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Bishop-Phelps-Bollobás property for positive operators when the domain is L_∞

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We prove that the class of positive operators from $L_\infty(\mu)$ to Y has the Bishop-Phelps-Bollobás property for any positive measure μ, whenever Y is a uniformly monotone Banach lattice with a weak unit. The same result also holds for the pair (c_0, Y) for any uniformly monotone Banach lattice Y. Further we show that these results are optimal in case that Y is strictly monotone.

Keywords: Banach space, operator, Bishop-Phelps-Bollobás theorem, Bishop-Phelps-Bollobás property.

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1. Introduction

In 1961 Bishop and Phelps proved that every continuous linear functional on a Banach space can be approximated by norm attaining functionals [10]. Since then a lot of attention has been devoted to extend Bishop-Phelps theorem in the setting

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of operators on Banach spaces (see [2]). On the other hand Bollobás proved a "quantitative version" of that result in 1970 [11]. Before stating this result we introduce some notation. By B_X, S_X and X^* we denote the closed unit ball, the unit sphere and the topological dual of a Banach space X, respectively. If X and Y are both real or complex Banach spaces, $L(X,Y)$ denotes the space of (bounded linear) operators from X to Y, endowed with its usual operator norm.

Bishop-Phelps-Bollobás theorem (see [12, Theorem 16.1] or [13, Corollary 2.4]). Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{2}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

In 2008, Acosta *et al.* defined the Bishop-Phelps-Bollobás property for operators between Banach spaces [4].

Definition 1.1 ([4, Definition 1.1]). Let X and Y be either real or complex Banach spaces. The pair (X,Y) is said to have the *Bishop-Phelps-Bollobás property for operators* if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_{L(X,Y)}$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $T \in S_{L(X,Y)}$ satisfying the following conditions

$$\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

Since then a number of interesting results related to this property have been obtained (see [3]).

Very recently in [8], the authors introduced the notion of Bishop-Phelps-Bollobás property for positive operators between two Banach lattices. Let us mention that the only difference between this property and the previous one is that in the new property the operators appearing in Definition 1.1 are positive. In the same paper it is proved that the pairs $(c_0, L_1(\nu))$ and $(L_\infty(\mu), L_1(\nu))$ have the Bishop-Phelps-Bollobás property for positive operators for any positive measures μ and ν (see [8, Theorems 1.7 and 1.6]).

In this paper we show a far reaching extension of those results. More precisely we prove that the pair (c_0, Y) has the Bishop-Phelps-Bollobás property for positive operators whenever Y is a uniformly monotone Banach lattice (see Corollary 3.3). We also show that the pair $(L_\infty(\mu), Y)$ has the Bishop-Phelps-Bollobás property for positive operators for any positive measure μ if Y is a uniformly monotone Banach lattice with a weak unit (see Corollary 2.6). Note that the last assumption is very mild. For instance, separable Banach lattices have a weak unit (see [9, Lemma 3, p. 367]).

We also remark that not every Banach function space Y satisfies that the pair (c_0, Y) has the Bishop-Phelps-Bollobás property for positive operators (see [8, Example 1.8]). It is worth also to mention that it is not known whether or not the pair (c_0, ℓ_1) has the Bishop-Phelps-Bollobás property for operators in the real case. For some partial results when the domain is c_0 see [15, 7, 6] and [5]. The paper [16] contains a positive result for the pair $(L_\infty(\mu), Y)$, whenever Y is a uniformly convex Banach space.

In the last section of the paper we show under very mild assumptions that anytime that a pair of Banach lattices (X, Y) has the BPBp for positive operators, then Y is indeed uniformly monotone, whenever X admits non trivial M-summands and Y is strictly monotone (see Proposition 4.3). As a consequence, in case that Y is strictly monotone, the main results of Secs. 2 and 3 provide characterizations indeed (see Corollary 4.4).

We remark that throughout this paper we consider only real Banach spaces.

2. Bishop–Phelps–Bollobás Property for Positive Operators from L_∞ to a Uniformly Monotone Banach Lattice

We begin by recalling some definitions and the appropriate notion of Bishop-Phelps-Bollobás property for positive operators. For the terminology and basic facts related to Banach lattices see, for instance, [1, 9, 17].

An *ordered vector space* is a real vector space X equipped with an order relation $\leq$ that is compatible with the algebraic structure of X. An ordered vector space is a *Riesz space* if every pair of vectors has a least upper bound and a greatest lower bound. In a Riesz space X, given two elements x and y in X, we denote by $x \wedge y$ and $|x|$ the infimum of x and y and the supremum of x and $-x$, respectively. A norm $\| \ \|$ on a Riesz space X is said to be *a lattice norm* whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. *A normed Riesz space* is a Riesz space equipped with a lattice norm. A *Banach lattice* is a normed Riesz space whose norm is complete. A positive element e in a Banach lattice X is a *weak unit* if $x \wedge e = 0$ for some $x \in X$ implies that $x = 0$.

A Banach lattice E is *uniformly monotone* if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that whenever $x \in S_E$, $y \in E$ and $x, y \geq 0$ the condition $\|x + y\| \leq 1 + \delta(\varepsilon)$ implies that $\|y\| \leq \varepsilon$. A Banach lattice X is said to be *order continuous* whenever $(x_\alpha) \downarrow 0$ in X implies $(\|x_\alpha\|) \to 0$.

