JOURNAL OF TIME SERIES ANALYSIS
J. Time Ser. Anal. **42**: 431–441 (2021)
Published online 6 January 2021 in Wiley Online Library
(wileyonlinelibrary.com) DOI: 10.1111/jtsa.12576

ORIGINAL ARTICLE

IDENTIFIABILITY OF STRUCTURAL SINGULAR VECTOR AUTOREGRESSIVE MODELS

BERND FUNOVITS[a,b] AND ALEXANDER BRAUMANN[c]

[a] *Faculty of Social Sciences, Discipline of Economics, University of Helsinki, Helsinki, Finland*
[b] *Institute of Statistics and Mathematical Methods in Economics, Econometrics and System Theory, TU Wien, Vienna, Austria*
[c] *TU Braunschweig, Institute for Mathematical Stochastics, Braunschweig, Germany*

We generalize well-known results on structural identifiability of vector autoregressive (VAR) models to the case where the innovation covariance matrix has reduced rank. Singular structural VAR models appear, for example, as solutions of rational expectation models where the number of shocks is usually smaller than the number of endogenous variables, and as an essential building block in dynamic factor models. We show that order conditions for identifiability are misleading in the singular case and we provide a rank condition for identifiability of the noise parameters. Since the Yule–Walker equations may have multiple solutions, we analyse the effect of restricting system parameters on over- and underidentification in detail and provide easily verifiable conditions.

Received 23 August 2019; Revised 14 September 2020; Accepted 26 November 2020

Keywords: Stochastic singularity; structural vector autoregressive models; identifiability.

MOS subject classification: 62M10, 37M10.

JEL. C32; C50.

1. INTRODUCTION

Singular structural vector autoregressive (SVAR) models play an important role in macroeconomic modelling. To introduce the subject, we succinctly discuss generalized dynamic factor models (GDFM) and dynamic stochastic general equilibrium (DSGE) models, and their relation to singular SVAR models.

In the literature on GDFMs (Forni *et al.*, 2000, 2005; Bai and Ng, 2007; Deistler *et al.*, 2010), singular VAR models are the essential building block connecting static factors (a static transformation of the denoised observables) to the uncorrelated lower-dimensional shocks. Chen *et al.* (2011) and Deistler *et al.* (2011) treat canonical forms of singular VAR models, that is, they focus on the reduced form. In Forni *et al.* (2009), it is demonstrated that dynamic factor models (and consequently singular VAR models) are useful for structural modelling. In this article, we provide results regarding identifiability of singular SVAR models and thus analyse Step C in Forni *et al.* (2009, p. 1332) in more detail.

A key issue in the econometric treatment of DSGE models is caused by the fact that the number of exogenous shocks driving the system is often strictly smaller than the number of endogenous variables. This is known as the stochastic singularity problem (DeJong and Dave, 2011, p. 184f.) and investigated in, for example, Ruge-Murcia (2007). The relationship between DSGE and SVAR models is analysed in DeJong and Dave (2011), Giacomini (2013), Kilian and Lütkepohl (2017, Chapter 6.2), and most recently by Lippi (2019). It has been acknowledged (Kilian and Lütkepohl, 2017, p. 177) that the usual strategies[1] for solving this rank deficiency problem are not satisfactory. Thus, one way forward would be the estimation of singular SVAR models.

* Correspondence to: Bernd Funovits, Faculty of Social Sciences, Discipline of Economics, University of Helsinki, P. O. Box 17 (Arkadiankatu 7), Helsinki FIN-00014, Finland. Email: bernd.funovits@helsinki.fi

[1] Kilian and Lütkepohl (2017) (i) adding measurement noise as, for example, in Sargent (1989) or Ireland (2004), and discussed in Lippi (2019), (ii) reducing the number of observables (Bouakez *et al.*, 2005) and (iii) augmenting the number of economically interpretable shocks (Ingram *et al.*, 1994; Leeper and Sims, 1994).

The singularity of the innovation covariance matrix has two possible consequences for the restrictions imposed by the modeller.[2] On the one hand, the restrictions imposed by the modeller might contradict the restrictions that are implicit due to the singularity structure of the innovation covariance matrix. On the other hand, the restrictions imposed by the modeller might already be contained in the restrictions that are implicit due to the singularity structure of the innovation covariance matrix and are therefore redundant. These cases must be taken into account when analysing identifiability properties of singular SVAR models. Moreover, restrictions on the system parameters are not necessarily over-identifying when the innovation covariance matrix is singular because the Yule–Walker (YW) equations might have multiple solutions.

The rest of this article is structured as follows. In Section 2, we specify the model, we introduce restrictions on model parameters in a general fashion, and we define notions which will be necessary later. As preparation for the main results, we collect well-known facts on singular VAR models in reduced form (in particular regarding the possible non-singularity of the Toeplitz matrix appearing in the YW equations) in Section 3. In Section 4, we analyse restrictions on the noise and system parameters and provide results as to how a singular innovation covariance matrix needs to be taken into account for identifiability analysis. We illustrate that the usual order condition may be misleading in the singular case with a (stochastically singular) DSGE model and provide easily verifiable conditions for under- and over-identification when the YW equations have multiple solutions. All proofs are deferred to the Appendix.

The following notation is used in the article. We use z as a complex variable as well as the backward shift operator on a stochastic process, that is, $z(y_t)_{t\in\mathbb{Z}} = (y_{t-1})_{t\in\mathbb{Z}}$. For a (matrix) polynomial $p(z)$, we denote by $\deg(p(z))$ the highest degree of $p(z)$. The transpose of an $(m \times n)$ dimensional matrix A is represented by A'. We use $\mathrm{vec}(A) \in \mathbb{R}^{nm\times 1}$ to stack the columns of A into a column vector and $\mathrm{vech}(A) \in \mathbb{R}^{\frac{n(n+1)}{2}\times 1}$ to stack the lower-triangular elements of an n-dimensional square matrix A analogously. The n-dimensional identity matrix is denoted by I_n. The inequality ">0" refers to positive definiteness in the context of matrices. For the span of the row space and the column space of A, we write $span_R(A)$ and $span_C(A)$ respectively, and the projection of A on $span_R(B)$, $B \in \mathbb{R}^{r\times n}$, is $Proj_R(A|B)$ and the projection of A on $span_C(D)$, $D \in \mathbb{R}^{m\times s}$, is $Proj_C(A|D)$. We use $\mathbb{E}(\cdot)$ for the expectation of a random variable with respect to a given probability space.

