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Strongly zero product determined Banach algebras ☆

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ABSTRACT

C^*-algebras, group algebras, and the algebra $\mathcal{A}(X)$ of approximable operators on a Banach space X having the bounded approximation property are known to be zero product determined. In this paper we give a quantitative estimate of this property by showing that, for the Banach algebra A, there exists a constant α with the property that for every continuous bilinear functional $\varphi\colon A \times A \to \mathbb{C}$ there exists a continuous linear functional ξ on A such that

$$\sup_{\|a\|=\|b\|=1} |\varphi(a,b) - \xi(ab)| \leq \alpha \sup_{\substack{\|a\|=\|b\|=1,\\ ab=0}} |\varphi(a,b)|$$

in each of the following cases: (i) A is a C^*-algebra, in which case $\alpha = 8$; (ii) $A = L^1(G)$ for a locally compact group G, in which case $\alpha = 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}$; (iii) $A = \mathcal{A}(X)$ for a Banach space X having property $(\mathbb{A})$ (which is a rather strong approximation property for X), in which case $\alpha =$

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$60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}C^2$, where C is a constant associated with the property ($\mathbb{A}$) that we require for X.

1. Introduction

Let A be a Banach algebra. Then $\pi\colon A \times A \to A$ denotes the product map, we write A^* for the dual of A, and $\mathcal{B}^2(A, \mathbb{C})$ for the space of continuous bilinear functionals on A.

The Banach algebra A is said to be *zero product determined* if every $\varphi \in \mathcal{B}^2(A, \mathbb{C})$ with the property

$$a, b \in A,\ ab = 0 \ \Rightarrow\ \varphi(a, b) = 0 \tag{1}$$

belongs to the space

$$\mathcal{B}^2_\pi(A, \mathbb{C}) = \{\xi \circ \pi : \xi \in A^*\}.$$

This concept implicitly appeared in [1] as an additional outcome of the so-called property $\mathbb{B}$ which was introduced in that paper, and was the basis of subsequent Jordan and Lie versions (see [2–4]). For a comprehensive survey of the theory of the zero product determined Banach algebras we refer the reader to [10]. The algebra A is said to have *property* $\mathbb{B}$ if every $\varphi \in \mathcal{B}^2(A, \mathbb{C})$ satisfying (1) belongs to the closed subspace $\mathcal{B}^2_b(A, \mathbb{C})$ of $\mathcal{B}^2(A, \mathbb{C})$ defined by

$$\mathcal{B}^2_b(A, \mathbb{C}) = \{\psi \in \mathcal{B}^2(A, \mathbb{C}) : \psi(ab, c) = \psi(a, bc)\ \forall a, b, c \in A\}.$$

In [1] it was shown that this class of Banach algebras is wide enough to include a number of examples of interest: C^*-algebras, the group algebra $L^1(G)$ of any locally compact group G, and the algebra $\mathcal{A}(X)$ of approximable operators on any Banach space X.

Throughout, we confine ourselves to Banach algebras having a bounded left approximate identity. Then $\mathcal{B}^2_\pi(A, \mathbb{C}) = \mathcal{B}^2_b(A, \mathbb{C})$ (Proposition 2.1), and hence A is a zero product determined Banach algebra if and only if A has property $\mathbb{B}$. For example, this applies to C^*-algebras, group algebras and the algebra $\mathcal{A}(X)$ on any Banach space X having the bounded approximation property, so that all of them are zero product determined Banach algebras.

For each $\varphi \in \mathcal{B}^2(A, \mathbb{C})$, the distance from φ to $\mathcal{B}^2_\pi(A, \mathbb{C})$ is

$$\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\right) = \inf\left\{\|\varphi - \psi\| : \psi \in \mathcal{B}^2_\pi(A, \mathbb{C})\right\},$$

which can be easily estimated through the constant

$$|\varphi|_b = \sup\{|\varphi(ab,c)-\varphi(a,bc)| : a,b,c \in A,\ \|a\|=\|b\|=\|c\|=1\}$$

(Proposition 2.1 below). Our purpose is to estimate $\operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\big)$ through the constant

$$|\varphi|_{zp} = \sup\{|\varphi(a,b)| : a,b \in A,\ \|a\|=\|b\|=1,\ ab=0\}.$$

Note that A is zero product determined precisely when

$$\varphi \in \mathcal{B}^2(A,\mathbb{C}),\ |\varphi|_{zp}=0 \ \Rightarrow\ \varphi \in \mathcal{B}^2_\pi(A,\mathbb{C}). \tag{2}$$

We call the Banach algebra A *strongly zero product determined* if condition (2) is strengthened by requiring that there is a distance estimate

$$\operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\big) \le \alpha\, |\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A,\mathbb{C}) \tag{3}$$

for some constant α; in this case, the optimal constant α for which (3) holds will be denoted by α_A. The inequality $|\varphi|_{zp} \le \operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A,\mathbb{C})\big)$ is always true (Proposition 2.1 below). We also note that A has property $\mathbb{B}$ exactly in the case when

$$\varphi \in \mathcal{B}^2(A,\mathbb{C}),\ |\varphi|_{zp}=0 \ \Rightarrow\ |\varphi|_b = 0,$$

and the algebra A is said to have the *strong property* $\mathbb{B}$ if there is an estimate

$$|\varphi|_b \le \beta\, |\varphi|_{zp} \quad \forall \varphi \in \mathcal{B}^2(A,\mathbb{C}) \tag{4}$$

for some constant β; in this case, the optimal constant β for which (4) holds will be denoted by β_A. The inequality $|\varphi|_{zp} \le M\, |\varphi|_b$ is always true for some constant M (Proposition 2.1 below). The spirit of this concept first appeared in [6], and was subsequently formulated in [14] and refined in [15]. This property has proven to be useful to study the hyperreflexivity of the spaces of continuous derivations and, more generally, continuous cocycles on A (see [7,8,13–15]).

From [5, Corollary 1.3], we obtain the following result.

Theorem 1.1. *Let A be a C^*-algebra. Then A is strongly zero product determined, has the strong property $\mathbb{B}$, and $\alpha_A, \beta_A \le 8$.*

It is shown in [15] that each group algebra has the strong property $\mathbb{B}$ and so (by Corollary 2.2 below) it is also strongly zero product determined. In Theorem 3.3 we prove that, for each group G,

$$\alpha_{L^1(G)} \le \beta_{L^1(G)} \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}.$$

This gives a sharper estimate for the constant of the strong property $\mathbb{B}$ of $L^1(G)$ to the one given in [15, Theorem 3.4]. The estimates given in Theorems 1.1 and 3.3 can be used to sharp the upper bound given in [15, Theorem 4.4] for the hyperreflexivity constant of $\mathcal{Z}^n(A, X)$, the space of continuous n-cocycles from A into X, where A is a C^*-algebra or the group algebra of a group with an open subgroup of polynomial growth and X is a Banach A-bimodule for which the n^{th} Hochschild cohomology group $\mathcal{H}^{n+1}(A, X)$ is a Banach space.

Finally, in Theorem 4.1 we prove that the algebra $\mathcal{A}(X)$ is strongly zero product determined for each Banach space X having property $(\mathbb{A})$ (which is a rather strong approximation property for the space X). Further, we will use this result to show that the space $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ is hyperreflexive for each Banach $\mathcal{A}(X)$-bimodule Y.

There is no reason for an arbitrary zero product Banach algebra to be strongly zero product determined. However, as yet, we do not know an example of a zero product determined Banach algebra which is not strongly zero product determined.

Throughout, our reference for Banach algebras, and particularly for group algebras, is the monograph [11].

2. Elementary estimates

In the following result we gather together some estimates that relate the seminorms $\operatorname{dist}\big(\cdot, \mathcal{B}^2_\pi(A, \mathbb{C})\big)$, $|\cdot|_b$, and $|\cdot|_{zp}$ on $\mathcal{B}^2_\pi(A, \mathbb{C})$ to each other.

Proposition 2.1. *Let A be a Banach algebra with a left approximate identity of bound M. Then $\mathcal{B}^2_\pi(A, \mathbb{C}) = \mathcal{B}^2_b(A, \mathbb{C})$ and, for each $\varphi \in \mathcal{B}^2(A, \mathbb{C})$, the following properties hold:*

(i) *The distance* $\operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\big)$ *is attained;*
(ii) $\frac{1}{2} |\varphi|_b \le \operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\big) \le M |\varphi|_b$;
(iii) $|\varphi|_{zp} \le \operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\big)$.

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a left approximate identity of bound M.

(i) Let (ξ_n) be a sequence in A^* such that

$$\operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\big) = \lim_{n\to\infty} \|\varphi - \xi_n \circ \pi\| .$$

For each $n \in \mathbb{N}$ and $a \in A$, we have

$$|\xi_n(e_\lambda a)| = |(\xi_n \circ \pi)(e_\lambda, a)| \le M \|\xi_n \circ \pi\| \|a\| \quad \forall \lambda \in \Lambda$$

and hence, taking limit in the above inequality and using that $\lim_{\lambda\in\Lambda} e_\lambda a = a$, we see that $|\xi_n(a)| \le M \|\xi_n \circ \pi\| \|a\|$, which shows that $\|\xi_n\| \le M \|\xi_n \circ \pi\|$. Further, since

$$\|\xi_n \circ \pi\| \le \|\varphi - \xi_n \circ \pi\| + \|\varphi\| \quad \forall n \in \mathbb{N},$$

it follows that the sequence $(\|\xi_n\|)$ is bounded. By the Banach–Alaoglu theorem, the sequence (ξ_n) has a weak*-accumulation point, say ξ, in A^*. Let $(\xi_\nu)_{\nu\in N}$ be a subnet of (ξ_n) such that w*-$\lim_{\nu\in N}\xi_\nu=\xi$. The task is now to show that

$$\|\varphi-\xi\circ\pi\|=\operatorname{dist}\left(\varphi,\mathcal{B}^2_\pi(A,\mathbb{C})\right).$$

For each $a,b\in A$ with $\|a\|=\|b\|=1$, we have

$$|\varphi(a,b)-\xi_\nu(ab)|\leq\|\varphi-\xi_\nu\circ\pi\|\quad\forall\nu\in N,$$

and so, taking limits on both sides of the above inequality and using that

$$\lim_{\nu\in N}\xi_\nu(ab)=\xi(ab)$$

and that $(\|\varphi-\xi_\nu\circ\pi\|)_{\nu\in N}$ is a subnet of the convergent sequence $(\|\varphi-\xi_n\circ\pi\|)$, we obtain

$$|\varphi(a,b)-\xi(ab)|\leq\operatorname{dist}\left(\varphi,\mathcal{B}^2_\pi(A,\mathbb{C})\right).$$

This implies that $\|\varphi-\xi\circ\pi\|\leq\operatorname{dist}\left(\varphi,\mathcal{B}^2_\pi(A,\mathbb{C})\right)$, and the converse inequality $\operatorname{dist}\left(\varphi,\mathcal{B}^2_\pi(A,\mathbb{C})\right)\leq\|\varphi-\xi\circ\pi\|$ trivially holds.

