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The geometric origin of perspectivist science in G.W. Leibniz. Analysis based on unpublished manuscripts

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Abstract

The perspectivist research carried out by G.W. Leibniz between 1679 and 1686 in the field of geometry is analysed. This work is reflected in six, as yet unpublished, texts, of which the main three are analysed: *Constructio et usus scalae perspectivae*, *Origo regularum artis perspectivae* and *Scientia perspectiva*. The philosophical perspectivism advocated by the German thinker is widely known, but his geometric research on perspective is much less so. This article seeks to remedy this situation. The first of these writings (*Constructio et usus scalae perspectivae*) includes Leibniz's experimentation with the perspectivist methodology of scales. Then, in *Origo regularum artis perspectivae quales*, Leibniz constructs his perspectivist *regula generalis*. Finally, Leibniz wrote *Scientia perspectiva and* readdresses the main rule of perspective and experiments with the theoretical limits of the analysis carried out in this discipline. Primarily, he supposed a 'minimum distance' between the elements that constitute it, and then theorised an 'infinite interval' between these same elements.
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Zusammenfassung

Die perspektivische Forschung von G.W. Leibniz zwischen 1679 und 1686 auf dem Gebiet der Geometrie wird analysiert. Diese Arbeit spiegelt sich in sechs noch unveröffentlichten Arbeiten wider, von denen die drei wichtigsten analysiert werden: Constructio et usus scalae perspektive, Origo regulumum artis perspektive quales und Scientia perspektive. Die erste dieser Arbeiten enthält Leibniz 'Experimente mit der perspektivischen Methodik von Skalen. Dann konstruiert Leibniz in Origo regularum artis perspektivae quales seine perspektivische regula generalis. Schließlich schreibt Leibniz Scientia perspektive. Obwohl es das einzige ist, das für die Veröffentlichung konzipiert zu sein scheint, war es es nie. In Scientia perspektive spricht Leibniz die "allgemeine Regel" der Perspektive neu an und experimentiert mit den theoretischen Grenzen der in dieser Disziplin durchgeführten Analyse. In erster Linie nahm er einen "Mindestabstand" zwischen den Elementen an, aus denen er besteht, und theoretisiert dann ein "unendliches Intervall" zwischen denselben Elementen.
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1. Introduction: the research on perspective

Between 1679 and 1686, shortly after his Parisian period (1672-1676), Leibniz began his geometric research on perspective. During this time, he wrote: *Constructio et usus scalae perspectivae*; *Fundamentum perspectivae meo marte investigatum; Auxilia Calculi*; *Origo regularum artis perspectivae quales sine libro ac magistro inveni*; *Si omnia punta objecti referantur ad planum spectatoris parallelum*; and *Scientia perspectiva*.[1] In this set of writings, which remain unpublished,[2] Leibniz did research on linear perspective, namely, 'the art of using geometry for constructing images obtained by a central projection'.[3] The definition offered in *Scientia perspectiva* establishes the objective of the study: 'Scientia perspectiva est, ars objecti apparentiam in Tabula exhibendi [. . .]'.[4] Analysis of this research is the subject of this paper.

Leibniz's first contacts with geometric research related to perspective occurred during his stay in Paris (1672-1676). During this stay, Leibniz studied the works of B. Pascal (1623-1662) on conic sections.[5] In these works, the French geometer accepted that the geometric-pictorial study of perspective could be reduced to the projective analysis of conic sections.[6] Each section is a different image of the base of the cone obtained by central projection. However, this is not completely true; geometric research on perspective is not completely comparable to the projective analysis of cones. Perspective methods are orientated towards

[1] The manuscripts of these texts are part of the collection of the G.W. Leibniz Bibliothek (Hannover); these manuscripts are part of the Leibniz Handschriften, from now on designated as LH. *Constructio et usus scalae perspectivae* (LH 35 XI, 17, 23); *Fundamentum perspectivae meo marte investigatum* (LH 35 XI, 17, 21)*; Auxilia Calculi* (LH 35 XI, 17, 24); *Origo regularum artis perspectivae quales sine libro ac magistro inveni* (LH 35 XI, 17, 19-20); *Si omnia punta objecti referantur ad planum spectatoris parallelum* (LH 35 XI 1, 13); *Scientia perspectiva* (LH 35 XI 1, 9-10). These manuscripts can be found at http://digitale-sammlungen.gwlb.de/start/. They are currently being transcribed by *Mathesis* group; their results are updated in the following webpage: http://mathesis.altervista.org/perspective-and-projective-geometry/. The existence of these writings and their dating was reported for the first time by Javier Echeverría in (Echeverría, 1983). As indicated by J. Echeverría, *Scientia perspectiva* could have been written around 1695. However, the unity of perspectivist research, and Leibniz's biographical vicissitudes, make it more plausible to suppose that this paper was written around 1686. Before Leibniz began his trip to Italy.

[2] Transcriptions made by Javier Echeverría and reviewed by Ricardo Rodríguez Hurtado.

[3] This characterisation of geometric research on perspective is taken from the work by K. Andersen, *The Geometry of an Art*: "From the Renaissance onwards, the branch of perspective dealing with representation on a two-dimensional surface was divided into various subdisciplines. The one dealing with the problem of depicting straight lines and lengths was called *linear perspective*. In general, I use the word 'perspective' to mean linear perspective, or, more precisely, the art of using geometry for constructing images obtained by a central projection". (Andersen, 2007, XX). In this article the word 'perspective' is used in this same sense.

[4] 'The perspective science is the art of exhibiting on the Tabula the object appearance [. . .]' (LH 35 XI 1, 9).

[5] During these years, Leibniz had at his disposal the missing *Traité des coniques* by B. Pascal, but also the *Essai pour les coniques*. For Leibniz's relationship with these works see (Debuiche, 2016). In addition to the texts analysed in this work, two writings could prove relevant to the matter in question: *Ducere minimum ad sectionem conicae ex puncto plano* (LH 35 XI, 17, 17-18) and *Conica pascaliana* (LH 35 XI, 17, 3).

[6] Only the first part of the missing *Traité des coniques* is preserved. While Leibniz had at his disposal the treatise, he copied *Generatio Conisectionum* (Leibniz, 1923, series VII, volume 7, 2019, 584-591). In *Generatio Conisectionum*, in the corollary that follows the sixth definition, Pascal structures the reduction of the geometric analysis of perspective to the projective analysis of conic sections, see (Leibniz, 1923, series VII, volume 7, 2019, 585-586). According to that corollary, the following consider the ellipse, the parabola and the hyperbola as optical appearances of the base of the cone. In them, the sections of the cone are named as appearances on a plane of representation. Girard Desargues had included at the end of the *Brouillon Project* the basis of this same interpretation, namely, the application of cone analysis to perspective, see (Leibniz, 1923, series VII, volume 7, 2019, 593). For an analysis of Leibniz's relationship with the work of Desargues, see (Echeverría, 1994). This question has also been addressed by Valérie Debuiche in (Debuiche, 2013).

structuring the dimensions of pictorial representations according to the viewer's optical distortion. The relationship between the representation and the object represented is projective; however, this does not nullify the particularity of perspective studies. The proportional structuring of the dimensions of the objects represented is related to the distance between the viewer, the plane of representation and the objects. Projective analysis does not deal with this aspect of research.

After his time in Paris, during his early years in Hanover (1676-1686), Leibniz studied a variety of treatises devoted to perspective. It was during this time that he became fully aware of the meaning of this model of study and decided to do his own work on the subject. During these years, Leibniz was director of the library of the Hanoverian court and had at his disposal a large number of works dedicated to this discipline. Of these, he studied the following: *Manière universelle de Mr Desargues* (Bosse, 1648); *La perspective pratique* (1642-1649) (Dubreuil, 1642–1647); and *La perspective speculative et pratique* (Migon and Aleaume, 1643) edited by E. Migon, based on the work by J. Aleaume (Leibniz, 1923, series VIII, volume 1, 2009, 210-235).[7] Against the theoretical backdrop of the methods presented in these works, in 1679 Leibniz began his own research. Then, between 1679 and 1686, he wrote the six texts mentioned above. Our article is divided into two sections: in the first, Leibniz's research on perspective is contextualised; in the second, three of the writings of the German author are analysed.

2. The context of Leibniz's research on perspective

This section introduces three contextual questions that frame Leibniz's research on perspective: first, its position within Leibniz's theoretical approach; second, the treatises on perspective that Leibniz had at his disposal to study this discipline; third, the theoretical context from which the German thinker approached the discipline.

2.1. The conceptualisations of section and magnitude: the deficiencies of the geometric theory

During his years in Paris, Leibniz encountered Cartesian geometry, the core of the mathematical debate since the publication of *La Géométrie* (1637). From the very beginning, Leibniz was critical of the French thinker's mathematical work. His works on the quadrature of the circle (1675-1676) are an excellent example of this stance.[8] However, his disagreement with the Cartesian mathematical approach is especially evident in his works on the foundations of geometry. In his first years in Hanover, Leibniz devoted much of his intellectual efforts to reconceptualising the foundations of geometry. In August 1679, he wrote an important essay entitled *Characteristica geometrica*.[9] In a letter to C. Huygens (1629-1685), Leibniz presented the objective of this work:

> [...] je croy qu'il nous faut encor une autre Analyse proprement geometrique ou lineaire, qui nous exprime directement *situm* comme l'Algebre exprime *magnitudinem*. Et je croy d'en avoir le moyen, et qu'on pourroit representer des figures et meme des machines et mouvemens en caracteres, comme l'Algebre represente

[7] Although these marginal notes appear to date back to the Parisian period (1672-1676), an investigation carried out by the G.W. Leibniz Bibliothek on its bibliographical collections refuted this dating. This investigation verified that the three aforementioned treatises were in Hanover when Leibniz arrived at the court (1676). In addition to these marginal notes, Leibniz's papers also include a set of study notes on Dubreuil's work (LH 35 XI, 17, 7; LH 35 XI, 17, 8; LH 35 XI, 17, 12; LH 35 XI, 17, 13).

[8] For a work on Leibniz's analysis of the curves, see (Knobloch, 2013). Knobloch highlights the universality of the analytical methodology sought by the young intellectual in this work.

[9] This early essay on geometry, can be found in (Leibniz, 1995, 124-233). For an explanation of the essay, see (De Risi, 2007, 132-140).

