Karapınar et al. *Advances in Difference Equations* (2021) 2021:339
https://doi.org/10.1186/s13662-021-03491-w

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theory in the setting of $(\alpha, \beta, \psi, \phi)$-interpolative contractions

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Abstract

In this manuscript we introduce the notion of $(\alpha, \beta, \psi, \phi)$-interpolative contraction that unifies and generalizes significant concepts: Proinov type contractions, interpolative contractions, and ample spectrum contraction. We investigate the necessary and sufficient conditions to guarantee existence and uniqueness of the fixed point of such mappings.

MSC: Primary 47H10; 47H09; secondary 54H25

Keywords: Interpolative contraction; Kannan contraction; Proinov contraction; Ample spectrum contraction; Fixed point

1 Introduction

For many years researchers in the field of fixed point theory have known and employed contractivity conditions of the type:

$$\psi\bigl(d(\mathcal{T}u, \mathcal{T}v)\bigr) \le \phi\bigl(d(u, v)\bigr) \quad \text{for all } u, v \in \mathcal{X},$$

where $(\mathcal{X}, d)$ is a metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a self-mapping, and $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are two auxiliary functions satisfying certain general conditions whose main aim is to help in the task of proving existence and, in most of cases, uniqueness of fixed points of the mapping $\mathcal{T}$ (Boyd and Wong [1], Geraghty [2], Amini-Harandi and Petruşel [3], Jleli and Samet [4], Wardowski [5], etc.) This is only a simple, but interesting, way to generalize the original Banach's contractivity condition that firstly appeared in [6]. After that, many other ways to generate the Banach contractive mapping principle have appeared [6], based mainly on three distinct directions: on the abstract metric structure of the underlying space, on the formulation of novel increasingly general contractive conditions, and also on the introduction of algebraic (no necessarily metric) structures (binary relations, partial orders [7], fuzzy metrics [8, 9], etc.)

In the line of research regarding the generalization of contractive conditions, it is worth highlighting the evolution of two distinct models. On the one hand, there are models that are only based on the usage of two main terms: the distance between any two points in space and the distance between their respective images. Along this line, it is worth highlighting the efforts due to Boyd and Wong [1], etc. More recently, Khojasteh et al. [10]

introduced the notion of *simulation function*, which was later subtly modified by Roldán López de Hierro et al. in [11]. After a series of successive research papers deepening this area (see [12–14]), this last author and Shahzad presented the notion of *ample spectrum contraction* in [15] with a double aim: to analyze the essential properties, from an abstract point of view, that any contraction that may arise in the future must satisfy and generalize as many results as possible in the field of fixed point theory.

On the other hand, the efforts made in the line of research making new terms intervene in the contractive condition (for example, the distance between any point and its image) are also noteworthy. In this line we can highlight the results given by Ćirić, Kannan [16, 17], Hardy-Rogers [18], etc. Inspired by Kannan contractions and the notion of interpolation triple of type γ described in [19], Karapınar introduced in [20] the *interpolative Kannan type contractions*. Such advance was later extended in [21] with the help of simulation functions (see [10, 11]). Very recently, Aydi et al. [22] have presented the *ω-interpolative Ćirić–Reich–Rus-type contractions* by including in the contractivity condition some new terms and algebraic structures. This last class of contractive applications inspires some of the content that we will present in this paper.

Returning to the first commented line of research (employing only two terms in the contractive condition), last year Proinov [23] published a manuscript in which new results about existence and uniqueness of fixed points were demonstrated. This paper attracted much attention among researchers in this area, essentially due to the weak hypotheses that were used to develop the main results. For example, it was demonstrated that the involved Picard sequence was a Cauchy sequence without the need to establish monotonic conditions on the functions ψ and ϕ that appeared in the contractive condition. Furthermore, it was shown that a large number of contractions introduced by other authors (Amini-Harandi and Petrusel, Moradi, Geragthy, Jleli and Samet, Wardowski) could be seen as Proinov contractions.

In this paper, drawing inspiration from the above-mentioned works, we introduce a new class of contractions that generalize the Proinov contractions. Namely, we use a Ćirić–Reich–Rus type contractive condition in which new terms appear (distinct to the two used in the first discussed line of research). For this class of operators, we introduce new results about existence and uniqueness of fixed points. Furthermore, before that, we point out that this class properly extends the family of Proinov contractions since, specifically, we proved that all Proinov contractions are particular cases of the Roldán López de Hierro and Shahzad ample spectrum contractions.

2 Preliminaries

In this section we introduce the notation we are going to use, and we present the three classes of contractions in which this paper is inspired.

2.1 Algebraic background

The framework in which we will introduce our main results from now on is a metric space $(\mathcal{X}, d)$ endowed with a self-mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$. An element $u \in \mathcal{X}$ such that $\mathcal{T}u = u$ is a *fixed point of* $\mathcal{T}$. The set of all fixed points of $\mathcal{T}$ is denoted by $\mathrm{Fix}(\mathcal{T})$.

Henceforth, let $\mathbb{N} = \{1, 2, 3, \dots\}$ stand for the set of all positive integer numbers. Following [24], a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{X}$ satisfying that $u_k \neq u_p$ for all $k, p \in \mathbb{N}$, $k \neq p$, is called an *infinite* sequence. If $\{u_k\}_{k\in\mathbb{N}}$ is given by $u_{k+1} = \mathcal{T}u_k$ for all $k \in \mathbb{N}$, then that sequence is known as a *Picard sequence of* $\mathcal{T}$.

Proposition 1 ([18], Proposition 2) *Let $\{u_k\}_{k\in\mathbb{N}}$ be a Picard sequence in a metric space $(\mathcal{X},d)$ such that $\{d(u_k,u_{k+1})\}\to 0$. If there are $k_1,k_2\in\mathbb{N}$ such that $k_1<k_2$ and $u_{k_1}=u_{k_2}$, then there are $k_0\in\mathbb{N}$ and $u\in\mathcal{X}$ such that $u_k=u$ for all $k\geq k_0$ (that is, $\{u_k\}_{k\in\mathbb{N}}$ is constant from a term onwards). In such a case, u is a fixed point of the self-mapping for which $\{u_k\}_{k\in\mathbb{N}}$ is a Picard sequence.*

