Wormholes in spacetime with torsion

Luis A. Anchordoqui*

Departamento de Física, Universidad Nacional de La Plata

C.C. 67, 1900, La Plata, Buenos Aires, Argentina

Abstract

Analytical wormhole solutions in U_4 theory are presented. It is also discussed whether the introduction of extremely short range repulsive forces, related to the spin angular momentum of matter, could be the "carrier" of exoticity that threads the wormhole throat.

PACS *number(s):* 04.20Jb

*Electronic address: doqui@venus.fisica.unlp.edu.ar

I. INTRODUCTION

Wormhole physics has crept back into the literature since the analysis of classical transversable wormholes performed by Morris and Thorne [1] and by Morris, Thorne and Yurtserver [2] [1]. These *wormhole spacetimes* require gravitational sources that violate all known energy conditions in order to support them [4]. Such a kind of source has been christened exotic. Another exciting feature of this unconventional arena is the possibility of constructing time machines. As was shown in [2] the generic relative motion of one mouth of a Lorentzian wormhole transforms its throat into a time tunnel, violating Hawking's chronology protection conjecture [5]. A different way to construct a time machine was proposed by Frolov and Novikov considering the interaction of a wormhole with a generic gravitational field in its neighbourhood [6].

A fair amount of work has been done to understand the requirements of matter that violates the energy conditions. Quantum field theory that allows local violations of the weak energy condition in the form of locally negative energy densities and fluxes, seems a plausible theorethical background [7]. Within classical General Relativity, non static Lorentzian wormhole solutions, which do not require more than a "flash" of exotic matter in order to support the wormhole throat, have been found [8]. Another original attempt to construct wormhole geometries has been suggested by Visser using "Schwarzschild surgery" [9]. Finally, the study of alternative theories of gravity has been also followed as a suitable technique to tackle the violation of the energy conditions. Among them, we could mention, $R+R^2$ theories [10], Moffat's nonsymmetric theory [11], Einstein-Gauss-Bonnet theory [12], and Brans-Dicke theory [13–15]. In this direction, we shall look for static wormhole solutions in U_4 theory [16]. We also examine the stress-energy tensor that might give rise to these geometries.

---

[1] A classical example of non traversible wormhole is the Schwarzschild wormhole , which "pinches off" before any signal can travel through it, yielding two singularities [3].

II. WORMHOLES IN U_4 THEORY

The differentiable spacetime manifold of Eistein–Cartan theory has an asymmetric affine conection[2]. Its antisymetric part, the torsion tensor $S_{ij}{}^{k} \equiv \Gamma^{k}_{ij}$, is linked to the spin angular momentum of matter $\tau_{ij}{}^{k}$. The field equations of U_4 theory are found to be,

$$R_{ij} - \frac{1}{2} g_{ij} R_{k}{}^{k} = 8\pi \Sigma_{ij} \tag{1}$$

$$S_{ij}{}^{k} + \delta^{k}_{i} S_{jl}{}^{l} - \delta^{k}_{j} S_{il}{}^{l} = 8\pi \tau_{ij}{}^{k} \tag{2}$$

where δ^{j}_{i} is the Kronecker delta, g_{ij} the metric tensor with signature $(+,-,-,-)$, and $R_{ij} = R_{kij}{}^{k}$, with $R_{ijk}{}^{l}$ the curvature tensor of the Riemann-Cartan connection $\Gamma^{k}_{ij} = \{^{k}_{ij}\} + S_{ij}{}^{k} - S_{j}{}^{k}{}_{i} + S^{k}{}_{ij}$. $\{^{k}_{ij}\}$ is the Christoffel symbol of the metric and Σ^{ij} is the canonical energy-momentum tensor of matter. If one substitutes eq. (2) in eq. (1), after a bit of algebra one arrives at the combined field equation,

$$R^{ij}(\{\}) - \frac{1}{2} g^{ij} R_{k}{}^{k}(\{\}) = 8\pi \tilde{\sigma}^{ij} \tag{3}$$

with

$$\tilde{\sigma}^{ij} \equiv \sigma^{ij} + \kappa\, [-4\tau^{ik}{}_{[l}\tau^{jl}{}_{k]} - 2\tau^{ikl}\tau^{j}{}_{kl} + \tau^{kl\,i}\tau_{kl}{}^{j} + 1/2\, g^{ij}\,(\,4\,\tau_{m}{}^{k}{}_{[l}\tau^{ml}{}_{k]} + \tau^{m\,kl}\tau_{m\,kl}\,)] \tag{4}$$

$\{\}$ means that the quantities have been computed for the Riemannian part, $\{^{k}_{ij}\}$, of the affine connection and are the same as in General Relativity [17].