> les nombres ou grandeurs; et je Vous envoye un essay, qui me paroist considerable (Leibniz, 1923, series III, vol. 2, 1987, 847).[10]

In contrast to the Cartesian algebraic model, centred on the determination of magnitudes (analysis of quantitative relations), Leibniz wanted to construct a qualitative model, based on analysis of position (*situm*). Huygens rejected this innovation and this was a serious blow to Leibniz' geometrical project. After the essay of 10 June 1679, Leibniz slowed down the development of his geometric theory. Between 1679 and 1686, he wrote texts mainly dedicated to redefining some geometric notions and compilation texts (De Risi, 2007, 74-75). At that time, Leibniz's theory was deficient in its conceptualisation of section (De Risi, 2007, 68-69). Perspective is particularly useful in this area; however, he considered pictorial representation as a section. Furthermore, Leibniz's geometric theory does not sufficiently conceptualise the metric question. A question that allows him to address perspective (Debuiche, 2016, 64-65; De Risi, 2007, 75). The determination of a constant and regulated proportion between the magnitudes of the different objects represented is part of this discipline.

Geometric research on perspective thus maintained a continuity with the previous geometric reflection. It allowed Leibniz to reflect on section through the concept of pictorial representation, and on magnitude through the analysis of proportion. However, just as linear perspective is not reduced to the projective analysis of conic sections, this research is not comparable to geometric characteristic. In the same way that linear perspective presupposes the projective character between the image and the object, Leibniz's research on perspective presupposes this geometric project. However, perspective continues to focus on the construction of pictorial representations.

2.2. *The bibliographical collections of the Hanover court: a privileged position for the study of perspective*

As director of the library of the Hanoverian court, Leibniz had at his disposal an extensive bibliography on perspective.[11] The institution held works dedicated to this discipline by classical authors as *L'architettura* (Alberti, 1565) and *Les quatre lievres d'Albert Durer* (Durer, 1614).[12] It also had mathematical courses that incorporated it in the form of a lesson. Among French intellectuals of the 17th century, it was common to introduce perspectivist questions into these courses. The second volume of *Cursus seu mundus mathematicus* (Dechales, 1674) is a good example of this.[13] *Ars Magna lucis et umbrae in mundo* (1646)

[10] '[...] I believe that what is necessary is another inherently geometric or linear Analysis which expresses *situm* directly like Algebra expresses *magnitudinem*. And I believe that I have the means, and that figures and machines and movements in characters can be represented, like Algebra represents numbers and sizes; and I send you a test that seems significant to me.' The test mentioned in the letter can be found in (Leibniz, 1923, series III, volume 2, 1987, 851-860).

[11] This review was possible thanks to the research and cataloguing carried out by the G.W. Leibniz Bibliothek. The information used in this section comes from an online search tool that provides access to this catalogue, which can be found at: http://www.leibnizcentral.de/. This bibliographic information comes from the database, *Leibniz' Arbeitsbibliothek*, within *Datenbanken* ('Leibniz' Arbeitsbibliothek. Datenbanken', Leibniz central, 5 May 2018). This information includes the books that the Hanover library housed in its collections when Leibniz arrived at the court and began working as its director.

[12] The library also had works on perspective by authors as important as Guidobaldo del Monte and Salomon de Caus. It did not, however, have their works dedicated to the discipline. The Hanover library also had a copy of *De re aedificatoria* (1565) by L.B. Alberti, and there was also evidence of a missing copy of his *De pictura* (1435).

[13] In the Hanover library, there are only three of the six volumes of P. Hérigone's mathematics course (*Cursus matematici*). The second, containing his reflections on perspective, is not there. A similar situation occurs with the works of Rohault (1620) and Ozanam (1640-1717). All of these mathematical courses included a section dedicated to the study of perspective, see (Andersen, 2007, 403-410).

(Kircher, 1671),[14] by the German humanist A. Kircher (1602-1680), also sets aside a section for it. Noteworthy in the library's collections were the presence of two significant treatises on this discipline: *Le due regole della prospettiva* (1642) (Da Vignola, 1573), and *La perspective curieuse* (1652), by J.F. Niceron. In *Nouveaux Essais*, Leibniz mentions the method of anamorphosis presented in this last work (Leibniz, 1923, series VI, volume 6, 1990, 258). In the Hanover library, there is a significant number of Italian works available: *Dispareri in materia d'architettvra, et perspettiva* (Di Martino, 1572); *Lo inganno de gl'occhi* (Accolti, 1625); and *La pratica di prospettiva* (1625) (Sirigatti, 1614). The vanguard of perspectivist research during the 17th century, however, was to be found in the French tradition (including the southern part of the Netherlands).[15] Thus, in addition to the aforementioned mathematical courses, this library has works as *Opera mathematica traictans de gemoetrie, perspective, architecture et fortification* (Marolois, 1662) and *Examen du libre des recreations mathematiques* (Mydorge, 1639). There are also most of the works of A. Bosse (1604-1676) and J. Dubreuil (1602-1670). In addition to those mentioned, *Manière universelle de Mr. Desargues* and *La perspective pratique*, the Hanoverian court library holds his painting, sculpture and architecture works.[16] It should be noted that until the end of his life Leibniz updated this list of works. With his research on the subject already carried out, he acquired a copy of *Traité de Perspective* (Lamy, 1701). During these same years, he also acquired *Perspectiva pictorum et architectorum* (Pozzo, 1693–1698).[17]

With this extensive volume of works on perspective at his disposal, Leibniz focused his study on *Manière universelle de Mr Desargues*, *La perspective pratique* and *La perspective speculative et pratique*.[18] What did Leibniz seek in them?

2.3. The meaning of the study of the treatises on perspective: the controversy over the authorship of the method of scales

These three treatises are the main references for one of the most important public controversies affecting the study of perspective: the controversy over the authorship of the method of scales. At the beginning of the sixteenth century, engineers such as S. Caus, J.L. Vaulezrd and J. Aleaume began to use scales in the construction of representations.[19] Each author created his own scales, but all of them had the same aim, namely, to represent distances. The use of scales acquired became more systematic in *Exemple de l'une des manières universelles* (1636)[20] by Desargues. The relationship between the *scale of measures* and the *scale of distances* proposed by the French geometer enabled the building of any image in perspective (Desargues, 1987, 152). It is not worth describing the Arguesian method in greater detail, as Leibniz criticised it strongly. He accused it of not being suitable for the purpose and of not being duly justified (Leibniz, 1923, series VIII, volume 1, 2009, 210). In addition, Leibniz's version of the method of scales would be analysed

[14] In this extensive compendium, the initial parts of book II are devoted to the perspectivist discipline. In *Drôle de pensée* (1675), Leibniz refers to 'Cabinet' by A. Kircher, also to 'La Lanterne Magique' (Leibniz, 1923, series IV, volume 1, 1983, 567-568).

[15] This division follows (Andersen, 2007, 401-403).

[16] Bosse, A., 1643. La pratique du trait a preuves de Mr Desargues Lyonnois pour la coupe des pierres en l'architecture. Des Hayes, Paris; Bosse, A., 1652. Kunstbüchlein handelt Von der Radier und Etzkunst [...]. Pillenhofer, Nuremberg; Du Breuil., J., 1674. L'Art universel des fortifications, Trait. 1/6. Paris.

[17] Leibniz's relationship with these works can be confirmed in his correspondence: with *Traité de Perspective*, in (Leibniz, 1923, series I, volume 20, 2006, 5) and with *Perspectiva pictorum et architectorum*, in (Leibniz, 1923, series I, volume 20, 2006, 277).

[18] Leibniz's notes on these three treatises can be found in (Leibniz, 1923, series VIII, volume 1, 2009, 210-226, *Manière universelle de Mr Desargues*), (Leibniz, 1923, series VIII, volume 1, 2009, 228-232, *La perspective speculative et pratique*), and (Leibniz, 1923, series VIII, volume 1, 2009, 233-234, *La perspective pratique*).

[19] For a review of the methods of scales of S. Caus, J.L. Vaulezrd and J. Aleaume, see (Andersen, 2007, 410-427).

[20] For an english translation of *Exemple de l'une des manières universelels du SGDL touchant la practique de la perspective*, see (Desargues, 1987, 144-160).

in *Constructio et usus scalae perspectivae*. A brief description of the dispute over the authorship of the method, however, enables us to understand Leibniz's approach to perspective studies.

The presentation of the Arguesian method by J. Dubreuil in *La perspective pratique* (1642) was the trigger for this disagreement. In addition to his discontent over the description of his methodology, Desargues was especially annoyed because Dubreuil used in his treatise the same plates as *Exemple de l'une des manières universelles*.[21] When Dubreuil learned about the Lyonnais geometer's opinion, a tense public dispute began. Since then, the circle of French perspectivists was divided in two. Supporters of both groups accused the others of plagiarism.[22] The dispute was fuelled by other contentious issues in which Desargues also participated. The intervention by Jean Beaugrand, the royal secretary (with responsibility for the royal privileges of publication), was decisive. Publicly opposed to the Lyonnais geometer since the publication of the *Brouillon Project* (1639), Jean Beaugrand promoted the publication of *La perspective speculative et pratique* (1643). In this treatise, the mathematician E. Migon reproduces the method of the former royal engineer, J. Aleaume (1562-1627). Dubreuil's supporters turned to *La perspective speculative et pratique* to justify their allegation of plagiarism against Desargues. Some years later, A. Bosse published *Manière universelle de Mr Desargues* (1648). With this work Bosse puts the method of his teacher, Desargues, on a different level from that of Aleaume. Leibniz's study of this treatise ended with the aforementioned criticism of the Arguesian method, in which he accused the French geometer's method of being inappropriate for the invention.

Leibniz's study of these three treatises (*La perspective pratique*, *La perspective speculative et pratique* y *Manière universelle de Mr Desargues*) shows his interest in the method of scales. But it is *Constructio et usus scalae perspectivae*, the first of Leibniz's writings on perspective, which definitively confirms it.

3. Leibniz's principal manuscripts on perspective

Leibniz's continuation with geometric characteristic (in his 1679 version), his access to an extensive bibliography dedicated to perspective, and his study of the treatises involved in the dispute regarding the authorship of the method of scales were the three contextual fundamentals of his research. Of the six writings that make up this study, however, the following are analysed: *Constructio et usus scalae perspectivae*; *Origo regularum artis perspectivae quales sine libro ac magistro inveni*; and *Scientia perspectiva*. These three are the most relevant writings; the other materials should be considered as annotations.

