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ULBOSSYN UALIKHANOVA

Gravity theories based on torsion: theoretical and observational constraints

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Supervisors: PhD Manuel Hohmann
Associate professor, Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, Estonia

PhD Laur Järv
Associate professor, Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, Estonia

Opponents: DSc Alexey Golovnev
Senior lecturer, Centre for Theoretical Physics, The British University in Egypt (BUE), Cairo, Egypt

PhD Jose Beltrán Jiménez
Junior research fellow, Departamento de Física Fundamental, Instituto Universitario de Física Fundamental y Matemáticas (IUFFyM), Universidad de Salamanca, Salamanca, Spain

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III M. Hohmann, C. Pfeifer, U. Ualikhanova, J.L. Said "Propagation of gravitational waves in symmetric teleparallel gravity theories"
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IV M. Hohmann, L. Järv, U. Ualikhanova "Covariant formulation of scalar-torsion gravity" *Phys. Rev.* D **97**, 104011 (2018)
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V M. Hohmann, L. Järv, U. Ualikhanova "Dynamical systems approach and generic properties of $f(T)$ cosmology"*Phys. Rev.* D **96**, 043508 (2017)
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Author's contribution

I, Ulbossyn Ualikhanova, have calculated and checked each and every equation in the papers. I wrote most of the manuscript for Reference I and partially the manuscripts, mostly working on calculations, for References II, III, IV, V. In addition I participated in all discussions, contributed to implementation of the research and to the analysis of the results. I thank my supervisors, Manuel Hohmann and Laur Järv for their guidance, teaching the methods of calculations, checking the results and quick feedback.

I have presented the results of all papers by giving a talk at the conferences and seminars:

1. Physics Department of Federico II University, Seminar (Naples, 2019).
2. 10th Alexander Friedmann International Seminar (St. Petersburg, 2019).
3. Geometric Foundations of Gravity 2019 (Tartu, 2019).
4. The 29th Workshop on General Relativity and Gravitation in Japan (Kobe, 2019).
5. Fifteenth Marcel Grossmann Meeting (Rome, 2018).
6. Teleparallel Gravity Workshop in Tartu (Tartu, 2018).
7. Gravity Malta: Gravitational waves, black holes and fundamental physics (Malta, 2018).
8. The 27th Workshop on General Relativity and Gravitation in Japan (Higashi Hiroshima, 2017).
9. Tartu-Tuorla cosmology meeting 2017 (Tartu, 2017).
10. Geometric Foundations of Gravity (Tartu, 2017).

Chapter 1

Introduction

Contents

1.1 Motivation

General Relativity (GR), which relates the gravitational effects to spacetime curvature, has been very successful in describing a wide range of phenomena. However, general relativity by itself does not provide any explanation for a number of observations in modern cosmology: the homogeneity of the cosmic microwave background, the accelerating expansion of the universe, the origin of structure and the motion of galaxies and galactic clusters. These observations can be explained by introducing mechanisms known as inflation, dark energy and dark matter, but the precise nature of these has remained unknown. Further, attempts to quantize general relativity have so far been unsuccessful.

The aforementioned reasons have motivated the study of a large number of possible extensions and modifications of general relativity [1], like f(R), f(R,G) and similar theories. However, in this thesis we do not study the modifications of GR itself, but a number of the theories that differ from GR by underlying connection.

We study first the teleparallel gravity [2, 3, 4], where the torsion free, metric compatible Levi-Civita connection of general relativity with its curvature is replaced by the curvature free, metric compatible Weitzenböck connection with its torsion. Teleparallel gravity describes gravity as a force, distinguishes between gravitational and inertial effects, which raises hope that the purely gravitational force part will be accessible to quantization. It can be considered as a gauge theory for the translation group, and therefore could be more in line with other gauge theories [5, 6, 3].

We also consider the symmetric teleparallel gravity, a class of theories, which use the curvature and torsion free symmetric teleparallel connection that is not metric compatible to describe gravity. Despite the difference in mathematical foundations, these two alternative geometries can be used to equivalently describe the dynamics of general relativity, thus can be referred as (symmetric)teleparallel equivalent of GR, i.e. TEGR or STEGR.

On the theoretical side an important criterion of viability is local Lorentz invariance. Teleparallel gravity is usually formulated in the formalism of tetrad and spin connection, the latter being independent of the former. In the teleparallel equivalent of general relativity the spin connection does not affect the tetrad field equations, and can be chosen to be zero arbitrarily [3]. Interpolating this property to the extensions like $f(T)$ or scalar-torsion gravity leads to a problematic result, for the action fails to be locally Lorentz invariant [7, 8], violating the basics of the tetrad formalism. It was argued that therefore these theories implied preferred frame effects, acausality, and were inhabitated by extra spurious degrees of freedom [9, 10, 11, 12]. The Lorentz invariance issue is fixed in the covariant formulation of the theory [13], which allows nontrivial spin connection compatible with vanishing curvature, i.e., flat spin connection.

After accepting nonvanishing spin connection there arises an obvious question how to determine it. An answer to the latter came only recently in the context of $f(T)$ gravity. Namely, variation of the action with respect to spin connection by carefully maintaining the flatness property yields an equation which can fix the remaining six components of the spin connection [14, 15]. This equation involves only the first derivatives of the spin connection, so one may ask whether the spin connection is an independent dynamical quantity in $f(T)$ gravity. One can not set the spin connection arbitrarily to zero, but for a given tetrad must make sure the spin connection satisfies the respective condition. As a pleasant byproduct it turns out that when the condition on the spin connection is satisfied, the antisymmetric part of the tetrad field equations vanishes automatically [14]. It is remarkable that this feature also holds in much more general theories of torsion [16, 17].

On the observational side our main focus will be on the gravitational wave tests and the solar system tests. The recent discovery of gravitational waves has opened a new era of observations in gravitational physics. Besides the possibility of gravitational astronomy, it allows to test gravity theories in regimes which have so far been inaccessible to experiments. Whereas GW observations have continued to be confirmed, the first three-detector observation by LIGO and VIRGO holds important significance in that such measurements allow for signal localization and, more to the purpose of this work, constraints on the six potential polarization modes of metric theories of gravity [18]. Moreover there has been the first multi messenger observations [19] which constrain the difference of the propagation velocity between GW and electromagnetic waves in vacuum, which can be different from zero in various modified theories of gravity [20, 21, 22, 23, 24, 25, 26]. Thus GW observations offer the possibility for strong constraints on theories predicting extra modes and a propagation velocity different from the speed of light, and so may be the route to reducing the variety of potential gravitational theories [27].

An important question is the consistency of any new theory with observations in the solar system. For those systems where gravity is still sufficiently weak, a theoretical description making use of a post-Newtonian approximation (PN) is usually performed. Our primary goal is to calculate the PN limit of teleparallel gravity theories using the parameterized post-Newtonian (PPN) formalism [28] which allows a characterization of gravity by ten parameters, which have been measured in high precision experiments. To do so, an adaptation of the classical PPN formalism to tetrad based theories is required. A possible adaptation can be derived from a similar approach to the PN limit of scalar-tetrad theories [29], by omitting the scalar field part. Further, it needs to be adapted to the covariant formulation of teleparallel gravity [13, 14, 30, 31], which we will use in our calculations, and in which also a flat spin connection appears as a dynamical field. The purpose of doing it is thus twofold. Our main aim is to put forward a general method for calculating the PN limit of teleparallel gravity theories in their covariant formulation, by expanding the tetrad components in a pure spacetime basis and expressing them in terms of the PN potentials and a number of constants, which are then determined by solving the field equations. The second aim is to use this general method in order to determine the PN limit of a general class of teleparallel gravity theories [32, 33]. This class is chosen to be very generic, such as to encompass a large number of theories discussed in the literature, while at the same time being prototypical for applying our formalism to even more general theories.

Finally, we use the dynamical systems approach to obtain a qualitative assessment of the behavior of solutions in a model, without delving into the often almost impossible task to find the analytic form of the solutions. While dynamical systems have been helpful in uncovering the main features of solutions in particular models [34, 35, 36, 37, 38, 39], there have been only a few papers attempting a more systematic analysis of generic $f(T)$ cosmology [35, 40, 41, 42]. Our present study aims at completing this task by deriving the general expressions for de Sitter fixed points, acceleration, phantom dark energy, and finite time singularities.

1.2 Aim of the thesis and of the overview article

As briefly mentioned in the Motivation, an important question is to check the viability of a new theory.

- In order to test the consistency and viability of a large class of cosmological models in the solar system we make use of the parametrized post-Newtonian (PPN) formalism. In particular, we calculate the post-Newtonian limit of the general class of teleparallel theories, whose action is given by a free function of three scalar quantities [32, 33]. This general class of teleparallel theories encompasses both the new relativity class of theories and the wide class of $f(T)$ theories.

- We test the most general class of teleparallel gravity theories whose action is quadratic in the torsion tensor, known as new general relativity and the most general class of symmetric teleparallel gravity theories whose action is quadratic in the nonmetricity tensor by deriving the propagation velocity of gravitational waves, which has been measured for the first time by Advanced LIGO when the gravitational wave signal GW170817 with optical follow-up is received. Further, we derive the polarization of gravitational waves in these theories, which can also be measured from combined LIGO and VIRGO observations.

- We put forward the covariant formulation of the a generalized form of scalar-torsion gravity $f(T, \phi)$ in order to fix the local Lorentz invariance issue.

- We use the method of dynamical systems to describe a wide range of phenomena in cosmology, like acceleration, phantom dark energy, and finite time singularities in $f(T)$ theories.

1.3 Structure of the thesis

The overview article contains the current Motivation, which also includes the mathematical notions and connections. Chapter Motivation is followed by two chapters introducing definition, geometry and field equations of the teleparallel gravity theories in Chapter 2 with torsion and in Chapter 3 with nonmetricity. Then in Chapter 4 we review the phenomenology, including gravitational wave observations, solar system tests and dynamical systems approach and test the aforementioned theories making use of them. The paper I "Parametrized post-Newtonian limit of general teleparallel gravity theories" is attached at the Chapter 5. The paper II "Propagation of gravitational waves in teleparallel gravity theories" is attached at the Chapter 6. The paper III "Propagation of gravitational waves in symmetric teleparallel gravity theories" is attached at the Chapter 7. The paper IV "Covariant formulation of scalar-torsion gravity" is attached at the Chapter 8. The paper V "Dynamical systems approach and generic properties of $f(T)$ cosmology" is attached at the Chapter 9. The overview article ends with the Summary. Each of the chapters is preceded by a local Table of Contents.

1.4 Mathematical notions

1. In the thesis we denote:

    - Lorentz indices with uppercase Latin letters $A, B, \ldots = 0, \ldots, 3$,
    - spacetime indices with lowercase Greek letters $\mu, \nu, \ldots = 0, \ldots, 3$,
    - spatial indices with lowercase Latin letters $i, j, \ldots = 1, \ldots, 3$.

2. We use the following abbreviation:

    - round brackets for symmetrization of indices $(\mu\nu) = \frac{1}{2}(\mu\nu + \nu\mu)$,
    - square brackets for antisymmetrization $[\mu\nu] = \frac{1}{2}(\mu\nu - \nu\mu)$,
    - the fixed indices, those not used in the (anti)symmetrization, are distinguished by vertical lines. For example, symmetrization over μ, γ; ν remains fixed $(\mu|\nu|\gamma) = \frac{1}{2}(\mu\nu\gamma + \gamma\nu\mu)$.

3. In our convention, we use the Minkowski metric η_{AB} and $\eta_{\mu\nu}$ with signature $(-,+,+,+)$.

4. In the majority of cases,

    - we will use geometrised units for the speed of light $c = 1$ and the Newtonian constant $G_N = 1$.
    - we will treat energy momentum tensor as a perfect fluid.

Remarks

- The bullet (•) denotes quantities related to the teleparallel spin connection.
- Open circle (∘) denotes quantities related to the Levi-Civita connection.
- Cross (×) denotes quantities related to the symmetric teleparallel connection.

1.5 Connections

We start with a brief review of the general geometry, particular cases of which we use in this thesis. The fundamental fields defining the geometry are a Lorentzian metric $g_{\mu\nu}$ and an affine connection with coefficients $\Gamma^{\rho}{}_{\mu\nu}$. Using the metric, the affine connection can be decomposed into

$$\Gamma^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu}, \tag{1.5.1}$$

consisting of the Levi-Civita connection (known also as Christoffel and Riemannian connection)

$$\overset{\circ}{\Gamma}{}^{\mu}{}_{\sigma\rho} \equiv \frac{1}{2} g^{\mu\tau} \left(\partial_{\sigma} g_{\tau\rho} + \partial_{\rho} g_{\tau\sigma} - \partial_{\tau} g_{\sigma\rho}\right) , \tag{1.5.2}$$

the contortion

$$K^{\rho}{}_{\mu\nu} = \frac{1}{2} \left(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}\right) , \tag{1.5.3}$$

and the disformation

$$L^{\rho}{}_{\mu\nu} = \frac{1}{2} Q^{\rho}{}_{\mu\nu} - Q_{\mu}{}^{\rho}{}_{\nu} - Q_{\nu}{}^{\rho}{}_{\mu} \,. \tag{1.5.4}$$

The last two are defined by torsion

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = -2\Gamma^{\rho}{}_{[\mu\nu]} \tag{1.5.5}$$

and nonmetricity

$$Q_{\rho\mu\nu} = \nabla_{\rho} g_{\mu\nu} = \partial_{\rho} g_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\rho} g_{\alpha\nu} - \Gamma^{\alpha}{}_{\nu\rho} g_{\mu\alpha} \,. \tag{1.5.6}$$

It is helpful to remark that the torsion, as well as curvature

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\lambda}{}_{\rho\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\rho\mu} + \Gamma^{\tau}{}_{\rho\nu}\Gamma^{\lambda}{}_{\tau\mu} - \Gamma^{\tau}{}_{\rho\mu}\Gamma^{\lambda}{}_{\tau\nu} \tag{1.5.7}$$

are properties of the connection. Nonmetricity is not a property of the connection (alone), but a property of the metric-affine geometry, since it depends on both, metric and connection.

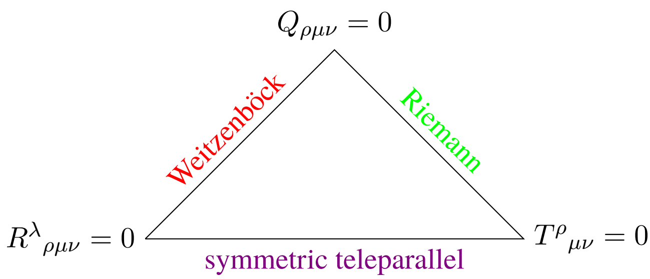


FIG. 1. Subclasses of metric-affine geometry, depending on the properties of connection.

Vanishing curvature gives teleparallel geometry, while vanishing torsion presents torsion free geometry and vanishing nonmetricity leads to Riemann-Cartan geometry. We can restrict the metric-affine geometry by their combination as well.

For example, vanishing torsion and nonmetricity condition leads to the Levi-Civita connection, known as the general relativity connection, which is

1. symmetric $\mathring{\Gamma}^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\nu\mu}$, i.e.

$$\mathring{T}^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\nu\mu} - \mathring{\Gamma}^{\rho}{}_{\mu\nu} = 0\,, \tag{1.5.8}$$

2. compatible with the metric g (metricity condition)

$$\mathring{\nabla}_{\rho} g_{\mu\nu} \equiv \partial_{\rho} g_{\mu\nu} - \mathring{\Gamma}^{\alpha}{}_{\mu\rho} g_{\alpha\nu} - \mathring{\Gamma}^{\alpha}{}_{\nu\rho} g_{\mu\alpha} = 0 \tag{1.5.9}$$

and has non-vanishing curvature

$$\mathring{R}^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\mathring{\Gamma}^{\lambda}{}_{\rho\nu} - \partial_{\nu}\mathring{\Gamma}^{\lambda}{}_{\rho\mu} + \mathring{\Gamma}^{\tau}{}_{\rho\nu}\mathring{\Gamma}^{\lambda}{}_{\tau\mu} - \mathring{\Gamma}^{\tau}{}_{\rho\mu}\mathring{\Gamma}^{\lambda}{}_{\tau\nu}\,. \tag{1.5.10}$$

In the following chapters we discuss other two possible combinations.

Chapter 2

Teleparallel gravity

Contents

2.1 Geometry with torsion

The second possible combination, an alternative to general relativity, is to assume vanishing curvature and nonmetricity from the beginning,

$$\overset{\bullet}{R}{}^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\rho\nu} - \partial_\nu \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\rho\mu} + \overset{\bullet}{\Gamma}{}^{\tau}{}_{\rho\nu} \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\tau\mu} - \overset{\bullet}{\Gamma}{}^{\tau}{}_{\rho\mu} \overset{\bullet}{\Gamma}{}^{\lambda}{}_{\tau\nu} = 0 \,. \tag{2.1.1}$$

This case we work in the Weitzenböck geometry. Weitzenböck connection has non-vanishing torsion

$$T^{\rho}{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} = -2\overset{\bullet}{\Gamma}{}^{\rho}{}_{[\mu\nu]} \,. \tag{2.1.2}$$

Let us begin with a brief outline of the geometry of the theories we consider in this chapter. The fundamental variables in teleparallel theories of gravity, following their covariant formulation [13, 14, 30, 31], are a tetrad $\theta^A{}_\mu$ and a curvature free Lorentz spin connection $\overset{\bullet}{\omega}{}^A{}_{B\mu}$. We denote the inverse tetrad by $e_A{}^\mu$, which satisfies

$$\theta^A{}_\mu e_A{}^\nu = \delta^\nu_\mu, \qquad \theta^A{}_\mu e_B{}^\mu = \delta^A_B. \tag{2.1.3}$$

Via these variables one defines the spacetime metric with its inverse

$$g_{\mu\nu} = \eta_{AB}\theta^A{}_\mu \theta^B{}_\nu, \qquad g^{\mu\nu} = \eta^{AB} e_A{}^\mu e_B{}^\nu \tag{2.1.4}$$

and, conversely, the Minkowski metric

$$\eta_{AB} = e_A{}^\mu e_B{}^\nu g_{\mu\nu}, \qquad \eta^{AB} = \theta^A{}_\mu \theta^B{}_\nu g^{\mu\nu} \,. \tag{2.1.5}$$

One can also raise or lower indices by making use of metrics

$$e_{A\mu} = g_{\mu\nu} e_A{}^\nu, \qquad \theta^{A\mu} = g^{\mu\nu}\theta^A{}_\nu \,, \tag{2.1.6}$$

$$\theta^A{}_\mu = \eta^{AB} e_{B\mu}, \qquad e_A{}^\mu = \eta_{AB}\theta^{B\mu} \,. \tag{2.1.7}$$

The tetrad corresponding to a given metric is not defined uniquely, but only up to a local Lorentz transformation which transforms the spin connection as well,

$$\theta'^A{}_\mu = \Lambda^A{}_B \theta^B{}_\mu \,, \quad \overset{\bullet}{\omega}{}'^A{}_{B\mu} = \Lambda^A{}_C \, \overset{\bullet}{\omega}{}^C{}_{D\mu} \, \Lambda_B{}^D + \Lambda^A{}_C \, \partial_\mu \Lambda_B{}^C \,, \tag{2.1.8}$$

here $\Lambda_A{}^B$ is the inverse of the Lorentz transformation matrix $\Lambda^A{}_B$. The transformation (2.1.8) just reflects the possibility to switch between different local observers. Demanding that the spin connection vanishes is a particular gauge choice and in general means picking a specific (class of) observer(s) among the others.

Covariant derivative acting on the Lorentz or spacetime indices is given by

$$\overset{\bullet}{D}_\mu \theta^A{}_\nu = \partial_\mu \theta^A{}_\nu + \overset{\bullet}{\omega}{}^A{}_{B\mu}\theta^B{}_\nu \,, \tag{2.1.9}$$

$$\overset{\bullet}{\nabla}_\mu e_B{}^\nu = \partial_\mu e_B{}^\nu + \overset{\bullet}{\Gamma}{}^\nu{}_{\rho\mu} e_B{}^\rho \,. \tag{2.1.10}$$

The relation between Weitzenböck connection $\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu}$ and spin connection $\overset{\bullet}{\omega}{}^A{}_{B\nu}$ is given by

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = e_A{}^\rho \overset{\bullet}{D}_\nu \theta^A{}_\mu = e_A{}^\rho \left(\partial_\nu \theta^A{}_\mu + \overset{\bullet}{\omega}{}^A{}_{B\nu} \theta^B{}_\mu \right) \,, \tag{2.1.11}$$

and, conversely, we have

$$\overset{\bullet}{\omega}{}^A{}_{B\mu} = \theta^A{}_\nu \overset{\bullet}{\nabla}_\mu e_B{}^\nu = \theta^A{}_\nu \left(\partial_\mu e_B{}^\nu + \overset{\bullet}{\Gamma}{}^\nu{}_{\rho\mu} e_B{}^\rho \right) \,. \tag{2.1.12}$$

In particular, expression (2.1.11) is equivalent to the identity

$$0 = \partial_\mu \theta^A{}_\nu + \overset{\bullet}{\omega}{}^A{}_{B\mu} \theta^B{}_\nu - \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} \theta^A{}_\rho \,. \tag{2.1.13}$$

One can see from the covariant derivative of the Minkowski metric (2.1.5) and (2.1.12)

$$\begin{aligned} \overset{\bullet}{\nabla}_\sigma \eta_{AB} &= e_B{}^\nu g_{\mu\nu} \overset{\bullet}{\nabla}_\sigma e_A{}^\mu + e_A{}^\mu g_{\mu\nu} \overset{\bullet}{\nabla}_\sigma e_B{}^\nu \\ &= e_{B\mu} \overset{\bullet}{\nabla}_\sigma e_A{}^\mu + e_{A\nu} \overset{\bullet}{\nabla}_\sigma e_B{}^\nu = \overset{\bullet}{\omega}_{BA\mu} + \overset{\bullet}{\omega}_{AB\mu} = 0 \end{aligned} \tag{2.1.14}$$

that the spin connection is antisymmetric in its internal indices, i.e. $\overset{\bullet}{\omega}_{AB\mu} = -\overset{\bullet}{\omega}_{BA\mu}$.

Making use of (2.1.11) the covariant formulation of the torsion (2.1.2) can be rewritten as

$$T^A{}_{\mu\nu} = \overset{\bullet}{D}_\mu \theta^A{}_\nu - \overset{\bullet}{D}_\nu \theta^A{}_\mu = \partial_\mu \theta^A{}_\nu - \partial_\nu \theta^A{}_\mu + \overset{\bullet}{\omega}{}^A{}_{B\mu} \theta^B{}_\nu - \overset{\bullet}{\omega}{}^A{}_{B\nu} \theta^B{}_\mu \,, \tag{2.1.15}$$

while the curvature (1.5.7) as

$$\overset{\bullet}{R}{}^A{}_{B\mu\nu} = \partial_\mu \overset{\bullet}{\omega}{}^A{}_{B\nu} - \partial_\nu \overset{\bullet}{\omega}{}^A{}_{B\mu} + \overset{\bullet}{\omega}{}^A{}_{C\mu} \overset{\bullet}{\omega}{}^C{}_{B\nu} - \overset{\bullet}{\omega}{}^A{}_{C\nu} \overset{\bullet}{\omega}{}^C{}_{B\mu} = 0 \,. \tag{2.1.16}$$

Note, that tetrads can be used to convert the spacetime indices into internal indices and vice versa, e.g. one can transform the torsion components $T^A{}_{\mu\nu}$ to the purely spacetime index components $T^\rho{}_{\mu\nu} = e_A{}^\rho T^A{}_{\mu\nu}$.

The pure tetrad formulation of teleparallel gravity neglects the spin connection,

$$\overset{\bullet}{\omega}{}^A{}_{B\mu} = 0 \rightarrow \begin{cases} \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = e_A{}^\rho \partial_\nu \theta^A{}_\mu, \\ T^\rho{}_{\mu\nu} = 2 e_A{}^\rho \partial_{[\mu} \theta^A{}_{\nu]}. \end{cases}$$

As a consequence, torsion tensor does not transform as a tensor under local Lorentz transformations, which is a violation of local Lorentz invariance. However, this approach is physically meaningful, if we are interested in the solutions of the field equations. It correlates with the assumption that the spin connection does not represent gravitation, but only inertial effects. As a result, there exists a proper Lorentz frame (also known as the Weitzenböck gauge) where the inertial effects are eliminated, the spin connection vanishes, and the field equations become simpler to solve. For example, we have used this feature in the study of the solar system tests (see Chapter 5) and the gravitational waves solutions (see Chapters 6 and 7). So, both formulations, the pure tetrad and covariant one, lead to the same results for aforementioned solutions.

To present a teleparallel gravitational theory we need an action functional constructed from the torsion tensor. The quantity under the action integral should remain invariant (transform as a scalar up to a boundary term) under general spacetime coordinate transformations and local Lorentz transformations. To construct such a scalar there are three possible contractions of the torsion tensor:

$$\mathcal{T}_1 = T^{\mu\nu\rho}T_{\mu\nu\rho}\,, \quad \mathcal{T}_2 = T^{\mu\nu\rho}T_{\rho\nu\mu}\,, \quad \mathcal{T}_3 = T^{\mu}{}_{\mu\rho}T_{\nu}{}^{\nu\rho}\,, \tag{2.1.17}$$

which are quadratic and leave parity even (do not involve the Levi-Civita totally antisymmetric symbol).

2.2 Variations

To derive the field equations we need the variations of the action components. The variation of the fundamental variables are given by [14]

$$\delta_\theta \theta^A{}_\mu = \delta\theta^A{}_\mu\,, \tag{2.2.1a}$$

$$\delta_\theta e_A{}^\mu = -e_A{}^\nu e_B{}^\mu \delta\theta^B{}_\nu\,, \tag{2.2.1b}$$

$$\delta_\theta \theta = \theta\, e_A{}^\mu \delta\theta^A{}_\mu\,, \tag{2.2.1c}$$

where $\theta = det(\theta^A{}_\mu)$.

Then the variation of the torsion tensor

$$T^\rho{}_{\mu\nu} = 2e_A{}^\rho \left(\partial_{[\mu}\theta^A{}_{\nu]} + \overset{\bullet}{\omega}{}^A{}_{B[\mu}\theta^B{}_{\nu]}\right) \tag{2.2.2}$$

with respect to the tetrad can be calculated as

$$\delta_\theta T^\rho{}_{\mu\nu} = -2e_B{}^\rho e_A{}^\sigma \delta\theta^B{}_\sigma \left(\partial_{[\mu}\theta^A{}_{\nu]} + \overset{\bullet}{\omega}{}^A{}_{B[\mu}\theta^B{}_{\nu]}\right) + 2e_A{}^\rho \left(\partial_{[\mu}\delta\theta^A{}_{\nu]} + \overset{\bullet}{\omega}{}^A{}_{B[\mu}\delta\theta^B{}_{\nu]}\right)\,. \tag{2.2.3}$$

Making use of $T^{\sigma}{}_{\mu\nu} = e_A{}^{\sigma} T^A{}_{\mu\nu}$ and $\overset{\bullet}{D}_{\nu}\theta^A{}_{\mu} = \partial_{\nu}\theta^A{}_{\mu} + \overset{\bullet}{\omega}{}^A{}_{B\nu}\theta^B{}_{\mu}$ we get a compacted form

$$\delta_{\theta} T^{\rho}{}_{\mu\nu} = -e_A{}^{\rho} T^{\sigma}{}_{\mu\nu}\delta\theta^A{}_{\sigma} + 2e_A{}^{\rho}\overset{\bullet}{D}_{[\mu}\delta\theta^A{}_{\nu]} \,. \tag{2.2.4}$$

The variation of the torsion with respect to the spin connection is given by

$$\delta_{\omega} T^{\rho}{}_{\mu\nu} = 2e_A{}^{\rho}\delta\overset{\bullet}{\omega}{}^A{}_{B[\mu}\theta^B{}_{\nu]} \,. \tag{2.2.5}$$

In particular, for the trace we have

$$\delta_{\omega} T^{\nu}{}_{\nu\mu} = e_A{}^{\nu}\theta^B{}_{\mu}\delta\overset{\bullet}{\omega}{}^A{}_{B\nu} \,. \tag{2.2.6}$$

We can now obtain the variation of the quadratic terms with respect to the tetrad

$$\delta_{\theta}\left(T^{\mu\nu\rho}T_{\mu\nu\rho}\right) = -4T^{\rho\sigma\mu}T_{\rho\sigma\nu}e_A{}^{\nu}\delta\theta^A{}_{\mu} - 4e_A{}^{\rho}T_{\rho}{}^{\mu\nu}\overset{\bullet}{D}_{\nu}\delta\theta^A{}_{\mu} \,, \tag{2.2.7a}$$

$$\delta_{\theta}\left(T^{\mu\nu\rho}T_{\rho\nu\mu}\right) = 4T^{[\sigma\mu]\rho}T_{\rho\sigma\nu}e_A{}^{\nu}\delta\theta^A{}_{\mu} - 4T^{[\mu}{}_{|\rho|}{}^{\nu]}e_A{}^{\rho}\overset{\bullet}{D}_{\nu}\delta\theta^A{}_{\mu} \,, \tag{2.2.7b}$$

$$\delta_{\theta}\left(T^{\mu}{}_{\mu\rho}T^{\nu}{}_{\nu}{}^{\rho}\right) = 4T^{\rho}{}_{\rho}{}^{[\sigma}T^{\mu]}{}_{\sigma\nu}e_A{}^{\nu}\delta\theta^A{}_{\mu} + 4T^{\rho}{}_{\rho}{}^{[\mu}e_A{}^{\nu]}\overset{\bullet}{D}_{\nu}\delta\theta^A{}_{\mu} \tag{2.2.7c}$$

and with respect to the spin connection

$$\delta_{\omega}\left(T^{\mu\nu\rho}T_{\mu\nu\rho}\right) = -4T_{\rho}{}^{\sigma\mu}e_A{}^{\rho}\theta^B{}_{\sigma}\delta\overset{\bullet}{\omega}{}^A{}_{B\mu} \,, \tag{2.2.8a}$$

$$\delta_{\omega}\left(T^{\mu\nu\rho}T_{\rho\nu\mu}\right) = 4T^{[\mu}{}_{|\rho|}{}^{\sigma]}e_A{}^{\rho}\theta^B{}_{\sigma}\delta\overset{\bullet}{\omega}{}^A{}_{B\mu} \,, \tag{2.2.8b}$$

$$\delta_{\omega}\left(T^{\mu}{}_{\mu\rho}T^{\nu}{}_{\nu}{}^{\rho}\right) = 2T^{\rho}{}_{\rho}{}^{\nu}e_A{}^{\mu}\theta^B{}_{\nu}\delta\overset{\bullet}{\omega}{}^A{}_{B\mu} \,. \tag{2.2.8c}$$

In the following sections, we briefly review the theories we test to check their viability, using the dynamical systems approach, the gravitational wave tests and the solar system tests. In particular, we display their action and field equations needed to the analysis.

2.3 Generic $\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ theories

Let us write a generic action given by two parts,

$$S = S_g[\theta, \overset{\bullet}{\omega}] + S_m[\theta, \chi] \,, \tag{2.3.1}$$

where S_g is the gravitational part, S_m is the matter part, and χ denotes an arbitrary set of matter fields. The variation of the matter action S_m with respect to the tetrad $\theta^A{}_{\mu}$ can be written in the general form

$$\delta_{\theta} S_m = -\int_M \Theta_A{}^{\mu}\delta\theta^A{}_{\mu}\,\theta\, d^4x \,. \tag{2.3.2}$$

Here θ is the determinant of the tetrad. Further, $\Theta_A{}^\mu$ denotes the energy-momentum tensor, which we assume to be symmetric, $\Theta_{[\mu\nu]} = 0$, by imposing local Lorentz invariance on the matter action.

The gravitational part of the action S_g can be defined via the free function $\mathcal{F}$,

$$S_g[\theta, \omega] = \frac{1}{2\kappa^2} \int_M \mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)\, \theta\, d^4x, \tag{2.3.3}$$

which depends on the three scalar quantities, which are parity-even and quadratic in the torsion, given by the formula above (2.1.17).

This action defines a generic class of teleparallel gravity theories, which has been discussed before in the literature [32, 33], and shall serve both as a generic example and starting point for further extensions in future work. By variation of the total action (2.3.1) with respect to the tetrad we find the gravitational field equations[1]

$$\begin{aligned}\kappa^2\Theta_{\mu\nu} = {}& \frac{1}{2}\mathcal{F} g_{\mu\nu} + 2\overset{\circ}{\nabla}{}^\rho \left(\mathcal{F}_{,1} T_{\nu\mu\rho} + \mathcal{F}_{,2} T_{[\rho\mu]\nu} + \mathcal{F}_{,3} T^\sigma{}_{\sigma[\rho} g_{\mu]\nu}\right) \\ & + \mathcal{F}_{,1} T^{\rho\sigma}{}_\mu \left(T_{\nu\rho\sigma} - 2T_{[\rho\sigma]\nu}\right) - \frac{1}{2}\mathcal{F}_{,3} T^\sigma{}_{\sigma\rho}\left(T^\rho{}_{\mu\nu} + 2T_{(\mu\nu)}{}^\rho\right) \\ & + \frac{1}{2}\mathcal{F}_{,2}\left[T_\mu{}^{\rho\sigma}\left(2T_{\rho\sigma\nu} - T_{\nu\rho\sigma}\right) + T^{\rho\sigma}{}_\mu\left(2T_{[\rho\sigma]\nu} - T_{\nu\rho\sigma}\right)\right],\end{aligned} \tag{2.3.4}$$

where $\mathcal{F}_{,i} = \partial\mathcal{F}/\partial\mathcal{T}_i$ with $i = 1, 2, 3$ and $\overset{\circ}{\nabla}$ is the covariant derivative with respect to the Levi-Civita connection of the metric $g_{\mu\nu}$. The antisymmetric part of these field equations is identical to the connection field equations obtained by variation with respect to the spin connection.

2.4 New general relativity

In the new general relativity (NGR) class of teleparallel gravity theories [43] the Lagrangian is given by the general linear combination of quadratic torsion invariants

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = t_1\mathcal{T}_1 + t_2\mathcal{T}_2 + t_3\mathcal{T}_3 \tag{2.4.1}$$

with constant coefficients t_i. It is remarkable that, unless we introduce higher derivatives or scalar fields, the general teleparallel Lagrangian of the action (2.6.2) reduce to the case of NGR at perturbative level. The choice of the parameters $t_1 = \frac{1}{4}, t_2 = \frac{1}{2}, t_3 = -1$ in (2.4.1) yields the teleparallel formulation of general relativity, which is called teleparallel equivalent of general relativity (TEGR) [4].

[1] Since we are interested in the study of the PPN limit and the GWs of the given theories (see Chapters 5 and 6), the gravitational field equations (in the Weitzenböck gauge) are the only field equations we require.

2.5 $f(T)$ theories

Another important class of theories is given by the so-called $f(T)$ class of theories, whose Lagrangian is given by

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = f(T)\,, \quad T = \frac{1}{4}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_2 - \mathcal{T}_3\,. \tag{2.5.1}$$

Here T is the torsion scalar, which constitutes the Lagrangian of the TEGR. Torsion scalar can be defined by

$$T = \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\mu}{}_{\rho\mu}T^{\nu\rho}{}_{\nu}\,, \tag{2.5.2}$$

or, equivalently, by

$$T = \frac{1}{2}T^{\rho}{}_{\mu\nu}S_{\rho}{}^{\mu\nu}\,, \tag{2.5.3}$$

with the superpotential

$$S_{\rho}{}^{\mu\nu} = K^{\mu\nu}{}_{\rho} - \delta^{\mu}_{\rho}T_{\sigma}{}^{\sigma\nu} + \delta^{\nu}_{\rho}T_{\sigma}{}^{\sigma\mu}\,. \tag{2.5.4}$$

The variations of the torsion scalar take the form

$$\delta_{\theta}T = -2S^{\rho\sigma\mu}T_{\rho\sigma\nu}e_{A}{}^{\nu}\delta\theta^{A}{}_{\mu} - 2S_{\rho}{}^{\mu\nu}e_{A}{}^{\rho}\overset{\bullet}{D}_{\nu}\delta\theta^{A}{}_{\mu}\,, \tag{2.5.5a}$$

$$\delta_{\omega}T = \left(T^{\mu}{}_{\rho\sigma}e_{A}{}^{\rho}e_{B}{}^{\sigma} - 2T^{\rho}{}_{\rho\nu}e_{A}{}^{\mu}e_{B}{}^{\nu}\right)\delta\overset{\bullet}{\omega}{}^{AB}{}_{\mu}\,. \tag{2.5.5b}$$

The tetrad field equations of these theories are given by its symmetric part

$$\frac{1}{2}fg_{\mu\nu} + \overset{\circ}{\nabla}_{\rho}\left(f_{T}S_{(\mu\nu)}{}^{\rho}\right) - \frac{1}{2}f_{T}S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} = \kappa^{2}\Theta_{\mu\nu} \tag{2.5.6}$$

and the antisymmetric part

$$0 = \partial_{[\rho}f_{T}T^{\rho}{}_{\mu\nu]}\,. \tag{2.5.7}$$

where subscriptl denote the derivative $f_T = \frac{df}{dT}$. In the last term the notation means that one first needs to antisymmetrize with respect all three lower indices and then sum over with the repeating upper index.

The connection field equations are

$$0 = -\overset{\bullet}{\nabla}_{[\mu}\overset{\bullet}{\nabla}_{\nu]}f_{T} + \partial_{[\nu}f_{T}\overset{\bullet}{K}{}^{\rho}{}_{\mu]\rho} = \frac{3}{2}\partial_{[\rho}f_{T}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu]}\,, \tag{2.5.8}$$

i.e. are equivalent to the antisymmetric part of the tetrad field equations (2.5.8).

2.6 Scalar-torsion gravity

For convenience, the comma notation will denote the partial derivatives, e.g., $\partial_\mu \phi \equiv \phi_{,\mu}$. We consider the scalar-torsion gravity model given by the action

$$S = S_g[\theta, \overset{\bullet}{\omega}, \phi] + S_m[\theta, \chi]\,. \tag{2.6.1}$$

Here the variation of the matter part of the action S_m with respect to the tetrad $\theta^A{}_\mu$ is of the same form as it given in (2.3.2). For the gravitational part we choose the action

$$S_g = \frac{1}{2\kappa^2}\int_M d^4x\,\theta\left[f(T,\phi) + Z(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right]\,, \tag{2.6.2}$$

which depends on two arbitrary functions f and Z of the torsion scalar and a scalar field ϕ, while $2\kappa^2 = 16\pi G_N$ sets the Newtonian gravitational constant.

We can denote the kinetic term of the scalar field by [44]

$$X = -\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\,. \tag{2.6.3}$$

as a matter of convenience and find its variation with respect to the scalar field and the tetrad

$$\delta_\phi X = -g^{\mu\nu}\phi_{,\nu}\delta\phi_{,\mu}\,, \tag{2.6.4a}$$
$$\delta_\theta X = g^{\mu\nu}\phi_{,\nu}\phi_{,\rho}e_a{}^\rho\delta\theta^a{}_\mu\,. \tag{2.6.4b}$$

From the variation of the gravitational part (2.6.2) with respect to the tetrad $\theta^a{}_\mu$ we obtain the symmetric part of the tetrad field equations

$$\frac{1}{2}fg_{\mu\nu} + \overset{\circ}{\nabla}_\rho\left(f_T S_{(\mu\nu)}{}^\rho\right) - \frac{1}{2}f_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - Z\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}Zg_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} = \kappa^2\Theta_{\mu\nu} \tag{2.6.5}$$

and the antisymmetric part of the tetrad field equations

$$0 = \partial_{[\rho}f_T T^\rho{}_{\mu\nu]}\,. \tag{2.6.6}$$

In the last term the notation means that one first needs to antisymmetrize with respect all three lower indices and then sum over with the repeating upper index.

Variation of the gravitational part (2.6.2) with respect to the spin connection yields the connection field equations

$$0 = -\overset{\bullet}{\nabla}_{[\mu}\overset{\bullet}{\nabla}_{\nu]}f_T + \partial_{[\nu}f_T\overset{\bullet}{K}{}^\rho{}_{\mu]\rho} = \frac{3}{2}\partial_{[\rho}f_T\overset{\bullet}{T}{}^\rho{}_{\mu\nu]}\,. \tag{2.6.7}$$

One can see that the antisymmetric part of the tetrad field equations (2.6.7) is equivalent to the connection field equatuions (2.6.7).

Variation of the gravitational part (2.6.2) of the action with respect to the scalar field ϕ yields the scalar field equation

$$f_\phi - Z_\phi g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2Z\overset{\circ}{\Box}\phi = 0\,, \tag{2.6.8}$$

where $f_\phi = \dfrac{df}{d\phi}$ and $\overset{\circ}{\Box} = g^{\mu\nu}\overset{\circ}{\nabla}_\mu\overset{\circ}{\nabla}_\nu$ is the d'Alembert operator.

Chapter 3

Symmetric teleparallel gravity

Contents

In the paper 7 we study symmetric teleparallel gravity, which differs from the teleparallel gravity by the underlying geometry, to compare the results from the gravitational waves in theories with torsion to those with nonmetricity.

3.1 Geometry with nonmetricity

In the theories with nonmetricity we assume vanishing curvature and torsion from the beginning

$$\overset{\times}{R}{}^{\lambda}{}_{\rho\mu\nu} = \partial_\mu \overset{\times}{\Gamma}{}^{\lambda}{}_{\rho\nu} - \partial_\nu \overset{\times}{\Gamma}{}^{\lambda}{}_{\rho\mu} + \overset{\times}{\Gamma}{}^{\tau}{}_{\rho\nu} \overset{\times}{\Gamma}{}^{\lambda}{}_{\tau\mu} - \overset{\times}{\Gamma}{}^{\tau}{}_{\rho\mu} \overset{\times}{\Gamma}{}^{\lambda}{}_{\tau\nu} \equiv 0, \tag{3.1.1a}$$

$$\overset{\times}{T}{}^{\rho}{}_{\mu\nu} = \overset{\times}{\Gamma}{}^{\rho}{}_{\nu\mu} - \overset{\times}{\Gamma}{}^{\rho}{}_{\mu\nu} \equiv 0, \tag{3.1.1b}$$

but nonvanishing nonmetricity

$$Q_{\alpha\mu\nu} = \overset{\times}{\nabla}_\alpha g_{\mu\nu} \tag{3.1.2}$$

which implies

$$Q_\alpha{}^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} Q_{\alpha\rho\sigma} = -\overset{\times}{\nabla}_\alpha g^{\mu\nu} \,. \tag{3.1.3}$$

The nonmetricity is symmetric in its second and third index, $Q_{\alpha\mu\nu} = Q_{\alpha\nu\mu}$. Some helpful expressions that we will use in our calculations later are

$$Q_\alpha = g^{\mu\nu} Q_{\alpha\mu\nu} \,, \tag{3.1.4a}$$

$$\tilde{Q}_\alpha = g^{\mu\nu} Q_{\mu\nu\alpha} \,. \tag{3.1.4b}$$

The most general connection which satisfies the assumptions (3.1.1a) and (3.1.1b) is generated by a coordinate transformation defined by functions $\xi^\mu(x)$ in the form [45, 16]

$$\overset{\times}{\Gamma}{}^{\mu}{}_{\nu\sigma} = \frac{\partial x^\mu}{\partial \xi^\rho} \partial_\nu \partial_\sigma \xi^\rho \,. \tag{3.1.5}$$

It further follows that it is always possible to find coordinates such that

$$\overset{\times}{\Gamma}{}^{\alpha}{}_{\mu\nu} \equiv 0, \tag{3.1.6}$$

not only at a single point, but in an open neighborhood. This particular choice of coordinates is known as the coincident gauge [46], and will be used throughout this work. Note that this uniquely determines the coordinate system (x^μ) we use, up to linear transformations of the form

$$x^\mu \mapsto \tilde{\xi}^\mu(x) = \tilde{\xi}^\mu(x_0) + (x^\nu - x_0^\nu)\, \partial_\nu \tilde{\xi}^\mu \Big|_{x=x_0} , \tag{3.1.7}$$

so that $\partial_\mu \partial_\nu \tilde{\xi}^\alpha \equiv 0$. It follows that we have no further gauge freedom left to impose conditions on the metric degrees of freedom, except at a single point, as it is conventionally the case, e.g., in general relativity. In the coincident gauge covariant derivatives are replaced by partial derivatives, so that the nonmetricity reads

$$Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu} \,. \tag{3.1.8}$$

3.2 Newer general relativity

We consider the "newer GR" model given by action for the metric, the coordinate functions ξ^μ and the matter fields χ^I [47, 16, 46]

$$S[g_{\mu\nu}, \xi^\sigma, \chi^I] = S_g[g_{\mu\nu}, \xi^\sigma] + S_m[g_{\mu\nu}, \chi^I]\,, \quad S_g = -\int_M \frac{\sqrt{-g}}{2k^2} \mathbb{Q} d^4x\,. \quad (3.2.1)$$

Here the matter part S_m does not depend on the affine connection $\overset{\times}{\Gamma}{}^\alpha{}_{\mu\nu}[\xi]$, but only on the metric and a set of matter fields .