In case that (Ω, μ) is a measure space, we denote by $L^0(\mu)$ the space of (equivalence classes of μ-a.e. equal) real-valued measurable functions on Ω. We say that a Banach space X is a *Banach function space* on (Ω, μ) if X is an ideal in $L^0(\mu)$ and whenever $x, y \in X$ and $|x| \leq |y|$ a.e., then $\|x\| \leq \|y\|$.

An operator $T : X \to Y$ between two ordered vector spaces is called *positive* if $x \geq 0$ implies $Tx \geq 0$.

Let us note that in case that Y is a uniformly monotone Banach lattice it follows from the definition that the function δ satisfies $\delta(t) \leq t$ for every positive real t.

Definition 2.1 ([8, Definition 1.3]). Let X and Y be Banach lattices. The pair (X, Y) is said to have the *Bishop-Phelps-Bollobás property for positive operators* if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in S_{L(X,Y)}$, such that $S \geq 0$, if $x_0 \in S_X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and a positive operator $T \in S_{L(X,Y)}$ satisfying the following conditions

$$\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

Remark 2.2. In case that the pair (X, Y) satisfies the previous definition, if the element x_0 is positive, then the element u_0 can also be chosen positive.

Proof. Assume that (X, Y) has the BPBp for positive operators and assume that $S \in S_{L(X,Y)}$ and $x_0 \in S_X$ satisfy the assumptions in Definition 2.1 and x_0 is also positive. So there exists a pair $(T, u_0) \in S_{L(X,Y)} \times S_X$ such that T is positive and satisfying

$$\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

We will check that the positive element $|u_0|$ also satisfies the desired conditions.

Note that from triangle inequality we have $\||u_0| - |x_0|\| \le \|u_0 - x_0\| < \varepsilon$, so since x_0 is positive we conclude $\||u_0| - x_0\| \le \|u_0 - x_0\| < \varepsilon$. On the other hand since the operator T is positive, $|T(u_0)| \le T(|u_0|)$. Hence $1 = \|T(u_0)\| \le \|T(|u_0|)\| \le \|T\| = 1$, so $\|T(|u_0|)\| = 1$. Therefore the element $|u_0| \in S_X$ satisfies the desired conditions. □

Next we show some technical results that will be useful later. Throughout the rest of the section, if (Ω, μ) is a measure space, we denote by $\mathbb{1}$ the constant function equal to 1 on Ω. Since an element f in $B_{L_\infty(\mu)}$ satisfies that $|f| \le \mathbb{1}$ a.e., it is clear that a positive operator from $L_\infty(\mu)$ to any other Banach lattice satisfies the next assertion.

Lemma 2.3. *Let μ be a positive measure and T a positive operator from $L_\infty(\mu)$ to some Banach lattice Y. Then $\|T\| = \|T(\mathbb{1})\|$.*

The next result extends [8, Lemma 1.5], where the analogous result was proved for $L_1(\mu)$.

Lemma 2.4. *Let Y be a uniformly monotone Banach function space and $0 < \varepsilon < 1$. Assume that f_1 and f_2 are positive elements in Y such that*

$$\|f_1 + f_2\| \le 1 \quad \textit{and} \quad \frac{1}{1 + \delta(\frac{\varepsilon}{3})} \le \|f_1 - f_2\|,$$

where δ is the function satisfying the definition of uniform monotonicity for Y. Then there are two positive functions h_1 and h_2 in Y with disjoint supports satisfying that

$$\|h_1 + h_2\| = 1 \quad \textit{and} \quad \|h_i - f_i\| < \varepsilon \quad \textit{for} \quad i = 1, 2.$$

Proof. Assume that Y is a Banach function space on the measure space (Ω, μ). We consider the partition of Ω given by the following sets

$$C_1 = \{t \in \Omega : f_2(t) \le f_1(t)\} \quad \text{and} \quad C_2 = \{t \in \Omega : f_1(t) < f_2(t)\}.$$

Clearly C_1 and C_2 are measurable sets. The function h given by $h = 2\big(f_1 \chi_{C_2} + f_2 \chi_{C_1}\big)$ belongs to Y since Y is a Banach function space on (Ω, μ). It is also clear that h is positive and it is satisfied that

$$|f_1 - f_2| + h = f_1 + f_2. \tag{2.1}$$

Clearly we have that

$$\|f_1 + f_2\| \le 1 \le \left(1 + \delta\left(\frac{\varepsilon}{3}\right)\right) \|f_1 - f_2\| = \left(1 + \delta\left(\frac{\varepsilon}{3}\right)\right) |||f_1 - f_2|||.$$

By using that Y is uniformly monotone, from (2.1) it follows that

$$\|h\| \le \frac{\varepsilon}{3}|||f_1 - f_2||| = \frac{\varepsilon}{3}\|f_1 - f_2\| \le \frac{\varepsilon}{3}\|f_1 + f_2\| \le \frac{\varepsilon}{3}.$$

As a consequence

$$\|f_1\chi_{C_2}\| \le \frac{\|h\|}{2} \le \frac{\varepsilon}{6} \quad \text{and} \quad \|f_2\chi_{C_1}\| \le \frac{\varepsilon}{6}. \tag{2.2}$$