This section is divided into three parts: the first analyses *Constructio et usus scalae perspectivae*; the second *Origo regularum artis perspectivae*; and the third *Scientia perspectiva*.

3.1. *Initial interest in perspective:* Constructio et usus scalae perspectivae

In this section, *Constructio et usus scalae perspectivae* is analysed. The section is divided into three parts: in the first, the basic procedure of the method of scales introduced in the writing is presented (this presentation also serves to characterise the theoretical meaning of the method); in the second, the perspective exercise carried out by Leibniz is reconstructed; and in the third, the first formulation of the general rule of perspective is set out.

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21 One of the notes found in Leibniz's papers on perspective mentions Dubreuil's presentation of the Arguesian method (LH 35 XI 17, 8). Other notes refer to other passages in that same work (LH 35 XI 17 and 12).

22 For an overview of the dispute, see (Andersen, 2007, 448-451; Taton, 1971, 46-51). In addition to the marginal notes on the treatises of Aleaume, Dubreuil and Bosse, a brief note by Leibniz bears witness to his knowledge of this dispute. In this note, the German thinker wrote the following: 'Des Argues a publié à ce qu'on croit l'Optique d'Aleaume comme la sienne. Mons. l'Abbé Mariotte'. (Leibniz, 1923, series VIII, volume 1, 2009, 227).

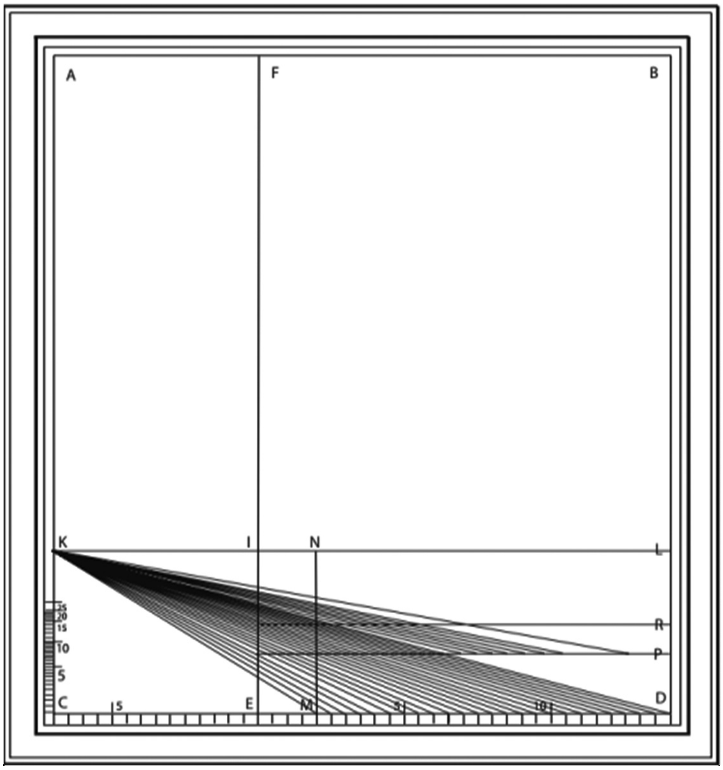

Plate 1. Marginal notes in *La perspective speculative et pratique* (Leibniz, 1923, series VIII, volume 1, 2009, 228).

3.1.1. *Discussion on the method of scales: Leibniz versus Aleaume*

Despite his exhaustive study of *Manière universelle de Mr Desargues*, Leibniz turned his initial attention to *La perspective speculative et pratique*.[23] As with the method of scales developed by Aleaume (*Plate 1*), the method in *Constructio et usus scalae perspectivae* also determines the distances between transversals through the sections produced by a bundle of orthogonal lines on a vertical centre line.[24] In *La perspective speculative et pratique*, point K (on line *KL*, the horizon) constitutes the origin of the bundle of orthogonal lines, line *FE* is the vertical centre line (on it is I, the point of view). This way of determining the distance between transversals makes it possible to specify the vertical scale (*DL or CK*) on the frame of the plane of representation. Unlike the horizontal scale *CD*, divided by units of the same value, the units that divide *CK* and *LD* vary. They become smaller as they approach the horizontal (*KL*).

In a similar way, Leibniz places the origin of the bundle of orthogonal lines at point O (identifying this O as *oculus*),[25] considering *NP* as the vertical centre line. As with the previous method, a point is marked at each of the intersections of the orthogonal lines with the vertical centre line (*Figure 1*). In this way, Leibniz's method intersects $O1$, $O2$, $O3$, $O4$ with *NP*, with this last line marked as follows: (1), (2), (3),

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[23] The dedication that Leibniz places in the study of Bosse's work could be set against that placed in Migon in the marginal notes of these works. Respectively, (Leibniz, 1923, series VIII, volume 1, 2009, 210-226) and (Leibniz, 1923, series VIII, volume 1, 2009, 228-232).

[24] The transversal lines are the straight lines parallel to the ground line. In accordance with this, in *Plate 1* (*La perspective speculative et pratique*), they are those parallel to *CD*. Neither the transversals nor the verticals, perpendicular to the previous, converge at one point. The other parallels do.

[25] This difference with respect to the previous method, in which the point of view or *oculus* was placed at I, anticipated the important theoretical divergence that would soon separate these two methodologies.

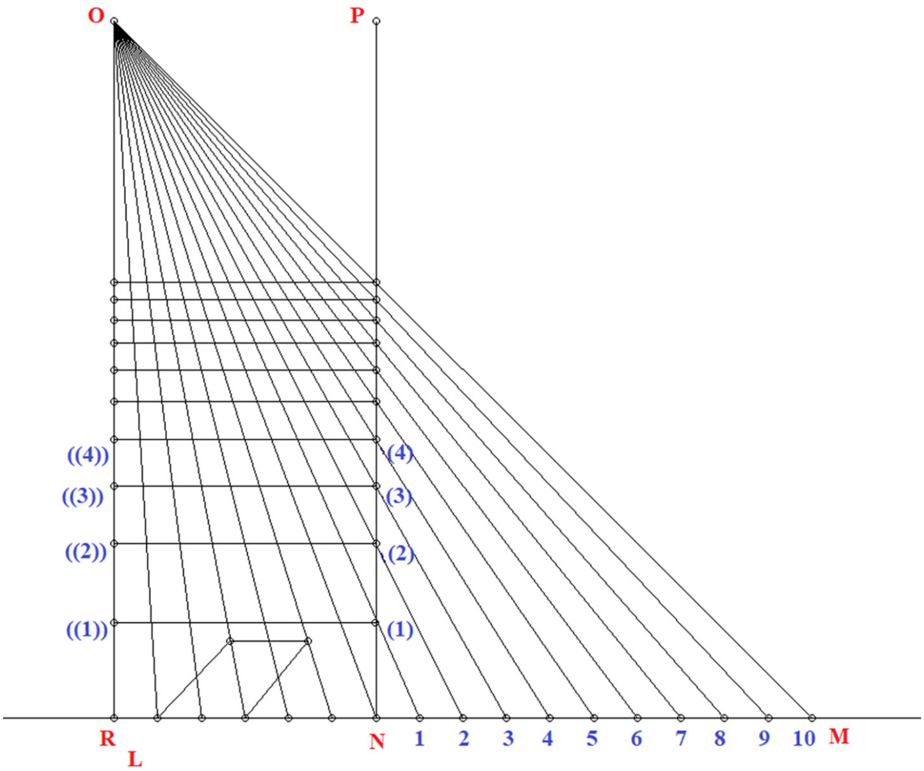

Figure 1. *Constructio et usus scalae perspectivae*.

(4). As also with Aleaume's method, from each of these points one of the transversals starts. From (1), (2), (3), (4), the lines go straight to *OL*, with this last line marked as follows: ((1)), ((2)), ((3)), ((4)). This then produces the distances that we wanted to find: (1) ((1))-(2) ((2)); (2) ((2))-(3) ((3)); (4)-((4)).[26] This way of determining the distances between transversals relates the method of scales of Aleaume with *Constructio et usus scalae perspectivae*; however, these methods differ in their objectives. While Aleaume sought to create a coordinate system, Leibniz developed a purely pictorial method.

This method illustratively identifies the objective of the linear perspective. Geometric research on perspective manages to represent the magnitudes of objects proportionally. The determination of the distances between the transversals shows the relationship between the units of measure that constitute the magnitudes of the objects represented. Thus, for example, the units that divide segment *RN* are larger than the units that divide segment (4)-((4)) (*Figure 1*). Even being the same size, *RN* is divided into six units and (4)-((4)) is divided into ten. The units of measurement that make up these two segments are in a ratio of 10:6, that is, each unit of *RN* is 1.6 times larger than each unit of (4)-((4)). This ratio between units makes (4)-((4)) larger than *RN*, that is to say, that this ratio *represents* the *actual* magnitude of (4)-((4)) with respect to *RN*.

3.1.2. The pictorial procedure: an exercise in perspective

In *Constructio et usus scalae perspectivae* Leibniz constructs the image of a solid, specifically a cube. This exercise gave a new meaning to the conceptual structure shared with Aleaume.

26 Unlike Leibniz's method, the Aleaume procedure marked the transversals on the right side of the frame. It should also be noted that the Aleaume method started at *MN*. Although *FE* is a vertical centre line, Migon chose to take (the orthogonal line) *KM* as the beginning of the progression. The intersection of the next orthogonal line (starting from *K*) with *MN* will give the value of the first transversal. For a more complete explanation of Aleaume's method, see (Andersen, 2007, 420-421).

