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# A New Method to Solve Fuzzy Interval Flexible Linear Programming Using a Multi-Objective Approach

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## ABSTRACT

Recently fuzzy interval flexible linear programs have attracted many interests. These models are an extension of the classical linear programming which deal with crisp parameters. However, in most of the real-world applications, the nature of the parameters of the decision-making problems is generally imprecise. Such uncertainties can lead to increased complexities in the related optimisation efforts. Simply ignoring these uncertainties is considered undesired as it may result in inferior or wrong decisions. Therefore, inexact linear programming methods are desired under uncertainty. In this paper, we concentrate a fuzzy flexible linear programming model with flexible constraints and the interval objective function and then propose a new solving approach based on solving an associated multi-objective model. Finally, numerical example is included to illustrate the mentioned solving process.

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Multi-objective linear programming; fuzzy interval flexible linear programming; interval linear programming; interval arithmetic; flexible constraints

## 1. Introduction

Fuzzy sets theory has been extensively employed in linear programming. The main objective in fuzzy linear programming is to find the best solution possible with imprecise, vague, uncertain or incomplete information. There are many sources of imprecision in fuzzy linear programming. The sources of imprecision in fuzzy linear programming vary. For example, sometimes constraint satisfaction limits are vague and other times coefficient variables are not known precisely. The research on fuzzy linear programming has risen highly since Bellman and Zadeh proposed the concept of decision making in fuzzy environment. Zimmermann [1] introduced the first formulation of fuzzy linear programming to address the impreciseness and vagueness of the parameters in linear programming problems with fuzzy constraints and objective functions. There are generally four fuzzy linear programming classifications in the literature.

Zimmerman [2] has classified fuzzy linear programming problems into two categories: symmetrical and non-symmetrical models. In a symmetrical fuzzy decision, there is no difference between the weight of the objectives and constraints while in the asymmetrical

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fuzzy decision, the objectives and constraints are not equally important and have different weights [3].

Leung [4] has classified fuzzy linear programming problems into four categories: a precise objective and fuzzy constraints; a fuzzy objective and precise constraints; a fuzzy objective and fuzzy constraints; and robust programming.

Luhandjula [5] has classified fuzzy linear programming problems into three categories: flexible programming; mathematical programming with fuzzy parameters and fuzzy stochastic programming.

Inuiguchi et al. [6] have classified fuzzy linear programming problems into six categories: flexible programming; possibilistic programming; possibilistic linear programming using fuzzy max; robust programming; possibilistic programming with fuzzy preference relations and possibilistic linear programming with fuzzy goals.

Delgado et al. [7] studied a general model for fuzzy linear programming problems which simultaneously involved in the constraints set both fuzzy numbers and fuzzy constraints. Mahdavi-Amiri and Nasseri [8] proposed a fuzzy linear programming model where a linear ranking function was used to rank order trapezoidal fuzzy numbers. They established the dual problem of the linear programming problem with trapezoidal fuzzy variables and deduced some duality results to solve the fuzzy linear programming problem directly with the primal simplex tableau. Mahdavi-Amiri and Nasseri [9] developed some methods for solving fuzzy linear programming problems by introducing and solving certain auxiliary problems. They apply a linear ranking function to order trapezoidal fuzzy numbers and deduce some duality results by establishing the dual problem of the linear programming problem with trapezoidal fuzzy variables. Wu [10] derived the optimality conditions for fuzzy linear programming problems by proposing two solution concepts based on similar solution concept, called the non-dominated solution, in the multiobjective programming problem. Inuiguchi and Ramik [6] and Peidro et al. have developed a number of fuzzy linear programming models to solve problems ranging from supply chain management to product development. Then Verdegay in [11] used the duality results to solve the original fuzzy linear programming. After that, Nasseri et al. in [12] introduced an equivalent fuzzy linear model for the flexible linear programming problems and proposed a fuzzy primal simplex algorithm to solve these problems. Recently, Attari and Nasseri [13] introduced a concept of feasibility and efficiency of solution for the fuzzy mathematical programming problems. The suggested algorithm needs to solve two classical associated linear programming problems to achieve an optimal flexible solution.

Interval linear programming, based on interval analysis, was proved to be an effective approach in dealing with uncertainties. Interval linear programming did not require distributional information and would not lead to complicated intermediate models. However, it was to be noted that the outputs of Interval linear programming were with lower and upper bounds, and thus could not reflect the distribution of uncertainty within the lower and upper bounds [14]. In some methods, Interval Linear Programming (ILP) model transformed into two sub-model whereas their optimal solutions formed a set which is called solution space of ILP model. The optimal solution set of ILP is determined by the best and worst model constraints, when the feasible solution components of best model are positive. In the Best and Worst Cases (BWC) method presented by Tong, the ILP model transformed into two sub-models [15,16], which consist of the largest and smallest feasible regions, so the BWC method introduces exact bounds of objective function values. A given point

is feasible of ILP model, if it satisfies best model constraints and it is optimal of ILP model, if it is optimal solution of arbitrary characteristic model of ILP model. Chinnec and Ramadan developed BWC method when ILP model includes equality constraints [17], and a new method for solving proposed by Huang and More [18]. Part of solution space of BWC and ILP methods may be infeasible. To ensure that solutions are absolutely feasible, Zhou et al. exhibited Modified Interval Linear Programming (MILP) method, by adding an extra constraint to the second sub-model. Some of the solutions which are obtained by the MILP method may be non-optimal. Also, among methods for solving ILP model, a Two-Step Method (TSM) had been presented by Huang et al. [18]. Solution space of the TSM method may be included infeasible solutions. To eliminate infeasible solutions from solution space of the TSM method, some methods are proposed. Wang and Huang added extra constraints to the second sub-model of TSM to ensure feasibility of solutions (namely ITSM). Part of solution space of ITSM is not optimal. Recently, Mishmast Nehi and Allahdadi [17,19] modified and improved the Tong method, which was unable to get optimal response on some issues. In this study, we give a generalised form of these problems in two ways: in first way, we consider the flexibility condition for the constraints, and in second way we consider the multi-objective case for the objective. In this sense, we introduce a new extended model and then propose a method for solving the proposed model.

