}$$

By simple differentiation with respect to the source one infers from this expression the bosonization recipe for j_μ given in eq.(1).

The $d = 3, U(1)$, massive case

The fermion determinant in $d = 3$ dimensional space-time cannot be computed in a closed form. One can however consider an approximation approach which in the present case can be envisaged as an expansion in inverse powers of the fermion mass, $1/m$. Indeed, following refs.[17]-[21] one gets an expression containing parity violating contributions as well as parity conserving terms which can be computed order by order in $1/m$

$$\ln \det(i\partial\!\!\!/ + m + b\!\!\!/) = \pm\frac{i}{16\pi}\int \epsilon_{\mu\nu\alpha} f^{\mu\nu} b^\alpha d^3x \; + I_{PC}[b_\mu] + O(\partial^2/m^2), \tag{25}$$

$$I_{PC}[b_\mu] = -\frac{1}{24\pi m}\int d^3x\, f^{\mu\nu} f_{\mu\nu} + \ldots \tag{26}$$

Using this result to the leading order in $1/m$, the corresponding partition function Z_F can be written in the form

$$Z_F \simeq \int Db_\mu DA_\mu \exp[-\int \left(\mp\frac{i}{8\pi}\epsilon^{\mu\alpha\nu}b_\mu\partial_\alpha b_\nu + A_\mu\epsilon_{\mu\nu\alpha}f_{\nu\alpha}\right)d^3x\,] \tag{27}$$

where $\simeq$ indicates that the identity is valid to the lowest order in $1/m$. Again, the integration over b_μ is quadratic so that it can be trivially performed so that one finally obtains

$$Z_F \simeq \int DA_\mu \exp[\pm\frac{i}{2}\int d^3x\,\epsilon^{\mu\alpha\nu}A_\mu\partial_\alpha A_\nu] \tag{28}$$

or

$$Z_F \simeq Z_{CS} \tag{29}$$

where Z_{CS} denotes the partition function for a pure Abelian Chern-Simons theory. Equation (29) establishes the connection between a theory of free fermions and the pure Chern-Simons theory in $2+1$ dimensions, to the lowest order in inverse powers of the fermion mass. This last restrictions implies that our results are valid only for long distances in contrast with $1+1$ bosonization which is in a sense a short-distance result. This peculiarity makes the comparison of the commutator algebra, which tests short distances, somehow hazardeous.

As in the previous example, we could have added an external source s_μ for the fermion current in the original fermionic Lagrangian. After the trivial shift $b_\mu + s_\mu \to b_\mu$ in (17) we end with an identity of the type (28)-(29) but now in the presence of sources:

$$\begin{aligned} Z_F[s] &= \int \mathcal{D}\bar\psi\mathcal{D}\psi \exp[-\int \left(\bar\psi(i\partial\!\!\!/ + m)\psi + s_\mu j_\mu\right)d^3x\,] \\ &\simeq \int DA_\mu \exp[\pm\frac{i}{2}\int(\epsilon^{\mu\alpha\nu}A_\mu\partial_\alpha A_\nu + \sqrt{2/\pi}s_\mu\epsilon^{\mu\alpha\nu}\partial_\alpha A_\nu)\,d^3x\,]. \end{aligned} \tag{30}$$

From this, we confirm the bosonization rule for the fermion current given by eq.(2).

Symmetries

Let us discuss at this point the issue of symmetries in our bosonization approach. We have started from a free fermionic Lagrangian invariant under global $U(N)$ rotations. Through transformations (12), which are the local counterpart of those global rotations, we have forced the appearence of gauge degrees of freedom in the effective Lagrangian. Although trivial at the start (they enter through a flat connection) these degrees of freedom become non-trivial once flatness is implemented via a Lagrange multiplier. So, the bosonic equivalent of the original fermionic Lagrangian is a Lagrangian for a Lagrange multiplier whose character (scalar, vector, in general a rank $d-1$ completely antisymmetric field which in the dualization approach to bosonization is taken as a Kalb-Ramond gauge potential [2]) depends on the space-time dimensionality. How does the imposed local symmetry reflect in the resulting bosonic model? The bosonic field ϕ enters in our approach through the delta function ensuring flatness,

$$\delta[\epsilon_{\mu_1\mu_2\ldots\mu_d} f_{\mu_1\mu_2}] = \int D\phi \exp(-\frac{1}{\sqrt{\pi}} tr \int d^d x \epsilon_{\mu_1\mu_2\ldots\mu_d} \phi_{\mu_3\ldots\mu_d} f_{\mu_1\mu_2}). \tag{31}$$