The gravitational part S_g is given by the most general action quadratic in the nonmetricity, where the nonmetricity scalar

$$\mathbb{Q} = Q_\alpha{}^{\mu\nu} P^\alpha{}_{\mu\nu} \quad (3.2.2)$$

is defined via the nonmetricity conjugate

$$P^\alpha{}_{\mu\nu} = c_1 Q^\alpha{}_{\mu\nu} + c_2 Q_{(\mu}{}^\alpha{}_{\nu)} + c_3 Q^\alpha g_{\mu\nu} + c_4 \delta^\alpha_{(\mu} \tilde{Q}_{\nu)} + \frac{c_5}{2} \left(\tilde{Q}^\alpha g_{\mu\nu} + \delta^\alpha_{(\mu} Q_{\nu)} \right), \quad (3.2.3)$$

unless one introduces also derivatives [48] .

By variation of the total action with respect to the metric, one obtains the field equations

$$\frac{2}{\sqrt{-g}} \overset{\times}{\nabla}_\alpha (\sqrt{-g} P^\alpha{}_{\mu\nu}) + P_{\mu\sigma\rho} Q_\nu{}^{\sigma\rho} - 2 Q_{\rho\mu}{}^\sigma P^\rho{}_{\nu\sigma} - \frac{1}{2} \mathbb{Q} g_{\mu\nu} = k^2 \Theta_{\mu\nu}\,, \quad (3.2.4)$$

where the energy-momentum tensor $\Theta_{\mu\nu}$ is derived from the matter action S_m. To obtain the second set of field equations, we vary the total action with respect to the components of the connection generating coordinate functions ξ^μ. Note that this is equivalent to performing a restricted variation of the flat, symmetric connection $\overset{\times}{\Gamma}{}^\alpha{}_{\mu\nu}$, which must be of the form $\delta \overset{\times}{\Gamma}{}^\alpha{}_{\mu\nu} = \overset{\times}{\nabla}_\mu \overset{\times}{\nabla}_\nu \delta \xi^\alpha$ in order to keep the vanishing torsion and curvature, $\delta \overset{\times}{T}{}^\alpha{}_{\mu\nu} \equiv 0$ and $\delta \overset{\times}{R}{}^\alpha{}_{\beta\mu\nu} \equiv 0$. After twice performing integration by parts, carefully taking into account the terms arising from $\overset{\times}{\nabla}_\mu \sqrt{-g}$ due to the nonmetricity, this yields the field equations

$$\overset{\times}{\nabla}_\mu \overset{\times}{\nabla}_\nu \left(\sqrt{-g} P^{\mu\nu}{}_\alpha \right) = 0\,. \quad (3.2.5)$$

Note that their right hand side vanishes, since we have assumed no direct coupling of the matter to the flat, symmetric connection, and so the hypermomentum vanishes. The symmetric teleparallel equivalent of general relativity (STEGR) is included in the Newer GR class of gravity theories for the choice of the parameters $c_1 = -\frac{1}{4}, c_2 = \frac{1}{2}, c_3 = \frac{1}{4}, c_4 = 0$ and $c_5 = -\frac{1}{2}$ [16, 49].

Chapter 4

Phenomenology

Contents

A significant task is to verify the new theory on the viability and the consistency with observations. In order to do so, we make use of the parameterized post-Newtonian (PPN) formalism, the Newman–Penrose (NP) formalism and the dynamical systems approach as a tool. While the PPN formalism is used to test theories in the solar system, the NP formalism provides testing of these theories on GW polarizations. The dynamical systems approach is a tool used in cosmology to obtain a qualitative assessment of the behavior of solutions in a model, without delving into the almost impossible task to find the analytic form of the solutions.

4.1 Solar System Tests

We briefly review in this section the PPN formalism and our results (see the attached paper 5), which we received by applying this formalism to the to the generic $\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ theories (see Sec. 2.3). An important ingredient of the PPN formalism is the assumption that the matter which acts as the source of the gravitational field is given by a perfect fluid, whose velocity in a particular, fixed frame of reference is small, measured in units of the speed of light, and that all physical quantities relevant for the solution of the gravitational field equations can be expanded in orders of this velocity.

We choose to work in the Weitzenböck gauge $\omega^A{}_{B\mu} \equiv 0$. The tetrad is expanded as

$$\theta^A{}_\mu = \Delta^A{}_\mu + \tau^A{}_\mu = \Delta^A{}_\mu + \overset{1}{\tau}{}^A{}_\mu + \overset{2}{\tau}{}^A{}_\mu + \overset{3}{\tau}{}^A{}_\mu + \overset{4}{\tau}{}^A{}_\mu + \mathcal{O}(5)\,. \quad (4.1.1)$$

where

$$\Delta^A{}_\mu = \mathrm{diag}(1,1,1,1)\,, \quad \Delta_A{}^\mu = \mathrm{diag}(1,1,1,1) \quad (4.1.2)$$

is the diagonal background tetrad. The Minkowski metric is

$$\eta_{AB} = \mathrm{diag}(-1,1,1,1)\,, \quad \eta_{\mu\nu} = \eta_{AB}\Delta^A{}_\mu\Delta^B{}_\nu = \mathrm{diag}(-1,1,1,1)\,. \quad (4.1.3)$$

We also use the notational conventions

$$\tau^\mu{}_\nu = \Delta_A{}^\mu\tau^A{}_\nu\,, \quad \tau_{A\mu} = \eta_{AB}\tau^B{}_\mu\,, \quad \tau_{\mu\nu} = \eta_{\mu\rho}\tau^\rho{}_\nu = \Delta^A{}_\mu\tau_{A\nu}\,, \quad (4.1.4)$$

i.e., indices of the perturbation τ are transformed with the background tetrad and raised and lowered with the corresponding Minkowski metric. A detailed analysis shows that the only relevant, non-vanishing components of the field variables we need to determine in this article are given by

$$\overset{2}{\tau}{}_{00}\,, \quad \overset{2}{\tau}{}_{ij}\,, \quad \overset{3}{\tau}{}_{0i}\,, \quad \overset{3}{\tau}{}_{i0}\,, \quad \overset{4}{\tau}{}_{00}\,. \quad (4.1.5)$$

Using the expansion (4.1.1) and the components listed above we can expand all geometric quantities appearing in the field equations up to their relevant velocity orders. This concerns in particular the metric

$$g_{\mu\nu} = \eta_{AB}\theta^A{}_\mu\theta^B{}_\nu\,, \quad (4.1.6)$$

whose background solution follows from the diagonal background tetrad $\Delta^A{}_\mu$ to be a flat Minkowski metric, $\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu}$, and whose perturbation around this background is given by

$$\overset{2}{g}_{00} = 2\overset{2}{\tau}_{00}\,, \quad \overset{2}{g}_{ij} = 2\overset{2}{\tau}_{(ij)}\,, \quad \overset{3}{g}_{0i} = 2\overset{3}{\tau}_{(i0)}\,, \quad \overset{4}{g}_{00} = -(\overset{2}{\tau}_{00})^2 + 2\overset{4}{\tau}_{00}\,. \quad (4.1.7)$$

Since we choose to work in the Weitzenböck gauge the torsion tensor takes the form

$$T^\rho{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = 2e_A{}^\rho\partial_{[\mu}\theta^A{}_{\nu]}\,. \tag{4.1.8}$$

Using the tetrad expansion, we can expand the torsion tensor as well

$$\overset{2}{T}{}^0{}_{0i} = \overset{2}{\tau}_{00,i}\,, \quad \overset{2}{T}{}^i{}_{jk} = 2\delta^{il}\overset{2}{\tau}_{l[k,j]}\,, \quad \overset{3}{T}{}^i{}_{0j} = \delta^{ik}(\overset{2}{\tau}_{kj,0} - \overset{3}{\tau}_{k0,j})\,,$$
$$\overset{3}{T}{}^0{}_{ij} = 2\overset{3}{\tau}_{0[i,j]}\,, \quad \overset{4}{T}{}^0{}_{0i} = \overset{2}{\tau}_{00}\overset{2}{\tau}_{00,i} - \overset{3}{\tau}_{0i,0} + \overset{4}{\tau}_{00,i}\,. \tag{4.1.9}$$

For the energy-momentum tensor we use the standard perfect fluid form

$$\Theta_{00} = \rho\left(1 - \overset{2}{\tau}_{00} + v^2 + \Pi\right) + \mathcal{O}(6)\,, \tag{4.1.10a}$$
$$\Theta_{0i} = -\rho v_i + \mathcal{O}(5)\,, \tag{4.1.10b}$$
$$\Theta_{ij} = \rho v_i v_j + p\delta_{ij} + \mathcal{O}(6)\,. \tag{4.1.10c}$$

Finally, in order to expand also the gravitational side of the field equations (2.3.4), we need to introduce a suitable expansion for the free function $\mathcal{F}$ and its derivatives. For this purpose we use a Taylor expansion of the form

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = \mathcal{F}(0,0,0) + \sum_{i=1}^{3}\mathcal{F}_{,i}(0,0,0)\mathcal{T}_i + \mathcal{O}(\mathcal{T}^2)\,. \tag{4.1.11}$$

Higher orders beyond the linear approximation will not be required. We further introduce the notation $F = \mathcal{F}(0,0,0)$ and $F_{,i} = \mathcal{F}_{,i}(0,0,0)$ for the constant Taylor coefficients.

Results and discussion

We find the PPN parameters for the theory as

$$\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0\,, \tag{4.1.12}$$

from which we deduce that there is no violation of the conservation of total energy-momentum, as well as no preferred frame or preferred location effects; theories of this type are called fully conservative. The only non-trivial result is given by the PPN parameters

$$\beta = \frac{6F_{,1} + 3F_{,2} + 7F_{,3}}{4(2F_{,1} + F_{,2} + 2F_{,3})}\,, \qquad \gamma = \frac{F_{,3}}{2F_{,1} + F_{,2} + 2F_{,3}}\,. \tag{4.1.13}$$

More expressively, we find that their deviation from the general relativity values $\beta_{\text{GR}} = \gamma_{\text{GR}} = 1$ can be written in terms of a single constant ϵ by defining

$$\beta - 1 = -\frac{\epsilon}{2}\,, \qquad \gamma - 1 = -2\epsilon\,, \qquad \epsilon = \frac{2F_{,1} + F_{,2} + F_{,3}}{2(2F_{,1} + F_{,2} + 2F_{,3})}\,. \tag{4.1.14}$$

In particular, we obtain $\beta = \gamma = 1$ for $2F_{,1} + F_{,2} + F_{,3} = 0$, so that theories satisfying these conditions are indistinguishable from general relativity by measurements of the PPN parameters.

Here the specific examples:

(i) The deviation (4.1.14) of the PPN parameters in NGR is given by

$$\epsilon = \frac{2t_1 + t_2 + t_3}{2(2t_1 + t_2 + 2t_3)}\,. \tag{4.1.15}$$

This result agrees with the values obtained for β and γ in the original presentation [43] of the theory.

(ii) In any $f(T)$ type theories we find that the deviation (4.1.14) of the PPN parameters from their general relativity values vanishes identically, $\epsilon = 0$, hence cannot be distinguished from GR by their PPN parameters.

Comparison to observations

For the discussion of experimental bounds it is important to take into account that the deviations (4.1.14) of the PPN parameters from their general relativity values are not independent. This fact is relevant for most measurements of the PPN parameters, where the result depends on a linear combination of the parameters, such as the perihelion shift of Mercury or the Nordtvedt effect [28]. The latter is in particular remarkable, since from the values (4.1.13) follows $4\beta - \gamma = 3$, so that the Nordtvedt parameter [50, 51]

$$\eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2 \tag{4.1.16}$$

vanishes identically, indicating the absence of the Nordtvedt effect independently of the theory under consideration. Hence, lunar laser ranging experiments searching for the Nordtvedt effect will not be affected, and are thus insensitive to the modifications we discuss here.

For measurements of the PPN parameter γ alone, the most stringent bound is obtained from the Cassini tracking experiment [52], which yields the bound

$$\gamma - 1 = -2\epsilon \leq (2.1 \pm 2.3) \cdot 10^{-5}\,. \tag{4.1.17}$$

Comparable bounds on ϵ may be obtained from solar system ephemeris, which yields bounds on both γ and β [53].

4.2 Gravitational Waves

Recall that we consider minimal coupling between gravity and matter, i.e., coupling only through the metric seen as function of the tetrad, but not through the flat spin connection. This is the usual coupling prescription for non-spinning matter, which we will henceforth assume. It follows from this choice of the matter coupling that test particles follow the geodesics of the metric, and hence the autoparallel curves of its Levi-Civita connection. The effect of a gravitational wave on an ensemble of test particles, or any other type of gravitational wave detector, such as the mirrors of an interferometer, is therefore described by the corresponding geodesic deviation equation.

$$a_i = -\overset{\circ}{R}_{0i0j} x^j \,, \tag{4.2.1}$$

where $\overset{\circ}{R}_{0i0j} x^j$ are the six so-called electric components of the Riemann tensor, x^j are the spatial coordinates.

The NP formalism makes use of a set of a particular complex double null basis of the tangent space. The basis vectors are denoted by

$$\begin{aligned} l^\mu = (1,0,0,1)\,, \quad n^\mu = \frac{1}{2}(1,0,0,-1)\,, \\ m^\mu = \frac{1}{\sqrt{2}}(0,1,i,0)\,, \quad \bar{m}^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0)\,. \end{aligned} \tag{4.2.2}$$

in terms of the canonical basis vectors of the Cartesian coordinate system they can be defined as

$$l = \partial_0 + \partial_3, \quad n = \frac{1}{2}(\partial_0 - \partial_3), \quad m = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \quad \bar{m} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2)\,. \tag{4.2.3}$$

In the new basis the Minkowski metric takes the form

$$\eta_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} l^\mu & n^\mu & m^\mu & \bar{m}^\mu \end{array} \\ \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} l^\mu \\ n^\mu \\ m^\mu \\ \bar{m}^\mu \end{array} \,. \tag{4.2.4}$$

From the above, we can see that the only nonvanishing inner products of null vectors are $-l^\mu n_\mu = m^\mu \bar{m}_\mu = 1$ and all other naturally vanish.

Using the null basis Newman and Penrose introduced the coefficients, that encodes the Weyl tensor, trace-free Ricci tensor and Ricci scalar, to describe the gravitational radiation field. In the same way, the coefficients can be determined by the Riemann tensor.

We now consider a plane wave propagating in the positive x^3 direction, which corresponds to a single Fourier mode. The wave covector then takes the form $k_\mu = -\omega l_\mu$ and the metric perturbations can be written as

$$h_{\mu\nu} = H_{\mu\nu} e^{i\omega u}\,, \tag{4.2.5}$$

where we introduced the retarded time $u = x^0 - x^3$ and the wave amplitudes are denoted $H_{\mu\nu}$.

As shown in [54], the Riemann tensor of this plane wave is determined completely by the six electric components

$$\Psi_2 = -\frac{1}{6}\mathring{R}_{nlnl} = \frac{1}{12}\ddot{h}_{ll}\,, \quad \Psi_3 = -\frac{1}{2}\mathring{R}_{nln\bar{m}} = -\frac{1}{2}\overline{\mathring{R}_{nlnm}} = \frac{1}{4}\ddot{h}_{l\bar{m}} = \frac{1}{4}\overline{\ddot{h}_{lm}}\,,$$
$$\Psi_4 = -\mathring{R}_{n\bar{m}n\bar{m}} = -\overline{\mathring{R}_{nmnm}} = \frac{1}{2}\ddot{h}_{\bar{m}\bar{m}} = \frac{1}{2}\overline{\ddot{h}_{mm}}\,, \quad \Phi_{22} = -\mathring{R}_{nmn\bar{m}} = \frac{1}{2}\ddot{h}_{m\bar{m}}\,, \tag{4.2.6}$$

where, for example, $\mathring{R}_{nlnl} = \mathring{R}_{\mu\nu\rho\sigma} n^\mu l^\nu n^\rho l^\sigma$ is the electric component of the Riemann tensor in the null basis and dots denote derivatives with respect to u.

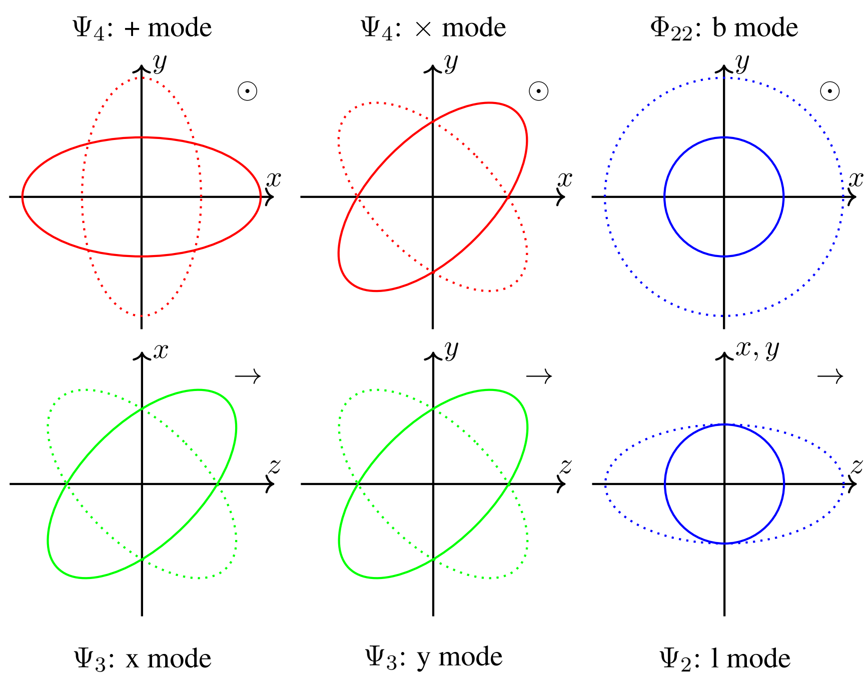


FIG. 2. Polarizations in the Newman-Penrose formalism

Results and discussion

In the attached papers we studied gravitational waves in the most general class of teleparallel gravity theories whose action is quadratic in the torsion tensor, known as new general relativity 6 and in the symmetric teleparallel gravity theories 7. Here we provide a short review of the results we received from the analysis of the possible gravitational wave polarizations in these theories.

E(2) classification:

- II_6 : all 6 modes are allowed.
- III_5 : there is no longitudinal mode $\Psi_2 \equiv 0$, all other modes are allowed.
- N_3: $\Psi_2 \equiv \Psi_3 \equiv 0$, tensor and breathing modes are allowed.
- N_2: $\Psi_2 \equiv \Psi_3 \equiv \Phi_{22} \equiv 0$, only tensor modes are allowed.
- O_1: $\Psi_2 \equiv \Psi_3 \equiv \Psi_4 \equiv 0$, $\Phi_{22} \not\equiv 0$, only breathing mode is allowed.
- O_0: $\Psi_2 \equiv \Psi_3 \equiv \Psi_4 \equiv \Phi_{22} \equiv 0$. No wave.

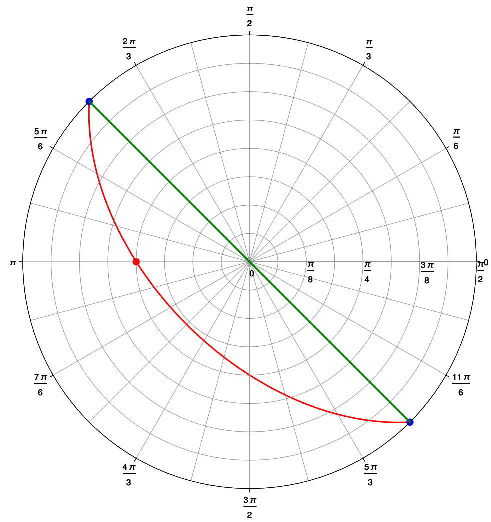
GW polarizations in symmetric teleparallel gravity theories

Inserting the wave ansatz (4.2.5) and writing the gravitational field strength tensor $E_{\mu\nu}$ in the Newman-Penrose basis, we find that the five component equations $0 = E_{ll} = E_{lm} = E_{l\bar{m}} = E_{mm} = E_{\bar{m}\bar{m}}$ are satisfied identically, while the remaining five component equations take the form

$$0 = E_{nn} = 2c_5\ddot{h}_{m\bar{m}} - 2(c_2 + c_4 + c_5)\ddot{h}_{ln}\,, \tag{4.2.7a}$$
$$0 = E_{nm} = -(c_2 + c_4)\ddot{h}_{lm}\,, \tag{4.2.7b}$$
$$0 = E_{n\bar{m}} = -(c_2 + c_4)\ddot{h}_{l\bar{m}}\,, \tag{4.2.7c}$$
$$0 = E_{m\bar{m}} = c_5\ddot{h}_{ll}\,, \tag{4.2.7d}$$
$$0 = E_{ln} = -(c_2 + c_4)\ddot{h}_{ll}\,. \tag{4.2.7e}$$

One can see that the parameters c_1, c_3 have no influence on the allowed polarizations, while depending on the parameters c_2, c_4, c_5 we obtain the E_2 class II_6, III_5, N_3 or N_2, with N_3 filling most of the parameter space.

$$c_2 = \sin\theta\cos\phi$$
$$c_4 = \sin\theta\sin\phi$$
$$c_5 = \cos\theta$$



- $\mathsf{N}_2 : c_2 + c_4 + c_5 = 0,\, c_5 \neq 0$
- N_3: $c_2 + c_4 \neq 0,\, c_2 + c_4 + c_5 \neq 0$
- $\mathsf{III}_5 : c_2 + c_4 = 0,\, c_5 \neq 0$
- $\mathsf{II}_6 : c_2 + c_4 = c_5 = 0$

FIG. 3. Visualization of the parameter space using polar coordinates.

We have also seen that there exists a four parameter family of theories besides STEGR which is of class N_2 and thus exhibits the same two tensor modes as in general relativity. Theories in this class therefore cannot be distinguished from general relativity by observing the polarizations of gravitational waves alone.

GW polarizations in the new general relativity

Note that the metric perturbation components $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + 2\varepsilon s_{\mu\nu}\,. \tag{4.2.8}$$

depend only on the symmetric perturbation of the tetrad, so that these are the only components whose presence or absence we must determine (see the attached paper 6). Inserting the wave ansatz (4.2.5) and writing the gravitational Euler-Lagrange tensor $E_{\mu\nu}$ in the Newman-Penrose basis, we find that the eight component equations

$$E_{ll} = E_{lm} = E_{ml} = E_{nl} = E_{mm} = E_{\bar{m}\bar{m}} = E_{l\bar{m}} = E_{\bar{m}l} = 0 \tag{4.2.9}$$

are satisfied identically, while the remaining eight component equations take the form

$$
\begin{aligned}
0 = E_{nn} &= (2c_1 + c_2 + c_3)\ddot{s}_{nl} + 2c_3\ddot{s}_{m\bar{m}} + (2c_1 + c_2 + c_3)\ddot{a}_{nl}\,, && \text{(4.2.10a)}\\
0 = E_{mn} &= (2c_1 + c_2)\ddot{s}_{ml} + (2c_1 - c_2)\ddot{a}_{ml}\,, && \text{(4.2.10b)}\\
0 = E_{\bar{m}n} &= (2c_1 + c_2)\ddot{s}_{\bar{m}l} + (2c_1 - c_2)\ddot{a}_{\bar{m}l}\,, && \text{(4.2.10c)}\\
0 = E_{nm} &= -c_3\ddot{s}_{ml} + (2c_2 + c_3)\ddot{a}_{ml}\,, && \text{(4.2.10d)}\\
0 = E_{n\bar{m}} &= -c_3\ddot{s}_{\bar{m}l} + (2c_2 + c_3)\ddot{a}_{\bar{m}l}\,, && \text{(4.2.10e)}\\
0 = E_{m\bar{m}} &= E_{\bar{m}m} = -c_3\ddot{s}_{ll}\,, && \text{(4.2.10f)}\\
0 = E_{ln} &= (2c_1 + c_2)\ddot{s}_{ll}\,. && \text{(4.2.10g)}
\end{aligned}
$$

where $a_{\mu\nu}$ is the antisymmetric part of the tetrad perturbation, which does not contribute to the geodesic deviation equation, and so we do not discuss it.
We have seen that depending on the parameters c_1, c_2, c_3 we obtain the E_2 class II_6, III_5, N_3 or N_2, with N_3 filling most of the parameter space.

$$
\begin{aligned}
c_1 &= \sin\theta\cos\phi\\
c_2 &= \sin\theta\sin\phi\\
c_3 &= \cos\theta
\end{aligned}
$$

N_2 : $2c_1 + c_2 + c_3 = 0,\ c_3 \neq 0$

N_3: $2c_1(c_2 + c_3) + c_2^2 \neq 0,$
$2c_1 + c_2 + c_3 \neq 0$

III_5 : $2c_1(c_2 + c_3) + c_2^2 = 0,$
$2c_1 + c_2 + c_3 \neq 0$

II_6: $2c_1 + c_2 = c_3 = 0$

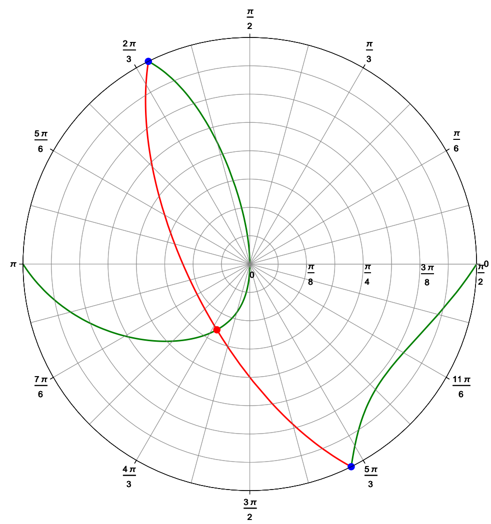


FIG. 4. Visualization of the parameter space using polar coordinates.

We have also seen that there exists a family of theories besides TEGR which is of class N_2 and thus exhibits the same two tensor modes as in general relativity. Theories in this class therefore cannot be distinguished from general relativity by observing the polarizations of gravitational waves alone.

4.3 Dynamical Systems Analysis

The method of dynamical systems is a widely used set of tools in cosmology to obtain a qualitative assessment of the behavior of solutions in a model, without delving into the often almost impossible task to find the analytic form of the solutions [55].

In the attached paper 9 "Dynamical systems approach and generic properties of $f(T)$ cosmology" we have derived a two-dimensional dynamical system from the flat FLRW cosmological field equations of a generic $f(T)$ gravity theory, where the matter content is given by a combination of dust and radiation. We have shown that the full cosmological dynamics of this model depend only on a single function $W(H) = f + 12H^2 f_T$ of the Hubble parameter H, which is derived from the function $f(T)$ defining the particular theory under consideration. Instead of choosing a particular form of $f(T)$, we have kept the function fully generic and derived a number of physically relevant properties of the whole family of $f(T)$ theories.

Our main result is comprised of numerous conditions on the Friedmann function $W(H)$, which determine the existence and stability of fixed points in the cosmological dynamics, the possibility of a bounce or turnaround, the existence and severity of finite time singularities, the existence of accelerating and decelerating phases of the expansion of the universe and transitions between them as well as the possibility of crossing the phantom divide. As a fully generic result, we have shown that there exist no periodic orbits in the phase space, and no oscillating universe solutions, independent of the choice of the function $f(T)$. Further, we have shown how points on the phase space and the shape of the Friedmann function $W(H)$ at these points can be related to observational cosmological parameters. Note that our chosen matter content manifestly satisfies all energy conditions, and that all features we discussed are direct consequences of the modified gravitational dynamics.

To illustrate our results and the general formalism, we have applied it to a generic power law model $f(T) = T + \alpha(-T)^n$. We have shown how the dynamics on the physical phase space depend on the constant parameters α and n of the model and displayed the phase diagrams for all qualitatively different values of these parameters. We have further characterized all possible trajectories in these phase spaces and their acceleration and effective dark energy. In particular, we have shown that it is not possible to dynamically cross the phantom divide $w_{DE} = -1$ in these models. We have finally shown that there are no trajectories that start from an initial accelerating period (which would be interpreted as inflation), become decelerating, and finally transition back to an accelerating de Sitter phase.

Summary

In the attached paper 8 “Covariant formulation of scalar-torsion gravity” we have presented a new class of theories in the covariant teleparallel framework, where the gravitational action depends on an arbitrary function of the torsion scalar and a scalar field, $f(T, \phi)$. This generic setup subsumes and generalizes a number of previously considered models, like $f(T)$ gravity and a scalar field nonminimally coupled to T, putting them in unified scheme so that they can be studied together. We derived the field equations for the tetrads, scalar field, and flat spin connection. The latter is especially important and until recently was missing in the covariant teleparallel picture. The spin connection equation turns out to be related to the antisymmetric part of the tetrad field equations and makes it to vanish identically. One also needs the spin connection equation when combining the field equations in order to show the matter energy-momentum conservation. As a matter of fact, the spin connection field equation contains only first order derivatives with respect to the spacetime coordinates, and provides a consistency condition that from the tetrad ansatz determines the six nontrivial spin connection components (remaining after imposing zero curvature). These six components can be interpreted as gauge degrees of freedom, since they can be absorbed into the tetrad by a suitable local Lorentz transformation.

In the paper 5 “Parametrized post-Newtonian limit of general teleparallel gravity theories” we derived the post-Newtonian limit of a general class of teleparallel gravity theories, whose action is given by a Lagrange function depending on three scalar quantities formed from the parity-even contractions of the torsion tensor [32, 33]. We found that the post-Newtonian limit of these theories is fully determined by a single constant, which is calculated from four Taylor coefficients of the Lagrange function at the zeroth and first order. Our results show that the class of theories we considered is fully conservative in the sense that it does not exhibit any preferred frame or preferred location effects, or violation of energy-momentum conservation, which is reflected by the fact that only the PPN parameters γ and β potentially deviate from their general relativity values. Further, due to the aforementioned fact that deviations of the PPN parameters from their general relativity values are governed by a single combination of the constant Taylor coefficients, large parts of the parameter space of possible theories are left with a post-Newtonian limit which is identical to that of general relativity, so that these theories are indistinguishable by solar system experiments at the respective post-Newtonian order. Further, we found that the Nordvedt effect is absent in the whole class of theories we considered. We then applied our findings to two particular subclasses of theories: new general relativity [43] and $f(T)$ gravity [56, 57]. In the former case the aforementioned Taylor coefficients are given by the three constant parameters which determine the new general relativity action, and our findings agree with the original calculation of γ and β from a static, spherically symmetric ansatz [43]. In the latter case we find that the post-Newtonian parameters are identical to those of general relativity, so that any $f(T)$ gravity theory is consistent with solar system observations.

In the papers 6 and 7 we studied the propagation of gravitational waves in the most general class of teleparallel gravity theories whose action is quadratic in the torsion tensor, known as new general relativity and the most general class of symmetric teleparallel gravity theories whose action is quadratic in the nonmetricity tensor. We made use of the Newman-Penrose formalism to derive the possible polarizations of gravitational waves. Our results show that the two tensor polarizations, which are present also in general relativity, are allowed for the whole class of theories we considered, while additional modes - two vector modes and up to two scalar modes - may be present for particular models within this class. We found that the TEGR and STEGR are not the unique theory exhibiting exactly two polarizations, but there is a one in NGR and four in STG parameter family of theories with the same property. It thus follows that observations of gravitational wave polarizations may only give partial results on the parameter space of these theories.

In the paper 9 "Dynamical systems approach and generic properties of $f(T)$ cosmology" we used the dynamical systems approach to determine the existence and stability of fixed points in the cosmological dynamics, the possibility of a bounce or turnaround, the existence and severity of finite time singularities, the existence of accelerating and decelerating phases of the expansion of the universe and transitions between them as well as the possibility of crossing the phantom divide in generalized $f(T)$ gravity theory. Depending on the model parameters, it is possible to have a bounce (from contraction to expansion) or a turnaround (from expansion to contraction), but cyclic or oscillating scenarios are prohibited. As an illustration of the formalism we consider power law $f(T) = T + \alpha(-T)^n$ models, and show that these allow only one period of acceleration and no phantom divide crossing. The formalism and generic results derived in this article can now be applied to any particular $f(T)$ gravity theory or class of such theories, in order to get a systematic overview of its cosmological behavior.

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Kokkuvõte (in Estonian)

Väändel baseeruvad gravitatsiooniteooriad: teoreetilised ja vaatluslikud piirangud

Lisatud artiklis 8 "Covariant formulation of scalar-torsion gravity" (Skalaar-väände gravitatsiooni kovariantne kirjeldus) me esitlesime kovariantses teleparalleelses raamistikus uut teooriate klassi, milles mõjufunktsionaal sõltub väändeskalaari ja skalaarvälja suvalisest funktsioonist $f(T, \phi)$. See üldine esitus sisaldab ja üldistab mitu varasemalt uuritud mudelit, seehulgas $f(T)$ gravitatsiooni ja väändeskalaariga mitteminimaalselt seotud skalaarvälja, korrastades neid üldises skeemis, ja võimaldades neid ühiselt uurida. Me tuletasime tetraadi, skalaarvälja ja tasase spinniseostuse väljavõrrandeid. Viimasel on eriti oluline roll, mida pandi tähele alles hiljuti kovariantse vaatenurga alt. Spinniseostuse väljavõrrand on seotud tetraadi väljavõrrandi antisümmeetrilise osaga ja lahendab selle identselt. Samuti on spinniseostuse väljavõrrand vajalik kõigi võrrandite kombinatsioonist mateeria energia-impulsi jäävusseaduse tuletamiseks. Õigupoolest sisaldab spinniseostuse väljavõrrand ainult esimest järku tuletisi aegruumi koordinaatide järgi ning esitab kooskõla tingimused, millega tuletatakse tetraadist kuus mittetriviaalset spinniseostuse komponenti (mis on veel alles kui eeldame nulliga võrduvat kõverust). Neid komponente võib interpreteerida kui kalibratsioonivabadusastmeid, kuna neid on võimalik absorbeerida tetraadi sisse sobiva lokaalse Lorentzi teisenduse abil.

Artiklis 5 "Parametrized post-Newtonian limit of general teleparallel gravity theories" (Üldiste teleperalleelsete gravitatsiooniteooriate post-Newtoni piir) tuletasime post-Newtoni piiri üldiste teleperalleelsete gravitatsiooniteooriate klassile, mille mõjufunktsionaali moodustab vaba funktsioon kolmest skalaarsest suurusest, mis koosnevad väändetensori paarsust säilitavatest ahenditest [32, 33]. Me leidsime, et nende teooriate post-Newtoni piiri määrab täielikult vaid üks konstant, mis tuleneb lagranžiaani Taylori rea neljast nullindat ja esimest järku koefitsiendist. Meie tulemused näitavad, et uuritud teooriate klass on täielikult konservatiivne, see tähendab, et ei esine eelistatud raami ega asukoha efekte või energia-impulsi jäävuse rikkumist, mida peegeldab fakt, et ainult PPN parameetrid γ ja β võivad erineda oma üldrelatiivsusteooria väärtustest. Lisaks järeldub sellest, et PPN parameetrite erinevus üldrelatiivsusteooriast sõltub vaid ühest Taylori koefitsientide kombinatsioonist ning suur osa teooriate parameetrite ruumist omab sama post-Newtoni piiri kui üldrelatiivsusteooria, nii et neid teooriaid ei ole võimalik eristada kasutades vastava post-Newtoni täpsusega katseid Päikesesüsteemis. Muuseas leidsime, et Nordtvedti efekt ei esine terves uuritud teooriate klassis. Järgmisena rakendasime oma tulemusi kahele teooriate alamklassile: uus üldrelatiivsusteooria [43] ja $f(T)$ gravitatsioon [56, 57]. Esimesel juhul Taylori koefitsiendid võrduvad kolme konstandiga, mis määravad uue üldrelatiivsusteooria mõjufunktsionaali, ja meie tulemus on võrdne γ ja β varasema arvutusega staatilise ja sfääriliselt sümmeetrilise geomeetria puhul [43]. Teisel juhul me leidsime, et post-Newtoni parameetrid on samad kui üldrelatiivsusteoorias, millest järeldub, et $f(T)$ on kooskõlas vaatlustega Päikesesüsteemis.

Artiklites 6 ja 7 uurisime gravitatsioonilainete levimist kõige üldisemas teleparalleelsete gravitatsiooniteooriate klassis, mille mõjufunktsionaal on teise astme polünoom väändetensorist ja mida nimetatakse uueks üldrelatiivsusteooriaks, ning kõige üldisemas sümmeetriliste teleparalleelsete gravitatsiooniteooriate klassis, mille mõjufunktsionaal on teise astme polünoom mittemeetrilisuse tensorist. Me kasutasime võimalike gravitatsioonilainete polarisatsioonide tuletamiseks Newman-Penrose formalismi. Meie tulemused näitavad, et kaks tensorpolarisatsiooni, mis esinevad ka üldrelatiivsusteoorias, on kooskõlas terve uuritud teooriateklassiga, kuid lisapolarisatsioonid, täpsemalt kaks vektorit ja kuni kaks skalaari, ilmuvad ainult konkreetsetes mudelites selles klassis. Me leidsime, et üldrelatiivsusteooria teleparalleelsed ekvivalendid pole ainsad teooriad, milles on täpselt kaks polarisatsiooni, kuna uue üldrelatiivsusteooria klassis on kahe parameetri teooriate pere ning sümmeetrilises teleparalleelses teoorias nelja parameetri teooriate pere, millel on sama omadus. Seega järeldub, et gravitatsioonilainete polarisatsioonid võimaldavad teha vaid osalisi kitsendusi nende teooriate parameetriruumi kohta.

Artiklis 9 “Dynamical systems approach and generic properties of $f(T)$ cosmology” ($f(T)$ kosmoloogia dünaamiliste süsteemide käsitlus ja üldised omadused) kasutasime dünaamilise süsteemi meetodit selleks, et uurida püsipunktide olemasolu ja stabiilsust kosmoloogilises dünaamikas, paisumise “põrke” ja “ümberpöörde” võimalikkust, lõpliku aja jooksul ilmuva singulaarsuse võimalikkust ja tõsidust, universumi kiireneva ja aeglustuva paisumise faaside olemasolu ja üleminekuid nende vahel ning fantoomipiiri ületamist üldises $f(T)$ gravitatsiooniteoorias. Sõltuvalt mudeli parameetritest on võimalik “põrge” (paisumisest kokkutõmbumisele) või “ümberpööre” (kokkutõmbumisest paisumisele), kuid korduv või võnkuv käitumine on keelatud. Formalismi illustreerimiseks uurisime astmeseaduse $f(T) = T + \alpha(-T)^n$ mudeleid ja näitasime, et neis esineb vaid üks kiirenevalt paisuva universumi ajajärk ja fantoompiiri ei ületata. Selles artiklis arendatud formalismi ja tulemusi on võimalik rakendada igale $f(T)$ gravitatsiooniteooriale või teooriateklassile nende kosmoloogilise käitumise süstemaatiliselt uurimiseks.

Attached publications

Chapter 5

Parametrized post-Newtonian limit of general teleparallel gravity theories

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Parametrized post-Newtonian limit of general teleparallel gravity theories

Ulbossyn Ualikhanova* and Manuel Hohmann†
Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, Wilhelm Ostwaldi 1, 50411 Tartu, Estonia

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We derive the post-Newtonian limit of a general class of teleparallel gravity theories, whose action is given by a free function of three scalar quantities obtained from the torsion of the teleparallel connection. This class of theories is chosen to be sufficiently generic in order to include the $f(T)$ class of theories as well as new general relativity as subclasses. To derive its post-Newtonian limit, we first impose the Weitzenböck gauge, and then introduce a post-Newtonian approximation of the tetrad field around a Minkowski background solution. Our results show that the class of theories we consider is fully conservative, with only the parameters β and γ potentially deviating from their general relativity values. In particular, we find that the post-Newtonian limit of any $f(T)$ theory is identical to that of general relativity, so that these theories cannot be distinguished by measurements of the post-Newtonian parameters alone.

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I. INTRODUCTION

General relativity is challenged both by observations in cosmology and by its theoretical tensions with quantum theory. These challenges have led to the development of a plethora of modified gravity theories. While most of these theories take the most well-known formulation of general relativity in terms of the curvature of a Levi-Civita connection as their starting point, there exist other formulations which may serve as possible starting points for modifications [1]. An important class of such modifications is based on the teleparallel equivalent of general relativity (TEGR) [2], and thus belongs to the class of teleparallel gravity theories [2–5]. The characteristic feature of these theories is to employ a flat, metric-compatible connection, whose torsion mediates the gravitational interaction.

A large class of modified teleparallel gravity theories is obtained by assuming a gravitational Lagrangian of the form $f(T)$ [6,7], where T is the torsion scalar appearing in the TEGR action [2]. Various phenomenological and theoretical aspects of these theories have been investigated, including their cosmological dynamics [8–10] and perturbations [11], gravitational waves [12–15], and degrees of freedom from a Hamiltonian analysis [16–20]. The rich phenomenology and generality of this class of gravity theories hence invite further investigations of the class of a whole, studying further phenomenological aspects.

Another line of studies has been devoted to theories in which the three scalar quantities, which may be obtained from contractions of the torsion tensor, are treated separately. An early contender of this class is given by new general relativity [21], whose Lagrangian is simply the general linear combination of these three terms, and thus can be understood as derived from a general, local, and linear constitutive relation [22,23]. Several aspects of these theories have been studied, such as the equivalence principle [24], gravitational waves [25], and Hamiltonian formulation [26]. Further relaxing the condition of linearity in the three scalar terms leads to an even more general class of teleparallel theories, whose action is given by a free function of three scalar quantities [27,28]. This general class of teleparallel theories, which encompasses both the new relativity class of theories and the wide class of $f(T)$ theories, will be the subject of our studies in this article.

While aiming to model the present observations in cosmology, any viable theory of gravity must of course also comply with observations on smaller scales, such as the Solar System, orbiting pulsars, and laboratory experiments. A commonly used framework which was developed for collectively deriving this local-scale phenomenology is the parametrized post-Newtonian (PPN) formalism [29–31]. It characterizes gravity theories by a set of ten parameters, which have been measured with high precision in various experiments. Because of its generality and the availability of numerous observations, the PPN formalism has become an important tool for assessing the viability of gravity theories.

In order to calculate the post-Newtonian limit of teleparallel theories of gravity, an adaptation of the classical PPN formalism to tetrad-based theories is required.

*ulbossyn.ualikhanova@ut.ee
†manuel.hohmann@ut.ee

A possible adaptation can be derived from a similar approach to the post-Newtonian limit of scalar-tetrad theories [32] by omitting the scalar field part. Further, it needs to be adapted to the covariant formulation of teleparallel gravity [20,33–35], which we will use in this article, and in which also a flat spin connection appears as a dynamical field. The purpose of this article is thus twofold. Our main aim is to put forward a general method for calculating the post-Newtonian limit of teleparallel gravity theories in their covariant formulation, by expanding the tetrad components in a pure spacetime basis and expressing them in terms of the post-Newtonian potentials and a number of constants, which are then determined by solving the field equations. The second aim is to use this general method in order to determine the post-Newtonian limit of a general class of teleparallel gravity theories [27,28]. This class is chosen to be very generic, such as to encompass a large number of theories discussed in the literature, while at the same time being prototypical for applying our formalism to even more general theories.