2. MODEL

Here, we start by defining the model, that is, the system and noise parameters as well as the stability, singularity, and researcher imposed restrictions. Next, we describe the observed quantities that are available to the econometrician; in our case the second moments. Lastly, we discuss the notion of identifiability, that is, the connection between the internal and external characteristics.

We consider a SVAR[3] system

$$\begin{aligned} A_0 y_t &= A_1 y_{t-1} + \cdots + A_p y_{t-p} + B\varepsilon_t, \\ &= \underbrace{(A_1, \ldots, A_p)}_{=A_+} x_{t-1} + B\varepsilon_t \end{aligned} \tag{1}$$

where $x_{t-1} = (y'_{t-1}, \ldots, y'_{t-p})'$ and where the dimension q of the white noise process (ε_t) of (economically) fundamental shocks with covariance matrix I_q is strictly smaller than n, the number of observed variables of y_t. The matrix $B \in \mathbb{R}^{n\times q}$ has full column rank. This implies that the covariance matrix Σ_u of the innovations $u_t = B\varepsilon_t$ is of rank $q < n$.

[2] A similar problem appears in Kilian and Lütkepohl (2017, Chapter 10.2) where it is emphasized that the reduced rank of a certain matrix appearing in cointegration analysis must be 'taken into account when determining the number of restrictions that have to be imposed for full identification of the structural shocks'.

[3] Most work on SVAR models is performed in the parametrization where $A_0 = I_n$ (Kilian and Lütkepohl, 2017, Chapter 8) and investigates how to estimate B. Since we also treat structural restrictions on A_+, rather than on the reduced form parameters $A_0^{-1}A_+$, we allow here for additional generality and treat the so-called AB-model (in the nomenclature of Lütkepohl (2005)). Except for Section 4.2, it is sufficient to set $A_0 = I_n$.

  J. Time Ser. Anal. 42: 431–441 (2021)
Journal of Time Series Analysis published by John Wiley & Sons Ltd. DOI: 10.1111/jtsa.12576

Furthermore, we assume that the matrices $A_i \in \mathbb{R}^{n\times n}$ are such that the stability condition

$$\det(a(z)) \neq 0, \; |z| \leq 1, \tag{2}$$

holds, where $a(z) = A_0 - A_1 z - \cdots - A_p z^p$, and that $\det(A_0) \neq 0$. Lastly, we assume that the system and noise parameters satisfy the restrictions

$$C_S \mathrm{vec}(A_+') = c_S \;\text{ and }\; C_N \mathrm{vec}\left(\begin{pmatrix} A_0 & B \end{pmatrix}\right) = c_N \tag{3}$$

where C_S and C_N are of dimensions $(r_S \times n^2 p)$ and $(r_N \times (n^2 + nq))$ respectively, describing the (*apriori* known) restrictions imposed by the modeller. To summarize, we define the internal characteristics that we would like to identify as the parameters $(A_+, (A_0, B))$ in system (1) which satisfy the restrictions imposed by (2) and (3).

Next, we discuss the external characteristics which are observed by the econometrician. The stationary solution of the system (1) (together with the restrictions imposed on the parameters) is called a singular VAR process. Having available all finite joint distributions of the singular VAR process corresponds to the maximal information we could possibly obtain regarding external characteristics. Another commonly used set of external characteristics is the second moment information contained in the singular VAR process, that is, the autocovariance function $\gamma(s) = \mathbb{E}(y_t y_{t-s}')$ or equivalently the spectral density $f(e^{-i\lambda}) = \frac{1}{2\pi}\sum_{s=-\infty}^{\infty} \gamma(s) e^{-is\lambda}$.

We follow Rothenberg (1971) to define identifiability of parametric models. Two internal characteristics $(A_+^{(1)}, (A_0^{(1)}, B^{(1)}))$ and $(A_+^{(2)}, (A_0^{(2)}, B^{(2)}))$ are called observationally equivalent if they imply the same external characteristics. An internal characteristic is globally identifiable if there is no other observationally equivalent internal characteristic. Likewise, an internal characteristic $(A_+, (A_0, B))$ is locally identifiable if there exists a neighbourhood around the parameter $(A_+, (A_0, B))$ corresponding to the internal characteristic such that there is no other observationally equivalent internal characteristic in this neighbourhood. In this article, we focus on identifiability from second moment information, that is, the external characteristics correspond to the spectral density of the observed process (y_t).

3. IDENTIFIABILITY ISSUES IN REDUCED FORM SINGULAR VAR MODELS

To prepare for the structural case where we will connect to external characteristics uniquely to the deep parameters, we review identifiability of the reduced form of singular VAR models, see also Anderson *et al.* (2012). In particular, we discuss the rank of finite sections of the covariance of the observed process and its relation to the rank of the innovation covariance matrix Σ_u. Moreover, we show how p, q and the left-kernel $L \in \mathbb{R}^{(n-q)\times n}$ of Σ_u, which are assumed to be known in the identifiability analysis in Section 4, can be obtained from the external characteristics.