The rest of this paper is organised as follows: In Section 2, we review the basic definitions and results on interval linear programming problem. Section 3 gives the definition of FFLP problem and proposes parametric approach to solve it. We give a new method for solving FFLP problem with multi-objective and interval objective function in Section 4. Section 5 is assigned to the illustrated example. Finally, conclusion is discussed in Section 6.

## 2. Interval Linear Programming Problems

In many real-world models, these coefficients are uncertain, so that they are bounded between upper and lower bounds. Therefore, in the formulation of research question in operations, if the data are in form of interval numbers, then the problem is an interval linear programming problem. In first time, Ben and Robbers presented the first interval linear programming model for interval constraints. Subsequently Huang and Moore introduced a new linear programming model in which all parameters and variables were interval. Generally, the solution method in these cases is the application of concepts that can turn the interval problem into problems with ordinary coefficients [14,18,19].

**Definition 2.1:** Given  $x^-$  and  $x^+ \in \mathbb{R}$  such that  $x^- \leq x^+$ , we define a closed interval  $x = [x^-, x^+]$  as the set  $\{x \in \mathbb{R} : x^- \leq x \leq x^+\}$ .

The values  $x^-$  and  $x^+$  are called the lower bound and upper bound of the interval  $x$ , respectively.

**Definition 2.2:** An interval  $[x, \bar{x}]$  with  $x^- = x^+$  is said to be degenerate.

Since a degenerate interval  $[x^-, x^+]$  only contains a single number, it is often identified with the number  $x$  itself, therefore it holds that  $x = [x, x]$ .

**Definition 2.3:** Given two matrices  $A^-$  and  $A^+ \in \mathbb{R}^{m \times n}$  such that  $A^- \leq A^+$ , we define a real interval matrix  $A = [A^-, A^+]$  as the set  $\{A \in \mathbb{R}^{m \times n} : A^- \leq A \leq A^+\}$ . The matrices  $A^-, A^+$

are called the lower bound matrix and the upper bound matrix of the interval matrix  $A$ , respectively.

The radius and centre of  $A$  are  $A_{\Delta} = \frac{1}{2}(A^{+} - A^{-})$  and  $A_C = \frac{1}{2}(A^{+} + A^{-})$ , respectively.

Thus  $A = [A^{-}, A^{+}] = [A_C - A_{\Delta}, A_C + A_{\Delta}]$ .

An interval vector  $I$  is introduced as the set  $\{I : I^{-} \leq I \leq I^{+}\}$  where  $I^{\pm} \in \mathbb{R}^n$  are crisp vector [20].

**Definition 2.4:** A general form of the Interval Linear Programming (ILP) model is defined as follows:

$$\begin{aligned} \max \quad & Z^{\pm} = \sum_{j=1}^n C_j^{\pm} x_j^{\pm} \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij}^{\pm} x_j^{\pm} \leq b_i^{\pm}, \quad i = 1, 2, \dots, m, \\ & x_j^{\pm} \geq 0, \quad j = 1, 2, \dots, n, \end{aligned} \quad (1)$$

where  $C_j^{\pm} \in [C_j^{-}, C_j^{+}]$ ,  $a_{ij}^{\pm} \in [a_{ij}^{-}, a_{ij}^{+}]$  and  $b_i^{\pm} \in [b_i^{-}, b_i^{+}]$  are interval numbers and  $x_j \in [x_j^{-}, x_j^{+}]$  is an  $n$ -dimensional interval decision vector.

**Theorem 2.1:** In the ILP model (1), the largest and smallest feasible regions are

$$\begin{aligned} \sum_{j=1}^n a_{ij}^{+} x_j \leq b_i^{-}, \quad i = 1, 2, \dots, m, \quad x_j \geq 0, \quad j = 1, 2, \dots, n \text{ and} \\ \sum_{j=1}^n a_{ij}^{-} x_j \leq b_i^{+}, \quad \forall i, \quad x_j \geq 0, \quad j = 1, 2, \dots, n, \text{ respectively.} \end{aligned}$$

**Proof:** The proof is straightforward by the common interval arithmetic. ■

**Definition 2.5:** A point  $y = (y_1, y_2, \dots, y_n)$  is said to be a feasible point of ILP model (1) if

$$\sum_{j=1}^n a_{ij}^{+} y_j \leq b_i^{-}, \quad i = 1, 2, \dots, m, \text{ and } y_j \geq 0, \quad j = 1, 2, \dots, n.$$

There are several methods for solving interval linear programming problems, one of which the Best and Worst Cases (BWC) method. The BWC method for solving linear interval programming problems in such a way that in general the linear programming problem with interval parameters turns into two optimistic and pessimistic linear programming models, where their solutions are the optimal interval of the main problem. This method examines the answers to the linear programming problems derived from the standard form [10,17].

Mentioned method transforms the ILP problem (1) into pessimistic and optimistic sub-problems, which are summarised as follows:

The pessimistic sub-problem:

$$\begin{aligned} \max \quad & Z^- = \sum_{j=1}^n C_j^- x_j, \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij}^+ x_j \leq b_i^-, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned} \quad (2)$$

The optimistic sub-problem:

$$\begin{aligned} \max \quad & Z^+ = \sum_{j=1}^n C_j^+ x_j, \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij}^- x_j \leq b_i^+, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned} \quad (3)$$

The optimal solutions to sub-problems (2) and (3) are in box form as follows:  $x^\pm = (x_1^\pm, x_2^\pm, \dots, x_n^\pm)$ , where for all  $j = 1, 2, \dots, n$ ,  $x_j^\pm = [x_j^-, x_j^+]$ . This box is the solution area which is introduced by Tong.