Now, since the l.h.s. in this identity is gauge invariant, one should define the r.h.s. accordingly. In the two abelian examples discussed above, the curvature $f_{\mu_1\mu_2}$ is gauge-invariant and then (31) does not force the Lagrange multiplier to have a definite transformation law. In particular, in the 2-dimensional case ϕ is just a scalar which does not partake of local gauge transformations. In the 3-dimensional case, it is reasonable to interpret ϕ_μ as a gauge connection $\phi_\mu \equiv A_\mu$ since the resulting effective bosonic action is (to order 1/m) a Chern-Simons action for A_μ which naturally possesses a gauge-invariance.

The issue is more subtle, in the non-abelian case, where $f_{\mu\nu}$ changes covariantly. One could in principle demand $\phi_{\mu_3\ldots\mu_d}$ to change covariantly in order to have a gauge invariant r.h.s. . In the $d=2$ case, this problem can be overcome just by fixing the gauge degrees of freedom associated with b_μ, as discussed in [22],[15]. In particular, in this last reference, in an approach similar to the one here presented, non-abelian two-dimensional bosonization is discussed in the spirit of smooth bosonization [14].

Concerning higher dimensions, already for $d=3$ the problem is more complicated. It is in handling this problem that the arbitrary functional $\hat{F}[g] = F[b]$ appearing in eq.(18) finds its relevance: since the $d=3$ non-abelian bosonization recipe summarized by eq.(3) leads to a bosonic model with non-abelian Chern-Simons dynamics, one should expect the bosonic partition function to possess gauge invariance. With this in mind, one can fix $F[b]$ so that the bosonic field ϕ_μ can be again taken as a gauge connection, this time taking values in the Lie algebra of $U(N)$ and the resulting theory then exhibits gauge-invariance. This can be achieved by chosing $F[b]$ in the form

$$F[b] = \exp[\frac{i}{12\pi}\,\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\alpha}b_\mu b_\nu b_\alpha] \tag{32}$$

Indeed, under gauge transformations

$$b_\mu \to b_\mu^h = h^{-1}b_\mu h + ih^{-1}\partial_\mu h \tag{33}$$

$F[b]$ then changes as

$$F[b] \to F[b^h] = F[b] \times \exp(2\pi i\omega[h]) \times \exp[\frac{i}{16\pi}\,\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\alpha}f_{\mu\nu}h\partial_\alpha h^{-1}] \tag{34}$$

with $\omega[h]$ the winding number of h,

$$\omega[h] = \frac{1}{24\pi^2}\,\mathrm{tr}\int d^3x\,\epsilon_{\mu\nu\alpha}h^{-1}\partial_\mu h h^{-1}\partial_\nu h h^{-1}\partial_\alpha h. \tag{35}$$

For compact non-abelian gauge groups (thus with $\Pi_3 = Z$) $\omega[h]$ takes integer values so the second factor in the r.h.s. of (34) is irrelevant. Concerning the last one, it precisely cancels out that arising from the delta function representation when the Lagrange multiplier $\phi_\mu \equiv A_\mu$ changes as a connection

$$A_\mu \to A_\mu^h = h^{-1}A_\mu h + ih^{-1}\partial_\mu h. \tag{36}$$

That is, one has

$$F[b^h] \times \exp(-\frac{1}{\sqrt{\pi}}tr\int d^3x\epsilon_{\mu\nu\alpha}f_{\mu\nu}^h A_\alpha^h) = F[b] \times \exp(-\frac{1}{\sqrt{\pi}}tr\int d^3x\epsilon_{\mu\nu\alpha}f_{\mu\nu}A_\alpha) \tag{37}$$