Lemma 1 ([23, 25, 26]) *Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in a metric space $(\mathcal{X},d)$ such that $\{d(u_k,u_{k+1})\}\to 0$ as $k\to+\infty$. If the sequence $\{u_k\}_{k\in\mathbb{N}}$ is not d-Cauchy, then there exist $e>0$ and two partial subsequences $\{u_{p(k)}\}_{k\in\mathbb{N}}$ and $\{u_{q(k)}\}_{k\in\mathbb{N}}$ of $\{u_k\}_{k\in\mathbb{N}}$ such that*

$$p(k)<q(k)<p(k+1) \quad \text{and} \quad e<d(u_{p(k)+1},u_{q(k)+1}) \quad \text{for all } k\in\mathbb{N},$$

$$\begin{aligned}\lim_{k\to+\infty} d(u_{p(k)},u_{q(k)}) &= \lim_{k\to+\infty} d(u_{p(k)+1},u_{q(k)}) = \lim_{k\to+\infty} d(u_{p(k)},u_{q(k)+1}) \\ &= \lim_{k\to+\infty} d(u_{p(k)+1},u_{q(k)+1}) = e.\end{aligned}$$

2.2 Roldán–Shahzad ample spectrum contractions

In [15], Roldán López de Hierro and Shahzad introduced a great class of contractions that generalized many previous kinds of contractions with a particular property: they only used the terms $d(u,v)$ and $d(\mathcal{T}u,\mathcal{T}v)$ on their contractivity conditions. In the following definitions, $\mathcal{S}$ represents a binary relation on $\mathcal{X}$ and $\mathcal{S}^*$ is given by $u\mathcal{S}^*v$ when $u,v\in\mathcal{X}$ and $u\neq v$.

Definition 1 Let $\{a_k\}$ and $\{b_k\}$ be two sequences of real numbers. We say that $\{(a_k,b_k)\}$ is a $(\mathcal{T},\mathcal{S}^*)$-*sequence* if there exist two sequences $\{u_k\},\{v_k\}\subseteq\mathcal{X}$ such that

$$u_k\mathcal{S}^*v_k, \qquad \mathcal{T}u_k\mathcal{S}^*\mathcal{T}v_k, \qquad a_k=d(\mathcal{T}u_k,\mathcal{T}v_k)>0 \quad \text{and} \quad b_k=d(u_k,v_k)>0$$

for all $k\in\mathbb{N}$.

Definition 2 ([15], Definition 4) We will say that $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ is an *ample spectrum contraction* if there exists a function $\varrho:\mathcal{A}\times\mathcal{A}\to\mathbb{R}$ such that $\mathcal{T}$ and ϱ satisfy the following four conditions:

($\mathcal{B}_1$) $\mathcal{A}$ is nonempty and $\{d(u,v)\in[0,+\infty): u,v\in\mathcal{X}, u\mathcal{S}^*v\}\subseteq\mathcal{A}$.

($\mathcal{B}_2$) If $\{u_k\}\subseteq\mathcal{X}$ is a Picard $\mathcal{S}$-nondecreasing sequence of $\mathcal{T}$ such that

$$u_k\neq u_{k+1} \quad \text{and} \quad \varrho\big(d(u_{k+1},u_{k+2}),d(u_k,u_{k+1})\big)\geq 0 \quad \text{for all } k\in\mathbb{N},$$

then $\{d(u_k,u_{k+1})\}\to 0$.

($\mathcal{B}_3$) If $\{(a_k,b_k)\}\subseteq\mathcal{A}\times\mathcal{A}$ is a $(\mathcal{T},\mathcal{S}^*)$-sequence such that $\{a_k\}$ and $\{b_k\}$ converge to the same limit $e\geq 0$ and verifying that $e<a_k$ and $\varrho(a_k,b_k)\geq 0$ for all $k\in\mathbb{N}$, then $e=0$.

($\mathcal{B}_4$) $\varrho(d(\mathcal{T}u,\mathcal{T}v),d(u,v))\geq 0$ for all $u,v\in\mathcal{X}$ such that $u\mathcal{S}^*v$ and $\mathcal{T}u\mathcal{S}^*\mathcal{T}v$.

In such a case, we will say that $\mathcal{T}$ is an *ample spectrum contraction with respect to $\mathcal{S}$ and ϱ*.

Some additional properties were taken into account in order to introduce very general results in the field of fixed point theory.

($\mathcal{B}_2'$) If $u_1, u_2 \in X$ are two points such that

$$\mathcal{T}^k u_1 \mathcal{S}^* \mathcal{T}^k u_2 \quad \text{and} \quad \varrho\left(d\left(\mathcal{T}^{k+1}u_1, \mathcal{T}^{k+1}u_2\right), d\left(\mathcal{T}^k u_1, \mathcal{T}^k u_2\right)\right) \geq 0 \quad \text{for all } n \in \mathbb{N},$$

then $\{d(\mathcal{T}^k u_1, \mathcal{T}^k u_2)\} \to 0$.

($\mathcal{B}_5$) If $\{(a_k, b_k)\}$ is a $(\mathcal{T}, \mathcal{S}^*)$-sequence such that $\{b_k\} \to 0$ and $\varrho(a_k, b_k) \geq 0$ for all $k \in \mathbb{N}$, then $\{a_k\} \to 0$.

Although it was not explicitly enunciated in [15], we state here the following result which is a simple consequence of the main theorems given in [15]. The trivial binary relation $\mathcal{S}_X$ on X is given by $x\mathcal{S}_X y$ for all $x, y \in X$.

Theorem 1 ([15]) *Each ample spectrum contraction w.r.t. $\mathcal{S}_X$ from a complete metric space into itself satisfying ($\mathcal{B}_2'$) and ($\mathcal{B}_5$) has a unique fixed point. In fact, each Picard sequence converges to such a fixed point.*

Proof The existence of fixed points follows from Theorem 2 in [15] using item (b) and the uniqueness follows from Theorem 3 in [15] because we assume condition ($\mathcal{B}_2'$). □

2.3 ω-Interpolative Ćirić–Reich–Rus-type contractions

The notion of ω-interpolative Ćirić–Reich–Rus-type contraction was firstly inspired by the celebrated Kannan type contractions. A self-mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ is a Kannan contraction if there is $k \in [0, 0.5)$ such that

$$d(\mathcal{T}u, \mathcal{T}v) \leq k\left(d(u, \mathcal{T}u) + d(v, \mathcal{T}v)\right) \quad \text{for all } u, v \in \mathcal{X},$$

where $(\mathcal{X}, d)$ is a metric space. The reason why Kannan contractions were considered as an outstanding generalization of the Banach contractions lies in the fact that Kannan contractions are not necessarily continuous. Having in mind such kind of contractions and the concept of interpolation triple of type γ described in [19], Karapınar introduced in [20] the notion of *interpolative Kannan type contraction*, which corresponds to mappings $\mathcal{T}: \mathcal{X} \to \mathcal{X}$, from a metric space $(\mathcal{X}, d)$ into itself, such that the following condition is fulfilled:

$$d(\mathcal{T}u, \mathcal{T}v) \leq k d(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{1-\alpha} \quad \text{for all } u, v \in \mathcal{X},$$

where α is a constant belonging to the interval $(0, 1)$. This author was able to prove that each interpolative Kannan type contraction defined on a complete metric space has a unique fixed point.

