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On nonstandard chemotactic dynamics with logistic growth induced by a modified complex Ginzburg–Landau equation

José Luis López

Departamento de Matemática Aplicada and Excellence Research Unit "Modeling Nature" (MNat), Facultad de Ciencias, Universidad de Granada, Granada, Spain

Correspondence
José Luis López, Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, Granada, 18194, Spain.
Email: jllopez@ugr.es

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Abstract

In this paper, we derive a variant of the classical Keller–Segel model of chemotaxis incorporating a growth term of logistic type for the cell population $n(t,x)$, say $\nu n(1-n)$ with $\nu > 0$, and a nonstandard chemical production–degradation mechanism involving first- and second-order derivatives of the logarithm of the cell density, say $f_{\lambda ab}(n, n_x, n_{xx}) = \lambda n + a\frac{n_{xx}}{n} + b\frac{n_x^2}{n^2}$ with $\lambda, a, b \in \mathbb{R}$, via the (n, S)-hydrodynamical system associated with a modified Ginzburg–Landau equation governing the evolution of the complex wavefunction $\psi = \sqrt{n}\, e^{iS}$. In a chemotactic context, $S(t,x)$ will play the role of the concentration of chemical substance. Then, after carrying out a detailed analysis of the modulational stability of uniform-in-space plane waves, dark soliton-shaped traveling wave densities of the former system are constructed from solitary wave solutions of the latter.

KEYWORDS
chemotaxis, Keller–Segel system, modified complex Ginzburg–Landau equation, modulational stability, solitary wave, soliton

1 | INTRODUCTION AND MAIN RESULTS

1.1 | Preliminaries

The complex Ginzburg–Landau equation, typically written as

$$i\partial_t\psi + \alpha\Delta_x\psi + \beta n\psi = i\sigma\psi \tag{1}$$

emerges as an extended version of the nonlinear Schrödinger equation with $\alpha, \beta \in \mathbb{C}$ and $\sigma \in \mathbb{R}$, where $n(t,x) = |\psi(t,x)|^2$ represents the probability density associated with the complex wave-function $\psi : [0,T] \times \mathbb{R}^N \to \mathbb{C}$. Indeed, if the parameters α, β are assumed to be real numbers and $\sigma = 0$, then the well-known nonlinear (cubic) Schrödinger equation is recovered, either in the focusing ($\beta > 0$) or defocusing ($\beta < 0$) form. As usual, ∂_t denotes the first-order time-derivative operator, Δ_x stands for the Laplace operator with respect to the position variable $x \in \mathbb{R}^N$ and $i^2 = -1$. Equation (1) has proved useful to describe a number of dissipative processes in superconductivity, superfluidity, hydrodynamics, phase transitions, chemical turbulence, Bose–Einstein condensation, pattern formation, and nonlinear oscillations, among other applications. More specifically, it governs the amplitude near the onset of fluid-mechanical instabilities (see, for instance, Refs. 1–3 and references therein).

The cubic nonlinearity present in Equation (1) sometimes appears augmented by additional gradient terms leading to the following generalized Ginzburg–Landau equation:

$$i\partial_t\psi + \alpha\Delta_x\psi + \beta n\psi = \left(i\sigma + \gamma\frac{|\nabla_x\psi|^2}{n} + \delta Q[n]\right)\psi, \tag{2}$$

where

$$Q[n] = -\frac{\Delta_x\sqrt{n}}{2\sqrt{n}} = \frac{|\nabla_x n|^2}{8n^2} - \frac{\Delta_x n}{4n} \tag{3}$$

denotes the quantum potential stemming from Bohmian mechanics. Equation (2) is commonly referred to as the modified complex Ginzburg–Landau equation[4–9] and has been revealed as a powerful tool in the modeling of collective motion in a superfluid background when considering vacuum dissipative effects, among other diffusive processes in nonequilibrium systems.[10–12]

In a recent work,[13] the author showed the accurate connection between a nonlinear Schrödinger–Doebner–Goldin equation[14–16] with focusing cubic interaction, namely,

$$i\partial_t\psi + \frac{1}{2}\Delta_x\psi + \lambda n\psi - i\frac{\tau}{2}\log\left(\frac{\psi}{\bar{\psi}}\right)\psi = \left(\frac{i}{2}\left(\frac{\Delta_x n}{n}\right) - \nabla_x\cdot\left(\frac{J}{n}\right) - \frac{|J|^2}{2n^2} - Q[n]\right)\psi, \tag{4}$$

and the parabolic–parabolic Keller–Segel system of chemotaxis

$$\begin{cases} \partial_t n = \Delta_x n - \nabla_x\cdot(n\nabla_x S) \\ \partial_t S = \Delta_x S + \lambda n - \tau S, \end{cases} \tag{5}$$

which describes the process through which a cell population moves in the direction of the concentration of some chemical agents released by themselves. In a biomechanical context, n and S represent the cell and chemical concentrations, respectively. The link between the Schrödinger model (4) and the macroscopic system (5) was established through the hydrodynamic laws governing the time evolution of the modulus square ($n = |\psi|^2$) and the argument ($S = \frac{i}{2}\log(\psi/\overline{\psi})$) of the wavefunction ψ. Here, we denoted $\overline{\psi}$ the complex conjugate of ψ,

$$J = \mathrm{Im}(\overline{\psi}\nabla_x\psi) \tag{6}$$

the quantum-mechanical electric current and $\mathrm{Im}(\phi)$ the imaginary part of a complex function ϕ.

1.2 | Literature review, aims, and scope

What we aim to do in this piece of work is extending this relationship to a wider class of quantum-mechanical equations which can give coverage to more sophisticated effects in the Keller–Segel framework, such as logistic growth of the cell population[17–24] (which may prevent the eventual blow-up of solutions) and nonstandard chemical production–degradation mechanisms[25] (which may guarantee the existence of several types of traveling wave profiles). By exploiting this connection, a family of sharp soliton-shaped traveling waves of a generalized Keller–Segel system

$$\begin{cases} \partial_t n = \Delta_x n - \nabla_x \cdot (n\nabla_x S) + \nu n(1-n) \\ \partial_t S = \Delta_x S + \lambda n + a\frac{\Delta_x n}{n} + b\frac{|\nabla_x n|^2}{n^2}, \end{cases} \tag{7}$$

will be constructed (see Theorems 1 and 3) by means of a suitable gauge transformation of the solutions to a diffusive correction of the modified complex Ginzburg–Landau equation (2).