In order to summarize what we have learned in the three dimensional case concerning symmetry, let us write the partition function (9) in terms of the bosonic field A_μ, once bosonization is achieved, in the form

$$Z_F = \int DA_\mu \times \exp(-S_B[A_\mu]) \tag{38}$$

with

$$\exp(-S_B[A_\mu]) = \int Db_\mu F[b]det(i\partial\!\!\!/ + m + b\!\!\!/)\exp(-\frac{1}{\sqrt{\pi}}tr\int d^3x\epsilon_{\mu\nu\alpha}f_{\mu\nu}A_\alpha) \tag{39}$$

Now, using eq.(37) as well as the invariance of the fermion determinant and the path-integral measure under the change $b_\mu \to b_\mu^h$, one trivially sees that

$$\exp(-S_B[A_\mu^h]) = \exp(-S_B[A_\mu]). \tag{40}$$

That is, the bosonic theory can be endowed, in 3 dimensions, with a local gauge symmetry and this is an exact result in the sense it does not arise from some approximation for the fermion determinant. Of course, the role of the functional $F[b]$ is crucial for this property. Presumably, there is always a choice for $F[b]$ such that, as it hapens in $d = 3$, the d-dimensional bosonic effective theory can be defined as a theory in which the bosonic field $\phi_{\mu_3 \ldots \mu_d}$ is a Kalb-Ramond gauge field.

Equation (40) can be seen as connecting the approach presented here with Faddeev-Shatashvili proposal for quantizing anomalous theories [11]. Indeed, as explained in [12], this proposal can be implemented by starting from a partition function where (classically redundant) gauge degrees of freedom are taken into account at the quantum level. It is in this way that gauge invariance of the effective action for chiral fermions is achieved and an identity analogous to (40) can be proved. From this, consistency of anomalous gauge theories can be proved, at least in the two dimensional case.

The analysis above shows how our approach, inspired in the bosonization proposal of ref.[2], sheds light on the issue of gauge-invariance of the resulting bosonic theory. This becomes a central point when deriving 3-dimensional non-abelian bosonization. Indeed, up to date, only the approach of refs.[4]-[5], based in the construction of an auxiliary master Lagrangian has led to a concrete recipe for bosonization of the fermionic current (eq.(3)). This recipe shows that, for large distances, the bosonic theory is endowed with a local gauge invariance and the precedent discussions clarifies why this is so. The derivation of the bosonization recipe (3) within the approach presented here is difficulted by the fact that non-linear terms prevent the trivial integration of the auxiliary field b_μ. This problem can be handled in the same way it was done in the interpolating Lagrangian approach [5] but we hope that a three dimensional analogous of Polyakov-Wiegman identity [23] may simplify the derivation following the lines of the present work.

The construction of the interpolating Lagrangian in [4]-[5] heavily depends on the fact that, for $d = 3$, a connection between selfdual and Maxwell-Chern-Simons models can be established [8]. For higher dimensions, we think that the method of ref.[2], presented here through an alternative route, is better adapted. One should note in particular the simplicity with which one can include sources for the fermion current since, after a shift in the auxiliary field b, the source s_μ always ends coupled to the bosonic field in the form

$$S_{source}[s] = tr \int d^d x \epsilon_{\mu_1 \mu_2 \dots \mu_d} s_{\mu_1} \partial_{\mu_2} \phi_{\mu_3 \dots \mu_d} + O(s^2) \tag{41}$$

this suggesting a bosonization rule of the form

$$\bar{\psi} \gamma_{\mu_1} t^a \psi \to \epsilon_{\mu_1 \mu_2 \dots \mu_d} \partial_{\mu_2} \phi^a_{\mu_3 \dots \mu_d}. \tag{42}$$

Of course, the construction of the actual bosonic model in which this fermion-boson mapping is realized will depend on the posibility of evaluating the fermionic determinant. For $d \geq 3$ one can only envisage approximations (like for example the $1/m$ expansion employed in the $d = 3$ case) but the basic steps leading to bosonization, summarized by eqs.(12)-(18), remain the same no matter the number of space-time dimensions.

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