One way to connect the observable characteristics to the internal characteristics is by using the YW equations[4], that is,

$$\bar{A}_+ \Gamma_p = \gamma_p \text{ and } \Sigma_u = \gamma(0) - \bar{A}_+ \gamma_p',$$

where $\bar{A}_+ = (\bar{A}_1, \ldots, \bar{A}_p) = A_0^{-1} A_+$ are the reduced form system parameters, $\Gamma_p = \begin{pmatrix} \gamma(0) & \gamma(1) & \ldots & \gamma(p-1) \\ \gamma(-1) & \gamma(0) & & \\ \vdots & & \ddots & \\ \gamma(-p+1) & & & \gamma(0) \end{pmatrix}$ and $\gamma_p = (\gamma(1), \ldots, \gamma(p))$. If Γ_p is invertible, there is a unique internal characteristic $(\bar{A}_+, \Sigma_u)$ for a given external characteristic $(\gamma(0), \ldots, \gamma(p))$. While for VAR models with non-singular innovation covariance matrix it can be shown (Hannan and Deistler, 2012, p. 112) that Γ_r is non-singular for all $r \in \mathbb{N}$, this is not the case for VAR models that have a singular innovation covariance matrix. Indeed, it is easy to see (Anderson and Deistler, 2009, Theorem 7) that, for $s > 0$, $rk(\Gamma_{p+s}) = rk(\Gamma_p) + s\cdot q$ holds. Even Γ_p might be rank deficient: Consider a solution $(\bar{A}_+^{(1)}, \Sigma_u)$ of the

[4] They are obtained by right-multiplying $(y_t', \ldots, y_{t-p}')$ on (1) and taking expectations.

wileyonlinelibrary.com/journal/jtsa
DOI: 10.1111/jtsa.12576 *Journal of Time Series Analysis* published by John Wiley & Sons Ltd.

YW equations and the polynomial matrix $U(z) = I_n + cc'z$ where $c' \in \mathbb{R}^{1\times n}$ is non-trivial and in the left-kernel of both $\bar{A}_p^{(1)}$ and $\overline{B} = A_0^{-1}B$. One can verify that $U(z)\bar{a}^{(1)}(z)$, where $\bar{a}^{(1)}(z)$ is the polynomial corresponding to $\bar{A}_+^{(1)}$, is also a polynomial matrix of degree p and solves the YW equations which implies that Γ_p has a non-trivial left-kernel. Note that the perpendicular of the projection is unique irrespective of how the projection itself is parametrized. More formally, $\Sigma_u^{(1)} = \gamma(0) - \bar{A}_+^{(1)}\gamma_p' = \gamma(0) - \bar{A}_+^{(2)}\gamma_p' = \Sigma_u^{(2)}$ holds even if $\bar{A}_+^{(1)} \neq \bar{A}_+^{(2)}$ for two solutions $(\bar{A}_+^{(1)}, \Sigma_u^{(1)})$ and $(\bar{A}_+^{(2)}, \Sigma_u^{(2)})$ of the YW equations.

Examining the ranks of Γ_r for some consecutive values of r, the integer-valued parameters q and p can be obtained. Having the rank q of the innovation covariance Σ_u available, it is straightforward to obtain (a basis of) the left-kernel $L \in \mathbb{R}^{(n-q)\times n}$ of Σ_u (Al-Sadoon, 2017).

For the remainder of this article, we will assume that p, q, and L are known by the practitioner (in addition to the other external characteristics).

4. IMPOSING STRUCTURAL RESTRICTIONS

We discuss identifiability of noise and system parameters in the case of singular SVAR models. First, we derive a condition which ensures that the modeller imposed restrictions on the noise parameters do not contradict the singularity of the innovation covariance matrix. Subsequently, we derive a rank condition similar to the previous literature and illustrate with a new-Keynesian DSGE model that the order condition does not provide useful information in the stochastically singular case. Second, we discuss whether researcher imposed restrictions on system parameters are under-, just- or over-identifying. In particular, we show that it is uncommon that researcher imposed restrictions do not solve the underidentification problem (if the number of restrictions is at least as large as the rank deficiency of $(I_n \otimes \Gamma_p)$).

We start with affine restrictions on the noise parameters (A_0, B) which appear in short-run restrictions, see Kilian and Lütkepohl (2017, Chapter 8) for the non-singular case. The conditions that we derive are local in nature. Next, we deal with the case where Γ_p may be singular and where affine restrictions on the elements in A_+ are imposed. These results concern global identifiability.

4.1. Affine Restrictions on the Noise Parameters

In the light of the discussion in Section 3, we start with a singular Σ_u and with researcher imposed restrictions given by

$$C_N \mathrm{vec}((A_0, B)) = c_N. \tag{4}$$

Here, $C_N = \begin{pmatrix} C_{A_0} & 0_{r_{A_0}\times nq} \\ 0_{r_B\times n^2} & C_B \end{pmatrix}$ is block-diagonal and has full row rank, and $c_N' = (c_{A_0}', c_B')$. To show the existence of a unique pair (A_0, B) for parametrizing $\Sigma_u = A_0^{-1}BB'(A_0')^{-1}$, one usually calls on the implicit functions theorem. While in the non-singular SVAR case the system of equations to be analysed always has at least one solution, it might happen in the singular SVAR case that the set of solutions of (4) (for which the restrictions imposed by the researcher are satisfied) is the empty set. Since the premises of the implicit function theorem are such that there must be at least one solution, one needs to make sure that the affine restrictions (4) imposed by the researcher do not contradict the singularity structure of the model. In the following, we will provide an analytical condition which implies and is implied by a non-empty solution set.

The linear dependence structure induced by the singularity of Σ_u implies $L(A_0^{-1}B) = 0$, where the rows of $L \in \mathbb{R}^{(n-q)\times n}$ span the left-kernel of Σ_u, which is equivalent to

$$(I_q \otimes L)\mathrm{vec}(A_0^{-1}B) = 0. \tag{5}$$

The condition for when the solution set of the joint system of restrictions given in (4) and (5) is non-empty is given in the following

Journal of Time Series Analysis published by John Wiley & Sons Ltd. DOI: 10.1111/jtsa.12576

Lemma 1. Let $L \in \mathbb{R}^{(n-q)\times n}$ be a basis of the left-kernel of Σ_u, define $\mathcal{N} := \{[(A_0^{-1}B)', I_q] \otimes LA_0^{-1}\}$, and let $M := C_N - Proj_R(C_N|\mathcal{N})$ be the perpendicular of the projection of C_N on the row-span of $\mathcal{N}$. The restrictions $C_N\text{vec}(A_0, B) = c_N$ are consistent with the singularity of Σ_u if and only if $\text{rk}(M) = \text{rk}\begin{pmatrix} M & c_N \end{pmatrix}$, that is, if and only if c_N is in the image of M.