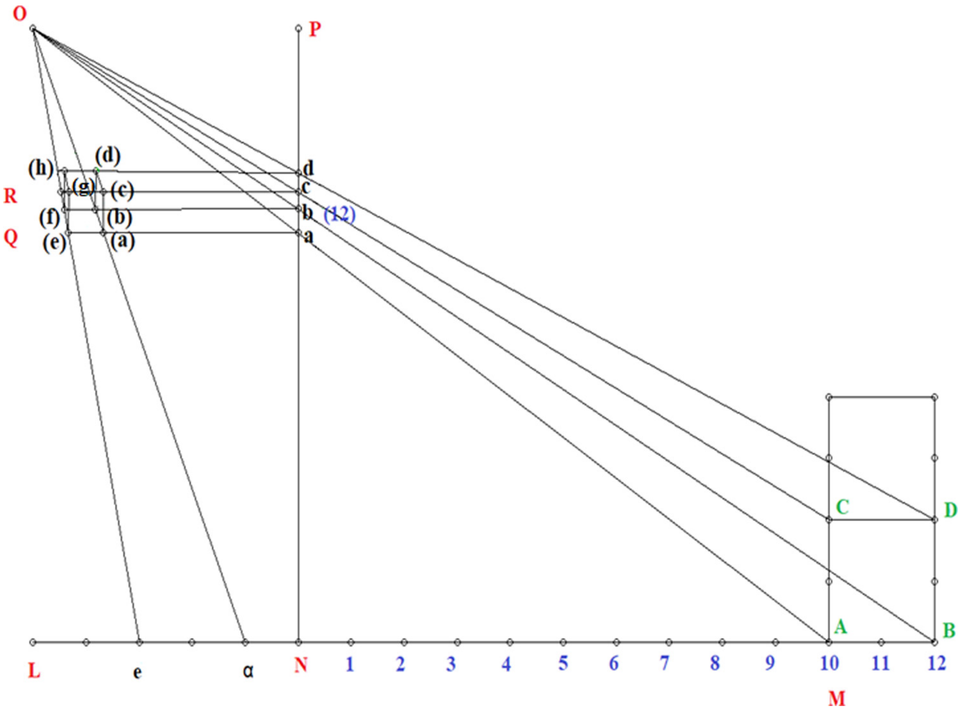

Figure 2. *Constructio et usus scalae perspectivae*.

Leibniz begins by placing the geometrical representation of the object (*ABCD*) on the right side of *NP*.[27] He then chooses to place the cube at a distance of 10 units, which is why he places *A* over this number on *NM* scale. Since each of the sides of this body measures 2 units, he places point *B* on number 12 (on *NM*). The intersection *OA*-*NP* then gives *a*, just as the *OB*-*NP* intersection gives *b*. By then, Leibniz has connected (on the left side of *NP*) *O* to *e* and *α*. These segments (*Oe* and *Oα*) mark the removal (*recessus*) of the sides of the cube. From *a*, he draws a line (perpendicular to *NP*) that intersects these last two lines: obtaining (*a*)(*e*) (the front side of the cube - parallel to *LM*). Similarly, from *b*, obtaining (*b*)(*f*) (the opposite side and parallel to the previous). Leibniz uses this procedure in a similar way to construct height. From *O*, he draws straight lines to *C* and *D* intersecting *NP* at *c* and *d* respectively. From these last points, he draws two lines (perpendicular to *NP*) towards the left. Points *c*, *d*, *g*,[28] *h* are at the intersection between these last lines (drawn from *c* and *d*, perpendicular to *NP*) and each of the (imaginary) lines drawn from *a*, *b*, *e*, *f* (parallel to *NP*). It only remains to unite these eight points to obtain the image of the cube (*f*,*e*,*a*,*b*/*h*,*d*,*c*,*g*) located at a distance of 10 units.

3.1.3. The first formulation of the main rule of perspective

Leibniz did not seem satisfied with his pictorial method and on the same folio he provided another text. In it, he defines various notions, such as ‘appearance,’ ‘horizon’ and ‘vertical,’ and analyses aspects of

[27] This way of proceeding is significant since the greatest achievement claimed by the Aleaume method is to have dispensed with geometrical representation. See (Andersen, 2007, 418-427). By contrast, Desargues and Bosse maintained, and even stressed, the importance of adequately building this representative dimension. See (Andersen, 2007, 427-447; Debuiche, 2013, 370).

[28] This letter does not appear in *Figure 2*. This should, however, respond to the size of the original figure, its small proportions would have prevented Leibniz from writing it.

his own method.[29] Although these last lines of reasoning cannot be followed (as they refer to letters and figures not included in the writing), they have a recognisable objective. With them Leibniz begins to work on the main rule of perspective. He formulates this rule in the following way: '*Regula generalis*: Elongatio, elevatio et declinatio objectivae divisae per numerum constantem, dant perspectivas' (LH 35 XI 17, 23).[30]

3.2. *The writing about the general rule of perspective:* Origo regularum artis perspectivae quales sine libro ac magistro inveni

In *Origo regularum artis perspectivae*, Leibniz resumed research on the *regula generalis*. However, the text begins by describing the theoretical model of perspective: firstly, it identifies its three conceptual elements, that is, the *tabula* (the representation plane), the *oculus* (the viewer position) and the *objectum* (the object represented); secondly, it presents the problem to which it responds, that is, how to determine the location of the image of a certain point on the representation plane.[31] After this presentation, the geometric-pictorial research began.

This section is divided into two parts: in the first, the geometric argumentation that justifies the rule of perspective is set out; in the second, the pictorial meaning of this rule is reconstructed, that is to say, its application to the construction of representations.

3.2.1. *The geometric argumentation: similarity and proportion between triangles*

Before reconstructing the geometric argumentation that follows this presentation, it is worth formulating the rule of perspective shown in *Origo regularum artis perspectivae*. Towards the end of the work, Leibniz expresses it as follows: 'Altitudo et latitudo quaesiti erunt inter se in composita ratione ex ratione altitudinis et latitudinis datorum directa, et longitudinem eorundem datorum reciproca' (LH 35 XI 17, 19-20).[32] This rule makes it possible to determine the values of height (*altitudo*) and latitude (*latitudo*) of the image. Before writing the aforementioned passage, in the middle of the writing, Leibniz characterises each of these parts separately. He writes: 'Primo: *Summa longitudinum est ad longitudinem oculi* (*ET* vel *OF*) *ut latitudo objecti* (*SN* vel *MT*) *est ad latitudinem apparentiae* (*TP*) *quaesitam*' (LH 35 XI 17, 19-20).[33] Immediately after: 'Secundo: *Summa longitudinem* (*EN*) *est ad longitudinem objecti* (*SM*), *ut altitude oculi* (*OH*) *est ad altitudinem apparentia* (*QL*) *quaesitam*' (LH 35 XI 17, 19-20).[34] The argumentation that leads Leibniz to these conclusions is reconstructed below. The procedural importance acquired by similarity in

[29] '*Aparentia* est repraesentatio objecti in Tabula; *Planum oculi* voco planum Tabullae parallelum per oculum; *Distantiam Tabulae* voco distantiam tabulae a plano oculi seu oculi a tabula; *Distantiam objecti* voco puncti objectivi distantiam a plano oculi. *Elongationem objecti* voco puncti objectivi distantiam a tabula; *Horizon* est planum certum semel erectum, quod solet intelligi horizonte reali parallelum; *Verticalis* est planum horizonti et tabulae normale'. (LH 35 XI 17, 23).

[30] '*General rule*: The objective extension, elevation and inclination divided by a constant number give the perspectives'. We thank Manuel Molina Sánchez, professor of Latin at Granada University, for his contributions to the translations of the latin texts.

[31] At the beginning, perspective was characterised as the construction of images by means of a central projection. When analysing the method of scales, the objective of the discipline was characterised in a more specific way. Geometric analysis of perspective focused on determining the distances (or ratios between units of measurement) in the representation. By contrast, in *Origo regularum artis perspectivae*, the construction of images focused on determining the positional values of the image point. This way of understanding research on perspective did not contradict the previous one, but it involved taking a different approach.

[32] 'Height and latitude will be found according to the following composition: in direct ratio to the height and latitude information, and in reciprocal ratio to the same information with respect to length.'

[33] 'First: *The longest length is at the length of the eye* (*ET or OF*) *what the latitude of the object* (*SN or MT*) *is at the latitude of the appearance* (*TP*) *sought*.' Where in the text '(*ET* vel *OF*)' appears, in the original it was '(*ET* vel *OP*).'

[34] 'Second: *The longest length* (*EN*) *is at the length of the object* (*SM*) *what the height of the eye* (*OH*) *is at the height of the appearance* (*QL*) *sought*.' Where in the text '(*TP*)' appears, in the original it was '(*TB*).'

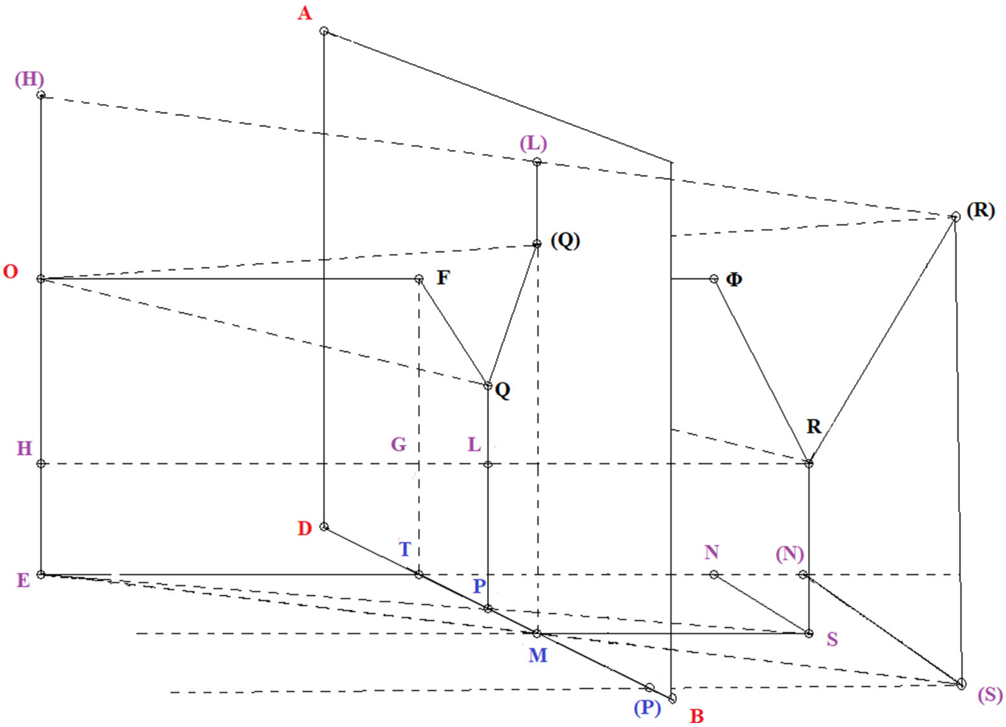

Figure 3. *Origo regularum artis perspectivae*.

this argumentation can be seen;[35] this relationship underpins the analysis of proportions embodied in the following construction.