**Theorem 2.2:** In solving process of the ILP model, if  $Z^*$  is the optimal objective value of model (1), and  $Z^{-*}$ ,  $Z^{+*}$  are the optimal objective value of the model (2) and model (3), respectively, then  $Z^* \in [Z^{-*}, Z^{+*}]$ .

**Proof:** Let us consider the problem (1), we prove that the solution of this model is in the interval  $[\underline{z}', \bar{z}']$ . If  $x^0$  is a solution given by the above model, then we will have it  $\sum_{j=1}^n a_{ij} x_j^0 \geq b_i$ ,  $i = 1, 2, \dots, m$ . On the other hand,

$$a_{ij} \leq \overline{a_{ij}} \xrightarrow{x_j^0 \geq 0} a_{ij} x_j^0 \leq \overline{a_{ij}} x_j^0 \quad \forall i \rightarrow \sum_{j=1}^n a_{ij} x_j^0 \leq \sum_{j=1}^n \overline{a_{ij}} x_j^0.$$

Given the above phrases and  $b_i \geq \underline{b}_i$  we have  $\underline{b}_i < b_i \leq \sum_{j=1}^n a_{ij} x_j^0 \leq \sum_{j=1}^n \overline{a_{ij}} x_j^0$ ,  $i = 1, 2, \dots, m$ . ■

Therefore, every solution to the problem (3) is a solution to the model (1). So, the feasible area for the problem (1) includes the feasible area of problem (3). We now prove that the optimal value of the model (3) is less than the optimal value of the model (1). If  $x^*$  is the optimal solution for the model (3):

$$\underline{c}_j \leq c_j \xrightarrow{x_j^* \geq 0} \underline{c}_j x_j^* \leq c_j x_j^* \quad \forall j \rightarrow \sum_{j=1}^n \underline{c}_j x_j^* \leq \sum_{j=1}^n c_j x_j^*.$$

If  $\underline{z}' = \sum_{j=1}^n \underline{c}_j x_j^*$  is the objective value of model (1), then we have  $\underline{z}' \leq z^*$  and if  $\underline{z}'$  is optimal solution of model (2), then  $\underline{z}' < \underline{z}$  and so  $\underline{z}' < z^*$ . Similarly, if  $\bar{x}$  is a solution of model

(3), then  $\sum_{j=1}^n \underline{a}_{ij}\tilde{x}_j \geq \bar{b}_i$  and  $\underline{a}_{ij} \leq a_{ij}\forall j \rightarrow \tilde{x}_j \geq 0 \rightarrow \underline{a}_{ij}\tilde{x}_j \leq a_{ij}\tilde{x}_j\forall j \rightarrow \sum_{j=1}^n \underline{a}_{ij}\tilde{x}_j \leq \sum_{j=1}^n a_{ij}\tilde{x}_j$ , and since  $\bar{b}_i \geq b_i$ . Thus we will have  $b_i \leq \bar{b}_i \leq \sum_{j=1}^n \underline{a}_{ij}\tilde{x}_j \leq \sum_{j=1}^n a_{ij}\tilde{x}_j$ .

Therefore, each feasible solution of model (2) is a feasible solution of model (3), or, in other words, the feasible area of the model (3) including the feasible area of the model (2). Now, we prove that the optimal value of the model (2) is greater than the optimal model (3). Now consider  $x''$  be the optimal solution of problem (2).  $c_j \leq \bar{c}_j \forall j \rightarrow x_j'' \geq 0 \rightarrow c_j x_j'' \leq \bar{c}_j x_j'' \forall j \rightarrow \sum_{j=1}^n c_j x_j'' \leq \sum_{j=1}^n \bar{c}_j x_j''$ . If  $z$  is the value of the objective function model (3) for the feasible solution,  $z < \bar{z}'$  and since  $x^*$  is the solution of model (2), should be  $z^* \leq z$ , as a result  $z^* \leq \bar{z}'$ . So we'll have it  $\underline{z}' \leq z^* \leq \bar{z}'$  and the theorem is completed.

### 3. Fuzzy Flexible Linear Programming

Let us consider a case where the decision maker assumes that there is a certain tolerance in the fulfillment of constraints. In other word, a certain degree of violation is allowed and this is created by the decision makers. The general form of the Fuzzy Flexible Linear Programming (FFLP) problems with fuzzy resources can be formulated as follows (see in [21] too):

$$\begin{aligned} \max \quad & z = f(x, C) = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & g_i(x) = \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned} \quad (4)$$

In the above model, the relation ' $\leq$ ' is called 'fuzzy less than or equal to' and it is assumed that the tolerance  $p_i$  for each constraint is given [22]. This means that the decision maker can accept a violation of each constraint up to degree  $p_i$ . In this case, constraint  $g_i(x) \leq b_i$  is equivalent to  $g_i(x) \leq b_i + \theta p_i$ , ( $i = 1, 2, \dots, m$ ), where  $\theta \in [0, 1]$ .