A short review of traveling waves in Keller–Segel systems is given below. For the standard Keller–Segel system with logistic growth term, namely,

$$\begin{cases} \partial_t n = \Delta_x n - \nabla_x \cdot (n\nabla_x S) + \nu n(1-n) \\ \partial_t S = \Delta_x S + \lambda n - \tau S, \end{cases} \tag{8}$$

the existence of traveling waves in different scenarios has been widely dealt with in recent years (see, for instance, Refs. 20, 26–28). More precisely, in[27] the authors show the existence of traveling waves for a range of constant sensibilities, as set out in Ref. 28 but for the parabolic–elliptic counterpart. For its part, Refs. 20, 26, and 58 address the issue of the existence of a minimal wave speed above which the system has a traveling wave solution. The same problem for the simpler case $\nu = 0$ was already analyzed in the seminal works.[29,30] As a matter of fact, it is known[31] that no bounded traveling waves solving (5) do exist due to the presence of a diffusive term in the equation for S. On the contrary, when the chemotactic sensitivity is logarithmic and the consumption mechanism is of the type $f(n, S) = -nS^m$, that is,

$$\begin{cases} \partial_t n = \Delta_x n - \nabla_x \cdot (n\nabla_x \log(S)) \\ \partial_t S = \Delta_x S - nS^m, \end{cases} \tag{9}$$

as raised for the first time in Ref. 57 and more recently in the review paper,[32] traveling waves of pulse and front type are shown to exist. This study is extended in Ref. 33 to drift mechanisms characterized by the presence of a kinetic term of the type nS^m in the equation for the cell density, too. Another source of interesting traveling wave profiles is the nonlinear modification of the diffusion mechanism underlying the chemotactic process by an appropriate flux limiter, as, for instance, in Refs. 34–39. In this situation, the cell density profiles are shown that they could develop discontinuities provided that a logarithmic chemosensitivity is prescribed or if porous media flows are considered. To the author's knowledge, a comprehensive study of solitonic patterns for logistic Keller–Segel type systems remains to be done. Our main goal in this regard consists in finding tanh-fronts (or, in other words, dark solitons in a quantum-mechanical context as evidenced by the modulational stability criterium stated in Theorem 2) associated with the evolution of the cell density n, which will be shown to move with a velocity proportional to the square root of the logistic parameter ν, in accordance with the theoretical results proved in Refs. 20, 26. To this aim, some $\log(n)$-gradients need to be incorporated to the production–degradation mechanisms of the usual chemotaxis systems (compare systems (19) and (8)), giving rise to a new family of models of Keller–Segel type which may deserve further exploration.

Our starting point is the following diffusive generalization of the modified complex Ginzburg–Landau equation (2):

$$i\partial_t\psi + \alpha\Delta_x\psi + \beta n\psi = \left(\sigma + \gamma\frac{|\nabla_x\psi|^2}{n} + \delta Q[n] + \kappa\nabla_x\cdot\left(\frac{J}{n}\right)\right)\psi := \mathcal{L}_{MCGL}[\psi], \tag{10}$$

with $\alpha, \beta, \gamma, \delta, \sigma, \kappa \in \mathbb{C}$. Notice that when the modulus-argument decomposition of the wavefunction

$$\psi = \sqrt{n}\, e^{iS} \tag{11}$$

is considered, the right-hand side of Equation (10) can be rewritten as

$$\mathcal{L}_{MCGL}[\psi] = \left(\sigma + F_{\kappa,\gamma}(S) - F_{\frac{\delta}{4},\frac{\delta}{8}-\frac{\gamma}{4}}(\log(n))\right)\psi, \tag{12}$$

with

$$F_{\alpha,\beta}(u) = \alpha\Delta_x u + \beta|\nabla_x u|^2. \tag{13}$$

Here, we used the fact that (see Ref. 40)

$$\frac{J}{n} = \mathrm{Im}\left(\frac{\nabla_x\psi}{\psi}\right) = \mathrm{Im}\left(\frac{\nabla_x n}{2n} + i\nabla_x S\right) = \nabla_x S. \tag{14}$$

The two $F_{\alpha,\beta}$-contributions in (12) adopt the form of viscous Hamilton–Jacobi operators acting on S and $\log(n)$, respectively. These fluxes are known to properly model the evolution of growing interfaces, for instance, the growth of various types of aggregates, fronts or tumors, where the Laplacian term describes relaxational diffusion while the gradient square term typically accounts for lateral growth (see Refs. 41, 42).

1.3 | Results

In what follows, the main purpose is to establish how an adequate choice of the coefficients involved in Equation (10) may induce extended chemotactic (Keller–Segel) dynamics with logistic growth for the hydrodynamical magnitudes n and S, as well as to construct exact solitary wave solutions whose density evolves as a solitonic traveling wave. As a matter of fact, the generalizations of the standard Keller–Segel system recently derived in Ref. 25, which contain nonstandard production–degradation chemotactic mechanisms having the general form

$$f_{abcd}(n, \nabla_x n, \Delta_x n, \nabla_x S) = a\frac{\Delta_x n}{n} + b\frac{|\nabla_x n|^2}{n^2} + c|\nabla_x S|^2 + d\frac{\nabla_x n \cdot \nabla_x S}{n}, \tag{15}$$

with $a, b, c, d \in \mathbb{R}$, are shown to fall in a natural way into the hydrodynamic evolution represented by Equation (10).

In what follows, we focus on the one-dimensional case ($x \in \mathbb{R}$) for simplicity and use the subscripts t or x to denote time or positional partial derivatives, respectively. Our main results are the following.

Theorem 1. *Let $\nu > 0$, $\lambda \in \mathbb{R}$, and*

$$\psi(t,x) = \sqrt{n(t,x)}\, e^{iS(t,x)}, \quad t > 0, \quad x \in \mathbb{R}, \tag{16}$$

be a solution of the modified complex Ginzburg–Landau equation (10), with $Q[n]$ and J defined as in (3) and (6), respectively. Then, for the following choice of the coefficients involved in Equation (10)

$$\alpha = -\gamma = \frac{1}{2}(1-i), \quad \beta = \lambda + \frac{\nu}{2}(1+i), \quad \sigma = \frac{\nu}{2}(1+i), \quad \delta \in \mathbb{R}, \quad \kappa = -1, \tag{17}$$

the macroscopic magnitudes $n(t,x)$ (i.e., the modulus square of the wavefunction ψ) and

$$\hat{S}(t,x) = S(t,x) + \frac{1}{2}\log(n(t,x)) \tag{18}$$

(a log-gauged argument of the wavefunction ψ) satisfy the following parabolic–parabolic hydrodynamic system of Keller–Segel type:

$$\begin{cases} n_t = n_{xx} - \left(n\hat{S}_x\right)_x + \nu n(1-n) \\ \hat{S}_t = \hat{S}_{xx} + \lambda n + \frac{\delta}{4}\left(\frac{n_{xx}}{n}\right) + \left(\frac{4-\delta}{8}\right)\frac{n_x^2}{n^2}. \end{cases} \tag{19}$$