(ii) For each $\lambda\in\Lambda$ define $\xi_\lambda\in A^*$ by

$$\xi_\lambda(a)=\varphi(e_\lambda,a)\quad\forall a\in A.$$

Then $\|\xi_\lambda\|\leq M\,\|\varphi\|$ for each $\lambda\in\Lambda$, so that $(\xi_\lambda)_{\lambda\in\Lambda}$ is a bounded net in A^* and hence the Banach–Alaoglu theorem shows that it has a weak*-accumulation point, say ξ, in A^*. Let $(\xi_\nu)_{\nu\in N}$ be a subnet of $(\xi_\lambda)_{\lambda\in\Lambda}$ such that w*-$\lim_{\nu\in N}\xi_\nu=\xi$. For each $a,b\in A$ with $\|a\|=\|b\|=1$, we have

$$|\varphi(e_\nu a,b)-\varphi(e_\nu,ab)|\leq M\,|\varphi|_b\quad\forall\nu\in N$$

and hence, taking limit and using that $(e_\nu a)_{\nu\in N}$ is a subnet of the convergent net $(e_\lambda a)_{\lambda\in\Lambda}$ and that $\lim_{\nu\in N}\varphi(e_\lambda,ab)=\xi(ab)$, we see that

$$|\varphi(a,b)-\xi(ab)|\leq M\,|\varphi|_b\,.$$

This gives $\|\varphi-\xi\circ\pi\|\leq M\,|\varphi|_b$, whence

$$\operatorname{dist}\left(\varphi,\mathcal{B}^2_\pi(A,\mathbb{C})\right)\leq M|\varphi|_b.$$

Set $\xi \in A^*$. For each $a, b, c \in A$ with $\|a\| = \|b\| = \|c\| = 1$, we have

$$\begin{aligned}|\varphi(ab, c) - \varphi(a, bc)| &= |\varphi(ab, c) - (\xi \circ \pi)(ab, c) + (\xi \circ \pi)(a, bc) - \varphi(a, bc)| \\ &\leq |\varphi(ab, c) - (\xi \circ \pi)(ab, c)| + |(\xi \circ \pi)(a, bc) - \varphi(a, bc)| \\ &\leq \|\varphi - \xi \circ \pi\| \, \|ab\| \, \|c\| + \|\varphi - \xi \circ \pi\| \, \|a\| \, \|bc\| \\ &\leq 2 \, \|\varphi - \xi \circ \pi\|\end{aligned}$$

and therefore $|\varphi|_b \leq 2 \, \|\varphi - \xi \circ \pi\|$. Since this inequality holds for each $\xi \in A^*$, it follows that

$$|\varphi|_b \leq 2 \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\right).$$

(iii) Let $a, b \in A$ with $\|a\| = \|b\| = 1$ and $ab = 0$. For each $\xi \in A^*$, we see that

$$|\varphi(a, b)| = |\varphi(a, b) - (\xi \circ \pi)(a, b)| \leq \|\varphi - \xi \circ \pi\|,$$

and consequently $|\varphi|_{zp} \leq \|\varphi - \xi \circ \pi\|$. Since the above inequality holds for each $\xi \in A^*$, we conclude that

$$|\varphi|_{zp} \leq \operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\right).$$

Finally, it is clear that $\mathcal{B}^2_\pi(A, \mathbb{C}) \subset \mathcal{B}^2_b(A, \mathbb{C})$. To prove the reverse inclusion take $\varphi \in \mathcal{B}^2_b(A, \mathbb{C})$. Then $|\varphi|_b = 0$, hence (ii) shows that $\operatorname{dist}\left(\varphi, \mathcal{B}^2_\pi(A, \mathbb{C})\right) = 0$, and (i) gives $\psi \in \mathcal{B}^2_\pi(A, \mathbb{C})$ such that $\|\varphi - \psi\| = 0$, which implies that $\varphi = \psi \in \mathcal{B}^2_\pi(A, \mathbb{C})$. □

The following result is an immediate consequence of assertion (ii) in Proposition 2.1.

Corollary 2.2. *Let A be a Banach algebra with a left approximate identity of bound M. Then A is a strongly zero product determined Banach algebra if and only if has the strong property $\mathbb{B}$, in which case*

$$\tfrac{1}{2}\beta_A \leq \alpha_A \leq M\beta_A.$$

Let X and Y be Banach spaces, and let $n \in \mathbb{N}$. We write $\mathcal{B}^n(X, Y)$ for the Banach space of all continuous n-linear maps from $X \times \overset{n}{\cdots} \times X$ to Y. As usual, we abbreviate $\mathcal{B}^1(X, Y)$ to $\mathcal{B}(X, Y)$, $\mathcal{B}(X, X)$ to $\mathcal{B}(X)$, and $\mathcal{B}(X, \mathbb{C})$ to X^*. The identity operator on X is denoted by I_X. Further, we write $\langle \cdot, \cdot \rangle$ for the duality between X and X^*. For each subspace E of X, $E^\perp$ denotes the annihilator of E in X^*.

For a Banach algebra A and a Banach space X, and for each $\varphi \in \mathcal{B}^2(A, X)$, we continue to use the notations

$$|\varphi|_b = \sup\left\{|\varphi(ab, c) - \varphi(a, bc)| : a, b, c \in A, \ \|a\| = \|b\| = \|c\| = 1\right\},$$
$$|\varphi|_{zp} = \sup\left\{|\varphi(a, b)| : a, b \in A, \ \|a\| = \|b\| = 1, \ ab = 0\right\}.$$

Proposition 2.3. *Let A be a Banach algebra with a left approximate identity of bound M and having the strong property $\mathbb{B}$. Let X be a Banach space, and let $\varphi \in \mathcal{B}^2(A, X)$. Then the following properties hold:*

(i) $|\varphi|_b \le \beta_A |\varphi|_{zp}$;
(ii) *If X is a dual Banach space, then there exists $\Phi \in \mathcal{B}(A, X)$ such that $\|\varphi - \Phi \circ \pi\| \le M\beta_A$.*

Proof. (i) For each $\xi \in X^*$, we have

$$|\xi \circ \varphi|_b \le \beta_A |\xi \circ \varphi|_{zp}.$$

It follows from the Hahn-Banach theorem that

$$|\varphi|_b = \sup\{|\xi \circ \varphi|_b : \xi \in X^*,\ \|\xi\| = 1\},$$
$$|\varphi|_{zp} = \sup\{|\xi \circ \varphi|_{zp} : \xi \in X^*,\ \|\xi\| = 1\}.$$

In this way we obtain (i).

(ii) Suppose that X is the dual of a Banach space X_*. Let $(e_\lambda)_{\lambda \in \Lambda}$ be a left approximate identity for A of bound M, and define a net $(\Phi_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{B}(A, X)$ by setting

$$\Phi_\lambda(a) = \varphi(e_\lambda, a) \quad \forall a \in A,\ \forall \lambda \in \Lambda.$$

Since each bounded subset of $\mathcal{B}(A, X)$ is relatively compact with respect to the weak* operator topology on $\mathcal{B}(A, X)$ and the net $(\Phi_\lambda)_{\lambda \in \Lambda}$ is bounded, it follows that there exist $\Phi \in \mathcal{B}(A, X)$ and a subnet $(\Phi_\nu)_{\nu \in N}$ of $(\Phi_\lambda)_{\lambda \in \Lambda}$ such that wo*-$\lim_{\nu \in N} \Phi_\nu = \Phi$. For each $a, b \in A$ with $\|a\| = \|b\| = 1$, and $x_* \in X_*$ with $\|x_*\| = 1$, we have

$$\left|\langle x_*, \varphi(e_\nu a, b)\rangle - \langle x_*, \varphi(e_\nu, ab)\rangle\right| \le \|\varphi(e_\nu a, b) - \varphi(e_\nu, ab)\| \le M\beta_A \quad \forall \nu \in N$$

and hence, taking limit and using that $(e_\nu a)_{\nu \in N}$ is a subnet of the net $(e_\lambda a)_{\lambda \in \Lambda}$ (which converges to a with respect to the norm topology) and that $\lim_{\nu \in N}\langle x_*, \varphi(e_\nu, ab)\rangle = \langle x_*, \Phi(ab)\rangle$ (by definition of Φ), we see that

$$\left|\langle x_*, \varphi(a, b) - \Phi(ab)\rangle\right| = M\beta_A.$$

This gives $\|\varphi - \Phi \circ \pi\| \le M\beta_A$. $\square$

3. Group algebras

In this section we prove that the group algebra $L^1(G)$ of each locally compact group G is a strongly zero product determined Banach algebra and we provide an estimate of the constants $\alpha_{L^1(G)}$ and $\beta_{L^1(G)}$. Our estimate of $\beta_{L^1(G)}$ improves the one given in [15].

For the basic properties of this important class of Banach algebras we refer the reader to [11, Section 3.3].