Our work is in line with a number of previous studies of the post-Newtonian limit of the related class of Poincaré gauge theories. For a more restricted class of teleparallel theories, which is included in the class of theories we study here, it has been shown that post-Newtonian effects only occur at higher perturbation orders than the ones considered in the PPN formalism [36–39]. More general classes of quadratic Poincaré gauge theories, in which both curvature and torsion are present, show deviations already at the PPN level, and may necessitate the use of additional PPN potentials and parameters beyond the standard formalism [40,41]. Note, however, that this will not be the case for the class of teleparallel gravity theories we consider in this article, for which the curvature of the considered connection vanishes identically.

The outline of this article is as follows: In Sec. II, we briefly review the dynamical variables and fields used in the covariant formulation of teleparallel gravity and display the class of theories we consider, together with their action and field equations. In Sec. III, we review the basic ingredients of the post-Newtonian (PPN) formalism, and show how it can be adapted to the field variables relevant for teleparallel gravity. We employ this formalism in order to solve the field equations for a general post-Newtonian matter distribution in Sec. IV. From this solution we obtain the post-Newtonian metric and PPN parameters in Sec. V, where we also compare our result with observations. Finally, in Sec. VI we discuss a number of specific examples. We end with a conclusion in Sec. VII.

In this article, we use uppercase latin letters $A, B, \ldots = 0, \ldots, 3$ for Lorentz indices, lowercase greek letters $\mu, \nu, \ldots = 0, \ldots, 3$ for spacetime indices, and lowercase latin letters $i, j, \ldots = 1, \ldots, 3$ for spatial indices. In our convention, the Minkowski metric η_{AB} and $\eta_{\mu\nu}$ has the signature $(-,+,+,+)$.

II. FIELD VARIABLES AND THEIR DYNAMICS

We start with a brief review of the underlying geometry and dynamics of the theories we consider in this article. The fundamental variables in teleparallel theories of gravity, following their covariant formulation [20,33–35], are a tetrad $\theta^A{}_\mu$ and a curvature-free Lorentz spin connection $\omega^A{}_{B\mu}$. We denote the inverse tetrad by $e_A{}^\mu$, which satisfies $\theta^A{}_\mu e_A{}^\nu = \delta^\nu_\mu$ and $\theta^A{}_\mu e_B{}^\mu = \delta^A_B$. Via these variables, one defines the metric

$$g_{\mu\nu} = \eta_{AB}\theta^A{}_\mu\theta^B{}_\nu \tag{1}$$

and the torsion

$$T^\rho{}_{\mu\nu} = e_A{}^\rho(\partial_\mu\theta^A{}_\nu - \partial_\nu\theta^A{}_\mu + \omega^A{}_{B\mu}\theta^B{}_\nu - \omega^A{}_{B\nu}\theta^B{}_\mu). \tag{2}$$

To give dynamics to these fundamental field variables, we consider an action given by two parts:

$$S[\theta,\omega,\chi] = S_g[\theta,\omega] + S_m[\theta,\chi], \tag{3}$$

where S_g is the gravitational part, S_m is the matter part, and χ denotes an arbitrary set of matter fields. The variation of the matter action S_m with respect to the tetrad $\theta^A{}_\mu$ can be written in the general form

$$\delta_\theta S_m = -\int_M \Theta_A{}^\mu \delta\theta^A{}_\mu \theta d^4x. \tag{4}$$

Here θ is the determinant of the tetrad. Further, $\Theta_A{}^\mu$ denotes the energy-momentum tensor, which we assume to be symmetric, $\Theta_{[\mu\nu]} = 0$, by imposing local Lorentz invariance on the matter action. For the remainder of this article, we will treat the matter source as a perfect fluid, as discussed in detail in Sec. III. Also note that here we have used the tetrad to change the index character, i.e., $\Theta_{\mu\nu} = \theta^A{}_\mu g_{\nu\rho}\Theta_A{}^\rho$.

The gravitational part of the action S_g is defined via the free function $\mathcal{F}$,

$$S_g[\theta,\omega] = \frac{1}{2\kappa^2}\int_M \mathcal{F}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\theta d^4x, \tag{5}$$

which depends on the three scalar quantities, which are parity-even and quadratic in the torsion, and take the forms

$$\mathcal{T}_1 = T^{\mu\nu\rho}T_{\mu\nu\rho}, \qquad \mathcal{T}_2 = T^{\mu\nu\rho}T_{\rho\nu\mu}, \qquad \mathcal{T}_3 = T^\mu{}_{\mu\rho}T_\nu{}^{\nu\rho}. \tag{6}$$

This action defines a generic class of teleparallel gravity theories, which has been discussed before in the literature [27,28] and shall serve both as a generic example and as a starting point for further extensions in future work.

By variation of the total action [Eq. (3)] with respect to the tetrad, we find the field equations

$$\kappa^2\Theta_{\mu\nu} = \frac{1}{2}\mathcal{F}g_{\mu\nu} + 2\overset{\circ}{\nabla}{}^\rho(\mathcal{F}_{,1}T_{\nu\mu\rho} + \mathcal{F}_{,2}T_{[\rho\mu]\nu} + \mathcal{F}_{,3}T^\sigma{}_{\sigma[\rho}g_{\mu]\nu}) + \mathcal{F}_{,1}T^{\rho\sigma}{}_\mu(T_{\nu\rho\sigma} - 2T_{[\rho\sigma]\nu}) + \frac{1}{2}\mathcal{F}_{,2}[T_\mu{}^{\rho\sigma}(2T_{\rho\sigma\nu} - T_{\nu\rho\sigma}) + T^{\rho\sigma}{}_\mu(2T_{[\rho\sigma]\nu} - T_{\nu\rho\sigma})] - \frac{1}{2}\mathcal{F}_{,3}T^\sigma{}_{\sigma\rho}(T^\rho{}_{\mu\nu} + 2T_{(\mu\nu)}{}^\rho), \tag{7}$$

where $\mathcal{F}_{,i} = \partial\mathcal{F}/\partial\mathcal{T}_i$ with $i = 1, 2, 3$, and $\overset{\circ}{\nabla}$ is the covariant derivative with respect to the Levi-Civita connection of the metric $g_{\mu\nu}$. These are the field equations we will be solving in the remainder of this article. For this purpose, we will make use of a post-Newtonian approximation of the teleparallel geometry, which will be detailed in the following section.

III. POST-NEWTONIAN APPROXIMATION

The main tool we use in this article is the parametrized post-Newtonian (PPN) formalism [29–31], which we briefly review in this section, taking into account that we intend to apply it to the class of extended teleparallel theories of gravity detailed in the preceding section. An important ingredient of the PPN formalism is the assumption that the matter which acts as the source of the gravitational field is given by a perfect fluid, whose velocity in a particular, fixed frame of reference is small, measured in units of the speed of light, and that all physical quantities relevant for the solution of the gravitational field equations can be expanded in orders of this velocity. In this section we discuss how this expansion in velocity orders proceeds for the quantities we need in our calculation in the following sections, in particular for the tetrad.

The starting point of our calculation is the energy-momentum tensor of a perfect fluid with rest energy density ρ, specific internal energy Π, pressure p, and four-velocity u^μ, which is given by

$$\Theta^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{8}$$

The four-velocity u^μ is normalized by the metric $g_{\mu\nu}$, so that $u^\mu u^\nu g_{\mu\nu} = -1$. We will now expand all dynamical quantities in orders $\mathcal{O}(n) \propto |\vec{v}|^n$ of the velocity $v^i = u^i/u^0$ of the source matter in a given frame of reference, starting with the field variables. We choose to work in the Weitzenböck gauge, and so we set $\omega^A{}_{B\mu} \equiv 0$. For the tetrad $\theta^A{}_\mu$, we assume an expansion around a flat diagonal background tetrad $\Delta^A{}_\mu = \mathrm{diag}(1, 1, 1, 1)$:

$$\theta^A{}_\mu = \Delta^A{}_\mu + \tau^A{}_\mu = \Delta^A{}_\mu + \overset{1}{\tau}{}^A{}_\mu + \overset{2}{\tau}{}^A{}_\mu + \overset{3}{\tau}{}^A{}_\mu + \overset{4}{\tau}{}^A{}_\mu + \mathcal{O}(5). \tag{9}$$

Here we have used overscript numbers to denote velocity orders; i.e., each term $\overset{n}{\tau}{}^A{}_\mu$ is of order $\mathcal{O}(n)$. Velocity orders beyond the fourth order are not considered and will not be relevant for our calculation.

For the tetrad perturbation $\tau^A{}_\mu$, it will turn out to be more convenient to lower the Lorentz index using the Minkowski metric η_{AB} and convert it into a spacetime index using the background tetrad $\Delta^A{}_\mu$, so that we introduce the perturbations

$$\tau_{\mu\nu} = \Delta^A{}_\mu\eta_{AB}\tau^B{}_\nu, \qquad \overset{n}{\tau}{}_{\mu\nu} = \Delta^A{}_\mu\eta_{AB}\overset{n}{\tau}{}^B{}_\nu. \tag{10}$$

A detailed analysis shows that not all components of the tetrad field need to be expanded to the fourth velocity order, while others vanish due to Newtonian energy conservation or time-reversal symmetry. The only relevant nonvanishing components of the field variables we need to determine in this article are given by

$$\overset{2}{\tau}{}_{00}, \quad \overset{2}{\tau}{}_{ij}, \quad \overset{3}{\tau}{}_{0i}, \quad \overset{3}{\tau}{}_{i0}, \quad \overset{4}{\tau}{}_{00}. \tag{11}$$

Using the expansion [Eq. (9)] and the components listed above, we can expand all geometric quantities appearing in the field equations up to their relevant velocity orders. This concerns in particular the metric, whose background solution follows from the diagonal background tetrad $\Delta^A{}_\mu$ to be a flat Minkowski metric, $\overset{0}{g}{}_{\mu\nu} = \eta_{\mu\nu}$, and whose perturbation around this background is given by

$$\overset{2}{g}{}_{00} = 2\overset{2}{\tau}{}_{00}, \quad \overset{2}{g}{}_{ij} = 2\overset{2}{\tau}{}_{(ij)}, \quad \overset{3}{g}{}_{0i} = 2\overset{3}{\tau}{}_{(i0)}, \quad \overset{4}{g}{}_{00} = -(\overset{2}{\tau}{}_{00})^2 + 2\overset{4}{\tau}{}_{00}. \tag{12}$$

For later use, we also write out the relevant torsion components, which take the forms

$$\overset{2}{T}{}^0{}_{0i} = \overset{2}{\tau}{}_{00,i}, \qquad \overset{2}{T}{}^i{}_{jk} = 2\delta^{il}\overset{2}{\tau}{}_{l[k,j]}, \qquad \overset{3}{T}{}^i{}_{0j} = \delta^{ik}(\overset{2}{\tau}{}_{kj,0} - \overset{3}{\tau}{}_{k0,j}), \qquad \overset{3}{T}{}^0{}_{ij} = 2\overset{3}{\tau}{}_{0[i,j]}, \qquad \overset{4}{T}{}^0{}_{0i} = \overset{2}{\tau}{}_{00}\overset{2}{\tau}{}_{00,i} - \overset{3}{\tau}{}_{0i,0} + \overset{4}{\tau}{}_{00,i}, \tag{13}$$

and which will be necessary for the decomposition of the field equations into velocity orders. Here we have made use of the additional assumption that the gravitational field is quasistatic, so that changes are only induced by the motion of the source matter. Time derivatives ∂_0 of the tetrad components are therefore weighted with an additional velocity order $\mathcal{O}(1)$.

Using the expansion [Eq. (12)] of the metric tensor, we can now also expand the energy-momentum tensor [Eq. (8)] into velocity orders. For this purpose, we must assign velocity orders also to the rest mass density, the specific internal energy and the pressure of the perfect fluid. Based on their orders of magnitude in the Solar System, one assigns velocity orders $\mathcal{O}(2)$ to ρ and Π, and $\mathcal{O}(4)$ to p.

The energy-momentum tensor [Eq. (8)] can then be expanded in the form

$$\Theta_{00} = \rho(1 + \Pi + v^2 - 2\overset{2}{\tau}_{00}) + \mathcal{O}(6), \tag{14a}$$

$$\Theta_{0j} = -\rho v_j + \mathcal{O}(5), \tag{14b}$$

$$\Theta_{ij} = \rho v_i v_j + p\delta_{ij} + \mathcal{O}(6). \tag{14c}$$

Finally, in order to expand also the gravitational side of the field equations [Eq. (7)], we need to introduce a suitable expansion for the free function $\mathcal{F}$ and its derivatives. For this purpose, we use a Taylor expansion of the form

$$\mathcal{F}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) = \mathcal{F}(0,0,0) + \sum_{i=1}^{3}\mathcal{F}_{,i}(0,0,0)\mathcal{T}_i + \mathcal{O}(\mathcal{T}^2). \tag{15}$$

Higher orders beyond the linear approximation will not be required. We further introduce the notation $F = \mathcal{F}(0,0,0)$ and $F_{,i} = \mathcal{F}_{,i}(0,0,0)$ for the constant Taylor coefficients. This will be used throughout the following sections.

IV. EXPANSION OF THE FIELD EQUATIONS AND SOLUTION

In order to discuss the post-Newtonian parameters, we need to expand the field equations to the required order in the perturbation and then make use of the post-Newtonian approximation. We will do so in the following sections. Further, we will make use of a generic *Ansatz* for the tetrad perturbations, which consists of post-Newtonian potentials and constant coefficients, which we will also determine here by solving the field equations. We proceed order by order. The zeroth order, which corresponds to the background solution around which we expand, is discussed in Sec. IV A. We then solve for the second order in Sec. IV B, the third order in Sec. IV C, and finally the fourth order in Sec. IV D.

A. Background field equations

We start our discussion with the zeroth order of the field equations [Eq. (7)]. From the expansion in Eq. (14), it follows that at the zeroth velocity order the energy-momentum tensor vanishes, $\overset{0}{\Theta}_{\mu\nu} = 0$, so that we are left with solving the vacuum field equations. Inserting our assumed background values $\overset{0}{\theta}{}^A{}_\mu = \Delta^A{}_\mu$ for the tetrad into the respective field equations [Eq. (7)], we find that they take the form

$$0 = \frac{1}{2}F\eta_{\mu\nu}. \tag{16}$$

It thus follows that the field equations are solved at the zeroth order only for theories which satisfy $F = 0$. This is a consequence of our assumption that the background solution is given by a flat Minkowski metric, which therefore excludes a cosmological constant. We will thus restrict ourselves to theories satisfying this restriction for the remainder of this article. This restriction will not be of importance for any actual phenomenology, since the effects of a nonvanishing cosmological constant in agreement with cosmological observations would be negligible on Solar System scales.

B. Second velocity order

We continue with expanding the gravitational part $E_{\mu\nu}$ of the field equations [Eq. (7)] in the perturbation $\tau_{\mu\nu}$ at the second velocity order. The corresponding components take the form

$$\overset{2}{E}_{00} = -(2F_{,1} + F_{,2} + F_{,3})\overset{2}{\tau}_{00,ii} + 2F_{,3}\overset{2}{\tau}_{i[i,j]j}, \tag{17a}$$

$$\overset{2}{E}_{ij} = 4F_{,1}\overset{2}{\tau}_{j[k,i]k} + 2F_{,2}(\overset{2}{\tau}_{i[k,j]k} + \overset{2}{\tau}_{k[j,i]k}) + F_{,3}[2\overset{2}{\tau}_{k[k,i]j} - \overset{2}{\tau}_{00,ij} + (\overset{2}{\tau}_{00,kk} + 2\overset{2}{\tau}_{k[l,k]l})\delta_{ij}]. \tag{17b}$$

It follows from their index structure that the tetrad components τ_{00}, τ_{ij} should transform as a scalar and a tensor, respectively, under spatial rotations [29,31]. Further using their respective velocity orders and their relation to the source matter, we can write down an *Ansatz* for the tetrad as

$$\overset{2}{\tau}_{00} = a_1 U, \qquad \overset{2}{\tau}_{ij} = a_2 U\delta_{ij} + a_3 U_{ij}. \tag{18}$$

Here a_i (and also the later appearing b_i, c_i) are constant coefficients, which we will determine by solving the field equations and by imposing gauge conditions, while U and U_{ij} are post-Newtonian functionals of the matter variables. These functionals are related to the matter variables by the differential relations

$$\nabla^2\chi = -2U, \quad U_{ij} = \chi_{,ij} + U\delta_{ij}, \quad \nabla^2 U = -4\pi\rho, \tag{19}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the spatial Laplace operator of the flat background metric, and χ is the so-called superpotential, which is auxiliary in the definition of U_{ij} [29]. For the sake of convenience, we will from now on rewrite the field equations making use of the shorthand notation $\overset{n}{\underline{E}}_{\mu\nu} = \overset{n}{E}_{\mu\nu} - \kappa^2\overset{n}{\Theta}_{\mu\nu} = 0$. Then, inserting the appropriate

Ansatz [Eq. (18)] for the tetrad into the field equations [Eq. (17)] at the second velocity order, and using the relations in Eq. (19), we obtain

$$\overset{2}{\underline{E}}_{00} = -[\kappa^2 - 4\pi a_1(2F_{,1} + F_{,2} + F_{,3}) + 8\pi(a_2 + a_3)F_{,3}]\rho, \tag{20a}$$

$$\overset{2}{\underline{E}}_{ij} = -[a_1 F_{,3} - (a_2 + a_3)(2F_{,1} + F_{,2} + 2F_{,3})] \times (4\pi\delta_{ij}\rho + U_{,ij}), \tag{20b}$$

where we can see that the terms contained in square brackets in front of the post-Newtonian functionals must be zero, in order for the equations to be solved for arbitrary matter distributions. Further, note that we obtain only two independent equations, while our *Ansatz* [Eq. (18)] contains three free constants. This is a consequence of the gauge freedom, which is related to the diffeomorphism invariance of the theory. We thus may choose a gauge by supplementing the system with one additional equation. The standard PPN gauge mandates that the coefficient in front of U_{ij} vanishes, and so we make the gauge choice $a_3 = 0$. Thus, we get for the coefficients

$$a_1 = \frac{2F_{,1} + F_{,2} + 2F_{,3}}{(2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})}\frac{\kappa^2}{4\pi},$$
$$a_2 = \frac{F_{,3}}{(2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})}\frac{\kappa^2}{4\pi}, \qquad a_3 = 0. \tag{21}$$

We will subsequently use this second-order solution in the remaining higher-order field equations.

C. Third velocity order

At the third velocity order in the perturbation expansion, we still work with linearized field equations, which are of the forms

$$\overset{3}{E}_{0i} = 2F_{,1}(\overset{2}{\tau}_{ij,0j} - \overset{3}{\tau}_{i0,jj}) + F_{,2}(\overset{2}{\tau}_{ji,0j} - \overset{3}{\tau}_{j0,ij} + 2\overset{3}{\tau}_{0[j,i]j}) + F_{,3}(\overset{2}{\tau}_{jj,0i} - \overset{3}{\tau}_{j0,ij}), \tag{22a}$$

$$\overset{3}{E}_{i0} = 2F_{,1}(2\overset{3}{\tau}_{0[j,i]j} - \overset{2}{\tau}_{00,0i}) + F_{,2}(2\overset{3}{\tau}_{[j|0|,i]j} + 2\overset{2}{\tau}_{[ij],0j} - \overset{2}{\tau}_{00,0i}) + F_{,3}(2\overset{2}{\tau}_{j[j,|0|i]} - \overset{2}{\tau}_{00,0i}). \tag{22b}$$

Observe that the components τ_{0i}, τ_{i0} must behave as vectors under spatial rotations, which are of the third velocity order, and so they can be expressed in terms of PPN potentials in the forms

$$\overset{3}{\tau}_{i0} = b_1 V_i + b_2 W_i, \qquad \overset{3}{\tau}_{0i} = b_3 V_i + b_4 W_i, \tag{23}$$

with the PPN vector potentials satisfying

$$\nabla^2 V_i = -4\pi\rho v_i, \qquad \nabla^2 W_i = -4\pi\rho v_i + 2U_{,0i}. \tag{24}$$

In this case, proceeding analogously to the equation (20), we obtain the third-order field equations

$$\overset{3}{\underline{E}}_{i0} = [\kappa^2 + 4\pi(b_1 + b_2)F_{,2} + 8\pi(b_3 + b_4)F_{,1}]\left(\rho v_i - \frac{U_{,0i}}{4\pi}\right), \tag{25a}$$

$$\overset{3}{\underline{E}}_{0i} = [\kappa^2 + 8\pi(b_1 + b_2)F_{,1} + 4\pi(b_3 + b_4)F_{,2}]\rho v_i + \bigg[(b_1 - b_2)F_{,3} - b_2(4F_{,1} + F_{,2}) + (b_1 - b_3 - b_4)F_{,2} + \frac{\kappa^2}{4\pi}\frac{F_{,3}}{2F_{,1} + F_{,2}}\bigg]U_{,0i}. \tag{25b}$$

We see that we obtain three independent equations, given by the vanishing of the square brackets, for the four coefficients $b_1, \ldots, b_4$. This is again a consequence of the gauge invariance which we encountered also for the second-order equations (20) and coefficients (18). We could thus fix the gauge also here by adding one more equation. However, we will proceed differently in this case, and leave one of the constant coefficients undetermined at this stage. The reason for this will become clear at the fourth velocity order, where this free constant will allow us to choose the standard PPN gauge by eliminating one more PPN potential. Choosing $b_4 = b_0$ as the undetermined parameter, we find

$$b_1 = -\frac{1}{(2F_{,1} + F_{,2})}\frac{\kappa^2}{4\pi}, \qquad b_2 = 0,$$
$$b_3 = -b_0 - \frac{1}{(2F_{,1} + F_{,2})}\frac{\kappa^2}{4\pi}, \qquad b_4 = b_0. \tag{26}$$

Again, we will make use of this (now only partial) solution in the fourth-order equations, which we address next.

D. Fourth velocity order

Finally, for the fourth order, we find that we need to consider only certain components of the field equations and the linear combinations thereof. In particular, we need the time component

$$\overset{4}{E}_{00} = (2F_{,1} + F_{,2} + F_{,3})\left[-\overset{4}{\tau}_{00,ii} + \overset{3}{\tau}_{0i,0i} + \overset{2}{\tau}_{00}\overset{2}{\tau}_{00,ii} + 2\overset{2}{\tau}_{ij}\overset{2}{\tau}_{00,ij} + \overset{2}{\tau}_{00,i}\left(\overset{2}{\tau}_{(ij),j} - \frac{\overset{2}{\tau}_{00,i}}{2} - \overset{2}{\tau}_{jj,i}\right)\right]$$
$$+ 2F_{,1}\overset{2}{\tau}_{ij,k}\overset{2}{\tau}_{i[k,j]} - F_{,2}\overset{2}{\tau}_{ij,k}(\overset{2}{\tau}_{k[j,i]} + \overset{2}{\tau}_{j[i,k]}) + \frac{F_{,3}}{2}\left(\overset{2}{\tau}_{ij,i}\overset{2}{\tau}_{kj,k} + \overset{2}{\tau}_{ii,j}\overset{2}{\tau}_{kk,j} + 2\overset{2}{\tau}_{ij,i}\overset{2}{\tau}_{jk,k}\right)$$
$$+ 2F_{,3}\left[\overset{4}{\tau}_{i[i,j]j} + \overset{2}{\tau}_{ij,k}\overset{2}{\tau}_{j[k,i]} + 2\overset{2}{\tau}_{00}\overset{2}{\tau}_{i[j,i]j} - \overset{2}{\tau}_{ii,j}\overset{2}{\tau}_{(jk),k} + \overset{2}{\tau}_{ij}\left(\overset{2}{\tau}_{j[k,i]k} + \overset{2}{\tau}_{k(i,j)k} - \overset{2}{\tau}_{kk,ij}\right)\right] \tag{27}$$

and the trace of the spatial part of the field equations

$$\overset{4}{E}_{ii} = 2(2F_{,1} + F_{,2} + 2F_{,3})\left(\overset{2}{\tau}_{i[i,j]}\overset{2}{\tau}_{jk,k} - \overset{4}{\tau}_{i[i,j]j}\right) - 2(F_{,1} + F_{,2} + F_{,3})\overset{2}{\tau}_{ij,k}\overset{2}{\tau}_{jk,i} - (2F_{,1} + F_{,2})\overset{2}{\tau}_{ij}\overset{2}{\tau}_{ij,kk}$$
$$+ 2F_{,3}\left[\overset{4}{\tau}_{00,ii} - \overset{3}{\tau}_{0i,0i} - \overset{2}{\tau}_{00,i}\overset{2}{\tau}_{ij,j} + \overset{2}{\tau}_{ii}\overset{2}{\tau}_{jk,jk} - \overset{2}{\tau}_{ij}\overset{2}{\tau}_{jk,ik} + \overset{2}{\tau}_{ji}\overset{2}{\tau}_{ij,kk} - \overset{2}{\tau}_{kk}\overset{2}{\tau}_{ii,jj} + \overset{2}{\tau}_{00,ii}\left(\overset{2}{\tau}_{00} + \overset{2}{\tau}_{jj}\right)\right]$$
$$+ (2F_{,1} + F_{,2} + 3F_{,3})\left[\overset{2}{\tau}_{ii,00} - \overset{3}{\tau}_{i0,i0} + 2\overset{2}{\tau}_{00,i}\overset{2}{\tau}_{j[j,i]} + 2\overset{2}{\tau}_{ij}\left(\overset{2}{\tau}_{kk,ij} - \overset{2}{\tau}_{k(i,j)k}\right)\right] + \frac{1}{2}(2F_{,1} + F_{,2} + F_{,3})\overset{2}{\tau}_{00,i}\overset{2}{\tau}_{00,i}$$
$$+ F_{,1}\left[2\overset{2}{\tau}_{ik}\overset{2}{\tau}_{ij,jk} + 2\overset{2}{\tau}_{kj,i}\overset{2}{\tau}_{ki,j} + \overset{2}{\tau}_{ij,k}\left(\overset{2}{\tau}_{ij,k} - 3\overset{2}{\tau}_{ik,j}\right)\right] + \frac{F_{,2}}{2}\left(\overset{2}{\tau}_{ij,k}\overset{2}{\tau}_{kj,i} + 2\overset{2}{\tau}_{ij}\overset{2}{\tau}_{ik,jk}\right) - 3F_{,3}\overset{2}{\tau}_{(ij)}\overset{2}{\tau}_{00,ij}$$
$$+ \left(2F_{,1} + F_{,2} + \frac{3}{2}c_3\right)\left[\overset{2}{\tau}_{ii,j}\left(2\overset{2}{\tau}_{kj,k} - \overset{2}{\tau}_{kk,j}\right) - \overset{2}{\tau}_{ij,i}\overset{2}{\tau}_{kj,k}\right] + \left(2F_{,1} + \frac{3}{2}F_{,2} + 2F_{,3}\right)\overset{2}{\tau}_{ij,k}\overset{2}{\tau}_{ji,k}. \tag{28}$$

In order to determine the post-Newtonian metric, we need to solve these equations for the tetrad component $\overset{4}{\tau}_{00}$. Note that this component should transform as a scalar under rotations, and thus we can consider an *Ansatz* of the form

$$\overset{4}{\tau}_{00} = c_1\Phi_1 + c_2\Phi_2 + c_3\Phi_3 + c_4\Phi_4 + c_5U^2 \tag{29}$$

with the fourth-order scalar potentials

$$\nabla^2\Phi_1 = -4\pi\rho v^2, \qquad \nabla^2\Phi_2 = -4\pi\rho U, \qquad \nabla^2\Phi_3 = -4\pi\rho\Pi, \qquad \nabla^2\Phi_4 = -4\pi p. \tag{30}$$

Finally, to eliminate the spatial component $\overset{4}{\tau}_{ij}$ of the tetrad, which appears in the field equations (27) and (28), but is not relevant for our calculation, we make use of the linear combination

$$\underline{\overset{4}{E}} = (2F_{,1} + F_{,2} + 2F_{,3})\overset{4}{E}_{00} + F_{,3}\overset{4}{E}_{ii} \tag{31}$$

and find

$$\underline{\overset{4}{E}} = (2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})\{2b_0U_{,00} + 4\pi[c_1\rho v^2 + (c_2 + 2c_5)\rho U + c_3\rho\Pi + c_4p] - 2c_5U_{,i}U_{,i}\}$$
$$+ \frac{\kappa^2}{4\pi}(2F_{,1} + F_{,2} + 2F_{,3})\left(U_{,00} + \frac{\kappa^2\rho U}{2F_{,1} + F_{,2} + 3F_{,3}}\right) - 3F_{,3}\kappa^2p - (2F_{,1} + F_{,2} + 3F_{,3})\kappa^2\rho v^2$$
$$- \kappa^2(2F_{,1} + F_{,2} + 2F_{,3})\left(\rho\Pi + \frac{\kappa^2}{32\pi^2}\frac{U_{,i}U_{,i}}{2F_{,1} + F_{,2}}\right). \tag{32}$$

In order to obtain the solution in the standard PPN gauge, the coefficient in front of the term $U_{,00}$ must vanish, since it does not correspond to any of the terms in the *Ansatz* [Eq. (29)] and would introduce a term violating the standard PPN gauge. Together with the remaining, independent terms, we then find the six independent equations

$$4\pi(2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})c_4 - 3\kappa^3F_{,3} = 0, \tag{33a}$$

$$4\pi(2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})c_3 - \kappa^2(2F_{,1} + F_{,2} + 2F_{,3}) = 0, \tag{33b}$$

$$4\pi(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})(c_2+2c_5)+\frac{\kappa^4}{4\pi}\frac{2F_{,1}+F_{,2}+2F_{,3}}{2F_{,1}+F_{,2}+3F_{,3}}=0, \tag{33c}$$

$$2(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})b_0+\frac{\kappa^2}{4\pi}(2F_{,1}+F_{,2}+2F_{,3})=0, \tag{33d}$$

$$(2F_{,1}+F_{,2}+3F_{,3})[4\pi(2F_{,1}+F_{,2})c_1-\kappa^2]=0, \tag{33e}$$

$$-2(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})c_5-\frac{\kappa^4}{32\pi^2}\frac{2F_{,1}+F_{,2}+2F_{,3}}{2F_{,1}+F_{,2}}=0. \tag{33f}$$

Solving these equations for the remaining six undetermined constants then yields their values:

$$b_0=-\frac{2F_{,1}+F_{,2}+2F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{8\pi},\qquad c_1=\frac{1}{(2F_{,1}+F_{,2})}\frac{\kappa^2}{4\pi},$$
$$c_2=-\frac{(2F_{,1}+F_{,2}-3F_{,3})(2F_{,1}+F_{,2}+2F_{,3})}{(2F_{,1}+F_{,2})^2(2F_{,1}+F_{,2}+3F_{,3})^2}\frac{\kappa^4}{32\pi^2},\qquad c_3=\frac{2F_{,1}+F_{,2}+2F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{4\pi},$$
$$c_4=\frac{3F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{4\pi},\qquad c_5=-\frac{2F_{,1}+F_{,2}+2F_{,3}}{(2F_{,1}+F_{,2})^2(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^4}{64\pi^2}. \tag{34}$$

With this result, we have fully solved the general field equations (7) at all velocity orders which are required to determine the PPN metric and hence the PPN parameters. This will be done in the following section.

V. PPN METRIC AND PARAMETERS

Using the solution obtained in the previous section, we can now finally determine the PPN metric, and hence the PPN parameters of the general class of teleparallel gravity theories we consider in this article. We will do so in three steps. In Sec. VA, we briefly recall the relevant tetrad components and display their solutions after inserting the constant coefficients we determined into the respective *Ansätze*. From these components, we derive the metric components in Sec. V B. Finally, in Sec. V C, we read off the PPN parameters. We compare this result to observations in Sec. V D, in order to obtain bounds on the class of theories we consider.

A. Post-Newtonian tetrad

We start by briefly recalling the tetrad components and displaying their solutions from Sec. IV. From the *Ansatz* [Eq. (18)] together with the solutions in Eq. (21) for the constant coefficients, we find the second-order components

$$\overset{2}{\tau}_{00}=\frac{2F_{,1}+F_{,2}+2F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{4\pi}U,\qquad \overset{2}{\tau}_{ij}=\frac{F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{4\pi}U\delta_{ij}. \tag{35}$$

We then come to the third-order *Ansatz* [Eq. (23)], together with the solution in Eq. (26) and the missing coefficient b_0 in the solution in Eq. (34). This yields the components

$$\overset{3}{\tau}_{i0}=-\frac{1}{2F_{,1}+F_{,2}}\frac{\kappa^2}{4\pi}V_i,\qquad \overset{3}{\tau}_{0i}=-\frac{\kappa^2}{8\pi}\frac{(2F_{,1}+F_{,2}+4F_{,3})V_i+(2F_{,1}+F_{,2}+2F_{,3})W_i}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}. \tag{36}$$

Finally, we recall the *Ansatz* [Eq. (29)] for the only fourth-order component we have to determine. With the solution in Eq. (34), we find

$$\overset{4}{\tau}_{00}=\frac{1}{2F_{,1}+F_{,2}}\frac{\kappa^2}{4\pi}\Phi_1-\frac{(2F_{,1}+F_{,2}-3F_{,3})(2F_{,1}+F_{,2}+2F_{,3})}{(2F_{,1}+F_{,2})^2(2F_{,1}+F_{,2}+3F_{,3})^2}\frac{\kappa^4}{32\pi^2}\Phi_2+\frac{2F_{,1}+F_{,2}+2F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{4\pi}\Phi_3$$
$$+\frac{3F_{,3}}{(2F_{,1}+F_{,2})(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^2}{4\pi}\Phi_4-\frac{2F_{,1}+F_{,2}+2F_{,3}}{(2F_{,1}+F_{,2})^2(2F_{,1}+F_{,2}+3F_{,3})}\frac{\kappa^4}{64\pi^2}U^2. \tag{37}$$

These are all tetrad components which are relevant to construct the post-Newtonian metric.

B. Post-Newtonian metric

In the next step, we calculate the post-Newtonian metric. For this purpose, we insert the tetrad components displayed in Sec. V A into the metric expansion in Eq. (12). We start with the second-order metric component

$$\overset{2}{g}_{00} = \frac{2F_{,1} + F_{,2} + 2F_{,3}}{(2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})} \frac{\kappa^2}{2\pi} U = 2GU, \tag{38}$$

which follows immediately from the second-order tetrad perturbation [Eq. (35)]. Here we have introduced the Newtonian gravitational constant G. Solving the normalization condition $G = 1$, as this is the conventional PPN choice of units and yields the relation

$$\kappa^2 = 4\pi \frac{(2F_{,1} + F_{,2})(2F_{,1} + F_{,2} + 3F_{,3})}{2F_{,1} + F_{,2} + 2F_{,3}}. \tag{39}$$

Using this normalization, we find for the remaining components

$$\overset{2}{g}_{ij} = \frac{2F_{,3}}{2F_{,1} + F_{,2} + 2F_{,3}} U\delta_{ij} \tag{40}$$

at the second order,

$$\overset{3}{g}_{0i} = -\frac{6F_{,1} + 3F_{,2} + 10F_{,3}}{2(2F_{,1} + F_{,2} + 2F_{,3})} V_i - \frac{1}{2} W_i \tag{41}$$

at the third order, and finally

$$\overset{4}{g}_{00} = \frac{1}{2F_{,1} + F_{,2} + 2F_{,3}} \bigg[-\frac{6F_{,1} + 3F_{,2} + 7F_{,3}}{2} U^2 + 2(2F_{,1} + F_{,2} + 3F_{,3})\Phi_1 - (2F_{,1} + F_{,2} - 3F_{,3})\Phi_2 + 2(2F_{,1} + F_{,2} + 2F_{,3})\Phi_3 + 6F_{,3}\Phi_4 \bigg] \tag{42}$$

at the fourth order. Further components will not be necessary in order to obtain the PPN parameters.

C. Post-Newtonian parameters

By comparing the metric components shown in Sec. V B with the standard PPN form of the metric [29,31], we find the PPN parameters for the theory as

$$\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0, \tag{43}$$

from which we deduce that there is no violation of the conservation of total energy-momentum, as well as no preferred frame or preferred location effects; theories of this type are called fully conservative. The only nontrivial result is given by the PPN parameters

$$\beta = \frac{6F_{,1} + 3F_{,2} + 7F_{,3}}{4(2F_{,1} + F_{,2} + 2F_{,3})}, \qquad \gamma = \frac{F_{,3}}{2F_{,1} + F_{,2} + 2F_{,3}}. \tag{44}$$

More expressively, we find that their deviation from the general relativity values $\beta_{\mathrm{GR}} = \gamma_{\mathrm{GR}} = 1$ can be written in terms of a single constant ϵ by defining

$$\beta - 1 = -\frac{\epsilon}{2}, \qquad \gamma - 1 = -2\epsilon, \qquad \epsilon = \frac{2F_{,1} + F_{,2} + F_{,3}}{2(2F_{,1} + F_{,2} + 2F_{,3})}. \tag{45}$$

In particular, we obtain $\beta = \gamma = 1$ for $2F_{,1} + F_{,2} + F_{,3} = 0$, so that theories satisfying these conditions are indistinguishable from general relativity by measurements of the PPN parameters. We will discuss this particular case later in Sec. VI, when we discuss specific examples.

D. Comparison to observations

For the discussion of experimental bounds, it is important to take into account that the deviations [Eq. (45)] of the PPN parameters from their general relativity values are not independent. This fact is relevant for most measurements of the PPN parameters, where the result depends on a linear combination of the parameters, such as the perihelion shift of Mercury or the Nordvedt effect [30]. The latter is in particular remarkable, since from the values in Eq. (44) $4\beta - \gamma = 3$ follows, so that the Nordvedt parameter [42,43]

$$\eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2 \tag{46}$$

vanishes identically, indicating the absence of the Nordvedt effect independently of the theory under consideration. Hence, lunar laser ranging experiments searching for the Nordvedt effect will not be affected, and are thus insensitive to the modifications we discuss here.

For measurements of the PPN parameter γ alone, the most stringent bound is obtained from the Cassini tracking experiment [44], which yields the bound

$$\gamma - 1 = -2\epsilon \leq (2.1 \pm 2.3) \times 10^{-5}. \tag{47}$$

Comparable bounds on ϵ may be obtained from Solar System ephemeris, which yields bounds on both γ and β [45].

This concludes our discussion of the PPN parameters for a general teleparallel theory. To illustrate our results, we will present the most commonly encountered examples in the following section.

VI. EXAMPLES

We now apply the general result we derived in the previous sections to a number of example theories. We start with a simple rewriting of the gravitational Lagrangian in its axial, vector, and tensor parts in Sec. VI A. In Sec. VI B, we then consider new general relativity, in which the general function $\mathcal{F}$ is replaced by a linear function of its three arguments. In Sec. VI C, we finally consider the $f(T)$ class of theories, where f is a function depending on the TEGR torsion scalar only.

A. $\mathcal{G}(T_{\mathrm{ax}}, T_{\mathrm{vec}}, T_{\mathrm{ten}})$ theories

We begin by noting that the theory of gravity given in Ref. [27] is identical to the class of theories we discussed here, since its action is of the same form:

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = \mathcal{G}(T_{\mathrm{ax}}, T_{\mathrm{vec}}, T_{\mathrm{ten}}), \tag{48}$$

with the torsion components

$$T_{\mathrm{ax}} = \frac{1}{18}(\mathcal{T}_1 - 2\mathcal{T}_2), \qquad T_{\mathrm{ten}} = \frac{1}{2}(\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3),$$
$$T_{\mathrm{vec}} = \mathcal{T}_3, \tag{49}$$

which are fully equivalent for expressing the action. It follows that the Taylor coefficients

$$G = \mathcal{G}|_{T=0}, \qquad G_{,a} = \left.\frac{\partial \mathcal{G}}{\partial T_{\mathrm{ax}}}\right|_{T=0},$$
$$G_{,t} = \left.\frac{\partial \mathcal{G}}{\partial T_{\mathrm{ten}}}\right|_{T=0}, \qquad G_{,v} = \left.\frac{\partial \mathcal{G}}{\partial T_{\mathrm{vec}}}\right|_{T=0} \tag{50}$$

are related by

$$F = G, \qquad F_{,1} = \frac{1}{18}G_{,a} + \frac{1}{2}G_{,t},$$
$$F_{,2} = -\frac{1}{9}G_{,a} + \frac{1}{2}G_{,t}, \qquad F_{,3} = G_{,v} - \frac{1}{2}G_{,t}. \tag{51}$$

Note in particular that $G_{,a}$ drops out whenever $F_{,1}$ and $F_{,2}$ appear only in the combination $2F_{,1} + F_{,2}$. Hence, the axial part does not contribute to the deviation [Eq. (45)] of the PPN parameters from their general relativity values, since

$$\epsilon = \frac{G_{,v} + G_{,t}}{4G_{,v} + G_{,t}} \tag{52}$$

contains only vectorial and tensorial parts. This agrees with earlier findings, that purely axial modifications show up only in higher post-Newtonian orders than considered in the PPN formalism [36–39].

B. New general relativity

Next, we consider the new general relativity (NGR) class of teleparallel gravity theories [21]. Its Lagrangian is given by the general linear combination

$$\mathcal{F}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = t_1\mathcal{T}_1 + t_2\mathcal{T}_2 + t_3\mathcal{T}_3 \tag{53}$$

with constant coefficients t_i. It thus follows immediately that the Taylor coefficients are given by $F = 0$ and $F_{,i} = t_i$, $i = 1, 2, 3$. The deviation [Eq. (45)] of the PPN parameters is thus given by

$$\epsilon = \frac{2t_1 + t_2 + t_3}{2(2t_1 + t_2 + 2t_3)}. \tag{54}$$

This result agrees with the values obtained for β and γ in the original presentation [21] of the theory.

C. $f(T)$ theories

Another important class of theories which is covered by the calculations we present in this article is given by the so-called $f(T)$ class of theories, whose Lagrangian is given by

$$\mathcal{F}(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) = f(T), \quad T = \frac{1}{4}\mathcal{T}_1 + \frac{1}{2}\mathcal{T}_2 - \mathcal{T}_3. \tag{55}$$

Here T is the torsion scalar which constitutes the Lagrangian of the teleparallel equivalent of general relativity (TEGR) [2]. For the Taylor coefficients we find $F = f(0)$, so that at the zeroth order we get the condition $F = f(0) = 0$. The remaining Taylor coefficients are given by $F_{,1} = \frac{1}{4}f'(0)$, $F_{,2} = \frac{1}{2}f'(0)$, and $F_{,3} = -f'(0)$. As a consequence, we find that the deviation in Eq. (45) of the PPN parameters from their general relativity values vanishes identically, $\epsilon = 0$, for any theories of this class. Hence, we find that any $f(T)$-type theories cannot be distinguished from general relativity by their PPN parameters.

VII. CONCLUSION

We derived the post-Newtonian limit of a general class of teleparallel gravity theories, whose action is given by a Lagrange function depending on three scalar quantities formed from the parity-even contractions of the torsion tensor [27,28]. We found that the post-Newtonian limit of these theories is fully determined by a single constant, which is calculated from four Taylor coefficients of the Lagrange function at the zeroth and first orders. The zeroth order, which plays the role of a cosmological constant, must be set to zero to achieve consistency between the background (vacuum) field equations and the post-Newtonian *Ansatz* of a flat Minkowski background (or at least sufficiently small such as not to affect the Solar System dynamics). The post-Newtonian parameters are then fully determined by the first-order Taylor coefficients. We displayed these coefficients in two different representations, both through the canonical contractions of the torsion tensor and its axial-vector-tensor decomposition.

Our results show that the class of theories we considered is fully conservative in the sense that it does not exhibit any preferred frame or preferred location effects, or violation of energy-momentum conservation, which is reflected by the fact that only the PPN parameters γ and β potentially deviate from their general relativity values. Further, due to the aforementioned fact that deviations of the PPN parameters from their general relativity values are governed by a single combination of the constant Taylor coefficients, large parts of the parameter space of possible theories are left with a post-Newtonian limit which is identical to that of general relativity, so that these theories are indistinguishable by Solar System experiments at the respective post-Newtonian order. Further, we found that the Nordvedt effect is absent in the whole class of theories we considered.

We then applied our findings to two particular subclasses of theories: new general relativity [21] and $f(T)$ gravity [6,7]. In the former case, the aforementioned Taylor coefficients are given by the three constant parameters which determine the new general relativity action, and our findings agree with the original calculation of γ and β from a static, spherically symmetric *Ansatz* [21]. In the latter case, we find that the post-Newtonian parameters are identical to those of general relativity, so that any $f(T)$ gravity theory is consistent with Solar System observations.