Theorem 2 (Criterium for modulational stability)**.** *Let $A_0 \in \mathbb{C}$, $\varepsilon > 0$, and*

$$\psi_\varepsilon(t,x) = A_0(1 + \varepsilon A(t,x))e^{i\lambda t} \tag{20}$$

be a family of $o(\varepsilon)$-solutions to the modified complex Ginzburg–Landau equation (10)–(17). Let also assume that the perturbation $A(t,x)$ admits the Fourier decomposition

$$A(t,x) = A_1\, e^{i(Kx+\Omega t)} + \overline{A_2}\, e^{-i(Kx+\overline{\Omega}t)}, \tag{21}$$

with amplitudes $A_1, A_2 \in \mathbb{C}$, wavenumber $K \in \mathbb{R}$, and angular frequency $\Omega \in \mathbb{C}$. Then, the following dispersion relation is satisfied:

$$\Omega^2 - i(2K^2 + \nu)\Omega - K^2\left(\left(1 + \frac{\delta}{4}\right)K^2 + \nu - \lambda\right) = 0. \tag{22}$$

As consequence, a sufficient and necessary condition for linear stability of the spatially uniform plane waves $\psi_0 = A_0\, e^{i\lambda t}$ with respect to transversally modulated perturbations (20)–(21) is the following:

$$(4 + \delta)K^2 + 4(\nu - \lambda) > 0. \tag{23}$$

Theorem 3. *Let $\delta = -\frac{5}{3}$ and $\lambda = -\frac{4\nu}{9}$ in (17). Then, the solitary wave*

$$\psi(t, x) = \sqrt{n(\xi)}\, e^{-i\frac{4\nu}{9}t}, \quad \xi = x - \upsilon t, \tag{24}$$

with

$$n(\xi) = \frac{1}{4}\left(1 - \tanh\left(\sqrt{\frac{\nu}{12}}\,\xi\right)\right)^2 \tag{25}$$

is a solution to Equations (10)–(17) whose probability density $n(\xi)$ travels with velocity

$$\upsilon = 5\sqrt{\frac{\nu}{12}}. \tag{26}$$

As consequence, the couple of functions consisting of (25) and

$$\hat{S}(t, x) = \frac{1}{2}\log(n(\xi)) - \frac{4\nu}{9}t \tag{27}$$

solves the Keller–Segel type system (19).

The rest of the paper is devoted to prove Theorem 1. It is structured as follows: In Section 2, we develop the formal connection between Equation (10) and the parabolic–parabolic Keller–Segel type model with logistic damping and chemical production–degradation mechanism ruled by gradient terms stated in (19) (an exhaustive analysis of the way back and forth relating the Schrödinger–Doebner–Goldin equation with its associated modulus-argument hydrodynamical system was already made in Ref. 40 in full rigour; see also Ref. 13). Section 3 explores the stability of uniform-in-space plane wave solutions to the modified complex Ginzburg–Landau equation established in Section 2 under small modulations of the amplitude. Finally, Section 4 is devoted to the construction of solitary wave profiles to Equations (10)–(17) which will provide tanh-shaped traveling wave cell densities $n(t, x)$ at the Keller–Segel level (in a parameter regime contemplated by the modulational stability criterium) as those given in (25), as well as chemical densities $\hat{S}(t, x)$ of $\log(n)$-type as those given in (27).

2 | DERIVATION OF THE LOGISTIC KELLER–SEGEL TYPE SYSTEM (19)

The main purpose of this section is to derive the logistic Keller–Segel type system (19) as the hydrodynamical model associated with a particular version of Equation (10). In what follows, we adopt the notation

$$\alpha = \alpha_r + i\alpha_i, \quad \beta = \beta_r + i\beta_i, \quad \sigma = \sigma_r + i\sigma_i, \quad \gamma = \gamma_r + i\gamma_i, \quad \delta = \delta_r + i\delta_i, \quad \kappa = \kappa_r + i\kappa_i \tag{28}$$

for the complex coefficients involved in Equation (10). The modulus-argument decomposition of the wavefunction given by (11) gives rise to the following expressions after simple calculus:

$$\begin{aligned} \psi_t &= \left(\frac{n_t}{2\sqrt{n}} + i\sqrt{n}\, S_t \right) e^{iS}, \quad |\psi_x|^2 = \frac{n_x^2}{4n} + nS_x^2, \\ \psi_{xx} &= \left(\frac{n_{xx}}{2\sqrt{n}} - \frac{n_x^2}{4n^{3/2}} - \sqrt{n}\, S_x^2 + i\left(\sqrt{n}\, S_{xx} + \frac{n_x S_x}{\sqrt{n}} \right) \right) e^{iS}. \end{aligned} \tag{29}$$

After inserting these formulas into Equation (10) and using (14) we find

$$\begin{aligned} in_t - 2nS_t &= 2(\kappa_r + i\kappa_i)nS_{xx} + 2(\alpha_r + \gamma_r + i(\alpha_i + \gamma_i))nS_x^2 + 2(\alpha_i - i\alpha_r)(nS_x)_x \\ &\quad + 2(\sigma_r + i\sigma_i)n(1-n) - 2(\beta_r - \sigma_r + i(\beta_i - \sigma_i))n^2 - \left(\alpha_r + \frac{\delta_r}{2} + i\left(\alpha_i + \frac{\delta_i}{2} \right) \right) n_{xx} \\ &\quad + \frac{1}{2}\left(\alpha_r + \gamma_r + \frac{\delta_r}{2} + i\left(\alpha_i + \gamma_i + \frac{\delta_i}{2} \right) \right) \frac{n_x^2}{n}. \end{aligned} \tag{30}$$

Then, taking imaginary parts we obtain

$$\begin{aligned} n_t &= -\left(\alpha_i + \frac{\delta_i}{2} \right) n_{xx} - 2\alpha_r (nS_x)_x + 2\sigma_i n(1-n) + 2\kappa_i nS_{xx} + 2(\alpha_i + \gamma_i)nS_x^2 - 2(\beta_i - \sigma_i)n^2 \\ &\quad + \frac{1}{2}\left(\alpha_i + \gamma_i + \frac{\delta_i}{2} \right) \frac{n_x^2}{n}. \end{aligned} \tag{31}$$

After making the following choices

$$\alpha_r = \frac{1}{2} \quad \alpha_i = -\gamma_i = -\frac{1}{2}, \quad \kappa_i = \delta_i = 0, \quad \sigma_i = \beta_i = \frac{\nu}{2}. \tag{32}$$

Equation (31) becomes the Keller–Segel evolution equation for the cell concentration $n(t, x)$ in the presence of a logistic source term, namely,

$$n_t = \frac{1}{2} n_{xx} - (nS_x)_x + \nu n(1-n) \tag{33}$$

for a given quantity $\nu > 0$ which will eventually denote the logistic growth rate. Furthermore, taking now real parts in Equation (30) we are led to

$$S_t = \left(\frac{1}{2} - \kappa_r\right) S_{xx} + \frac{n_x S_x}{2n} + \beta_r n - \sigma_r - \left(\gamma_r + \frac{1}{2}\right) S_x^2 + \left(\frac{\delta_r + 1}{4}\right) \frac{n_{xx}}{n} - \left(\frac{2\gamma_r + \delta_r + 1}{8}\right) \frac{n_x^2}{n^2}. \tag{34}$$