The exercise that opens *Origo regularum artis perspectivae* requires the determination of the location (height and latitude) of image point *Q* on the representation plane –*ADB* – (*Figure 3*).[36] To find it, Leibniz determines the magnitudes of segments *TP* (*altitudo*) and *QL* (*latitudo*). For this, he makes a convoluted geometric argumentation, based mainly on the similarities of triangles *ENS-ETP* (and *SPM*) and *OHR-QLR*.[37] Given these similarities, he establishes a series of proportionality relations that allow him to obtain the desired values. He begins by determining the latitude in order to later find the height. The first step of the first part of the argumentation is to draw the diagonal line that connects the location of the observer (E) to the objective point (S). On this line is image point P, representation on the *tabula* of objective point *S*.[38]

Having built this structure, Leibniz notes that the proportionality between the segment formed by *ET+MS* with respect to segment *SN* is equal to the proportionality that *TP* has with respect to *ET*.[39] Since *ET+SM*,

35 See (Debuiche, 2016, 63-65). In this work, Debuiche points out the importance of the concept of similarity in Leibniz's research on perspective.

36 The objective point that corresponds to this image point is point *R*.

37 The triangle *OHR* allows us to identify the conceptual elements of perspective: *O* indicates the viewer's eye (*oculus*), *H* designates the ground plane, that is, the plane where the viewer stands, and *R* is the objective point, thus, the point meant to be represented. According to the definitions, *OH* is the viewer's height. Moreover, *Q* is the perspective image of *R* and *L*, the perpendicular projection of *H* on the picture plane.

38 Points *E* and *S* are projections on the horizontal plane (lower than the ground plane) of *O* (*oculus*) and *R* (objective point), respectively. Similarly, *P* represents point *Q* on this plane.

39 'Jam ob triangual ETP, ENS similia esse patet ut EN, sive ut ET + MS est ad SN, ita ET ad quartam quaestitam TP'. (LH 35 XI, 17, 19-20). Keeping in mind that *ET+MS* is equal to *EN* helps us to understand this argumentation. Thus, in this way, we find that *EN* with respect to *SN* (triangle *ENS*) is the same as *ET* is with respect to *TP* (triangle *STP*). This same proportion occurs between

SN, and *ET* are known elements (by definition), the *TP* value can be determined.[40] Leibniz then goes on to analyse how to determine height (*LQ*). He begins by noting that *LQ* has the same proportion with respect to *OH* that *RL* has with respect to *RH*; this proportionality is also found between segments *SP* and *SE*, as well as between segments *SM* and *EN*. Since *EN* is the same as *ET+SM*, it can be said that *SM* has this same proportionality with respect to the sum of segments *ET* and *SM*. As a result of this chain of argumentation, the magnitude of *LQ* is concluded. Thus, *OH* has the same proportion with respect to *LQ* as *SM+ET* has with respect to *SM*, and the magnitudes of all these segments (different from *LQ*) are known.[41]

3.2.2. The first formalisation of the main rule of perspective

This construction/demonstration shows Leibniz's mathematical ability. Having obtained the results of the reasoning, he decides to formulate them again. Each segment is then given a meaning within the respective study model. In this way, instead of talking about segments, he goes on to refer to elements that intervene in the construction of representations. He presents the following formulas:

Application to the latitude: $\frac{S.Lo}{Lo.Oc} \sqcap \frac{Lat.Ob}{\boldsymbol{Lat.App}}$ $\qquad S.Lo \sqcap \frac{Lo.Oc., Lat.Ob}{\boldsymbol{Lat.Ap}}$

Application to the altitude: $\frac{S.Lo.}{Lo.Ob.} \sqcap \frac{alt.oc.}{\boldsymbol{alt.ap}}$ $\qquad S.Lo \sqcap \frac{Lo.Ob., alt.Oc}{\boldsymbol{alt.ap}}$

The nomenclature in these formulas is described below: **S.Lo** (*Summa longitudinum*): the distance between the position of the viewer and the location of objective point *EN*[42]; **Lo. Oc** (*longitudinem oculi*): the distance between the location of the viewer (*oculo*) and representation plane (*tabula*) *ET*; **Lat. Ob** (*latitudo objecti*): the distance between the main radius and objective point *NS*; **Lat App** (*latitudinem apparentiae*): the distance between the point of view and image point *TP*; **Lo. Ob** (*longitudinem objeti*): the distance between the objective point and representation plane *SM*; **Alt oc** (*altitude oculi*): the altitude of the eye *OH*; **Alt. Ap** (*altitudinem apparentia*): the distance between the image point and the point on the representation plane that marks ground line *QL*.[43] The magnitudes (*longitudinem*) of the different segments constitute the

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SM and *MP* (*SPM*). These proportion relations are correct given that *STP*, *ENS* and *SPM* are similar. Leibniz deals with these ratios of similarity/proportionality in *Auxilia Calculi* (LH 35 XI 17, 24). *Auxilia Calculi* does not, however, articulate the perspectivist significance of the question evident in *Origo regularum Artis Perspectivae*.

40 *ET* = distance from the viewer to the representation plane; *MS* = distance between the objective point and the representation plane; *EN* = *ET+SM*; *SN* = objective declination. The perspectivist exercise requires the construction of the representation (in this case, the apparent declination *TP*) with these values.

41 'Item ob triangula OHR, QLR similia est; LQ:OH: :RL:RH: :SP:SE: :SM:EN: :SM:SM + ET; sive erit ut SM+ET ad SM, ita OH ad quartam proportionalem LQ'. (LH 35 XI, 17, 19-20). This argumentation is intimately connected to the previous one (referring to declination). On this occasion, the result, instead of considering segment *SN*, introduces *ET*. That is, instead of the objective *declinatio* of point *Q* (*SN*), consider as the second element the distance to the objective point/representation plane *ABD* (*SM*). The rest of the argumentation consists of taking back the proportion relation sought, between the height of the viewer (a value determined from the beginning) and the height of the represented point, up to the initial relation. So that the aforementioned relations of similarity between *STP*, *ENS* and *SPM* can be used.

42 As in this case, where *S.Lo* is identified with segment *EN*, in each of the definitions, each of the terms is illustrated with its corresponding segment. These terms may, however, refer to different segments than those used to illustrate them.

43 Although it has previously been indicated, it is worth remembering that the horizontal plane where Leibniz situates *ENS* can be, and is in fact, located below the ground plane.

distance values acquired by these variables. After renaming $Lo.Oc.$ as a and $Alt.Oc.$ as b, after a series of transformations (not specified in the writing),[44] Leibniz arrives at:

$$\frac{\boldsymbol{lat.ap.}}{\boldsymbol{alt.ap.}} \sqcap \frac{a}{b}, \ \frac{lat.ob.}{lo.ob.}.$$

Having thus given a symbolic structure to his rule,[45] Leibniz writes: 'Altitudo et latitudo quaesiti erunt inter se in composita ratione ex ratione altitudinis et latitudinis datorum directa, et longitudinem eorundem datorum reciproca' (LH 35 XI 17, 19-20). In this way, he describes the relationship expressed in the formula; although this constructs the length in an articulated manner, that is, as a (viewer-plane of representation distance) divided by b (viewer height).

3.3. *The universality of perspective:* Scientia perspectiva

Origo regularum artis perspectivae constitutes a decisive step in Leibniz's research, but the most complete version of his analysis on perspective is in *Scientia perspectiva*. At the very beginning of *Scientia perspectiva*, Leibniz presents a classic definition of the discipline: '*Scientia perspectiva* est, ars objeti apparentiam in Tabula exhibendi [. . .]'.[46] Immediately afterwards, he identifies the elements of *ars*: the *tabula* (the representation plane); the object (the represented element); and the eye (the viewer's position). Unlike the introduction to *Origo regularum artis perspectivae*, he presents a number of variations over these elements: he allows the *tabula* to be flat, concave or convex (or mixed); he accepts any geometric element as a possible object of representation: he allows that the viewer can be placed in any position with respect to the representation plane. Leibniz also includes indications about the material support of the *tabula*, about its reflective or refractive nature; the possibilities of luminous application (its simplicity or multiplicity) and the distance of the focus; and the importance of correctly contrasting light and dark colours. Before starting the research, however, Leibniz simplified the theoretical scenario. He established the following assumptions: the elimination of any consideration of reflection and/or refraction; the choice of a non-curved representation plane; and, finally, the assumption that light rays are parallel to the representation plane. These conditions relax the statement that precedes them: 'Haec idea perspectivae vastissima est, et totam comprehendit Geometriam situs, quae scilicet a magnitudinis (praeterquam rectarum) et motus calculo abstinent' (LH 35 XI 1, 9).[47] The interpretation of this sentence offers the theoretical meaning of Leibniz's research. Why does the theoretical amplitude of the discipline depend on its analysis having to abstain

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[44] Given '$Lo.Oc \sqcap a, \ alt.oc \ \sqcap b$' (LH 35 XI, 17, 19-20), these are the intermediate (algebraic) steps:

$$\frac{a, Lat.Ob}{Lat.Ap} \sqcap \frac{Lo.Ob., b}{alt.ap}; \ \frac{a, Lat.Ob.}{b} \sqcap \frac{Lo.Ob., lat.Ap}{alt.ap}; \ \frac{lat.ap.}{alt.ap.} \sqcap \frac{a}{b}, \ \frac{lat.ob}{lo.ob}.$$

[45] It is relevant to point out that the spectator position has been here identified by $\frac{a}{b}$. Then, it is not a simple value, but a complex one.

[46] 'Scientia perspectiva est, ars objecti apparentiam in Tabula exhibendi, id est *data aliqua Tabula* (sive plana, sive superficie concava aut convexa, aut mixta) et objecto (sive id sit punctum, sive linea, superficies, solidum, cujus natura data est) *datoque eorum situ inter se*, *et cum oculo* (etiamsi distantiae quaedam ponantur infinitae aut infinite parvae; infinitae, ut si oculus a tabula vel objecto vel haec inter se infinite distare intelligantur; infinite parvae, si oculus incidat in Tabulam vel objectum, quo casu neutrum debet esse figura plana, vel objectum in tabulam, quo casu saltem non debent esse planae parallelae; item etiamsi oculus sit inter objectum et Tabulam; vel objectum inter tabulam et oculum) *datoque medio* ([...]) *et situ ac figura lucidi* ([...]) *ducendi in tabula lineas* ([...]) lineas objecti ([...]) *determinantes* ([...]) *et exprimendi superficies lucidas umbrosasque* ([...]) *repraesentes lucem et umbram objecti* (...)' (LH 35 XI 1, 9).