Our work invites numerous generalizations and extensions. In particular, one may consider more general theories, for example, one derived from a general constitutive relation [46], possibly including also parity-odd terms. Another possibility is to include a coupling to scalar fields [47–52], up to Horndeski-like teleparallel theories [53,54]. This would extend previous calculations of the PPN parameters for specific theories in this class [55–57]. Further, taking inspiration from the so-called trinity of gravity [1], one may consider extensions to the symmetric teleparallel equivalent of gravity [58], and apply the parametrized post-Newtonian formalism to generalized theories based on the symmetric teleparallel geometry [59–63]. Another possible extension would be studying the motion of compact objects at higher orders in the post-Newtonian expansion, in order to derive the emitted gravitational waves [64].

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Chapter 6

Propagation of gravitational waves in teleparallel gravity theories

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Contents

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Propagation of gravitational waves in teleparallel gravity theories

Manuel Hohmann,* Martin Krššák,† Christian Pfeifer,‡ and Ulbossyn Ualikhanova§
Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia

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We investigate the propagation of gravitational waves in the most general teleparallel gravity model with second order field equations as perturbations around the Minkowski background. We argue that in this case the most general Lagrangian at the first nonvanishing order of the perturbations is given by a linear combination of quadratic invariants and hence coincides with the well-known new general relativity model. We derive the linearized field equations and analyze them using the principal polynomial and the Newman-Penrose formalism. We demonstrate that all gravitational wave modes propagate at the speed of light, and there are up to six possible polarizations. We show that two tensorial modes of general relativity are always present, and the number of extra polarizations depends on the free parameters of the new general relativity model.

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I. INTRODUCTION

Modified gravity theories are a viable alternative to dark energy in addressing the problem of accelerated expansion of the Universe [1,2]. A novel class of modified gravity models that caught a lot of attention recently is the so-called modified teleparallel theories. These theories are motivated by the fact that the ordinary general relativity (GR) can be reformulated using the teleparallel geometry, resulting in a theory known as the teleparallel equivalent of general relativity (TEGR) or shortly just teleparallel gravity [3–10].

Whereas TEGR is equivalent to the ordinary formulation of GR in terms of curvature in all physical predictions, this equivalence is lost when we consider modified gravity theories based on these different underlying geometries. The most well-known example is the case of $f(T)$ gravity, constructed in analogy with $f(R)$ gravity, where the Lagrangian is taken to be an arbitrary function of the so-called torsion scalar, which defines the TEGR action [11–14]. When a nonlinear function f is considered, the resulting $f(T)$ theory represents a novel gravity model with rich dynamics distinctive from $f(R)$ gravity. See [15] for an extensive overview.

The recent discovery of gravitational waves [16,17] opened a new way to test various modified theories of gravity [18–24]. This motivates a study of gravitational waves in modified gravity theories and proper understanding of their fundamental properties. Particularly interesting are the questions about the number of polarization modes of gravitational waves and their corresponding propagation velocities. The case of $f(R)$ gravity is well-understood, and it has been shown that these theories all possess an additional massive scalar gravitational wave mode [25–28] compared to GR.

In the case of modified teleparallel theories, gravitational waves have been studied first in the case of $f(T)$ gravity [29,30], where, in contrast to the $f(R)$ case, it was shown there are no extra propagating gravitational modes compared to GR. As we will argue later, this follows from a simple observation that $f(T)$ gravity effectively reduces to TEGR at the perturbative level, and hence we obtain only the usual two GR polarizations. Only very recently [31,32], it was shown that new polarization modes appear if we extend $f(T)$ gravity by introducing scalar fields or higher-derivative terms of the torsion in the case of so-called $f(T, B)$ [33] and $f(T, T_G)$ [34] theories, where B is the boundary term relating the Riemannian curvature scalar with the torsion scalar and T_G is the teleparallel equivalent of the Gauss-Bonnet term.

In this paper we follow another approach and study gravitational waves propagating around the Minkowski background in the model known as new general relativity (NGR) [35], where the Lagrangian is taken to be a most general linear combination of quadratic parity preserving torsion invariants.[1] Our study is motivated by a simple observation that, unless we introduce higher derivatives or scalar fields, the most general teleparallel gravity Lagrangian at the perturbative level is given by the linear

*manuel.hohmann@ut.ee
†martin.krssak@ut.ee
‡christian.pfeifer@ut.ee
§ulbossyn.ualikhanova@ut.ee

[1]Note that sometimes "new general relativity" refers only to a special subclass of these theories in which only one of the three parameters we consider here is left free and two are fixed to a specific value.

combination of quadratic invariants of the torsion, i.e., NGR. For example, recently proposed $f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec})$ gravity [36] and the so-called axiomatic electrodynamics inspired models [37], which are both very general frameworks designed to include all previously studied teleparallel models as special instances, reduce to the case of NGR at the perturbative level.

We analyze the gravitational waves in the NGR model using two methods. First, we consider a perturbative analysis of the NGR model following the example of [38,39] and analyze the resulting linearized field equations by the method of the principal symbol. Second, we use the Newman-Penrose formalism [40] and classify the resulting polarizations according to the classification scheme introduced in [41,42]. We show that all gravitational modes propagate at the speed of light and derive how the number of polarization modes depends on the free paramaters of the NGR Lagrangian.

The outline of this paper is as follows. In Sec. II we briefly introduce teleparallel geometry and the NGR model as the most general teleparallel gravity at the perturbative level. In Sec. III we introduce the principal symbol and determine that all gravitational wave modes propagate at the speed of light. In Sec. IV we use the Newman-Penrose formalism to analyze the possible polarizations of gravitational waves and show how they depend on the free parameters of the NGR Lagrangian. We conclude this paper with a brief discussion and outlook in Sec. V.

In this article we use the following notation. Latin letters $a, b, \ldots$ are Lorentz indices, and Greek letters $\mu, \nu, \ldots$ are spacetime coordinate indices. The Minkowski metric is denoted by η and has components $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

II. LINEARIZED TELEPARALLEL GRAVITY

We begin this article with a short review of the required geometric notions in teleparallel gravity in Sec. II A. We then recall the NGR Lagrangian in Sec. II B, where we also derive the corresponding linearized field equations.

A. Teleparallel geometry

The fundamental variables in theories of gravity formulated in terms of teleparallelism are the tetrad 1-forms θ^a, their dual vector fields e_a and the curvature free spin connection $\omega^a{}_b$ generated by local Lorentz transformations $\Lambda^a{}_b$. In local coordinates on spacetime they can be expressed as

$$\theta^a = \theta^a{}_\mu \mathrm{d}x^\mu, \quad e_a = e_a{}^\mu \partial_\mu,$$
$$\omega^a{}_b(\Lambda) = \omega^a{}_{b\mu}(\Lambda)\mathrm{d}x^\mu = \Lambda^a{}_q \mathrm{d}(\Lambda^{-1})^q{}_b = \Lambda^a{}_q \partial_\mu (\Lambda^{-1})^q{}_b \mathrm{d}x^\mu. \tag{1}$$

Moreover the tetrad 1-forms and their duals satisfy

$$\theta^a(e_b) = \theta^a{}_\mu e_b{}^\mu = \delta^a_b, \qquad \theta^a{}_\mu e_a{}^\nu = \delta^\nu_\mu, \tag{2}$$

and define a Lorentzian spacetime metric via

$$g_{\mu\nu} = \eta_{ab}\theta^a{}_\mu \theta^b{}_\nu, \qquad g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu. \tag{3}$$

Tensor fields can be expressed either in a coordinate or tetrad basis. For a (1,1)-tensor Z we may for example write

$$Z = Z^\mu{}_\nu \mathrm{d}x^\nu \otimes \partial_\mu = Z^a{}_b \theta^b \otimes e_a. \tag{4}$$

Thus when we change an index from Latin to Greek, this operation is done via multiplication with $\theta^a{}_\mu$ or $e_a{}^\mu$, respectively.

The building block of Lagrange densities is the torsion of the spin connection given by

$$T^a = \mathrm{D}\theta^a = (\partial_\mu \theta^a{}_\nu + \omega^a{}_{b\mu}\theta^b{}_\nu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{5}$$

where the spin covariant derivative D ensures a covariant transformation behavior under local Lorentz transformations of the tetrad [43,44]. More precisely, consider a tetrad $\hat\theta^a$ which is related to the original tetrad by a local Lorentz transformation $\tilde\Lambda^a{}_b$, i.e., $\hat\theta^a = \tilde\Lambda^a{}_b \theta^b$. Then, the torsion tensor of the tetrads is related by $\hat T^a = \tilde\Lambda^a{}_b T^b$, where the connections are given in terms of two further Lorentz transformations $\hat\Lambda$ and Λ,

$$\hat\omega^a{}_b = \hat\Lambda^a{}_c \mathrm{d}(\hat\Lambda^{-1})^c{}_b,$$
$$\omega^a{}_b = (\tilde\Lambda^{-1})^a{}_c \hat\Lambda^c{}_d \mathrm{d}(\tilde\Lambda^e{}_b (\hat\Lambda^{-1})^d{}_e) = \Lambda^a{}_d \mathrm{d}(\Lambda^{-1})^d{}_b. \tag{6}$$

In particular when one considers $\tilde\Lambda = \hat\Lambda$ one chooses the so-called proper tetrad or a tetrad in the Weitzenbock gauge, for which $\omega^a{}_b = 0$ [44].

The components of the torsion in local coordinates are therefore canonically labeled by $T^a = \frac{1}{2} T^a{}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. In the following we will use the torsion components with spacetime indices only obtained via $T^\alpha{}_{\mu\nu} = T^a{}_{\mu\nu} e_a{}^\alpha$.

B. Lagrange density and field equations

We consider here the new general relativity (NGR) [35] model given by the action,

$$L_{\rm tot}(\theta, \partial\theta, \Lambda, \partial\Lambda, \Phi^I) = L(\theta, \partial\theta, \Lambda, \partial\Lambda) + L_{\rm M}(\theta, \Phi^I), \tag{7}$$

where $L_{\rm M}(\theta, \Phi^I)$ is the matter Lagrangian, which is constructed via the usual minimal coupling principle. The spacetime metrics appearing during that procedure are understood as functions of the tetrads. The gravitational Lagrangian is the most general Lagrange density quadratic in the torsion tensor,

$$\begin{aligned} L(\theta, \partial\theta, \Lambda, \partial\Lambda) &= |\theta|(c_1 T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + c_2 T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho + c_3 T^\rho{}_{\mu\rho} T^{\sigma\mu}{}_\sigma) \\ &= |\theta| G_{\alpha\beta}{}^{\mu\nu\rho\sigma} T^\alpha{}_{\mu\nu} T^\beta{}_{\rho\sigma}, \end{aligned} \tag{8}$$

where three real parameters c_1, c_2 and c_3 define different NGR theories. In the last equality we introduced the supermetric [45] or constitutive tensor [37,46],

$$G_{\alpha\beta}{}^{\mu\nu\rho\sigma} = c_1 g_{\alpha\beta} g^{\rho[\mu} g^{\nu]\sigma} - c_2 \delta_\beta^{[\mu} g^{\nu][\rho} \delta_\alpha^{\sigma]} - c_3 \delta_\alpha^{[\mu} g^{\nu][\rho} \delta_\beta^{\sigma]}, \quad (9)$$

which will turn out to be convenient for the following analysis. The appearing spacetime metric g is understood as a function of the tetrads (3). The teleparallel equivalent of general relativity (TEGR) is included in the NGR class of gravity theories for the choice $c_1 = \frac{1}{4}, c_2 = \frac{1}{2}$ and $c_3 = -1$.

To analyze the propagation of gravitational waves for NGR gravity around the Minkowski background we derive the linearized field equations of the theory. To do so we fix Cartesian coordinates $(x^\mu, \mu = 0, \dots, 3)$ and make the following perturbative ansatz for the tetrad and the Lorentz transformation defining the spin connection,

$$\theta^a{}_\mu = \delta^a_\mu + \varepsilon \mathrm{u}^a{}_\mu \quad (10a)$$

$$e_a{}^\mu = \delta^\mu_a + \varepsilon \mathrm{v}_a{}^\mu \quad (10b)$$

$$\Lambda^a{}_b = \delta^a_b + \varepsilon \mathrm{w}^a{}_b, \quad (10c)$$

where ε is a perturbation parameter. The duality between θ^a and e_a implies to first order in ε that $\mathrm{v}_a{}^\mu \delta^b_\mu = -\mathrm{u}^b{}_\nu \delta^\nu_a$, and Λ being a local Lorentz transformation implies that $\mathrm{w}_{ab} = -\mathrm{w}_{ba}$. The perturbative gauge transformations are

$$\hat{\theta}^a = \tilde{\Lambda}^a{}_b \theta^b \Rightarrow \hat{\mathrm{u}}^a{}_\mu = \tilde{\mathrm{w}}^a{}_\mu + \mathrm{u}^a{}_\mu$$
$$\Lambda^a{}_d = (\tilde{\Lambda}^{-1})^a{}_c \hat{\Lambda}^c{}_d \Rightarrow \hat{\mathrm{w}}^a{}_b = \tilde{\mathrm{w}}^a{}_b + \mathrm{w}^a{}_b. \quad (11)$$

Moreover changing the index type from Lorentz to spacetime, to first order in the perturbation, is done with $\delta^a{}_\mu$ or $\delta^\mu{}_a$, respectively, and raising and lowering any kind of index is done with the Minkowski metric η_{ab} or $\eta_{\mu\nu}$, respectively, or its inverse.

The torsion tensor can be expanded into the first order fields as

$$T^a{}_{\mu\nu} = 2\partial_{[\mu}\theta^a{}_{\nu]} + 2\omega^a{}_{b[\mu}\theta^b{}_{\nu]} = 2\varepsilon(\partial_{[\mu}\mathrm{u}^a{}_{\nu]} - \partial_{[\mu}\mathrm{w}^a{}_{\nu]}) + \mathcal{O}(\varepsilon^2). \quad (12)$$

In this order of the perturbation theory we transform the torsion components $T^a{}_{\mu\nu}$ to the purely spacetime index components $T^\alpha{}_{\mu\nu}$, which are used in the Lagrangian, and find the lowest order nonvanishing term in NGR Lagrangian (8):

$$\varepsilon^2 \mathcal{G}_{\alpha\beta}{}^{\mu\nu\rho\sigma}(\partial_\mu \mathrm{u}^\alpha{}_\nu - \partial_\mu \mathrm{w}^\alpha{}_\nu)(\partial_\rho \mathrm{u}^\beta{}_\sigma - \partial_\rho \mathrm{w}^\beta{}_\sigma) + \mathcal{O}(\varepsilon^3). \quad (13)$$

The expression $\mathcal{G}_{\alpha\beta}{}^{\mu\nu\rho\sigma}$ is the zeroth order of $G_{\alpha\beta}{}^{\mu\nu\rho\sigma}$; i.e., all metric components $g_{\mu\nu}$ in (9) are replaced by components of the Minkowski metric $\eta_{\mu\nu}$. Observe that the Lagrangian of every teleparallel theory of gravity, which is constructed from the torsion and the tetrad alone without involving higher derivatives of the tetrad, has a lowest order term of the kind (13).

The field equations to lowest nontrivial order are now easily obtained from the Euler-Lagrange equations. The Lagrangian only depends on the derivative of the fundamental variables u and w, and thus we find

$$0 = \partial_\lambda \frac{\partial L}{\partial \partial_\lambda \mathrm{u}^\tau{}_\kappa} \Leftrightarrow 0 = \mathcal{G}_{\tau\beta}{}^{\lambda\kappa\rho\sigma}\partial_\lambda(\partial_\rho \mathrm{u}^\beta{}_\sigma - \partial_\rho \mathrm{w}^\beta{}_\sigma), \quad (14)$$

$$0 = \partial_\lambda \frac{\partial L}{\partial \partial_\lambda \mathrm{w}^\tau{}_\kappa} \Leftrightarrow 0 = (\mathcal{G}_{\tau\beta}{}^{\lambda\kappa\rho\sigma} - \eta_{\gamma\tau}\eta^{\xi\kappa}\mathcal{G}_{\xi\beta}{}^{\lambda\gamma\rho\sigma})\partial_\lambda(\partial_\rho \mathrm{u}^\beta{}_\sigma - \partial_\rho \mathrm{w}^\beta{}_\sigma), \quad (15)$$

where we use the antisymmetry of $\mathrm{w}_{\mu\nu}$ in its indices to derive the second equation or, in other words, allowed only antisymmetric variations of w; note that due to our restriction (6) to flat spin connections this is essentially the linearized version of the restricted variation method introduced in [47]. Raising the index τ the equations can be written as

$$0 = \mathcal{G}^{\tau\beta\lambda\kappa\rho\sigma}\partial_\lambda\partial_\rho(\mathrm{u}_{\beta\sigma} - \mathrm{w}_{\beta\sigma}), \quad (16)$$

$$0 = \mathcal{G}^{[\tau|\beta\lambda|\kappa]\rho\sigma}\partial_\lambda\partial_\rho(\mathrm{u}_{\beta\sigma} - \mathrm{w}_{\beta\sigma}). \quad (17)$$

It is clear that these two sets of equations are not independent of each other, but the latter is the antisymmetric part of the former, a feature that has been discussed in the context of the covariant formulation of teleparallel theories of gravity [37,44]. Moreover it is clear that u and w are not independent variables of the theory.

To proceed we introduce the new gauge invariant [compare (11)] variable $\mathrm{x}_{\beta\sigma} = \mathrm{u}_{\beta\sigma} - \mathrm{w}_{\beta\sigma}$, which must satisfy the field equations,

$$0 = \mathcal{G}^{\tau\beta\lambda\kappa\rho\sigma}\partial_\lambda\partial_\rho \mathrm{x}_{\beta\sigma}. \quad (18)$$

For further simplification we decompose $\mathrm{x}_{\beta\sigma}$ into its symmetric and antisymmetric part $\mathrm{x}_{\beta\sigma} = s_{\beta\sigma} + a_{\beta\sigma}$, which allows us to analyze the field equations further. Using this decomposition and the explicit form of $\mathcal{G}$, see (9), they take the form

$$\begin{aligned} 0 = E^{\tau\kappa} = {} & \partial_\rho[(2c_1 - c_2)\partial^\rho a^{\tau\kappa} - (2c_1 - c_2)\partial^\kappa a^{\tau\rho} \\ & + (2c_2 + c_3)\partial^\tau a^{\rho\kappa}] + \partial_\rho[(2c_1 + c_2)\partial^\rho s^{\tau\kappa} \\ & - (2c_1 + c_2)\partial^\kappa s^{\tau\rho} + c_3(\eta^{\tau\kappa}(\partial^\rho s^\beta{}_\beta - \partial_\beta s^{\rho\beta}) \\ & - \eta^{\tau\rho}(\partial^\kappa s^\beta{}_\beta - \partial_\beta s^{\kappa\beta}))]. \end{aligned} \quad (19)$$

These equations can further be decomposed into a symmetric and into an antisymmetric part, which are independent and given by

$$0 = \partial_\rho[-(2c_1 + c_2 + c_3)\partial^{(\tau}a^{\kappa)\rho}] + \partial_\rho[(2c_1 + c_2)\partial^\rho s^{\tau\kappa} - (2c_1 + c_2 + c_3)\partial^{(\tau}s^{\kappa)\rho} + c_3(\eta^{\tau\kappa}(\partial^\rho s^\beta{}_\beta - \partial_\lambda s^{\rho\lambda}) - \eta^{\rho(\tau}\partial^{\kappa)}s^\beta{}_\beta)], \quad (20)$$

$$0 = \partial_\rho[(2c_1 - c_2)\partial^\rho a^{\tau\kappa} + (2c_1 - 3c_2 - c_3)\partial^{[\tau}a^{\kappa]\rho}] + \partial_\rho[(2c_1 + c_2 + c_3)\partial^{[\tau}s^{\kappa]\rho}))]. \quad (21)$$

Observe that for $(2c_1 + c_2 + c_3) = 0$ the symmetric and the antisymmetric field equations decouple. If one further demands that (21) vanishes identically, in addition $(2c_1 - c_2) = 0$ and $(2c_1 - 3c_2 - c_3) = 0$ have to be satisfied, which implies $c_1 = -\frac{1}{4}c_3$ and $c_2 = -\frac{1}{2}c_3$. Hence for all theories, whose Lagrangian is a multiple of the TEGR Lagrangian, the antisymmetric part of the field equations is satisfied trivially and only for those. We like to point out that linearized field equations in the case of TEGR have been studied in [39] and the fully general case, albeit in a different representation, in [48,49].

In the following we will deduce the propagation velocity and the polarization modes of the perturbations from these field equations.

III. PRINCIPAL POLYNOMIAL AND SPEED OF PROPAGATION

The propagation of waves satisfying a partial differential equation is determined by the principal symbol and principal polynomial of the field equations [50,51]. The vanishing of the principal polynomial defines the wave covectors k of the propagating degrees of freedom of the theory and thus their propagation velocity.

The principal symbol is the highest derivative term of the field equations where the partial derivatives are replaced by wave covectors $\partial \to ik$. Here this corresponds to considering the field equations in Fourier space. From (18) we find

$$0 = \mathcal{G}^{\tau\beta\lambda\kappa\rho\sigma}k_\lambda k_\rho \hat{x}_{\beta\sigma} = P^{\tau\beta\kappa\sigma}(k)\hat{x}_{\beta\sigma}, \quad (22)$$

where $\hat{x}_{\beta\sigma}$ is the Fourier transform of our original field variable $x_{\beta\sigma}$ and

$$P^{\tau\beta\kappa\sigma}(k) = \frac{c1}{2}\eta^{\tau\beta}(\eta(k,k)\eta^{\kappa\sigma} - k^\kappa k^\sigma) - \frac{c2}{4}(k^\beta k^\kappa \eta^{\sigma\tau} - k^\beta k^\tau \eta^{\kappa\sigma} + k^\sigma k^\tau \eta^{\beta\kappa} - \eta(k,k)\eta^{\beta\kappa}\eta^{\sigma\tau}) - \frac{c3}{4}(k^\tau k^\kappa \eta^{\sigma\beta} - k^\beta k^\tau \eta^{\kappa\sigma} + k^\sigma k^\beta \eta^{\tau\kappa} - \eta(k,k)\eta^{\tau\kappa}\eta^{\sigma\beta}). \quad (23)$$

The principal polynomial $P(k)$ is given by the determinant of the principal symbol, which is interpreted as a metric on the space of fields $y^{\tau\kappa} = P^{\tau\beta\kappa\sigma}(k)\hat{x}_{\beta\sigma}$.

From the antisymmetry of the field equations in the indices $\lambda\kappa$ and $\rho\sigma$ it is immediately clear that the principal symbol is degenerate as fields of the form $\hat{x}_{\beta\sigma} = k_\sigma V_\beta(k)$ solve the field equations trivially. This is a clear sign of the presence of gauge degrees of freedom in the theory. In order to derive the principal symbol we must restrict the field equations to the subspace of fields, on which they are nondegenerate. This feature is common in field theories with gauge degrees of freedom and appears also in general premetric theories of electrodynamics [52] for example.

The field equations can be seen as a map from the space of 4×4 matrices $\hat{x}_{\beta\sigma}$ to its duals. To identify the subspace $\mathcal{V}$ of all 4×4 matrices on which the field equations are nondegenerate we employ the following decomposition:

$$\hat{x}_{\beta\sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta\sigma}, \quad (24)$$

where the scalar U, the 1-form components V_α and W_α and the (0, 2)-tensor $Q_{\beta\sigma}$ satisfy the constraints

$$k_\alpha V^\alpha = 0, \quad k_\alpha W^\alpha = 0, \quad k_\alpha Q^\alpha{}_\beta = 0, \quad k_\alpha Q_\beta{}^\alpha = 0. \quad (25)$$

The 4 degrees of freedom U and V^α cannot be dynamical as they trivially solve the field equations. Remaining are 12 degrees of freedom, $4 - 1 = 3$ encoded in W_α and $16 - 7 = 9$ in $Q_{\alpha\beta}$, which span the subspace $\mathcal{V}$. Expanding $Q^{\tau\kappa}$ further into its symmetric traceless and antisymmetric part as well as its trace by writing $Q^{\tau\kappa} = S^{\tau\kappa} + A^{\tau\kappa} + \frac{1}{3}(\eta^{\tau\kappa} - \frac{k^\tau k^\kappa}{\eta(k,k)})Q^\sigma{}_\sigma$, and using (9), the Fourier space field equations become

$$0 = \hat{E}^{\tau\kappa} = (2c_1 + c_2 + c_3)\eta(k,k)k^\tau W^\kappa + (2c_1 + c_2)\eta(k,k)S^{\tau\kappa} + (2c_1 - c_2)\eta(k,k)A^{\tau\kappa} + \frac{1}{3}Q^\sigma{}_\sigma \eta(k,k)(2c_1 + c_2 + 3c_3)\left(\eta^{\kappa\tau} - \frac{k^\tau k^\kappa}{\eta(k,k)}\right), \quad (26)$$

where we use, for the sake of readability, the notation $\eta(k,k) = \eta^{\mu\nu}k_\mu k_\nu$. To analyze them further we observe that they decompose into their contractions with k, their trace, their symmetric traceless and antisymmetric part,

$$0 = \hat{E}^{\tau\kappa}k_\tau k_\kappa, \qquad 0 = \hat{E}^{\tau\kappa}k_\kappa, \quad (27a)$$

$$0 = \hat{E}^{\tau\kappa}k_\tau = (2c_1 + c_2 + c_3)\eta(k,k)^2 W^\kappa, \quad (27b)$$

$$0 = \hat{E}^\tau{}_\tau = (2c_1 + c_2 + 3c_3)\eta(k,k)Q^\tau{}_\tau, \quad (27c)$$

$$0 = \hat{E}^{[\tau\kappa]} - \frac{k^{[\tau}\hat{E}^{|\sigma|\kappa]}k_\sigma}{\eta(k,k)} = (2c_1 - c_2)\eta(k,k)A^{\tau\kappa}, \quad (27d)$$

$$0 = \hat{E}^{(\tau\kappa)} - k^{(\tau}\hat{E}^{|\sigma|\kappa)}k_\sigma - \frac{1}{3}\left(\eta^{\tau\kappa} - \frac{k^\tau k^\kappa}{\eta(k,k)}\right)\hat{E}^\sigma{}_\sigma = (2c_1 + c_2)\eta(k,k)S^{\tau\kappa}. \quad (27e)$$

The first two equations are satisfied trivially for any choice of parameters c_1, c_2 and c_3. The remaining four nontrivial field equations can be represented by a block diagonal matrix acting on a field space vector which is an element of $\mathcal{V}$:

$$\eta(k,k)\begin{pmatrix} (2c_1+c_2+c_3)\eta(k,k) & 0 & 0 & 0 \\ 0 & (2c_1+c_2+3c_3) & 0 & 0 \\ 0 & 0 & (2c_1-c_2) & 0 \\ 0 & 0 & 0 & (2c_1+c_2) \end{pmatrix}\begin{pmatrix} W^\kappa \\ Q^\tau{}_\tau \\ \hat{A}^{\tau\kappa} \\ \hat{S}^{\tau\kappa} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

Due to their simple nature the principal polynomial is now easily obtained as determinant of the above matrix,

$$P(k) = (2c_1+c_2+c_3)^3(2c_1+c_2+3c_3)(2c_1-c_2)^3(2c_1+c_2)^5(\eta(k,k))^{15}. \tag{29}$$

A necessary nontrivial solution to satisfy the field equations is that in which their wave covectors k are such that $P(k)=0$. From the above equation (29) it is evident that only null covectors of the Minkowski metric $\eta(k,k)=0$ realize this condition. Hence we find that for NGR theories of gravity, perturbations propagate with the speed of light determined by Maxwell electrodynamics on Minkowski spacetime.

We would like to remark that this feature can also already be seen from the decomposed Fourier space field equations (27b) to (27e). For all field equations there can only exist a nontrivial solution of the field W^κ, $Q^\tau{}_\tau$, $S^{\tau\kappa}$ or $A^{\tau\kappa}$ if and only if $\eta(k,k)$ vanishes, so all field modes in the theory are massless. For the W^κ mode we find a double pole in its propagator, which is consistent with [49,53]. For $\eta(k,k)\neq 0$ the only solution of the field equations is that the fields themselves vanish identically.

IV. NEWMAN-PENROSE FORMALISM AND POLARIZATIONS

We now focus on the polarization of gravitational waves. As we have seen in the previous section, gravitational waves in new general relativity are described by Minkowski null waves, independently of the choice of the parameters c_1, c_2, and c_3. This allows us to make use of the well-known Newman-Penrose formalism [40] in order to decompose the linearized field equations into components, which directly correspond to particular polarizations. We then employ the classification scheme detailed in [41,42], which characterizes the allowed polarizations of gravitational waves in a given gravity theory by a representation of the little group, which is the two-dimensional Euclidean group E(2) in case of null waves. In this section we determine the E(2) class of new general relativity for all possible values of the parameters c_1, c_2, and c_3.

The main ingredient of the Newman-Penrose formalism is the choice of a particular complex double null basis of the tangent space. In the following, we will use the notation of [54] and denote the basis vectors by l^μ, n^μ, m^μ, and $\bar{m}^\mu$. In terms of the canonical basis vectors of the Cartesian coordinate system they are defined as

$$l = \partial_0 + \partial_3, \qquad n = \frac{1}{2}(\partial_0 - \partial_3),$$
$$m = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \qquad \bar{m} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2). \tag{30}$$

We now consider a plane wave propagating in the positive x^3 direction, which corresponds to a single Fourier mode. The wave covector then takes the form $k_\mu = -\omega l_\mu$, and the symmetric and antisymmetric parts of the tetrad perturbations can be written in the form

$$s_{\mu\nu} = S_{\mu\nu}e^{i\omega u}, \qquad a_{\mu\nu} = A_{\mu\nu}e^{i\omega u}, \tag{31}$$

where we introduced the retarded time $u = x^0 - x^3$, and the wave amplitudes are denoted $S_{\mu\nu}$ and $A_{\mu\nu}$.

Recall that we consider minimal coupling between gravity and matter, i.e., coupling only through the metric seen as function of the tetrad but not through the flat spin connection. This is the usual coupling prescription for nonspinning matter, which we will henceforth assume. It follows from this choice of the matter coupling that test particles follow the geodesics of the metric and hence the autoparallel curves of its Levi-Civita connection. The effect of a gravitational wave on an ensemble of test particles, or any other type of gravitational wave detector, such as the mirrors of an interferometer, is therefore described by the corresponding geodesic deviation equation. The observed gravitational wave signal hence depends only on the Riemann tensor derived from the Levi-Civita connection. As shown in [42], the Riemann tensor of a plane wave is determined completely by the six so-called electric components. For the wave (31), these can be written as

$$\Psi_2 = -\frac{1}{6}R_{nlnl} = \frac{1}{12}\ddot{h}_{ll},$$
$$\Psi_3 = -\frac{1}{2}R_{nln\bar{m}} = -\frac{1}{2}\overline{R_{nlnm}} = \frac{1}{4}\ddot{h}_{l\bar{m}} = \frac{1}{4}\overline{\ddot{h}_{lm}},$$
$$\Psi_4 = -R_{n\bar{m}n\bar{m}} = -\overline{R_{nmnm}} = \frac{1}{2}\ddot{h}_{\bar{m}\bar{m}} = \frac{1}{2}\overline{\ddot{h}_{mm}},$$
$$\Phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2}\ddot{h}_{m\bar{m}}, \tag{32}$$

where dots denote derivatives with respect to u and the metric perturbation components $h_{\mu\nu}$ are derived from the perturbation ansatz (10) as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \varepsilon(\eta_{\mu\rho}u^{\rho}{}_{\nu} + \eta_{\nu\rho}u^{\rho}{}_{\mu}) = \eta_{\mu\nu} + 2\varepsilon s_{\mu\nu}. \tag{33}$$

Note that they depend only on the symmetric perturbation of the tetrad, so that these are the only components whose presence or absence we must determine. We now examine which of the components (32) may occur for gravitational waves satisfying the linearized field equations (19).

Inserting the wave ansatz (31) and writing the gravitational Euler-Lagrange tensor $E_{\mu\nu}$ in the Newman-Penrose basis, we find that the eight component equations,

$$E_{ll} = E_{lm} = E_{ml} = E_{nl} = E_{mm} = E_{\bar{m}\bar{m}} = E_{l\bar{m}} = E_{\bar{m}l} = 0, \tag{34}$$

are satisfied identically, while the remaining eight component equations take the form

$$0 = E_{nn} = (2c_1 + c_2 + c_3)\ddot{s}_{nl} + 2c_3\ddot{s}_{m\bar{m}} + (2c_1 + c_2 + c_3)\ddot{a}_{nl}, \tag{35a}$$

$$0 = E_{mn} = (2c_1 + c_2)\ddot{s}_{ml} + (2c_1 - c_2)\ddot{a}_{ml}, \tag{35b}$$

$$0 = E_{\bar{m}n} = (2c_1 + c_2)\ddot{s}_{\bar{m}l} + (2c_1 - c_2)\ddot{a}_{\bar{m}l}, \tag{35c}$$

$$0 = E_{nm} = -c_3\ddot{s}_{ml} + (2c_2 + c_3)\ddot{a}_{ml}, \tag{35d}$$

$$0 = E_{n\bar{m}} = -c_3\ddot{s}_{\bar{m}l} + (2c_2 + c_3)\ddot{a}_{\bar{m}l}, \tag{35e}$$

$$0 = E_{m\bar{m}} = E_{\bar{m}m} = -c_3\ddot{s}_{ll}, \tag{35f}$$

$$0 = E_{ln} = (2c_1 + c_2)\ddot{s}_{ll}. \tag{35g}$$

We now distinguish the following cases, which are also visualized in the diagram in Fig. 1, which we explain later in this section:

(i) $2c_1 + c_2 = c_3 = 0$: In this case Eqs. (35f) and (35g) are satisfied identically for arbitrary amplitudes S_{ll}. For waves of this type the corresponding component $R_{nlnl} = -6\Psi_2$ of the Riemann tensor, which describes a longitudinally polarized wave mode, is allowed to be nonzero. Following the classification detailed in [42], they belong to the E(2) class II_6 with six polarizations. This case corresponds to the two blue points in Fig. 1, which is actually a line in the three-dimensional parameter space and hence a single point in the projected parameter space shown in the diagram, which happens to lie on the cut $c_3 = 0$ and hence appears twice on the circular perimeter.

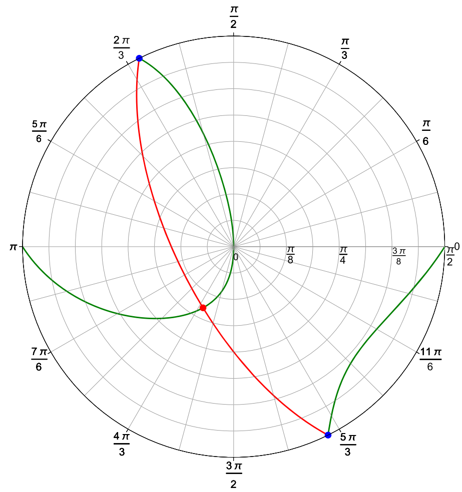

FIG. 1. Visualization of the parameter space using polar coordinates. The radial axis shows the zenith angle θ, whereas the (circular) polar axis shows the azimuth angle ϕ, following the definition (38). Blue points: $2c_1 + c_2 = c_3 = 0$, class II_6, 6 polarizations. Green line: $2c_1 + c_2 + c_3 \neq 0, 2c_1(c_2 + c_3) + c_2^2 = 0$, class III_5, 5 polarizations. White area: $2c_1(c_2 + c_3) + c_2^2 \neq 0$, $2c_1 + c_2 + c_3 \neq 0$, class N_3, 3 polarizations. Red line: $2c_1 + c_2 + c_3 = 0, c_3 \neq 0$, class N_2, 2 polarizations.

(ii) $2c_1(c_2 + c_3) + c_2^2 = 0$ and $2c_1 + c_2 + c_3 \neq 0$: It follows from the second condition that at least one of $2c_1 + c_2$ or c_3 must be nonzero. Hence, either Eq. (35f) or Eq. (35g) imposes the condition $S_{ll} = 0$, so that there is no longitudinal mode Ψ_2. The first condition is equivalent to a vanishing determinant of the matrix,

$$\begin{pmatrix} 2c_1 + c_2 & 2c_1 - c_2 \\ -c_3 & 2c_2 + c_3 \end{pmatrix}, \tag{36}$$

so that Eqs. (35b), (35c), (35d) and (35e) allow for nonvanishing solutions. This further implies that the two columns of this matrix are linearly dependent and hence proportional to each other. However, from the second condition further follows that neither column vanishes. Hence, at least one of the pairs (35b), (35c) and (35d), (35e) of equations must be nontrivial, with the coefficients of both the symmetric and the antisymmetric tetrad perturbation nonvanishing. Hence, nonvanishing solutions of these equations have both symmetric and antisymmetric contributions and, therefore, in particular nonvanishing S_{lm} and $S_{l\bar{m}}$; however, recall that the antisymmetric part does not contribute to the geodesic deviation equation, and so we do not discuss it here. It then follows that $R_{nln\bar{m}} = -2\Psi_3$, whose complex components describe two vector polarizations, is allowed to be nonzero. Waves of this type belong to the E(2) class III_5 encompassing five polarizations. This case is represented by the green line in Fig. 1.

(iii) $2c_1(c_2+c_3)+c_2^2 \neq 0$ and $2c_1+c_2+c_3 \neq 0$: In this case the only linearized field equation which allows for nonvanishing solutions is Eq. (35a). Here the only relevant component for the geodesic deviation is $S_{m\bar{m}}$, so that we can neglect the other terms. This component is allowed to be nonvanishing and hence allows a nonvanishing component $R_{nmn\bar{m}} = -\Phi_{22}$ of the Riemann tensor. The corresponding scalar wave mode is called the breathing mode. The remaining equations impose the condition $\Psi_2 = \Psi_3 = 0$, so that the longitudinal and vector modes are prohibited. This wave has the E(2) class N_3 and thus three polarizations. Almost all points of the parameter space, shown in white in Fig. 1, belong to this class.

(iv) $2c_1+c_2+c_3 = 0$ and $c_3 \neq 0$: It follows immediately from Eq. (35f) that $S_{ll} = 0$, so that the longitudinal mode Ψ_2 is prohibited. Taking the sum of the pairs (35b), (35c) and (35d), (35e) of equations and replacing c_2 by $-2c_1 - c_3$ one further finds that $S_{lm} = S_{l\bar{m}} = 0$, and hence also the vector modes Ψ_3 must vanish. Finally, Eq. (35a) imposes the condition $S_{m\bar{m}} = 0$, so that also the breathing mode Φ_{22} is prohibited. It thus follows that the only unrestricted electric components of the Riemann tensor are $R_{nmnm} = -\bar{\Psi}_4$ and its complex conjugate, corresponding to two tensor modes. The E(2) class of this wave is N_2, with two polarizations. This case is shown as a red line in Fig. 1. Note in particular that TEGR, marked as a red point, belongs to this class, as one would expect. This subclass corresponds to the so-called one-parameter family of teleparallel models and has received particular attention in previous studies [48]. It has been argued that this condition is necessary to avoid ghosts [49,53]. However, we will not address the question of ghosts in this article and leave this discussion for a separate study.

We have visualized the aforementioned cases in Fig. 1, which we constructed as follows. We first made use of our assumption that at least one of the parameters c_1, c_2, and c_3 is nonvanishing and introduced normalized parameters,

$$\tilde{c}_i = \frac{c_i}{\sqrt{c_1^2 + c_2^2 + c_3^2}}, \tag{37}$$

for $i = 1, 2, 3$. One easily checks that the E(2) classes we found only depend on these normalized parameters. We then introduced polar coordinates (θ, ϕ) on the unit sphere to express the parameters as

$$\tilde{c}_1 = \sin\theta\cos\phi, \quad \tilde{c}_2 = \sin\theta\sin\phi, \quad \tilde{c}_3 = \cos\theta. \tag{38}$$

As the E(2) class is the same for antipodal points on the parameter sphere, we restrict ourselves to the hemisphere $\tilde{c}_3 \geq 0$, and hence $0 \leq \theta \leq \frac{\pi}{2}$; this is equivalent to identifying antipodal points on the sphere and working with the projective sphere instead, provided that we also identify antipodal points on the equator $\tilde{c}_3 = 0$. We then considered (θ, ϕ) as polar coordinates on the plane in order to draw the diagram shown in Fig. 1. Note that antipodal points on the perimeter, such as the two blue points, are identified with each other as they describe the same class of theories.

This concludes our discussion of gravitational wave polarizations. We have seen that depending on the parameters c_1, c_2, and c_3 we obtain the E_2 class II_6, III_5, N_3 or N_2, with N_3 filling most of the parameter space. We have also seen that there exists a family of theories besides TEGR which is of class N_2 and thus exhibits the same two tensor modes as in general relativity. Theories in this class therefore cannot be distinguished from general relativity by observing the polarizations of gravitational waves alone.

V. CONCLUSION

We studied the propagation of gravitational waves in the most general class of teleparallel gravity theories whose action is quadratic in the torsion tensor, known as new general relativity. The wave we considered is modeled as a linear perturbation of a diagonal tetrad corresponding to a Minkowski background metric. We derived the principal polynomial of the linearized field equations and found that gravitational waves propagate at the speed of light; i.e., their wave covector must be given by a null vector of the Minkowski background. Further, we made use of the Newman-Penrose formalism to derive the possible polarizations of gravitational waves. Our results show that the two tensor polarizations, which are present also in general relativity, are allowed for the whole class of theories we considered, whereas additional modes—two vector modes and up to two scalar modes—may be present for particular models within this class. We found that the teleparallel equivalent of general relativity is not the unique theory exhibiting exactly two polarizations, but there is a one-parameter family of theories with the same property. It thus follows that observations of gravitational wave polarizations may only give partial results on the parameter space of these theories.

We remark that, although we restricted our analysis to theories whose action is quadratic in the torsion tensor, our results are valid for a significantly larger class of theories. This is due to the fact that the torsion is linear in the tetrad perturbations, so that the action is already quadratic in the perturbations. Hence, any higher order correction terms would have no influence on the linearized field equations. This observation agrees with previous results that there are no additional gravitational polarizations in $f(T)$ gravity compared to general relativity [29] as up to the required perturbation order the Lagrangian can be approximated as $f(T) = f(0) + f'(0)T + \mathcal{O}(T^2)$, which is equivalent to general relativity with a cosmological constant. An extension to the class of theories discussed in [36] is shown in [55].

Although higher order terms in the action do not influence the linear perturbations around a Minkowski background, they certainly have an influence on the cosmological dynamics of the theory and therefore on the expansion history of the Universe. This modified expansion history might thus also leave an imprint on the observed gravitational waves propagating in a cosmological background. An interesting extension of our work would be to study gravitational waves as a perturbation to a tetrad corresponding to a Friedmann-Lemaitre-Robertson-Walker metric, taking into account modifications of the background dynamics arising from higher order torsion terms. Note that such modifications do not show up in the quadratic action we considered in this article as all terms in the gravitational action become proportional to the square of the Hubble parameter in the case of cosmological symmetry, and so the action reduces to the teleparallel equivalent of general relativity, up to a constant factor.

Another possible class of extensions is to consider additional fields nonminimally coupled to torsion and to study their influence both on the speed and the polarization of gravitational waves. A canonical example is given by scalar torsion theories [56–59] constructed from the TEGR torsion scalar and an additional scalar field, where one would expect the presence of an additional scalar mode compared to general relativity as it is also the case for scalar curvature gravity. These theories can be extended by replacing the TEGR torsion scalar with the NGR torsion scalar which defined the Lagrangian considered in this article.