Now define $g_i = f_i\chi_{C_i}$ for $i = 1, 2$. Note that g_1 and g_2 are positive functions with disjoint supports, belong to Y and also satisfy

$$\|g_1 + g_2\| \le 1. \tag{2.3}$$

By (2.2) the function g_1 satisfies

$$\|g_1 - f_1\| = \|f_1\chi_{C_1} - f_1\| = \|f_1\chi_{C_2}\| \le \frac{\varepsilon}{6}. \tag{2.4}$$

By using the same argument we also obtain that

$$\|g_2 - f_2\| \le \frac{\varepsilon}{6}. \tag{2.5}$$

It is also satisfied that

$$\begin{aligned} \|g_1 + g_2\| &\ge \|f_1 + f_2\| - \|f_1 - g_1\| - \|f_2 - g_2\| \\ &\ge \|f_1 - f_2\| - \frac{\varepsilon}{3} \quad \text{(by (2.4) and (2.5))} \\ &\ge \frac{1}{1 + \delta\left(\frac{\varepsilon}{3}\right)} - \frac{\varepsilon}{3} \\ &\ge \frac{1}{1 + \frac{\varepsilon}{3}} - \frac{\varepsilon}{3} > 0. \end{aligned} \tag{2.6}$$

For $i = 1, 2$ we can define the function h_i by $h_i = \frac{g_i}{\|g_1+g_2\|}$. Clearly h_1 and h_2 are positive functions in Y with disjoint supports satisfying also that $\|h_1 + h_2\| = 1$.

For $i = 1, 2$ we also have that

$$\begin{aligned} \|h_i - g_i\| &= \left\| \frac{g_i}{\|g_1 + g_2\|} - g_i \right\| \\ &= \frac{\|g_i\|}{\|g_1 + g_2\|} |1 - \|g_1 + g_2\|| \\ &\le 1 - \|g_1 + g_2\| \quad \text{(by (2.3))} \\ &\le 1 - \frac{1}{1 + \delta\left(\frac{\varepsilon}{3}\right)} + \frac{\varepsilon}{3} \quad \text{(by (2.6))} \\ &< \frac{2\varepsilon}{3}. \end{aligned} \tag{2.7}$$

By using (2.7), (2.4) and (2.5) we can estimate the distance between h_i and f_i for $i = 1, 2$ as follows

$$\|h_i - f_i\| \leq \|h_i - g_i\| + \|g_i - f_i\| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{6} < \varepsilon.$$

This finishes the proof. □

Theorem 2.5. *The pair $(L_\infty(\mu), Y)$ has the Bishop–Phelps–Bollobás property for positive operators, for any positive measure μ, whenever Y is a uniformly monotone Banach function space. The function η satisfying Definition* 2.1 *depends only on the function δ satisfying the definition of uniform monotonicity for Y.*

Proof. Assume that (Ω_1, μ) is a measure space and Y is a Banach function space on (Ω_2, ν). Let $0 < \varepsilon < 1$ and δ be the function satisfying the definition of uniform monotonicity for the Banach function space Y. Choose a real number η such that $0 < \eta = \eta(\varepsilon) < \frac{\varepsilon}{18}$ and satisfying also

$$\frac{1}{1 + \delta\left(\frac{\varepsilon}{18}\right)} < \frac{1}{1 + \delta(\eta^2)} - 3\eta. \tag{2.8}$$

Assume that $f_0 \in S_{L_\infty(\mu)}, S \in S_{L(L_\infty(\mu),Y)}$ and S is a positive operator such that

$$\|S(f_0)\| > \frac{1}{1 + \delta(\eta^2)}.$$

We can assume without loss of generality that $|f_0| \leq 1$. We define the sets A, B and C given by

$$A = \{t \in \Omega_1 : -1 \leq f_0(t) < -1 + \eta\}, \quad B = \{t \in \Omega_1 : 1 - \eta < f_0(t) \leq 1\}$$

and

$$C = \{t \in \Omega_1 : |f_0(t)| \leq 1 - \eta\}.$$

Clearly $\{A, B, C\}$ is a partition of Ω_1 into measurable sets. We clearly have that $|f_0| + \eta\chi_C \in S_{L_\infty(\mu)}$. By using that S is a positive operator we have that

$$\begin{aligned} \|S(|f_0| + \eta\chi_C)\| &\leq 1 \\ &< \|S(f_0)\|(1 + \delta(\eta^2)) \\ &\leq \|S(|f_0|)\|(1 + \delta(\eta^2)). \end{aligned}$$

In view of the uniform monotonicity of Y the previous inequality implies that $\|S(\eta\chi_C)\| \leq \eta^2$ and so

$$\|S(\chi_C)\| \leq \eta. \tag{2.9}$$

From the definition of the sets A and B we have that $\|\chi_A + f_0\chi_A\|_\infty \leq \eta$ and $\|f_0\chi_B - \chi_B\|_\infty \leq \eta$ and so

$$\|S(\chi_A) + S(f_0\chi_A)\| \leq \eta \quad \text{and} \qquad \|S(f_0\chi_B) - S(\chi_B)\| \leq \eta. \tag{2.10}$$