Hereafter we shall consider a static spherically symmetric matter distribution represented by the spacetime metric,

$$ds^2 = e^{2\Phi} dt^2 - e^{2\Lambda} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{5}$$

The only surviving components of the energy momentum tensor are,

---

[2]In what follows, latin indices $i, j, \ldots$, run from 0 to 3 and $\hat{k}, \hat{l} \ldots$ from 1 to 3; square brackets denote antisymmetrization, and we set $G = c = 1$.

$$\Sigma_0{}^0 = \rho \qquad \Sigma_1{}^1 = \breve{\sigma}_{\text{R}} \qquad \Sigma_2{}^2 = \Sigma_3{}^3 = \breve{\sigma}_{\text{L}} \tag{6}$$

with ρ the mass-energy density, and $\Sigma_{\hat{l}\hat{k}}$ the $\hat{l}$-component of the force per unit area excerted across a surface normal to the $\hat{k}$ direction, yielding local radial and lateral stresses, $\breve{\sigma}_{\text{R}}$ and $\breve{\sigma}_{\text{L}}$ respectively. The spin tensor is given by,

$$\tau_{ij}{}^k = s_{ij} u^k \tag{7}$$

with the constraint,

$$s_{ij} u^j = 0 \tag{8}$$

where s_{ij} is the spin density and u^k the four velocity. Since the fluid is supposed to be static, the four velocity $u^k = \delta_0^k$.

The application of the field equations (3) leads to the following expressions,

$$\frac{1}{r^2} + e^{-2\Lambda}\left(\frac{2\Lambda'}{r} - \frac{1}{r^2}\right) = 8\pi\rho - 16\pi^2 s^2 \tag{9}$$

$$\frac{1}{r^2} - e^{-2\Lambda}\left(\frac{2\Phi'}{r} + \frac{1}{r^2}\right) = 8\pi\breve{\sigma}_{\text{R}} + 16\pi^2 s^2 \tag{10}$$

$$-e^{-2\Lambda}\left\{\Phi'' + \Phi'^2 - \Lambda'\Phi' + \frac{1}{r}(\Phi' - \Lambda')\right\} = 8\pi\breve{\sigma}_{\text{L}} + 16\pi^2 s^2 \tag{11}$$

with $s^2 = 2 s_{ij} s^{ij}$ the square of the spin. In order to solve the system we shall adopt as equation of sate $\rho = -\Pi$, being $\Pi = 1/3\, \Sigma_{\hat{k}}{}^{\hat{k}}$ the static-fluid pressure.

In the spirit of [19], we make the Ansatz $\Phi = -\alpha/r$, where α is a positive constant to be determined. This choice guarantees that the redshift function Φ is finite everywhere, and consequently there are no event horizons. Taking into account these conditions, the field equations can be rewritten in the compact form,

$$\frac{2e^{2\Lambda}}{r^2} + \Lambda'\left(\frac{4}{r} + \frac{\alpha}{r^2}\right) = \left(\frac{2}{r^2} + \frac{\alpha^2}{r^4}\right) \tag{12}$$

which integrates sraightforwardly to,

$$e^{-2\Lambda} = \frac{c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4 + c_5\varphi^5 + c_6\varphi^6 + c_7\varphi^7 + c_8\varphi^8 + c_9\varphi^9 + e^{2/\varphi}\varphi^8\mathcal{K}}{(4\varphi+1)^9} \tag{13}$$

with $\varphi = r/\alpha$, $\mathcal{K}$ an integration constant and the coefficients c_i are listed in Table I. It is easily seen, that as $\varphi \to \infty$, $e^{2\Lambda} \to 1$.

In order to fix the constant $\mathcal{K}$, we must select a value for the dimensionless radius ($\varphi_{\rm th}$) such that the "flaring out" condition:

$$\lim_{\varphi \to \varphi_{\rm th}^+} e^{-2\Lambda} = 0^+ \tag{14}$$

is fulfilled. Note that the absolute size of the throat still depends on α.

The aforementioned properties of Λ together with the definition of the redshift function Φ bear out that the metric tensor describes two asymptotically flat space-times joined by a throat. Since the properties of the solution pertaining to eq. (13) are similar to the ones that Kar and Sahdev have studied in good detail for General Relativity [19], we dwell on violations of the energy conditions in U_4 theory [20,21].