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Chapter 7

Propagation of gravitational waves in symmetric teleparallel gravity theories

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Contents

Propagation of gravitational waves in symmetric teleparallel gravity theories

Manuel Hohmann,* Christian Pfeifer,† and Ulbossyn Ualikhanova‡
Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia

Jackson Levi Said§
Department of Physics, University of Malta, Msida, MSD 2080, Malta

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Symmetric teleparallel gravity (STG) offers an interesting third geometric interpretation of gravitation besides its formulation in terms of a spacetime metric and Levi-Civita connection or its teleparallel formulation. It describes gravity through a connection which is not metric compatible, however is torsion and curvature free. We investigate the propagation velocity of the gravitational waves around Minkowski spacetime and their potential polarizations in a general class of STG theories, the so-called "newer general relativity" class. It is defined in terms of the most general Lagrangian that is quadratic in the nonmetricity tensor, does not contain its derivatives and is determined by five free parameters. In our work we employ the principal symbol method and the Newman-Penrose formalism, to find that all waves propagate with the speed of light, i.e., on the Minkowski spacetime light cone, and to classify the theories according to the number of polarizations of the waves depending on the choice of the parameters in the Lagrangian. In particular it turns out that there exist more theories than just the reformulation of general relativity which allow only for two polarization modes. We also present a visualization of the parameter space of the theory to better understand the structure of the model.

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I. INTRODUCTION

The observation of gravitational waves (GWs) has opened the possibility of a new window on strong field physics [1] that is not accessible by electromagnetic observations alone. While GW observations have continued to be confirmed, the first three-detector observation holds important significance in that such measurements allow for signal localization and, more to the purpose of this work, constraints on the six potential polarization modes of metric theories of gravity [2]. Moreover there has been the first multimessenger observations [3] which constrain the difference of the propagation velocity between GW and electromagnetic waves in vacuum, which is different from zero in various modified theories of gravity [4–10]. Thus GW observations offer the possibility for strong constraints on theories predicting extra modes and a propagation velocity different from the speed of light, and so may be the route to reducing the landscape of potential gravitational theories consistent with observation [11].

Viewed through the prism of the connection, metric theories of gravity can be classified into three broad classes of theories. The ones which use the Levi-Civita connection of the metric and its curvature, the ones which use the tetrads of a metric and their curvature free, metric-compatible, Weitzenböck connection with torsion and the ones which use a curvature and torsion free symmetric teleparallel connection that is not metric compatible. This classification nicely highlights the sometimes overlooked point that curvature is a property of the connection and not of the metric tensor or the manifold [12]. It becomes a property of the metric only through the use of the Levi-Civita connection. For the description of gravity it is remarkable that general relativity (GR) and the Einstein equations can be equivalently formulated in terms of either of the connections just mentioned [13–15], i.e., all three connections can be used to define Lagrangians whose Euler-Lagrange equations coincide with the Einstein equations for a particular choice of contributing terms.

Historically most used for the construction of GR and extended theories of gravity [16] is the Levi-Civita connection, resulting mainly in $f(R)$, $f(R,G)$ and similar theories. However, the use of torsion and nonmetricity allow for another kind of generalization [17]. In particular, the irreducible contributions of the Lagrangian of these two

*manuel.hohmann@ut.ee
†christian.pfeifer@ut.ee
‡ulbossyn.ualikhanova@ut.ee
§jsaid01@um.edu.mt

theories can be elevated to arbitrary coupling coefficients with a limit to their GR equivalent for a particular numerical choice. These two avenues of generalization are important because they may provide constraints on these novel and not extensively studied generalizations which may lead to a better understanding of the unique coincidence that GR appears to represent. Moreover, by altering the connection a new landscape of gravitational theories can be studied which differ from each other at a fundamental level in the classical regime [18].

GWs offer the possibility of a model independent test of the polarization modes a theory exhibits [19,20]. In principle, this provides a strict test of which theories are realistic in the strong field regime. Thus far, the topic has not been studied as well for STG theories, as for torsion based (or teleparallel) gravity theories. In teleparallel gravity [21,22], the propagation of GW modes has been shown to have a varied nature depending on the particular form the theory takes. This was first studied in Ref. [23] where it was found that the straightforward generalizations of the teleparallel equivalent of GR (TEGR), namely $f(T)$ theories, exhibit the same polarization structure as that of GR and thus is indistinguishable at the level of GW modes. The work then was further confirmed and expanded upon to encompass scalar fields and a generalized form of $f(R)$ gravity [24], the speed of the GWs and the effect of the three-detector observation was then studied in Ref. [25], which then culminated in the explicit expression of the modes in these extended teleparallel theories in Ref. [26]. In Refs. [27,28], the general scenario of decomposed Lagrangians of both the torsional and nonmetricity situations is considered with clear groundwork for further analysis in either theory. Another approach to the propagator of generalized symmetric teleparallel gravity theories including higher derivative orders and making use of the Barnes-Rivers formalism can be found in Ref. [29].

In the present study, we investigate the GW polarization modes of the massless contribution in the general form of the STG setting. As in the teleparallel setting, since the Lagrangian can be divided into irreducible contributors, it is interesting to understand the GW mode structure that this seemingly arbitrary landscape provides [14,17]. We then represent the resulting parameter space of this theory in a novel way, since the model has a lot of potential avenues to it.

The paper is organized as follows. In Sec. II we briefly introduce the key components of the model we are considering and form the linearized field equations. This is crucial to understanding the relevant contributions to the GW modes. In Fourier space, the field equations are then decomposed and the speed of GWs in STG is determine in Sec. III to determine the polarization states the Newman-Penrose formalism is considered in Sec. IV where these states are depicted. Lastly, we close with a discussion in Sec. V.

II. LINEARIZED GENERAL SYMMETRIC TELEPARALLEL GRAVITY THEORIES

Before we derive the speed and polarization of gravitational waves in symmetric teleparallel gravity, we need to derive its linearized field equations. This is done in two parts. In Sec. II A we briefly review the underlying spacetime geometry and its gauge aspects. We turn our focus to the dynamics of the theory in Sec. II B, where we review the action and field equations, which we then linearize after gauge fixing.

A. Geometry with nonmetricity

We start with a brief review of the underlying geometry involving nonmetricity, which we use in this article. The fundamental fields defining the geometry are a Lorentzian metric $g_{\mu\nu}$ and an affine connection $\Gamma^\mu{}_{\nu\rho}$. The connection is chosen to have vanishing curvature,

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\omega\rho}\Gamma^\omega{}_{\nu\sigma} - \Gamma^\mu{}_{\omega\sigma}\Gamma^\omega{}_{\nu\rho} \equiv 0, \quad (1)$$

and vanishing torsion

$$T^\mu{}_{\rho\sigma} = \Gamma^\mu{}_{\sigma\rho} - \Gamma^\mu{}_{\rho\sigma} \equiv 0. \quad (2)$$

It does, however, possess in general nonvanishing nonmetricity,

$$Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu}. \quad (3)$$

Indices are raised and lowered using the metric $g_{\mu\nu}$. Note that due to the presence of nonmetricity this implies

$$Q_\alpha{}^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}Q_{\alpha\rho\sigma} = -\nabla_\alpha g^{\mu\nu}. \quad (4)$$

The nonmetricity is obviously symmetric in its second and third index, $Q_{\alpha\mu\nu} = Q_{\alpha\nu\mu}$, which allows the definition of two different traces,

$$Q_\alpha = g^{\mu\nu}Q_{\alpha\mu\nu}, \qquad \tilde{Q}_\alpha = g^{\mu\nu}Q_{\mu\nu\alpha}. \quad (5)$$

The most general connection which satisfies the assumptions (1) and (2) is generated by a coordinate transformation defined by functions $\xi^\mu(x)$ in the form [14,30]

$$\Gamma^\mu{}_{\nu\sigma} = \frac{\partial x^\mu}{\partial \xi^\rho}\partial_\nu\partial_\sigma\xi^\rho. \quad (6)$$

It further follows that it is always possible to find coordinates such that

$$\Gamma^\alpha_{\mu\nu} \equiv 0, \quad (7)$$

not only at a single point, but in an open neighborhood. This particular choice of coordinates is known as the coincident gauge [17], and will be used throughout this work. Note that this uniquely determines the coordinate system (x^μ) we use, up to linear transformations of the form

$$x^\mu \mapsto \tilde{\xi}^\mu(x) = \tilde{\xi}^\mu(x_0) + (x^\nu - x_0^\nu)\partial_\nu \tilde{\xi}^\mu|_{x=x_0}, \tag{8}$$

so that $\partial_\mu \partial_\nu \tilde{\xi}^\alpha \equiv 0$. It follows that we have no further gauge freedom left to impose conditions on the metric degrees of freedom (d.o.f.), except at a single point, as it is conventionally the case, e.g., in general relativity. In the coincident gauge covariant derivatives are replaced by partial derivatives, so that the nonmetricity reads

$$Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}. \tag{9}$$

We will make use of this formula in the following, when we derive the linearized field equations.

B. Action and field equations

The starting point for the derivation of the linearized field equations is the "newer general relativity" action for the metric, the coordinate functions ξ^μ and the matter fields [14,17,31], which can be written in the form

$$S[g_{\mu\nu}, \xi^\sigma, \chi^I] = S_g[g_{\mu\nu}, \xi^\sigma] + S_m[g_{\mu\nu}, \chi^I],$$
$$S_g = -\int_M \frac{\sqrt{-g}}{2} \mathbb{Q} \mathrm{d}^4 x. \tag{10}$$

We assume that the matter part S_m of the action does not depend on the affine connection $\Gamma^\alpha{}_{\mu\nu}[\xi]$, but only on the metric $g_{\mu\nu}$ and a set of matter fields χ^I. The gravitational part S_g of the action is expressed in terms of the nonmetricity scalar $\mathbb{Q}$, seen as a function of the metric and the connection generating vector field, and is most conveniently defined via the nonmetricity conjugate

$$P^\alpha{}_{\mu\nu} = c_1 Q^\alpha{}_{\mu\nu} + c_2 Q_{(\mu}{}^\alpha{}_{\nu)} + c_3 Q^\alpha g_{\mu\nu} + c_4 \delta^\alpha_{(\mu} \tilde{Q}_{\nu)} + \frac{c_5}{2}(\tilde{Q}^\alpha g_{\mu\nu} + \delta^\alpha_{(\mu} Q_{\nu)}), \tag{11}$$

as

$$\mathbb{Q} = Q_\alpha{}^{\mu\nu} P^\alpha{}_{\mu\nu}. \tag{12}$$

This is the most general Lagrangian which is quadratic in the nonmetricity, unless one introduces also derivatives [29]. Choosing the parameters $c_1 = -\frac{1}{4}, c_2 = \frac{1}{2}, c_3 = \frac{1}{4}$, $c_4 = 0$, and $c_5 = -\frac{1}{2}$ one obtains the nonmetricity formulation of general relativity [14,32], which is usually called symmetric teleparallel equivalent of general relativity (STEGR). By variation of the total action with respect to the metric, one obtains the field equations

$$\frac{2}{\sqrt{-g}} \nabla_\alpha(\sqrt{-g} P^\alpha{}_{\mu\nu}) + P_{\mu\sigma\rho} Q_\nu{}^{\sigma\rho} - 2Q_{\rho\mu}{}^\sigma P^\rho{}_{\nu\sigma} - \frac{1}{2}\mathbb{Q} g_{\mu\nu} = \mathfrak{T}_{\mu\nu}, \tag{13}$$

where the energy-momentum tensor $\mathfrak{T}_{\mu\nu}$ is derived from the matter action S_m. To obtain the second set of field equations, we vary the total action with respect to the components of the connection generating coordinate functions ξ^μ. Note that this is equivalent to performing a restricted variation of the flat, symmetric connection $\Gamma^\alpha{}_{\mu\nu}$, which must be of the form $\delta\Gamma^\alpha{}_{\mu\nu} = \nabla_\mu \nabla_\nu \delta\xi^\alpha$ in order to keep the vanishing torsion and curvature, $\delta T^\alpha{}_{\mu\nu} \equiv 0$ and $\delta R^\alpha{}_{\beta\mu\nu} \equiv 0$. After twice performing integration by parts, carefully taking into account the terms arising from $\nabla_\mu \sqrt{-g}$ due to the nonmetricity, this yields the field equations

$$\nabla_\mu \nabla_\nu (\sqrt{-g} P^{\mu\nu}{}_\alpha) = 0. \tag{14}$$

Note that their right hand side vanishes, since we have assumed no direct coupling of the matter to the flat, symmetric connection, and so the hypermomentum vanishes. We remark that this second set of field equations can alternatively be obtained from the diffeomorphism invariance of the gravitational action, giving an equivalent of the Bianchi identities, and the matter action, giving the matter energy-momentum conservation. This shows that the field equations (13) and (14) are not independent, and reflects the presence of the gauge symmetry under diffeomorphisms. Hence, we may restrict ourselves to solving the metric field equations (13).

In order to linearize the metric field equations, we now adopt the coincident gauge $\Gamma^\alpha{}_{\mu\nu} \equiv 0$ and consider a small perturbation around a Minkowski background metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{15}$$

The nonmetricity tensor thus takes the form

$$Q_{\alpha\mu\nu} = \partial_\alpha h_{\mu\nu}. \tag{16}$$

Further, we restrict ourselves to the vacuum field equations, so that $\mathfrak{T}_{\mu\nu} \equiv 0$. Up to the linear order in the metric perturbations $h_{\mu\nu}$, the metric field equations (13) then reduce to

$$0 = 2c_1 \Box h_{\mu\nu} + (c_2 + c_4)\eta^{\alpha\sigma}(\partial_\alpha \partial_\mu h_{\sigma\nu} + \partial_\alpha \partial_\nu h_{\sigma\mu}) + 2c_3 \eta_{\mu\nu} \eta^{\tau\omega} \Box h_{\tau\omega} + c_5 \eta_{\mu\nu} \eta^{\omega\gamma} \eta^{\alpha\sigma} \partial_\alpha \partial_\omega h_{\sigma\gamma} + c_5 \eta^{\omega\sigma} \partial_\mu \partial_\nu h_{\omega\sigma}. \tag{17}$$

Note that up to higher order terms, indices are now raised and lowered by the Minkowski metric $\eta_{\mu\nu}$. This in particular applies to the d'Alembert operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. In the following, we will use the linearized equations (17) in order to derive properties of gravitational wave propagation.

III. PRINCIPAL POLYNOMIAL AND SPEED OF PROPAGATION

To determine the propagation speed of gravitational waves in nonmetricity theories of gravity we study the field equations in Fourier space. A necessary condition solutions of the field equations have to satisfy is, that the so called principal polynomial of the equations, as function of the wave covectors, has to vanish [33,34].

The field equations in Fourier space are

$$0 = \hat{E}_{\mu\nu} = (2c_1\eta^{-1}(k,k)\delta^\lambda_\mu\delta^\rho_\nu + (c_2+c_4)k^\lambda(k_\mu\delta^\rho_\nu + k_\nu\delta^\rho_\mu) + 2c_3\eta_{\mu\nu}\eta^{-1}(k,k)\eta^{\lambda\rho} + c_5(\eta_{\mu\nu}k^\lambda k^\rho + \eta^{\lambda\rho}k_\mu k_\nu))\hat{h}_{\lambda\rho}. \tag{18}$$

The principal polynomial of the equation is the determinant of the highest power in k term. To calculate this determinant we decompose the equations with help of a decomposition with respect to a gauge vector field κ^μ which is dual to k_μ, i.e., satisfies $\kappa^\mu k_\mu = 1$.

$$\hat{h}_{\lambda\rho} = S_{\lambda\rho} + 2k_{(\lambda}V_{\rho)} + \frac{1}{3}\left(\eta_{\lambda\rho} - \frac{k_\lambda k_\rho}{\eta^{-1}(k,k)}\right)T + \left(k_\lambda k_\rho - \frac{1}{4}\eta_{\lambda\rho}\eta^{-1}(k,k)\right)U, \tag{19}$$

where the divergence free symmetric traceless part $S_{\mu\nu}$ and the divergence free vector V_μ satisfy

$$\eta^{\lambda\rho}S_{\lambda\rho} = 0, \qquad k^\lambda S_{\lambda\rho} = 0, \qquad k^\rho V_\rho = 0. \tag{20}$$

The remaining scalars are the trace $T = \hat{h}_{\mu\nu}\eta^{\mu\nu}$ and the weighted double divergence $U = \frac{4}{3}\frac{h_{\mu\nu}k^\mu k^\nu}{\eta^{-1}(k,k)^2}$. Inserting this decomposition into the field equations yields

$$0 = \hat{E}_{\mu\nu} = 2c_1\eta^{-1}(k,k)S_{\mu\nu} + (2c_1+c_2+c_4)\eta^{-1}(k,k)2k_{(\mu}V_{\nu)} + \left(\frac{2}{3}(c_1+3c_3)\eta^{-1}(k,k)\eta_{\mu\nu} + \left(c_5 - \frac{2}{3}c_1\right)k_\mu k_\nu\right)T + \left(\frac{3}{4}\left(c_5 - \frac{2}{3}c_1\right)\eta^{-1}(k,k)^2\eta_{\mu\nu} + \frac{1}{2}(4c_1+3c_2+3c_4)\eta^{-1}(k,k)k_\mu k_\nu\right)U \tag{21}$$

To further analyse them we consider their contractions with k, their trace and their symmetric traceless part

$$0 = \hat{E}_{\mu\nu}k^\mu k^\nu = (2c_3+c_5)\eta^{-1}(k,k)^2T + \left(\frac{3}{4}c_5 + \frac{3}{2}(c_1+c_2+c_4)\right)\eta^{-1}(k,k)^3U, \tag{22}$$

$$0 = \hat{E}^\mu{}_\mu = (2c_1+8c_3+c_5)\eta^{-1}(k,k)T + \left(3c_5 + \frac{3}{2}(c_2+c_4)\right)\eta^{-1}(k,k)^2U, \tag{23}$$

$$0 = \hat{E}_{\mu\nu}k^\mu - \frac{k_\nu}{\eta^{-1}(k,k)}E_{\rho\sigma}k^\rho k^\sigma = (2c_1+c_2+c_4)\eta^{-1}(k,k)^2V_\nu, \tag{24}$$

$$0 = \hat{E}_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{k_\mu k_\mu}{\eta^{-1}(k,k)}\right)\hat{E}^\sigma{}_\sigma + \frac{1}{3}\left(\eta_{\mu\nu} - 4\frac{k_\mu k_\nu}{\eta^{-1}(k,k)}\right)\frac{\hat{E}_{\rho\sigma}k^\rho k^\sigma}{\eta^{-1}(k,k)} = 2c_1\eta^{-1}(k,k)S_{\mu\nu}. \tag{25}$$

To obtain the principal polynomial we can represent the decomposed equations as nearly diagonal matrix

$$\eta^{-1}(k,k)\begin{pmatrix} (2c_3+c_5)\eta^{-1}(k,k) & \left(\frac{3}{4}c_5+\frac{3}{2}(c_1+c_2+c_4)\right)\eta^{-1}(k,k)^2 & 0 & 0 \\ (2c_1+8c_3+c_5) & \left(3c_5+\frac{3}{2}(c_2+c_4)\right)\eta^{-1}(k,k) & 0 & 0 \\ 0 & 0 & (2c_1+c_2+c_4)\eta^{-1}(k,k) & 0 \\ 0 & 0 & 0 & 2c_1 \end{pmatrix}\begin{pmatrix} T \\ U \\ V_\nu \\ S_{\mu\nu}\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \tag{26}$$

and calculate its determinant

$$P(k) = (3\times 2^3)c_1^5(2c_1+c_2+c_4)^3(3c_5^2 - 4c_1^2 - 12c_3(c_2+c_4) - 4c_1(c_2+c_4+c_5+4c_3))\eta^{-1}(k,k)^{15}. \tag{27}$$

The necessary and nontrivial condition, solutions of the field equations have to satisfy, is, that their wave covectors k are such that $P(k) = 0$. From the above Eq. (27) we find that this implies $\eta^{-1}(k,k) = 0$ must be satisfied, i.e., all propagating modes propagate on the null cone of the Minkowski metric, or in other words, with the vacuum speed of light. We like to remark that this does not mean that necessarily all the modes must be propagating d.o.f. The conclusion here is only that if they are, then they are propagating with the speed of light.

In case one considers nonmetricity gravity theories with parameters c_1 to c_5, such that one ore more field equations (22)–(25) are solved trivially, for example $c_1 = 0$ for Eq. (25) or $2c_1 + c_2 + c_4 = 0$ for (24), the corresponding modes, for example the tensor or vector mode, can not be propagating d.o.f. of the theory. Their value must be defined by constraints which must be satisfied on initial data hypersurfaces. Such features become most visible in a full fledged Hamiltonian analysis of the theory in consideration, which shall be performed in the future.

To illustrate the statement just made we display the field equations for the values of the coefficients in the nonmetricity equivalent of general relativity $c_1 = -\frac{1}{4}, c_2 = \frac{1}{2}, c_3 = \frac{1}{4}, c_4 = 0$ and $c_5 = -\frac{1}{2}$

$$0 = 0, \tag{28}$$

$$0 = \hat{E}^{\mu}{}_{\mu} = \eta^{-1}(k,k)T - \frac{3}{4}\eta^{-1}(k,k)^2 U, \tag{29}$$

$$0 = \hat{E}_{\mu\nu}k^{\mu} - \frac{k_\nu}{\eta^{-1}(k,k)} E_{\rho\sigma}k^\rho k^\sigma = 0, \tag{30}$$

$$\begin{aligned} 0 &= \hat{E}_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{k_\mu k_\mu}{\eta^{-1}(k,k)}\right)\hat{E}^{\sigma}{}_{\sigma} \\ &\quad + \frac{1}{3}\left(\eta_{\mu\nu} - 4\frac{k_\mu k_\nu}{\eta^{-1}(k,k)}\right)\frac{\hat{E}_{\rho\sigma}k^\rho k^\sigma}{\eta^{-1}(k,k)} \\ &= \frac{1}{2}\eta^{-1}(k,k)S_{\mu\nu}. \end{aligned} \tag{31}$$

The vector modes V_μ can not be dynamical d.o.f. since their field equation is satisfied identically. The two scalar modes are coupled and the tensor modes decouple. For general relativity it is know that a thorough Hamilton analysis yields that only two propagating degrees of remain and all other are fixed by constraints.

As final remark of this section we would like to remark here that, as in the analysis of linearized teleparallel theories of gravity [27], higher order poles appear in the propagators of the scalar and vector modes due do the higher then linear appearance of $\eta^{-1}(k,k)$ in the Eqs. (22)–(24), which survive even in the non-metricity equivalent of general relativity for one of the scalar modes (29). On general grounds it is argued that the appearance of such terms signals the existence of ghost in the theory [35]. However the existence of such terms in the GR equivalent case shows that a more thorough analysis is required to identify if the ghost mode is coupling to the propagating field modes or not. The above mentioned complete Hamilton analysis of the theory considered here will also answer this question in the future.

IV. NEWMAN-PENROSE FORMALISM AND POLARIZATIONS

We now focus on the polarization of gravitational waves. As we have seen in the previous section, gravitational waves in quadratic symmetric teleparallel gravity are described by Minkowski null waves, independently of the choice of the parameters $c_1, \ldots, c_5$. This allows us to make use of the well-known Newman-Penrose formalism [36] in order to decompose the linearized field equations into components, which directly correspond to particular polarizations. We then employ the classification scheme detailed in [19,20], which characterizes the allowed polarizations of gravitational waves in a given gravity theory by a representation of the little group, which is the two-dimensional Euclidean group E(2) in case of null waves. In this section we determine the E(2) class of quadratic symmetric teleparallel gravity for all possible values of the parameters $c_1, \ldots, c_5$.

The main ingredient of the Newman-Penrose formalism is the choice of a particular complex double null basis of the tangent space. In the following, we will use the notation of [37] and denote the basis vectors by $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$. In terms of the canonical basis vectors of the Cartesian coordinate system they are defined as

$$\begin{aligned} l &= \partial_0 + \partial_3, & n &= \frac{1}{2}(\partial_0 - \partial_3), \\ m &= \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2), & \bar{m} &= \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2). \end{aligned} \tag{32}$$

We now consider a plane wave propagating in the positive x^3 direction, which corresponds to a single Fourier mode. The wave covector then takes the form $k_\mu = -\omega l_\mu$ and the metric perturbations can be written as

$$h_{\mu\nu} = H_{\mu\nu}e^{i\omega u}, \tag{33}$$

where we introduced the retarded time $u = x^0 - x^3$ and the wave amplitudes are denoted $H_{\mu\nu}$.

It follows from our choice of the matter coupling that test particles follow the geodesics of the metric, and hence the autoparallel curves of the Levi-Civita connection. The effect of a gravitational wave on an ensemble of test particles, or any other type of gravitational wave detector, therefore depends only on the Riemann tensor derived from the Levi-Civita connection. As shown in [20], the Riemann tensor of a plane wave is determined completely by the six so-called electric components. For the wave (33), these can be written as

$$\Psi_2 = -\frac{1}{6}R_{nlnl} = \frac{1}{12}\ddot{h}_{ll},$$
$$\Psi_3 = -\frac{1}{2}R_{nln\bar{m}} = -\frac{1}{2}\overline{R_{nlnm}} = \frac{1}{4}\ddot{h}_{l\bar{m}} = \frac{1}{4}\overline{\ddot{h}_{lm}},$$
$$\Psi_4 = -R_{n\bar{m}n\bar{m}} = -\overline{R_{nmnm}} = \frac{1}{2}\ddot{h}_{\bar{m}\bar{m}} = \frac{1}{2}\overline{\ddot{h}_{mm}},$$
$$\Phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2}\ddot{h}_{m\bar{m}}, \quad (34)$$

where dots denote derivatives with respect to u. We now examine which of the components (34) may occur for gravitational waves satisfying the linearized field equations (17).

Inserting the wave ansatz (33) and writing the gravitational field strength tensor $E_{\mu\nu}$ in the Newman-Penrose basis, we find that the five component equations

$$0 = E_{ll} = E_{lm} = E_{l\bar{m}} = E_{mm} = E_{\bar{m}\bar{m}}, \quad (35)$$

are satisfied identically, while the remaining five component equations take the form

$$0 = E_{nn} = 2c_5\ddot{h}_{m\bar{m}} - 2(c_2 + c_4 + c_5)\ddot{h}_{ln}, \quad (36a)$$

$$0 = E_{nm} = -(c_2 + c_4)\ddot{h}_{lm}, \quad (36b)$$

$$0 = E_{n\bar{m}} = -(c_2 + c_4)\ddot{h}_{l\bar{m}}, \quad (36c)$$

$$0 = E_{m\bar{m}} = c_5\ddot{h}_{ll}, \quad (36d)$$

$$0 = E_{ln} = -(c_2 + c_4)\ddot{h}_{ll}. \quad (36e)$$

Note in particular that the parameters c_1 and c_3 do not appear in these equations. This can be understood by taking a closer look at the linearized field equations (17). Here the constants c_1 and c_3 appear in front of terms of the form $\Box h_{\mu\nu}$, where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembert operator of the flat background. These terms vanish identically for the null wave (33), independently of the amplitudes $H_{\mu\nu}$, since the retarded time u is a light cone coordinate, and so $\Box e^{i\omega u} \equiv 0$. This can also be seen from the fact that the corresponding wave covector $k_\mu = -\omega l_\mu$ is null, i.e., $\eta^{-1}(k,k) = 0$, which is a necessary condition for solving the Eqs. (17) as shown in the preceding Sec. III. Hence, it is a direct consequence of the form of the propagator that the allowed polarizations depend only on the remaining parameters c_2, c_4, c_5. We now distinguish the following cases, which are also visualized in the diagram in Fig. 1 which we explain later in this section:

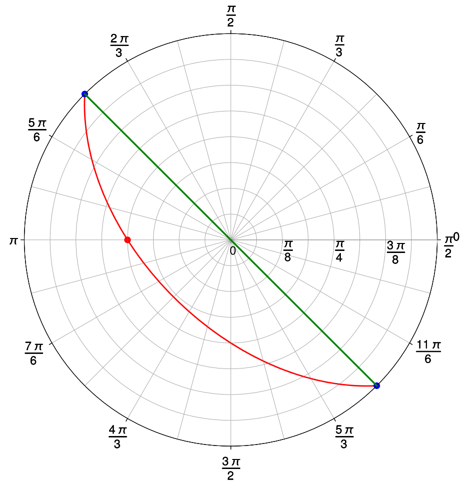

FIG. 1. Visualization of the parameter space. Blue Points: $c_2 + c_4 = c_5 = 0$, class II_6, 6 polarizations; green line: $c_2 + c_4 = 0, c_5 \neq 0$, class III_5, 5 polarizations; white area: $c_2 + c_4 \neq 0, c_2 + c_4 + c_5 \neq 0$, class N_3, 3 polarizations; red line: $c_2 + c_4 + c_5 = 0, c_5 \neq 0$, class N_2, 2 polarizations.

(i) $c_2 + c_4 = c_5 = 0$: In this case Eqs. (36d) and (36e) are satisfied identically for arbitrary amplitudes H_{ll}. For waves of this type the corresponding component $R_{nlnl} = -6\Psi_2$ of the Riemann tensor, which describes a longitudinally polarized wave mode, is allowed to be nonzero. Following the classification detailed in [20], they belong to the E(2) class II_6 with six polarizations. This case corresponds to the two blue points in Fig. 1, which is actually a line in the three-dimensional parameter space, and hence a single point in the projected parameter space shown in the diagram, which happens to lie on the cut $c_5 = 0$ and hence appears twice on the circular perimeter.

(ii) $c_2 + c_4 = 0$ and $c_5 \neq 0$: It follows from the second condition that Eq. (36d) prohibits a nonvanishing amplitude H_{ll}. Hence, there is no longitudinal mode Ψ_2. Equations (36b) and (36c) are satisfied identically for arbitrary amplitudes H_{lm} and $H_{l\bar{m}}$. It then follows that $R_{nln\bar{m}} = -2\Psi_3$, whose complex components describe two vector polarizations, is allowed to be nonzero. Waves of this type belong to the E(2) class III_5, and there are five polarizations. This case is represented by the green line in Fig. 1.

(iii) $c_2 + c_4 \neq 0$ and $c_2 + c_4 + c_5 \neq 0$: In this case it follows from Eqs. (36b), (36c), and (36e) that H_{ll}, H_{lm} and $H_{l\bar{m}}$ must vanish. Hence, the longitudinal mode Ψ_2 and vector modes Ψ_3 are prohibited. The remaining linearized field equation which allows for nonvanishing solutions is Eq. (36a). In particular, it allows for a non-vanishing amplitude $H_{m\bar{m}}$, and hence a nonvanishing component $R_{nmn\bar{m}} = -\Phi_{22}$ of the Riemann tensor. The corresponding scalar wave mode is called the breathing mode. This wave has the E(2) class N_3, exhibiting three polarizations. Almost all points of the parameter space, shown in white in Fig. 1, belong to this class.

(iv) $c_2 + c_4 + c_5 = 0$ and $c_5 \neq 0$: The linearized field equations (36) in the Newman-Penrose basis now yield the conditions $H_{ll} = H_{lm} = H_{l\bar{m}} = H_{m\bar{m}} = 0$. It thus follows that the longitudinal mode Ψ_2, the vector modes Ψ_3 and also the breathing mode Φ_{22} must vanish. The only unrestricted electric components of the Riemann tensor are therefore $R_{nmnm} = -\bar{\Psi}_4$ and its complex conjugate, corresponding to two tensor modes. The E(2) class of this wave is N_2, so that there are two polarizations. This case is shown as a red line in Fig. 1. Note in particular that STEGR, marked as a red point, belongs to this class, as one would expect.

We have visualized the aforementioned cases in Fig. 1, which we constructed as follows. We first made the assumption that at least one of the parameters c_2, c_4, c_5 is nonvanishing and introduced normalized parameters

$$\tilde{c}_i = \frac{c_i}{\sqrt{c_2^2 + c_4^2 + c_5^2}} \tag{37}$$

for $i = 2, 4, 5$. One easily checks that the E(2) classes we found only depend on these normalized parameters, except for the case $c_2 = c_4 = c_5 = 0$ belonging to class II_6. We then introduced polar coordinates (θ, ϕ) on the unit sphere to express the parameters as

$$\tilde{c}_2 = \sin\theta\cos\phi, \quad \tilde{c}_4 = \sin\theta\sin\phi, \quad \tilde{c}_5 = \cos\theta. \tag{38}$$

Since the E(2) class is the same for antipodal points on the parameter sphere, we restrict ourselves to the hemisphere $\tilde{c}_5 \geq 0$, and hence $0 \leq \theta \leq \frac{\pi}{2}$; this is equivalent to identifying antipodal points on the sphere and working with the projective sphere instead, provided that we also identify antipodal points on the equator $\tilde{c}_5 = 0$. We then considered (θ, ϕ) as polar coordinates on the plane in order to draw the diagram shown in Fig. 1. Note that antipodal points on the perimeter, such as the two blue points, are identified with each other, since they describe the same class of theories; in fact, these blue points correspond to a straight line passing through and including the origin $c_2 = c_4 = c_5 = 0$.

This concludes our discussion of gravitational wave polarizations. We have seen that the parameters c_1, c_3 have no influence on the allowed polarizations, while depending on the parameters c_2, c_4, c_5 we obtain the E_2 class II_6, III_5, N_3 or N_2, with N_3 filling most of the parameter space. We have also seen that there exists a four parameter family of theories besides STEGR which is of class N_2 and thus exhibits the same two tensor modes as in general relativity. Theories in this class therefore cannot be distinguished from general relativity by observing the polarizations of gravitational waves alone.

V. CONCLUSION

We studied the propagation of gravitational waves in the most general class of symmetric teleparallel gravity theories whose action is quadratic in the nonmetricity tensor. The wave we considered is modeled as a linear perturbation of a Minkowski background metric in the coincident gauge, in which the coefficients of the flat, symmetric connection vanish. We derived the principal polynomial of the linearized field equations and found that gravitational waves propagate at the speed of light, i.e., their wave covector must be given by a null covector of the Minkowski spacetime background. Further, we made use of the Newman-Penrose formalism to derive the possible polarizations of gravitational waves. Our results show that the two tensor polarizations, which are present also in general relativity, are allowed for the whole class of theories we considered, while additional modes—two vector modes and up to two scalar modes—may be present for particular models within this class. We found that the symmetric teleparallel equivalent of general relativity is not the unique theory exhibiting exactly two polarizations, but there is a four parameter family of theories with the same property. It thus follows that observations of gravitational wave polarizations may only give partial results on the parameter space of these theories.

We remark that although we restricted our analysis to theories whose action is quadratic in the nonmetricity tensor, our results are valid for a significantly larger class of theories. This is due to the fact that the nonmetricity is linear in the metric perturbations, so that the action is already quadratic in the perturbations. Hence, any higher order correction terms would have no influence on the linearized field equations. This is shown, e.g., in [27] for the polarizations of gravitational waves in a more general class of theories, whose Lagrangian is defined by a free function of the five scalar terms quadratic in nonmetricity considered in this article.

Another possible class of extensions is to consider additional fields nonminimally coupled to nonmetricity and to study their influence both on the speed and the polarization of gravitational waves. A canonical example is given by scalar-nonmetricity theories [18,30] constructed from the STEGR nonmetricity scalar and an additional scalar field, where one would expect the presence of an additional scalar mode compared to general relativity as it is also the case for scalar-curvature gravity. These theories can be extended further by replacing the STEGR nonmetricity scalar with the general quadratic nonmetricity scalar which defined the Lagrangian considered in this article.

Finally, another interesting extension of our work would be to study gravitational waves as a perturbation to a Friedmann-Lemaitre-Robertson-Walker metric. One may expect that in this case also nonmetricity terms of higher then quadratic order in the Lagrangian would affect the result, as they would lead to modifications of the background dynamics. This modified expansion history might thus also leave an imprint on the observed gravitational waves propagating in a cosmological background.

In conclusion, the formulation of theories of gravity in the symmetric teleparallel/nonmetricity language allows for promising extensions of GR which are consistent with the basic gravitational wave observations. An analysis of further observables in this particular class of theories, like the calculation of PPN parameters, rotational curves of galaxies and the cosmological expansion of the universe, will explore their viability further in the future.

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Chapter 8

Covariant formulation of scalar-torsion gravity

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Contents

Covariant formulation of scalar-torsion gravity

Manuel Hohmann,* Laur Järv,† and Ulbossyn Ualikhanova‡
Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia

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We consider a generalized teleparallel theory of gravitation, where the action contains an arbitrary function of the torsion scalar and a scalar field, $f(T,\phi)$, thus encompassing the cases of $f(T)$ gravity and a nonminimally coupled scalar field as subclasses. The action is manifestly Lorentz invariant when besides the tetrad one allows for a flat but nontrivial spin connection. We derive the field equations and demonstrate how the antisymmetric part of the tetrad equations is automatically satisfied when the spin connection equation holds. The spin connection equation is a vital part of the covariant formulation, since it determines the spin connection associated with a given tetrad. We discuss how the spin connection equation can be solved in general and provide the cosmological and spherically symmetric examples. Finally, we generalize the theory to an arbitrary number of scalar fields.

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I. INTRODUCTION

General relativity, which encodes the effects of gravity in the curvature, has been remarkably successful in describing a wide range of phenomena. However, to give a satisfactory account of cosmology would necessitate an *ad hoc* inclusion of major extra matter ingredients responsible for triggering the rapid expansion of the early Universe (inflation), for the observed structure formation and galactic rotations (dark matter), as well as for the accelerated expansion of the present Universe (dark energy). These problems keep motivating the study of extensions of general relativity, whereby many different theories and approaches have been proposed and considered [1–3]. Perhaps the most popular and promising of those are generalizing the gravitational action to be some function of the curvature scalar, $f(R)$, and the inclusion of a nonminimally coupled scalar field (scalar-tensor gravity). Such models arise naturally when quantum effects are taken into account [4], but they also happen to be favored in, e.g., describing the spectrum of fluctuations from inflation [5–7] and naturally permit the effective barotropic index $w_{\mathrm{eff}} < -1$, as recent observations seem to suggest [8,9].

From the geometric point of view, curvature is not a property of spacetime *per se* but a property of the chosen connection. General relativity adopts the Levi-Cività connection, which implies vanishing torsion and nonmetricity but allows nontrivial curvature. An alternative approach, first probed by Einstein himself [10], would be to take the Weitzenböck connection, which sets curvature and nonmetricity to zero but allows nontrivial torsion. The ensuing theory where R is replaced by the torsion scalar T in the action is known as teleparallel equivalent of general relativity [11–14], since its observational predictions exactly match those of general relativity. Things get more interesting, however, when one considers an extended theory, e.g., generalizing the action to $f(T)$ [15,16] or introducing a nonminimally coupled scalar field [17]. It turns out that the extended theories based on torsion differ from their counterparts based on curvature. This realization launched a flurry of studies regarding dark energy, inflation, black holes, and other solutions and properties of the extended teleparallel theories [18].

There was a catch, though. Teleparallel gravity is usually formulated in the formalism of a tetrad and spin connection, the latter being independent of the former. In the teleparallel equivalent of general relativity, the spin connection does not affect the tetrad field equations and can be chosen arbitrarily [12]. Interpolating this property to the extensions like $f(T)$ or scalar-torsion gravity leads to a problematic result, for the action fails to be locally Lorentz invariant [19,20], violating the basics of the tetrad formalism. It was argued that therefore these theories implied preferred frame effects and acausality and were inhabited by extra spurious degrees of freedom (d.o.f.) [21–24]. The Lorentz invariance issue is fixed in the covariant formulation of the theory [25], which allows a nontrivial spin connection compatible with vanishing curvature, i.e., flat spin connection (but the question of the d.o.f. is still under investigation [26,27]).

After accepting a nonvanishing spin connection, there arises the obvious question of how to determine it. An answer to the latter came only recently in the context of

*manuel.hohmann@ut.ee
†laur.jarv@ut.ee
‡ulbossyn.ualikhanova@ut.ee

$f(T)$ gravity. Namely, variation of the action with respect to the spin connection by carefully maintaining the flatness property yields an equation which fixes the remaining six components of the spin connection [28,29]. This equation involves only the first derivatives of the spin connection, so one may ask whether the spin connection is an independent dynamical quantity in $f(T)$ gravity. One cannot set the spin connection arbitrarily to zero but for a given tetrad must make sure the spin connection satisfies the respective condition. As a pleasant byproduct, it turns out that when the condition on the spin connection is satisfied, the antisymmetric part of the tetrad field equations vanishes automatically [28]. It is remarkable that this feature also holds in much more general theories of torsion [30,31].

In the current paper, we consider a generalized form of scalar-torsion gravity where the action involves an arbitrary function of the torsion scalar and a scalar field, $f(T,\phi)$, thus including both the $f(T)$ and nonminimally coupled scalar models as particular subcases. After recalling the main geometric formulas and defining the action in Sec. II, we derive the field equations by varying the action with respect to the tetrad, the flat spin connection, and the scalar field in Sec. III. The spin connection equation generalizes the result found for $f(T)$ gravity [28,29] and shares the property that it automatically makes the antisymmetric part of the tetrad equations be identically satisfied, as we demonstrate explicitly. In this section, we also show how the spin connection equation is instrumental in guaranteeing the conservation of matter energy momentum and how the field configurations with constant T and ϕ reduce the equations to those of general relativity. In Sec. IV, we discuss different possibilities of how to solve the spin connection equation and concisely present the examples of simple diagonal tetrads with the associated spin connection corresponding to cosmologies of spatially flat, spherical, and hyperbolic homogeneous and isotropic spacetimes; Kasner anisotropic spacetime; as well as a general static spherically symmetric spacetime. Later in Sec. V we develop a further generalization of the theory to multiple scalar fields and give the respective field equations. The article ends in Sec. VI with a summary and discussion.

II. SCALAR-TORSION MODEL

We start our discussion with a brief outline of the scalar-torsion model we consider in this article. In Sec. II A, we define the kinematic variables of the theory and briefly review the definition of the terms we will use in the action. The action itself is presented in Sec. II B. Finally, in Sec. II C, we list a number of special cases of our model that have been discussed in the literature.

A. Kinematic variables

We derive our model from the covariant formulation of teleparallel gravity [25,28], in which the basic variables in the gravity sector are a tetrad $h^a{}_\mu$ and a spin connection $\overset{\bullet}{\omega}{}^a{}_{b\mu}$, and augment these by adding a scalar field ϕ. (Here, the greek indices correspond to the spacetime coordinates, while the latin indices pertain to an orthonormal frame with Lorentzian metric η_{ab}.) For a given spacetime metric,

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu, \tag{1}$$

the corresponding tetrad is not defined uniquely, but only up to a local Lorentz transformation which transforms the spin connection as well,

$$h'^a{}_\mu = \Lambda^a{}_b h^b{}_\mu, \qquad \overset{\bullet}{\omega}{}'^a{}_{b\mu} = \Lambda^a{}_c \overset{\bullet}{\omega}{}^c{}_{c\mu} \Lambda_b{}^d + \Lambda^a{}_c \partial_\mu \Lambda_b{}^c; \tag{2}$$

here, $\Lambda_a{}^b$ is the inverse of the Lorentz transformation matrix $\Lambda^a{}_b$. The transformation (2) just reflects the possibility of switching between different local observers. Demanding that the spin connection vanishes is a particular gauge choice and in general means picking a specific (class of) observer(s) among the others. The bullet (•) denotes quantities related to the teleparallel spin connection, which is chosen to be flat, i.e., having vanishing curvature,

$$\overset{\bullet}{R}{}^a{}_{b\mu\nu} = \partial_\mu \overset{\bullet}{\omega}{}^a{}_{b\nu} - \partial_\nu \overset{\bullet}{\omega}{}^a{}_{b\mu} + \overset{\bullet}{\omega}{}^a{}_{c\mu} \overset{\bullet}{\omega}{}^c{}_{b\nu} - \overset{\bullet}{\omega}{}^a{}_{c\nu} \overset{\bullet}{\omega}{}^c{}_{b\mu}. \tag{3}$$

The spin connection defines a spacetime connection with connection coefficients

$$\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu} = h_a{}^\rho \overset{\bullet}{D}_\nu h^a{}_\mu = h_a{}^\rho (\partial_\nu h^a{}_\mu + \overset{\bullet}{\omega}{}^a{}_{b\nu} h^b{}_\mu), \tag{4}$$

where $\overset{\bullet}{D}_\mu$ is the gauge covariant Fock-Ivanenko derivative and $h_a{}^\mu$ denotes the inverse tetrad, which satisfies $h^a{}_\mu h_b{}^\mu = \delta^a_b$ and $h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu$. The connection coefficients $\overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu}$ are defined such that the total covariant derivative of the tetrad vanishes (metricity condition),

$$0 = \overset{\bullet}{\nabla}_\mu h^a{}_\nu = \partial_\mu h^a{}_\nu + \overset{\bullet}{\omega}{}^a{}_{b\mu} h^b{}_\nu - \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} h^a{}_\rho. \tag{5}$$

Note that we adopt the convention that the last index on the connection coefficients is the "derivative" index, while the first pair of indices is the "endomorphism" indices. This connection in general has nonvanishing torsion

$$T^\rho{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^\rho{}_{\nu\mu} - \overset{\bullet}{\Gamma}{}^\rho{}_{\mu\nu}. \tag{6}$$

We further use an open circle (∘) to denote quantities related to the Levi-Cività connection $\overset{\circ}{\nabla}_\mu$ of the metric (1), the connection coefficients of which are given by

$$\overset{\circ}{\Gamma}{}^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{7}$$

In contrast to the teleparallel connection, it has vanishing torsion but in general nonvanishing curvature. The difference

$$K^{\rho}{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}) \quad (8)$$

is called the contortion tensor. For convenience, we will denote the partial derivatives using the comma notation, e.g., $\partial_\mu\phi \equiv \phi_{,\mu}$.