Throughout this section, $\mathbb{T}$ denotes the circle group, and we consider the normalized Haar measure on $\mathbb{T}$. We write $A(\mathbb{T})$ and $A(\mathbb{T}^2)$ for the Fourier algebras of $\mathbb{T}$ and $\mathbb{T}^2$, respectively. For each $f \in A(\mathbb{T})$, $F \in A(\mathbb{T}^2)$, and $j, k \in \mathbb{Z}$, we write $\widehat{f}(j)$ and $\widehat{F}(j,k)$ for the Fourier coefficients of f and F, respectively. Let $\mathbf{1}, \zeta \in A(\mathbb{T})$ denote the functions defined by

$$\mathbf{1}(z) = 1, \quad \zeta(z) = z \quad \forall z \in \mathbb{T}.$$

Let $\Delta\colon A(\mathbb{T}^2) \to A(\mathbb{T})$ be the bounded linear map defined by

$$\Delta(F)(z) = F(z,z) \quad \forall z \in \mathbb{T},\ \forall F \in A(\mathbb{T}^2).$$

For $f, g \in A(\mathbb{T})$, let $f \otimes g\colon \mathbb{T}^2 \to \mathbb{C}$ denote the function defined by

$$(f \otimes g)(z,w) = f(z)g(w) \quad \forall z, w \in \mathbb{T},$$

which is an element of $A(\mathbb{T}^2)$ with $\|f \otimes g\| = \|f\|\,\|g\|$.

Lemma 3.1. *Let $\Phi\colon A(\mathbb{T}^2) \to \mathbb{C}$ be a continuous linear functional, and let the constant $\varepsilon \geq 0$ be such that*

$$f, g \in A(\mathbb{T}),\ fg = 0 \ \Rightarrow\ |\Phi(f \otimes g)| \leq \varepsilon \|f\|\,\|g\|.$$

Then

$$|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \leq \|\Phi|_{\ker \Delta}\|\, 2 \sin \tfrac{\pi}{10} + 60\sqrt{27}\left(1 + \sin \tfrac{\pi}{10}\right)\varepsilon.$$

Proof. Set

$$E = \left\{e^{\theta i} : -\tfrac{1}{5}\pi \leq \theta \leq \tfrac{1}{5}\pi\right\},$$
$$W = \left\{(z,w) \in \mathbb{T}^2 : zw^{-1} \in E\right\},$$

and let $F \in A(\mathbb{T}^2)$ be such that

$$F(z,w) = 0 \quad \forall (z,w) \in W. \tag{5}$$

Our objective is to prove that

$$|\Phi(F)| \leq 30\sqrt{27}\,\|F\|\,\varepsilon. \tag{6}$$

For this purpose, we take

$$\begin{aligned} a &= e^{\frac{1}{15}\pi i}, \\ A &= \left\{ e^{\theta i} : 0 < \theta \le \tfrac{1}{15}\pi \right\}, \\ B &= \left\{ e^{\theta i} : \tfrac{2}{15}\pi < \theta \le \tfrac{29}{15}\pi \right\}, \\ U &= \left\{ e^{\theta i} : -\tfrac{1}{30}\pi < \theta < \tfrac{1}{30}\pi \right\}, \end{aligned}$$

and we define functions $\omega, \upsilon \in A(\mathbb{T})$ by

$$\omega = 30\, \chi_A * \chi_U, \quad \upsilon = 30\, \chi_B * \chi_U.$$

We note that

$$\begin{aligned} \{z \in \mathbb{T} : \omega(z) \neq 0\} = AU = \left\{ e^{\theta i} : -\tfrac{1}{30}\pi < \theta < \tfrac{1}{10}\pi \right\}, \\ \{z \in \mathbb{T} : \upsilon(z) \neq 0\} = BU = \left\{ e^{\theta i} : \tfrac{1}{10}\pi < \theta < \tfrac{59}{30}\pi \right\}, \end{aligned}$$

and, with $\|\cdot\|_2$ denoting the norm of $L^2(\mathbb{T})$,

$$\begin{aligned} \|\omega\| &\le 30\, \|\chi_A\|_2\, \|\chi_U\|_2 = 30\, \frac{1}{\sqrt{30}}\, \frac{1}{\sqrt{30}} = 1, \\ \|\upsilon\| &\le 30\, \|\chi_B\|_2\, \|\chi_U\|_2 = 30\, \frac{\sqrt{27}}{\sqrt{30}}\, \frac{1}{\sqrt{30}} = \sqrt{27}. \end{aligned}$$

Since

$$\bigcup_{k=0}^{29} a^k A = \mathbb{T}, \quad \bigcup_{k=2}^{28} a^k A = B,$$

it follows that

$$\sum_{k=0}^{29} \delta_{a^k} * \chi_A = \sum_{k=0}^{29} \chi_{a^k A} = \mathbf{1}, \quad \sum_{k=2}^{28} \delta_{a^k} * \chi_A = \sum_{k=2}^{28} \chi_{a^k A} = \chi_B,$$

and thus, for each $j \in \mathbb{Z}$, we have

$$\sum_{k=j}^{j+29} \delta_{a^k} * \omega = 30\delta_{a^j} * \sum_{k=0}^{29} \delta_{a^k} * \chi_A * \chi_U = 30\delta_{a^j} * \mathbf{1} * \chi_U = \mathbf{1}, \tag{7}$$

$$\sum_{k=j+2}^{j+28} \delta_{a^k} * \omega = 30\delta_{a^j} * \sum_{k=2}^{28} \delta_{a^k} * \chi_A * \chi_U = 30\delta_{a^j} * \chi_B * \chi_U = \delta_{a^j} * \upsilon. \tag{8}$$

If $j \in \mathbb{Z}$, $k \in \{j-1, j, j+1\}$, and $z, w \in \mathbb{T}$ are such that $(\delta_{a^j} * \omega)(z)(\delta_{a^k} * \omega)(w) \neq 0$, then

$$zw^{-1} \in a^j AU \big(a^k AU\big)^{-1} \subset a^{j-k} \left\{ e^{\theta i} : -\tfrac{2}{15}\pi < \theta < \tfrac{2}{15}\pi \right\} \subset E,$$

whence $\{(z,w)\in\mathbb{T}^2 : (\delta_{a^j}*\omega)\otimes(\delta_{a^k}*\omega)(z,w)\neq 0\}\subset W$ and (5) gives

$$F(\delta_{a^j}*\omega)\otimes(\delta_{a^k}*\omega)=0. \tag{9}$$

Since $AU\cap BU=\varnothing$, it follows that $\omega\upsilon=0$, and therefore

$$(\delta_{a^k}*\omega)(\delta_{a^k}*\upsilon)=0 \quad \forall k\in\mathbb{Z}. \tag{10}$$

From (7), (8), and (9) we deduce that

$$\begin{aligned}
F &= F\sum_{j=0}^{29}\sum_{k=j-1}^{j+28}(\delta_{a^j}*\omega)\otimes(\delta_{a^k}*\omega)\\
&=\sum_{j=0}^{29}\sum_{k=j-1}^{j+1}F(\delta_{a^j}*\omega)\otimes(\delta_{a^k}*\omega)+\sum_{j=0}^{29}\sum_{k=j+2}^{j+28}F(\delta_{a^j}*\omega)\otimes(\delta_{a^k}*\omega)\\
&=\sum_{j=0}^{29}\sum_{k=j+2}^{j+28}F(\delta_{a^j}*\omega)\otimes(\delta_{a^k}*\omega)=\sum_{j=0}^{29}F(\delta_{a^j}*\omega)\otimes(\delta_{a^j}*\upsilon).
\end{aligned}$$

As

$$F=\sum_{j,k=-\infty}^{\infty}\widehat{F}(j,k)\zeta^j\otimes\zeta^k$$

we have

$$F=\sum_{j,k=-\infty}^{\infty}\sum_{l=0}^{29}\widehat{F}(j,k)\big(\zeta^j(\delta_{a^l}*\omega)\big)\otimes\big(\zeta^k(\delta_{a^l}*\upsilon)\big),$$

so that

$$\Phi(F)=\sum_{j,k=-\infty}^{\infty}\sum_{l=0}^{29}\widehat{F}(j,k)\Phi\Big(\big(\zeta^j(\delta_{a^l}*\omega)\big)\otimes\big(\zeta^k(\delta_{a^l}*\upsilon)\big)\Big).$$

By (10), for each $j,k,l\in\mathbb{Z}$,

$$\big(\zeta^j(\delta_{a^l}*\omega)\big)\big(\zeta^k(\delta_{a^l}*\upsilon)\big)=0$$

and therefore

$$\begin{aligned}
\left|\Phi\left(\big(\zeta^j(\delta_{a^l}*\omega)\big)\otimes\big(\zeta^k(\delta_{a^l}*\upsilon)\big)\right)\right| &\leq \varepsilon\left\|\zeta^j(\delta_{a^l}*\omega)\right\|\left\|\zeta^k(\delta_{a^l}*\upsilon)\right\|\\
&=\varepsilon\left\|\omega\right\|\left\|\upsilon\right\|\leq\sqrt{27}\,\varepsilon.
\end{aligned}$$

We thus get

$$\begin{aligned}|\Phi(F)| &= \sum_{j,k=-\infty}^{\infty} \sum_{l=0}^{29} \left|\widehat{F}(j,k)\right| \left|\Phi\Big(\big(\zeta^j(\delta_{a^l} * \omega)\big) \otimes \big(\zeta^k(\delta_{a^l} * \upsilon)\big)\Big)\right| \\ &\le \sum_{j,k=-\infty}^{\infty} \sum_{l=0}^{29} \left|\widehat{F}(j,k)\right| \sqrt{27}\,\varepsilon = 30\sqrt{27}\,\|F\|\,\varepsilon,\end{aligned}$$

and (6) is proved.