We clearly obtain that

$$\begin{aligned}\|S(\chi_B) - S(\chi_A)\| &\geq \|S(f_0\chi_B) + S(f_0\chi_A)\| - \|S(f_0\chi_B) - S(\chi_B)\| \\ &\quad - \|S(\chi_A) + S(f_0\chi_A)\| \\ &\geq \|S(f_0)\| - \|S(f_0\chi_C)\| - 2\eta \quad (\text{by } (2.10)) \\ &> \frac{1}{1+\delta(\eta^2)} - \|S(\chi_C)\| - 2\eta \\ &\geq \frac{1}{1+\delta(\eta^2)} - 3\eta \quad (\text{by } (2.9)) \\ &> \frac{1}{1+\delta\left(\frac{\varepsilon}{18}\right)}. \end{aligned} \tag{2.11}$$

Since S is a positive operator and $\|S(\chi_A)+S(\chi_B)\| \leq 1$, in view of (2.11) we can apply Lemma 2.4. Hence there are two positive functions h_1 and h_2 in Y satisfying the following conditions

$$\|h_1 - S(\chi_A)\| < \frac{\varepsilon}{6}, \quad \|h_2 - S(\chi_B)\| < \frac{\varepsilon}{6}, \tag{2.12}$$

$$\text{supp } h_1 \cap \text{supp } h_2 = \varnothing \quad \text{and} \quad \|h_1 + h_2\| = 1. \tag{2.13}$$

Hence

$$\begin{aligned}\|S(\chi_A)\chi_{\Omega_2\setminus \text{supp } h_1}\| &= \|(h_1 - S(\chi_A))\chi_{\Omega_2\setminus \text{supp } h_1}\| \\ &\leq \|h_1 - S(\chi_A)\| \\ &< \frac{\varepsilon}{6} \quad (\text{by } (2.12)) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\|S(\chi_B)\chi_{\Omega_2\setminus \text{supp } h_2}\| &= \|(h_2 - S(\chi_B))\chi_{\Omega_2\setminus \text{supp } h_2}\| \\ &\leq \|h_2 - S(\chi_B)\| \\ &< \frac{\varepsilon}{6} \quad (\text{by } (2.12)). \end{aligned} \tag{2.15}$$

Now we define the operator $U : L_\infty(\mu) \to Y$ as follows

$$U(f) = S(f\chi_A)\chi_{\text{supp } h_1} + S(f\chi_B)\chi_{\text{supp } h_2} \quad (f \in L_\infty(\mu)).$$

Since Y is a Banach function space and $S \in L(L_\infty(\mu), Y)$, U is well-defined and belongs to $L(L_\infty(\mu), Y)$. The operator U is positive since S is positive. It also satisfies that

$$\begin{aligned}\|U - S\| &= \sup\{\|S(f\chi_A)\chi_{\Omega_2\setminus \text{supp } h_1} + S(f\chi_B)\chi_{\Omega_2\setminus \text{supp } h_2} + S(f\chi_C)\| : f \in B_{L_\infty(\mu)}\} \\ &\leq \sup\{\|S(f\chi_A)\chi_{\Omega_2\setminus \text{supp } h_1}\| + \|S(f\chi_B)\chi_{\Omega_2\setminus \text{supp } h_2})\| + \|S(f\chi_C)\| : f \in B_{L_\infty(\mu)}\}\end{aligned}$$

$$\leq \|S(\chi_A)\chi_{\Omega_2\setminus \text{supp } h_1}\| + \|S(\chi_B)\chi_{\Omega_2\setminus \text{supp } h_2})\| + \|S(\chi_C)\|$$
$$< \frac{\varepsilon}{3} + \eta < \frac{\varepsilon}{2} \quad \text{(by (2.14), (2.15) and (2.9))}. \tag{2.16}$$

Hence

$$|\|U\| - 1| < \frac{\varepsilon}{2}, \tag{2.17}$$

so $U \neq 0$.

Finally we define $T = \frac{U}{\|U\|}$. Since U is a positive operator, T is also positive. Of course $T \in S_{L(L_\infty(\mu),Y)}$ and also satisfies

$$\begin{aligned}\|T - S\| &\leq \|T - U\| + \|U - S\| \\ &< \left\|\frac{U}{\|U\|} - U\right\| + \frac{\varepsilon}{2} \quad \text{(by (2.16))} \\ &= |1 - \|U\|| + \frac{\varepsilon}{2} \\ &< \varepsilon \quad \text{(by (2.17))}.\end{aligned} \tag{2.18}$$

The function f_1 given by $f_1 = \chi_B - \chi_A + f_0\chi_C$ belongs to $S_{L_\infty(\mu)}$ and satisfies that

$$\|f_1 - f_0\|_\infty = \|\chi_B - \chi_A + f_0\chi_C - f_0\|_\infty \leq \eta < \varepsilon. \tag{2.19}$$

We clearly have that

$$U(f_1) = U(\chi_B - \chi_A + f_0\chi_C) = S(\chi_B)\chi_{\sup h_2} - S(\chi_A)\chi_{\sup h_1}.$$

Since U is a positive operator it satisfies that

$$\|U\| = \|U(\mathbb{1})\| = \|S(\chi_A)\chi_{\text{supp } h_1} + S(\chi_B)\chi_{\text{supp } h_2}\|.$$

For each $t \in \Omega_2$ we obtain that

$$\begin{aligned}|(U(\mathbb{1}))(t)| &= |(S(\chi_A)\chi_{\sup h_1} + S(\chi_B)\chi_{\sup h_2})(t)| \\ &= |(-S(\chi_A)\chi_{\sup h_1} + S(\chi_B)\chi_{\sup h_2})(t)| \quad \text{(by (2.13))} \\ &= |(U(\chi_B - \chi_A + f_0\chi_C)(t)|| \\ &= |(U(f_1))(t)|.\end{aligned}$$

Since Y is a Banach function space we conclude that

$$\|U\| = \|U(\mathbb{1})\| = \|U(f_1)\|.$$

By (2.19) and (2.18), since T attains its norm at f_1, the proof is finished □

The previous result was proved in [8, Theorem 1.6] in case that the range is a L_1 space, so Theorem 2.5 is already a far reaching extension of that result.