Other significant advance in this line of study was the so-called *α-admissible interpolative Rus–Reich–Ćirić type $\mathcal{Z}$-contractions* which involve a contractivity condition as follows:

$$\zeta\left(\alpha(u, v) d(\mathcal{T}u, \mathcal{T}v), \psi\left(R(u, v)\right)\right) \geq 0 \quad \text{for all } u, v \in \mathcal{X} \text{ with } u \neq \mathcal{T}u,$$

where

$$R(u, v) = d(u, v)^{\beta} d(u, \mathcal{T}u)^{\gamma} d(v, \mathcal{T}v)^{1-\gamma-\beta} \quad \text{for all } u, v \in \mathcal{X},$$

$\zeta \in \mathcal{Z}$ is a simulation function (see [10, 11]), ψ and α are appropriate functions, and β and γ are positive constants such that $\gamma + \beta < 1$ (see [21] for more details).

Finally, it is worth mentioning the class of contractions that inspire this work. Following Definition 2 in [22], given a metric space $(\mathcal{X}, d)$, a function $\omega : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$, two positive real $\beta, \gamma > 0$ such that $\beta + \gamma < 1$, a function $\psi \in \Psi$, we say that a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is an *ω-interpolative Ćirić–Reich–Rus-type contraction* if it verifies

$$\omega(u, v)d(\mathcal{T}u, \mathcal{T}v) \leq \psi\left(d(u, v)^{\beta} d(u, \mathcal{T}u)^{\gamma} d(v, \mathcal{T}v)^{1-\gamma-\beta}\right)$$

for all $u, v \in \mathcal{X} \setminus \operatorname{Fix}(\mathcal{T})$. For the sake of completeness, we clarify that here Ψ represents the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following condition:

$$\sum_{p=1}^{+\infty} \psi^{p}(s) < +\infty \quad \text{for all } s > 0.$$

Theorem 2 ([22, Theorem 3]) *Suppose that a continuous self-mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is ω-orbital admissible also an ω-*interpolative Ćirić–Reich–Rus-type contraction *on a complete metric space $(\mathcal{X}, d)$. If there is a point $u \in \mathcal{X}$ such that $\omega(u, \mathcal{T}u) \geq 1$, then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.*

The notion of ω-orbital admissible mapping can be found in [22, 27].

2.4 Proinov contractions

Very recently, Proinov announced some results which unify many known results.

Theorem 3 (Proinov [23], Theorem 3.6) *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$\psi\left(d(\mathcal{T}u, \mathcal{T}v)\right) \leq \phi\left(d(u, v)\right) \quad \text{for all } u, v \in \mathcal{X} \text{ with } d(\mathcal{T}u, \mathcal{T}v) > 0, \tag{2.1}$$

where the functions $\psi, \phi : (0, +\infty) \to \mathbb{R}$ satisfy the following conditions:

(a_1) *ψ is nondecreasing;*

(a_2) *$\phi(s) < \psi(s)$ for any $s > 0$;*

(a_3) *$\limsup_{s \to e^+} \phi(s) < \lim_{s \to e^+} \psi(s)$ for any $e > 0$.*

Then $\mathcal{T}$ has a unique fixed point $v_0 \in \mathcal{X}$ and the iterative sequence $\{\mathcal{T}^k u\}_{k \in \mathbb{N}}$ converges to v_0 for every $u \in \mathcal{X}$.

3 $(\alpha, \beta, \psi, \phi)$-Interpolative contractions

As we have commented in the introduction, Theorem 3 has attracted the attention of many researchers in the field of fixed point theory due to the very weak hypotheses that are assumed in the statement in order to guarantee the existence and uniqueness of fixed points of the involved operator $\mathcal{T}$. Furthermore, this author also showed in [23] that many fixed point results introduced along the last years in nonlinear analysis can be easily deduced from Theorem 3. Due to its great impact, we will say that a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ from a metric space $(\mathcal{X}, d)$ into itself is a *Proinov contraction* if there are two functions $\psi, \phi : (0, +\infty) \to \mathbb{R}$, satisfying the conditions (a_1), (a_2), and (a_3), such that contractivity condition (2.1) holds. Theorem 3 guarantees that each Proinov contraction from a complete metric space into itself has a unique fixed point.[1] Inspired by this class of contractions and also by ω-interpolative Ćirić–Reich–Rus-type contractions [22], in this paper we are going to study the existence and uniqueness of fixed points of self-mapping satisfying a more general contractivity condition.

Definition 3 Let $(\mathcal{X}, d)$ be a metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be an $(\alpha, \beta, \psi, \phi)$-*interpolative contraction* if there exist $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and a pair of functions $\psi, \phi : (0, +\infty) \to \mathbb{R}$ such that:

$$\psi\big(d(\mathcal{T}u, \mathcal{T}v)\big) \leq \phi\big(d(u,v)^{1-\alpha-\beta} d(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{\beta}\big) \tag{3.1}$$

for any $u, v \in \mathcal{X} \setminus \operatorname{Fix}(\mathcal{T})$ with $d(\mathcal{T}u, \mathcal{T}v) > 0$.

Although at a first sight contractivity condition (3.1) seems to be similar to (2.1), we would like to highlight that they are definitively distinct in nature. The main difference lies in the fact that contractive condition (2.1) only uses the terms $d(u, v)$ and $d(\mathcal{T}u, \mathcal{T}v)$, while in condition (3.1) the terms $d(u, \mathcal{T}u)$ and $d(v, \mathcal{T}v)$ also appear. This fact forces us to assume that the initial points u and v that are used in contractivity condition (3.1) are not fixed points of the mapping $\mathcal{T}$, since the function ϕ is not defined for $t = 0$. Furthermore, in such a case, if $\alpha = 0$ and u is a fixed point of $\mathcal{T}$, the argument of ϕ in the right-hand side of (3.1) could contain an algebraic indetermination of type 0^0. To show that this difference is very important, we start our main results by showing that each Proinov contraction is an ample spectrum contraction.

Theorem 4 *Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a Proinov contraction from a metric space $(\mathcal{X}, d)$ into itself associated with the functions $\psi, \phi : (0, +\infty) \to \mathbb{R}$. Let us define*:

- *the binary relation $\mathcal{S}_X$ on X given by $u\mathcal{S}_X v$ for all $u, v \in \mathcal{X}$,*
- $A = (0, +\infty)$,
- $\varrho_{\psi,\phi} : (0, +\infty) \times (0, +\infty) \to \mathbb{R}$,

$$\varrho_{\psi,\phi}(t, s) = \phi(s) - \psi(t) \quad \text{for all } t, s \in (0, +\infty).$$

Then $\mathcal{T}$ is an ample spectrum contraction with respect to $\mathcal{S}$ and $\varrho_{\psi,\phi}$ satisfying additional properties $(\mathcal{B}_2')$ and $(\mathcal{B}_5)$.