Let $f \in A(\mathbb{T})$ be such that $f(z) = 0$ for each $z \in E$, and define the function $F\colon \mathbb{T}^2 \to \mathbb{C}$ by

$$F(z,w) = f(zw^{-1})w = \sum_{k=-\infty}^{\infty} \widehat{f}(k) z^k w^{-k+1} \quad \forall z, w \in \mathbb{T}.$$

Then $F \in A(\mathbb{T}^2)$, $\|F\| = \|f\|$, $\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F \in \ker \Delta$, and

$$(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F)(z,w) = \Big(1 - \widehat{f}(1)\Big) z + \Big(-1 - \widehat{f}(0)\Big) w - \sum_{k \ne 0,1} \widehat{f}(k) z^k w^{-k+1},$$

which certainly implies that

$$\|\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F\| = \left|1 - \widehat{f}(1)\right| + \left|-1 - \widehat{f}(0)\right| + \sum_{k \ne 0,1} \left|\widehat{f}(k)\right| = \|\zeta - \mathbf{1} - f\|.$$

According to (6), we have

$$\begin{aligned}|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| &\le |\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F)| + |\Phi(F)| \\ &\le \|\Phi\,|_{\ker \Delta}\|\,\|\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta - F\| + 30\sqrt{27}\,\|F\|\,\varepsilon \\ &= \|\Phi\,|_{\ker \Delta}\|\,\|\zeta - \mathbf{1} - f\| + 30\sqrt{27}\,\|f\|\,\varepsilon \\ &\le \|\Phi\,|_{\ker \Delta}\|\,\|\zeta - \mathbf{1} - f\| + 30\sqrt{27}\big(\|\zeta - \mathbf{1} - f\| + 2\big)\varepsilon\end{aligned}$$

(as $\|f\| \le \|\zeta - \mathbf{1} - f\| + \|\zeta - \mathbf{1}\|$). Further, this inequality holds for each function from the set $\mathcal{I}$ consisting of all functions $f \in A(\mathbb{T})$ such that $f(z) = 0$ for each $z \in E$. Consequently,

$$|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \le \|\Phi\,|_{\ker \Delta}\|\,\mathrm{dist}(\zeta - \mathbf{1}, \mathcal{I}) + 30\sqrt{27}\,(\mathrm{dist}(\zeta - \mathbf{1}, \mathcal{I}) + 2)\,\varepsilon.$$

On the other hand, it is shown at the beginning of the proof of [9, Corollary 3.3] that

$$\mathrm{dist}(\zeta - \mathbf{1}, \mathcal{I}) \le 2 \sin \tfrac{\pi}{10},$$

and we thus get

$$|\Phi(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \le \|\Phi\,|_{\ker \Delta}\|\, 2\sin\tfrac{\pi}{10} + 30\sqrt{27}\,\big(2\sin\tfrac{\pi}{10} + 2\big)\,\varepsilon,$$

which completes the proof. □

Lemma 3.2. *Let* $\Phi\colon A(\mathbb{T}^2) \to \mathbb{C}$ *be a continuous linear functional, and let the constant* $\varepsilon \ge 0$ *be such that*

$$f, g \in A(\mathbb{T}),\ fg = 0 \ \Rightarrow\ |\Phi(f \otimes g)| \le \varepsilon \|f\| \|g\| .$$

Then

$$\left|\Phi\big(F - \mathbf{1} \otimes \Delta F\big)\right| \le 60\sqrt{27}\, \frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\, \varepsilon \|F\|$$

for each $F \in A(\mathbb{T}^2)$.

Proof. Fix $j, k \in \mathbb{Z}$. We claim that

$$\left|\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k})\right| \le \|\Phi\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1 + \sin\tfrac{\pi}{10}\right)\varepsilon. \tag{11}$$

Of course, we are reduced to proving (11) for $j \ne 0$. We define $d_j\colon A(\mathbb{T}) \to A(\mathbb{T})$, and $D_j, L_k\colon A(\mathbb{T}^2) \to A(\mathbb{T}^2)$ by

$$d_j f(z) = f(z^j) \quad \forall f \in A(\mathbb{T}),\ \forall z \in \mathbb{T}$$

and

$$D_j F(z, w) = F(z^j, w^j), \quad L_k F(z, w) = F(z, w) w^k \quad \forall F \in A(\mathbb{T}^2),\ \forall z, w \in \mathbb{T},$$

respectively. Further, we consider the continuous linear functional $\Phi \circ L_k \circ D_j$. If $f, g \in A(\mathbb{T})$ are such that $fg = 0$, then $(d_j f)(\zeta^k d_j g) = \zeta^k d_j(fg) = 0$, and so, by hypothesis,

$$|\Phi \circ L_k \circ D_j(f \otimes g)| = \left|\Phi(d_j f \otimes \zeta^k d_j g)\right| \le \varepsilon \|d_j f\| \left\|\zeta^k d_j g\right\| = \varepsilon \|f\| \|g\| .$$

By applying Lemma 3.1, we obtain

$$\begin{aligned}\left|\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k})\right| &= |\Phi \circ L_k \circ D_j(\zeta \otimes \mathbf{1} - \mathbf{1} \otimes \zeta)| \\ &\le \|\Phi \circ L_k \circ D_j\,|_{\ker\Delta}\|\, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1 + \sin\tfrac{\pi}{10}\right)\varepsilon.\end{aligned}$$

We check at once that $(L_k \circ D_j)(\ker\Delta) \subset \ker\Delta$, which gives

$$\|\Phi \circ L_k \circ D_j\,|_{\ker\Delta}\| \le \|\Phi\,|_{\ker\Delta}\| ,$$

and therefore (11) is proved.

Take $F \in A(\mathbb{T}^2)$. Then

$$F = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\zeta^j \otimes \zeta^k$$

and

$$\Delta F = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\zeta^{j+k}.$$

Consequently,

$$\Phi(F - \mathbf{1} \otimes \Delta F) = \sum_{j,k=-\infty}^{\infty} \widehat{F}(j,k)\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k}),$$

and (11) gives

$$\begin{aligned} |\Phi(F - \mathbf{1} \otimes \Delta F)| &\leq \sum_{j,k=-\infty}^{\infty} \left|\widehat{F}(j,k)\right| \left|\Phi(\zeta^j \otimes \zeta^k - \mathbf{1} \otimes \zeta^{j+k})\right| \\ &\leq \sum_{j,k=-\infty}^{\infty} \left|\widehat{F}(j,k)\right| \left[\|\Phi\mid_{\ker\Delta}\| \, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right] \qquad (12) \\ &= \|F\| \left[\|\Phi\mid_{\ker\Delta}\| \, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right]. \end{aligned}$$

In particular, for each $F \in \ker\Delta$, we have

$$\|\Phi(F)\| \leq \|F\| \left[\|\Phi\mid_{\ker\Delta}\| \, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right].$$

Thus

$$\|\Phi\mid_{\ker\Delta}\| \leq \|\Phi\mid_{\ker\Delta}\| \, 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon,$$

so that

$$\|\Phi\mid_{\ker\Delta}\| \leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,\varepsilon.$$

Using this estimate in (12), we obtain

$$\begin{aligned} |\Phi(F - \mathbf{1} \otimes \Delta F)| &\leq \|F\| \left[60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,\varepsilon 2\sin\tfrac{\pi}{10} + 60\sqrt{27}\left(1+\sin\tfrac{\pi}{10}\right)\varepsilon\right] \\ &= \|F\| \, 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,\varepsilon \end{aligned}$$

for each $F \in A(\mathbb{T}^2)$, which completes the proof. □

Theorem 3.3. *Let G be a locally compact group. Then the Banach algebra $L^1(G)$ is strongly zero product determined and*

$$\alpha_{L^1(G)} \leq \beta_{L^1(G)} \leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}.$$

Proof. On account of Corollary 2.2, it suffices to prove that $L^1(G)$ has the strong property $\mathbb{B}$ with

$$\beta_{L^1(G)} \leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}, \tag{13}$$

because $L^1(G)$ has an approximate identity of bound 1. For this purpose set $\varphi \in \mathcal{B}^2(L^1(G),\mathbb{C})$.

Let $t \in G$, and let δ_t be the point mass measure at t on G. We define a contractive homomorphism $T\colon A(\mathbb{T}) \to M(G)$ by

$$T(u) = \sum_{k=-\infty}^{\infty} \widehat{u}(k)\delta_{t^k} \quad \forall u \in A(\mathbb{T}).$$

Take $f, h \in L^1(G)$ with $\|f\| = \|h\| = 1$, and define a continuous linear functional $\Phi\colon A(\mathbb{T}^2) \to \mathbb{C}$ by

$$\Phi(F) = \sum_{(j,k)\in\mathbb{Z}^2} \widehat{F}(j,k)\varphi(f * \delta_{t^j}, \delta_{t^k} * h) \quad \forall F \in A(\mathbb{T}^2).$$

Further, if $u, v \in A(\mathbb{T})$, then

$$\Phi(u\otimes v) = \sum_{(j,k)\in\mathbb{Z}^2} \widehat{u}(j)\widehat{v}(k)\varphi(f * \delta_{t^j}, \delta_{t^k} * h) = \varphi(f * T(u), T(v) * h);$$

in particular, if $uv = 0$, then $(f * T(u)) * (T(v) * h) = f * T(uv) * h = 0$, and so

$$\begin{aligned}|\Phi(u\otimes v)| &= |\varphi(f * T(u), T(v) * h)| \leq |\varphi|_{zp}\,\|f * T(u)\|\,\|T(v) * h\| \\ &\leq |\varphi|_{zp}\,\|u\|\,\|v\|.\end{aligned}$$

By applying Lemma 3.2 with $F = \zeta \otimes \mathbf{1}$, we see that

$$|\varphi(f * \delta_t, h) - \varphi(f, \delta_t * h)| = |\Phi(\zeta\otimes\mathbf{1} - \mathbf{1}\otimes\zeta)| \leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,|\varphi|_{zp}.$$

We now take $g \in L^1(G)$ with $\|g\| = 1$. By multiplying the above inequality by $|g(t)|$, we arrive at

$$|\varphi(g(t)f * \delta_t, h) - \varphi(f, g(t)\delta_t * h)| \leq 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,|\varphi|_{zp}\,|g(t)|. \tag{14}$$

Since the convolutions $f * g$ and $g * h$ can be expressed as

$$f * g = \int_G g(t) f * \delta_t \, dt,$$

$$g * h = \int_G g(t) \delta_t * h \, dt,$$

where the expressions on the right-hand side are considered as Bochner integrals of $L^1(G)$-valued functions of t, it follows that

$$\varphi(f * g, h) - \varphi(f, g * h) = \int_G \left[\varphi(g(t) f * \delta_t, h) - \varphi(f, g(t)\delta_t * h) \right] dt.$$

From (14) we now deduce that

$$\begin{aligned} |\varphi(f * g, h) - \varphi(f, g * h)| &\leq \int_G |\varphi(g(t) f * \delta_t, h) - \varphi(f, g(t)\delta_t * h)| \, dt \\ &\leq 60\sqrt{27} \, \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}} \, |\varphi|_{zp} \int_G |g(t)| \, dt \\ &= 60\sqrt{27} \, \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}} \, |\varphi|_{zp} \, . \end{aligned}$$