Extensions of the Raychaudhuri eq. as well as the Hawking-Penrose singularity theorems [4] have been already performed [18]. The inequality,

$$\left(\tilde{\sigma}^{ij} - \frac{1}{2}\, g^{ij}\, \tilde{\sigma}_k{}^k \right) \xi_i\, \xi_j \geq 0 \tag{15}$$

that must hold for all timelike unit vectors ξ^i, generalizes the energy conditions in U_4 theory. Recalling that we are dealing with a fluid at rest, the usual inequality $\rho \geq 0$ should be replaced by,

$$\rho - 8\pi s^2 \geq 0 \tag{16}$$

It is a straightforward derivation to prove that a wormhole threaded with a distribution of "ordinary" matter with square spin effects overwhelming the mass terms, does not necessarily "pinch off". Hehl, Heyde and Kerlick [18] have estimated the required critical density $\rho_{\rm C}$ for such a situation. In the case of a spin fluid of neutrons with isotropic pressure, $\rho_{\rm C} \approx 10^{54}$ g cm^{-3}, which is orders of magnitude greater than the density at the center of the most massive neutron stars [22]. In the case of electrons, the estimated value is $\rho_{\rm C} \approx 10^{47}$ g cm^{-3}. These huge mass densities could not have been present but only in the very early beginning, after the bang.

III. OUTLOOK

It was shown that U_4 theory admits wormhole solutions. In spite of the very restrictive conditions upon the matter density, the very early universe might provide a fruitfull scenario for wormholes to rise. If so, inflation has been already suggested as the natural scheme for the enlargement of submicroscopic wormholes [23]. Whereas, the universe becomes colder and the matter density spreads out triggering a gravitational collapse of the wormhole throat, unless an alternative mechanism could play the paramount role of exoticity. All these, remain to be explored.

ACKNOWLEDGMENTS

I am grateful to S. Perez Bergliaffa and D. F. Torres for illuminating discussions and critical reading of the manuscript. Comments by R. Borzi are also acknowledged. This work has been partially suported by FOMEC.

TABLES

TABLE I. Coefficients of the function $e^{-2\Lambda}$

c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
2	70	1071	9310	48077	506829	$\frac{965581}{2}$	262144

REFERENCES

[1] M. Morris and K. Thorne, Am. J. Phys. **56**, 395 (1988).

[2] M. Morris, K. Thorne and U. Yurtserver, Phys. Rev. Lett. **61**, 1446 (1988).

[3] R. Fuller and J. Wheeler, Phys. Rev. D **128**, 9191 (1962).

[4] S. Hawking and G. Ellis, *The large scale structure of space time*, (Cambridge Univ. Press, Cambridge, 1973).

[5] S. W. Hawking, Phys. Rev. D **46**, 603 (1992).

[6] V. P. Frolov, I. D. Novikov, Phys. Rev. D **42**, 1057 (1990).

[7] H. Epstein, V. Glasser, and A. Jaffe, Nuovo Cimento **36**, 2296 (1965).

[8] S. Kar, Phys. Rev. D **49**, 862 (1994)

[9] M. Visser, Phys. Rev. D **39**, 3182 (1989); Nucl. Phys. **B328**, 203 (1989).

[10] D. Hochberg, Phys. Lett. **B251**, 349 (1990).

[11] J. Moffat and T. Svodoba, Phys. Rev. D **44**, 429 (1991).

[12] B. Bhawal and S. Kar, Phys.Rev. D **46**, 2464 (1992).

[13] A. Agnese and M. La Camera, Phys. Rev. D **51**, 2011 (1995).

[14] L. A. Anchordoqui, S. E. Perez Bergliaffa, D. F. Torres, gr-qc 9610070.

[15] P. S. Letelier, A. Wang, Phys. Rev. D **48**, 631 (1993).

[16] For reviews see, e.g., F. W. Hehl, P. Heyde, G. D. Kerlick, Rev. Mod. Phys. **48** 393 (1976); F. W. Hehl, B. K. Datta, J. Math. Phys. **12**, 1334 (1971).

[17] It is important to stress, that in U_4 theory, eqs. (3) and (4) hold in a microscopic domain, besides, the quantities σ^{ij} and $\tau_{ij}{}^{k}$ are microscopically fluctuating. Thus, one has to performed a spacetime average of eqs. (3) and (4) in order to obtain an equation for

bulk matter. Even after taking this average, the energy momentum tensor is corrected by spin square terms, which are negligible at normal matter densities [18].

[18] F. W. Hehl, P. Heyde, G. D. Kerlick, Phys. Rev. D **10**, 1066 (1974).

[19] S. Kar and D. Sahdev, Phys. Rev D **52**, 2030 (1995).

[20] An "absurdly benign" wormhole can be threaded in U_4 theory, fixing the parameters in the solutions obtained by Prasanna. A. R. Prasanna, Phys. Rev. D **11**, 2076 (1975).

[21] Static spherically symmetric vacuum solutions endowed with torsion, generated by Brans-Dicke scalar were computed by Kim and Cho. These solutions are not by themselves wormholes, however a Visser-type wormhole can be constructed with one of them. S. W. Kim, B. H. Cho, Phys. Lett. A **124**, 243 (1987).

[22] G. D. Kerlick, Ap. J. **185**, 631 (1973).

[23] T. A. Roman, Phys. Rev. D **47**, 1370 (1993).