Our purpose now is to obtain a version of Theorem 2.5 for some abstract Banach lattices. In order to get this result, we note that every uniformly monotone Banach lattice is order continuous (see [9, Theorem 21, p. 371] and [17, Proposition 1.a.8]). It is also known that any order continuous Banach lattice with a weak unit is order isometric to a Banach function space (see [17, Theorem 1.b.14]). From Theorem 2.5 and the previous argument we deduce the following result.

Corollary 2.6. *The pair $(L_\infty(\mu), Y)$ has the Bishop-Phelps-Bollobás property for positive operators, for any positive measure μ, whenever Y is a uniformly monotone Banach lattice with a weak unit. Moreover, the function η satisfying Definition* 2.1 *depends only on the function δ satisfying the definition of uniform monotonicity for Y.*

3. Bishop–Phelps–Bollobás Property for Positive Operators in Case that the Domain is c_0

In this section we show parallel results to Theorem 2.5 and Corollary 2.6 in case that the domain space is c_0. We begin with a technical result whose proof is straightforward.

Lemma 3.1. *Let Y be a Banach lattice and $T \in L(c_0, Y)$ be a positive operator. Then the following assertions are satisfied*:

(1) $\|T\| = \sup\{\|T(\sum_{k=1}^n e_k)\| : n \in \mathbb{N}\} = \sup\{\|T(\sum_{k=1}^n e_k)\| : n \geq N\}$, *for all natural numbers N.*
(2) $\sup\{\|T(x\chi_C)\| : x \in B_{c_0}\} = \sup\{\|T(\chi_{C\cap\{k\in\mathbb{N}:k\leq n\}})\| : n \in \mathbb{N}, n \geq N\}$ *for all $C \subset \mathbb{N}$ and all positive integers N.*

Theorem 3.2. *The pair (c_0, Y) satisfies the Bishop–Phelps–Bollobás property for positive operators, for any uniformly monotone Banach function space Y. Moreover, the function η satisfying Definition* 2.1 *depends only on the function δ satisfying the definition of uniform monotonicity for Y.*

Proof. The proof of this result is similar to the proof of Theorem 2.5. We include the details of the proof for the sake of completeness.

Assume that Y is a Banach function space on the measure space (Ω, μ).

Let $0 < \varepsilon < 1$, and assume that Y satisfies the definition of uniform monotonicity with the function δ. Choose a positive real number such that $\eta < \frac{\varepsilon}{18}$ satisfying also

$$\frac{1}{1+\delta\left(\frac{\varepsilon}{18}\right)} < \frac{1}{1+\delta(\eta^2)} - 3\eta. \tag{3.1}$$

Assume that $x_0 \in S_{c_0}, S \in S_{L(c_0,Y)}$ and S is a positive operator such that

$$\|S(x_0)\| > \frac{1}{1+\delta(\eta^2)}.$$

Consider the sets A, B and C defined by

$$A = \{k \in \mathbb{N} : -1 \leq x_0(k) < -1 + \eta\}, \quad B = \{k \in \mathbb{N} : 1 - \eta < x_0(k) \leq 1\}$$

and

$$C = \{k \in \mathbb{N} : |x_0(k)| \leq 1 - \eta\}.$$

Clearly $\{A, B, C\}$ is a partition of $\mathbb{N}$. Also the subsets A and B are finite sets, so χ_A and χ_B belong to c_0. For a positive integer n denote by $C_n = C \cap \{k \in \mathbb{N} : k \leq n\}$. We clearly have that $|x_0| + \eta\chi_{C_n} \in S_{c_0}$ for all $n \in \mathbb{N}$. Since S is a positive operator for each positive integer n it is satisfied that

$$\begin{aligned} \|S(|x_0| + \eta\chi_{C_n})\| &\leq 1 \\ &< \|S(x_0)\|(1 + \delta(\eta^2)) \\ &\leq \|S(|x_0|)\|(1 + \delta(\eta^2)). \end{aligned}$$

In view of the uniform monotonicity of Y the previous inequality implies that $\|S(\eta\chi_{C_n})\| \leq \eta^2$ for all $n \in \mathbb{N}$ and so

$$\|S(\chi_{C_n})\| \leq \eta, \quad \forall\, n \in \mathbb{N}.$$

From Lemma 3.1 we deduce that

$$\|S(x\chi_C)\| \leq \sup\{\|S(\chi_{C_n})\| : n \in \mathbb{N}\} \leq \eta, \quad \forall\, x \in B_{c_0}. \tag{3.2}$$