Proof We check all the assumptions given in Definition 2.

$(\mathcal{B}_1)$ It is trivial because $A = (0, +\infty)$.

$(\mathcal{B}_2)$ Let $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ be a Picard $\mathcal{S}$-nondecreasing sequence of $\mathcal{T}$ such that

$$u_k \neq u_{k+1} \quad \text{and} \quad \varrho_{\psi,\phi}\big(d(u_{k+1}, u_{k+2}), d(u_k, u_{k+1})\big) \geq 0 \quad \text{for all } k \in \mathbb{N}.$$

Then Lemma 2 implies that $\{d(u_k, u_{k+1})\}_{k\in\mathbb{N}} \to 0$.

[1] Notice that this author forgot to include the completeness of the metric space in his main results in [23].

($\mathcal{B}_2'$) Let $u_1, v_1 \in \mathcal{X}$ be two initial points such that the Picard sequences $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$, defined by $u_{k+1} = \mathcal{T}u_k$ and $v_{k+1} = \mathcal{T}v_k$ for all $k \in \mathbb{N}$, satisfy

$$u_k \mathcal{S}^* v_k \quad \text{and} \quad \varrho_{\psi,\phi}\big(d(u_{k+1}, v_{k+1}), d(u_k, v_k)\big) \geq 0 \quad \text{for all } k \in \mathbb{N}.$$

Therefore, for all $k \in \mathbb{N}$,

$$0 \leq \varrho_{\psi,\phi}\big(d(u_{k+1}, v_{k+1}), d(u_k, v_k)\big) = \phi\big(d(u_k, v_k)\big) - \psi\big(d(u_{k+1}, v_{k+1})\big).$$

Taking into account condition (a_2) and the fact that $d(u_k, v_k) > 0$, the previous inequality means that

$$\psi\big(d(u_{k+1}, v_{k+1})\big) \leq \phi\big(d(u_k, v_k)\big) < \psi\big(d(u_k, v_k)\big) \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

Since the function ψ is nondecreasing, then $d(u_{k+1}, v_{k+1}) < d(u_k, v_k)$ for all $k \in \mathbb{N}$. Let $e = \lim_{k\to+\infty} d(u_k, v_k) \geq 0$. Notice that $e < d(u_k, v_k)$ for all $k \in \mathbb{N}$. To prove that $e = 0$, assume that $e > 0$. In this case the following limits are equal:

$$\lim_{k\to+\infty} \psi\big(d(u_k, v_k)\big) = \lim_{k\to+\infty} \psi\big(d(u_{k+1}, v_{k+1})\big) = \lim_{s\to e^+} \psi(s) \geq \psi(e).$$

Then (3.2) leads to

$$\lim_{k\to+\infty} \phi\big(d(u_k, v_k)\big) = \lim_{s\to e^+} \psi(s),$$

which contradicts (a_3) because

$$\lim_{s\to e^+} \psi(s) = \lim_{k\to+\infty} \phi\big(d(u_k, v_k)\big) \leq \lim_{s\to e^+} \phi(s).$$

As a consequence, $e = 0$, that is, $\{d(u_k, v_k)\} \to 0$.

($\mathcal{B}_2$) It directly follows from ($\mathcal{B}_2'$) by choosing an arbitrary point $u_1 \in \mathcal{X}$ and using $v_1 = \mathcal{T}u_1 = u_2$ in ($\mathcal{B}_2'$).

($\mathcal{B}_3$) Let $\{(a_k, b_k)\} \subseteq A \times A$ be a $(\mathcal{T}, \mathcal{S}^*)$-sequence such that $\{a_k\}$ and $\{b_k\}$ converge to the same limit $e \geq 0$ and verifying that $e < a_k$ and $\varrho_{\psi,\phi}(a_k, b_k) \geq 0$ for all $k \in \mathbb{N}$. Therefore, for all $k \in \mathbb{N}$, $0 \leq \varrho_{\psi,\phi}(a_k, b_k) = \phi(b_k) - \psi(a_k)$, so $\psi(a_k) \leq \phi(b_k)$. Since $a_k > 0$ and $b_k > 0$, then (a_2) implies that

$$\psi(a_k) \leq \phi(b_k) < \psi(b_k) \quad \text{for all } k \in \mathbb{N}. \tag{3.3}$$

As ψ is nondecreasing, $a_k < b_k$, so $e < a_k < b_k$ for all $k \in \mathbb{N}$. Taking into account that

$$\lim_{k\to+\infty} \psi(a_k) = \lim_{k\to+\infty} \psi(b_k) = \lim_{s\to e^+} \psi(s) \geq \psi(e),$$

property (3.3) leads to

$$\lim_{k\to+\infty} \phi(b_k) = \lim_{s\to e^+} \psi(s).$$

However, if $e > 0$, we deduce a contradiction with (a_3) because

$$\lim_{s\to e^+} \psi(s) = \lim_{k\to+\infty} \phi(b_k) \le \lim_{s\to e^+} \phi(s).$$

Therefore, $e = 0$.

$(\mathcal{B}_4)$ Let $u, v \in \mathcal{X}$ be such that $u\mathcal{S}^* v$ and $\mathcal{T}u\mathcal{S}^*\mathcal{T}v$, that is, $u \neq v$ and $\mathcal{T}u \neq \mathcal{T}v$. Therefore contractivity condition (2.1) implies that

$$\varrho_{\psi,\phi}\big(d(\mathcal{T}u, \mathcal{T}v), d(u, v)\big) = \phi\big(d(u, v)\big) - \psi\big(d(\mathcal{T}u, \mathcal{T}v)\big) \ge 0.$$

$(\mathcal{B}_5)$ Let $\{(a_k, b_k)\}$ be a $(T, \mathcal{S}^*)$-sequence such that $\{b_k\} \to 0$ and $\varrho_{\psi,\phi}(a_k, b_k) \ge 0$ for all $k \in \mathbb{N}$. We can repeat the argument of (3.3), so we deduce that $a_k < b_k$ for all $k \in \mathbb{N}$. Hence $\{a_k\} \to 0$. □

Corollary 1 *Each Proinov contraction is an ample spectrum contraction.*

Corollary 2 *Theorem 3 directly follows from Theorem 1.*

Theorem 4 implies that $(\alpha, \beta, \psi, \phi)$-interpolative contractions belong to a category of contractions that are very distinct in nature to the class of Proinov contractions: each Proinov contraction is a particular case of an ample spectrum contraction; however, for the moment, it is an open problem to study the possible relationship between $(\alpha, \beta, \psi, \phi)$-interpolative contractions and ample spectrum contractions.