Thus problem (4) can be equivalently considered as the following fuzzy inequality constraints (see also in [16]):

$$\begin{aligned} \max \quad & z = f(x, C) = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & g_i(x) = \sum_{j=1}^n a_{ij} x_j \leq \tilde{b}_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n. \end{aligned} \quad (5)$$

In model (5),  $\tilde{b}_i$  is a fuzzy number with the following membership function:

$$\mu_{\tilde{b}_i}(x) = \begin{cases} 1, & x \leq b_i, \\ 1 - (x - b_i)/p_i, & b_i \leq x \leq b_i + p_i, \\ 0, & x \geq b_i + p_i. \end{cases} \quad (6)$$

Verdegay [11] proved that Problem (4) is equivalent to the crisp parametric LP problem when the membership functions of the fuzzy constraints are continuous and non-increasing functions. According to this non-symmetric approach, the membership function of fuzzy inequality constraints of problem (4) can be modelled as follows:

$$\mu_i(g_i(x)) = \begin{cases} 1, & g_i(x) \leq b_i, \\ 1 - (g_i(x) - b_i)/p_i, & b_i \leq g_i(x) \leq b_i + p_i, \\ 0, & g_i(x) \geq b_i + p_i, \end{cases} \quad (7)$$

In this case, the membership function of all constraints of the problem (4) according to the Bellman and Zadeh operator is given by

$$\mu(g(x)) = \min\{\mu_1(g_1(x)), \mu_2(g_2(x)), \dots, \mu_m(g_m(x))\}. \quad (8)$$

Assuming,  $\alpha = \min\{\mu_1(g_1(x)), \mu_2(g_2(x)), \dots, \mu_m(g_m(x))\}$ , then Problem (4) is equivalent to

$$\begin{aligned} \max \quad & z = f(x, C) = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \mu_i(g_i(x)) \geq \alpha, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n, \quad \alpha \in [0, 1]. \end{aligned} \quad (9)$$

Consider the circumstances that the decision maker seeks to achieve the optimal answer with different degrees of validity in different constraints, according to a priority among the constraints. Clearly, the Verdegay's approach or single-parameter method is rejected in this case. By introducing various parameters for different constraints and using this multi-parameter approach, the decision-maker's need and appeal will be easily resolved. The following is a description of this method [13].

Consider the linear programming problem (9), the general form of the fuzzy linear programming problem is modified in this way:

$$\begin{aligned} \max \quad & z = f(x, C) = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \mu_i(g_i(x)) \geq \alpha_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n, \quad \alpha_i \in [0, 1]. \end{aligned} \quad (10)$$

Now, by substituting membership function (7) into problem (10), the following crisp parametric LP problem is achieved:

$$\begin{aligned} \max \quad & z = f(x, C) = \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & g_i(x) = (Ax)_i - b_i \leq (1 - \alpha_i)p_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \alpha_i \in (0, 1], \quad j = 1, 2, \dots, n. \end{aligned} \quad (11)$$

Note that for each  $\alpha_i \in (0, 1], i = 1, 2, \dots, m$ , an optimal solution is obtained. This indicates that the solution with  $\alpha$  grade of membership function is actually fuzzy.



Let's start with the following definitions below to continue the article.

**Definition 3.1:** Let  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, 1]^m$  be a vector, and

$$X_{\bar{\alpha}} = \{x \in \mathbb{R}^n | x \geq 0, \mu_i\{g_j(x, a_i) \leq 0\} \geq \alpha_i, \quad i = 1, 2, \dots, m\}.$$

Then, a vector  $x \in X_{\bar{\alpha}}$  is called an  $\bar{\alpha}$ -feasible solution of model (5).

Following proposition enables us to define feasible set of model (5) as an intersection of all  $\alpha$ -cuts corresponding to fuzzy constraints.

**Proposition 3.1:** Let  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, 1]^m$ , then  $X_{\bar{\alpha}} = \bigcap_{i=1}^m X_{\alpha_i}^i$ , where

$X_{\alpha_i}^i = \{x \in \mathbb{R}^n | x \geq 0, \mu_i\{g_j(x, a_i) \leq 0\} \geq \alpha_i\}$ , for  $i \in I = \{1, \dots, m\}$  (namely,  $X_{\alpha_i}^i$  is the  $\alpha$ -cuts of the  $i$ th constraint).

**Proof:** For  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, 1]^m$ , let  $x \in X_{\bar{\alpha}}$ . Therefore,  $\mu_i\{g_j(x, a_i) \leq 0\} \geq \alpha_i$  and from  $X_{\alpha_i}^i = \{x \in \mathbb{R}^n | x \geq 0, \mu_i\{g_j(x, a_i) \leq 0\} \geq \alpha_i\}$ , we have  $x \in X_{\alpha_i}^i, i \in I$ .

Therefore,  $x \in \bigcap_{i=1}^m X_{\alpha_i}^i$ . Also, if  $x \in \bigcap_{i=1}^m X_{\alpha_i}^i$ , we have  $x \in X_{\alpha_i}^i, i \in I$ , thus  $\mu_i\{g_j(x, a_i) \leq 0\} \geq \alpha_i$  and hence,  $x \in X_{\bar{\alpha}}$ .

Therefore, the proof is completed. ■

**Proposition 3.2:** Let  $\bar{\alpha}' = (\alpha'_1, \dots, \alpha'_m)$  and  $\bar{\alpha}'' = (\alpha''_1, \dots, \alpha''_m)$ , where  $\alpha'_i \leq \alpha''_i$  for all  $i$ . Then,  $\bar{\alpha}''$ -feasibility of  $x$  implies the  $\bar{\alpha}'$ -feasibility of it.

**Proof:** The proof is straightforward.

For a given  $\alpha \in (0, 1]$ , let a solution  $x \in \mathbb{R}^n$  be ordinary  $\alpha$ -feasible to problem (4) a solution in which has the same satisfaction degree in all of constraints. It means that  $\mu_i\{g_j(x, a_i) \leq 0\} \geq \alpha_i$ , or  $x \in X_{\alpha_i}^i$ , for all  $i \in I$ . If  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, 1]^m$ , then  $x \in X_{\bar{\alpha}}$ , which implies that the  $\bar{\alpha}$ -feasibility of problem (5) can be understood as a special case of the  $\bar{\alpha}$ -feasibility. Therefore, we have the next result. ■

**Remark 3.1:** If problem (5) is not infeasible, we immediately conclude that  $X_{\bar{\alpha}}$  is not empty.

**Definition 3.2:** Let ' $\preceq$ ' be a fuzzy extension of relation ' $\leq$ ' and a solution  $X = (x_1, \dots, x_n)^T \in \mathbb{R}^n$  be an  $\bar{\alpha}$ -feasible to problem (5), where  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m) \in (0, 1]^m$  and let  $f(x, C)$  be an objective function in the form of maximisation. Then,  $X = (x_1, \dots, x_n)$ , where  $x_j \in \mathbb{R}^n$  is an  $\bar{\alpha}$ -efficient solution to problem (5), if there is no  $x' \in X_{\bar{\alpha}}$  so that  $f(x, C) < f(x', C)$ .