Equation (34) can be rewritten in a more recognizable way after performing the change of unknown functions

$$\hat{n} = n, \quad \hat{S} = S + \frac{1}{2} \log(\hat{n}), \tag{35}$$

and selecting

$$\gamma_r = -\frac{1}{2}, \quad \sigma_r = \frac{\nu}{2}, \quad \beta_r = \lambda + \frac{\nu}{2}, \quad \kappa_r = -1, \tag{36}$$

where $\lambda \in \mathbb{R}$ stands for the rate of chemical production ($\lambda > 0$) or consumption ($\lambda < 0$). Indeed, the transformations stated in (35) yield

$$\begin{cases} \hat{n}_t = \hat{n}_{xx} - (\hat{n}\hat{S}_x)_x + \nu \hat{n}(1 - \hat{n}) \\ \hat{S}_t = \hat{S}_{xx} + \lambda \hat{n} + \frac{\delta_r}{4}\left(\frac{\hat{n}_{xx}}{\hat{n}}\right) + \left(\frac{4-\delta_r}{8}\right)\frac{\hat{n}_x^2}{\hat{n}^2}, \end{cases} \tag{37}$$

which is just the system (19) stated in Theorem 1.

As in Ref. 25, we remark that the coefficients of the terms $\frac{\hat{n}_{xx}}{\hat{n}}$ and $\frac{\hat{n}_x^2}{\hat{n}^2}$ cannot vanish simultaneously, so that the chemical production/degradation mechanism that appears in the right-hand side of the second equation in (19) is genuinely nonstandard. When $\lambda = \nu = 0$, the system (19) coincides with that derived in Ref. 25 from Doebner–Goldin theory. According to the results there stated, traveling wave solutions with the form

$$\hat{n}(t, x) = n_0 \cosh\left(k_0(x - vt)\right)^2, \tag{38}$$

$$\hat{S}(t, x) = 2 \log(\cosh(k_0(x - vt))) + \frac{k_0^2}{2}(\delta_r + 4)t + \frac{v}{2} \cdot x \tag{39}$$

can be found in this situation. Notice that the particular case $\delta_r = -4$ corresponds to a logarithmic modulation of the chemical diffusion according to the size of the cell population, namely, the second equation in (19) becomes $\hat{S}_t = (\hat{S} - \log(\hat{n}))_{xx} + \lambda \hat{n}$, resulting in the killing of the explicit time dependence in (39). The two other distinguished cases are $\delta_r = 0$ and $\delta_r = 4$, which make one of the density gradient terms in the right-hand side of the equation for $\hat{S}$ vanish, thereby simplifying as far as possible the chemotactic production/consumption mechanism up to $f(\hat{n}, \hat{n}_x) = \lambda \hat{n} + \frac{\hat{n}_x^2}{2\hat{n}^2}$ or $f(\hat{n}, \hat{n}_{xx}) = \lambda \hat{n} + \frac{\hat{n}_{xx}}{\hat{n}}$.

In the next sections, we are concerned with the special case $\nu \neq 0$, which does correspond with the presence of a logistic damping term in the equation for $\hat{n}$ (cf. (19)). At variance with (38), we will see how this term contributes to produce tanh-shaped solitary waves.

3 | MODULATIONAL STABILITY

We are now concerned with the linear stability analysis of plane wave solutions (see, for instance, Refs. 43–49) to Equations (10)–(17), specifically those which are uniform in space and oscillatory in time: $\psi(t,x) = A_0\,\mathrm{e}^{-i\omega t}$, where $A_0 \in \mathbb{C}$ is the (constant) wave amplitude and ω stands for the angular frequency. These profiles are easily seen to solve our modified complex Ginzburg–Landau equation provided that the following amplitude and dispersion relations are fulfilled:

$$|A_0|^2 = 1, \quad \omega + \lambda = 0. \tag{40}$$

We then consider a family of perturbed waves as those given in (20), where the perturbations are supposed to be sufficiently small (i.e., those terms of order ε^2 are assumed to be negligible). Before inserting (20) into Equations (10)–(17), we collect some useful identities in the following result:

Lemma 1. *Let ψ_ε be as in (20) and $n_\varepsilon = |\psi_\varepsilon|^2$. Then, for any complex functions A, F, the following relations hold true:*

(i) $\varepsilon(1+\varepsilon A)\mathrm{Re}(\frac{F}{1+\varepsilon A}) = \varepsilon\mathrm{Re}(F) + o(\varepsilon^2)$.
(ii) $\varepsilon(1+\varepsilon A)\mathrm{Im}(\frac{F}{1+\varepsilon A}) = \varepsilon\mathrm{Im}(F) + o(\varepsilon^2)$.
(iii) $(1+\varepsilon A)|1+\varepsilon A|^2 = 1 + \varepsilon(2A+\overline{A}) + o(\varepsilon^2)$.
(iv) $Q[n_\varepsilon](1+\varepsilon A) = -\frac{\varepsilon}{2}\mathrm{Re}(A_{xx}) + o(\varepsilon^2)$.
(v) $(S_\varepsilon)_{xx}(1+\varepsilon A) = \varepsilon\mathrm{Im}(A_{xx}) + o(\varepsilon^2)$.

Proof. (i) follows easily from the fact that

$$\begin{aligned}
\mathrm{Re}\left((1+\varepsilon A)\mathrm{Re}\left(\frac{F}{1+\varepsilon A}\right)\right) &= \mathrm{Re}(F) + \mathrm{Im}(1+\varepsilon A)\mathrm{Im}\left(\frac{F}{1+\varepsilon A}\right)\\
&= \mathrm{Re}(F) + \varepsilon\mathrm{Im}(A)\mathrm{Im}\left(\frac{F}{1+\varepsilon A}\right),\\
\mathrm{Im}\left((1+\varepsilon A)\mathrm{Re}\left(\frac{F}{1+\varepsilon A}\right)\right) &= \mathrm{Im}(F) - \mathrm{Re}(1+\varepsilon A)\mathrm{Im}\left(\frac{F}{1+\varepsilon A}\right)\\
&= \mathrm{Im}(F) - (1+\varepsilon\mathrm{Re}(A))\mathrm{Im}\left(\frac{F}{1+\varepsilon A}\right),
\end{aligned} \tag{41}$$

which yields

$$\begin{aligned}
\varepsilon(1+\varepsilon A)\mathrm{Re}\left(\frac{F}{1+\varepsilon A}\right) &= \varepsilon\mathrm{Re}(F) + i\varepsilon\mathrm{Im}\left(F - \frac{F}{1+\varepsilon A}\right)\\
&\quad + \varepsilon^2\mathrm{Im}\left(\frac{F}{1+\varepsilon A}\right)(\mathrm{Im}(A) - i\mathrm{Re}(A)) = \varepsilon\mathrm{Re}(F) + o(\varepsilon^2).
\end{aligned} \tag{42}$$

(ii) is proved analogously.