4 Fixed point theorems for $(\alpha, \beta, \psi, \phi)$-interpolative contractions

In this section we introduce some fixed point results for $(\alpha, \beta, \psi, \phi)$-interpolative contractions. The following lemma is important in order to show later that a Picard sequence is Cauchy.

Lemma 2 *Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an $(\alpha, \beta, \psi, \phi)$-interpolative contraction from a metric space $(\mathcal{X}, d)$ into itself, and suppose that the functions ϕ and ψ satisfy, at least, one of the following conditions*:

$(\mathbb{A})$ *It holds, at the same time,*

- (a_1) *ψ is nondecreasing,*
- (a_2) *$\phi(s) < \psi(s)$ for all $s > 0$,*
- (a_3) *$\limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s)$ for all $e > 0$.*

$(\mathbb{B})$ *If $\{s_n\} \subset (0, +\infty)$ is a sequence such that $\psi(s_{k+1}) \le \phi(s_k^{1-\beta} s_{k+1}^{\beta})$ for all $k \in \mathbb{N}$, then $\{s_k\} \to 0$.*

Then $\mathcal{T}$ is an asymptotically regular mapping.

Proof Let $u \in \mathcal{X}$ be arbitrary, and let us define $u_1 = u$ and $u_{k+1} = \mathcal{T}u_k$ for all $k \in \mathbb{N}$. If there is $k_0 \in \mathbb{N}$ such that $u_{k_0} = u_{k_0+1}$, then u_{k_0} is a fixed point of $\mathcal{T}$. In such a case, $\{d(u_k, u_{k+1})\}_{k\ge k_0} = \{0\} \to 0$. On the contrary, suppose that $u_k \neq u_{k+1}$ for all $k \in \mathbb{N}$. Then each u_k is not a fixed point of $\mathcal{T}$ and also

$$d(u_k, u_{k+1}) > 0 \quad \text{and} \quad d(\mathcal{T}u_k, \mathcal{T}u_{k+1}) > 0 \quad \text{for all } k \in \mathbb{N}.$$

Applying contractivity condition (3.1), we deduce that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
\psi\left(d(u_{k+1},u_{k+2})\right) &= \psi\left(d(\mathcal{T}u_k,\mathcal{T}u_{k+1})\right) \\
&\leq \phi\left(d(u_k,u_{k+1})^{1-\alpha-\beta} d(u_k,\mathcal{T}u_k)^{\alpha} d(u_{k+1},\mathcal{T}u_{k+1})^{\beta}\right) \\
&= \phi\left(d(u_k,u_{k+1})^{1-\alpha-\beta} d(u_k,u_{k+1})^{\alpha} d(u_{k+1},u_{k+2})^{\beta}\right) \\
&= \phi\left(d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta}\right).
\end{aligned}$$

If we define $s_k = d(u_k,u_{k+1})$ for all $k \in \mathbb{N}$, the previous inequality means that the sequence $\{s_k\}$ satisfies $\psi(s_{k+1}) \leq \phi(s_k^{1-\beta} s_{k+1}^{\beta})$ for all $k \in \mathbb{N}$. Under condition $(\mathbb{B})$, we deduce that $\{d(u_k,u_{k+1})\} = \{s_k\} \to 0$, and this completes the proof.

Next, suppose that $(\mathbb{A})$ holds. Since $d(u_k,u_{k+1}) > 0$ and $d(u_{k+1},u_{k+2}) > 0$, property (a_2) guarantees that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
\psi\left(d(u_{k+1},u_{k+2})\right) &\leq \phi\left(d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta}\right) \\
&< \psi\left(d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta}\right).
\end{aligned} \tag{4.1}$$

Taking into account that the function ψ is nondecreasing, it follows that

$$d(u_{k+1},u_{k+2}) < d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta} \quad \text{for all } k \in \mathbb{N},$$

which immediately leads to

$$d(u_{k+1},u_{k+2}) < d(u_k,u_{k+1}) \quad \text{for all } k \in \mathbb{N}.$$

As the sequence of positive real numbers $\{d(u_k,u_{k+1})\}_{k\in\mathbb{N}}$ is strictly decreasing, it is convergent. Then there is $e \geq 0$ such that

$$\left\{d(u_k,u_{k+1})\right\} \to e \quad \text{and} \quad d(u_k,u_{k+1}) > e \quad \text{for all } k \in \mathbb{N}.$$

To prove that $e = 0$, we assume, by contradiction, that $e > 0$. In this case, both sequences,

$$\left\{d(u_k,u_{k+1})\right\}_{k\in\mathbb{N}} \quad \text{and} \quad \left\{d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta}\right\}_{k\in\mathbb{N}},$$

are strictly decreasing and converging to e (notice that if $\beta = 0$, then $d(u_{k+1},u_{k+2})^{\beta} = 1$ for all $k \in \mathbb{N}$). As ψ is nondecreasing, the following limits exist and they are equal:

$$\begin{aligned}
\psi(e) \leq \lim_{s\to e^+} \psi(s) &= \lim_{k\to+\infty} \psi\left(d(u_{k+1},u_{k+2})\right) \\
&= \lim_{k\to+\infty} \psi\left(d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta}\right).
\end{aligned}$$

As a consequence, letting $k \to +\infty$ in (4.1), the following limit exists and it is finite:

$$\lim_{s\to e^+} \psi(s) = \lim_{k\to+\infty} \phi\left(d(u_k,u_{k+1})^{1-\beta} d(u_{k+1},u_{k+2})^{\beta}\right).$$

However, this fact contradicts the property (a_3) because

$$\lim_{s\to e^+}\psi(s)=\lim_{k\to+\infty}\phi\big(d(u_k,u_{k+1})^{1-\beta}d(u_{k+1},u_{k+2})^{\beta}\big)\le\limsup_{s\to e^+}\phi(s)<\lim_{s\to e^+}\psi(s).$$

As a consequence, $e=0$ and $\mathcal{T}$ is an asymptotically regular mapping. □

Remark 1

1 Notice that the previous result shows that condition (ii) is not necessary in [23, Lemma 3.2] when we suppose that (iii) holds.
2 We include assumption $(\mathbb{B})$ in the previous result because it can be easy to check in practical examples. For instance, if there is $\lambda\in[0,1)$ such that $\psi(s)=s$ and $\phi(s)=\lambda s$ for all $s\in(0,+\infty)$, then $(\mathbb{B})$ trivially holds and we do not need to check the properties given in $(\mathbb{A})$.

Our first main theorem in this paper is the following result.