[47] 'This idea of perspective, which certainly departs from the calculation of magnitude (except straight lines) and movement, is vast and encompasses the entire *Geometriam situ*'.

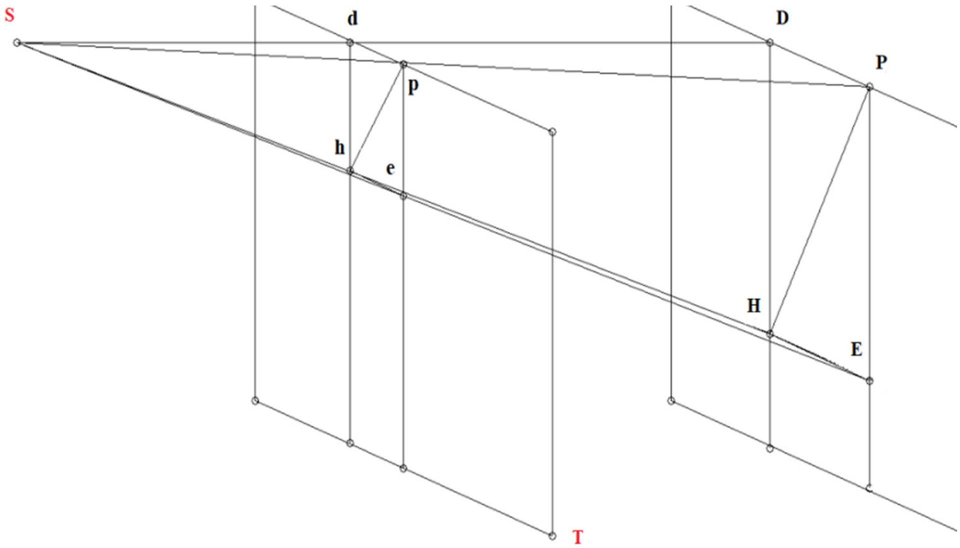

Figure 4. *Scientia perspectiva*.

from the calculation of magnitudes and movement? The analysis of *Scientia perspectiva* seeks to answer this question.

This section is divided into four parts: in the first, the simplification of the general rule of perspective is presented; in the second, the minimum distance hypothesis is analysed; in the third, the infinite interval hypothesis is discussed; and in the fourth, the question about the amplitude of the theoretical meaning of the discipline is answered.

3.3.1. The simplification of the general rule of perspective

In the first place, Leibniz reworks the main rule of perspective.[48] This time, he constructs a simpler argumentation based on the similarities of triangles *SPD-Spd* and *SPE-Spe* (*Figure 4*). He designates *SP* as *Radio principali* and *P* as the main point, that is, the point of view. Given these similarities between triangles, Leibniz notes the following relations: pd : *PD* : : $\underline{Sp}$: *SP*; $\underline{pe}$: *PE* : : $\underline{Sp}$: *SP*.[49] In accordance with them, taking *P* as a reference, the position of *h* can be determined. The *declinatio* (*dextrum sive sinistrum*) of *h* is determined by $\underline{pd}$: *PD* : : $\underline{Sp}$: *SP*; its *elevatio* (*supra vel infra*) is obtained from $\underline{pe}$: *PE* : : $\underline{Sp}$: *SP*. These relations connect the viewer-*tabula*/image-object (*SP*; *Sp*) distances to the positioning values (*declinatio*: *PD* (*pd*); *elevatio*: *PE* (*pd*)). As in *Origo regularum artis perspectivae*, to make this relation work, a certain number of values must be known. However, the simplicity of the argumentation in *Scientia perspectiva* allows Leibniz to explore the limits of the research. He then proposes two hypotheses: a minimum distance between viewer-*tabula*-object; and the extension of parallel lines in an infinite interval.

3.3.2. The minimum distance hypothesis: the construction of the formulas of declination and elevation

Having verified that the behaviour of these pairs of triangles (*SPD-Spd* and *SPE-Spe*) makes it possible to determine the image point, Leibniz reworks the research. He goes on to analyse how to build the perspective image of a solid object. He presents then three different planes: one horizontal, which passes through

[48] Thus, he expresses the rule when ending this new argumentation: 'Itaque *inclinationes vel declinationes punctorum apparentium sunt ad inclinationes vel declinationes punctorum objectivorum, ut distantia spectatoris a plano tabulae ad distantiam spectatoris a plano objectivo*. Distantia autem Spectatoris intelligi potest portio radii principalis intercepta'. (LH 35 XI 1, 9).

[49] 'Est autem $\underline{pd}$: *PD* : : $\underline{sp}$: *SP*. ob triangula similia $\underline{Spd}$, *SPD*. Eodem modo $\underline{pe}$: *PE* : : $\underline{sp}$: *SP* ob triangula similia $\underline{Spe}$, *SPE*'. (LH 35 XI 1, 9).

the eye; one primary (*objectivum primarium*), parallel to the *tabula* passing through a fixed point of the object;[50] and, one vertical, perpendicularly passing through the two previous planes.[51] At the same time as he presents these methodological elements, he introduces the case of a *distantia minima* (or infinitely close distance). For this analysis, Leibniz constructs the following terminology: $\boldsymbol{a}$ = the distance between viewer and representation plane; $\boldsymbol{b}$ = the distance between viewer and primary objective plane; $\boldsymbol{l}$ = the distance between the objective point and the primary objective plane; $\boldsymbol{E}$= the distance between objective point and horizontal plane; $\boldsymbol{D}$ = the distance between objective point and vertical plane. In accordance with this terminology, he states:

- the height or elevation results in: $\boldsymbol{e} = \dfrac{\boldsymbol{a}}{\boldsymbol{b} \mp \boldsymbol{l}}\boldsymbol{E}$
- the latitude or declination results in: $\boldsymbol{d} = \dfrac{\boldsymbol{a}}{\boldsymbol{b} \mp \boldsymbol{l}}\boldsymbol{D}$

When introducing the nomenclature Leibniz considers that all of these distances have a *minimum magnitude*; this initial *minimum distances* should be considered as a starting point.[52] Departing from this minimum distance planes begin to detach from each other; according to this removal, the values of elevation (e) and declination (d) modifies. As it can be noticed, b ∓ l plays a significant role in these modifications;[53] b ∓ l is the factor that determines the reciprocity of the proportionality between the objective values (E, D) and apparent (e, d) values. Once this idea of an initial moment has been conveyed, Leibniz considers that they all these distances are simply *known values*.[54] Depending on the values of these variables, the solid is represented in a different way.

3.3.3. *The infinite interval hypothesis: the difference between the referent and its images*

After constructing these two formulas, Leibniz presents the notion of point of view. He characterises it in the following way:

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[50] This *objective primary plane* would correspond to plane O in *Figure 4*. As now Leibniz works with a solid (not just with a point), this plane cut this volume.

[51] ‘In objecto intelligantur tria plana, unum horizonti parallelum transiens per oculum, quod vocemus horizontale, alterum Tabellae parallelum transiens per unum aliquod punctum fixum objecti quod vocemus objectivum primarium, tertium verticale transiens per oculum et punctum primarium, perpendiculare horizonti et tabulae’. LH 35 XI 1, 10. Example of these type of planes can be found in (Migon and Aleaume, 1643, 7-9).

[52] One might even suppose that the minimum distance between the elements stated would give an exact correspondence of the object with its images. Given this closeness, the representation of the object would coincide with the same object. In addition, these formulas give significant results, such as these: when $b \mp l = 1$, then, $e = a$ and $d = a\ D$; when $b \mp l = 0$, both e and d give infinity.

[53] ‘Apparentia lineae rectae est recta, nisi tunc cum omnia ejus puncta in radium aliquem visionis incidunt, tunc apparentia rectae est punctum quod fit quando plura puncta eandem habent inclinationem et declinationem (ab eodem latere), seu cum inclinationes et declinationes proportionaliter crescunt cum ipsis $b \mp l$ seu longitudinibus distantia oculi a plano objectivo primario auctis’. (LH 35 XI 1, 10). B (distance spectator-plane *primarium objectivum*) and l (distance objective point-plane *primarium objectivum*) play a similar to *SM+ET* in *Origo regularum artis aerspectivae*.

[54] When concluding the argumentation, he presents it like this: ‘[...] data distantia oculi et tabellae a, distantia oculi et plani objectivi primarii b, distantia puncti objectivi propositi et plani objectivi primarii (seu longitudine verař L., distantia puncti objectivi propositi et plani verticalis (seu declinatione sive latitudine vera) D. distantia puncti objectivi propositi et plani horizontis oculi (seu inclinatione sive elevatione vera) E [...]’. (LH 35 XI 1, 9-10). Previously, when presenting this nomenclature, he supposed a minimum distance between all of these elements: ‘Datur distantia minima spectatoris et Tabellae; item spectatoris et plani objectivi primarii, prior vocetur a., posterior b.; jam distantia minima puncti objectivi a plano objectivo primario vocetur l. seu longitudo, et distantia minima puncti objectivi a plano objectivo primario erit *b+l* (posito planum objectivum primarium esse proprius oculo Sit remotius erit *b-l*). Distantia minima puncti objectivi a plano horizontali seu inclinatio vel elevatio vera vocetur E’. (LH 35 XI 1, 9-10).

> Hinc cum parallelae intelligi possint convergentes infinito ab hinc intervallo, patet etiam pucntum quo recta ipsis parallela per oculum ducta Tabellae occurrit, eorum apparentiam communem esse, ac proinde parallelarum apparentias esse convergentes, modo ipsis Tabulae parallelae non sint (LH 35 XI 1, 10).[55]

Straight lines are represented in perspective as converging lines at one point; this convergence point is the point of view. This arrangement of parallels on the plane of representation structures the images in perspective. All parallel rays are represented as converging lines with one exception. Those parallel to the plane of representation are still represented as parallel: these are the transversals (cf. 3.1.1). In the first illustration, the point of view is represented; in the second illustration, the transversals are shown. The characterisation of the point of view as the meeting point in the infinity of the parallels comes from Arguesian perspective.[56] However, in *Scientia perspectiva*, Leibniz does not follow the Arguesian method. After this characterisation, he begins to work using a fairly classic method.