We thus get

$$|\varphi|_b \leq 60\sqrt{27} \, \frac{1 + \sin \frac{\pi}{10}}{1 - 2 \sin \frac{\pi}{10}} \, |\varphi|_{zp} \, ,$$

and (13) is proved. □

4. Algebras of approximable operators

Let X be a Banach space. Then we write $\mathcal{F}(X)$ for the two-sided ideal of $\mathcal{B}(X)$ consisting of finite-rank operators, and $\mathcal{A}(X)$ for the closure of $\mathcal{F}(X)$ in $\mathcal{B}(X)$ with respect to the operator norm. For each $x \in X$ and $\phi \in X^*$, we define $x \otimes \phi \in \mathcal{F}(X)$ by $(x \otimes \phi)(y) = \langle y, \phi \rangle x$ for each $y \in X$. A *finite, biorthogonal system* for X is a set

$$\big\{ (x_j, \phi_k) : j, k = 1, \dots, n \big\}$$

with $x_1, \dots, x_n \in X$ and $\phi_1, \dots, \phi_n \in X^*$ such that

$$\langle x_j, \phi_k \rangle = \delta_{j,k} \quad \forall j, k \in \{1, \dots, n\}.$$

Each such system defines an algebra homomorphism

$$\theta\colon \mathbb{M}_n \to \mathcal{F}(X), \quad (a_{j,k}) \mapsto \sum_{j,k=1}^{n} a_{j,k} x_j \otimes \phi_k,$$

where $\mathbb{M}_n$ is the full matrix algebra of order n over $\mathbb{C}$. The identity matrix is denoted by I_n.

The Banach space X is said to have *property* $(\mathbb{A})$ if there is a directed set Λ such that, for each $\lambda \in \Lambda$, there exists a finite, biorthogonal system

$$\left\{(x_j^\lambda, \phi_k^\lambda) : j, k = 1, \dots, n_\lambda\right\}$$

for X with corresponding algebra homomorphism $\theta_\lambda\colon \mathbb{M}_{n_\lambda} \to \mathcal{F}(X)$ such that:

(i) $\lim_{\lambda\in\Lambda} \theta_\lambda(I_{n_\lambda}) = I_X$ uniformly on the compact subsets of X;
(ii) $\lim_{\lambda\in\Lambda} \theta_\lambda(I_{n_\lambda})^* = I_{X^*}$ uniformly on the compact subsets of X^*;
(iii) for each index $\lambda \in \Lambda$, there is a finite subgroup G_λ of the group of all invertible $n_\lambda \times n_\lambda$ matrices over $\mathbb{C}$ whose linear span is all of $\mathbb{M}_{n_\lambda}$, such that

$$\sup_{\lambda\in\Lambda} \sup_{t\in G_\lambda} \|\theta_\lambda(t)\| < \infty. \tag{15}$$

Property $(\mathbb{A})$ forces the Banach algebra $\mathcal{A}(X)$ to be amenable. For an exhaustive treatment of this topic (including a variety of interesting examples of spaces with property $(\mathbb{A})$) we refer to [12, Section 3.3].

The notation of the above definition will be standard for the remainder of this section. Furthermore, our basic reference for this section is the monograph [12].

Theorem 4.1. *Let X be a Banach space with property $(\mathbb{A})$. Then the Banach algebra $\mathcal{A}(X)$ is strongly zero product determined. Specifically, if C denotes the supremum in (15), then*

$$\tfrac{1}{2}\beta_{\mathcal{A}(X)} \le \alpha_{\mathcal{A}(X)} \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2.$$

Proof. For each $\lambda \in \Lambda$ we define $\Phi_\lambda\colon \ell^1(G_\lambda) \to \mathcal{F}(X)$ by

$$\Phi_\lambda(f) = \sum_{t\in G_\lambda} f(t)\theta_\lambda(t) \quad \forall f \in \ell^1(G_\lambda).$$

We claim that Φ_λ is an algebra homomorphism. It is clear the Φ_λ is a linear map and, for each $f, g \in \ell^1(G_\lambda)$, we have

$$\begin{aligned}\Phi_\lambda(f*g) &= \sum_{t\in G_\lambda}(f*g)(t)\theta_\lambda(t) = \sum_{t\in G_\lambda}\sum_{s\in G_\lambda} f(s)g(s^{-1}t)\theta_\lambda(t)\\ &= \theta_\lambda\left(\sum_{t\in G_\lambda}\sum_{s\in G_\lambda} f(s)g(s^{-1}t)t\right) = \theta_\lambda\left(\sum_{s\in G_\lambda} f(s)s\sum_{t\in G_\lambda} g(s^{-1}t)s^{-1}t\right)\\ &= \theta_\lambda\left(\sum_{s\in G_\lambda} f(s)s\sum_{r\in G_\lambda} g(r)r\right) = \theta_\lambda\left(\sum_{s\in G_\lambda} f(s)s\right)\theta_\lambda\left(\sum_{r\in G_\lambda} g(r)r\right)\\ &= \Phi_\lambda(f)\Phi_\lambda(g).\end{aligned}$$

Of course, Φ_λ is continuous because $\ell^1(G_\lambda)$ is finite-dimensional, and, further, for each $f\in\ell^1(G_\lambda)$, we have

$$\|\Phi_\lambda(f)\| \le \sum_{t\in G_\lambda}|f(t)|\,\|\theta_\lambda(t)\| \le \sum_{t\in G_\lambda}|f(t)|\,C = C\,\|f\|_1 .$$

Hence $\|\Phi_\lambda\|\le C$.

Let $\varphi\in\mathcal{B}^2(\mathcal{A}(X),\mathbb{C})$. Let us prove that

$$\left|\varphi(S\theta_\lambda(t),\theta_\lambda(t^{-1})T)-\varphi(S\theta_\lambda(I_{n_\lambda}),\theta_\lambda(I_{n_\lambda})T)\right| \le \beta_{\ell^1(G_\lambda)}C^2\,\|S\|\,\|T\|\,|\varphi|_{zp} \tag{16}$$

for all $\lambda\in\Lambda$, $S,T\in\mathcal{A}(X)$, and $t\in G_\lambda$. For this purpose, take $\lambda\in\Lambda$ and $S,T\in\mathcal{A}(X)$, and define $\varphi_\lambda\colon \ell^1(G_\lambda)\times\ell^1(G_\lambda)\to\mathbb{C}$ by

$$\varphi_\lambda(f,g)=\varphi(S\Phi_\lambda(f),\Phi_\lambda(g)T)\quad \forall f,g\in\ell^1(G_\lambda).$$

Then φ_λ is continuous and, for each $f,g\in\ell^1(G_\lambda)$ such that $f*g=0$, we have $(S\Phi_\lambda(f))(\Phi_\lambda(g)T)=S(\Phi_\lambda(f*g))T=0$ and therefore

$$|\varphi_\lambda(f,g)|\le|\varphi|_{zp}\,\|S\Phi_\lambda(f)\|\,\|\Phi_\lambda(g)T\|\le|\varphi|_{zp}\,C^2\,\|S\|\,\|T\|\,\|f\|_1\,\|g\|_1 ,$$

whence

$$|\varphi_\lambda|_{zp}\le C^2\,\|S\|\,\|T\|\,|\varphi|_{zp} .$$

For each $t\in G_\lambda$, we have

$$\begin{gathered}\left|\varphi_\lambda(\delta_t,\delta_{t^{-1}})-\varphi_\lambda(\delta_{I_{n_\lambda}},\delta_{I_{n_\lambda}})\right| = \left|\varphi_\lambda(\delta_{I_{n_\lambda}}*\delta_t,\delta_{t^{-1}})-\varphi_\lambda(\delta_{I_{n_\lambda}},\delta_t*\delta_{t^{-1}})\right|\le\\ |\varphi_\lambda|_b\le\beta_{\ell^1(G_\lambda)}\,|\varphi_\lambda|_{zp}\le\beta_{\ell^1(G_\lambda)}C^2\,\|S\|\,\|T\|\,|\varphi|_{zp} ,\end{gathered}$$

which gives (16).

The projective tensor product $\mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X)$ becomes a Banach $\mathcal{A}(X)$-bimodule for the products defined by

$$R \cdot (S \otimes T) = (RS) \otimes T, \quad (S \otimes T) \cdot R = S \otimes (TR) \quad \forall R, S, T \in \mathcal{A}(X).$$

We define a continuous linear functional $\widehat{\varphi} \in \big(\mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X)\big)^*$ through

$$\langle S \otimes T, \widehat{\varphi} \rangle = \varphi(S, T) \quad \forall S, T \in \mathcal{A}(X).$$

For each $\lambda \in \Lambda$, set $P_\lambda = \theta_\lambda(I_{n_\lambda})$ and

$$D_\lambda = \frac{1}{|G_\lambda|} \sum_{t \in G_\lambda} \theta_\lambda(t) \otimes \theta_\lambda(t^{-1}).$$

Then $(P_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for $\mathcal{A}(X)$ and $(D_\lambda)_{\lambda\in\Lambda}$ is an approximate diagonal for $\mathcal{A}(X)$ (see [12, Theorem 3.3.9]), so that $(\|S \cdot D_\lambda - D_\lambda \cdot S\|)_{\lambda\in\Lambda} \to 0$ for each $S \in \mathcal{A}(X)$.