B. Action

We now come to the action of our model, which we write in the form

$$S = S_g[h^a{}_\mu, \overset{\bullet}{\omega}{}^a{}_{b\mu}, \phi] + S_m[h^a{}_\mu, \chi], \quad (9)$$

where S_g denotes the gravitational part, while S_m denotes the matter part, and matter fields are collectively denoted by χ. For the gravitational part, we choose the action

$$S_g = \frac{1}{2\kappa^2}\int_M d^4x h[f(T,\phi) + Z(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}], \quad (10)$$

which depends on two free functions f and Z, while $2\kappa^2 = 16\pi G_N$ sets the Newtonian gravitational constant. Here, $h = \det(h^a{}_\mu)$ denotes the determinant of the tetrad, while T is the torsion scalar defined as

$$T = \frac{1}{2}T^\rho{}_{\mu\nu}S_\rho{}^{\mu\nu}, \quad (11)$$

with the superpotential

$$S_\rho{}^{\mu\nu} = K^{\mu\nu}{}_\rho - \delta^\mu_\rho T_\sigma{}^{\sigma\nu} + \delta^\nu_\rho T_\sigma{}^{\sigma\mu}. \quad (12)$$

Here, the reader should be alerted that we use a convention where the definition of the torsion scalar (11) contains a factor $\frac{1}{2}$, while the superpotential (12) does not carry such a factor [28]. Other authors, often in the field of $f(T)$ gravity and cosmology, use a different convention by including the factor $\frac{1}{2}$ in the definition of the superpotential and leaving the torsion scalar without (e.g., Ref. [16]). Finally, in the literature on the teleparallel equivalent of general relativity, there is yet another convention which puts the factor $\frac{1}{2}$ directly into the gravitational action, while keeping the definitions of the torsion scalar and superpotential free of it [12,25]. One should be careful, as these choices affect the respective factors in the field equations as well.

The gravitational part S_g of the action is complemented by a matter part S_m, which is assumed to depend only on the tetrad $h^a{}_\mu$ and a set of matter fields χ, the precise content of which is not relevant for the purpose of this article. The only relevant quantity is the energy-momentum tensor $\Theta_a{}^\mu$ defined from the variation of the matter action with respect to the tetrad,

$$\delta_h S_m = -\int_M d^4x h\Theta_a{}^\mu \delta h^a{}_\mu. \quad (13)$$

We demand that S_m is invariant under local Lorentz transformations (2), which then implies that the energy-momentum tensor is symmetric [12],

$$0 = \Theta_{[\mu\nu]} = h^a{}_{[\mu}g_{\nu]\rho}\Theta_a{}^\rho. \quad (14)$$

Further, we demand that S_m is invariant under diffeomorphisms, which implies that $\Theta_{\mu\nu}$ is covariantly conserved with respect to the Levi-Cività connection,

$$\overset{\circ}{\nabla}_\mu\Theta^{\mu\nu} = 0. \quad (15)$$

We will discuss this aspect further in Sec. III E.

C. Special cases

Let us note that the action (10) encompasses several previously studied scalar-torsion theories as subclasses, e.g.:

(i) teleparallel equivalent of general relativity with a minimally coupled scalar field (quintessence),

$$f(T,\phi) = T - 2\kappa^2V(\phi), \qquad Z(\phi) = \kappa^2; \quad (16)$$

(ii) $f(T)$ gravity [15,16],

$$f(T,\phi) = f(T), \qquad Z(\phi) = 0; \quad (17)$$

(iii) minimally coupled scalar field in $f(T)$ gravity [32]

$$f(T,\phi) = f(T) - 2\kappa^2V(\phi), \qquad Z(\phi) = \kappa^2; \quad (18)$$

(iv) teleparallel dark energy [17]

$$f(T,\phi) = (1 + 2\kappa^2\xi\phi^2)T - 2\kappa^2V(\phi),$$
$$Z(\phi) = \kappa^2; \quad (19)$$

(v) generalized teleparallel dark energy [33]

$$f(T,\phi) = (1 + 2\kappa^2\xi f(\phi))T - 2\kappa^2V(\phi),$$
$$Z(\phi) = \kappa^2, \quad (20)$$

or [34]

$$f(T,\phi) = (1 + 2\kappa^2\xi f(\phi))F(T) - 2\kappa^2V(\phi),$$
$$Z(\phi) = \kappa^2; \quad (21)$$

(vi) Brans-Dicke–like action with constant kinetic term coupling [23]

$$f(T,\phi) = f(\phi)T - 2\kappa^2V(\phi), \quad Z(\phi) = \omega, \quad (22)$$

or with dynamical kinetic term coupling [35]

$$f(T,\phi) = \phi T - 2\kappa^2 V(\phi), \quad Z(\phi) = \frac{\omega(\phi)}{\phi}; \quad (23)$$

(vii) the scalar-torsion equivalent to $F(T)$ gravity [23]

$$f(T,\phi) = \frac{dF}{d\phi}T - \phi\frac{dF}{d\phi} - F(\phi), \qquad Z(\phi) = 0. \quad (24)$$

The actions (20)–(23) can be transformed into each other by a suitable redefinition of the scalar field ϕ [35,36]. In the last case (24), the scalar field is actually a nondynamical auxiliary field [23]. For the theories based only on the torsion scalar (11) and without introducing derivative couplings between the scalar field and torsion [37] or a nonstandard kinetic term for the scalar field [38], the action (10) is in the most general form. As a remark, we note that, in contrast to the scalar-curvature theories, the scalar-torsion action (10) is not invariant under conformal rescalings of the tetrad, for these introduce a coupling between the scalar field and vector torsion [39], leading to a much broader class of theories [40,41].

III. EQUATIONS OF THE THEORY

With the preliminaries in place, we will now go on to vary the action (10) with respect to the tetrad in Sec. III A, the flat spin connection in Sec. III B, and the scalar field in Sec. III C and write down the ensuing field equations. In Sec. III D, we show in detail that the condition arising from the variation with respect to the spin connection is equivalent to the antisymmetric part of the tetrad field equations. In Sec. III E, we demonstrate that combining the equations also leads to an expression for the matter energy-momentum conservation, and the spin connection equation is instrumental in guaranteeing that. Finally, in Sec. III F, we explicate how the symmetric part of the tetrad field equations reduces to general relativity for configurations with a constant torsion scalar and scalar field.

A. Tetrad field equation

We start with the derivation of the tetrad field equation, which is obtained by variation of the action (9) with respect to the tetrad $h^a{}_\mu$. From the variation of the gravitational part (10), we obtain

$$\begin{aligned}
\delta_h S_g &= \frac{1}{2\kappa^2}\int_M h\{[(f + Zg^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma})h_a{}^\mu - 2S^{\rho\sigma\mu}T_{\rho\sigma\nu}h_a{}^\nu f_T - 2Zg^{\mu\nu}\phi_{,\nu}\phi_{,\rho}h_a{}^\rho]\delta h^a{}_\mu - 2f_T S_\rho{}^{\mu\nu}h_a{}^\rho \dot{D}_\nu \delta h^a{}_\mu\}d^4x \\
&= \frac{1}{2\kappa^2}\int_M h\left[(f + Zg^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma})h_a{}^\mu - 2S^{\rho\sigma\mu}T_{\rho\sigma\nu}h_a{}^\nu f_T - 2Zg^{\mu\nu}\phi_{,\nu}\phi_{,\rho}h_a{}^\rho + \frac{2}{h}\dot{D}_\nu(hf_T S_\rho{}^{\mu\nu}h_a{}^\rho)\right]\delta h^a{}_\mu d^4x \\
&= \frac{1}{2\kappa^2}\int_M h[(f + Zg^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma})g_{\mu\nu} + 2S^{\rho\sigma}{}_\mu(K_{\rho\nu\sigma} - T_{\rho\sigma\nu})f_T - 2Z\phi_{,\mu}\phi_{,\nu} + 2\mathring{\nabla}_\rho(f_T S_{\nu\mu}{}^\rho)]g^{\mu\tau}h_a{}^\nu \delta h^a{}_\tau d^4x. \qquad (25)
\end{aligned}$$

From the second line, one can find the left-hand side of the tetrad field equations with one Lorentz index and one spacetime index; however, we skip this step here and proceed with the third line, which is already written in lower spacetime indices only. Using the definitions (12) of the superpotential and (8) of the contortion, as well as the energy-momentum tensor (13), the resulting field equation can be written as

$$\begin{aligned}
&\frac{1}{2}fg_{\mu\nu} + \mathring{\nabla}_\rho(f_T S_{\nu\mu}{}^\rho) + \frac{1}{2}f_T\left(T^\rho{}_{\rho\sigma}T^\sigma{}_{\mu\nu} + 2T^\rho{}_{\rho\sigma}T_{(\mu\nu)}{}^\sigma - \frac{1}{2}T_{\mu\rho\sigma}T_\nu{}^{\rho\sigma} + T_{\mu\rho\sigma}T^{\rho\sigma}{}_\nu\right) \\
&- Z\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}Zg_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} = \kappa^2\Theta_{\mu\nu}. \qquad (26)
\end{aligned}$$

Let us remark that, contrary to the $f(R)$ theories, these equations contain derivatives of the tetrad (or metric) only up to the second order. Therefore, several issues and subtleties characteristic of $f(R)$ gravity [1] do not arise. This nice feature has been one of the motivations to study $f(T)$ gravity [18] and persists in the more general $f(T,\phi)$ models considered here as well.

In analogy to the teleparallel equivalent of general relativity [12], one may notice that the left-hand side of the tetrad field equations allows for the definition of a conserved gravitational energy-momentum pseudotensor. Given a minimally coupled scalar field, this can further be split into contributions from the tetrad and the scalar field. However, in the case of nonminimal coupling, the energy momenta of the tetrad and scalar fields are entwined, since the function f contains both of them.

B. Connection field equation

We now come to the variation of the action (9) with respect to the spin connection $\dot{\omega}^a{}_{b\mu}$. Here, we follow the constrained variation prescription [28], where the variation is restricted to a gauge covariant derivative of the form

$\delta\dot{\omega}^{ab}{}_{\mu} = -\dot{D}_{\mu}\lambda^{ab}$ with $\lambda^{(ab)} = 0$, such that the varied spin connection remains flat, $\delta_{\omega}\dot{R}^{a}{}_{b\mu\nu} = 0$. Note that the spin connection enters the action (10) only through the torsion scalar, the variation of which can be written in the very simple form [28]

$$\begin{aligned}\delta_{\omega}T &= \delta_{\omega}(-\mathring{R} + 2\mathring{\nabla}_{\mu}T_{\nu}{}^{\nu\mu}) = 2\mathring{\nabla}_{\mu}\delta_{\omega}T_{\nu}{}^{\nu\mu}\\ &= 2\mathring{\nabla}_{\mu}(h_{a}{}^{\nu}h_{b}{}^{\mu}\delta\omega^{ab}{}_{\nu}),\end{aligned} \tag{27}$$

using the relation between the torsion scalar T and the Ricci scalar $\mathring{R}$ and the fact that the latter depends only on the tetrad but not on the teleparallel spin connection. We then obtain the variation of the gravitational part of the action as

$$\begin{aligned}\delta_{\dot{\omega}}S_{g} &= \frac{1}{2\kappa^{2}}\int_{M} hf_{T}\delta_{\omega}T d^{4}x\\ &= -\frac{1}{\kappa^{2}}\int_{M} hf_{T}\mathring{\nabla}_{\mu}(h_{a}{}^{\nu}h_{b}{}^{\mu}\dot{D}_{\nu}\lambda^{ab})d^{4}x\\ &= \frac{1}{\kappa^{2}}\int_{M} h\partial_{\nu}f_{T}h_{a}{}^{\mu}h_{b}{}^{\nu}\dot{D}_{\mu}\lambda^{ab}d^{4}x\\ &= \frac{1}{\kappa^{2}}\int_{M} h(-\dot{\nabla}_{\mu}\dot{\nabla}_{\nu}f_{T} + K^{\rho}{}_{\mu\rho}\partial_{\nu}f_{T})h_{a}{}^{\mu}h_{b}{}^{\nu}\lambda^{ab}d^{4}x,\end{aligned} \tag{28}$$

where we have twice performed integration by parts. Due to the antisymmetry of λ^{ab}, as well as the relation

$$\dot{\nabla}_{\mu}\dot{\nabla}_{\nu}f_{T} - \dot{\nabla}_{\nu}\dot{\nabla}_{\mu}f_{T} = T^{\rho}{}_{\nu\mu}\partial_{\rho}f_{T} \tag{29}$$

for the commutator of covariant derivatives acting on the scalar function f_T, the field equation reads

$$0 = -\dot{\nabla}_{[\mu}\dot{\nabla}_{\nu]}f_{T} + \partial_{[\nu}f_{T}K^{\rho}{}_{\mu]\rho} = \frac{3}{2}\partial_{[\rho}f_{T}T^{\rho}{}_{\mu\nu]}. \tag{30}$$

In the last term, the notation means that one first needs to antisymmetrize with respect all three lower indices and then sum over with the repeating upper index. By expanding the torsion into the tetrad and the spin connection, one can also write this equation as

$$\partial_{\mu}f_{T}[\partial_{\nu}(hh_{[a}{}^{\mu}h_{b]}{}^{\nu}) + 2hh_{c}{}^{[\mu}h_{[a}{}^{\nu]}\dot{\omega}^{c}{}_{b]\nu}] = 0, \tag{31}$$

which is of the same form as the corresponding equation in $f(T)$ gravity [28,29]. However, note that in the scalar-torsion model the function f also depends on the scalar field ϕ, so the derivative reads

$$\partial_{\mu}f_{T} = f_{TT}\partial_{\mu}T + f_{T\phi}\partial_{\mu}\phi, \tag{32}$$

whereas the second term is not present in $f(T)$ gravity. This second term vanishes if and only if the scalar field is minimally coupled, $f_{T\phi} = 0$, or in a field configuration with a uniform scalar field.

Equation (31) contains only the first derivatives of the spin connection which appear since we are taking the derivatives of the torsion scalar (11) in f_T. Therefore, this equation can be interpreted as a condition to determine the spin connection components associated with a given tetrad. It is a feature characteristic of the generalized teleparallel framework, since in the $f(R)$ and scalar-curvature theories the spin connection is completely determined by the tetrad via the Levi-Cività prescription.

C. Scalar field equation

Finally, we come to the scalar field equation. Variation of the gravitational part (10) of the action with respect to the scalar field yields

$$\begin{aligned}\delta_{\phi}S_{g} &= \frac{1}{2\kappa^{2}}\int_{M} h[(f_{\phi} + Z_{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu})\delta\phi + 2Zg^{\mu\nu}\phi_{,\nu}\delta\phi_{,\mu}]d^{4}x\\ &= \frac{1}{2\kappa^{2}}\int_{M} h[f_{\phi} + Z_{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2g^{\mu\nu}\mathring{\nabla}_{,\mu}(Z\phi_{,\nu})]\delta\phi d^{4}x,\end{aligned} \tag{33}$$

where we have performed integration by parts to arrive at the second line. The corresponding variation of the matter part of the action vanishes, $\delta_{\phi}S_{m} = 0$, since we do not consider any direct coupling between the scalar field and matter fields. Hence, the field equation does not contain a source term and can finally be brought into the form

$$f_{\phi} - Z_{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2Z\mathring{\Box}\phi = 0, \tag{34}$$

where $\mathring{\Box} = g^{\mu\nu}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}$ is the d'Alembert operator.

Let us note that, contrary to the scalar-curvature theories, the second derivatives of the tetrad (or metric) do not appear in the scalar field equation, and thus the procedure of "debraiding" (cf. Ref. [42]) is not necessary. As a consequence, when the matter Lagrangian does not explicitly involve the scalar field, the scalar field equation remains without a matter contribution as a source term, e.g., the trace of the matter energy-momentum tensor. Therefore, in scalar-torsion gravity, the chameleon screening mechanism does not work as in the scalar-curvature theories [43,44], unless one introduces a coupling of the scalar field to matter [45] or a boundary term [39–41].

D. Relation between field equations

It has recently been shown for $f(T)$ gravity and more general teleparallel gravity theories with second order field equations that the antisymmetric part of the tetrad field equations is identical to the connection field equations, so the flat spin connection is a pure gauge d.o.f. corresponding to the local Lorentz invariance of the action [28–31]. We now show that the same holds true also for the class of scalar-torsion theories presented here. For this purpose, we consider the antisymmetric part of the tetrad field equations (26), which reads

$$\begin{aligned}0 &= 2\overset{\circ}{\nabla}_\rho(f_T S_{[\nu\mu]}{}^\rho) + f_T(T^\rho{}_{\rho\sigma}T^\sigma{}_{\mu\nu} + T^{\rho\sigma}{}_{[\nu}T_{\mu]\rho\sigma}) \\ &= 3\partial_{[\rho}f_T T^\rho{}_{\mu\nu]} + f_T(3\overset{\circ}{\nabla}_{[\rho}T^\rho{}_{\mu\nu]} + T^\rho{}_{\rho\sigma}T^\sigma{}_{\mu\nu} - T^{\rho\sigma}{}_{[\mu}T_{\nu]\rho\sigma}),\end{aligned} \tag{35}$$

where we have used the definitions (12) of the superpotential and (8) of the contortion. For the first term in brackets, we now expand the covariant derivative into Christoffel symbols,

$$\overset{\circ}{\nabla}_{[\rho}T^\rho{}_{\mu\nu]} = \partial_{[\rho}T^\rho{}_{\mu\nu]} + \overset{\circ}{\Gamma}{}^\rho{}_{\sigma[\rho}T^\sigma{}_{\mu\nu]} - \overset{\circ}{\Gamma}{}^\sigma{}_{[\mu\rho}T^\rho{}_{|\sigma|\nu]} - \overset{\circ}{\Gamma}{}^\sigma{}_{[\nu\rho}T^\rho{}_{\mu]\sigma}, \tag{36}$$

where the last two terms vanish due to the symmetry of the Christoffel symbols in their lower indices, since the Levi-Cività connection is torsion free. For the remaining term, we express the Levi-Cività connection through the teleparallel connection and the contortion,

$$\overset{\circ}{\Gamma}{}^\rho{}_{\sigma[\rho}T^\sigma{}_{\mu\nu]} = \overset{\bullet}{\Gamma}{}^\rho{}_{\sigma[\rho}T^\sigma{}_{\mu\nu]} - K^\rho{}_{\sigma[\rho}T^\sigma{}_{\mu\nu]}. \tag{37}$$

A direct calculation then shows that

$$\partial_{[\rho}T^\rho{}_{\mu\nu]} + \overset{\bullet}{\Gamma}{}^\rho{}_{\sigma[\rho}T^\sigma{}_{\mu\nu]} = \overset{\bullet}{R}{}^\rho{}_{[\mu\nu]\rho} = 0, \tag{38}$$

which vanishes, since the teleparallel spin connection and hence its spacetime connection are flat,

$$\overset{\bullet}{R}{}^\rho{}_{\sigma\mu\nu} = h_a{}^\rho h^b{}_\sigma \overset{\bullet}{R}{}^a{}_{b\mu\nu} = 0. \tag{39}$$

We are thus left with the only remaining term, which reads

$$\overset{\circ}{\nabla}_{[\rho}T^\rho{}_{\mu\nu]} = -K^\rho{}_{\sigma[\rho}T^\sigma{}_{\mu\nu]} = -\frac{1}{3}(T^\rho{}_{\mu\nu}T^\sigma{}_{\sigma\rho} - T^{\rho\sigma}{}_{[\mu}T_{\nu]\rho\sigma}). \tag{40}$$

Hence, the brackets in the second line of the antisymmetric field equation (35) vanish. This equation thus reduces to

$$\partial_{[\rho}f_T T^\rho{}_{\mu\nu]} = 0, \tag{41}$$

which indeed agrees with the connection field equations (30).

We are finally left with the symmetric part of the field equations:

$$\begin{aligned}&\frac{1}{2}fg_{\mu\nu} + \overset{\circ}{\nabla}_\rho(f_T S_{(\mu\nu)}{}^\rho) + \frac{1}{2}f_T\left(2T^\rho{}_{\rho\sigma}T_{(\mu\nu)}{}^\sigma - \frac{1}{2}T_{\mu\rho\sigma}T_\nu{}^{\rho\sigma} + T^{\rho\sigma}{}_{(\mu}T_{\nu)\rho\sigma}\right) \\ &\quad - Z\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}Zg_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} = \kappa^2\Theta_{\mu\nu}.\end{aligned} \tag{42}$$

Using the relation

$$S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} = -2T^\rho{}_{\rho\sigma}T_{(\mu\nu)}{}^\sigma + \frac{1}{2}T_{\mu\rho\sigma}T_\nu{}^{\rho\sigma} - T^{\rho\sigma}{}_{(\mu}T_{\nu)\rho\sigma}, \tag{43}$$

it can also be written as

$$\begin{aligned}&\frac{1}{2}fg_{\mu\nu} + \overset{\circ}{\nabla}_\rho(f_T S_{(\mu\nu)}{}^\rho) - \frac{1}{2}f_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - Z\phi_{,\mu}\phi_{,\nu} \\ &\quad + \frac{1}{2}Zg_{\mu\nu}g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} = \kappa^2\Theta_{\mu\nu}.\end{aligned} \tag{44}$$

These equations remain to be solved independently from the antisymmetric part (41).

To recap, the dynamical equations for the theory (9) are the ten symmetric equations (44) for the tetrad components, the scalar field equation (34), and the matter equations of motion (not specified here). Demanding flatness constrains the spin connection so that only six components (or combinations of components) remain free. These six freedoms in the spin connection get fixed by the conditions (31) [or in an equivalent form (41)]. The ten dynamical tetrad components match the number of independent metric components and describe gravity, while the other six tetrad components characterize the frame of the local observer. Choosing a particular local observer fixes these six tetrad components, which in turn completely fixes the spin connection, the latter encoding the inertial effects in the observer frame [29].

E. Conservation of matter energy momentum

We finally show that the covariant conservation of the energy-momentum tensor can also be derived from the gravitational field equations. For this purpose, we take the covariant divergence of the symmetric field equation (44), which reads

$$\begin{aligned}\kappa^2\mathring{\nabla}^\mu\Theta_{\mu\nu} = &-Z'g^{\mu\rho}\phi_{,\mu}\phi_{,\rho}\phi_{,\nu} - Z\mathring{\Box}\phi\phi_{,\nu} - Z\phi_{,\mu}\mathring{\nabla}^\mu\mathring{\nabla}_\nu\phi + \frac{1}{2}Z'g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma}\phi_{,\nu} + Z\phi_{,\rho}\mathring{\nabla}_\nu\mathring{\nabla}^\rho\phi + \frac{1}{2}f_\phi\phi_{,\nu}\\ &+ f_T\mathring{\nabla}^\mu\left(\mathring{\nabla}_\rho S_{(\mu\nu)}{}^\rho - \frac{1}{2}S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} + \frac{1}{2}Tg_{\mu\nu}\right)\\ &+ \mathring{\nabla}^\mu\mathring{\nabla}_\rho f_T S_{(\mu\nu)}{}^\rho - \mathring{\nabla}^\mu f_T\left(\frac{1}{2}S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - \mathring{\nabla}^\rho S_{(\mu\nu)\rho} - \mathring{\nabla}^\rho S_{(\rho\nu)\mu}\right). \end{aligned} \tag{45}$$

In the first line, two terms cancel since the Levi-Cività connection is torsion free, which implies

$$\mathring{\nabla}_\mu\mathring{\nabla}_\nu\psi - \mathring{\nabla}_\nu\mathring{\nabla}_\mu\psi = 0 \tag{46}$$

for any scalar function ψ, in particular also for $\psi = \phi$. The remaining terms in the first line can then be written as

$$\frac{1}{2}\phi_{,\nu}(f_\phi - Z'g^{\mu\rho}\phi_{,\mu}\phi_{,\rho} - 2Z\mathring{\Box}\phi), \tag{47}$$

and one recognizes that the term in brackets is simply the left-hand side of the scalar field equation (34) and thus vanishes. For the second line, one once again makes use of the geometric identity (56) to realize that the term in brackets is simply the Einstein tensor; its covariant divergence vanishes due to the Bianchi identity. We are left with the third line. For its first term, one finds the identity

$$\begin{aligned}\mathring{\nabla}^\mu\mathring{\nabla}_\rho f_T S_{(\mu\nu)}{}^\rho &= \frac{1}{2}(\mathring{\Box}f_T T^\rho{}_{\rho\mu} - \mathring{\nabla}_\mu\mathring{\nabla}_\rho f_T T^{\mu\rho}{}_\nu\\ &\quad - \mathring{\nabla}_\mu\mathring{\nabla}_\nu f_T T_\rho{}^{\rho\mu})\\ &= -\frac{3}{2}\mathring{\nabla}^\mu\mathring{\nabla}_{[\rho}f_T T^\rho{}_{\mu\nu]}, \end{aligned}\tag{48}$$

where we have once more used the symmetry (46) of the Levi-Cività connection, now with $\psi = f_T$. For the last two terms in the third line, we find

$$\begin{aligned}\mathring{\nabla}^\rho S_{(\mu\nu)\rho} + \mathring{\nabla}^\rho S_{(\rho\nu)\mu} = &-\frac{3}{2}\mathring{\nabla}_{[\rho}T^\rho{}_{\mu\nu]}\\ &+\frac{1}{2}(\mathring{\nabla}_\rho T_{\mu\nu}{}^\rho + \mathring{\nabla}_\mu T^\rho{}_{\rho\nu} - g_{\mu\nu}\mathring{\nabla}_\rho T_\sigma{}^{\sigma\rho}). \end{aligned}\tag{49}$$

Contracting the terms in brackets with $\mathring{\nabla}^\mu f_T$, we obtain

$$\begin{aligned}&\frac{1}{2}\mathring{\nabla}^\mu f_T(\mathring{\nabla}_\rho T_{\mu\nu}{}^\rho + \mathring{\nabla}_\mu T^\rho{}_{\rho\nu} - g_{\mu\nu}\mathring{\nabla}_\rho T_\sigma{}^{\sigma\rho})\\ &\quad = -\frac{3}{2}\mathring{\nabla}_{[\rho}f_T\mathring{\nabla}^\mu T^\rho{}_{\mu\nu]}. \end{aligned}\tag{50}$$

Now, the terms (48) and (50) are combined with the covariant divergence of the connection field equation (30) and hence vanish. Using the identity (40), the energy-momentum conservation law (45) reduces to

$$\kappa^2\mathring{\nabla}^\mu\Theta_{\mu\nu} = -\frac{1}{2}\mathring{\nabla}^\mu f_T(S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - T^\rho{}_{\mu\nu}T^\sigma{}_{\sigma\rho} + T^{\rho\sigma}{}_{[\mu}T_{\nu]\rho\sigma}). \tag{51}$$

It is helpful to realize that the term in brackets can be written as

$$-\frac{1}{2}(S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - T^\rho{}_{\mu\nu}T^\sigma{}_{\sigma\rho} + T^{\rho\sigma}{}_{[\mu}T_{\nu]\rho\sigma}) = S_{\rho\sigma\mu}K^{\rho\sigma}{}_\nu. \tag{52}$$

Note that the contortion tensor (8) is antisymmetric in its first two indices. Hence, only the part of the superpotential contributes, which is likewise antisymmetric in its first two indices. Finally, we find that

$$\mathring{\nabla}_\mu f_T S_{[\rho\sigma]}{}^\mu = -\frac{3}{2}\mathring{\nabla}_{[\mu}f_T T^\mu{}_{\rho\sigma]}, \tag{53}$$

which vanishes due to the connection field equations (30). Hence, the right-hand side of the conservation equations (45) vanishes, as one would expect. Let us note that the spin connection equation played a role in providing that.

F. Reduction to general relativity

The field equations (41) and (44) have the interesting property that for particular solutions they reduce to the field equations of general relativity. These solutions must satisfy the conditions that both T and ϕ are constant, and $f_\phi(T,\phi) = 0$ for these constant values. Note that a constant torsion scalar does not require a constant tetrad and spin connection; there are field configurations where those variables cancel each other in the torsion scalar. If both T and ϕ are constant with respect to spacetime, i.e., their derivatives with respect to spacetime directions vanish, the same holds for any function of these variables, and thus in particular for f and its derivatives. As an immediate consequence, the antisymmetric part (41) of the field equations is solved identically. Further, the scalar field equation (34) reduces to $f_\phi = 0$ and is solved due to our assumptions. It remains to show that the symmetric part (44) reduces to the general relativity field equations. For this purpose, we write these equations in the form

$$\frac{1}{2}fg_{\mu\nu} + f_T\left(\overset{\circ}{\nabla}_\rho S_{(\mu\nu)}{}^\rho - \frac{1}{2}S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma}\right) = \kappa^2\Theta_{\mu\nu}, \quad (54)$$

where we used the constancy of ϕ to omit the scalar field kinetic terms and the constancy of f_T to remove its contribution to the derivative in the second term. We can now split the first term to obtain

$$\frac{1}{2}(f - f_T T)g_{\mu\nu} + f_T\left(\overset{\circ}{\nabla}_\rho S_{(\mu\nu)}{}^\rho - \frac{1}{2}S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} + \frac{1}{2}Tg_{\mu\nu}\right) = \kappa^2\Theta_{\mu\nu}. \quad (55)$$

One further uses the identity

$$\overset{\circ}{\nabla}_\rho S_{(\mu\nu)}{}^\rho - \frac{1}{2}S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} + \frac{1}{2}Tg_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}, \quad (56)$$

so the field equations finally take the form

$$\frac{1}{2}(f - f_T T)g_{\mu\nu} + f_T\left(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2}\overset{\circ}{R}g_{\mu\nu}\right) = \kappa^2\Theta_{\mu\nu}. \quad (57)$$

Note that the coefficients of the metric and the Einstein tensor on the left-hand side are both constant with respect to spacetime, due to our assumptions, and can thus be related to effective cosmological and gravitational constants. It follows that the metric (1) satisfies the Einstein equations corresponding to these two parameters.

IV. DETERMINING THE SPIN CONNECTION

After deriving the field equations, the aim of this section is to analyze the connection equation (31) and provide a sample of tetrads, spin connections, and scalar fields which solve it and thus can serve as a starting point for finding solutions to the remaining field equations (44) and (34). We start with a few general considerations in Sec. IV A, before providing a number of specific examples. In particular, we discuss Friedmann-Lemaître-Robertson-Walker spacetimes in Sec. IV B, Kasner spacetimes in Sec. IV C, and spherically symmetric spacetimes in Sec. IV D.

A. General considerations

In principle, there is a reasonable way how to approach the set of field equations. First, one should take an Ansatz for the tetrad (which may implicitly relate to a certain Lorentz observer) as well as for the scalar field. Second, demanding that the curvature (3) vanishes, and the conditions (31) are satisfied when the Ansatz is substituted in, should determine the form of the spin connection associated with that Ansatz. Then, the tetrad Ansatz and the spin connection can be substituted into the tetrad field equations (44) and the scalar field equations (34) to be solved. Finally, one may use local Lorentz transformations (2) to find other equivalent forms of the solution (corresponding to different local observers).

Let us look more closely at the conditions on the spin connection (31) or (41). We may encounter several different situations, some of which have been noted before for $f(T)$ gravity [29] and have actually come up even before while trying to satisfy the antisymmetric part of the tetrad field equations [46,47]. First, if f_T is constant, i.e., $f_{TT} = f_{T\phi} \equiv 0$, the constraint is identically satisfied. This case pertains to taking the theory to be the teleparallel equivalent of general relativity (with the optional scalar field minimally coupled). There, the flat spin connection can be chosen completely arbitrarily, but there can be other principles to constrain it besides the field equations [48,49].

Second, we may choose the spin connection such that $\partial_\mu T = 0$ and $\partial_\mu\phi = 0$. This is related to a possible strategy of looking for solutions in $f(T)$ gravity where one tries to Lorentz transform the Ansatz tetrad of interest into a frame where the torsion scalar T or its derivative vanishes [46,47,50–52]. The strategy is good, since in that particular frame the connection condition is automatically satisfied for any $\overset{\bullet}{\omega}{}^a{}_{b\mu}$, including also a zero spin connection. Therefore, omitting the spin connection in such a frame while solving the tetrad field equations is a consistent move. However, the drawback is that the tetrad field equations reduce to those of the teleparallel equivalent of general relativity, as we have briefly shown in Sec. III F. Hence, with this method, one only recovers the solutions already present in general relativity. But this method is still a nice way to learn about universal solutions, i.e., solutions which are common to the whole $f(T,\phi)$ family of theories.

Third, due to the properties of the Ansatz, it might be possible to solve the spin connection condition independently of the function f. This can happen when $\partial_\mu T$ and $\partial_\mu\phi$ summed over with the terms in the brackets in Eq. (31) yield zero. This is a much more interesting option, since the whole set of equations is not necessarily reduced to the teleparallel equivalent of general relativity, for the tetrad and scalar field equations still involve the function f. Thus, these solutions may extend the repertoire of general relativity. A feasible way to realize this situation is when the Ansatz depends on one particular coordinate y, like time $y = t$ in cosmology or radial distance $y = r$ in static, spherically symmetric spacetime. Then, the two scalars T and ϕ, and hence f_T, depend only on this coordinate. As a consequence, $\partial_\mu f_T \propto \partial_\mu y$, and the particular choice of the function f becomes irrelevant for solving the connection field equations. From a geometric point of view, the expression in the brackets in Eq. (31) can be interpreted as a set of six vectors labeled by the six possible values of the antisymmetric index pair $[ab]$, and the equations are solved if these vectors are tangent to the hypersurfaces of constant y. In the following subsections, we illustrate how this works in a few examples. In principle, it may happen that the situation can be realized with more general Ansätze as well, due to some underlying symmetry.

Finally, in the more general case, it might not be possible to solve the spin connection equation independently of the function f. In the computationally worst case, one may have to tackle all Eqs. (31), (44), and (34) simultaneously. With the nonminimally coupled scalar field, there might also exist some particularly amenable forms of the function $f(T,\phi)$ which could allow cancellation in (32) and thus turn out to be helpful in finding the solutions. Note that also in this situation the aforementioned geometric interpretation holds, but the hypersurfaces are defined by constant values of f_T and thus depend on the choice of the function f.

In the following subsections, we present some tetrads in their diagonal form together with a nonvanishing spin connection which satisfies the condition (31).

B. Friedmann-Lemaître-Robertson-Walker spacetimes

Homogeneous and isotropic cosmological spacetimes are described by the metric in the Friedmann-Lemaître-Robertson-Walker (FLRW) form

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right], \tag{58}$$

written in spherical coordinates t, r, ϑ, φ. Here, $k \in \{-1, 0, 1\}$ determines the sign of the spatial curvature. Note that in all three cases the cosmological symmetry imposes that the scalar field is evolving homogeneously, $\phi = \phi(t)$, while the matter energy-momentum tensor must be given by an ideal fluid,

$$\Theta^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}, \tag{59}$$

with energy density $\rho = \rho(t)$, pressure $p = p(t)$, and four-velocity $u^\mu = \partial_t$ normalized by the metric, $g_{\mu\nu}u^\mu u^\nu = 1$. A canonical choice for the tetrad is the diagonal one,

$$h^a{}_\mu = \mathrm{diag}\left(1, \frac{a(t)}{\sqrt{1-kr^2}}, a(t)r, a(t)r\sin\vartheta\right). \tag{60}$$

Following the procedure detailed in the general discussion, one then determines the spin connection by solving the antisymmetric part (30) of the field equations. Depending on the value of k, one may find different solutions:

(i) In the case $k = 0$, one may use the spin connection [25]

$$\dot\omega^1{}_{2\vartheta} = -\dot\omega^2{}_{1\vartheta} = -1, \qquad \dot\omega^1{}_{3\varphi} = -\dot\omega^3{}_{1\varphi} = -\sin\vartheta,$$
$$\dot\omega^2{}_{3\varphi} = -\dot\omega^3{}_{2\varphi} = -\cos\vartheta. \tag{61}$$

One then finds that the remaining tetrad field equations are given by

$$\frac{1}{2}f + 6f_T H^2 - \frac{1}{2}Z\dot\phi^2 = \kappa^2\rho, \tag{62}$$

$$\frac{1}{2}f + 2f_{T\phi}H\dot\phi - 24f_{TT}\dot H H^2 + 6f_T H^2 + 2f_T\dot H + \frac{1}{2}Z\dot\phi^2 = -\kappa^2 p, \tag{63}$$

where $H = \frac{\dot a}{a}$ is the Hubble parameter and a dot denotes the derivative with respect to t. Note that Minkowski spacetime is included as the special case $a = 1$.

(ii) For $k = 1$, a viable spin connection is given by [53]

$$\dot\omega^1{}_{2\vartheta} = -\dot\omega^2{}_{1\vartheta} = -\sqrt{1-r^2}, \qquad \dot\omega^1{}_{2\varphi} = -\dot\omega^2{}_{1\varphi} = -r\sin\vartheta, \qquad \dot\omega^1{}_{3\vartheta} = -\dot\omega^3{}_{1\vartheta} = r,$$
$$\dot\omega^1{}_{3\varphi} = -\dot\omega^3{}_{1\varphi} = -\sqrt{1-r^2}\sin\vartheta, \qquad \dot\omega^2{}_{3r} = -\dot\omega^3{}_{2r} = -\frac{1}{\sqrt{1-r^2}}, \qquad \dot\omega^2{}_{3\varphi} = -\dot\omega^3{}_{2\varphi} = -\cos\vartheta. \tag{64}$$

In this case, one obtains the remaining tetrad field equations

$$\frac{1}{2}f + 6f_T H^2 - \frac{1}{2}Z\dot\phi^2 = \kappa^2\rho, \tag{65}$$

$$\frac{1}{2}f + 2f_{T\phi}H\dot\phi - 24f_{TT}\left(\dot H + \frac{1}{a^2}\right)H^2 + 6f_T H^2 + 2f_T\left(\dot H - \frac{1}{a^2}\right) + \frac{1}{2}Z\dot\phi^2 = -\kappa^2 p. \tag{66}$$

(iii) Finally, for $k = -1$, one may use the spin connection [53]

$$\dot\omega^0{}_{1r} = \dot\omega^1{}_{0r} = \frac{1}{\sqrt{1+r^2}}, \qquad \dot\omega^0{}_{2\vartheta} = \dot\omega^2{}_{0\vartheta} = r, \qquad \dot\omega^0{}_{3\varphi} = \dot\omega^3{}_{0\varphi} = r\sin\vartheta,$$
$$\dot\omega^1{}_{2\vartheta} = -\dot\omega^2{}_{1\vartheta} = -\sqrt{1+r^2}, \qquad \dot\omega^1{}_{3\varphi} = -\dot\omega^3{}_{1\varphi} = -\sqrt{1+r^2}\sin\vartheta, \qquad \dot\omega^2{}_{3\varphi} = -\dot\omega^3{}_{2\varphi} = -\cos\vartheta. \tag{67}$$

Here, the remaining tetrad field equations are given by

$$\frac{1}{2}f + 6f_T H\left(H - \frac{1}{a}\right) - \frac{1}{2}Z\dot{\phi}^2 = \kappa^2\rho, \quad (68)$$

$$\frac{1}{2}f + 2f_{T\phi}\left(H - \frac{1}{a}\right)\dot{\phi} - 24f_{TT}\left(\dot{H} + \frac{H}{a}\right)\left(H - \frac{1}{a}\right)^2 + 6f_T H\left(H - \frac{1}{a}\right) + 2f_T\left(\dot{H} + \frac{1}{a^2}\right) + \frac{1}{2}Z\dot{\phi}^2 = -\kappa^2 p. \quad (69)$$

In all three cases, the scalar field equation reduces to

$$f_\phi - 2Z\ddot{\phi} - 6ZH\dot{\phi} - Z_\phi\dot{\phi}^2 = 0. \quad (70)$$

One easily checks that the three mentioned spin connections satisfy the condition (31) for the connection, due to the reasons outlined as the third option in Sec. IV A. Let us emphasize that the tetrad (60) alone with a vanishing spin connection does not solve the condition (31). It is worth noting that the structures of the spin connection components as well as the tetrad field equations in the $k = +1$ (64) and $k = -1$ (67) cases are different. For example, in the latter case, there are nontrivial time components which do not appear in the former case. In the $k = 1$ case, it was found that a certain local Lorentz rotation can make a transformation into a local frame, where all the spin connection components vanish, but the tetrad components become nondiagonal and more complicated [47]. In the $k = -1$ case, the same can happen, but one needs to employ a Lorentz boost [53]. The remaining tetrad and scalar field equations we displayed above are invariant under this simultaneous transformation of the tetrad and the spin connection.

C. Kasner spacetimes

Homogeneous but in general anisotropic Kasner spacetime can be realized by choosing the tetrad

$$h^a{}_\mu = \mathrm{diag}(1, a(t), b(t), c(t)), \quad (71)$$

in Cartesian coordinates. This, together with a vanishing spin connection,

$$\overset{\bullet}{\omega}{}^a{}_{b\mu} = 0, \quad (72)$$

makes the connection condition (31) satisfied, which vindicates the respective studies in $f(T)$ gravity [54–57]. But for more general anisotropic models, one has to check the compatibility of the spin connection.

In the special case $a = b = c$, the tetrad (71) reduces to another viable tetrad to describe FLRW spacetime with $k = 0$,

$$h^a{}_\mu = \mathrm{diag}(1, a(t), a(t), a(t)). \quad (73)$$

Along with a vanishing spin connection (72), one finds that the condition (31) is easily satisfied. Since here assuming the vanishing spin connection yields the correct result, the former studies of the spatially flat cosmology in $f(T)$ and scalar-torsion gravity remain valid; see, e.g., Refs. [32,58–62] for a selection of latest works.

D. Spherically symmetric spacetimes

A general static spherically symmetric spacetime can be represented by a diagonal tetrad

$$h^a{}_\mu = \mathrm{diag}(A(r), B(r), r, r\sin\vartheta). \quad (74)$$

This, together with the flat spin connection [25]

$$\overset{\bullet}{\omega}{}^1{}_{2\vartheta} = -\overset{\bullet}{\omega}{}^2{}_{1\vartheta} = -1, \qquad \overset{\bullet}{\omega}{}^1{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{1\varphi} = -\sin\vartheta, \qquad \overset{\bullet}{\omega}{}^2{}_{3\varphi} = -\overset{\bullet}{\omega}{}^3{}_{2\varphi} = -\cos\vartheta \quad (75)$$

and a generic spherically symmetric Ansatz for the scalar field, $\phi = \phi(r)$, satisfies the condition for the connection (31). As with the previous cases presented in spherical coordinates, it is possible by a Lorentz transformation to find a frame where the spin connection vanishes, while the tetrad becomes nondiagonal [46]. Thus, the earlier research on the spherically symmetric systems in $f(T)$ gravity based on this nondiagonal tetrad [46,47,63–67] has a valid starting point, but albeit sees physics in a particular frame. Assuming the diagonal tetrad (74) must happen in conjunction with the nonzero spin connection (75), like in Ref. [68], otherwise one is led to wrong results.