Clearly, any  $\bar{\alpha}$ -efficient solution to the FFLP is a  $\bar{\alpha}$ -feasible solution to the FFLP with some additional properties.

#### 4. Solve FFLP Problem with Interval Multi-objective Function

In this section, we will present a new approach to solve the Fuzzy Flexible Linear Programming (FFLP) problem which is defined in (4) with interval multi-objective functions. We first

need to denote the form of mentioned problem as follows:

$$\begin{aligned} \max \quad & Z^\pm(x) = \{z^\pm_1(x), z^\pm_2(x), \dots, z^\pm_p(x)\} \\ \text{s.t.} \quad & g_j(x) = \sum_{j=1}^n a_{ij}x_j \leq b_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n, \end{aligned} \quad (12)$$

where  $x = (x_1, x_2, \dots, x_n)^T$  is a real vector of decision variables, and where  $Z^\pm(x)$  is an interval multi-objective function that is the objective coefficients is interval numbers.  $a_{ij}$  shows a coefficient matrix as  $A = [a_{ij}]$ , where  $A$  is an  $m \times n$ -dimensional matrix of interval technical coefficients. Objective functions and constraints where  $i \in \{1, \dots, m\}$  possess continuous property up to the second derivatives. Also, ' $\leq'$ ' denote a fuzzy extension of ' $\leq'$ ' on  $\mathbb{R}$  which is used to compare the left and right sides of fuzzy constraints [13].

In general, model (12) is not well defined due to the following reasons:

We cannot maximise the interval and multi-objective quantity  $Z^\pm(x)$ .

The constraint  $g_j(x) = \sum_{j=1}^n a_{ij}x_j \leq b_i, i = 1, 2, \dots, m$ , do not result in a crisp feasible set.

We first need to solve fuzzy flexible linear programming problem with multi-objectives. We show that this problem will reduce to one objective function by use of weighted technique for objective function. In the weighted method as well as used in [9], we assign  $k$  th value function equal to  $w_k$  that these  $w_k$  should be positive [23].

In other words, to find efficient solutions to the following multi-objective issues.

$$\begin{aligned} \max \quad & Z^\pm = \sum_{k=1}^p w_k Z^\pm_k(x) \\ \text{s.t.} \quad & g_j(x) = \sum_{j=1}^n a_{ij}x_j \leq b_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n, \end{aligned} \quad (13)$$

where  $0 \leq w_1, w_2, \dots, w_p \leq 1$  such that  $w_1 + w_2 + \dots + w_p = 1$  are the weights of the mentioned functions, which are determined by the decision maker.

It's important to have a few points in weighting:

The weight of each target  $w_i$  is between 0 and 1 and the total weight must be 1.

All target functions are Max or Min.

The coefficients of the decision variables in each objective function with the other objective function must be both scalable and therefore of a large category.

Now, by using BWC method transformed the ILP problem (13) into pessimistic and optimistic sub-problems, which are summarised as follows (see in [24] for more details):

The optimistic sub-problem:

$$\begin{aligned} \max \quad & Z^+ = \sum_{k=1}^p w_k Z^\pm_k(x) \\ \text{s.t.} \quad & g_j(x) = \sum_{j=1}^n a_{ij}x_j \leq b_i, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, \quad j = 1, 2, \dots, n, \end{aligned} \quad (14)$$

The pessimistic sub-problem:

$$\begin{aligned}
 \max \quad & Z^- = \sum_{k=1}^p w_k Z_{K^\pm}^-(x) \\
 \text{s.t.} \quad & g_i(x) = \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m, \\
 & x_j \geq 0, \quad j = 1, 2, \dots, n.
 \end{aligned} \tag{15}$$

Now, since constraints of problem are flexible, first, we select one of the above subproblems and describe the solving process, and then solve the following problem using the approach outlined below. Let us select sub-problem (14) for solving process. Therefore, in order to obviate those mentioned restrictions, we introduce the following problem:

$$\begin{aligned}
 \max \quad & Z^+ = \sum_{k=1}^p w_k Z_{K^\pm}^+(x) \\
 \text{s.t.} \quad & \mu_i\{g_i(x) \leq b_i\} \geq \alpha_i, \quad i = 1, 2, \dots, m, \\
 & x_j \geq 0, 0 \leq \alpha_i \leq 1, \quad j = 1, 2, \dots, n,
 \end{aligned} \tag{16}$$

where  $Z^+$  means the corresponding crisp value of interval function  $Z^\pm$ . To motivate for a meaningful choice of membership function for each fuzzy constraints, it is argued that if  $g_i(x) \leq b_i$ , then the  $i$ th constraint is absolutely satisfied, where as if  $g_i(x) \geq b_i + p_i$ , where  $p_i$  the predefined maximum tolerance from zero, as determined by the decision marker, then the  $i$ th constraint is absolutely violated. For  $g_i(x) \in (b_i, b_i + p_i)$ , the membership function is monotonically decreasing. If this decrease is along a linear function, then it makes sense to choose the membership function of the  $i$ th constraint ( $i = 1, 2, \dots, m$ ) as

$$\mu_i(g_i(x)) = \begin{cases} 1, & g_i(x) \leq b_i, \\ 1 - (g_i(x) - b_i)/p_i, & b_i \leq g_i(x) \leq b_i + p_i, \\ 0, & g_i(x) \geq b_i + p_i. \end{cases} \tag{17}$$