Remark 1. When we consider the SVAR setting in which $A_0 = I_n$, we only need to check whether c_B is contained in the column space of $C_B - Proj(C_B|(I_q \otimes L))$.

The singularity of Σ_u restricts the set of admissible restrictions on the parameter space. If C_N does not 'interfere' with the singularity restrictions, that is, if C_N lies in the orthogonal complement of $span_R(\mathcal{N})$ or expressed differently if $Proj_R(C_B|\mathcal{N}) = 0$, then $M = C_N$ and condition $\text{rk}(M) = \text{rk}\begin{pmatrix} M & c_N \end{pmatrix}$ is satisfied.

Proposition 1. Let A_0 and B be $(n \times n)$ and $(n \times q)$-dimensional matrices of full column rank, let $n > q$, and let $C_N\text{vec}(A_0, B) = c_N$ hold. For given Σ_u, the matrix (A_0, B) is the unique solution of $\Sigma_u = A_0^{-1}B(A_0^{-1}B)'$ if and only if c_N is in the image of $M = C_N - Proj_R(C_N|\mathcal{N})$ and the matrix $\begin{pmatrix} -2D_n^+(\Sigma_u \otimes A_0^{-1}) & 2D_n^+(A_0^{-1}B \otimes A_0^{-1}) \\ C_{A_0} & 0 \\ 0 & C_B \end{pmatrix}$ is of (full column) rank $n^2 + nq$.

Remark 2. Considering for simplicity the case where $A_0 = I_n$ and following Rothenberg (1971), the restrictions imposed on the structural parameter B are $C_B\text{vec}(B) = c_B$ as well as $(I_q \otimes L)vec(B) = 0$ which suggests that the matrix $\frac{\partial}{\partial(vec(B))'}\begin{pmatrix} \text{vec}h(BB') - \text{vech}(\Sigma_u) \\ C_B\text{vec}(B) - c_B \\ (I_q \otimes L)\text{vec}(B) \end{pmatrix}$ needs to be of rank nq. However, it is not necessary to include $(I_q \otimes L)$ in Proposition 1 because $(I_q \otimes L)\text{vec}(B) = 0$ is already implied by the fact that $BB' = \Sigma_u$. Put differently, the inequality $rk\begin{pmatrix} 2D_n^+(B \otimes I_n) \\ C_B \\ (I_q \otimes L) \end{pmatrix} \leq rk\begin{pmatrix} 2D_n^+(B \otimes I_n) \\ C_B \end{pmatrix}$ holds.

Remark 3. If $q < n$, the usual order condition requiring that the number of rows in $\begin{pmatrix} 2D_n^+(B \otimes I_n) \\ C_B \end{pmatrix}$ be larger than or equal to the number of columns is not useful. Consider the case where there are no researcher imposed restrictions. While the order condition is satisfied for $q \leq \frac{n+1}{2}$, the matrix $D_n^+(B \otimes I_n)$ of dimension $\left(\frac{n(n+1)}{2} \times nq\right)$ is of course rank deficient with co-rank $\frac{q(q-1)}{2}$.

Remark 4. The rank of the matrix $\begin{pmatrix} 2D_n^+(B \otimes I_n) \\ C_B \end{pmatrix}$ drops if some restrictions in C_B are already implied by the singularity structure of Σ_u, that is, if for the r-th row $[C_B]_{[r,\bullet]} \subseteq \text{span}_\text{R}(I_q \otimes L)$ holds. Thus, the $\frac{q(q-1)}{2}$ additional restrictions which are necessary to obtain a matrix $\begin{pmatrix} 2D_n^+(B \otimes I_n) \\ C_B \end{pmatrix}$ of full column rank must not be contained in the row space of $D_n^+(B \otimes I_n)$.

4.1.1. Illustration

To illustrate Proposition 1, we discuss a version of the new-Keynesian monetary business cycle model (Lubik and Schorfheide, 2003; Castelnuovo, 2013) featuring a 'supply-shifting' shock in the new-Keynesian Phillips curve (NKPC). We thus consider the model

$$\begin{aligned} \pi_t &= \beta\mathbb{E}_t(\pi_{t+1}) + \kappa x_t + \epsilon_t^\pi \\ x_t &= \mathbb{E}_t(x_{t+1}) - \tau(R_t - \mathbb{E}_t(\pi_{t+1})) \\ R_t &= \phi\mathbb{E}_t(\pi_{t+1}) + \epsilon_t^R. \end{aligned}$$

DOI: 10.1111/jtsa.12576 *Journal of Time Series Analysis* published by John Wiley & Sons Ltd.

where (π_t, x_t, R_t) denote inflation, output gap, and nominal interest rate in log-deviation from a unique steady state. The conditional expectations are to be understood as linear projections on the space spanned by present and past components of the uncorrelated shocks ε_t^π and ε_t^R which are white noise processes (whose variance is normalized to one for the sake of simplicity). The parameters of the model are the subjective time preference factor $\beta \in (0,1)$, $\phi \geq 0$ the elasticity of the interest response of the central bank, and the slope parameters κ and τ.