For each $\lambda \in \Lambda$ and $S, T \in \mathcal{A}(X)$, (16) shows that

$$\begin{aligned} &|\langle S \cdot D_\lambda \cdot T, \widehat{\varphi} \rangle - \varphi(SP_\lambda, P_\lambda T)| \\ &= \left| \frac{1}{|G_\lambda|} \sum_{t \in G_\lambda} \left[\varphi(S\theta_\lambda(t), \theta_\lambda(t^{-1})T) - \varphi(S\theta_\lambda(I_{n_\lambda}), \theta_\lambda(I_{n_\lambda})T) \right] \right| \\ &\le \beta_{\ell^1(G_\lambda)} C^2 \|S\| \|T\| |\varphi|_{zp} \end{aligned}$$

and Theorem 3.3 then gives

$$|\langle S \cdot D_\lambda \cdot T, \widehat{\varphi} \rangle - \varphi(SP_\lambda, P_\lambda T)| \le 60\sqrt{27} \, \frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}} \, C^2 \|S\| \|T\| |\varphi|_{zp} . \tag{17}$$

For each $\lambda \in \Lambda$, define $\xi_\lambda \in \mathcal{A}(X)^*$ by

$$\langle T, \xi_\lambda \rangle = \langle D_\lambda \cdot T, \widehat{\varphi} \rangle \quad \forall T \in \mathcal{A}(X).$$

Note that

$$\|\xi_\lambda\| \le \|\widehat{\varphi}\| \|D_\lambda\| \le \|\varphi\| C^2 \quad \forall \lambda \in \Lambda$$

and therefore $(\xi_\lambda)_{\lambda\in\Lambda}$ is a bounded net in $\mathcal{A}(X)^*$. By the Banach–Alaoglu theorem the net $(\xi_\lambda)_{\lambda\in\Lambda}$ has a weak*-accumulation point, say ξ, in $\mathcal{A}(X)^*$. Take a subnet $(\xi_\nu)_{\nu\in N}$ of $(\xi_\lambda)_{\lambda\in\Lambda}$ such that w*-$\lim_{\nu\in N} \xi_\nu = \xi$. Take $S, T \in \mathcal{A}(X)$. For each $\nu \in N$, we have

$$\begin{gathered} \varphi(SP_\nu, P_\nu T) - \xi_\lambda(ST) = \\ \varphi(SP_\nu, P_\nu T) - \langle S \cdot D_\nu \cdot T, \widehat{\varphi} \rangle + \langle (S \cdot D_\nu - D_\nu \cdot S) \cdot T, \widehat{\varphi} \rangle \end{gathered}$$

so that (17) gives

$$|\varphi(SP_\nu, P_\nu T) - \langle ST, \xi_\lambda\rangle| \le$$
$$60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,\|S\|\,\|T\|\,|\varphi|_{zp} + \|\varphi\|\,\|S\cdot D_\nu - D_\nu\cdot S\|\,\|T\|\,.$$

Taking limits on both sides of the above inequality, and using that $(SP_\nu)_{\nu\in N} \to S$, $(P_\nu T)_{\nu\in N} \to T$, and $(\|S\cdot D_\nu - D_\nu\cdot S\|)_{\nu\in N} \to 0$, we see that

$$|\varphi(S,T) - \langle ST, \xi\rangle| \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,\|S\|\,\|T\|\,|\varphi|_{zp}\,.$$

We thus get

$$\operatorname{dist}\big(\varphi, \mathcal{B}^2_\pi(\mathcal{A}(X),\mathbb{C})\big) \le 60\sqrt{27}\,\frac{1+\sin\frac{\pi}{10}}{1-2\sin\frac{\pi}{10}}\,C^2\,|\varphi|_{zp}\,,$$

which proves the theorem. □

The hyperreflexivity of the space $\mathcal{Z}^n(A,X)$ of continuous n-cocycles from A into X, where A is a C^*-algebra or a group algebra and X is a Banach A-bimodule has been already studied in [15, Theorem 4.4]. We conclude this section with a look at the hyperreflexivity of the space $\mathcal{Z}^n(\mathcal{A}(X),Y^*)$. For this purpose we introduce some terminology.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Set

$$L_X = \sup\big\{\|a\cdot x\| : x\in X,\ a\in A,\ \|x\| = \|a\| = 1\big\}$$

and

$$R_X = \sup\big\{\|x\cdot a\| : x\in X,\ a\in A,\ \|x\| = \|a\| = 1\big\}.$$

For each $n\in\mathbb{N}$, let $\delta^n\colon \mathcal{B}^n(A,X)\to\mathcal{B}^{n+1}(A,X)$ be the n-coboundary operator defined by

$$\begin{aligned}(\delta^n T)(a_1,\dots,a_{n+1}) = a_1\cdot T(a_2,\dots,a_{n+1}) &\\ +\sum_{k=1}^{n}(-1)^k T(a_1,\dots,a_k a_{k+1},\dots,a_{n+1}) &\\ +(-1)^{n+1}T(a_1,\dots,a_n)\cdot a_{n+1} &\end{aligned}$$

for all $T\in\mathcal{B}^n(A,X)$ and $a_1,\dots,a_{n+1}\in A$. Further, $\delta^0\colon X\to\mathcal{B}(A,X)$ is defined by

$$(\delta^0 x)(a) = a\cdot x - x\cdot a \quad \forall x\in X,\ \forall a\in A.$$

The space of continuous n-cocycles, $\mathcal{Z}^n(A,X)$, is defined as $\ker\delta^n$. The space of continuous n-coboundaries, $\mathcal{N}^n(A,X)$, is the range of δ^{n-1}. Then $\mathcal{N}^n(A,X)\subset\mathcal{Z}^n(A,X)$, and the quotient $\mathcal{H}^n(A,X) = \mathcal{Z}^n(A,X)/\mathcal{N}^n(A,X)$ is the n^{th} Hochschild cohomology group. For each $T \in \mathcal{B}^n(A,X)$, the constant

$$\operatorname{dist}_r\big(T, \mathcal{Z}^n(A,X)\big) := \sup_{\|a_1\|=\cdots=\|a_n\|=1} \inf\big\{\|T(a_1,\ldots,a_n) - S(a_1,\ldots,a_n)\| : S \in \mathcal{Z}^n(A,X)\big\}$$

is intended to estimate the usual distance from T to $\mathcal{Z}^n(A,X)$, and, in accordance with [14,15], the space $\mathcal{Z}^n(A,X)$ is called hyperreflexive if there exists a constant K such that

$$\operatorname{dist}\big(T, \mathcal{Z}^n(A,X)\big) \le K \operatorname{dist}_r\big(T, \mathcal{Z}^n(A,X)\big) \quad \forall T \in \mathcal{B}^n(A,X).$$

The inequality $\operatorname{dist}_r\big(T, \mathcal{Z}^n(A,X)\big) \le \operatorname{dist}\big(T, \mathcal{Z}^n(A,X)\big)$ is always true.

Proposition 4.2. *Let A be a C-amenable Banach algebra, and let X be a Banach A-bimodule. Then there exist projections $P, Q \in \mathcal{B}(X^*)$ onto $(X \cdot A)^\perp$ and $(A \cdot X)^\perp$, respectively, with $\|P\| \le 1 + R_X C$, $\|Q\| \le 1 + L_X C$, and such that*

$$\operatorname{dist}\big(T, \mathcal{Z}^1(A,X^*)\big) \le C\big(R_X + L_X\|P\| + \|P\|\|Q\|\big)\|\delta^1 T\|$$

for all $T \in \mathcal{B}(A,X^*)$. In particular, if the module X is essential, then

$$\operatorname{dist}\big(T, \mathcal{Z}^1(A,X^*)\big) \le R_X C\|\delta^1 T\|$$

for all $T \in \mathcal{B}(A,X^*)$.

Proof. The Banach algebra A has a virtual diagonal D with $\|\mathrm{D}\| \le C$. This is an element $\mathrm{D} \in (A\widehat{\otimes}A)^{**}$ such that, for each $a \in A$, we have

$$a \cdot \mathrm{D} = \mathrm{D} \cdot a \quad \text{and} \quad a \cdot \widehat{\pi}^{**}(\mathrm{D}) = a. \tag{18}$$

Here, the Banach space $A\widehat{\otimes}A$ turns into a contractive Banach A-bimodule with respect to the operations defined through

$$(a \otimes b)c = a \otimes bc, \ \ c(a \otimes b) = ca \otimes b \quad \forall a,b,c \in A,$$

and both $(A\widehat{\otimes}A)^{**}$ and A^{**} are considered as dual A-bimodules in the usual way. The map $\widehat{\pi}\colon A\widehat{\otimes}A \to A$ is the projective induced product map defined through

$$\widehat{\pi}(a \otimes b) = ab \quad \forall a,b \in A.$$

For each $\varphi \in \mathcal{B}^2(A,\mathbb{C})$ there exists a unique element $\widehat{\varphi} \in (A\widehat{\otimes}A)^*$ such that

$$\widehat{\varphi}(a \otimes b) = \varphi(a,b) \quad \forall a,b \in A,$$

and we use the formal notation

$$\int_{A\times A} \varphi(u,v)\, d\mathrm{D}(u,v) := \langle \widehat{\varphi}, \mathrm{D}\rangle.$$

Using this notation, the properties (18) can be written as

$$\int_{A\times A} \varphi(au,v)\, d\mathrm{D}(u,v) = \int_{A\times A} \varphi(u,va)\, d\mathrm{D}(u,v) \tag{19}$$

and

$$\int_{A\times A} \langle auv, \xi\rangle\, d\mathrm{D}(u,v) = \langle a, \xi\rangle \tag{20}$$

for all $\varphi \in \mathcal{B}^2(A,\mathbb{C})$, $a \in A$, and $\xi \in A^*$; further, it will be helpful noting that

$$\left| \int_{A\times A} \varphi(u,v)\, d\mathrm{D}(u,v) \right| \leq \|\mathrm{D}\| \|\widehat{\varphi}\| \leq C\|\varphi\|. \tag{21}$$

We proceed to define the projections P and Q. For this purpose we first define $P_0, Q_0 \in \mathcal{B}(X^*)$ by

$$\langle x, P_0\xi\rangle = \int_{A\times A} \langle x\cdot(uv), \xi\rangle\, d\mathrm{D}(u,v),$$
$$\langle x, Q_0\xi\rangle = \int_{A\times A} \langle (uv)\cdot x, \xi\rangle\, d\mathrm{D}(u,v)$$

for all $x \in X$ and $\xi \in X^*$, and set

$$P = I_{X^*} - P_0, \quad Q = I_{X^*} - Q_0.$$

From (21) we obtain $\|P_0\| \leq R_X C$ and $\|Q_0\| \leq L_X C$, so that $\|P\| \leq 1 + R_X C$ and $\|Q\| \leq 1 + L_X C$.