Now, in order to find maximum efficient solution, i.e. an  $\bar{\alpha}$ -efficient solution with  $\bar{\alpha} \geq \alpha$ ,  $i = 1, 2, \dots, m$ , we perform the following two-phase approach. To express this two-phase approach to the above problem, let us consider the problem (13) and implement a two-phase approach for this sub-problem, and then, with the resumption of the approach discussed below, we solve the second problem. In the two phase approach, Equation (7) is solved in Phase I, while in Phase II a solution is obtained which has higher satisfaction degrees than the previous solution. Thus by using this two-phase approach, we achieve a better utilisation of available resources. Further the solution resulting by this approach is always an  $\bar{\alpha}$ -efficient solution. Let us consider the definition and substituting in the problem (13) achieve the parametric linear programming that solved by linear techniques.

By substituting membership function (7) into problem (14), the following crisp parametric LP problem is achieved:

$$\begin{aligned} \max \quad & Z^+ = \sum_{k=1}^p w_k Z^+_{K(x)} \\ \text{s.t.} \quad & g_j(x) = (Ax)_j - b_j \leq (1 - \alpha_j) p_j, \quad i = 1, 2, \dots, m, \\ & x_j \geq 0, 0 \leq \alpha_i \leq 1, \quad j = 1, 2, \dots, n. \end{aligned} \quad (18)$$

Let us call the problem (15) as Phase I problem.

Let  $\bar{\alpha}^0 = (\alpha^0_1, \dots, \alpha^0_m)$  and  $(x^{+*}, C^{\pm} x^{+*})$  be the optimal solution of pessimistic sub-problem of Phase I with  $\bar{\alpha}^0$  degree of efficiency. Set  $\alpha_i^* = \mu_i\{g_j(x^*, a_j) \leq 0\} \geq \alpha_i^0, i = 1, 2, \dots, m$ .

In Phase II, we solve the following problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^m \alpha_i \\ \text{s.t.} \quad & \sum_{k=1}^p w_k Z^+_{K(x)} \geq \sum_{k=1}^p w_k Z^+_{K(x^{+*})} \\ & x_j \geq 0, \alpha_i^* \leq \alpha_i \leq 1, \quad j = 1, 2, \dots, n. \end{aligned} \quad (19)$$

**Theorem 4.1:** The optimal solution  $x^{**}$  to problem (19) is a maximum  $\bar{\alpha}$ -efficient solution to problem (16).

The process of the parametric approach for implementing the second sub-problem and the final solution is obtained.

Due to theorem, and optimal value and optimal solution of two sub-problems, interval solution of problem (12), is equal to:

#### Algorithm 4.1:

**Assumption 4.1:** Consider a Fuzzy Flexible Linear Programming (FFLP) problem is given to solve and obtain optimise  $Z$  such as problem (12).

**Step 1:** Using weighted method for problem (13) that transformed the multi-objectives into one target for objective function.

**Step 2:** By use of the best and worst cases (BWC) method, obtained the corresponding crisp objective function for the objective function of model (13), and achieved two sub-problems.

**Step 3:** Consider one of the two sub-problems and go to step 4.

**Step 4:** In Phase I, obtain the corresponding Multi-Parametric Linear Programming (MPLP) problem for problem (16) based on Equation (7).

**Step 5:** Solve the MPLP problem (18) and obtain optimal solution of problem.

**Step 6:** Based on the optimal solution of MPLP problem in step 5, obtain the MPLP problem of Phase II such as problem (19), and solve it.

**Step 7:** Consider second sub-problem (15) and go to step 4.

Now, we are a place to illustrate our suggested algorithm in the next section.

## 5. Numerical Examples

In this section, we solve the FFLP problem which is multi-objective and has interval coefficients in objective function by use of the proposed approach which is introduced in the last section.

**Example 5.1:** Consider the following interval multi-objective linear programming problems with flexible constraints.

$$\begin{aligned}
 \max \quad & z_1 = [1, 3]x_1 + [-1, 1.5]x_2 \\
 \max \quad & z_2 = [0.5, 2]x_1 + [-1.5, -1]x_2 \\
 \text{s.t.} \quad & 1.5x_1 + 2x_2 \leq 4, \\
 & 2x_1 + 3x_2 \leq 1, \\
 & x_1 \geq 0, \quad x_2 \geq 0,
 \end{aligned} \tag{20}$$

where  $p_1 = 2$  and  $p_2 = 5$  are predefined maximum tolerance.

**Step 1:** By considering the weights as  $w_1 = \frac{1}{2}$  and  $w_2 = \frac{1}{2}$  for the objective function, where  $\sum_{i=1}^3 w_i = 1$ , and then by use of weighted method reduce above multi-objective in one objective function in form of  $Z = w_1z_1 + w_2z_2 = [\frac{3}{4}, \frac{5}{2}]x_1 + [\frac{-5}{4}, \frac{1}{4}]x_2$ , and we can rewrite problem (20) as follows:

$$\begin{aligned}
 \max \quad & Z^\pm = [\frac{3}{4}, \frac{5}{2}]x_1 + [\frac{-5}{4}, \frac{1}{4}]x_2 \\
 \text{s.t.} \quad & 1.5x_1 + 2x_2 \leq 4, \\
 & 2x_1 + 3x_2 \leq 12, \\
 & x_1 \geq 0, \quad x_2 \geq 0,
 \end{aligned} \tag{21}$$