V. GENERALIZATION TO MULTIPLE SCALAR FIELDS

In the previous sections of this article, we have considered a class of scalar-torsion theories of gravity obtained by (in general nonminimally) coupling a scalar field to teleparallel gravity. In this section, we extend our findings to multiple scalar fields. For this purpose, we consider a multiplet $\boldsymbol{\phi} = (\phi^A)$, $A = 1, \ldots, N$ of N scalar fields. We then replace the action (9) by the more general form

$$S = \frac{1}{2\kappa^2}\int_M [f(T, \boldsymbol{\phi}) + Z_{AB}(\boldsymbol{\phi})g^{\mu\nu}\phi^A_{,\mu}\phi^B_{,\nu}]\theta d^4x + S_m[\theta^a, \chi^I]. \quad (76)$$

This action differs from the single field case in two ways. First, in order to potentially render all scalar fields dynamical, there must be a kinetic term involving all fields. A natural generalization of the single scalar field kinetic term is to equip the parameter function Z with two scalar field indices, such that it is symmetric in these indices, $Z_{[AB]} = 0$; any antisymmetric part would cancel due to the contraction with a symmetric tensor composed from the derivatives of the scalar fields. Second, the two parameter functions f and Z_{AB} now depend on all scalar fields, and hence on the field multiplet ϕ. This must be taken into account when calculating variations and derivatives of the parameter functions. [In special cases like $f(T,\boldsymbol{\phi}) = f(\boldsymbol{\phi})T$, it is possible to redefine the scalar fields so that only one of them is nonminimally coupled, as pointed out in the multiscalar-curvature theory [69], but we will not delve into this option here.]

It is now straightforward to derive the field equations from the action (76), essentially following the same steps as shown explicitly in Sec. III. We will not repeat these steps here and only display the field equations in their final form. We start with the symmetric part (44), which generalizes to

$$\frac{1}{2}fg_{\mu\nu} + \overset{\circ}{\nabla}_\rho(f_T S_{(\mu\nu)}{}^\rho) - \frac{1}{2}f_T S_{(\mu}{}^{\rho\sigma}T_{\nu)\rho\sigma} - Z_{AB}\phi^A_{,\mu}\phi^B_{,\nu} + \frac{1}{2}Z_{AB}\phi^A_{,\rho}\phi^B_{,\sigma}g^{\rho\sigma}g_{\mu\nu} = \kappa^2\Theta_{\mu\nu}, \tag{77}$$

where only the terms originating from the kinetic energy of the scalar fields are visibly affected. These do not appear in the antisymmetric part (41) of the tetrad field equations, which hence retain their form,

$$\partial_{[\rho}f_T T^\rho{}_{\mu\nu]} = 0. \tag{78}$$

This in particular implies that the connections given in Sec. IV remain valid also in the case of multiple scalar fields.

Finally, the generalization of the scalar field equation (34) requires more attention, since derivatives of Z_{AB} now carry different types of indices, and the correct indices must be chosen in contractions. Starting from the action (76), we find the scalar field equations

$$f_{\phi^A} - (2Z_{AB,\phi^C} - Z_{BC,\phi^A})g^{\mu\nu}\phi^B_{,\mu}\phi^C_{,\nu} - 2Z_{AB}\overset{\circ}{\square}\phi^B = 0. \tag{79}$$

We conclude with the remark that, although the form of the field equations is mostly unchanged compared to the single field case, there is an implicit and less apparent change coming from the fact that the parameter functions f and Z_{AB}, and hence their derivatives appearing in the field equations, depend on all scalar fields.

VI. SUMMARY AND DISCUSSION

In this paper, we have presented a new class of theories in the covariant teleparallel framework, where the gravitational action depends on an arbitrary function of the torsion scalar and a scalar field, $f(T,\phi)$. This generic setup subsumes and generalizes a number of previously considered models, like $f(T)$ gravity and a scalar field nonminimally coupled to T, putting them in a unified scheme so that they can be studied together. We derived the field equations for the tetrads, scalar field, and flat spin connection. The latter is especially important and until recently was missing in the covariant teleparallel picture. The spin connection equation turns out to be related to the antisymmetric part of the tetrad field equations and makes it vanish identically. One also needs the spin connection equation when combining the field equations in order to show the matter energy-momentum conservation.

As a matter of fact, the spin connection field equation contains only first order derivatives with respect to the spacetime coordinates and provides a consistency condition that from the tetrad Ansatz determines the six nontrivial spin connection components (remaining after imposing zero curvature). These six components can be interpreted as gauge d.o.f., since they can be absorbed into the tetrad by a suitable local Lorentz transformation.

Solving the spin connection equations is not an easy matter, though. In a simple case, the solution Ansatz can reduce the equations to be those of the teleparallel equivalent to general relativity, where the spin connection can be fixed arbitrarily, but then the possibilities of the wider generalized theory remain unexplored. As we explain, for certain symmetric configurations, it is possible to solve the spin connection equation independent of the function f and illustrate this by the examples of cosmological and spherically symmetric spacetimes. These results can be used as a starting point for integrating the tetrad field equations, whereby one typically needs to specify the form of the function f. In light of this understanding, not all previous results in $f(T)$ or scalar-torsion gravity can be automatically be taken with trust; one must check whether the assumed spin connection is consistent with the tetrad.

Our work leaves a number of possibilities for further investigations and generalizations. In particular, it invites studies of the phenomenology of the class of theories we discussed here, such as their post-Newtonian limit or gravitational waves. Also, one may derive the cosmological field equations and perform an analysis of the possible solutions, employing the method of dynamical systems. Also, fundamental questions, such as the number of propagating d.o.f., may be addressed, e.g., by performing a Hamiltonian analysis.

Another straightforward possibility is to consider more general action functionals, such as a Lagrangian given by an arbitrary function depending on the torsion scalar, the scalar field, and its kinetic term and also involving a coupling to vector torsion [70]. Particular subclasses of such a model, where the scalar field couples to different terms constructed from the underlying teleparallel geometry by a small number of free functions, similar to the case of scalar-tensor gravity [71], are also worth studying [72]. One may also pose the question of what is the most general theory coupling one or more scalar fields to torsion and investigate its generic properties [73].

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Chapter 9

Dynamical systems approach and generic properties of $f(T)$ cosmology

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Contents

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Dynamical systems approach and generic properties of $f(T)$ cosmology

Manuel Hohmann,[*] Laur Järv,[†] and Ulbossyn Ualikhanova[‡]
Laboratory of Theoretical Physics, Institute of Physics, University of Tartu,
W. Ostwaldi 1, 50411 Tartu, Estonia
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We present a systematic analysis of the dynamics of flat Friedmann-Lemaître-Robertson-Walker cosmological models with radiation and dust matter in generalized teleparallel $f(T)$ gravity. We show that the cosmological dynamics of this model are fully described by a function $W(H)$ of the Hubble parameter, which is constructed from the function $f(T)$. After reducing the phase space to two dimensions, we derive the conditions on $W(H)$ for the occurrence of de Sitter fixed points, accelerated expansion, crossing the phantom divide, and finite time singularities. Depending on the model parameters, it is possible to have a bounce (from contraction to expansion) or a turnaround (from expansion to contraction), but cyclic or oscillating scenarios are prohibited. As an illustration of the formalism we consider power law $f(T) = T + \alpha(-T)^n$ models, and show that these allow only one period of acceleration and no phantom divide crossing.

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I. INTRODUCTION

General relativity (GR), which relates the effects of gravity to spacetime curvature, has been highly successful in describing a wide range of phenomena. Teleparallel gravity [1–3] employs torsion instead of curvature, conceptually distinguishes between gravitation and inertia, and builds up its theoretical formulation more in line with gauge theories [2,4,5]. Despite the difference in mathematical setup and interpretation, teleparallel gravity is equivalent to general relativity in all physical predictions. This follows from the correspondence of the respective field equations, since the curvature scalar R in the Einstein-Hilbert action of general relativity differs from the torsion scalar T in the action of teleparallel equivalent of general relativity (TEGR) only by a total divergence term, $R = -T - 2\nabla^\mu T^\lambda{}_{\mu\lambda}$, where $T^\lambda{}_{\mu\lambda}$ are the components of the torsion tensor.

In searching for good models to describe the phenomena of dark energy, dark matter, and inflation, many researchers have looked beyond GR, generalizing its Lagrangian to an arbitrary function of curvature, $f(R)$, leading to fourth-order field equations [6,7]. In the same vein the Lagrangian of teleparallel gravity has been generalized to $f(T)$ [8,9]. The ensuing field equations are of second order, and we get a new class of theories essentially different from their counterparts based on curvature.

The original approach to $f(T)$ gravity had a problem in which the action failed to be invariant under the local Lorentz transformation of the tetrad fields [10,11]. This made the theory subject to preferred frame of reference effects, spurious degrees of freedom, and acausality [12–15]. The issue can be remedied by realizing that the Weitzenböck connection originally used in teleparallel gravity is not the most general connection consistent with nonzero torsion and vanishing curvature; one can also allow purely inertial spin connection [16,17]. This leads to a covariant approach to $f(T)$ gravity whereby one tackles the field equations by invoking a reference tetrad which encodes the inertial effects [18,19].

It is remarkable that while many solutions in $f(T)$ gravity need to be reconsidered in view of the covariant approach, the diagonal tetrad corresponding to flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe is already "proper" and consistent with the covariant view [18]. Already, the first studies of FLRW cosmology in $f(T)$ gravity have pointed out the possibility that it can naturally lead to accelerated expansion of the universe without any extra matter component, thus being interesting to model dark energy and inflation [8,9,20,21]. Later a number of works have focused upon various cosmological aspects of $f(T)$ models, from background evolution and growth of perturbations to comparison with observational data; see Refs. [22,23] for reviews.

The method of dynamical systems is a widely used set of tools in cosmology to obtain a qualitative assessment of the behavior of solutions in a model, without delving into the often almost impossible task to find the analytic form of the solutions. While dynamical systems have been helpful in uncovering the main features of solutions in particular models [21,22,24–27], there have been only a few papers attempting a more systematic analysis of generic $f(T)$ cosmology [24,28–30]. Our present study aims at completing this task by deriving the general expressions for de Sitter fixed points, acceleration, phantom dark energy, and finite time singularities. The method and formulas we present can be easily applied to study specific models or

[*]manuel.hohmann@ut.ee
[†]laur.jarv@ut.ee
[‡]ulbossyn.ualikhanova@ut.ee

for a heuristic construction of phenomenologically desirable scenarios.

The key in the dynamical systems analysis is choosing suitable dynamical variables, as also highlighted by recent insights in the study of $f(R)$ gravity [31,32]. One typically adopts Hubble-rescaled dimensionless variables evolving in dimensionless expansion-logarithmic time parameter $N = \ln a$. This leads to an at first three-dimensional phase space for flat FLRW $f(T)$ cosmology with radiation and dust matter components [21,24–29]. There is a further redundancy. In $f(T)$ gravity the Einstein equations are of second order, so in cosmology there are in principle five dynamical quantities $(a, H, T, \rho_r, \rho_m)$. Not all of them are independent, as in the equations the flat FLRW the scale factor a occurs only within the Hubble parameter $H = \frac{\dot{a}}{a}$, the density of radiation ρ_r and dust matter ρ_m are related to H and $f(T)$ by a Friedmann constraint equation, and in addition there is a geometric identity $T = -6H^2$. Therefore the physical phase space is two-dimensional, spanned by two variables given by combinations of the quantities mentioned before. The situation can be contrasted with generic $f(R)$ gravity where the Einstein equations are of fourth order and in cosmology there are six dynamical quantities $(a, H, R, \dot{R}, \rho_r, \rho_m)$. Again a is subsumed into H, and one quantity can be expressed via others by the Friedmann constraint, but the geometric identity $R = 6\dot{H} + 12H^2$ does not reduce R to H. So the phase space of flat FLRW $f(R)$ cosmology with radiation and dust is four-dimensional [7,31,32].

As in Refs. [24,27,30] we reduce the flat FLRW $f(T)$ cosmology phase space to two dimensions. Our choice of the dynamical variables allows a straightforward physical interpretation of results. Taking the Hubble parameter H to be one of the variables makes the fixed points correspond to de Sitter (or Minkowski) spacetime. The second variable given by the ratio of radiation energy density to overall matter energy density makes the flow from radiation to dust matter domination in expanding universe graphic on the other axis. We follow the evolution of the system in basic cosmological time in order to study both expanding and contracting phases under the same footing.

To illustrate our general results we consider a simple class of models $f(T) = T + \alpha(-T)^n$ as an example. These models allow the cosmic evolution from radiation domination through matter domination to dark energy domination eras [8,9], superbounce [33], initial singularity crossing [34], future sudden singularities [35], but no phantom crossing [36]. Its fixed points have been studied in Hubble-rescaled variables [21,24,27,28,30], and some analytic solutions are also known [37]. Constraints from various sets of observations can be found in Refs. [8,36,38–49]. We show how these features are reflected with our method and a comprehensive picture emerges.

The outline of the paper is as follows. In Sec. II we briefly review the action and cosmological field equations of $f(T)$ gravity and show that they are fully defined in terms of the Friedmann function $W(H)$. We then cast these equations into the form of a dynamical system in Sec. III, and read off a number of properties of this system: its boundaries and fixed points, as well as the possibility of bounces, turnarounds, and oscillating universe solutions. We then discuss finite time singularities in Sec. IV. Observable properties, in particular, the accelerating expansion of the universe and the properties of dark energy, are delineated in Sec. V. In Sec. VI we apply our general formalism to a generic power law model and show how its parameters influence the properties of the dynamical system. We end with a conclusion in Sec. VII. In order to collect and summarize the results obtained on the physical phase space, we provide a graphical index of phase space points in Appendix.

II. ACTION AND COSMOLOGICAL FIELD EQUATIONS

In this section we briefly review the cosmological dynamics of $f(T)$ gravity, starting from the most general action and cosmological field equations. We show that if we express them through the Hubble parameter of a FLRW spacetime, they take the form of one constraint equation and one dynamical equation. We further display these field equations for the special case in which the matter content of the universe is constituted by both dust matter and radiation and show their consistency with the corresponding continuity equations for this choice of the matter content.

The starting point of our derivation is the action functional of $f(T)$ gravity [2,8,9,22],

$$S = \frac{1}{16\pi G}\int |e| f(T) d^4x, \tag{1}$$

with an arbitrary function $f(T)$ of the torsion scalar

$$T = \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\mu}{}_{\rho\mu}T^{\nu\rho}{}_{\nu}. \tag{2}$$

The dynamical variable is given by the tetrad field e^i_μ, in terms of which the torsion tensor is expressed as

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = e^{\rho}_i(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu + \omega^i{}_{j\mu}e^j_\nu - \omega^i{}_{j\nu}e^j_\mu), \tag{3}$$

where the flat spin connection $\omega^i{}_{j\mu}$ is introduced in order to render the theory covariant under local Lorentz transformations. For our cosmological setting, we assume a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe, for which we can choose the tetrad to be

$$e^i_\mu = \mathrm{diag}(1, a, a, a) \tag{4}$$

with the usual cosmological scale factor $a = a(t)$. One finds that in this case the tetrad is "proper" and the inertial spin connection vanishes [18]. The torsion scalar reduces to

$$T = -6\frac{\dot{a}^2}{a^2} = -6H^2, \tag{5}$$

where H is the Hubble parameter. From our assumption of cosmological symmetry, i.e., homogeneity and isotropy, further follows that the matter energy-momentum tensor must take the form of a perfect fluid,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}, \tag{6}$$

where ρ stands for energy density, p denotes pressure, and the four-velocity

$$u^\mu = \partial_t \tag{7}$$

is normalized by the metric $g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu$. From the action (1) then follow the cosmological field equations [8,9]

$$12H^2 f_T + f = 16\pi G\rho, \tag{8a}$$

$$48H^2\dot{H}f_{TT} - (12H^2 + 4\dot{H})f_T - f = 16\pi G p, \tag{8b}$$

where subscripts denote derivatives, i.e.,

$$f_T = \frac{df}{dT}, \qquad f_{TT} = \frac{d^2 f}{dT^2}. \tag{9}$$

In the remainder of this article, without a loss of generality we will write $f(T) = T + F(T)$. In this parametrization the cosmological field equations read

$$6H^2 + 12H^2 F_T + F = 16\pi G\rho, \tag{10a}$$

$$4\dot{H}(12H^2 F_{TT} - F_T - 1) = 16\pi G(\rho + p). \tag{10b}$$

Note that if $F = 0$, these equations reduce to the usual Friedmann equations in TEGR and GR. Also note that if $F_T = \frac{F}{2T}$, i.e., $F \sim \sqrt{-T}$, the cosmological equations are still identical to the TEGR and GR case. However, the full field equations receive corrections, which possibly influence the evolution of perturbations of the cosmological background.

In order to discuss solutions to these field equations, we finally also need an equation of state for the matter content. Here we will use the simple assumption that the matter content is constituted by two components: dust and radiation. The density and pressure thus take the form

$$\rho = \rho_m + \rho_r, \qquad p = p_m + p_r, \tag{11}$$

where the equation of state is given by

$$p_m = 0, \qquad p_r = \frac{1}{3}\rho_r. \tag{12}$$

From these relations follow the matter continuity equations

$$\dot{\rho}_m = -3H\rho_m, \qquad \dot{\rho}_r = -4H\rho_r. \tag{13}$$

The cosmological field equations then finally take the form

$$W = 16\pi G(\rho_m + \rho_r), \tag{14a}$$

$$-\dot{H}\frac{W_H}{3H} = 16\pi G\left(\rho_m + \frac{4}{3}\rho_r\right). \tag{14b}$$

Here we have introduced the Friedmann function

$$W(H) = F + 6H^2 + 12H^2 F_T, \tag{15}$$

keeping in mind the relation (5) between T and H, and where the subscript stands for differentiation $W_H = dW/dH$. As we will see in the following, the function $W(H)$ encodes the main cosmological features of any given $f(T)$ gravity model. Equations (13) and (14) are the basis of the current study. Note that they are not independent of each other. For our analysis, we have to remove this redundancy and apply the constraint equation in order to obtain an unconstrained dynamical system. This will be done in the following section.

III. DYNAMICAL SYSTEMS APPROACH

We will now cast the cosmological equations of motion (14) and (13) detailed in the previous section into the language of dynamical systems and derive some of its properties. We start by determining the phase space of the dynamical system and its evolution equations in Sec. III A. Since this phase space will turn out to be unbounded in one direction, we perform a coordinate transformation in Sec. III B, which maps the system into a compact region. We then discuss three particular features of the dynamical system: its fixed points in Sec. III C, the possibility of crossing the line $H = 0$ in Sec. III D, and the possibility of an oscillating universe in Sec. III E.

A. Phase space and evolution equations

In the previous section we have seen that the Hubble parameter H and the energy densities ρ_m and ρ_r are not independent due to the algebraic Friedmann constraint (14a). The physical phase space of our dynamical system is thus a hypersurface of codimension 1 in the space

$$\{(H, \rho_m, \rho_r) | H \in (-\infty, \infty), \rho_m \in [0, \infty), \rho_r \in [0, \infty)\}, \tag{16}$$

which we parametrize as follows. We introduce a new variable

$$X = \frac{\rho_r}{\rho_r + \rho_m} \tag{17}$$

for the ratio of radiation to the total energy density. The original densities are then recovered as

$$\rho_r = X(\rho_r + \rho_m), \qquad \rho_m = (1 - X)(\rho_r + \rho_m), \tag{18}$$

where the total energy density on the right hand side is determined in terms of H by the Friedmann constraint (14). One easily sees that the physical phase space is restricted to $H \in (-\infty, \infty), X \in [0, 1]$. Another bound follows from the Friedmann constraint and the validity of the null energy condition, which implies that the total matter energy density $\rho_m + \rho_r$ must be non-negative and finite. From the Friedmann constraint (14) it follows that this is equivalent to $W(H) \geq 0$. The physical phase space is thus finally given by

$$\mathcal{P} = \{(H, X) | -\infty < H < \infty, \quad 0 \leq X \leq 1, \\ 0 \leq W(H) < \infty\}. \tag{19}$$

We now discuss the dynamics of the new variables. Taking the time derivative of the definition (17) and using the continuity equations (13), one finds

$$\dot{X} = HX(X - 1). \tag{20}$$

Similarly, we can solve the Friedmann equation (14a) for $\dot{H}$ and use the definition (17) to obtain

$$\dot{H} = -(X + 3)H\frac{W}{W_H} = -\frac{(X + 3)H}{(\ln W)_H}, \tag{21}$$

where we have tacitly assumed that the factor W_H/H by which we divided is nonzero and finite since otherwise the division would be ill-defined, or defined only via a suitable limiting procedure. We have to keep this in mind later when we will be discussing these limiting cases. Equations (20) and (21) define our dynamical system. Note that our choice for the dynamical variables to describe the two-dimensional phase space is different from Refs. [24,27,30].

We also remark that our model includes the two special cases in which the matter content is given by pure dust matter, $\rho_r = 0$, and pure radiation, $\rho_m = 0$. These are obtained by restricting the phase space to $X = 0$ or $X = 1$, respectively. From the continuity equation in the form (20) it follows that any dynamics that start on these subspaces will also remain there, so they can be treated as dynamical systems on their own. This will be considered next, when we compactify the phase space and discuss its boundary.

B. Compactified phase space and its boundary

For various applications, such as drawing phase diagrams and discussing limiting points of trajectories, it is more convenient if the dynamical system is defined on a compact domain. In order to map our dynamical system into a compact domain, we replace the Hubble parameter H with a new variable Y, which we define such that

$$Y = \frac{H}{\sqrt{1 + H^2}} \Leftrightarrow H = \frac{Y}{\sqrt{1 - Y^2}}. \tag{22}$$

The domain of these variables in given by $Y \in (-1, 1)$, $X \in [0, 1]$. We then express the dynamical equations in terms of the new variables, which yields

$$\dot{Y} = -\frac{(X + 3)Y(1 - Y^2)W}{W_H}, \tag{23a}$$

$$\dot{X} = \frac{XY(X - 1)}{\sqrt{1 - Y^2}}. \tag{23b}$$

This rescaled system will be used later when we show phase diagrams for particular functions W.

Using the new variables (Y, X), we can now discuss the boundary $\partial\mathcal{P}$ of the phase space. This is important since there may exist trajectories, whose limiting points are located on the boundary, and which may or may not be part of the physical phase space (16). This concerns, in particular, points at which H or W (and hence the total matter energy density) diverges, so the Friedmann equations (14) become singular, and thus invalid. From the definition (19) of the physical phase space it follows that we have the following components of the boundary $\partial\mathcal{P}$:

(i) The boundary along the line $X = 0$ belongs to the physical phase space. It contains those points of the phase space where the radiation energy density ρ_r vanishes, and the matter content of the universe is given by dust matter, or we have a vacuum. From the dynamical equation (20) it follows that $\dot{X}$ vanishes for $X = 0$, so any trajectories starting from this boundary stay on the boundary. Later in Sec. III C we see that regular de Sitter fixed points can reside on this boundary.

(ii) The same properties hold for the boundary along the line $X = 1$, which contains those points of the phase space where the dust energy density ρ_m vanishes, and the matter content of the universe is given by radiation, or we have a vacuum. Later in Sec. III C we see that regular de Sitter fixed points can reside also on this boundary.

(iii) Lines with constant $H = H^\star$ and $X \in [0, 1]$, where $W(H^\star) = 0$ and the sign of W is different for $H < H^\star$ and $H > H^\star$, also correspond to the boundaries, which are part of the physical phase space. They represent vacuum solutions of the field equations. Note that in this case the whole line $H = H^\star, X \in [0,1]$ represents a single point $(H, \rho_m, \rho_r) = (H^\star, 0, 0)$ of the original phase space. These boundaries always contain regular de Sitter fixed points, as we will see in a deeper discussion in Sec. III C.

(iv) Similarly, lines with constant $H = H^\star$ and $X \in [0,1]$, where $W \to \infty$, are also boundaries. In contrast to the previous case, they do not belong to the physical phase space, since the total matter energy density $\rho_m + \rho_r$ diverges. However, as we will see in Sec. III C, there are trajectories which approach these boundaries in infinite time, and they contain points which can be regarded as (singular) fixed points.

(v) Finally, the lines $Y = \pm 1, X \in [0,1]$ corresponding to $H \to \pm\infty$ also can be treated as boundaries of the (compactified) phase space, provided that W is nonnegative in the corresponding limit. These boundaries are relevant as they may contain limit points of trajectories that are reached in finite time, which correspond to particular types of singularities as discussed in Sec. IV B.

Note that it is also possible that $\dot{H}$ diverges, for example, at points (H, X) where $W_H = 0$, and that in this case trajectories reaching this point cannot be continued. However, these can still be regarded as parts of the physical phase space if the physical quantities (H, ρ_m, ρ_r) constituting the original phase space remain finite. These points are known as sudden singularities and are discussed in detail in Sec. IV C.

Since the boundary $\partial\mathcal{P}$ may contain a number of interesting points as briefly mentioned above and further discussed in the remaining sections of this article, it will turn out to be more convenient to study the dynamics on the compactified phase space $\bar{\mathcal{P}} = \mathcal{P} \cup \partial\mathcal{P}$.

C. Fixed points and their stability

We now come to the discussion of fixed points of the dynamical system defined by equations (20) and (21) in Sec. III A. Recall that the fixed points of a dynamical system are points $(H^\star, X^\star)$ in its phase space at which the flow of the dynamics vanishes. For the dynamical system we consider here this amounts to the conditions $\dot{X} = 0$ and $\dot{H} = 0$. From equation (20) one easily understands that $\dot{X} = 0$ if either $X = 0$, $X = 1$, or $H = 0$. The condition for $\dot{H} = 0$ given by equation (21) requires a more careful treatment, as it depends on the Friedmann function $W(H)$. We can distinguish the following cases:

(i) For $W \to 0$ it is obvious that $\dot{H} \to 0$ in the case that W_H remains finite. However, in the case that W_H either vanishes or diverges for $W = 0$, we also obtain a fixed point. To see this, note that $\ln W \to -\infty$ when $W \to 0$. However, this implies that also $(\ln W)_H \to \pm\infty$, where the sign depends on the direction of the limit. Hence, from $W \to 0$ always follows $\dot{H} \to 0$. From the Friedmann constraint (14a) it further follows that the energy density of dust matter and radiation vanishes, so these fixed points correspond to vacuum solutions. As we will discuss in Sec. V B, they are de Sitter vacuum solutions for $H^\star \neq 0$ and Minkowski vacuum solutions for $H^\star = 0$, which follows from the fact that for a constant Hubble parameter $\dot{a}/a = H = H^\star$ the scale factor behaves as

$$a(t) \sim \exp(H^\star t), \tag{24}$$

and so is constant, or exponentially increasing or decreasing, depending on the sign of $H^\star$.

(ii) Following the same line of argumentation, we can also consider the case that $W \to \infty$ for a finite value of $H^\star$. In this case we find $\ln W \to \infty$, which again implies $(\ln W)_H \to \pm\infty$, so also in this case $\dot{H} \to 0$ and we obtain a fixed point. Note that these points are not part of the physical phase space $\mathcal{P}$ defined by equation (19), but lie on the boundary $\partial\mathcal{P}$. We call them singular fixed points in order to distinguish them from regular fixed points which belong to the physical phase space. Here for $H^\star < 0$ the radiation and dust matter content get compressed to infinite density after infinite time in the future. Analogously for $H^\star > 0$ the radiation and matter started from an infinite density state infinite time ago in the past. Note that also in this case the scale factor approaches asymptotically the exponential behavior (24) in the vicinity of the fixed point.

(iii) We are left with the case that $W \to W^\star > 0$ remains finite. In this case we can still obtain a fixed point of the equation (21) if $W_H \to \pm\infty$ diverges. This condition is necessary and sufficient for $H \neq 0$. For $H = 0$ the weaker condition that W_H/H diverges is both necessary and sufficient. However, if we map this fixed point into the original phase space with variables (H, ρ_m, ρ_r), we find the paradox situation in which $\rho_m + \rho_r > 0$ is constant, since $W = W^\star > 0$ is also constant at a fixed point though, in general, $H = H^\star \neq 0$. This contradicts the continuity equations (13). The reason for this contradiction is the fact that we divided the dynamical equation (14b) by an infinite quantity, and hence generated a previously nonexisting solution. However, these points are still relevant in a suitable limit since they turn out to correspond to a certain class of finite time singularities, as shown in Sec. IV D.

Note that the existence and number of fixed points satisfying these conditions depends on the Friedmann function W, and hence on the choice of the function F.

We will not make such a choice here and discuss the generic properties of the aforementioned fixed points.

In order to discuss the stability of a fixed point $(H^\star, X^\star)$, we introduce small perturbations (h, x) around the fixed point such that

$$H = H^\star + h, \qquad X = X^\star + x, \tag{25}$$

and then linearize the dynamical equations in h and x. This leads to a linear system of the form

$$\begin{pmatrix} \dot{h} \\ \dot{x} \end{pmatrix} = J \cdot \begin{pmatrix} h \\ x \end{pmatrix}, \qquad J = \begin{pmatrix} \frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial X} \\ \frac{\partial \dot{X}}{\partial H} & \frac{\partial \dot{X}}{\partial X} \end{pmatrix}\Bigg|_{H=H^\star, X=X^\star}, \tag{26}$$

where the partial derivatives are given by

$$\frac{\partial \dot{H}}{\partial H} = -(X+3)\frac{WW_H + HW_H^2 - HWW_{HH}}{W_H^2}, \qquad \frac{\partial \dot{H}}{\partial X} = -H\frac{W}{W_H}, \tag{27a}$$

$$\frac{\partial \dot{X}}{\partial H} = X(X-1), \qquad \frac{\partial \dot{X}}{\partial X} = (2X-1)H. \tag{27b}$$

It follows immediately that $\partial\dot{H}/\partial X = 0$ whenever $\dot{H} = 0$. A detailed treatment is necessary for $\partial\dot{H}/\partial H$. For this purpose we write W in the form

$$W \approx W^\star + c|H - H^\star|^b \tag{28}$$

in the vicinity of the critical value $H^\star$, where $W^\star$, c, b are constants. This approximation covers all cases we mentioned before, and we find the following behavior:

(i) To study the case $W \to 0$, we set $W^\star = 0$ and $b > 0$. Note that we have $W_H \to \pm\infty$ for $b < 1$ and $W_H \to 0$ for $b > 1$, while W_H remains finite for $b = 1$. In all three cases we find

$$\dot{H}_H^\star = \lim_{H \to H^\star} \frac{\partial \dot{H}}{\partial H} = -\frac{(X+3)H^\star}{b}. \tag{29}$$

(ii) To model the case $W \to \infty$, we consider $b < 0$, and can likewise set $W^\star = 0$. We find the same limit (29).

(iii) Finally, we consider the case $W^\star > 0$ with $b > 0$. If $H^\star \neq 0$, W_H must diverge in order to obtain a fixed point. This is the case for $b < 1$. However, in this case $\partial\dot{H}/\partial H$ also diverges, so a linear approximation cannot be used to determine the stability of the fixed point, and one must explicitly study the behavior of $\dot{H}$ near the fixed point. If $H^\star = 0$, we only need $b < 2$ in order to obtain a fixed point. For $1 < b < 2$ we find that $\partial\dot{H}/\partial H$ likewise diverges, while for $0 < b < 1$ we obtain $\partial\dot{H}/\partial H \to 0$, and so also in these cases the linearized system is not sufficient. Finally, a special case is given by $H^\star = 0$ and $b = 1$, in which we find

$$\dot{H}_H^\star = \lim_{H \to H^\star} \frac{\partial \dot{H}}{\partial H} = -\frac{(X+3)W^\star}{c}. \tag{30}$$

From this analysis it follows that the Jacobi matrix simplifies significantly at a fixed point,

$$J = \begin{pmatrix} \dot{H}_H^\star & 0 \\ X^\star(X^\star - 1) & (2X^\star - 1)H^\star \end{pmatrix}, \qquad \dot{H}_H^\star = -A(X^\star + 3), \tag{31}$$

where the constant A follows from either formula (29) or (30), depending on the nature of the fixed point. Note that $A > 0$ in the cases $W \to 0$ at $H^\star > 0$, $W \to \infty$ at $H^\star < 0$ and $W > 0$, $W_H > 0$ at $H^\star = 0$. Similarly, we find $A < 0$ in the cases $W \to 0$ at $H^\star < 0$, $W \to \infty$ at $H^\star > 0$ and $W > 0$, $W_H < 0$ at $H^\star = 0$. In these cases a linear approximation is sufficient in order to determine the stability of the fixed points. We thus calculate the eigenvalues of J for these cases only, and distinguish between the three different conditions obtained from $\dot{X} = 0$.

(i) $H^\star = 0$: The Jacobi matrix and eigenvalues reduce to

$$J = \begin{pmatrix} -A(X^\star + 3) & 0 \\ X^\star(X^\star - 1) & 0 \end{pmatrix}, \qquad \lambda_1 = -A(X^\star + 3), \qquad \lambda_2 = 0. \tag{32}$$

Note that one eigenvalue vanishes, whose corresponding eigenvector is given by ∂_X. This relates to the fact that in this case all points with $H = 0$ and $0 \leq X \leq 1$ are nonisolated fixed points. The stability is determined by the remaining eigenvalue, which is positive for $W_H < 0$, which yields a repeller, and negative for $W_H > 0$, which yields an attractor.

(ii) $X^\star = 0$: The Jacobi matrix and eigenvalues are given by

$$J = \begin{pmatrix} -3A & 0 \\ 0 & -H^\star \end{pmatrix}, \quad \lambda_1 = -H^\star, \quad \lambda_2 = -3A. \tag{33}$$

Both eigenvalues are negative for $W \to 0$ at $H^\star > 0$, which yields an attractor, and positive for $W \to 0$ at $H^\star < 0$, which yields a repeller. For $W \to \infty$ the eigenvalues have opposite signs; we find a saddle point.

(iii) $X^\star = 1$: In this case the Jacobi matrix and eigenvalues take the form

$$J = \begin{pmatrix} -4A & 0 \\ 0 & H^\star \end{pmatrix}, \quad \lambda_1 = H^\star, \quad \lambda_2 = -4A. \tag{34}$$

Now the situation is reversed compared to the previous case. Both eigenvalues are negative for $W \to \infty$ at $H^\star < 0$, which yields an attractor, and positive for $W \to \infty$ at $H^\star > 0$, which yields a repeller. For $W \to 0$ the eigenvalues have opposite signs; we find a saddle point.

In all other cases the linearized analysis is not sufficient, and the sign of $\dot{H}$ must be studied explicitly in the vicinity of the fixed point using the full, nonlinear equations of motion. We can summarize our findings as follows. We start with a classification of regular fixed points, which are elements of the physical phase space $\mathcal{P}$:

Statement 1. A point $(H^\star, X^\star)$ is a regular fixed point of the dynamical system if it satisfies one of the following criteria:

(i) In the case $X^\star = 0$, $H^\star > 0$, $W^\star = 0$, it is an isolated attractor. The corresponding solution is an expanding de Sitter vacuum solution with scale factor (24).

(ii) In the case $X^\star = 0$, $H^\star < 0$, $W^\star = 0$, it is an isolated repeller. The corresponding solution is a contracting de Sitter vacuum solution with scale factor (24).

(iii) In the case $X^\star = 1$, $H^\star \neq 0$, $W^\star = 0$, it is an isolated saddle point. The corresponding solution is physically equivalent to either of the two aforementioned cases.

(iv) Points with $0 \le X^\star \le 1$, $H^\star = 0$, $W^\star > 0$, and $W^\star_H > 0$ are non-isolated attractors. The corresponding solution is a static universe with Minkowski geometry, but non-vanishing matter content.

(v) Points with $0 \le X^\star \le 1$, $H^\star = 0$, $W^\star > 0$ and $W^\star_H < 0$ are non-isolated repellers. The corresponding solution is a static universe as in the aforementioned case.

(vi) Fixed points, whose stability cannot be determined from a linearized analysis, are given by:

(a) $H^\star = 0$ and $W^\star = 0$; this is a Minkowski vacuum solution.

(b) $H^\star = 0$, $W^\star > 0$, and $W^\star_H$ diverges; also, this is a static universe with nonvanishing matter content.

(c) $H^\star = 0$, $W^\star > 0$, and $W^\star_H = 0$ such that $H/W_H \to 0$; this case corresponds to a finite time singularity of type IV, as shown in Sec. IV D. ■

There are a number of fixed points, which either lie outside the physical phase space (16), or do not correspond to solutions of the Friedmann equations (14) in terms of the original variables (H, ρ_m, ρ_r), since they are obtained from a singular coordinate transformation. Here we find the following conditions:

Statement 2. A point $(H^\star, X^\star)$ is an irregular fixed point of the dynamical system if it satisfies one of the following criteria:

(i) In the case $X^\star = 1$, $H^\star < 0$, $W^\star \to \infty$, it is an isolated attractor. Trajectories approaching this point undergo an exponential decreasing of the scale factor (24), while the matter density and pressure grow exponentially.

(ii) In the case $X^\star = 1$, $H^\star > 0$, $W^\star \to \infty$, it is an isolated repeller. Trajectories originating from this point undergo an exponential growth of the scale factor (24), while the matter density and pressure decrease exponentially.

(iii) In the case $X^\star = 0$, $H^\star \neq 0$, $W^\star \to \infty$, it is an isolated saddle point. Note that this point is neither a physical solution, nor approached by any trajectories.

(iv) Fixed points, whose stability cannot be determined from a linearized analysis, are given by

(a) $X^\star \in \{0, 1\}$, $H^\star \neq 0$, $W^\star > 0$, and $W^\star_H$ diverges; even though this point lies inside the physical phase space spanned by the variables (H, X), it does not have a corresponding solution in the original matter variables (H, ρ_m, ρ_r), as it originates from a singular coordinate transformation.

(b) $H^\star = 0$ and $W^\star \to \infty$; this point does not belong to the physical phase space and corresponds to a static universe with an infinite matter density. ■

An overview of all conditions listed in Statements 1 and 2, ordered by the properties of points in the compactified phase space, is given in Fig. 3 in Appendix A.

D. Possibility of bounce and turnaround

We now come to the discussion of bounces and turnarounds, i.e., transitions between expanding and contracting phases of the evolution of the universe. Note that for any such transition we have $\dot{a} = 0$, and hence $H = 0$. Thus, these kind of transitions can occur only if $H = 0$ lies inside the physical phase space (19) given by the condition that the total matter energy density is positive, and further require $\dot{H} \neq 0$. The former is the case if and only if

$$W|_{H=0} \ge 0. \tag{35}$$

A bounce is given when $\dot{H}|_{H=0} > 0$, while a turnaround is characterized by $\dot{H}|_{H=0} < 0$. From the dynamics (21) of the Hubble parameter it follows that $\dot{H}$ is nonzero and finite at $H = 0$ if and only if

$$\lim_{H\to 0}\frac{H}{(\ln W)_H} = \frac{1}{(\ln W)_{HH}}\bigg|_{H=0} \tag{36}$$

is finite, and hence, in particular, $(\ln W)_H \to 0$ for $H \to 0$. Explicitly calculating the derivatives of $\ln W$ then shows that $\dot{H} \neq 0$ if and only if $W > 0$, $W_H = 0$, and $W_{HH} \neq 0$ at $H = 0$, and that the sign of W_{HH} determines the sign of $\dot{H}$. We thus conclude and summarize as follows:

Statement 3. At $H = 0$ we have $\dot{H} \neq 0$ if and only if $W > 0$, $W_H = 0$, and $W_{HH} \neq 0$, where

(i) for $W_{HH} < 0$ we have $\dot{H} > 0$ and hence a bounce,
(ii) for $W_{HH} > 0$ we have $\dot{H} < 0$ and hence a turnaround. ∎

Examples of bouncing cosmologies in $f(T)$ gravity have been discussed in Refs. [33,50,51].

E. Impossibility of cyclic and oscillating universes

Another interesting aspect, which is closely related to the existence of bounces and turnarounds as discussed in the previous section, is the possibility of cyclic universe solutions. Conventionally, these are defined as periodic solutions for the scale factor $a(t + t_0) = a(t)$, and thus, in particular, imply that also the Hubble parameter $H(t + t_0) = H(t)$ is periodic and has both positive and negative phases during each period. This means that in a cyclic universe both bounces and turnarounds occur periodically. However, this can immediately be excluded using Statement 3, since the conditions for a bounce and a turnaround are mutually exclusive and cannot be simultaneously satisfied for any given $f(T)$ theory of gravity. Note that this property is even more restrictive and prohibits any solutions in which the scale factor shows an oscillating behavior in the sense that the dynamics change more than once between expansion and contraction.

We can also relax the periodicity condition and demand only that $H(t + t_0) = H(t)$ is periodic. This allows for a periodic growth of the scale factor, $a(t + t_0) = \lambda a(t)$ with constant λ. However, one easily sees that also this is not possible. Recall from Sec. III A that the sign of $\dot{H}$ given by equation (21) is independent of $X \in [0, 1]$. Any line of constant H can therefore be crossed in only one direction, with either increasing or decreasing H, but not in both directions, as it would be necessary for a periodic orbit with variable H. Periodic orbits with constant H and only variable X are likewise excluded, since for any fixed H the sign of $\dot{X}$ is also independent of X, and the same argument holds. Finally, oscillating behavior of the Hubble parameter H is also excluded, which follows from the same argumentation as for excluding the oscillating behavior of a. We summarize:

Statement 4. Periodic and oscillating orbits in the (H, X) phase space, as well as cyclic and oscillating universe solutions, are not possible. ∎

We finally remark that this very general result does not depend in any way on the choice of the function $f(T)$ in the action. It does, however, depend on the matter content, which we have fixed to dust and radiation. Exotic matter, which would allow for transitions between positive and negative matter densities, could potentially lead to oscillating behavior. However, we will not consider exotic matter here, and conclude our discussion of the phase space of $f(T)$ gravity and its basic properties. Another important aspect is the existence and classification of finite time singularities. We present an exhaustive treatment in the following section.

IV. FINITE TIME SINGULARITIES

The dynamical systems approach detailed in the previous section now allows us to discuss the possibility of finite time singularities [52,53] in $f(T)$ gravity. Note that there are different types of singularities, which can be distinguished by the behavior of H and $\dot{H}$ near the singularity. This will be explained in detail in Sec. IV A. We then describe three types of singularities: Those for which both H and $\dot{H}$ become infinite are discussed in Sec. IV B. The case where $\dot{H}$ diverges at a finite value of H is studied in Sec. IV C. Finally, in Sec. IV D we consider the case in which both H and $\dot{H}$ remain finite, but higher time derivatives of H diverge.

A. Types of singularities

We start with a brief review of the possible types of finite time singularities, studied in detail in Refs. [35,54]. For a singularity occurring at time $t^\diamond$, it is conventional to approximate the Hubble parameter close to the singularity by the asymptotic behavior [53]

$$H(t) \approx H^\diamond + \frac{h}{|t - t^\diamond|^k} \tag{37}$$

with real constants $H^\diamond$, h, k. Different types of singularities are distinguished by the value of the parameter k. Classically, one considers four types of singularities, which are denoted as follows [52]:

Type I: For $k \geq 1$ both H and $\dot{H}$ diverge for $t \to t^\diamond$; in this case we can set $H^\diamond = 0$ without loss of generality. By integrating the relation (37), one can see that the logarithm of the scale factor $\ln a$ also diverges at the singularity, and so either $a \to 0$ or $a \to \infty$. If this singularity occurs in the past of an expanding universe, it is called a big bang. A future expanding singularity of this type is known as a big rip, while a future collapsing singularity is called a big crunch.

Type II: In the range $-1 < k < 0$ the Hubble parameter $H \to H^\diamond$ stays finite, but its derivative $\dot{H}$ diverges at the singularity. These singularities are called sudden singularities.

Type III: If the singularity parameter is in the interval $0 < k < 1$, we have a similar behavior to the case of a type I singularity; both H and $\dot{H}$ diverge. The only difference between these two types lies in the fact that for a singularity of type III the scale factor remains finite.

Type IV: Finally, for $k < -1$ with $k \notin \mathbb{Z}$ both H and $\dot{H}$ remain finite at the singularity, but higher time derivatives of H diverge.