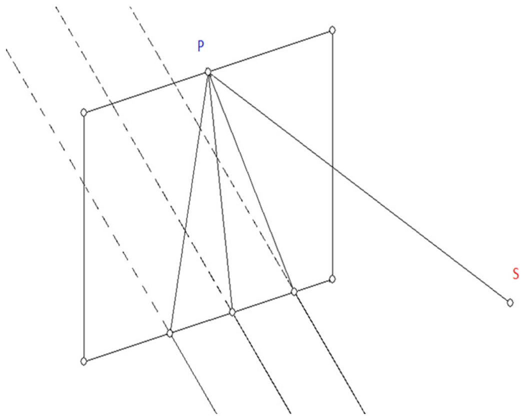

Illustration 1

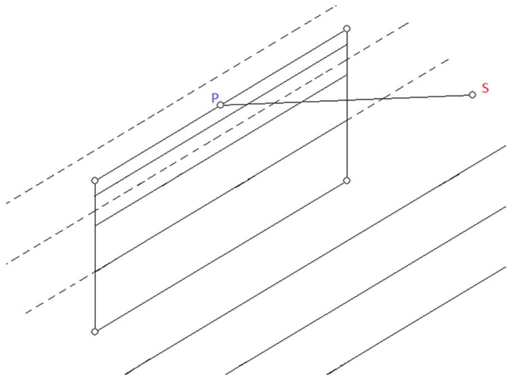

Illustration 2

3.3.3.1. The representative proportion: parallelism as a basis of similarity To represent the triangle *ABC* on *TTO* (*Figure 5*), Leibniz uses principal lines (*PM*, *PE* and *PN*)[57] and points of distance (H, η and V). Although Leibniz refers to the classical definition of point of distance, he constructs these three points using angular ratios.[58] From each of the lines connecting the vertices of the original figure and the representation plane (*AE*, *BM* and *CN*), Leibniz constructs a 90° angle; from each of these three angles he draws a line that cut the ground line (*TO*).[59] At each of these intersections, he places one of the points (H, η and V). This

55 'When the parallels are thought of as convergent in an infinite interval, the point where these parallels, starting from the eye, find the *Tabula* is a common appearance to all of them, and thus the appearance of the parallels is convergent (provided that these parallels do not respect the *Tabulae*)'.

56 It is worth mentioning that Leibniz transcribes the definition of point of view that Desargues makes at the end of the *Brouillon Project* (included in: A VII 7, 111).

57 M, E and N are perpendicular projections of the vertices of the original triangle, B, A and C, respectively. When connecting these points (M,E, N) to the point of view (P), we obtain, by definition, the main ones.

58 The definition of point of distance refers to the distance between the viewer and the *tabula*. Therefore, the point of distance is the point that is as far from the point of view on the horizon as the viewer is from the *tabula*. This definition can be found in the marginal notes on *La perspective speculative et pratique* (Leibniz, 1923, series VIII, volume 1, 2009, 228) and in *De la perspective* (LH 35 XI 1,2). However, although H, η and V perform the function of points of distance, they are constructed angularly. In fact, when constructed from 90° on lines perpendicular to the *tabula* (*AE*, *BM* and *CN*), these points (H, η and V) would only find it at infinity. In this case, they could be considered as centres of convergence.

59 Although in the initial construction of the rule (from *SPD-Spd*/*SPE-Spe*) Leibniz considers it an achievement to have gotten rid of the ground line, at this time he uses it again (for reasons of convenience). It should, however, be borne in mind that neither in

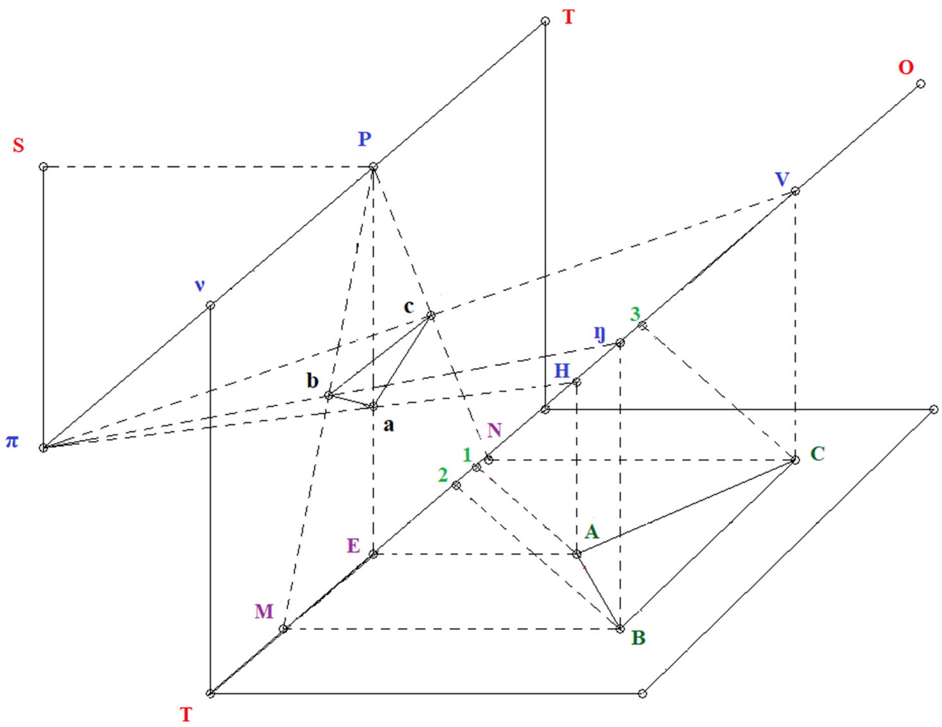

Figure 5. *Scientia perspectiva*.

results in the following triangles: *AEH*; *BMŋ*; *CNV*. On the other side of the representation plane, where he places the viewer (*S*), he repeats the procedure. Now taking *SP* as reference, he constructs a 90° angle. From this angle, he draws a line to the representation plane, intersecting it at π. Thus, he constructs $SP\pi$. From each of the previous points of distance (H, ŋ, V), he draws a straight line to π. In the places where each of these lines intersect its respective principal line (*Hπ-EP*; *ŋπ-MP*; *Vπ-NP*), the corresponding image point is located (a, b and c, respectively). All that remains is to join these points to obtain the image of the triangle in perspective.

Having made the image of *ABC* in perspective in this way, Leibniz continues constructing his methodology. On each of the initial lines (*AE*, *BM*, *CN*), he creates an angle of 45°; from each one of these angles, he draws a straight line to the ground line, marking on it, 1, 2 and 3, respectively. He also performs this process on the side of the viewer's plane, marking point ν. The previously used triangles (*AEH*, *BMŋ*, *CNV*, *SPπ*) end up as follows: on the side of *ABC*: *BMŋ* = *BM2* + *B2ŋ*; *AEH* =*AE1* + *A1H*; *CNV*= *CH3*+ *C3V*.; on the side of *S*: $SP\pi = SP\nu + S\nu\pi$. As in *Origo regularum artis perspectivae*, the similarity between triangles serves as a basis for analysing proportionality. In *Scientia perspectiva*, however, Leibniz uses parallelisms to ascertain the adequacy of this relationship of similarity/proportion.[60] At first, Leibniz tries with triangles $SP\pi$ and *AEH* (*Figure 5*); but, this does not conclude his argumentation. However, when working on the *Figure 6*, making the image in perspective of a parallelepiped (A, B, C), he returns to this argumentation. When analysing the relationship between $SP\pi$ and one of the angles on the object side, *CNV* (previously *C3V*), Leibniz reaches the following conclusion. With *SP*, $S\nu$ and $S\pi$ being respectively parallel to *CN*,

Origo regularum artis perspectivae nor *Scientia perspectiva* does he consider it necessary for the horizontal plane to correspond to the ground. It could be below this level, but also above.

60 In a more conventional way, in *Origo regularum artis perspectivae*, Leibniz presents an analysis centred on the eye-*tabula*/image-object intersection. The analysis of the parallelism between triangles constitutes an original innovation of the German thinker. This way of demonstrating the foundations of the perspectivist methodology is not common.

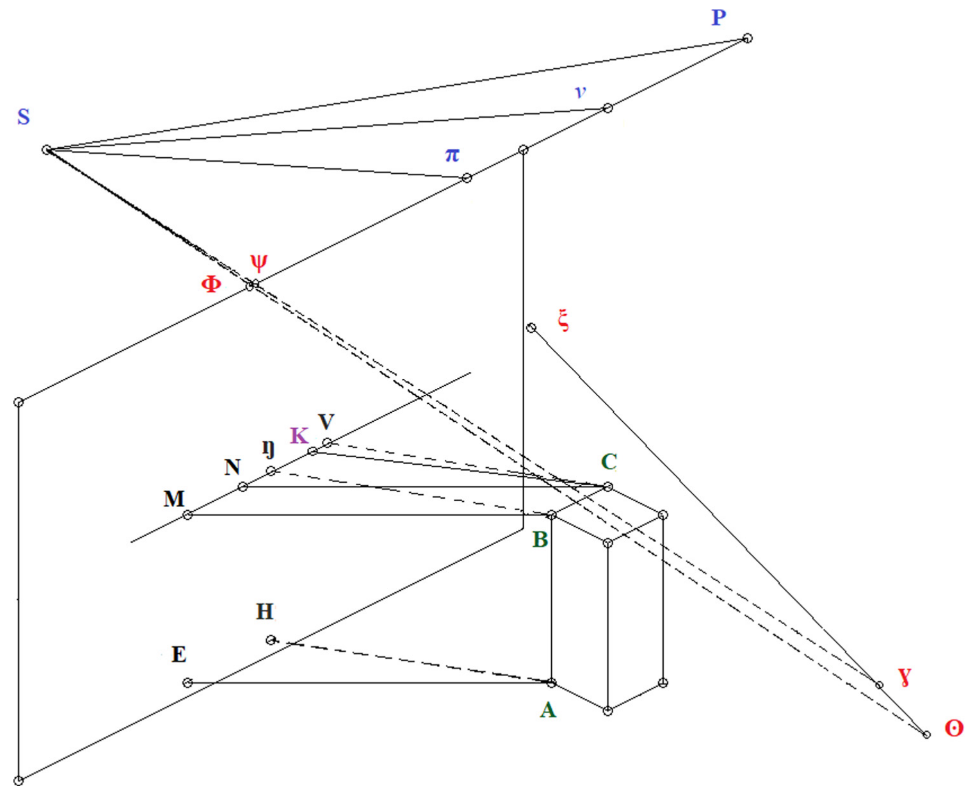

Figure 6. *Scientia perspectiva*.