We claim that

$$a\cdot P_0\xi = P_0(a\cdot\xi) = a\cdot\xi, \tag{22}$$
$$P_0\xi\cdot a = P_0(\xi\cdot a) \tag{23}$$

for all $a \in A$ and $\xi \in X^*$. Indeed, for $a \in A$, $\xi \in X^*$, and each $x \in X$, (19) and (20) gives

$$\begin{aligned}\langle x, a\cdot P_0\xi\rangle &= \langle x\cdot a, P_0\xi\rangle = \int\limits_{A\times A}\langle x\cdot(auv),\xi\rangle\, d\mathrm{D}(u,v)\\ &= \langle x\cdot a,\xi\rangle = \langle x, a\cdot\xi\rangle,\\ \langle x, P_0(a\cdot\xi)\rangle &= \int\limits_{A\times A}\langle x\cdot(uv), a\cdot\xi\rangle\, d\mathrm{D}(u,v)\\ &= \int\limits_{A\times A}\langle x\cdot(uva),\xi\rangle\, d\mathrm{D}(u,v)\\ &= \int\limits_{A\times A}\langle x\cdot(auv),\xi\rangle\, d\mathrm{D}(u,v) = \langle x, a\cdot\xi\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle x, P_0\xi\cdot a\rangle = \langle a\cdot x, P_0\xi\rangle &= \int\limits_{A\times A}\langle (a\cdot x)\cdot(uv),\xi\rangle\, d\mathrm{D}(u,v)\\ &= \int\limits_{A\times A}\langle x\cdot(uv),\xi\cdot a\rangle\, d\mathrm{D}(u,v) = \langle x, P_0(\xi\cdot a)\rangle,\end{aligned}$$

which proves (22) and (23). From (22) we deduce that

$$\langle x\cdot a, P\xi\rangle = \langle x, a\cdot\xi - a\cdot P_0\xi\rangle = 0,$$

and so $P\xi \in (X\cdot A)^\perp$. Further, if $\xi\in(X\cdot A)^\perp$, then

$$\langle x, P_0\xi\rangle = \int\limits_{A\times A}\langle \underbrace{x\cdot(uv)}_{\in X\cdot A},\xi\rangle\, d\mathrm{D}(u,v) = 0,$$

and so $P\xi = \xi$. The operator P is a projection onto $(X\cdot A)^\perp$. From (22) we deduce immediately that

$$P(A\cdot X^*) = \{0\}. \tag{24}$$

The operator Q can be handled in much the same way as P, and we obtain

$$\begin{aligned}&Q_0\xi\cdot a = Q_0(\xi\cdot a) = \xi\cdot a,\\ &a\cdot Q_0\xi = Q_0(a\cdot\xi)\end{aligned}$$

for all $a\in A$ and $\xi\in X^*$, the operator Q is a projection onto $(A\cdot X)^\perp$, and

$$Q(X^*\cdot A) = \{0\}. \tag{25}$$

Set $T \in \mathcal{B}(A, X^*)$, and define $\phi \in X^*$ by

$$\langle x, \phi\rangle = \int\limits_{A\times A} \langle x, u \cdot T(v)\rangle \, d\mathrm{D}(u, v) \quad \forall x \in X.$$

For each $x \in X$ and $a \in A$ we have

$$\langle x, P_0 T(a)\rangle = \int\limits_{A\times A} \langle x \cdot (uv), T(a)\rangle \, d\mathrm{D}(u, v) = \int\limits_{A\times A} \langle x, (uv) \cdot T(a)\rangle \, d\mathrm{D}(u, v)$$

and

$$\begin{aligned}\langle x, (\delta^0\phi)(a)\rangle &= \langle x, a \cdot \phi - \phi \cdot a\rangle = \langle x \cdot a - a \cdot x, \phi\rangle \\ &= \int\limits_{A\times A} \langle x \cdot a - a \cdot x, u \cdot T(v)\rangle \, d\mathrm{D}(u, v) \\ &= \int\limits_{A\times A} \langle x, (au) \cdot T(v) - u \cdot T(v) \cdot a\rangle \, d\mathrm{D}(u, v) \\ &= \int\limits_{A\times A} \langle x, u \cdot T(va) - u \cdot T(v) \cdot a\rangle \, d\mathrm{D}(u, v),\end{aligned}$$

so that

$$\begin{aligned}\langle x, (P_0 T - \delta^0\phi)(a)\rangle &= \int\limits_{A\times A} \langle x, u \cdot (\delta^1 T)(v, a)\rangle \, d\mathrm{D}(u, v) \\ &= \int\limits_{A\times A} \langle x \cdot u, (\delta^1 T)(v, a)\rangle \, d\mathrm{D}(u, v).\end{aligned}$$

From the latter identity and (21) we conclude that

$$|\langle x, (P_0 T - \delta^0\phi)(a)\rangle| \le C R_X \|\delta^1 T\| \|a\| \|x\|,$$

whence

$$\|P_0 T - \delta^0\phi\| \le C R_X \|\delta^1 T\|. \tag{26}$$

Write $S = PT$. From (22) and (23) it follows that $\delta^1 S(a, b) = P\delta^1 T(a, b)$, and so

$$\|\delta^1 S\| \le \|P\| \|\delta^1 T\|. \tag{27}$$

We now define $\psi \in X^*$ by

$$\langle x, \psi \rangle = \int\limits_{A \times A} \langle x, S(u) \cdot v \rangle \, d\mathrm{D}(u,v) \quad \forall x \in X.$$

For each $x \in X$ and $a \in A$ we have

$$\langle x, Q_0 S(a) \rangle = \int\limits_{A \times A} \langle (uv) \cdot x, S(a) \rangle \, d\mathrm{D}(u,v) = \int\limits_{A \times A} \langle x, S(a) \cdot (uv) \rangle \, d\mathrm{D}(u,v)$$

and

$$\begin{aligned} \langle x, (\delta^0 \psi)(a) \rangle &= \langle x, a \cdot \psi - \psi \cdot a \rangle = \langle x \cdot a - a \cdot x, \psi \rangle \\ &= \int\limits_{A \times A} \langle x \cdot a - a \cdot x, S(u) \cdot v \rangle \, d\mathrm{D}(u,v) \\ &= \int\limits_{A \times A} \langle x, a \cdot S(u) \cdot v - S(u) \cdot (va) \rangle \, d\mathrm{D}(u,v) \\ &= \int\limits_{A \times A} \langle x, a \cdot S(u) \cdot v - S(au) \cdot v \rangle \, d\mathrm{D}(u,v), \end{aligned}$$

and hence

$$\begin{aligned} \langle x, (Q_0 S + \delta^0 \psi)(a) \rangle &= \int\limits_{A \times A} \langle x, (\delta^1 S)(a,u) \cdot v \rangle \, d\mathrm{D}(u,v) \\ &= \int\limits_{A \times A} \langle v \cdot x, (\delta^1 S)(a,u) \rangle \, d\mathrm{D}(u,v). \end{aligned}$$

From the latter identity and (21) we conclude that

$$|\langle x, (Q_0 S + \delta^0 \psi)(a) \rangle| \le C L_X \|\delta^1 S\| \|a\| \|x\|.$$

Thus $\|Q_0 S + \delta^0 \psi\| \le C L_X \|\delta^1 S\|$ and (27) then gives

$$\|Q_0 S + \delta^0 \psi\| \le C L_X \|P\| \|\delta^1 T\|. \tag{28}$$

Our next goal is to estimate $\|QPT\|$. For each $u, v, a \in A$, we have

$$\delta^1 T(a, uv) = a \cdot T(uv) - T(auv) + T(a) \cdot (uv),$$

(23) and (24) gives

$$P(\delta^1 T(a, uv)) = \underbrace{P(a \cdot T(uv))}_{=0} - PT(auv) + PT(a) \cdot (uv),$$

and finally (25) yields

$$QP(\delta^1 T(a, uv)) = -QPT(auv) + \underbrace{Q(PT(a)\cdot(uv))}_{=0} = -QPT(auv).$$

We thus get

$$\begin{aligned}\langle x, QPT(a)\rangle &= \int\limits_{A\times A} \langle x, QPT(auv)\rangle \, d\mathrm{D}(u,v)\\ &= \int\limits_{A\times A} \langle x, -QP(\delta^1 T)(a, uv)\rangle \, d\mathrm{D}(u,v)\end{aligned}$$

and (21) implies

$$|\langle x, QPT(a)\rangle| \le C\|QP(\delta^1 T)\|\|x\|\|a\| \le C\|Q\|\|P\|\|\delta^1 T\|\|x\|\|a\|.$$

Hence

$$\|QPT\| \le C\|Q\|\|P\|\|\delta^1 T\|. \tag{29}$$

Finally, since

$$T - \delta^0\phi + \delta^0\psi = QPT + (P_0 T - \delta^0\phi) + (Q_0 PT + \delta^0\psi),$$

(26), (28), and (29) show that

$$\begin{aligned}\|T - \delta^0\phi + \delta^0\psi\| &\le \|P_0 T - \delta^0\phi\| + \|Q_0 PT + \delta^0\psi\| + \|QPT\|\\ &\le CR_X\|\delta^1 T\| + CL_X\|P\|\|\delta^1 T\| + C\|Q\|\|P\|\|\delta^1 T\|.\end{aligned}$$

Since $-\delta^0\phi + \delta^0\psi \in \mathcal{Z}^1(A, X^*)$, it follows that

$$\operatorname{dist}\big(T, \mathcal{Z}^1(A, X^*)\big) \le CR_X\|\delta^1 T\| + CL_X\|P\|\|\delta^1 T\| + C\|Q\|\|P\|\|\delta^1 T\|$$

as required. □

Corollary 4.3. *Let A be a C-amenable Banach algebra, let X be a Banach A-bimodule, and let $n \in \mathbb{N}$. Then*

$$\operatorname{dist}\big(T, \mathcal{Z}^n(A, X^*)\big) \le 2(n + L_X)(1 + R_X)C^3\|\delta^n T\|$$

for each $T \in \mathcal{B}^n(A, X^*)$.

Proof. Of course, we need only consider the case where A is a non-zero Banach algebra, which implies that $C \geq 1$.

Suppose that $n = 1$, and $T \in \mathcal{B}(A, X^*)$. By Proposition 4.2,

$$\begin{aligned} \operatorname{dist}\big(T, \mathcal{Z}^1(A, X^*)\big) &\leq C\big(R_X + L_X(1 + R_X C) + (1 + L_X C)(1 + R_X C)\big)\|\delta^1 T\| \\ &\leq 2(1 + L_X)(1 + R_X)C^3\|\delta^1 T\|, \end{aligned}$$

as $C \geq 1$.