For specific parameter values $(\beta, \phi, \tau, \kappa) = \left(\frac{4}{5}, \frac{39}{38}, \frac{3}{4}, \frac{1}{2}\right)$, we solve this system of equations involving conditional expectations of future endogenous variables (Sims, 2001; Funovits, 2017) and obtain the unique causal stationary solution $\begin{pmatrix} R_t \\ \pi_t \\ x_t \end{pmatrix} = B \begin{pmatrix} \varepsilon_t^R \\ \varepsilon_t^\pi \end{pmatrix}$, where $B = \begin{pmatrix} 1 & 0 \\ -\kappa\tau & 1 \\ -\tau & 0 \end{pmatrix}$, of the DSGE model described above. The innovation covariance matrix is obviously singular. The restrictions on B are described by $C_B \mathrm{vec}(B) = c_B$, with $C_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $c_B = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. To apply Proposition 1, we need to check the condition $\mathrm{rk}(M) = \mathrm{rk}\begin{pmatrix} M & c_B \end{pmatrix}$ of Lemma 1. The perpendicular of the projection of C_B on the row-span of $(I_2 \otimes L)$ for $L = \begin{pmatrix} \tau & 0 & 1 \end{pmatrix}$, is given by $M = \begin{pmatrix} \frac{1}{1+\tau^2} & 0 & -\frac{\tau}{1+\tau^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1+\tau^2} & 0 & -\frac{\tau}{1+\tau^2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\tau}{1+\tau^2} & 0 & \frac{\tau^2}{1+\tau^2} \end{pmatrix}$. For any value $\tau \in \mathbb{R} \setminus \{0\}$ the relation $\mathrm{rk}(M) = \mathrm{rk}\begin{pmatrix} M & c_B \end{pmatrix}$ is satisfied. We can now apply Proposition 1 and check the rank of $\begin{pmatrix} 2D_3^+(B \otimes I_3) \\ C_B \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -\kappa\tau & 1 & 0 & 1 & 0 & 0 \\ -\tau & 0 & 1 & 0 & 0 & 0 \\ 0 & -2\kappa\tau & 0 & 0 & 2 & 0 \\ 0 & -\tau & -\kappa\tau & 0 & 0 & 1 \\ 0 & 0 & -2\tau & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is equal to 6 for any values κ and τ.

4.2. Affine Restrictions on the System Parameters

We now focus on imposing linear restrictions on the structural parameters A_+ in the case where Γ_p is singular. Thus, without restrictions on A_+, there are multiple observationally equivalent solutions of the YW equations (one particular solution plus the left kernel of Γ_p). We will start by considering the case where $A_0 = I_n$ (such that the reduced form parameters $\bar{A}_+$ coincide with A_+). This simplifies the discussion and allows us to illustrate why the identifiability problem (for A_0 not necessarily equal to the identity matrix) can 'generically' be solved by (the right number of) arbitrary restrictions on A_+.[5]

Two aspects deserve special attention. First, the particular solutions (canonical representatives of the equivalence class of observational equivalence) introduced in Deistler *et al.* (2011) and Chen *et al.* (2011) can be obtained by choosing a particular set of restrictions on $\mathrm{vec}(A_+')$. Second, singular SVAR models are special in the sense that some researcher imposed restrictions are not over-identifying in the sense that imposing them does not restrict the feasible covariance structures. In Lemma 2 we provide a condition for checking whether the researcher imposed restrictions on A_+ are over-identifying.

To simplify discussion, we note that vectorizing the (transposed) YW equations leads to $(I_n \otimes \Gamma_p)\mathrm{vec}(A_+') = \mathrm{vec}(\gamma_p')$. In Deistler *et al.* (2011), the authors choose the first linearly independent rows of Γ_p as a basis of the

[5] To be more precise, it can be considered uncommon that $s \cdot n$, where s is the dimension of the kernel of Γ_p, 'random' restrictions on $\mathrm{vec}(A_+')$ do not solve the identifiability problem.

row space (or equivalently column space) of Γ_p to define a particular solution of the YW equations. To fix ideas, consider a Γ_p whose first $(np - s)$ linearly independent rows are selected by premultiplying S_1' of dimension $((np - s) \times np)$, containing only zeros and ones, and denote by S_2' the $(s \times np)$-dimensional matrix containing zeros and ones such that $S_1' S_2 = 0$. A basis of the column space thus consists of the columns of $\Gamma_p S_1$, that is, the elements $S_2' A_+'$ are restricted to zero. Restricting each column of A_+' to be orthogonal to the columns of S_2 therefore results in a unique solution of the YW equations, that is, the matrix in brackets in $\begin{bmatrix}(I_n \otimes \Gamma_p) \\ (I_n \otimes S_2')\end{bmatrix} vec(A_+') = \begin{pmatrix}\mathrm{vec}(\gamma_p') \\ 0_{n\times 1}\end{pmatrix}$ is of full rank. We denote the unique solution of the equation above by $\widehat{\mathrm{vec}(A_+')}$.

In Chen *et al.* (2011), the authors choose the minimum norm solution of the YW equations as the particular solution. Let $I_n \otimes \left[\begin{pmatrix} V_1 & V_2\end{pmatrix}\begin{pmatrix} D_{11} & 0_{(n^2p-s)\times s} \\ 0_{s\times(n^2p-s)} & 0_{s\times s}\end{pmatrix}\begin{pmatrix} V_1' \\ V_2'\end{pmatrix}\right]$ be the singular value decomposition (SVD)[6] of $(I_n \otimes \Gamma_p)$ of rank $n^2p - ns = n \cdot rk(\Gamma_p)$. The particular solution is such that coordinates corresponding to the basis vectors V_2 are set equal to zero. Put differently, $\mathrm{vec}(A_+')$ is required to be orthogonal to the columns of $(I_n \otimes V_2)$, that is, $\begin{bmatrix}(I_n \otimes \Gamma_p) \\ (I_n \otimes V_2')\end{bmatrix} \mathrm{vec}(A_+') = \begin{pmatrix}\mathrm{vec}(\gamma_p') \\ 0_{s\times 1}\end{pmatrix}$. We denote the unique solution of the equation above by $\widetilde{\mathrm{vec}(A_+')}$.