From the definition of the sets A and B we also have that $\|\chi_A + x_0\chi_A\|_\infty \leq \eta$ and $\|x_0\chi_B - \chi_B\|_\infty \leq \eta$, so

$$\|S(\chi_A) + S(x_0\chi_A))\| \leq \eta \quad \text{and} \quad \|S(x_0\chi_B) - S(\chi_B)\| \leq \eta. \tag{3.3}$$

We clearly obtain that

$$\begin{aligned} \|S(\chi_B) - S(\chi_A)\| &\geq \|S(x_0\chi_B) + S(x_0\chi_A)\| - \|S(x_0\chi_B) - S(\chi_B)\| \\ &\quad - \|S(\chi_A) + S(x_0\chi_A)\| \\ &\geq \|S(x_0)\| - \|S(x_0\chi_C)\| - 2\eta \quad (\text{by } (3.3)) \\ &> \frac{1}{1 + \delta(\eta^2)} - \|S(x_0\chi_C)\| - 2\eta \\ &\geq \frac{1}{1 + \delta(\eta^2)} - 3\eta \quad (\text{by } (3.2)) \\ &> \frac{1}{1 + \delta\left(\frac{\varepsilon}{18}\right)}. \end{aligned} \tag{3.4}$$

Since S is a positive operator and $\|S(\chi_A) + S(\chi_B)\| \leq 1$, in view of (3.4) we can apply Lemma 2.4. Hence there are two positive functions g_1 and g_2 in Y satisfying the following conditions:

$$\|g_1 - S(\chi_A)\| < \frac{\varepsilon}{6}, \quad \|g_2 - S(\chi_B)\| < \frac{\varepsilon}{6}, \tag{3.5}$$

$$\text{supp } g_1 \cap \text{supp } g_2 = \varnothing \quad \text{and} \quad \|g_1 + g_2\| = 1. \tag{3.6}$$

Hence

$$\begin{aligned} \|S(\chi_A)\chi_{\Omega\backslash \text{supp } g_1}\| &= \|(g_1 - S(\chi_A))\chi_{\Omega\backslash \text{supp } g_1}\| \\ &\le \|g_1 - S(\chi_A)\| \\ &< \frac{\varepsilon}{6}. \quad (\text{by (3.5)}) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|S(\chi_B)\chi_{\Omega\backslash \text{supp } g_2}\| &= \|(g_2 - S(\chi_B))\chi_{\Omega\backslash \text{supp } g_2}\| \\ &\le \|g_2 - S(\chi_B)\| \\ &< \frac{\varepsilon}{6} \quad (\text{by (3.5)}). \end{aligned} \tag{3.8}$$

Now we define the operator $R : c_0 \to Y$ as follows

$$R(x) = S(x\chi_A)\chi_{\text{supp } g_1} + S(x\chi_B)\chi_{\text{supp } g_2} \quad (x \in c_0).$$

Since Y is a Banach function space and $S \in L(c_0, Y)$, R is well-defined and belongs to $L(c_0, Y)$. The operator R is positive since S is positive. By using Lemma 3.1, for any element $x \in B_{c_0}$ we have that

$$\begin{aligned} \|(R - S)(x)\| &= \|S(x\chi_A)\chi_{\Omega\backslash \text{supp } g_1} + S(x\chi_B)\chi_{\Omega\backslash \text{supp } g_2} + S(x\chi_C)\| \\ &\le \|S(\chi_A)\chi_{\Omega\backslash \text{supp } g_1}\| + \|S(\chi_B)\chi_{\Omega\backslash \text{supp } g_2})\| + \|S(x\chi_C)\| \\ &< \frac{\varepsilon}{3} + \eta < \frac{\varepsilon}{2} \quad (\text{by (3.7), (3.8) and (3.2)}). \end{aligned}$$

As a consequence

$$\|R - S\| < \frac{\varepsilon}{2} \quad \text{and} \quad |\|R\| - 1| < \frac{\varepsilon}{2}, \tag{3.9}$$

so $R \neq 0$.

We put $T = \frac{R}{\|R\|}$. The operator T is positive since R is positive. Of course $T \in S_{L(c_0,Y)}$. Now we estimate the distance from T to S as follows

$$\begin{aligned} \|T - S\| &\le \|T - R\| + \|R - S\| \\ &< \left\| \frac{R}{\|R\|} - R \right\| + \frac{\varepsilon}{2} \quad (\text{by (3.9)}) \\ &= |1 - \|R\|| + \frac{\varepsilon}{2} \\ &< \varepsilon \quad (\text{by (3.9)}). \end{aligned} \tag{3.10}$$

The element u_0 given by $u_0 = \chi_B - \chi_A + x_0\chi_C$ belongs to the unit sphere of c_0 since $x_0 \in S_{c_0}$. It also satisfies

$$\|u_0 - x_0\|_\infty \leq \eta < \varepsilon. \tag{3.11}$$

Since g_1 and g_2 have disjoint supports, Y is a Banach function space and R is a positive operator, we also have that

$$\begin{aligned} \|R(u_0)\| &= \|S(-\chi_A)\chi_{\text{supp } g_1} + S(\chi_B)\chi_{\text{supp } g_2}\| \\ &= \|S(\chi_A)\chi_{\text{supp } g_1} + S(\chi_B)\chi_{\text{supp } g_2}\| \\ &= \|R(\chi_{A\cup B})\| = \|R\|. \end{aligned}$$

Hence the operator T also attains its norm at u_0. In view of (3.10) and (3.11) the proof is finished. □

Corollary 3.3. *The pair* (c_0, Y) *has the Bishop–Phelps–Bollobás property for positive operators, for any uniformly monotone Banach lattice* Y. *Moreover, the function* η *satisfying Definition* 2.1 *depends only on the function* δ *satisfying the definition of uniform monotonicity for* Y.