CK and *CV* (being all these lines perpendicular to the representation plane); with P, ν and π resting on a parallel to the line on which N, K and V lie: the analysis of proportions justifies (and articulates) the correspondence between elements on one side and on the other side of the plane.[61] This allows analysis of the type: given the ratio between $S\nu$ and νP; the magnitude of *NK* also being known (by definition similar to $\nu\pi$); the value of *CK* can be obtained. The proportion between *NK* and *CK* is the same as that between the previous segments ($S\nu$ and πP). The correspondence between both parts of the plane is justified in this way; it is confirmed that any variation, on any of the sides of the *tabula*, modifies the representation. However, after following this line of argumentation, Leibniz varies his argumentation.

3.3.3.2. Centres of convergence and relative places: Leibnizian experimentation After articulating the parallelism and proportionality between the triangles on either side of the plane of representation, Leibniz changes the meaning of the text. He then introduces a variation in his argumentation: Θ and ɣ, two new centres of convergence (*duo diversarum convergentiarum centra Θ et ɣ*). Connected to S, from these centres, he projects the images (*relative places*) of the objective points on the representation plane (*Figure 6*).[62]

61 In fact, in this way, in addition to confirming the similarity between both triangles, Leibniz also makes sure that they are properly arranged. They make up a perspective exercise.

62 Leibniz defines the relative points (*loca relativa*), similar to *EH*, in the following way: 'Generaliter regula perspectivae totius sic exprimetur: *Loca relativa* voco puncta quibus parallelae inter se per puncta realia seu objectiva ductae occurrunt Tabulae. Jam si oculi pariter ac punctorum objectivorum, dentur bina cujusque loca relativa secundum diversos parallelismos, et ab uno quoque loco relativo puncti objectivi, ducatur in Tabula recta ad locum relativum oculi ejusdem parallelismos, quae rectae vocentur *radii relative*' dico *apparentiam* puncti objectivi fore intersectionem duorum radiorum relativorum ad hoc punctum pertinentium'. (LH 35 XI 1, 10).

In the case of A, from $\varTheta$ and ɣ, the lines that intersect the plane at E and H (ɣ[A]E and $\varTheta$[A]H) are drawn. These centres of convergence intersect the representation plane at $\varPhi$ and ψ ($S\varTheta$, at $\varPhi$; Sɣ, at ψ). In accordance with this construction, Leibniz performs the same procedure with which he constructed *abc* (*Figure 3*). In this way, he finds point a (the image of A) as the intersection of $H\varPhi$ and $E\psi$. But, this time Leibniz is interested in experimenting with the limits of his argumentation.

The introduction of a new centre of convergence, ξ, leads him to redefine $S\varTheta$ɣ. After having introduced it, $S\varTheta\xi$ acquires prominence in the argumentation. Unlike $S\varTheta$ and Sɣ, which cut the plane at $\varPhi$ and ψ, ξ intersects the *tabula*. Given this situation, Leibniz notes the following intersection relations: when ɣ and $\varTheta$ tend to S, they intersect the plane at ψ and $\varPhi$; when $\varTheta$ and S converge on ɣ, they intersect the *tabula* at ξ and ψ; when S and ɣ converge on $\varTheta$, they intersect the plane at $\varPhi$ and ξ. Depending on the centre of convergence that governs each case, the intersections are different. To this renewed theoretical scenario, Leibniz introduces the previous construction: the intersection at a of $H\varPhi$ and $E\psi$. Furthermore he concludes the following: $E\psi$ and $H\varPhi$ intersect at a; $H\xi$ and $a\psi$ intersect at E; $a\varPhi$ and $E\xi$ intersect at H.[63] Each of the intersections, governed by the different centres of convergence, offers a different representation, although *Figure 6* only illustrates the representation of point a. Each of these centres of convergence can be considered as a point of view; however, these centres of convergence can be considered as alternative points of view. When one of them governs, this is the main point of the representation.

3.3.4. The conceptual conditions of perspective: the basis of its universality

The two theoretical scenarios proposed at the end of *Scientia perspectiva* are related to the deficiencies in the geometric theory pointed out by V. de Risi. On the one hand, the minimum distance hypothesis helped Leibniz to experiment with multiple intersections of planes on a volume; on the other hand, the infinite interval hypothesis served to conceptualise the representative mediation of the magnitude. In addition, these two theoretical scenarios allowed the German thinker to draw two conclusions. First, experimentation with the different planes allowed Leibniz to calculate the general rule of perspective in two formulas, that is, to mathematically calculate the proportional and reciprocal relationship between representational values and distances. Second, the categorisation of representative mediation allowed him to characterise place as a part of the dimension of perceptual appearance; depending on the place of the viewer (or the centre of convergence that it governs), the magnitudes of the objects are represented in a different way (according to different relative places). These conclusions explain why perspective abstains from calculating magnitudes and movement.

The formulas of the apparent elevation and declination make it possible to understand the abstention from the calculation of magnitudes, as perspective only deals with the appearance of these magnitudes. The consideration of *place* as a characteristic dimension of representative appearance gives meaning to the abstention from the calculation of movement, because all places and/or centres of convergence are alternatively points of view. However, these abstentions are better understood when considered together. In summary, the occupation of a place involves proportionally and reciprocally modifying the magnitude of the perceived object. Movement (changing position) involves renewing the perceptual modification. It thus prevents analysis of the general rule of perspective that affects any *place*. This scenario makes it possible

[63] '[. . .] et cum ergo puncti A, tria sint loca relativa a, E, H, secundum convergentias S, ɣ, Θ, et rectae ex ɣ et Θ ad S secant tabulam in ψ et Φ, ex Θ et S ad ɣ secant tabulam in ξ et ψ, ex S et ɣ ad Θ secant tabulam in Φ et ξ, itaque quemadmodum Eψ et HΦ secant se in a, ita Hξ et aψ secabunt se in E, ac denique aΦ et $E\xi$ secabunt se in H. (an hoc fortasse Hexagrammum Pascalii constat enim ex tribus punctis sex in eodem plano: ψ, Φ, ξ et a, E, H). Hic videndum an non juvetur constructio assumto puncto ξ, et consideratione universalitatis, quod scil. puncta S, ψ, Φ, ξ arbitraria, et quomodo referantur ad ipsa a, H, E uno ut a semper manente'. (LH 35 XI 1, 10). It is worth remembering that the previous points of distance (H, ŋ and V), in accordance with the angle ratios with which they were constructed, encountered the ground line, such as ξ, at infinity. In addition, the final reference to Pascal's mystic hexagram is thought-provoking. Although it is not unintelligible, this would have deserved a more detailed explanation by Leibniz.

to understand why perspective presupposes *geometria situm*. Proportional and reciprocal modification (depending on the distances or place) of the magnitude of the objects represented presupposes the existence of those objects. Linear perspective studies the representative modification of them.

4. Conclusion

There are three contextual elements that make it possible to understand Leibniz's research on linear perspective: the first, the state of his geometric theory in 1679; the second, his position as director of the Hanover library; and the third, his notes on *Manière universelle de Mr Desargues*, *La perspective pratique* and *La perspective speculative et pratique*. The first and third elements are the most decisive. On the one hand, Leibniz's research on perspective sought to solve deficiencies in his geometric theory; on the other hand, the study of these treatises indicates the theoretical beginning of the research. *Manière universelle de Mr Desargues*, *La perspective pratique* and *La perspective speculative et pratique* figure in the dispute about the authorship of the method of scales. The first of Leibniz's writings, *Constructio et usus scalae perspectivae*, creates a version of this methodology. However, Leibniz soon perceived the limitations of the use of scales, since *Constructio et usus scalae perspectivae* formulates the general rule of perspective.

The formulation of this general rule decisively changes the direction of Leibniz's research, moving it away from technical issues. In *Origo regularum artis perspectivae quales sine libro ac magistro inveni*, Leibniz demonstrates geometrically the pictorial meaning of this rule. Once this demonstration is complete, he simplifies it substantially in *Scientia perspectiva*. In this last writing, Leibniz presents the general rule in the following way: 'Itaque inclinationes vel declinationes punctorum apparentium sunt ad inclinationes vel declinationes punctorum objectivorum, ut distantia spectatoris a plano tabulae ad distantiam spectatoris a plano objectivo' (LH 35 XI 1, 9).[64] The hypotheses that follow this simplification of the rule of perspective have the same theoretical importance as the rule itself. The minimum distance allowed Leibniz to construct the mathematical formulas of apparent elevation and declination; the infinite interval allowed him, through the notion of centre of convergence, to categorise representative mediation. These two hypotheses also allowed him to develop the deficient aspects of his geometric theory. The minimum distance helped him to experiment with the intersections of planes on a solid; the infinite interval made it possible to reflect on the modification of the magnitude in the representative appearance.

The general rule of perspective explains, therefore, the construction of a pictorial representation. The place of the viewer, on one side, and the original dimensions of the object, on the other, determine the values of this representation. In this representative construction, two variables are involved: place (or distance) and the original dimensions of the object. In the same year that he wrote *Scientia perspectiva*, Leibniz used the terminology expressed in this discipline in *Discours de Métaphysique* (1686). In this text, he maintains the following: 'But although they all express the same phenomena, it does not follow that their expressions are perfectly similar; it is sufficient that they are proportional. In just the same way, several spectators believe that they are seeing the same thing and agree among themselves about it, even though each sees and speaks in accordance with his view' (Leibniz, 1989, 47).[65] This type of passage has led numerous researchers to study Leibniz's philosophical perspectivism. Among them is, for example, Deleuze who maintains that perspective expresses 'la condition sous laquelle apparaît au sujet la vérité d'une variation' (Deleuze, 1988, 27). Other researchers have addressed the question from readings closer to mathematics (see: Serres, 1968, 162; De Risi, 2007, 334-335); and others have emphasised the more strictly philosophical aspect (see: Pape, 1994; Bouquiaux, 2006; Nicolás, 2016; Nicolás, 2017). None of these works, however, have directly taken into account Leibniz's research on linear perspective. The conclusions drawn in this work pave the way

[64] 'Thus, the inclinations or declinations of the apparent points are with respect to the inclinations or declinations of the objective points, such as the distance of the viewer from the tabulae plane with respect to the distance of the viewer from the objective plane'.
[65] This text can be found in (Leibniz, 1923, series VI, volume 4, 1999, 1550-1551).

for further research to be carried out on the documentary analysis of the connection between geometry and metaphysics in Leibniz's intellectual development.

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