Theorem 5 *Let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ be an (α,β,ψ,ϕ)-interpolative contraction from a complete metric space $(\mathcal{X},d)$ into itself, and suppose that the functions ϕ and ψ satisfy the following properties*:

(a_1) *ψ is nondecreasing,*
(a_2) *$\phi(s)<\psi(s)$ for all $s>0$,*
(a_3) *$\limsup_{s\to e^+}\phi(s)<\lim_{s\to e^+}\psi(s)$ for all $e>0$.*

Then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u\in\mathcal{X}$.

Proof First of all, let us prove that $\mathcal{T}$ is not fixed-points free. Let $u\in\mathcal{X}$ be an arbitrary initial point, and let us consider the Picard sequence $\{u_k\}_{k\in\mathbb{N}}$ of $\mathcal{T}$ starting from u, that is, define $u_1=u$ and $u_{k+1}=\mathcal{T}u_k$ for all $k\in\mathbb{N}$. If there is $k_0\in\mathbb{N}$ such that $u_{k_0}=u_{k_0+1}$, then u_{k_0} is a fixed point of $\mathcal{T}$. In this case, the first part of the proof is finished. Next suppose that $u_k\neq u_{k+1}$ for all $k\in\mathbb{N}$. Hence Lemma 2 implies that

$$\lim_{k\to+\infty}d(u_k,u_{k+1})\to 0\quad\text{and}\quad d(u_k,u_{k+1})>0\quad\text{for all }k\in\mathbb{N}.\tag{4.2}$$

Furthermore, Proposition 1 guarantees that each two terms of the Picard sequence $\{u_k\}_{k\in\mathbb{N}}$ are distinct, that is, $u_{k_1}\neq u_{k_2}$ for any $k_1,k_2\in\mathbb{N}$ such that $k_1\neq k_2$.

In order to prove that $\{u_k\}_{k\in\mathbb{N}}$ is a d-Cauchy sequence, we reason by contradiction. Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is not d-Cauchy. In such a case, Lemma 1 states that there exist $e>0$ and two partial subsequences $\{u_{p(k)}\}_{k\in\mathbb{N}}$ and $\{u_{q(k)}\}_{k\in\mathbb{N}}$ of $\{u_k\}_{k\in\mathbb{N}}$ such that

$$p(k)<q(k)<p(k+1)\quad\text{and}\quad e<d(u_{p(k)+1},u_{q(k)+1})\quad\text{for all }k\in\mathbb{N},\tag{4.3}$$

$$\begin{aligned}\lim_{k\to+\infty}d(u_{p(k)},u_{q(k)})&=\lim_{k\to+\infty}d(u_{p(k)+1},u_{q(k)})=\lim_{r\to+\infty}d(u_{p(k)},u_{q(k)+1})\\&=\lim_{r\to+\infty}d(u_{p(k)+1},u_{q(k)+1})=e.\end{aligned}\tag{4.4}$$

Applying contractivity condition (3.1) and property (a_2), we deduce that, for all $k\in\mathbb{N}$,

$$\psi\big(d(u_{p(k)+1},u_{q(k)+1})\big)=\psi\big(d(\mathcal{T}u_{p(k)},\mathcal{T}u_{q(k)})\big)$$

$$\begin{aligned}&\leq \phi\big(d(u_{p(k)},u_{q(k)})^{1-\alpha-\beta}d(u_{p(k)},\mathcal{T}u_{p(k)})^{\alpha}d(u_{q(k)},\mathcal{T}u_{q(k)})^{\beta}\big)\\&=\phi\big(d(u_{p(k)},u_{q(k)})^{1-\alpha-\beta}d(u_{p(k)},u_{p(k)+1})^{\alpha}d(u_{q(k)},u_{q(k)+1})^{\beta}\big)\\&<\psi\big(d(u_{p(k)},u_{q(k)})^{1-\alpha-\beta}d(u_{p(k)},u_{p(k)+1})^{\alpha}d(u_{q(k)},u_{q(k)+1})^{\beta}\big).\end{aligned}$$

Since ψ is nondecreasing, then, for all $k\in\mathbb{N}$,

$$d(u_{p(k)+1},u_{q(k)+1})<d(u_{p(k)},u_{q(k)})^{1-\alpha-\beta}d(u_{p(k)},u_{p(k)+1})^{\alpha}d(u_{q(k)},u_{q(k)+1})^{\beta},$$

which, in particular, means that, for all $k\in\mathbb{N}$,

$$\begin{aligned}&e<d(u_{p(k)+1},u_{q(k)+1})\\&\quad<d(u_{p(k)},u_{q(k)})^{1-\alpha-\beta}d(u_{p(k)},u_{p(k)+1})^{\alpha}d(u_{q(k)},u_{q(k)+1})^{\beta}.\end{aligned}\tag{4.5}$$

Taking into account (4.2)–(4.4), notice that

$$\lim_{k\to+\infty}d(u_{p(k)},u_{p(k)+1})^{\alpha}=\begin{cases}1, & \text{if } \alpha=0,\\0, & \text{if } \alpha>0;\end{cases}$$

$$\lim_{k\to+\infty}d(u_{q(k)},u_{q(k)+1})^{\beta}=\begin{cases}1, & \text{if } \beta=0,\\0, & \text{if } \beta>0.\end{cases}$$

If $\alpha>0$ or $\beta>0$, then, letting $k\to+\infty$ in (4.5), we observe that

$$0<e\leq\lim_{k\to+\infty}\big[d(u_{p(k)},u_{q(k)})^{1-\alpha-\beta}d(u_{p(k)},u_{p(k)+1})^{\alpha}d(u_{q(k)},u_{q(k)+1})^{\beta}\big]=0,$$

which contradicts the fact that $e>0$. To avoid this contradiction, assume that $\alpha=\beta=0$. In such a case, contractivity condition (3.1) is given by

$$\psi\big(d(\mathcal{T}u,\mathcal{T}v)\big)\leq\phi\big(d(u,v)^{1-\alpha-\beta}\big)\quad\text{for all } u,v\in\mathcal{X}\setminus\operatorname{Fix}(\mathcal{T})\text{ such that }\mathcal{T}u\neq\mathcal{T}v.$$