Proof. By using the same argument of Corollary 2.6, in view of Theorem 3.2, we obtain the statement when Y is uniformly monotone Banach lattice with a weak unit. Note also that in Theorem 3.2 the function η appearing in Definition 2.1 depends only on the function δ satisfying the definition of uniform monotonicity for the Banach function space on the range.

Assume now that Y is a uniformly monotone Banach lattice and the function δ satisfying the definition of uniform monotonicity for Y. Since c_0 is separable, if $T \in L(c_0, Y)$, the space $T(c_0)$ is also separable. It is well known that the Banach lattice X generated by $T(c_0)$ is a separable Banach lattice of Y and so, it also satisfies the definition of uniform monotonicity with the function δ. By [9, Lemma 3, p. 367] X has a weak unit. By the previous arguments, the pair (c_0, X) satisfies the Bishop–Phelps–Bollobás property for positive operators with a function η depending only on δ. As a consequence, the pair (c_0, Y) also has the Bishop–Phelps–Bollobás property for positive operators. □

Kim proved that the pair (c_0, Y) has the Bishop–Phelps–Bollobás property for operators when Y is uniformly convex (see [15, Corollary 2.6]). Note that the previous result is a version of that result for positive operators.

4. Results are Optimal When the Range is Strictly Monotone

Our intention now is to show that under some mild assumption on the range space, Corollaries 2.6 and 3.3 are optimal. For this end we need some preliminary results.

Lemma 4.1. *Let M, N and Y be normed spaces. Assume that $T \in L(M \oplus_\infty N, Y)$ and $m + n \in S_{M\oplus_\infty N}$ satisfy that $\|n\| < 1$ and $\|T(m+n)\| = \|T\|$. Then $\|T(m)\| = \|T\|$.*

Proof. If we put $t = \|n\|$, then $0 \le t < 1$. In case that $t = 0$ it is clear that T attains its norm at m. Otherwise $0 < t < 1$, the element $m + \frac{1}{t}n$ belongs to the unit sphere of $M \oplus_\infty N$ and

$$m + n = (1-t)m + t\left(m + \frac{1}{t}n\right).$$

Since $\|T(m+n)\| = \|T\|$, and the function $x \mapsto \|T(x)\|$ is convex on $M \oplus N$, we conclude that $\|T(m)\| = \|T\|$. □

Next result is probably known, but we did not find a reference in the literature and we include it for the sake of completeness.

Proposition 4.2. *Let Y be a Banach lattice. The following conditions are equivalent.*

(1) *Y is uniformly monotone.*
(2) *For every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ satisfying*

$$u \in Y, v \in S_Y, \quad 0 \le u \le v \quad \textit{and} \quad \|v - u\| > 1 - \eta(\varepsilon) \Rightarrow \|u\| \le \varepsilon.$$

(3) *For every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ satisfying*

$$u, v \in Y, \quad 0 \le u \le v \quad \textit{and} \quad \|v - u\| > (1 - \eta(\varepsilon))\|v\| \Rightarrow \|u\| \le \varepsilon\|v\|.$$

Moreover, if (2) is satisfied, Y is uniformly monotone with $\delta(\varepsilon) = \eta(\frac{\varepsilon}{2})$. In case that Y is uniformly monotone with $\delta(\varepsilon)$ conditions (2) and (3) are satisfied with $\eta(\varepsilon) = \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}$.

Proof. (1) ⇒ (2)

Assume that Y is uniformly monotone with $\delta(\varepsilon)$. Assume that $0 < \varepsilon < 1$ and u and v are elements in Y such that

$$\|v\| = 1, \quad 0 \le u \le v \quad \text{and} \quad \|v - u\| > 1 - \eta(\varepsilon) = 1 - \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}.$$

Put $x = v - u, y = u$. So $x, y \ge 0$ and

$$\begin{aligned}\|x + y\| = \|v\| = 1 &= (1+\delta(\varepsilon))\left(\frac{1}{1+\delta(\varepsilon)}\right)\\ &= (1+\delta(\varepsilon))\left(1 - \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)}\right)\\ &< (1+\delta(\varepsilon))\|v-u\|\\ &= (1+\delta(\varepsilon))\|x\|.\end{aligned}$$

Hence from uniform monotonicity of Y, we conclude $\|u\| = \|y\| \leq \varepsilon \|x\| \leq \varepsilon$.