While the coordinate representations $\widehat{\mathrm{vec}(A_+')}$ and $\widetilde{\mathrm{vec}(A_+')}$ usually differ, $(I_n \otimes x_{t-1}')\widehat{\mathrm{vec}(A_+')}$ and $(I_n \otimes x_{t-1}')\widetilde{\mathrm{vec}(A_+')}$ represent the same projection (component wise on the space spanned by the columns of Γ_p or equivalently on the space spanned by the components of x_{t-1}). By construction, we have that $span_C(\Gamma_p) = span_C(V_1) = span_C(\Gamma_p S_1)$ and, in particular, that the rank of the projection of $\Gamma_p S_1$ on $span_C(\Gamma_p)$ is equal to the rank of Γ_p. This projection idea can be used to investigate whether researcher imposed restrictions on the system parameters are 'true' restrictions (in the sense that they restrict the possible covariance structures of the model) and whether the restrictions are sufficient to guarantee a unique solution. Let $C_S \mathrm{vec}(A_+') = 0$, where $C_S \in \mathbb{R}^{r_S \times n^2 p}$ is of full row rank, be the researcher imposed restrictions and denote the (right-) kernel of C_S by $S_A \in \mathbb{R}^{n^2p\times(n^2p - r_S)}$. If $span_C((I_n \otimes \Gamma_p)S_A) \supseteq span_C(I_n \otimes V_1)$, then the researcher imposed restrictions are not over-identifying in the sense that without them the same set of covariance structures are feasible. To investigate the validity of this inclusion of spaces, we define the SVD of

$$\underbrace{(I_n \otimes \Gamma_p)S_A}_{=n^2p\times(n^2p-r)} = \begin{pmatrix}\tilde{U}_1 & \tilde{U}_2\end{pmatrix}\begin{pmatrix}\tilde{D}_{11} & 0 \\ 0 & 0_{\tilde{s}\times\tilde{s}}\end{pmatrix}\begin{pmatrix}\tilde{V}_1' \\ \tilde{V}_2'\end{pmatrix}. \tag{6}$$

If $span_C((I_n \otimes \Gamma_p)S_A) \supseteq span_C(I_n \otimes V_1)$ holds, then we can express the column space of $(I_n \otimes V_1)$ in terms of the columns of $((I_n \otimes \Gamma_p)S_A)$ and, in other words, the projection of $(I_n \otimes V_1)$ on the column space of $((I_n \otimes \Gamma_p)S_A)$ must coincide with $(I_n \otimes V_1)$. Expressed in terms of SVDs, this leads to

Lemma 2. In the case $A_0 = I_n$, the restrictions described by the matrix C_S are not over-identifying if and only if

$$[I_{n^2p} - \tilde{U}_1 \tilde{U}_1'](I_n \otimes V_1) = 0, \tag{7}$$

where $\tilde{U}_1$ is obtained from (6). There is a unique solution of the YW equations if and only if the right-kernel of $(I_n \otimes \Gamma_p)S_A$ is trivial.

Returning to the general case where A_0 is not necessarily equal to the identity matrix, we will now show that it is in general enough to impose as many restrictions as there are basis vectors in the kernel of $(I_n \otimes \Gamma_p)$. Notice that $C_S vec(A_+') = [C_S(A_0 \otimes I_{np})]vec(\bar{A}_+')$ such that for given A_0, the restrictions on the parameters $\bar{A}_+$ can be obtained straight-forwardly from the ones on A_+.

[6] (V_1, V_2) are an orthonormal eigenbasis describing the image and the kernel of Γ_p respectively, and D_{11} is a diagonal matrix with positive diagonal elements.

DOI: 10.1111/jtsa.12576 *Journal of Time Series Analysis* published by John Wiley & Sons Ltd.

To provide some intuition for the following result, we consider a quite special example where counting the number of restrictions for deducing identifiability of the system parameters does not suffice. Consider $\Gamma_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $C_S = I_3 \otimes (0,1,0)$, such that $S_A = \left(I_3 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ and $(I_n \otimes \Gamma_p) S_A = \left(I_3 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$. Even though the order condition (that the rank deficiency of $(I_n \otimes \Gamma_p)$ is equal to the number of restrictions) is satisfied, they are not sufficient for obtaining a unique solution of the YW equations. Indeed, $[I_{n^2p} - \tilde{U}_1 \tilde{U}_1'](I_n \otimes V_1) = \left(I_3 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \left(I_3 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \neq 0$ and the right-kernel of $(I_n \otimes \Gamma_p) S_A = \left(I_3 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$ is non-empty. The non-generic nature of this example is summarized in

Proposition 2. Let $C_S \in \mathbb{R}^{ns \times n^2p}$ be of full row rank and let Γ_p be singular with rank deficiency equal to s. The set of restrictions $\{ C_S \in \mathbb{R}^{ns \times n^2p} \mid (7) \text{ does not hold} \}$ is of Lebesgue measure zero in $\mathbb{R}^{ns \times n^2p}$. A generic, randomly chosen restriction C_S can thus be used to obtain a unique solution of the system of equations $\begin{bmatrix} (I_n \otimes \Gamma_p) \\ C_S \end{bmatrix} \mathrm{vec}(A_+') = \begin{pmatrix} \mathrm{vec}(\gamma_p') \\ 0_{s\times 1} \end{pmatrix}$ and the system parameters are globally identified.

Notice, however, that solving the identifiability problem for A_+ by restricting the transfer function $a(z)^{-1}b$ (e.g. by restricting the long-run coefficients in $k(1) = a(1)^{-1}b$) is not possible. Since two observationally equivalent pairs $(a^{(1)}(z), b^{(1)})$ and $(a^{(2)}(z), b^{(2)})$ have the same transfer function by definition, restricting $a(z)^{-1}b$ directly has the effect of either excluding the whole equivalence class or not providing additional information for distinguishing different pairs $(a(z), b)$ with the same transfer function.