Although this condition is more general than condition (3.1) given in [23], the same arguments of the proof of Lemma 3.3 in that paper show that $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{X},d)$. Hence, we have proved that, in any case, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is d-Cauchy in the complete metric space $(\mathcal{X},d)$. As a consequence, there is $v\in\mathcal{X}$ such that $\{u_k\}\to v$, that is,

$$\lim_{k\to+\infty}d(u_k,v)=0.$$

We claim that $v\in\operatorname{Fix}(\mathcal{T})$. To prove it, assume by contradiction that $v\in\mathcal{X}\setminus\operatorname{Fix}(\mathcal{T})$, that is, $d(v,\mathcal{T}v)>0$. As the sequence $\{u_k\}_{k\in\mathbb{N}}$ is infinite, then there is $k_0\in\mathbb{N}$ such that $u_k\neq v$ and $u_k\neq\mathcal{T}v$ for all $k\geq k_0$. Using contractivity condition (3.1) and property (a_2), we deduce that, for all $k\geq k_0$,

$$\begin{aligned}\psi\big(d(u_{k+1},\mathcal{T}v)\big)&=\psi\big(d(\mathcal{T}u_k,\mathcal{T}v)\big)\\&\leq\phi\big(d(u_k,v)^{1-\alpha-\beta}d(u_k,\mathcal{T}u_k)^{\alpha}d(v,\mathcal{T}v)^{\beta}\big)\end{aligned}$$

$$= \phi\left(d(u_k,v)^{1-\alpha-\beta} d(u_k,u)^{\alpha} d(v,\mathcal{T}v)^{\beta}\right)$$
$$< \psi\left(d(u_k,v)^{1-\alpha-\beta} d(u_k,u)^{\alpha} d(v,\mathcal{T}v)^{\beta}\right).$$

Since ψ is nondecreasing, we deduce that

$$d(u_{k+1},\mathcal{T}v) < d(u_k,v)^{1-\alpha-\beta} d(u_k,u)^{\alpha} d(v,\mathcal{T}v)^{\beta} \quad \text{for all } k \geq k_0. \tag{4.6}$$

Since $1-\alpha-\beta > 0$, then $\lim_{k\to+\infty} d(u_k,v)^{1-\alpha-\beta} = 0$, so $\lim_{k\to+\infty} d(u_{k+1},\mathcal{T}v) = 0$. Hence the uniqueness of the limit of the sequence $\{u_k\}_{k\in\mathbb{N}}$ in the metric space $(\mathcal{X},d)$ guarantees that $\mathcal{T}v = v$, which is a contradiction because we have assumed that v is not a fixed point of $\mathcal{T}$.

This contradiction finally proves that v is a fixed point of $\mathcal{T}$ and, also, that the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to such a fixed point whatever the initial point $u \in \mathcal{X}$. □

In general, the uniqueness of the fixed point is not guaranteed. In fact, contractivity condition (3.1) that an (α,β,ψ,ϕ)-interpolative mapping satisfies cannot be employed to establish a relationship between any two distinct fixed points. The problem of condition

$$\psi\left(d(\mathcal{T}u,\mathcal{T}v)\right) \leq \phi\left(d(u,v)^{1-\alpha-\beta} d(u,\mathcal{T}u)^{\alpha} d(v,\mathcal{T}v)^{\beta}\right)$$

is that the functions $\psi,\phi : (0,+\infty) \to \mathbb{R}$ are not necessarily defined for $t = 0$. Therefore, if u (or v) is a fixed point of $\mathcal{T}$, then the right-hand side of the previous inequality is not well defined. On the one hand, when ψ is nondecreasing, the limit

$$\lim_{s\to 0^+} \psi(s)$$

exists but it could take the value $-\infty$ when the function ψ is not bounded from below. Avoiding this last case, it seems reasonable to assume that ψ is well defined on $t = 0$. On the other hand, there is not a clear constraint about the possible value of $\phi(0)$ because the condition $\phi < \psi$ does not lead to an additional condition unless we assume that both functions ϕ and ψ would be continuous at $t = 0$. In the following result we illustrate a simple way to deduce the uniqueness of the fixed point by assuming an additional condition.

Theorem 6 *Given a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ from a complete metric space $(\mathcal{X},d)$ into itself, suppose that there are two real numbers $\alpha,\beta \in [0,1)$ such that $\alpha + \beta < 1$ and a pair of functions $\psi,\phi : [0,+\infty) \to \mathbb{R}$ satisfying*

$$\psi\left(d(\mathcal{T}u,\mathcal{T}v)\right) \leq \phi\left(d(u,v)^{1-\alpha-\beta} d(u,\mathcal{T}u)^{\alpha} d(v,\mathcal{T}v)^{\beta}\right)$$

for any $u,v \in \mathcal{X}$ with $d(\mathcal{T}u,\mathcal{T}v) > 0$ (we agree here that $d(u,\mathcal{T}u)^{\alpha} = 1$ if $\alpha = 0$ and u is a fixed point of $\mathcal{T}$, and similarly with $d(v,\mathcal{T}v)^{\beta}$). Additionally, assume that the following conditions are fulfilled:

(a_1) *ψ is nondecreasing on $(0,+\infty)$;*
(a_2) *$\phi(s) < \psi(s)$ for all $s > 0$;*
(a_3) *$\limsup_{s\to e^+} \phi(s) < \lim_{s\to e^+} \psi(s)$ for all $e > 0$;*
(a_4) *if $s_0 \in [0,+\infty)$ verifies that $\psi(s_0) \leq \phi(0)$, then $s_0 = 0$.*

Then $\mathcal{T}$ has a unique fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to such a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof Let us consider the restricted functions $\psi' = \psi|_{(0,+\infty)}$, $\phi' = \phi|_{(0,+\infty)} : (0,+\infty) \to \mathbb{R}$. Then $\mathcal{T}$ is an $(\alpha,\beta,\psi',\phi')$-interpolative mapping. Since ψ' and ϕ' satisfy the auxiliary conditions (a_1), (a_2), and (a_3), Theorem 5 guarantees that the mapping $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$. To prove the uniqueness of the fixed point, suppose that there exist two distinct fixed points $u, v \in \mathcal{X}$ of $\mathcal{T}$. Then $d(\mathcal{T}u,\mathcal{T}v) = d(u,v) > 0$. In this case, the contractivity condition implies that

$$\psi\big(d(u,v)\big) = \psi\big(d(\mathcal{T}u,\mathcal{T}v)\big) \le \phi\big(d(u,v)^{1-\alpha-\beta} d(u,\mathcal{T}u)^{\alpha} d(v,\mathcal{T}v)^{\beta}\big). \tag{4.7}$$

We can reason as follows to obtain a contradiction.

- If $\alpha > 0$, then $d(u,\mathcal{T}u)^\alpha = 0$, and if $\beta > 0$, then $d(v,\mathcal{T}v)^\beta = 0$. In both cases, $\psi(d(u,v)) \le \phi(0)$. However, assumption (a_4) would imply that $d(u,v) = 0$, which contradicts the fact that $d(u,v) > 0$.
- If $\alpha = \beta = 0$, then (4.7) means that $\psi(d(u,v)) \le \phi(d(u,v))$, which contradicts (a_2) because $d(u,v) > 0$.

In any case, the obtained contradiction leads to the uniqueness of the fixed point of $\mathcal{T}$. □

Next we show some consequences of the previous main results.