(2) $\Rightarrow$ (3)

Trivial

(3) $\Rightarrow$ (1)

Assume that $0 < \varepsilon < 1$, $x \in S_Y$, $y \in Y$ satisfy that $x, y \geq 0$ and $\|x + y\| \leq 1 + \delta(\varepsilon) = 1 + \eta(\frac{\varepsilon}{2})$. Put $u = y, v = x + y$. So $0 \leq u \leq v$ and

$$\begin{aligned} \|v - u\| = \|x\| = 1 &> \left(1 - \eta\left(\frac{\varepsilon}{2}\right)\right)\left(1 + \eta\left(\frac{\varepsilon}{2}\right)\right) \\ &\geq \left(1 - \eta\left(\frac{\varepsilon}{2}\right)\right)\|x + y\| \\ &= \left(1 - \eta\left(\frac{\varepsilon}{2}\right)\right)\|v\|. \end{aligned}$$

Hence from the assumptions we conclude $\|y\| = \|u\| \leq \frac{\varepsilon}{2}\|v\| \leq \frac{\varepsilon}{2}(1 + \eta(\frac{\varepsilon}{2})) \leq \varepsilon$. So Y is uniformly monotone with $\delta(\varepsilon) = \eta(\frac{\varepsilon}{2})$. $\square$

Proposition 4.3. *Let X be a Banach lattice, M and N be nonzero Banach sublattices of X such that $X = M \oplus_\infty N$ and the canonical projections from X to M and N are positive operators. If Y is a strictly monotone Banach lattice and the pair (X, Y) has the Bishop–Phelps–Bollobás property for positive operators then Y is uniformly monotone.*

Proof. We will show that Y satisfies condition 2) in Proposition 4.2. Let $0 < \varepsilon < 1$. Let us take elements u and v in Y such that

$$0 \leq u \leq v, \quad \|v\| = 1 \quad \text{and} \quad \|v - u\| > 1 - \eta(\varepsilon),$$

where η is the function satisfying the definition of BPBp for positive operators for the pair (X, Y).

Since M and N are nonzero Banach sublattices, there are positive elements $m_0 \in S_M$ and $n_0 \in S_N$. By using the Hahn-Banach theorem for Banach lattices and positive elements (see [18, Theorem 39.3]), there are positive functionals $m_0^* \in S_{M^*}$ and $n_0^* \in S_{N^*}$ such that $m_0^*(m_0) = 1 = n_0^*(n_0)$.

Let P and Q be the canonical projections from X to M and N, respectively. Consider the operator S from X to Y given by

$$S(x) = m_0^*(P(x))(v - u) + n_0^*(Q(x))u, \quad (x \in X).$$

We have that $v - u$ and u are positive elements in Y, the functionals m_0^* and n_0^* are positive on M and N, respectively, and the projections P and Q are positive operators on X. Therefore S is a positive operator from X to Y.

By using the assumptions on X, the fact that the functionals m_0^* and n_0^* belong to the unit sphere of M^* and N^* respectively, since $v - u$ and u are positive elements in Y, for any element $x \in B_X$ we have that

$$\begin{aligned} \|S(x)\| &\leq \| \|P(x)\|(v-u) + \|Q(x)\|u\| \\ &\leq \|v\| = 1. \end{aligned} \tag{4.1}$$

Since $m_0 + n_0 \in S_X$ and $S(m_0 + n_0) = v \in S_Y$, we deduce that $S \in S_{L(X,Y)}$. Note also that $\|S(m_0)\| = \|v - u\| > 1 - \eta(\varepsilon)$. By using that the pair (X, Y) has the BPBp for positive operators and Remark 2.2, there exists a positive operator $T \in S_{L(X,Y)}$ and a positive element $x_1 \in S_X$ satisfying

$$\|T(x_1)\| = 1, \quad \|T - S\| < \varepsilon \quad \text{and} \quad \|x_1 - m_0\| < \varepsilon.$$

If we write $m_1 = P(x_1)$ and $n_1 = Q(x_1)$, since $\|n_1\| = \|Q(x_1 - m_0)\| \leq \|x_1 - m_0\| < \varepsilon < 1$, from Lemma 4.1 we conclude that $\|T(m_1)\| = 1$.

Since T and P are positive operators and x_1 is a positive element in X, we have that

$$1 = \|T(m_1)\| \leq \|T(m_1 + n_0)\| \leq 1.$$

By using that Y is strictly monotone we obtain that $T(n_0) = 0$. As a consequence,

$$\|u\| = \|S(n_0)\| = \|(S - T)(n_0)\| \leq \|S - T\| < \varepsilon.$$

In view of Proposition 4.2 we proved that Y is uniformly monotone. □

As a consequence of Proposition 4.3 and Corollaries 3.3 and 2.6 we deduce the following result, which is a version of the one obtained by Kim that asserts that a Banach space Y is uniformly convex whenever it is strictly convex and the pair (c_0, Y) has the Bishop-Phelps–Bollobás property for operators ([15, Theorem 2.7]).

It is worth to point out that a Banach lattice is strictly monotone whenever it is strictly convex. Analogously uniform convexity implies uniform monotonicity (see [14, Theorem 1], for instance). Note that the converse results do not hold since every $L_1(\mu)$ such that $\dim L_1(\mu) > 1$ is uniformly monotone, but it is not strictly convex.

Corollary 4.4. (1) *Let Y be a strictly monotone Banach lattice. Then the pair (c_0, Y) has the BPBp for positive operators if and only if Y is uniformly monotone.*

(2) *Let μ be a positive measure such that $\dim L_\infty(\mu) > 1$ and let Y be a strictly monotone Banach lattice. If the pair $(L_\infty(\mu), Y)$ has the BPBp for positive operators, then Y is uniformly monotone. In case that Y has a weak unit the converse is also true.*

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