(iii) follows from a straightforward computation by just expanding

$$(1+\varepsilon A)|1+\varepsilon A|^2 = (1+\varepsilon A)^2(\overline{1+\varepsilon A}). \tag{43}$$

We now prove (iv). It is a simple matter to check that

$$\begin{aligned} Q[n_\varepsilon] = \frac{(n_\varepsilon)_x^2}{8n_\varepsilon^2} - \frac{(n_\varepsilon)_{xx}}{4n_\varepsilon} &= \varepsilon^2 \frac{\left(\mathrm{Re}\left(\overline{A}_x(1+\varepsilon A)\right)\right)^2}{2|1+\varepsilon A|^4} - \varepsilon \frac{\mathrm{Re}\left(\overline{A}_{xx}(1+\varepsilon A) + \varepsilon |A_x|^2\right)}{2|1+\varepsilon A|^2} \\ &= -\varepsilon \frac{\mathrm{Re}\left(\overline{A}_{xx}(1+\varepsilon A)\right)}{2|1+\varepsilon A|^2} + o(\varepsilon^2). \end{aligned} \tag{44}$$

Hence,

$$Q[n_\varepsilon](1+\varepsilon A) = -\frac{\varepsilon}{2}(1+\varepsilon A)\mathrm{Re}\left(\frac{A_{xx}}{1+\varepsilon A}\right) = -\frac{\varepsilon}{2}\mathrm{Re}(A_{xx}) + o(\varepsilon^2) \tag{45}$$

by virtue of (i) with $F = A_{xx}$.

Finally, to prove (v) we first compute the perturbed current density

$$J_\varepsilon = \mathrm{Im}\left(\overline{\psi_\varepsilon}(\psi_\varepsilon)_x\right) = \mathrm{Im}\left(\varepsilon A_x(\overline{1+\varepsilon A})\right), \tag{46}$$

so that

$$(S_\varepsilon)_x = \frac{J_\varepsilon}{n_\varepsilon} = \varepsilon \frac{\mathrm{Im}\left(A_x(\overline{1+\varepsilon A})\right)}{|1+\varepsilon A|^2}. \tag{47}$$

After differentiating the above expression we find

$$(S_\varepsilon)_{xx} = \varepsilon \frac{\mathrm{Im}\left(A_{xx}(\overline{1+\varepsilon A})\right)}{|1+\varepsilon A|^2} + o(\varepsilon^2), \tag{48}$$

which yields

$$(S_\varepsilon)_{xx}(1+\varepsilon A) = \varepsilon(1+\varepsilon A)\mathrm{Im}\left(\frac{A_{xx}}{1+\varepsilon A}\right) + o(\varepsilon^2) = \varepsilon \mathrm{Im}(A_{xx}) + o(\varepsilon^2) \tag{49}$$

again by virtue of (ii) with $F = A_{xx}$. This ends the proof. ∎

We are now in a position to investigate the equation to be satisfied by the perturbation $A(t,x)$ in order that (20) be a solution to Equations (10)–(17). Indeed, after substitution we find

$$iA_t + \frac{1}{2}(1-i)A_{xx} + (\nu(1+i) - 2\omega)\mathrm{Re}(A) + \frac{\delta_r}{2}\mathrm{Re}(A_{xx}) + \mathrm{Im}(A_{xx}) = 0, \tag{50}$$

where we used the identities established in Lemma 1 as well as the dispersion relation for ω (cf. (40)). Separating the real and imaginary parts in Equation (50), we obtain

$$\mathrm{Im}(A_t) - \left(\frac{1+\delta_r}{2}\right)\mathrm{Re}(A_{xx}) - \frac{3}{2}\mathrm{Im}(A_{xx}) + (2\omega - \nu)\mathrm{Re}(A) = 0 \tag{51}$$

and

$$\mathrm{Re}(A_t) - \frac{1}{2}\mathrm{Re}(A_{xx}) + \frac{1}{2}\mathrm{Im}(A_{xx}) + \nu\mathrm{Re}(A) = 0, \tag{52}$$

respectively. We now look for perturbation profiles with the form (21). On one hand, plugging (21) into Equation (51) leads to

$$(p+q)A_1 + (p-q)A_2 = 0, \tag{53}$$

with

$$p = \left(\frac{1+\delta_r}{2}\right)K^2 + 2\omega - \nu, \quad q = \Omega - \frac{3i}{2}K^2. \tag{54}$$

On the other hand, plugging (21) into Equation (52) yields

$$(r + iK^2)A_1 + rA_2 = 0, \tag{55}$$

with

$$r = \frac{K^2}{2} + \nu + i\left(\Omega - \frac{K^2}{2}\right). \tag{56}$$

A necessary condition for the existence of nontrivial solutions to the linear system composed of Equations (53) and (55) is

$$2qr - iK^2(p-q) = 0, \tag{57}$$

which becomes

$$\Omega^2 - l\Omega - m = 0 \tag{58}$$

after some algebraic manipulations, where

$$l = i(2K^2 + \nu), \quad m = K^2\left(\left(1 + \frac{\delta_r}{4}\right)K^2 + \nu + \omega\right). \tag{59}$$

The solutions to Equations (58)–(59) are given by

$$\Omega_\pm = \frac{i}{2}(2K^2 + \nu) \pm \frac{1}{2}\sqrt{\delta_r K^4 + 4\omega K^2 - \nu^2}. \tag{60}$$

The asymptotic behavior of the perturbation $A(t,x)$ is the same in both cases, as it is determined by just the imaginary parts of $\Omega_{\pm}$. Indeed, substitution into (21) leads to

$$|A(t,x)| \leq (|A_1| + |A_2|)e^{-\mathrm{Im}(\Omega)t}. \tag{61}$$

We close this section by showing the conditions under which $\mathrm{Im}(\Omega) > 0$, in which case the perturbation vanishes for long times and stability under modulation is guaranteed. First, for those values of K for which $\delta_r K^4 + 4\omega K^2 - \nu^2 > 0$, it is immediately deduced that $\mathrm{Im}(\Omega_{\pm}) = K^2 + \frac{\nu}{2} > 0$ and we are done with the stability property. Otherwise, we would find

$$\Omega_{\pm} = \frac{i}{2}\left(2K^2 + \nu \pm \sqrt{\nu^2 - \delta_r K^4 - 4\omega K^2}\right). \tag{62}$$

As a consequence, a necessary and sufficient condition for modulational stability is given by the following dispersion relation:

$$2K^2 + \nu - \sqrt{\nu^2 - \delta_r K^4 - 4\omega K^2} > 0 \Longleftrightarrow (4 + \delta_r)K^2 + 4(\omega + \nu) > 0. \tag{63}$$

Indeed, in this regime the nonlinearities present in Equations (10)–(17) do not reveal strong enough to propitiate deviations from the spatially uniform plane wave. This closes the proof of Theorem 2 and paves the way to the presence of dark solitonic structures in our model,[43,50] to which we devote the following sections.