Corollary 3 *Let $(\mathcal{X}, d)$ be a complete metric space, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. Suppose that there exist a constant $\alpha \in (0,1)$ and two functions $\psi, \phi : (0,+\infty) \to \mathbb{R}$, satisfying (a_1), (a_2), and (a_3), such that*

$$\psi\big(d(\mathcal{T}u,\mathcal{T}v)\big) \le \phi\big(d(u,v)^{1-\alpha} d(u,\mathcal{T}u)^{\alpha}\big)$$
$$\text{for any } u, v \in \mathcal{X}\setminus \operatorname{Fix}(\mathcal{T}) \text{ with } d(\mathcal{T}u,\mathcal{T}v) > 0.$$

Then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof It follows from Theorem 5 applied to the case $\beta = 0$. □

We can easily deduce the Proinov theorem even if we assume that $u, v \in \mathcal{X}\setminus \operatorname{Fix}(\mathcal{T})$.

Corollary 4 *Let $(\mathcal{X}, d)$ be a complete metric space, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. Suppose that there exist two functions $\psi, \phi : (0,+\infty) \to \mathbb{R}$, satisfying (a_1), (a_2), and (a_3), such that*

$$\psi\big(d(\mathcal{T}u,\mathcal{T}v)\big) \le \phi\big(d(u,v)\big) \quad \text{for any } u, v \in \mathcal{X}\setminus \operatorname{Fix}(\mathcal{T}) \text{ with } d(\mathcal{T}u,\mathcal{T}v) > 0.$$

Then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof It follows from Theorem 5 applied to the case $\alpha = \beta = 0$. □

Corollary 5 *Let $(\mathcal{X}, d)$ be a complete metric space, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. Suppose that there exist a constant $\alpha \in (0, 0.5)$ and two functions $\psi, \phi : (0, +\infty) \to \mathbb{R}$, satisfying (a_1), (a_2), and (a_3), such that*

$$\psi\big(d(\mathcal{T}u, \mathcal{T}v)\big) \le \phi\big(d(u,v)^{1-2\alpha} d(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{\alpha}\big)$$
$$\text{for any } u, v \in \mathcal{X} \setminus \operatorname{Fix}(\mathcal{T}) \text{ with } d(\mathcal{T}u, \mathcal{T}v) > 0.$$

Then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof It corresponds to the case $\alpha = \beta$ in Theorem 5. □

Corollary 6 *Let $(\mathcal{X}, d)$ be a complete metric space, let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping, and let $\phi : (0, +\infty) \to \mathbb{R}$ be a function satisfying the following conditions:*

(a_2) *$\phi(s) < s$ for all $s > 0$;*

(a_3) *$\limsup_{s\to e^+} \phi(s) < e$ for all $e > 0$.*

If we assume that there exist two constants $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$ and

$$d(\mathcal{T}u, \mathcal{T}v) \le \phi\big(d(u,v)^{1-\alpha-\beta} d(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{\beta}\big)$$
$$\text{for any } u, v \in \mathcal{X} \setminus \operatorname{Fix}(\mathcal{T}) \text{ with } d(\mathcal{T}u, \mathcal{T}v) > 0,$$

then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof It is only necessary to apply Theorem 5 when $\psi(s) = s$ for all $s \in (0, +\infty)$. □

Corollary 7 *Let $(\mathcal{X}, d)$ be a complete metric space, let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping, and let $\psi : (0, +\infty) \to \mathbb{R}$ be a function satisfying the following conditions:*

(a_1) *ψ is nondecreasing on $(0, +\infty)$;*

(a_2) *$s < \psi(s)$ for all $s > 0$.*

If we assume that there exist two constants $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$ and

$$\psi\big(d(\mathcal{T}u, \mathcal{T}v)\big) \le d(u,v)^{1-\alpha-\beta} d(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{\beta}$$
$$\text{for any } u, v \in \mathcal{X} \setminus \operatorname{Fix}(\mathcal{T}) \text{ with } d(\mathcal{T}u, \mathcal{T}v) > 0,$$

then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k\in\mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof We use the function $\phi : (0, +\infty) \to \mathbb{R}$ defined by $\phi(s) = s$ for all $s \in (0, +\infty)$. Notice that, in this case, for all $e > 0$,

$$\limsup_{s\to e^+} \phi(s) = e < \psi(e) \le \lim_{s\to e^+} \psi(s),$$

so assumption (a_3) can be deduced from (a_1) and (a_2). Therefore, it is only necessary to apply Theorem 5 when $\phi(s) = s$ for all $s \in (0, +\infty)$. □

Corollary 8 *Let $(\mathcal{X}, d)$ be a complete metric space, and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. Assume that there exist three constants $\alpha, \beta, k \in (0,1)$ such that $\alpha + \beta < 1$ and*

$$d(\mathcal{T}u, \mathcal{T}v) \leq k(u, \mathcal{T}u)^{\alpha} d(v, \mathcal{T}v)^{\beta}$$
$$\text{for any } u, v \in \mathcal{X} \setminus \operatorname{Fix}(\mathcal{T}) \text{ with } d(\mathcal{T}u, \mathcal{T}v) > 0,$$

then $\mathcal{T}$ has a fixed point and the Picard sequence $\{\mathcal{T}^k u\}_{k \in \mathbb{N}}$ converges to a fixed point of $\mathcal{T}$ whatever the initial point $u \in \mathcal{X}$.

Proof This result follows from Theorem 5 by using $\psi(s) = s$ and $\phi(s) = ks$ for all $s \in (0, +\infty)$. Such functions clearly satisfy conditions (a_1), (a_2), and (a_3). □

5 Conclusions and prospect work

Inspired by two well-known families of contractions, namely, Proinov contractions and ω-interpolative Ćirić–Reich–Rus-type contractions, in this paper we have introduced the notion of $(\alpha, \beta, \psi, \phi)$-*interpolative contraction* as a way to take advantage of the best properties of both classes of contractive mappings. Furthermore, we have shown that this new kind of contractions is distinct in nature to the previous definitions, which provides substantial added value to our proposal compared to the above-mentioned families.

In prospect work we will study the possible relationships between this novel class of contractions and other families of contractive mappings, mainly in order to search for new applications, and we will also try to generalize them by employing more general conditions, auxiliary functions, and algebraic structures.

Acknowledgements
The authors thank their universities. A.F. Roldán López de Hierro is grateful to Junta de Andalucía by project FQM-365 of the Andalusian CICYE and to Ministerio de Ciencia e Innovación by project PID2020-119478GB-I00.

Funding
We declare that funding is not applicable for our paper.

Availability of data and materials
The data and material used to support the findings of this study are included within the article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Publisher's Note

Received: 4 May 2021 Accepted: 5 July 2021 Published online: 16 July 2021

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