5. CONCLUSION

In this article, we generalize the well-known identifiability results for SVAR models to the case of a singular innovation covariance matrix. The first main difference to the regular case is that the restrictions on the noise parameters (A_0, B) might contradict the singularity of the innovation covariance matrix. Moreover, the researcher imposed restrictions might already be contained in the restrictions implied by the singularity of the innovation covariance matrix and therefore do not have any further ‘identifying effect’. The second main difference pertains mainly to restrictions on the structural system parameters A_+. We provide conditions under which the researcher imposed restrictions on these parameters are over-identifying and show that underidentification can be considered an unusual case when the rank deficiency coincides with the number of restrictions.

ACKNOWLEDGEMENTS

Financial support by the Research Funds of the University of Helsinki as well as by funds of the Oesterreichische Nationalbank (Austrian Central Bank, Anniversary Fund, project number: 17646) is gratefully acknowledged. My research is also funded by the Research Funds of the University of Helsinki. In Wiley’s author services dashboard, “Helsingin Yliopiston Tiedesäätiö” showed up which I subsequently chose. One could also indicate the internal project number WBS772563109.

DATA AVAILABILITY STATEMENT

There is no data involved in this study.

SUPPORTING INFORMATION

Additional Supporting Information may be found online in the supporting information tab for this article.

REFERENCES

Al-Sadoon MM. 2017. A unifying theory of tests of rank. *Journal of Econometrics* **199**(1):49–62.

Anderson BDO, Deistler M. 2009. Properties of zero-free spectral matrices. *IEEE Transactions on Automatic Control* **54**(10):2365–2375.

Anderson BDO, Deistler M, Chen W, Filler A. 2012. Autoregressive models of singular spectral matrices. *Automatica* **48**(11):2843–2849.

Bai J, Ng S. 2007. Determining the number of primitive shocks in factor models. *Journal of Business and Economic Statistics* **25**:52–60.

Bouakez H, Cardia E, Ruge-Murcia FJ. 2005. Habit formation and the persistence of monetary shocks. *Journal of Monetary Economics* **52**(6):1073–1088.

Castelnuovo E. 2013. Monetary policy shocks and financial conditions: a Monte Carlo experiment. *Journal of International Money and Finance* **32**:282–303.

Chen W, Anderson BDO, Deistler M, Filler A. 2011. Solutions of Yule–Walker equations for singular AR processes. *Journal of Time Series Analysis* **32**:531–538.

Deistler M, Anderson BDO, Filler A, Zinner C, Chen W. 2010. Generalized linear dynamic factor models – an approach via singular autoregressions. *European Journal of Control* **16**(3):211–224.

Deistler M, Filler A, Funovits B. 2011. AR systems and AR processes: the singular case. *Communications in Information and Systems* **11**(3):225–236.

DeJong DN, Dave C. 2011. *Structural Macroeconmetrics*, Princeton University Press: Princeton.

Forni M, Giannone D, Lippi M, Reichlin L. 2009. Opening the black box: Structural factor models with large cross sections. *Econometric Theory* **25**(5):1319–1347.

Forni M, Hallin M, Lippi M, Reichlin L. 2000. The generalized dynamic factor model: identification and estimation. *Review of Economics and Statistics* **82**(4):540–554.

Forni M, Hallin M, Lippi M, Reichlin L. 2005. The generalized dynamic factor model: one-sided estimation and forecasting. *Journal of the American Statistical Association* **100**:830–840.

Funovits B. 2017. The full set of solutions of linear rational expectations models. *Economics Letters* **161**:47–51.

Giacomini, R. (2013). The relationship between DSGE and VAR models. Technical report, CEMMAP working paper.

Hannan EJ, Deistler M. 2012. *The Statistical Theory of Linear Systems*, SIAM Classics in Applied Mathematics: Philadelphia.

Ingram BF, Kocherlakota NR, Saven NE. 1994. Explaining business cycles: a multiple-shock approach. *Journal of Monetary Economics* **34**(3):415–428.

Ireland PN. 2004. A method for taking models to the data. *Journal of Economic Dynamics and Control* **28**:1205–1226.

Kilian L, Lütkepohl H. 2017. *Structural Vector Autoregressive Analysis*, Cambridge University Press: Cambridge.

Leeper EM, Sims CA. 1994. Toward a modern macroeconomic model usable for policy analysis. *NBER Macroeconomics Annual* **9**:81–140.

Lippi M. 2019. *Validating DSGE models with VARS and large-dimensional dynamic factor models*. Technical report, Institute for Economics and Finance: Einaudi.

Lubik TA, Schorfheide F. 2003. Computing sunspot equilibria in linear rational expectations models. *Journal of Economic Dynamics and Control* **28**(2):273–285.

Lütkepohl H. 1996. *Handbook of Matrices*, John Wiley & Sons Ltd: West Sussex, UK.

Lütkepohl H. 2005. *New Introduction to Multiple Time Series Analysis*, Springer: Berlin.

Rothenberg TJ. 1971. Identification in parametric models. *Econometric Theory* **39**(3):577–591.

Ruge-Murcia FJ. 2007. Methods to estimate dynamic stochastic general equilibrium models. *Journal of Economic Dynamics and Control* **31**(8):2599–2636.

Sargent TJ. 1989. Two models of measurements and the investment accelerator. *Journal of Political Economy* **97**(2):251–287.

Sims CA. 2001. Solving linear rational expectations models. *Computational Economics* **20**:1–20.