4 | SOLITARY WAVE SOLUTIONS OF EQUATIONS (10)–(17)

The purpose of this section is to construct solitary wave solutions to the one-dimensional modified complex Ginzburg–Landau equation (10) for the choice of coefficients stated in (17), which gives rise to the hydrodynamic (Keller–Segel) system (19) via the following modulus-argument decomposition of the wavefunction (cf. (35)):

$$\psi = \sqrt{n}\,\mathrm{e}^{i\left(S + \frac{1}{2}\log(n)\right)}. \tag{64}$$

We are especially interested in showing the persistence of those soliton profiles peculiar to the nonlinear Schrödinger equation, namely, probability densities of sech/tanh type. To this aim, we consider the following ansatz profile:

$$\psi(t,x) = A(\xi)\,\mathrm{e}^{i(kx - \omega t)}, \quad \xi = x - vt, \tag{65}$$

which straightforwardly leads to the relations $S_{xx} = 0$ and $Q = -\frac{A''}{2A}$. Here, we denoted $'$ the derivative with respect to the variable ξ specific to the traveling wave. After inserting (65) into Equations (10)–(17) and then separating the imaginary and real parts we obtain

$$\left(A'' + \frac{(A')^2}{A}\right) + 2(v - k)A' + \nu A(1 - A^2) = 0 \tag{66}$$

and

$$\left(\frac{1+\delta_r}{2}\right)A'' + \frac{(A')^2}{2A} + kA' + \left(\omega - \frac{\nu}{2}\right)A + \left(\lambda + \frac{\nu}{2}\right)A^3 = 0, \tag{67}$$

respectively. Equation (66) can be easily rewritten as

$$n'' + 2un' + 2\nu n(1-n) = 0, \tag{68}$$

where $n = A^2$ and

$$u = v - k. \tag{69}$$

Notice that (68) is exactly the equation satisfied by those traveling waves of the Fisher–Kolmogorov–Petrovsky–Piscounov (FKPP) equation[51,52]

$$n_t = \frac{1}{2}n_{xx} + \nu n(1-n) \tag{70}$$

moving with velocity u. Furthermore, Equation (67) reads as follows in terms of n:

$$\left(\frac{1+\delta_r}{4}\right)n'' - \frac{\delta_r(n')^2}{8n} + \frac{kn'}{2} + \left(\omega - \frac{\nu}{2}\right)n + \left(\lambda + \frac{\nu}{2}\right)n^2 = 0. \tag{71}$$

Thus, our goal will consist in finding soliton profiles which solve Equations (68) and (71) simultaneously, as well as the eventual relations among the parameters for this to happen.

4.1 | Solving Equation (68)

Although a relevant family of explicit solutions to Equation (70) has been found elsewhere,[53,54,56] here we adapt to our context the fundamental computational steps of the homogeneous balance method (see, for instance, Ref. 55 and references therein) for the sake of self-consistency. The main idea consists of searching for solutions with the shape

$$n(\xi) = (f(\phi))'' + A(f(\phi))' + B = f''(\phi)(\phi')^2 + f'(\phi)\phi'' + Af'(\phi)\phi' + B, \tag{72}$$

where $\phi = \phi(\xi)$ is a function to be properly determined later.

Lemma 2. *Let $\nu > 0$. Then, the function*

$$n(\xi) = \frac{1}{4}\left(1 - \tanh\left(\sqrt{\frac{\nu}{12}}\,\xi\right)\right)^2, \tag{73}$$

with

$$\xi = x - 5\sqrt{\frac{\nu}{12}}\,t \tag{74}$$

is a soliton-shaped traveling wave of the FKPP equation (68) which decreases from the equilibrium point $n = 1$ to the equilibrium point $n = 0$.

Proof. It is a simple matter to check that

$$\begin{aligned} n' &= f'''(\phi)(\phi')^3 + f''(\phi)(3\phi'\phi'' + A(\phi')^2) + f'(\phi)(\phi''' + A\phi''), \\ n'' &= f^{(iv)}(\phi)(\phi')^4 + f'''(\phi)(6(\phi')^2\phi'' + A(\phi')^3) \\ &\quad + f''(\phi)(3(\phi'')^2 + 4\phi'\phi''' + 3A\phi'\phi'') + f'(\phi)(\phi^{(iv)} + A\phi'''). \end{aligned} \tag{75}$$

As a consequence, after inserting the above expressions in Equation (68) and rearranging terms we obtain

$$\begin{aligned} &\frac{1}{2}f^{(iv)}(\phi)(\phi')^4 + f'''(\phi)\left(u(\phi')^3 + \frac{1}{2}(6(\phi')^2\phi_{xx} + A(\phi')^3)\right) + f''(\phi)\big(u(3\phi'\phi'' + A(\phi')^2) + \nu(1-2B)(\phi')^2\big) \\ &+ \frac{1}{2}f''(\phi)(4\phi'\phi''' + 3A\phi'\phi'' + 3(\phi'')^2) + f'(\phi)\left(u(\phi''' + A\phi'') + \frac{1}{2}(\phi^{(iv)} + A\phi''') + \nu(1-2B)(\phi'' + A\phi')\right) \\ &- \nu f''(\phi)^2(\phi')^4 - \nu f'(\phi)^2((\phi'')^2 + A^2(\phi')^2 + 2A\phi'\phi'') - 2\nu f'(\phi)f''(\phi)(\phi')^2(\phi'' + A\phi') + \nu B(1-B) = 0. \end{aligned} \tag{76}$$

The strategy is then based on the cancelation of the highest power of ϕ', namely, $(\phi')^4$, which leads to the relation

$$\frac{1}{2}f^{(iv)}(\phi) - \nu f''(\phi)^2 = 0. \tag{77}$$

It is straightforward to see that the function

$$f(\phi) = -\frac{3}{\nu}\log(\phi) \tag{78}$$

solves Equation (77). Now, taking into account that (78) gives rise to the relations

$$f'(\phi)^2 = \frac{3}{\nu}f''(\phi), \quad f'(\phi)f''(\phi) = \frac{3}{2\nu}f'''(\phi), \tag{79}$$