**Step 2:** By use of the best and worst cases (BWC) method convert the interval linear problem (21) into two sub-problems as follow:

$$\begin{aligned}
 \max \quad & Z^+ = \frac{5}{2}x_1 + \frac{1}{4}x_2 \\
 \text{s.t.} \quad & 1.5x_1 + 2x_2 \leq 4, \\
 & 2x_1 + 3x_2 \leq 12, \\
 & x_1 \geq 0, \quad x_2 \geq 0,
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 \max \quad & Z^- = \frac{3}{4}x_1 + \frac{-5}{4}x_2 \\
 \text{s.t.} \quad & 1.5x_1 + 2x_2 \leq 4, \\
 & 2x_1 + 3x_2 \leq 12, \\
 & x_1 \geq 0, \quad x_2 \geq 0,
 \end{aligned} \tag{23}$$

**Step 3:** Consider the sub-problem (22) and continue.

**Table 1.** Requirement per automobile.

	$M_1$	$M_2$	$M_3$
$A_1$	[2000, 2100]	[8000, 9000]	[4000, 4500]
$A_2$	[3000, 3200]	[1000, 1200]	0
$A_3$	[4000, 5000]	[4000, 4600]	[2000, 2400]

**Table 2.** Unit profit.

	$A_1$	$A_2$	$A_3$
Profit/unit	[5000, 5120]	[10,000, 12,100]	[12,000, 13,500]

**Step 4:** Obtain the corresponding multi-parametric linear programming (MPLP) problem for problem (22) based on Equation (7)

$$\begin{aligned}
 \max \quad & Z^+ = \frac{5}{2}x_1 + \frac{1}{4}x_2 \\
 \text{s.t.} \quad & 1.5x_1 + 2x_2 \leq 4 + 2(1 - \alpha_1), \\
 & 2x_1 + 3x_2 \leq 12 + 5(1 - \alpha_2), \\
 & x_1 \geq 0, x_2 \geq 0, \quad \alpha_1, \alpha_2 \in [0, 1],
 \end{aligned} \tag{24}$$

**Step 5:** By solving the above problem achieve  $x^* = (3.333, 0)$  be an  $(0.5, 0.4)$ – efficient solution with  $C^T x^* = 8.333$  as an optimal value of problem (24).

**Step 6:** Obtain the MPLP problem of Phase II such as problem (25) based on optimal solution of problem (24).

$$\begin{aligned}
 \max \quad & \alpha_1 + \alpha_2 \\
 \text{s.t.} \quad & \frac{5}{2}x_1 + \frac{1}{4}x_2 \geq 8.333, \\
 & 1.5x_1 + 2x_2 \leq 4 + 2(1 - \alpha_1), \\
 & 2x_1 + 3x_2 \leq 12 + 5(1 - \alpha_2), \\
 & x_1 \geq 0, x_2 \geq 0, 0.5 \leq \alpha_1^* \leq 1, 0.4 \leq \alpha_2^* \leq 1,
 \end{aligned} \tag{25}$$

An optimal solution to the above problem is  $x^{**} = (3.332, 0)$ , also  $C^T x^* = C^T x^{**} = 8.333$ , and we have

$$\mu_1(g_1(x^{**}, a_1)) = 1, \mu_2(g_2(x^{**}, a_2)) = 0.5.$$

**Step 7:** Consider second sub-problem (23) and solve it. Then we achieve that an optimal solution to the above problem is  $x^{**} = (3.332, 0)$ , also  $C^T x^* = C^T x^{**} = 2.5$ , and  $\mu_1(g_1(x^{**}, a_1)) = 1, \mu_2(g_2(x^{**}, a_2)) = 0.5$ .

Finally, with regard to Theorem 2.2, we can obtain  $Z^* = [2.5, 8.333]$  that is an interval optimal value of problem (20), and higher satisfaction in membership function in  $\mu_1$ .

**Example 5.2** An automobile factory produces three models  $A_1, A_2$  and  $A_3$ . Three types of raw materials  $M_1, M_2$  and  $M_3$  are required to manufacture them. The amounts (in kg) of the materials are given in Table 1.

Based on market analysis, the expected unit profits of  $A_1, A_2$  and  $A_3$  are given in Table 2.

**Table 3.** Per unit harmful pollutant.

	$A_1$	$A_2$	$A_3$
Pollutant/unit	[1000, 1050]	[2000, 2100]	[1000, 1150]

**Table 4.** Unit cost.

	$A_1$	$A_2$	$A_3$
Cost/unit	[1000, 1120]	[3000, 3090]	[4000, 4140]

According to the monthly production report, the per unit harmful pollutant is given in Table 3.

The unit cost of the automobile is listed in Table 4.

The decision makers of the factory attempt to achieve three goals on a weekly basis as follows:

To maximise the profit.

To minimise the generation of the harmful pollutant.

To minimise the production cost.

The three goals are constrained by the following capacities on a weekly basis:

Considering the cost of  $M_1$  is very high, the usage of  $M_1$  is required to be less than 40,000 kg and this amount is ultimately allowed in a rate of 50,000 kg.

Since the shortage of  $M_2$  is often a concern, the usage of  $M_2$  only is required to be less than 50,000 kg and the maximum amount can be increased to 5000 kg.

The usage of  $M_3$  cannot be more than 50,000 kg.

By the demand of automobiles in the market, it is required to produce at least 3 units of  $A_1$  and 5 units of  $A_3$  per week.