APPENDIX A: PROOF OF LEMMA 1

We write $C_N = \begin{pmatrix} C_{A_0} & 0_{r_{A_0}\times nq} \\ 0_{r_B\times n^2} & C_B \end{pmatrix}$ as orthogonal sum, that is,

$$C_N = Proj_R(C_N|\mathcal{N}) + \underbrace{(C_N - Proj_R(C_N|\mathcal{N}))}_{=M},$$

and substitute it into equation (4) such that $Proj_R(C_N|\mathcal{N})\text{vec}(A_0, B) + M\text{vec}(A_0, B) = c_N$. To fulfil the singularity restrictions of Σ_u, equation (5) needs to hold. Elementary calculations show that $\mathcal{N}\text{vec}(A_0, B) = 0$ under (5) which implies $Proj_R(C_N|\mathcal{N})\text{vec}(A_0, B) = 0$ because $Proj_R(C_N|\mathcal{N})$ projects C_N onto $span_R(\mathcal{N})$. The system of equations $M\text{vec}(A_0, B) = c_N$ has a solution if and only if $\text{rk}(M) = \text{rk}\begin{pmatrix} M & c_N \end{pmatrix}$.

APPENDIX B: PROOF OF PROPOSITION 1

Consider the following system of equations:

$$\varphi_1(\text{vec}(A_0, B)) := \text{vech}(A_0^{-1}BB'A_0'^{-1}) - \text{vech}(\Sigma_u) = 0,$$

$$\varphi_2(\text{vec}(A_0, B)) := C_N\text{vec}(A_0, B) - c_N = \begin{pmatrix} C_{A_0} & 0_{r_{A_0}\times nq} \\ 0_{r_B\times n^2} & C_B \end{pmatrix}\text{vec}(A_0, B) - c_N = 0.$$

Following Rothenberg (1971, Theorem 6), the equations $\varphi(\text{vec}(A_0, B)) = \begin{pmatrix} \varphi_1(\text{vec}(A_0, B)) \\ \varphi_2(\text{vec}(A_0, B)) \end{pmatrix} = 0 \in \mathbb{R}^{\left(\frac{n(n+1)}{2}+r_N\right)\times 1}$ have a unique solution in an open set around $\text{vec}(A_0, B) \in \mathbb{R}^{(n^2+nq)\times 1}$ if the $\left(\frac{n(n+1)}{2} + r_N\right) \times (n^2 + nq)$ dimensional matrix $\frac{\partial\varphi}{\partial\text{vec}(A_0,B)'}$ has full column rank $n^2 + nq$. Note that $\varphi(\text{vec}(A_0, B)) = 0$ holds if and only if c_N is in the image of $M = C_N - Proj_R(C_N|\mathcal{N})$ according to Lemma 1. The matrix $\frac{\partial\varphi}{\partial\text{vec}(A_0,B)'}$ can be calculated using standard rules for matrix differentiation (Lütkepohl, 1996) as

$$\begin{aligned}
\frac{\partial\varphi_1}{\partial\text{vec}(A_0)'}(\text{vec}(A_0, B)) &= D_n^+\frac{\partial\text{vec}(A_0^{-1}BB'A_0'^{-1})}{\partial\text{vec}(A_0^{-1}B)'}\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(A_0)'} \\
&= D_n^+\left((I\otimes A_0^{-1}B)K_{nq}\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(A_0^{-1}B)'} + (A_0^{-1}B\otimes I)\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(A_0^{-1}B)'}\right)\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(A_0)'} \\
&= -2D_n^+(\Sigma_u\otimes A_0^{-1}),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\varphi_1}{\partial\text{vec}(B)'}(\text{vec}(A_0, B)) = \frac{\partial\text{vech}(A_0^{-1}BB'A_0'^{-1})}{\partial\text{vec}(B)'} &= D_n^+\frac{\partial\text{vec}(A_0^{-1}BB'A_0'^{-1})}{\partial\text{vec}(A_0^{-1}B)'}\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(B)'} \\
&= D_n^+\left((I\otimes A_0^{-1}B)K_{nq}\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(A_0^{-1}B)'} + (A_0^{-1}B\otimes I)\frac{\partial\text{vec}(A_0^{-1}B)}{\partial\text{vec}(A_0^{-1}B)'}\right)A_0^{-1} \\
&= 2D_n^+(A_0^{-1}B\otimes A_0^{-1}),
\end{aligned}$$

and $\frac{\partial\varphi_2}{\partial\text{vec}(A_0,B)'}(\text{vec}(A_0, B)) = C_N$. Here, D_n^+ is the pseudo-inverse of the duplication matrix D_n which fulfils $D_n\text{vech}(A) = \text{vec}(A)$ for a matrix $A \in \mathbb{R}^{n\times n}$, and $K_{nm} \in \mathbb{R}^{nm\times nm}$ is a commutation matrix such that $\text{vec}(B') = K_{nm}\text{vec}(B)$ for $B \in \mathbb{R}^{n\times m}$, see for example, Lütkepohl (2005, pp. 662 and 663).

APPENDIX C: PROOF OF PROPOSITION 2

Let S_A of dimension $(n^2p \times n(np - s))$ denote the matrix obtained as the orthogonal complement of C_A. Since $(I_n \otimes \Gamma_p) = \left(I_n \otimes \begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} D_{11} & 0_{(n^2p-s)\times s} \\ 0_{s\times(n^2p-s)} & 0_{s\times s} \end{pmatrix}\begin{pmatrix} V_1' \\ V_2' \end{pmatrix}\right)$, it is obvious that $(I_n \otimes \Gamma_p)S_A$ does not have full rank if and only if $(I_n \otimes V_1')S_A$ is of reduced rank (smaller than $n^2p - ns$). For given Γ_p, the elements in the matrix of restrictions C_A (and therefore also the ones in S_A) are free (up to the requirement that the rows of C_A be linearly independent). The determinant $\det((I_n \otimes V_1')S_A)$ is thus a multivariate polynomial in the elements of S_A. This determinant is either identically zero or zero only on a set of Lebesgue measure zero. Since for $S_A = (I_n \otimes V_1)$ the determinant is equal to one, the determinant is not identically zero.

DOI: 10.1111/jtsa.12576 *Journal of Time Series Analysis* published by John Wiley & Sons Ltd.