Equation (76) becomes

$$\begin{aligned} &f'''(\phi)\left(u - \frac{5A}{2}\right)(\phi')^3 + f''(\phi)\big(u(3\phi'\phi'' + A(\phi')^2) + \nu(1-2B)(\phi')^2\big) \\ &+ \frac{1}{2}f''(\phi)(4\phi'\phi''' - 9A\phi'\phi'' + 3(\phi'')^2 - 6A^2(\phi')^2) \\ &+ f'(\phi)\left(u(\phi''' + A\phi'') + \frac{1}{2}(\phi^{(iv)} + A\phi''') + \nu(1-2B)(\phi'' + A\phi')\right) \\ &+ \nu B(1-B) = 0. \end{aligned} \tag{80}$$

We finally consider the ansatz profile

$$\phi(\xi) = 1 + e^{\theta\xi}, \quad \xi = x - ut, \tag{81}$$

so that (72) reads

$$\begin{aligned} n(\xi) &= -\frac{3\theta}{\nu}\frac{e^{\theta\xi}}{(1+e^{\theta\xi})^2}(\theta + A(1+e^{\theta\xi})) + B \\ &= -\frac{3\theta}{4\nu}\left(\theta\,\text{sech}^2\left(\frac{\theta\xi}{2}\right) + 2A\tanh\left(\frac{\theta\xi}{2}\right)\right) + B - \frac{3A\theta}{2\nu}. \end{aligned} \tag{82}$$

The relations among the different parameters involved in (82), namely, A, B, θ, and u, are now obtained by making the coefficients of the successive derivatives of f in (80) vanish get

$$f''' : \left(u - \frac{5A}{2}\right)(\phi')^3 = 0, \tag{83}$$

$$f'' : \frac{3}{2}(\phi'')^2 + 2\phi'\phi''' + 3\left(u - \frac{3A}{2}\right)\phi'\phi'' + \left(Au + \nu(1-2B) - 3A^2\right)(\phi')^2 = 0, \tag{84}$$

$$f' : \frac{1}{2}\phi^{(iv)} + \left(u + \frac{A}{2}\right)\phi''' + (Au + \nu(1-2B))\phi'' - \nu(1-2B)A\phi' = 0, \tag{85}$$

$$\text{Independent term} : B = 0 \,\text{or}\, B = 1. \tag{86}$$

When substituting (81) into Equations (84)–(85), we obtain

$$\theta^2 + 6A\theta - A^2 + 2\nu(1-2B) = 0, \tag{87}$$

$$\theta^3 + 6A\theta^2 + (5A^2 + 2\nu(1-2B))\theta + 2\nu(1-2B)A = 0. \tag{88}$$

Now, combining the fact that

$$u = \frac{5A}{2} \tag{89}$$

(as sheds from (83)) with Equation (88) we get

$$3A^2\theta + \nu(1-2B)A = 0, \tag{90}$$

hence

$$\theta = -\frac{\nu(1-2B)}{3A}. \tag{91}$$

We can then insert (91) into Equation (87) to deduce that $A^2 = \theta^2$, which yields $A = -\theta$ according to Equation (88). Thus, solutions to Equation (68) can be constructed with the following form (cf. (82)):

$$n(\xi) = \frac{1}{4}\left(1 + \tanh\left(\frac{\theta\xi}{2}\right)\right)^2 + B, \tag{92}$$

where we used that

$$\theta^2 = \frac{\nu(1-2B)}{3} \tag{93}$$

as follows from (91) and the fact that $A = -\theta$. Finally, for $\nu > 0$ fixed, formula (93) imposes the choice $B = 0$. Also, A must be chosen with the positive sign so as to have a positive velocity (cf. (89)), namely,

$$A = \sqrt{\frac{\nu}{3}} = -\theta. \tag{94}$$

Consequently, (92) finally reads as stated in the lemma. ■

Remark 1. We call the reader's attention to the fact that the well-known minimal velocity condition for the existence of traveling waves $n(x - ct)$ to the FKPP equation $n_t = n_{xx} + n(1 - n)$, namely, $c \geq 2$, here reads $u \geq \sqrt{2\nu}$, which is fulfilled without any further restriction on ν, as follows from the expressions (89) and (94) yielding $u = 5\sqrt{\frac{\nu}{12}}$.

4.2 | Compatibility with Equation (71)

In this section, we face the problem of finding a suitable choice of the parameters involved in (92), already shown to solve Equation (68), which makes it into a solution of Equation (71), too.

Lemma 3. *The following are sufficient and necessary conditions for (73)–(74) to solve Equation (71):*

(i) $k = 0$ *in (65).*
(ii) $\delta = -\frac{5}{3}$ *in (10).*
(iii) $\omega = \frac{4\nu}{9} = -\lambda$ *in (65) and (10), respectively.*

Proof. From (73), we have

$$n'(\xi) = \frac{\theta}{4}\left(1 - \tanh^2\left(\frac{\theta\xi}{2}\right)\right)\left(1 + \tanh\left(\frac{\theta\xi}{2}\right)\right),$$

$$n''(\xi) = \frac{\theta^2}{8}\left(1 - \tanh^2\left(\frac{\theta\xi}{2}\right)\right)^2 - \theta\tanh\left(\frac{\theta\xi}{2}\right)n'(\xi). \tag{95}$$

The following explicit expressions are also of interest before substitution of (73)–(74) into Equation (71):

$$\begin{aligned}
n(\xi)^2 &= \frac{1}{16}\left(\tanh^4\left(\frac{\theta\xi}{2}\right) + 4\tanh^3\left(\frac{\theta\xi}{2}\right) + 6\tanh^2\left(\frac{\theta\xi}{2}\right) + 4\tanh\left(\frac{\theta\xi}{2}\right) + 1\right),\\
n(\xi)^3 &= \frac{1}{64}\left(\tanh^6\left(\frac{\theta\xi}{2}\right) + 6\tanh^5\left(\frac{\theta\xi}{2}\right) + 15\tanh^4\left(\frac{\theta\xi}{2}\right) + 20\tanh^3\left(\frac{\theta\xi}{2}\right)\right.\\
&\quad \left. + 15\tanh^2\left(\frac{\theta\xi}{2}\right) + 6\tanh\left(\frac{\theta\xi}{2}\right) + 1\right),\\
n'(\xi)^2 &= \frac{\theta^2}{16}\left(\tanh^6\left(\frac{\theta\xi}{2}\right) + 2\tanh^5\left(\frac{\theta\xi}{2}\right) - \tanh^4\left(\frac{\theta\xi}{2}\right) - 4\tanh^3\left(\frac{\theta\xi}{2}\right)\right.\\
&\quad \left. - \tanh^2\left(\frac{\theta\xi}{2}\right) + 2\tanh\left(\frac{\theta\xi}{2}\right) + 1\right),\\
n(\xi)n'(\xi) &= \frac{\theta}{16}\left(-\tanh^5\left(\frac{\theta\xi}{2}\right) - 3\tanh^4\left(\frac{\theta\xi}{2}\right) - 2\tanh^3\left(\frac{\theta\xi}{2}\right) + 2\tanh^2\left(\frac{\theta\xi}{2}\right)\right.\\
&\quad \left. + \tanh\left(\frac{\theta\xi}{2}\right) + 3\right),\\
n(\xi)n''(\xi) &= \frac{\theta^2}{32}\left(3\tanh^6\left(\frac{\theta\xi}{2}\right) + 8\tanh^5\left(\frac{\theta\xi}{2}\right) + 3\tanh^4\left(\frac{\theta\xi}{2}\right) - 8\tanh^3\left(\frac{\theta\xi}{2}\right)\right.\\
&\quad \left. - 7\tanh^2\left(\frac{\theta\xi}{2}\right) + 1\right).
\end{aligned} \tag{96}$$