In order to find the optimal quantities of  $A_1, A_2$  and  $A_3$  per week, this problem is modelled as the following MOILP problem:

$$\begin{aligned}
 \max \quad & z_1 = [5000, 5120]x_1 + [10000, 12100]x_2 + [12000, 13500]x_3 \\
 \min \quad & z_2 = [1000, 1050]x_1 + [2000, 2100]x_2 + [1000, 1150]x_3 \\
 \min \quad & z_3 = [1000, 1120]x_1 + [3000, 3090]x_2 + [4000, 4140]x_3 \\
 \text{s.t.} \quad & [2000, 2100]x_1 + [3000, 3200]x_2 + [4000, 5000]x_3 \leq 40000, \\
 & [8000, 9000]x_1 + [1000, 1200]x_2 + [4000, 4600]x_3 \leq 50000, \\
 & [4000, 4500]x_1 + [2000, 2400]x_2 \leq 50000, \\
 & x_1 \geq 3, \\
 & x_3 \geq 5, \\
 & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0,
 \end{aligned} \tag{26}$$

**Table 5.** Some typical  $\bar{\alpha}$ -efficient solution of sub-problem 1.

a	b	c	d	e	f
$\bar{\alpha}$	(0.5,0.6)	(0.8,0.2)	(0.5,0.8)	(0.2,0.8)	(0.5,0.2)
$Cx$	44,997	41,657	44,997	47,224	44,997
$x_1$	3	3	3	3	3
$x_2$	6.3333	5.33	6.3333	7	6.3333
$x_3$	5	5	5	5	5

Take  $w_1 = 0.4, w_2 = 0.3$  and  $w_3 = 0.3$ . Then the interval weighted sum secularisation of the MOILP problem with respect to ' $\leq$ ' is as follows:

$$\begin{aligned}
 \max Z &= [1349, 1448]x_1 + [2443, 3340]x_2 + [3213, 3900]x_3 \\
 \text{s.t.} \quad & [2000, 2100]x_1 + [3000, 3200]x_2 + [4000, 5000]x_3 \leq 40000, \\
 & [8000, 9000]x_1 + [1000, 1200]x_2 + [4000, 4600]x_3 \leq 50000, \\
 & [4000, 4500]x_1 + [2000, 2400]x_2 \leq 50000, \tag{27} \\
 & x_1 \geq 3, \\
 & x_3 \geq 5, \\
 & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0,
 \end{aligned}$$

Now, we solve the problem (27) by using the solving technique of interval problems and solving algorithm in this paper and the given tolerances  $p_1 = 10000, p_2 = 5000$ . We will simplify the first sub-problem based on mentioned solving algorithm steps in (4.1) as follows:

$$\begin{aligned}
 \max \quad & Z^+ = 1448x_1 + 3340x_2 + 3900x_3 \\
 \text{s.t.} \quad & 2000x_1 + 3000x_2 + 4000x_3 \leq 40000 + 10000(1 - \alpha_1), \\
 & 8000x_1 + 1000x_2 + 4000x_3 \leq 50000 + 5000(1 - \alpha_2), \\
 & 4000x_1 + 2000x_2 \leq 50000, \tag{28} \\
 & x_1 \geq 3, \\
 & x_3 \geq 5, \\
 & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \alpha_1, \alpha_2 \in [0, 1],
 \end{aligned}$$

Some  $\bar{\alpha}$ -efficient solution with satisfaction degrees which decision maker can be found in Table 5.

Let  $x^* = (3, 6.3333, 5)$  be (0.5, 0.6)-efficient solution with  $C^T x^* = 44997$  as an optimal value of problem (28). In step 6, we need to solve the following linear problem:

$$\begin{aligned}
 \max \quad & \sum_{i=1}^2 \alpha_i \\
 \text{s.t.} \quad & 1448x_1 + 3340x_2 + 3900x_3 \geq 44997, \\
 & 2000x_1 + 3000x_2 + 4000x_3 \leq 40000 + 10000(1 - \alpha_1)
 \end{aligned}$$



$$\begin{aligned}
8000x_1 + 1000x_2 + 4000x_3 &\leq 50000 + 5000(1 - \alpha_2) \\
4000x_1 + 2000x_2 &\leq 50000, \\
0.5 \leq \alpha_1 \leq 1, 0.6 \leq \alpha_2 &\leq 1, \\
x_1 &\geq 3, \\
x_3 &\geq 5, \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0,
\end{aligned} \tag{29}$$

An optimal solution to the above problem is  $x^{**} = (3, 6.3332, 5)$ , also  $C^T x^* = C^T x^{**} = 44997$  we have

$$\mu_1(g_1(x^{**}, a_1)) = 0.5, \mu_2(g_2(x^{**}, a_2)) = 1.$$

Using the approach, we can get an optimal solution  $x^*$  which not only achieves the optimal objective value but also give a higher value in  $\mu_2$ .

Now, we use all these steps to solve the second sub-problem. Finally, by solving the second sub-problem obtain that if  $x^* = (3, 3.33, 5)$  be (0.5, 0.4)-efficient solution with

$$C^T x^* = C^T x^{**} = 26219 \text{ and } x^* = (3, 0, 6.2784),$$

$\mu_1(g_1(x^{**}, a_1)) = 1, \mu_2(g_2(x^{**}, a_2)) = 1$ . Finally, with regard to Theorem 2.2 optimal objective value of problem (27) is  $Z^* = [18595, 44997]$ .

## 6. Conclusion

In this paper, two main contributions are appeared. First, considering the feasibility for the constraints and second, an extra condition for the objective function where we assumed a multi-objective cases. Based on the generalised form of the problem, we suggested a new two-phase method. We saw that it was observed that using this concept as a generalisation of parametric the approach in linear programming provides a more appropriate tool for modelling real problems and improving the solving process. Also, in the process of solving a weighty technique for the multi-objective linear programming problem, it was suggested. This approach will be useful in obtaining flexible responses with a degree of satisfaction determined by the decision maker for fuzzy mathematical programming. There are still other approaches, such as for instance Rough Sets (see in [25]), to deal with the problem approached in the research. The second to indicate that readers interested in new Fuzzy Optimisation problems could consult that paper (see in [26]).

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No potential conflict of interest was reported by the author(s).

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