The Banach space $\mathcal{B}^n(A, X^*)$ is a Banach A-bimodule with respect to the operations

$$(a \cdot T)(a_1, \ldots, a_n) = a \cdot T(a_1, \ldots, a_n)$$

and

$$\begin{aligned} (T \cdot a)(a_1, \ldots, a_n) = T(aa_1, \ldots, a_n) & \\ + \sum_{k=1}^{n-1}(-1)^k T(a, a_1, \ldots, a_k a_{k+1}, \ldots, a_n) & \\ + (-1)^n T(a, a_1, \ldots, a_{n-1}) \cdot a_n & \end{aligned}$$

for all $T \in \mathcal{B}^n(A, X^*)$, and $a, a_1, \ldots, a_n \in A$. Let

$$\Delta^1 \colon \mathcal{B}(A, \mathcal{B}^n(A, X^*)) \to \mathcal{B}^2(A, \mathcal{B}^n(A, X^*))$$

be the 1-coboundary operator. We also consider the maps

$$\begin{aligned} \tau_1^n \colon \mathcal{B}^{1+n}(A, X^*) &\to \mathcal{B}(A, \mathcal{B}^n(A, X^*)), \\ \tau_2^n \colon \mathcal{B}^{2+n}(A, X^*) &\to \mathcal{B}^2(A, \mathcal{B}^n(A, X^*)) \end{aligned}$$

defined by

$$\begin{aligned} (\tau_1^n T)(a)(a_1, \ldots, a_n) &= T(a, a_1, \ldots, a_n), \\ (\tau_2^n T)(a, b)(a_1, \ldots, a_n) &= T(a, b, a_1, \ldots, a_n). \end{aligned}$$

Then:

- τ_1^n and τ_2^n are isometric isomorphisms;
- $\Delta^1 \circ \tau_1^n = \tau_2^n \circ \delta^{n+1}$;
- $\tau_1^n \mathcal{Z}^{n+1}(A, X^*) = \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))$.

For each $T \in \mathcal{B}^{1+n}(A, X^*)$ we have

$$\begin{aligned} \operatorname{dist}\big(T, \mathcal{Z}^{n+1}(A, X^*)\big) &= \operatorname{dist}\big(\tau_1^n T, \tau_1^n \mathcal{Z}^{n+1}(A, X^*)\big) \\ &= \operatorname{dist}\big(\tau_1^n T, \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))\big). \end{aligned} \tag{30}$$

Our next objective is to apply Proposition 4.2 to estimate the distance of the last term in (30). To this end, we realize that $\mathcal{B}^n(A, X^*)$ is a dual Banach A-bimodule by setting

$$Y = \underbrace{A\widehat{\otimes}\cdots\widehat{\otimes}A}_{n\text{-times}}\widehat{\otimes}X.$$

Then:

- Y is a Banach A-bimodule with respect to the operations

$$(a_1 \otimes \cdots \otimes a_n \otimes x) \cdot a = a_1 \otimes \cdots \otimes a_n \otimes (x \cdot a)$$

  and

$$\begin{aligned} a \cdot (a_1 \otimes \cdots \otimes a_n \otimes x) &= (aa_1) \otimes \cdots \otimes a_n \otimes x \\ &+ \sum_{k=1}^{n-1} (-1)^k a \otimes a_1 \otimes \cdots \otimes (a_k a_{k+1}) \otimes \cdots \otimes a_n \otimes x \\ &+ (-1)^n a \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes (a_n \cdot x) \end{aligned}$$

  for all $a, a_1, \ldots, a_n \in A$, and $x \in X$;
- we have the estimates

$$L_Y \leq n + L_X, \quad R_Y \leq R_X;$$

- the Banach A-bimodule $\mathcal{B}^n(A, X^*)$ is isometrically isomorphic to the Banach A-bimodule Y^* through the duality

$$\langle a_1 \otimes \cdots \otimes a_n \otimes x, T\rangle = \langle x, T(a_1, \ldots, a_n)\rangle$$

  for all $T \in \mathcal{B}^n(A, X^*)$, $a_1, \ldots, a_n \in A$, and $x \in X$.

Proposition 4.2 now leads to

$$\begin{aligned} \operatorname{dist}\big(\tau_1^n T, \mathcal{Z}^1(A, \mathcal{B}^n(A, X^*))\big) &= \operatorname{dist}\big(\tau_1^n T, \mathcal{Z}^1(A, Y^*)\big) \\ &\leq 2(1 + L_Y)(1 + R_Y)C^3 \left\|\Delta^1 \tau_1^n T\right\| \\ &\leq 2(1 + n + L_X)(1 + R_X)C^3 \left\|\Delta^1 \tau_1^n T\right\| \\ &= 2(1 + n + L_X)(1 + R_X)C^3 \left\|\tau_2^n \delta^{n+1} T\right\| \\ &= 2(1 + n + L_X)(1 + R_X)C^3 \left\|\delta^{n+1} T\right\|. \end{aligned}$$

Combining (30) with the inequality above, we obtain precisely the estimate of the corollary. □

Theorem 4.4. *Let X be a Banach space with property $(\mathbb{A})$, let Y be a Banach $\mathcal{A}(X)$-bimodule, and let $n \in \mathbb{N}$. Then the space $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ is hyperreflexive. Specifically, if C denotes the supremum in (15), then*

$$\operatorname{dist}\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big) \leq$$
$$(n + L_Y)(1 + R_Y)C^6 2^n \big(C^2 \beta_{\mathcal{A}(X)} + (C + 1)^2\big)^{n+1} \operatorname{dist}_r\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big)$$

for each $T \in \mathcal{B}^n(\mathcal{A}(X), Y^*)$, where

$$\beta_{\mathcal{A}(X)} \leq 120\sqrt{27}\,\frac{1 + \sin\frac{\pi}{10}}{1 - 2\sin\frac{\pi}{10}}\,C^2.$$

Proof. From Theorem 4.1 we see that $\mathcal{A}(X)$ has the strong property $\mathbb{B}$ and the estimate for $\beta_{\mathcal{A}(X)}$ holds.

The Banach algebra $\mathcal{A}(X)$ has an approximate identity of bound C. Further, for each $T \in \mathcal{F}(X)$ there exists $S \in \mathcal{F}(X)$ such that $ST = TS = T$, and [14, Proposition 5.4] then shows that $\mathcal{A}(X)$ has bounded local units.

By [12, Theorem 3.3.9], $\mathcal{A}(X)$ is C^2-amenable, and Corollary 4.3 now gives

$$\operatorname{dist}\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big) \leq 2(n + L_Y)(1 + R_Y)C^6\|\delta^n T\|$$

for each $T \in \mathcal{B}^n(\mathcal{A}(X), Y^*)$. This estimate shows that the map

$$\mathcal{B}^n(\mathcal{A}(X), Y^*)/\mathcal{Z}^n(\mathcal{A}(X), Y^*) \to \mathcal{N}^{n+1}(\mathcal{A}(X), Y^*)$$
$$T + \mathcal{Z}^n(\mathcal{A}(X), Y^*) \mapsto \delta^n T$$

is an isomorphism, hence $\mathcal{N}^{n+1}(\mathcal{A}(X), Y^*)$ is closed in $\mathcal{B}^{n+1}(\mathcal{A}(X), Y^*)$ and this implies that the n^{th} Hochschild cohomology group $\mathcal{H}^{n+1}(\mathcal{A}(X), Y^*)$ is a Banach space. By applying [15, Theorem 4.3] we obtain the hyperreflexivity of the space $\mathcal{Z}^n(\mathcal{A}(X), Y^*)$ as well as the statement about the estimate of $\operatorname{dist}\big(T, \mathcal{Z}^n(\mathcal{A}(X), Y^*)\big)$. □

Declaration of competing interest

There is no competing interest.

References

[1] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Maps preserving zero products, Stud. Math. 193 (2009) 131–159.
[2] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Zero Lie product determined Banach algebras, Stud. Math. 239 (2017) 189–199.
[3] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Zero Lie product determined Banach algebras, II, J. Math. Anal. Appl. 474 (2) (2019) 1498–1511.
[4] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Zero Jordan product determined Banach algebras, J. Aust. Math. Soc. (2020) 1–14, https://doi.org/10.1017/S1446788719000478.

[5] J. Alaminos, J. Extremera, M.L.C. Godoy, A.R. Villena, Hyperreflexivity of the space of module homomorphisms between non-commutative L^p-spaces, J. Math. Anal. Appl. 498 (2) (2021) 124964.
[6] J. Alaminos, J. Extremera, A.R. Villena, Approximately zero product preserving maps, Isr. J. Math. 178 (2010) 1–28.
[7] J. Alaminos, J. Extremera, A.R. Villena, Hyperreflexivity of the derivation space of some group algebras, Math. Z. 266 (2010) 571–582.
[8] J. Alaminos, J. Extremera, A.R. Villena, Hyperreflexivity of the derivation space of some group algebras, II, Bull. Lond. Math. Soc. 44 (2012) 323–335.
[9] G.R. Allan, T.J. Ransford, Power-dominated elements in a Banach algebra, Stud. Math. 94 (1) (1989) 63–79.
[10] M. Brešar, Zero Product Determined Algebras, Frontiers in Mathematics, Birkhäuser, Basel, 2021.
[11] H.G. Dales, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs, New Series, vol. 24, Oxford Science Publications, the Clarendon Press, Oxford University Press, New York, 2000.
[12] V. Runde, Amenable Banach algebras. A Panorama, Springer Monographs in Mathematics, Springer-Verlag, New York, 2020.
[13] E. Samei, Reflexivity and hyperreflexivity of bounded n-cocycles from group algebras, Proc. Am. Math. Soc. 139 (2011) 163–176.
[14] E. Samei, J. Soltani Farsani, Hyperreflexivity of the bounded n-cocycle spaces of Banach algebras, Monatshefte Math. 175 (2014) 429–455.
[15] E. Samei, J. Soltani Farsani, Hyperreflexivity constants of the bounded n-cocycle spaces of group algebras and C^*-algebras, J. Aust. Math. Soc. 109 (2020) 112–130.