Inserting the above formulas into Equation (71) and equating powers of the same order in $\tanh(\frac{\theta\xi}{2})$, we arrive at the following relations after some direct but lengthy computations:

$$\tanh^6\left(\frac{\theta\xi}{2}\right) : 3 + 2\delta + \frac{6}{\nu}\left(\lambda + \frac{\nu}{2}\right) = 0, \tag{97}$$

$$\tanh^5\left(\frac{\theta\xi}{2}\right) : (4 + 3\delta)\theta - 2k + \frac{18}{\nu}\left(\lambda + \frac{\nu}{2}\right)\theta = 0, \tag{98}$$

$$\tanh^4\left(\frac{\theta\xi}{2}\right) : (3 + 4\delta)\theta^2 - 12k\theta + 8\left(\omega - \frac{\nu}{2}\right) + 30\left(\lambda + \frac{\nu}{2}\right) = 0, \tag{99}$$

$$\tanh^3\left(\frac{\theta\xi}{2}\right) : -(2+\delta)\theta^2 - 2k\theta + 8\left(\omega - \frac{\nu}{2}\right) + 10\left(\lambda + \frac{\nu}{2}\right) = 0, \tag{100}$$

$$\tanh^2\left(\frac{\theta\xi}{2}\right) : -(7+6\delta)\theta^2 + 8k\theta + 48\left(\omega - \frac{\nu}{2}\right) + 30\left(\lambda + \frac{\nu}{2}\right) = 0, \tag{101}$$

$$\tanh\left(\frac{\theta\xi}{2}\right) : -2\delta\theta^2 + 4k\theta + 32\left(\omega - \frac{\nu}{2}\right) + 12\left(\lambda + \frac{\nu}{2}\right) = 0, \tag{102}$$

$$\tanh^0\left(\frac{\theta\xi}{2}\right) : \theta^2 + 12k\theta + 8\left(\omega - \frac{\nu}{2}\right) + 2\left(\lambda + \frac{\nu}{2}\right) = 0. \tag{103}$$

When using in (98) the information provided by (97), the parameter k can be identified in terms of θ as

$$k = -\left(\frac{5+3\delta}{2}\right)\theta. \tag{104}$$

Now, plugging (104) and (97) into (99), we obtain the following formula for ω after rearranging terms and simplifying:

$$\omega = \frac{\nu(3+\delta)}{3}. \tag{105}$$

At this stage, the conditions (100) and (101) become redundant. In the same way as before, plugging now (104) and (97) into (102) we find that

$$\delta = -\frac{5}{3}, \tag{106}$$

which makes (103) also redundant. As a consequence, it is easily deduced from (97) that

$$\lambda = -\frac{4\nu}{9}. \tag{107}$$

Finally, the expressions for k and ω (cf. (104) and (105)) can be restated as

$$k = 0, \quad \omega = -\lambda. \tag{108}$$

■

Also, (69) along with (89) and (94) provide the allowed values for the wave velocity v established in (26). This concludes the proof of Theorem 3.

Remark 2. Notice that the values for δ, λ, and ω found in Lemma 3(ii) and (iii) do satisfy the criterium of modulational stability established in Theorem 2.

5 | CONCLUDING REMARKS

In this paper, we have derived a parabolic–parabolic Keller–Segel type model with logistic source and nonstandard chemical production–degradation mechanism, namely,

$$\begin{cases} n_t = n_{xx} - (nS_x)_x + \nu n(1-n) \\ S_t = S_{xx} + f_{\lambda\delta}(n, n_x, n_{xx}), \end{cases} \tag{109}$$

where

$$f_{\lambda\delta}(n, n_x, n_{xx}) = \lambda n + \frac{\delta}{4}\left(\frac{n_{xx}}{n}\right) + \left(\frac{4-\delta}{8}\right)\frac{n_x^2}{n^2}. \tag{110}$$

Indeed it has been shown that, up to a gauge transformation of logarithmic type in S, the system (109)–(110) coincides with the modulus-argument hydrodynamic system associated with a diffusive version of the modified complex Ginzburg–Landau equation (Theorem 1). Then, a suitable choice of the involved parameters satisfying the modulational stability criterium (Theorem 2), namely, $\delta = -\frac{5}{3}$ and $\omega = -\lambda = \frac{4\nu}{9}$, has led us to the eventual construction of a nonlocalized solitary wave solution to this equation, whose modulus square n is an antikink-shaped traveling wave solution of the first equation in (109):

$$n(\xi) = \frac{1}{4}\left(1 - \tanh\left(\sqrt{\frac{\nu}{12}}\,\xi\right)\right)^2, \quad \xi = x - 5\sqrt{\frac{\nu}{12}}\,t \tag{111}$$

(see (25) in Theorem 3), and whose argument S, which plays the role of the chemical concentration in the Keller–Segel context, goes as $\log(n)$:

$$S(t,x) = \frac{1}{2}\log(n(\xi)) - \frac{4\nu}{9}t \tag{112}$$

(see (27) in Theorem 3). In this situation, the chemical production–degradation function (110) under the influence of (112) can be red as

$$f(n, S_x, S_{xx}) = -\frac{5}{6}S_{xx} + \frac{7}{6}S_x^2 - \frac{4\nu}{9}n, \tag{113}$$

which inserted into the second equation in (109) gives rise to

$$S_t = \frac{1}{6}S_{xx} + \frac{7}{6}S_x^2 - \frac{4\nu}{9}n. \tag{114}$$

As a consequence, $\lambda < 0$ can be interpreted as the consumption rate of the chemical agent while the production term is represented by a Hamilton–Jacobi type nonlinearity of order gradient square, instead of the Malthusian growth characteristic of the standard Keller–Segel dynamics. In a biomechanical setting, viscous Hamilton–Jacobi fluxes of the type $F_{\alpha,\beta}(u) = \alpha u_{xx} + \beta u_x^2$ are known to model the evolution of growing interfaces, where the Laplacian term describes relaxational diffusion while the gradient square term typically accounts for lateral growth.

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