In principle, it is also possible to consider singularities with a more general asymptotic behavior of the Hubble parameter than the power law (37); however, we do not consider such general singularities here, and restrict ourselves to the four aforementioned types. Moreover, since singularities of type I and type III differ only by the asymptotic behavior of the scale factor a, which is not explicit in our dynamical system, we will treat them together. By solving the asymptotic behavior (37) for the time t and doing the same with its time derivative, we can express $\dot{H}$ through H in the vicinity of the singularity. Note that by definition of the constants we have

$$\frac{H - H^\diamond}{h} > 0, \tag{38}$$

and so we can write

$$\dot{H} \approx \pm kh\left(\frac{H - H^\diamond}{h}\right)^{1+\frac{1}{k}}, \tag{39}$$

where the positive sign holds for future singularities $t < t^\diamond$, while the negative sign holds for past singularities $t > t^\diamond$. In the following, we use the abbreviation $W^\diamond = W(H^\diamond)$, as well as similar abbreviations for the derivatives of W at the singularity. We do not *a priori* demand that these derivatives exist at the singularity itself, but only in a neighborhood of the singularity, and then derive suitable limit values. In the following sections we give a detailed discussion of all singularity conditions. All conditions are also summarized in Fig. 3 in Appendix in graphical form.

B. Singularities of type I and III: $H \to \pm\infty$ and $\dot{H} \to \pm\infty$

The first case we discuss is $k > 0$, where both H and $\dot{H}$ diverge at the singularity, and we set $H^\diamond = 0$. From the asymptotic behavior (39) it follows that the condition for a singularity can be expressed as

$$\begin{aligned} 0 &= \pm\frac{1}{k}\lim_{H\to\pm\infty}\left(\frac{h}{H}\right)^{\frac{1}{k}} = \lim_{H=\pm\infty}\frac{H}{\dot{H}} = -\lim_{H\to\pm\infty}\frac{W_H}{(X+3)W} \\ &= -\lim_{H\to\pm\infty}\frac{(\ln W)_H}{(X+3)}, \end{aligned} \tag{40}$$

where the sign under the limit depends on whether one discusses a singularity for an expanding universe, $H \to \infty$, or collapsing universe $H \to -\infty$. We further see that also the type of the singularity, which is determined by the value of k, can be seen from the asymptotic behavior of $(\ln W)_H$ by taking the logarithm under the limit (40). This can be summarized as follows:

Statement 5. A finite time singularity with $H \to \pm\infty$ and $\dot{H} \to \pm\infty$ exists if and only if

$$\lim_{H\to\pm\infty}(\ln W)_H = 0, \tag{41}$$

where the positive sign corresponds to an expanding universe, while the negative sign corresponds to a collapsing universe. The singularity parameter $k > 0$ is given by

$$k = -\lim_{H\to\pm\infty}\frac{\ln|H|}{\ln|(\ln W)_H|}, \tag{42}$$

with the same sign as above. The singularity lies in the past if asymptotically $W_H > 0$, $\mathrm{sgn}\dot{H} = -\mathrm{sgn}H$, and in the future if asymptotically $W_H < 0$, $\mathrm{sgn}\dot{H} = \mathrm{sgn}H$. ■

C. Singularities of type II: Finite H but $\dot{H} \to \pm\infty$

We then discuss the case $-1 < k < 0$, which is also called a sudden singularity, and which occurs when $\dot{H}$ diverges for finite $H = H^\diamond$. This case occurs when there exists $H^\diamond$ such that

$$0 = \lim_{H\to H^\diamond}\frac{1}{\dot{H}} = -\lim_{H\to H^\diamond}\frac{W_H}{(X+3)HW}. \tag{43}$$

Recall from the discussion of fixed points in Sec. III C that $\dot{H} \to 0$ whenever $W \to 0$ or $W \to \infty$. Hence, we can exclude these cases here and study only the case of a finite limit $W \to W^\diamond > 0$. We distinguish the following two cases:

(i) $H^\diamond \neq 0$: In order for $\dot{H}$ to become singular, the numerator W_H of (39) must vanish for $H = H^\diamond$. This is the case if and only if $W_H \to W_H^\diamond = 0$.

(ii) $H^\diamond = 0$: The case is similar to the aforementioned one, but the condition for a sudden singularity is more restrictive and reads

$$\lim_{H\to 0}\frac{W_H}{H} \to 0, \tag{44}$$

which is the case if and only if both W_H and W_{HH} vanish at $H = 0$.

In order to determine the singularity parameter k, we make use of these conditions, which allow us to approximate the Friedmann function W near the singularity as

$$W \approx W^\diamond + \epsilon\left(\frac{H - H^\diamond}{c}\right)^b, \tag{45}$$

where $\operatorname{sgn} c = \operatorname{sgn} h$ is chosen, the expression inside the brackets becomes positive, $\epsilon = \pm 1$ is a sign, and we require $b > 1$ for $H^\diamond \neq 0$ and $b > 2$ for $H^\diamond = 0$. In this approximation the time derivative of the Hubble parameter becomes

$$\dot{H} \approx -\epsilon(X+3)HW^\diamond \frac{c}{b}\left(\frac{H-H^\diamond}{c}\right)^{1-b}. \tag{46}$$

We now have to distinguish two different cases. For $H^\diamond \neq 0$ we find that near the singularity $(H^\diamond, X^\diamond)$ we have

$$\dot{H} \approx -\epsilon(X^\diamond+3)H^\diamond W^\diamond \frac{c}{b}\left(\frac{h}{c}\right)^{1-b}\left(\frac{H-H^\diamond}{h}\right)^{1-b}. \tag{47}$$

By comparison with the general form (39) we immediately see $k = -b^{-1}$ from the exponent. The sign in equation (39) can be understood from

$$\operatorname{sgn}\left[-\epsilon\frac{(X^\diamond+3)H^\diamond W^\diamond c}{khb}\left(\frac{h}{c}\right)^{1-b}\right] = \operatorname{sgn}(\epsilon H^\diamond) = \begin{cases} 1 & \text{for future singularities,} \\ -1 & \text{for past singularities,}\end{cases} \tag{48}$$

where we have simply left out any positive, constant factors.

For $H^\diamond = 0$ we find the approximation

$$\dot{H} \approx -\epsilon(X^\diamond+3)W^\diamond \frac{c^2}{b}\left(\frac{h}{c}\right)^{2-b}\left(\frac{H}{h}\right)^{2-b}. \tag{49}$$

In this case we see $k = (1-b)^{-1}$, and the sign in equation (39) is given by

$$\operatorname{sgn}\left[-\epsilon\frac{(X^\diamond+3)W^\diamond c^2}{khb}\left(\frac{h}{c}\right)^{2-b}\right] = \operatorname{sgn}(\epsilon H) = \begin{cases} 1 & \text{for future singularities,} \\ -1 & \text{for past singularities,}\end{cases} \tag{50}$$

where in addition we used $\operatorname{sgn} h = \operatorname{sgn} H$ in this case.

We finally remark on the sign ϵ which appears in the results (48) and (50). Since $W^\diamond_H = 0$, the Friedmann function W must have either an extremal point or an inflection point at $H = H^\diamond$. In case of a maximum (minimum), ϵ is positive (negative) on both sides of the singularity. If W has an inflection point, ϵ differs on both sides of the singularity. We conclude and summarize:

Statement 6. A sudden singularity occurs at $H = H^\diamond$ if and only if $W^\diamond > 0$, $W^\diamond_H = 0$, and

(i) either $H^\diamond \neq 0$, in which case the singularity parameter is $k = -b^{-1}$ with $b > 1$,

(ii) or $H^\diamond = 0$ and $W^\diamond_{HH} = 0$, in which case $k = (1-b)^{-1}$ with $b > 2$,

where b can be determined from the ansatz (45). The singularity occurs in the future in the following cases:

(a) W has a local minimum at $H^\diamond > 0$,

(b) W has a local maximum at $H^\diamond < 0$,

(c) W has a rising inflection point at $H^\diamond$ and the singularity is approached from $|H| > |H^\diamond|$,

(d) W has a falling inflection point at $H^\diamond$ and the singularity is approached from $|H| < |H^\diamond|$.

The singularity occurs in the past if any of the aforementioned conditions is satisfied for $-W$ instead of W. ■

D. Singularities of type IV: finite H and $\dot{H}$

We finally come to the case $k < -1$ with $k \notin \mathbb{Z}$, which is the most subtle type of singularity, since both H and $\dot{H}$ remain finite, while higher time derivatives of H diverge. In order to study these singularities, we approximate the Hubble parameter as

$$H = H^\diamond + he^z \tag{51}$$

near the singularity, where we made use of the positivity condition (38). We then find that near the singularity

$$\frac{d}{dz}\ln|\dot{H}| \approx \frac{d}{dz}\ln|kh(e^z)^{1+\frac{1}{k}}| = 1 + \frac{1}{k}. \tag{52}$$

The dynamical equation (21) for $\dot{H}$ yields

$$\begin{aligned}\frac{d}{dz}\ln|\dot{H}| &= he^z\left(\frac{1}{H^\diamond + he^z} - [\ln|(\ln W)_H|]_H\right) \\ &= (H-H^\diamond)\left(\frac{1}{H} - [\ln|(\ln W)_H|]_H\right).\end{aligned} \tag{53}$$

The asymptotic behavior is obtained by approaching the singularity $H \to H^\diamond$; we conclude

$$1 + \frac{1}{k} = \lim_{H\to H^\diamond}(H-H^\diamond)\left(\frac{1}{H} - [\ln|(\ln W)_H|]_H\right). \tag{54}$$

For the values of k we consider in this section, the limit must be an element of the set

$$(0,1)\setminus\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}. \tag{55}$$

It is helpful to distinguish two different cases:

(i) For $H^\diamond \neq 0$ the term $1/H$ does not contribute to the limit. In this case the singularity condition reads

$$\lim_{H\to H^\diamond}(H-H^\diamond)[\ln|(\ln W)_H|]_H \in (-1,0)\setminus\left\{-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \ldots\right\}. \tag{56}$$

(ii) For $H^\diamond = 0$ the contribution of the term $1/H$ must be taken into account; the singularity condition becomes

$$\lim_{H\to H^\diamond}(H-H^\diamond)[\ln|(\ln W)_H|]_H \in (0,1)\backslash\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right\}. \tag{57}$$

Note that in order to satisfy these conditions, it is in particular necessary that the limit is finite and nonzero, which requires the asymptotic behavior

$$[\ln|(\ln W)_H|]_H = \frac{W_{HH}}{W_H} - \frac{W_H}{W} \sim \frac{1}{H-H^\diamond} \tag{58}$$

near the singularity. To achieve this behavior, we use the same approximation (45) as in the previous case of a sudden singularity. For $b > 0$ we obtain

$$\begin{aligned}\lim_{H\to H^\diamond}&(H-H^\diamond)[\ln|(\ln W)_H|]_H \\ &= -1 + \lim_{H\to H^\diamond} bW^\diamond\left[W^\diamond + \epsilon\left(\frac{H-H^\diamond}{c}\right)^b\right]^{-1} \\ &= b-1.\end{aligned} \tag{59}$$

For $H^\diamond \neq 0$ we thus require $b \in (0,1)\backslash\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$, which implies that W_H diverges for $H \to H^\diamond$. Similarly, for $H^\diamond = 0$ we require $b \in (1,2)\backslash\{\frac{3}{2},\frac{4}{3},\frac{5}{4},\ldots\}$. $W_H \to W_H^\diamond = 0$, while W_{HH} diverges for $H \to H^\diamond$. This can be summarized as follows:

Statement 7. Singularities with finite H and $\dot{H}$ occur at $H = H^\diamond$ if and only if $W^\diamond > 0$ and

(i) either $H^\diamond \neq 0$ and W_H diverges, in which case the singularity parameter is $k = b^{-1}$,

(ii) or $H^\diamond = 0$, $W_H^\diamond = 0$, and W_{HH} diverges, in which case $k = (1-b)^{-1}$,

where b is given by the asymptotic behavior (45), provided that $k \neq \mathbb{Z}$. The conditions for future and past singularities are the same as in Statement 6. ■

We finally remark that the conditions for the existence of this type of singularity in particular imply the existence of fixed points, since they satisfy Condition vi in Statement 1. These singularities hence comprise a special class of fixed points which are reached in finite time.

This concludes our discussion of finite time singularities. In the next section we will shift our focus to another aspect of $f(T)$ cosmology and derive a number of observable parameters.

V. OBSERVATIONAL PROPERTIES

In the final section about the general dynamical system approach we discuss how to relate the dynamical system to physical properties and observables of the cosmological model. For this purpose we study, in particular, two properties, namely the accelerating expansion of the universe in Sec. V A and the barotropic index of an equivalent dark energy model and the possibility of crossing the phantom divide in Sec. V B. Specific phases of accelerating expansion, in particular inflation and the observed late time acceleration, are discussed in Sec. V C. We finally show how several observational parameters, such as the Hubble parameter, deceleration parameter, and density parameters, can be read off from the dynamical system and further be used to constrain the Friedmann function $W(H)$ and select a particular phase space trajectory in Sec. V D.

A. Accelerating expansion

An important question about any $f(T)$ gravity model is whether it supports an epoch of accelerated expansion of the universe, and whether there are transitions between deceleration and acceleration. From the definition of the Hubble parameter it immediately follows that

$$\dot{H} = \frac{d}{dt}\frac{\dot{a}}{a} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2, \tag{60}$$

and hence the acceleration is given by

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H\left(H - (X+3)\frac{W}{W_H}\right). \tag{61}$$

We now focus on the transition between acceleration and deceleration, and thus in particular on phase space trajectories passing through the line where $\ddot{a} = 0$. Note that for $H = 0$ this condition implies $\dot{H} = 0$ and hence corresponds to a fixed point; there is no transition in this case. We thus find that transitions can occur only for $H \neq 0$ with

$$H = (X+3)\frac{W}{W_H} = \frac{X+3}{(\ln W)_H}. \tag{62}$$

One easily checks that this is possible only if W and W_H are finite and nonzero. To determine the direction of the transition, we further calculate the third derivative

$$2H\dot{H} + \ddot{H} = \frac{d}{dt}\frac{\ddot{a}}{a} = \frac{\dddot{a}a - \ddot{a}\dot{a}}{a^2} = \frac{\dddot{a}}{a} - H^3 - H\dot{H}. \tag{63}$$

We only need to study $\dddot{a}$ in the particular case $\ddot{a} = 0$ given by the relation (62). In this case the third derivative of the scale factor is given by

$$\left.\frac{\dddot{a}}{a}\right|_{\ddot{a}=0} = \frac{6H^3(X+1)W - H^5W_{HH}}{(X+3)W}. \tag{64}$$

We can thus summarize:

Statement 8. Transitions between acceleration and deceleration can occur only at phase space points satisfying

$H(\ln W)_H = X + 3$, following from equation (62), and the direction of the transition is determined by

$$\mathrm{sgn}\ddot{a} = \mathrm{sgn}\{H[6(X+1)W - H^2 W_{HH}]\}, \tag{65}$$

following from equation (64). ■

Studies of whether and how different models can incorporate accelerated expansion include Refs. [8,9,21,24,28,37,55–58].

B. Dark energy and the phantom divide

If we compare the cosmological field equations (14) with the corresponding equations for a generic dark energy model in general relativity, which are given by

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_{\mathrm{DE}}), \tag{66a}$$

$$\dot{H} = -4\pi G\left[\rho_m + \frac{4}{3}\rho_r + (1 + w_{\mathrm{DE}})\rho_{\mathrm{DE}}\right], \tag{66b}$$

then one can easily infer that $f(T)$ gravity can be described as an effective dark energy model, where the effective energy density of dark energy is given by

$$\rho_{\mathrm{DE}} = \frac{6H^2 - W}{16\pi G}, \tag{67}$$

while its effective barotropic index takes the form

$$\begin{aligned} w_{\mathrm{DE}} &= -1 - \frac{(X+3)}{3}\left(1 - 12\frac{H}{W_H}\right)\left(1 - 6\frac{H^2}{W}\right)^{-1} \\ &= -1 - \frac{X+3}{3}\frac{[\ln|W - 6H^2|]_H}{(\ln W)_H}. \end{aligned} \tag{68}$$

We are, in particular, interested in the question of whether the barotropic index is smaller or larger than -1, or whether there exists some $H = H^\times$ where it changes dynamically between these two possibilities. The critical value $w_{\mathrm{DE}} = -1$ discriminates between so-called phantom and non-phantom dark energy, and is hence also known as the "phantom divide." For this purpose it is sufficient to study the sign, zeroes, and poles of the second term. We can proceed similarly to the discussion of fixed points and singularities and distinguish a number of different cases. First, we consider $H^\times \neq 0$, and study the following three particular cases for $W^\times$:

(i) If W diverges, then also W_H diverges. Both terms in brackets in equation (68) approach 1, and the barotropic index approaches $-2 - \frac{X}{3} \neq -1$.

(ii) For $W \to 0$ one can easily see from the first expression for w_{DE} in equation (68) that $w_{\mathrm{DE}} \to -1$ unless $W_H \to 0$ and the corresponding factor in brackets diverges. However, one can see from the last expression in equation (68) that also in the case $W \to 0$ and $W_H \to 0$ the term $(\ln W)_H$ in the denominator diverges as discussed in Sec. III C, while the numerator stays finite. Hence, also in this case $w_{\mathrm{DE}} \to -1$. However, recall from Sec. III C that $W = 0$ implies $\dot{H} = 0$; no crossing can occur in this case.

(iii) If $W \to 6H^2$, it follows from an analogous argument that the denominator $[\ln|W - 6H^2|]_H$ diverges, and hence also w_{DE} diverges.

We thus see that none of these cases allow for a crossing of the phantom divide. For any other value of W, which is not covered by the aforementioned special cases, the second bracketed term in equation (68) is nonzero, finite, and does not equal 1. We then need to consult the value of W_H. Also, here there are three particular cases to be discussed:

(i) For $W_H \to 12H$ the first bracketed term in equation (68) vanishes and we obtain $w_{\mathrm{DE}} = -1$. Since $\dot{H} \neq 0$ in this case, as follows from Statement 1, this allows for a crossing of the phantom divide.

(ii) For $W_H \to 0$ the first bracketed term in equation (68) diverges, and so does w_{DE}.

(iii) Finally, if W_H diverges, the first bracketed term in equation (68) approaches 1, and we find $w_{\mathrm{DE}} \neq -1$.

Note that for all other values of W_H we likewise find $w_{\mathrm{DE}} \neq -1$; the only case we have found so far for crossing the phantom divide is the one involving $W_H = 12H$.

We are left with the case $H = 0$. Since we are interested in the possibility of crossing the phantom divide, we need to consider only such cases in which we obtain a finite, nonzero $\dot{H}$, i.e., a bounce or turnaround. These cases follow from the conditions given in Statement 3. In particular, we must have $W > 0$; the second bracketed term in equation (68) always approaches 1 for $H \to 0$. Thus, in order to cross the phantom divide, the first bracketed term must vanish. This is the case if and only if

$$\lim_{H\to 0}\frac{H}{W_H} = \frac{1}{W_{HH}|_{H=0}} = \frac{1}{12}, \tag{69}$$

i.e., if and only if $W > 0$, $W_H = 0$, and $W_{HH} = 12$ at $H = 0$.

We also determine in which direction the phantom divide is crossed. For this purpose we calculate the total time derivative $\dot{w}_{\mathrm{DE}}$. In the first crossing case, where $H^\times \neq 0$, we find

$$\dot{w}_{\mathrm{DE}} = -\frac{(X^\times + 3)^2(W^\times_{HH} - 12)(W^\times)^2}{432H^\times[W^\times - 6(H^\times)^2]}, \tag{70}$$

while in the second crossing case, for $H^\times = 0$, we obtain

$$\dot{w}_{\mathrm{DE}} = -\frac{(X^\times + 3)^2 W^\times W^\times_{HHH}}{864}. \tag{71}$$

We can summarize our findings as follows:

Statement 9. Crossing of the phantom divide occurs at $H = H^\times$ if and only if $W^\times > 0$ and

(i) either $H^\times \neq 0$, $W^\times \neq 6(H^\times)^2$ and $W_H^\times = 12H^\times$, in which case

$$\operatorname{sgn}\dot{w}_{\rm DE} = -\operatorname{sgn}\frac{(W_{HH}^\times - 12)}{H^\times[W^\times - 6(H^\times)^2]}, \tag{72}$$

(ii) or $H^\times = 0$, $W_H^\times = 0$, and $W_{HH}^\times = 12$, in which case

$$\operatorname{sgn}\dot{w}_{\rm DE} = -\operatorname{sgn}W_{HHH}^\times. \tag{73}$$

■

From the point of view of phantom dark energy and the divide line crossing, different models were considered in Refs. [55–57].

C. Inflation and late time acceleration

From our discussion of the effective dark energy content in the preceding section follows another interesting remark. We have seen that in the case $W = 0$, which implies $\dot{H} = 0$, we have $w_{\rm DE} = -1$. This leads to the following conclusion, using the effective dark energy density (67):

Statement 10. At fixed points $(H^\star, X^\star)$ with $W^\star = 0$ the solution becomes a de Sitter vacuum solution, i.e., $\rho_r = \rho_m = 0$ and $w_{\rm DE} = -1$, with cosmological constant

$$\Lambda = 8\pi G\rho_{\rm DE} = 3(H^\star)^2. \tag{74}$$

■

If $H > 0$, then this solution models the observed late time acceleration of the universe. Note that de Sitter fixed points of this type $W^\star = 0$ cannot be used to model inflation without invoking further mechanisms beyond the $f(T)$ dynamics we study here. To see this, recall from Statement 1 that for $W^\star = 0$ at $H^\star > 0$ there exists a saddle point at $X^\star = 1$ and an attractor at $X^\star = 0$. Any trajectories in the vicinity of these fixed points ultimately converge to the attractor, and thus never leave the accelerating de Sitter phase. Hence, there would be no exit from this type of inflation.

We finally remark that fixed points with $W^\star \to \infty$ could be potential candidates to model inflation. In the case $H^\star > 0$ we see from Statement 2 that there exists a repeller at $X^\star = 1$. This point, which is the limiting point of trajectories in their infinite past, corresponds to an infinite matter density. In this limit the scale factor a asymptotically becomes 0, with asymptotically constant Hubble parameter $H = H^\star$. However, note that of course our purely classical model breaks down as soon as densities become sufficiently high that the quantum nature of matter becomes relevant; one cannot extrapolate this trajectory into the infinite past. We would rather expect that inflation starts from a quantum regime.

D. Cosmological parameters

A number of observable parameters can be derived directly from the equations constituting the dynamical system. Most important are the density parameters, which are defined with the help of the critical density

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{75}$$

For different contributions to the matter density present in our model this yields the straightforward definitions

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{(1-X)W}{6H^2}, \qquad \Omega_r = \frac{\rho_r}{\rho_c} = \frac{XW}{6H^2}, \\ \Omega_{\rm DE} = \frac{\rho_{\rm DE}}{\rho_c} = 1 - \frac{W}{6H^2}. \tag{76}$$

Conventionally, one also defines a parameter Ω_k related to the spatial curvature; however, this parameter vanishes identically for our model, since we restrict ourselves to spatially flat FLRW spacetimes. As a consequence, the parameters satisfy the constraint equation

$$\Omega_m + \Omega_r + \Omega_{\rm DE} = 1, \tag{77}$$

which is simply a rewriting of the corresponding Friedmann equation (66), and which is in good agreement with current observations [59].

Another important set of observable parameters are of course the Hubble parameter H itself and the deceleration parameter q defined by

$$q = -\frac{\ddot{a}}{aH^2} = -1 - \frac{\dot{H}}{H^2} = -1 + (X+3)\frac{W}{HW_H}. \tag{78}$$

Their present values are related to a Taylor expansion of the scale factor $a(t)$ around the present time t_0 given by

$$a(t) = a_0\left[1 + H_0(t-t_0) - \frac{1}{2}H_0^2 q_0 (t-t_0)^2\right] + \mathcal{O}((t-t_0)^3). \tag{79}$$

From the present time values of these parameters we can derive two types of constraints. The Hubble parameter H_0 and the ratio between radiation and matter given by

$$\frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{X_0}{1-X_0} \tag{80}$$

fix a point (H_0, X_0) in phase space, and hence a particular trajectory. The total matter density further determines the present time value of W via

$$W_0 = 6H_0^2(\Omega_{m,0} + \Omega_{r,0}), \tag{81}$$

while the present time value of W_H follows from the deceleration parameter as

$$W_{H,0} = \frac{(X_0 + 3)W_0}{H_0(1 - q_0)}. \tag{82}$$

Hence, we obtain the first two coefficients in the Taylor expansion of the Friedmann function W around the present time value H_0 of the Hubble parameter.

With this we finish our discussion of observational properties, and thereby of generic $f(T)$ theories. We have now constructed a comprehensive set of tools to analyze any given $f(T)$ theory by the properties of the corresponding Friedmann function $W(H)$. In order to demonstrate the potential of these tools, we apply our formalism to a particular class of theories, for which $F(T)$ is given by a power law, in the next section.

VI. EXAMPLE: POWER LAW MODEL $F(T) = \alpha(-T)^n$

After discussing the general formalism we developed in the previous sections, we now apply this formalism to a specific model. In this model $F(T)$ is given by a power law $F(T) = \alpha(-T)^n$, where α and n are constant parameters. The main purpose of this section is to illustrate our formalism. We start with the definition of the power law model and a discussion of the allowed ranges and special values of its parameters in Sec. VI A. We then derive a number of properties of the general power law model, without restrictions on the constant parameters, using our general formalism. We discuss the physical phase space, its boundaries, and fixed points in Sec. VI B. Finite time singularities are discussed in Sec. VI C. We continue by discussing phases of accelerating expansion in Sec. VI D and the properties in Sec. VI E. The results obtained in Secs. VI B to VI E will then allow us to discuss the general dynamics and draw qualitative phase diagrams in Sec. VI F, as well as study the behavior of physical quantities along trajectories in Sec. VI G. Finally, in Sec. VI H we derive a number of observational parameters and discuss their values for commonly used values of the model parameters.

A. Definition and model parameters

We start with a discussion of the generic properties of the power law model $F(T) = \alpha(-T)^n$, where α and n are constant parameters whose values we leave arbitrary in this section. Note that $-T = 6H^2$ is never negative. We calculate the function

$$W = 6H^2 + (1 - 2n)\alpha(6H^2)^n, \tag{83}$$

its first derivative

$$W_H = 12H + 2n(1 - 2n)\alpha\frac{(6H^2)^n}{H}, \tag{84}$$

and second derivative

$$W_{HH} = 12[1 - n(1 - 2n)^2\alpha(6H^2)^{n-1}]. \tag{85}$$

It is important to distinguish a few special cases:

(i) For $\alpha = 0$ the model trivially reduces to teleparallel equivalent of general relativity.
(ii) In the case $n = 0$ we see that $F(T)$ is simply a cosmological constant. Hence, the model becomes equivalent to GR with a cosmological constant.
(iii) For $n = 1$ the total Lagrangian reads $(1 - \alpha)T$. This model is equivalent to general relativity, where the gravitational constant is rescaled by a factor $1 - \alpha$. Obviously, $\alpha = 1$ must be excluded in this case.
(iv) For $n = \frac{1}{2}$ the terms originating from $F(T)$ do not contribute to the cosmological field equations. Hence, we obtain the same cosmological dynamics as for general relativity.

In the following we will exclude these special cases from our analysis, since they would have to be treated separately but do not yield any new features beyond the exhaustively studied GR cosmology with a cosmological constant.

B. Physical phase space and fixed points

We now apply the general framework we developed in this article to this model. We first determine the physical phase space, which is given by the condition $W \geq 0$ obtained in Sec. III A. The boundary of the phase space is thus given by the condition that either $W = 0$ or $W \to \infty$, which are also fixed point conditions according to Statements 1 and 2 detailed in Sec. III C. It thus makes sense to study the fixed point conditions first. We start with the condition $W = 0$, which for the model we consider here has the solutions

$$H = 0 \quad \text{and} \quad (6H^2)^{1-n} = (2n - 1)\alpha. \tag{86}$$

The point $H = 0$, which is a solution only for $n > 0$, is special and will be treated separately. The second equation has a positive and a negative solution for $(2n - 1)\alpha > 0$ and $n \neq 1$. In the following we will denote the positive solution by $H = H^\star$. According to Conditions i, ii, and iii in Statement 1, we thus obtain different fixed points $(H, X) \in \mathcal{P}$: one attractor $(H^\star, 0)$, one repeller $(-H^\star, 0)$, and two saddle points $(\pm H^\star, 1)$. According to Statement 10 they all correspond to a de Sitter universe. Note that these are the only fixed points with $H \neq 0$, since the power law model does not satisfy any other condition listed in Statement 1 or 2, except for $H = 0$. The point $(H^\star, 0)$ should correspond to the same state as the stable de Sitter fixed point found in studies with Hubble-rescaled variables [21,24,27,28,30].

In the case $H = 0$ we see that Condition vi a in Statement 1 is satisfied for $n > 0$, while Condition iv b in Statement 2 is satisfied for $n < 0$. The first one is a regular vacuum fixed point, while the latter belongs to a divergent matter density, and hence lies on the boundary outside of the physical phase space. In both cases the stability cannot be determined from the linearized analysis. We will determine their stability from the phase diagrams in Sec. VI F.

From our analysis it follows that the physical phase space is bounded in H if either $\alpha > 0$ and $n > \frac{1}{2}$ or $\alpha < 0$ and $n < \frac{1}{2}$. Studying the sign of W in these cases yields different restrictions on the physical phase space:

(i) For $\alpha > 0$ and $n > 1$ we obtain $W \geq 0$ only for $|H| \leq H^\star$. The absolute value of the Hubble parameter thus has an upper bound.
(ii) For $\alpha > 0$ and $\frac{1}{2} < n < 1$ the physical phase space is given by $|H| \geq H^\star$, and so we receive a lower bound instead.
(iii) For $\alpha < 0$ and $n < \frac{1}{2}$ the absolute value of the Hubble parameter likewise has a lower bound given by $|H| \geq H^\star$.

Finally, for $(2n-1)\alpha \leq 0$, the physical phase space covers all of $H \in \mathbb{R}$.

C. Finite time singularities

We now come to the discussion of finite time singularities in the power law model, for which we proceed in the same way as for the fixed points. We start with the case $H \to \pm\infty$ as discussed in Sec. IV B, which belongs to the boundary of the physical phase space unless $\alpha > 1$ and $n > 1$. From Statement 5 it follows that we need to consider the asymptotic behavior of $(\ln W)_H$ for $H \to \pm\infty$. Note that in this limit we have $W \sim H^2$ if $n < 1$ and $W \sim H^{2n}$ if $n > 1$. In both cases, we find the asymptotic behavior

$$(\ln W)_H \sim \frac{1}{H}, \tag{87}$$

which corresponds to a finite time singularity of type I with parameter $k = 1$. For $H \to \infty$ we find $W_H > 0$; this is a past singularity, and hence a big bang. Conversely, for $H \to -\infty$, we find $W_H < 0$, and this is a future singularity for a collapsing universe, hence a big crunch.

We then come to sudden singularities, or singularities of type II, as discussed in Sec. IV C, which occur at finite H. Recall from Statement 6 that these occur only where $W^\diamond$ is nonzero and finite; we can exclude $H = 0$ for the power law model from our discussion. We are thus looking for points $H^\diamond \neq 0$ where $W^\diamond_H = 0$, as also required by Statement 6. This condition yields

$$(6H^2)^{1-n} = n(2n-1)\alpha, \tag{88}$$

where the right hand side is nonzero since we have already excluded those values for n and α for which it will vanish. In the following we will denote by $H^\diamond$ the positive solution of this equation, if it exists. This is the case for the following parameter ranges:

(i) For $\alpha > 0$ and $n < 0$ the physical phase space covers the whole range $H \in \mathbb{R}$, and hence also contains the singularity $H^\diamond$.
(ii) For $\alpha < 0$ and $0 < n < \frac{1}{2}$ we find the singularity at $H^\diamond < H^\star$; however, this point lies outside the physical phase space.
(iii) For $\alpha > 0$ and $\frac{1}{2} < n < 1$ we have qualitatively the same situation as in the aforementioned case, with a singularity at $H^\diamond < H^\star$ outside the physical phase space.
(iv) For $\alpha > 0$ and $n > 1$ the singularity also satisfies $H^\diamond < H^\star$, but in this case this point lies inside the physical phase space.

Hence, we need to discuss only the first and the last of these ranges. At the singularity we find that

$$W^\diamond = 6\left(1 - \frac{1}{n}\right)(H^\diamond)^2, \quad W^\diamond_H = 0, \quad W^\diamond_{HH} = 24(1-n). \tag{89}$$

In particular, we find that $W^\diamond_{HH}$ is finite and nonzero, and so the asymptotic behavior of the Friedmann function W is given by

$$W - W^\diamond \sim (H - H^\diamond)^2 \tag{90}$$

near the singularity. We hence obtain the singularity parameter $k = -\frac{1}{2}$, which is in the expected range for a singularity of type II. Finally, note that for $n < 0$ we have $W^\diamond_{HH} > 0$; W has a local minimum and we find a future sudden singularity at $H^\diamond > 0$, which is complemented by a past sudden singularity at $-H^\diamond$. The opposite time behavior is obtained in the case $n > 1$.

D. Accelerating expansion

As the next aspect we discuss the possibility of an accelerating expansion and the transition between accelerating and decelerating phases, noted already in the early papers [8,9]. For the acceleration we find the expression

$$\frac{\ddot{a}}{a} = H^2\left[1 - \frac{X+3}{2}\frac{6H^2 + (1-2n)\alpha(6H^2)^n}{6H^2 + n(1-2n)\alpha(6H^2)^n}\right]. \tag{91}$$

To determine the behavior of this function on the physical phase space, it is helpful to introduce the auxiliary functions

$$V = \frac{HW_H}{2} = 6H^2 + n(1-2n)\alpha(6H^2)^n,$$
$$U = \frac{W}{V} = \frac{6H^2 + (1-2n)\alpha(6H^2)^n}{6H^2 + n(1-2n)\alpha(6H^2)^n}. \tag{92}$$

With this definition it follows that

$$\frac{\ddot{a}}{a} = H^2\left[1 - \frac{X+3}{2}U\right], \tag{93}$$

and $\ddot{a} > 0$ if and only if $U < \frac{2}{X+3}$, where $\frac{2}{X+3}$ always takes values in the interval $[\frac{1}{2}, \frac{2}{3}]$. We have thus obtained a simple condition which determines the sign of the acceleration for the power law model. We will discuss this condition and its implications on the history of the universe in more detail in Sec. VI G.

E. Dark energy and the phantom divide

We start our discussion in this section with the possibility of crossing the phantom divide. For this purpose we check the conditions given in Statement 9 given in Sec. V B. Condition i is not satisfied since there is no $H^\times \neq 0$ for which $W_H^\times = 12H^\times$, as follows from equation (84). Also, Condition ii is not satisfied since $W_{HH} \neq 12$ at $H = 0$, independent of the parameters of the power law model, as follows from equation (85). Hence, there is no crossing of the phantom divide, as noted before in Ref. [36] via Statefinder and *Om* diagnostics.

The same result can also be seen from the barotropic index of the dark energy component, which is given by

$$\begin{aligned} w_{\mathrm{DE}} &= -1 + \frac{n}{3}(X+3)\frac{6H^2 + (1-2n)\alpha(6H^2)^n}{6H^2 + n(1-2n)\alpha(6H^2)^n} \\ &= -1 + \frac{n}{3}(X+3)U, \end{aligned} \tag{94}$$

where U is defined in equation (92). Note that $w_{\mathrm{DE}} = -1$ if and only if $W = 0$, and that this condition corresponds to fixed points according to our analysis in Sec. VI B. Hence, there are no transitions between $w_{\mathrm{DE}} < -1$ and $w_{\mathrm{DE}} > -1$, and thus no crossing of the phantom divide.

Using formula (94), we can also discriminate between phantom and nonphantom dark energy. One can easily see that $w_{\mathrm{DE}} < -1$ if and only if $nU < 0$, while for $nU > 0$ we find $w_{\mathrm{DE}} > -1$. If any of these conditions is satisfied for some point $(H, X) \in \mathcal{P}$, it is satisfied for all points on the trajectory through (H, X) since there is no crossing of the phantom divide. We can thus distinguish between phantom and nonphantom trajectories. We will do so in detail in Sec. VI G

F. General dynamics and phase diagrams

We now use the results on the physical phase space and the existence and behavior of fixed points and singularities obtained in Secs. VI B and VI C in order to discuss the general dynamics of the cosmological model for different values of the parameters n and α. Note that the only values of H at which the sign of $\dot{H}$ and $\dot{X}$, and hence the qualitative behavior of the system, can change are the values $0, \pm H^\diamond, \pm H^\star$. They divide the physical phase space into several regions, in which we now study the sign of $\dot{H}$, as well as the aforementioned physical quantities. The qualitative phase diagrams derived from our analysis are shown in Fig. 1, where we have used gray lines in order to mark the following distinguished values of H: a solid line marks $H = 0$, dashed lines mark the singularities $H = \pm H^\diamond$, and dotted lines mark the fixed points $H = \pm H^\star$. Note that all diagrams are symmetric under the transformation $H \mapsto -H, X \mapsto X, \dot{H} \mapsto \dot{H}, \dot{X} \mapsto -\dot{X}$. We will therefore only discuss the right half, $H \geq 0$, which corresponds to an expanding phase of the universe. Then the diagrams can be classified as follows:

(i) For $\alpha < 0$, $n > \frac{1}{2}$ shown in Fig. 1(b) and $\alpha > 0$, $0 < n < \frac{1}{2}$ shown in Fig. 1(d) the phase diagrams are qualitatively identical. The region $H > 0, 0 < X < 1$ is filled with trajectories which start from a big bang singularity at $(\infty, 1)$ and end at a static fixed point at (0,0). These are bounded by trajectories with $X \equiv 0$ and $X \equiv 1$, both of which start at the big bang (∞, X) and end at a static universe $(0, X)$. Finally, all points $(0, X)$ with $X \in [0, 1]$ are fixed points. This fact does not immediately become apparent from the phase diagrams, since $\dot{X} \sim H$ and $\dot{H} \sim H^2$; $\dot{H}/\dot{X} \sim H \to 0$ and trajectories become vertical near the fixed line $H = 0$. However, the velocity with which these trajectories are traversed converges to 0.

(ii) For $\alpha < 0$, $n < \frac{1}{2}$ shown in Fig. 1(a) and $\alpha > 0$, $\frac{1}{2} < n < 1$ shown in Fig. 1(e) there exists a critical value $H = H^\star$. Physical trajectories in the region $H > H^\star, 0 < X < 1$ start at the big bang singularity $(\infty, 1)$ and end at the attractive de Sitter fixed point $(H^\star, 0)$. Also, in this case there exist bounding trajectories with $X \equiv 0$ and $X \equiv 1$ going from (∞, X) to $(H^\star, X)$. Finally, there exists another bounding trajectory connecting the de Sitter saddle point $(H^\star, 1)$ to the attractive de Sitter fixed point $(H^\star, 0)$.

(iii) In the case $\alpha > 0$, $n < 0$ there exists another type of critical value $H = H^\diamond$ corresponding to a sudden singularity, which splits the physical phase space into different parts. Trajectories in the region $H > H^\diamond$ start at the big bang singularity $(\infty, 1)$ and reach the sudden singularity $(H^\diamond, X)$ at a finite value of X. Points with $0 < H < H^\diamond$ belong to trajectories starting at the static saddle point (0,1), which also reach the sudden singularity $(H^\diamond, X)$ at a finite value of X.

(iv) In the case $\alpha > 0$, $n > 1$ both types of critical values exist, with $0 < H^\diamond < H^\star$. Trajectories in the region $H^\diamond < H < H^\star$ start from the sudden singularity $(H^\diamond, X)$ at a finite value of X and approach the stable de Sitter fixed point $(H^\star, 0)$. In the region $0 < H < H^\diamond$ trajectories have the same starting condition but approach the static saddle point (0,0).

We can now also study the stability of the fixed points on the line $H = 0$, which we identified in Sec. VI B.

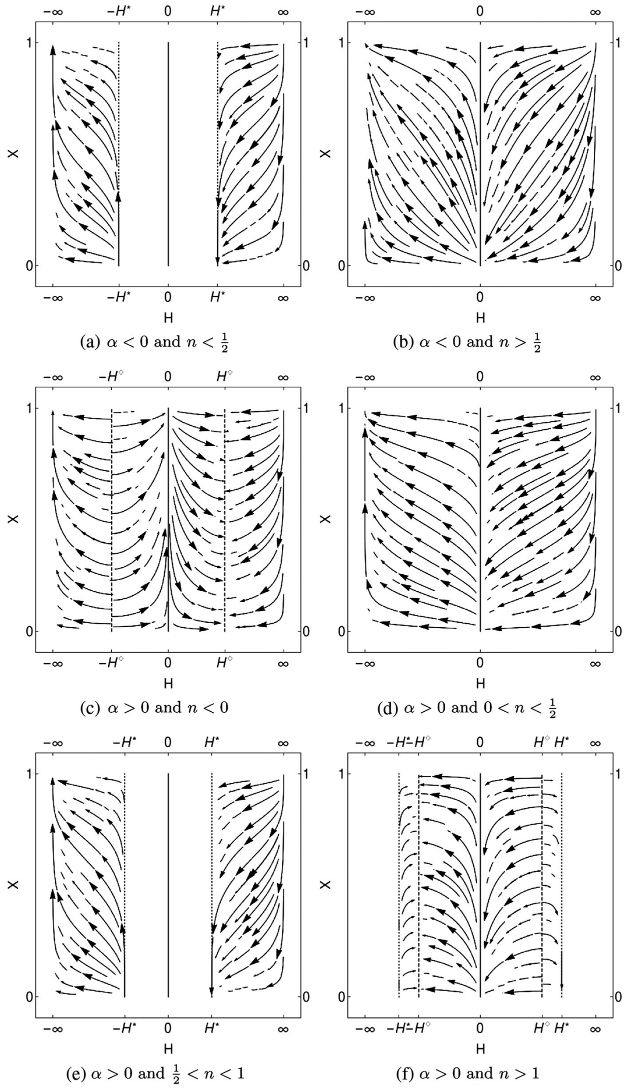

FIG. 1. Qualitative phase diagrams for the power law model. Distinguished values of H are marked by the following: a solid line marks $H = 0$, dashed lines mark the singularities $H = \pm H^{\diamond}$, and dotted lines mark the fixed points $H = \pm H^{\star}$.

Since the sign of $\dot{H}$ is the same on both sides $H > 0$ and $H < 0$ of these fixed points, it follows that trajectories on one side are traversed towards $H = 0$, while on the other side they are traversed away from $H = 0$. Hence, the fixed points on the line $H = 0$ are always saddle points.

G. Physical trajectories and their properties

Making use of the qualitative phase diagrams, we can now also study the behavior of the acceleration (91) and the barotropic index (94) of the effective dark energy along physical trajectories. In this discussion we restrict ourselves to trajectories in the interior of the phase space and exclude those which are located entirely on the boundary. Also, here we discuss only the case $H \geq 0$, using the fact that $\ddot{a}$ and w_{DE} do not change under a sign reversal $H \mapsto -H$. For this purpose we now take a closer look at the function $U(H)$ defined in equation (92). Note that

$$V^2 U_H = W_H V - W V_H = 12(n-1)^2(2n-1)\alpha H(6H^2)^n \tag{95}$$

has no zeroes for $H \neq 0$. Thus, U is monotonous whenever V is finite (and hence U is also defined). We distinguish the following cases, which are summarized in Fig. 2:

(i) For $\alpha < 0$ and $n < \frac{1}{2}$ as shown in Fig. 1(a) we have $U \to 0$ for the de Sitter fixed point $H \to H^\star$, where $\ddot{a} > 0$, and $U \to 1$ for $H \to \infty$; hence, $\ddot{a} < 0$. We thus have a transition from a decelerating to an accelerating phase. For $n < 0$ we have phantom dark energy, $w_{\mathrm{DE}} < -1$, while for $n > 0$ we obtain $w_{\mathrm{DE}} > -1$. Such dark energy behavior was noted in Ref. [36].

(ii) For $\alpha < 0$ and $\frac{1}{2} < n < 1$ as shown in Fig. 1(b) we find $U \to \frac{1}{n} > 1$ for $H \to 0$ and $U \to 1$ for $H \to \infty$. Hence, we have $1 < U < \frac{1}{n}$ everywhere, and thus $\ddot{a} < 0$. There is no accelerating phase. Further, $w_{\mathrm{DE}} > -1$; there is no phantom dark energy.

(iii) For $\alpha < 0$ and $n > 1$, which is also shown in Fig. 1(b), the limiting cases are given by $U \to 1$ for $H \to 0$ and $U \to \frac{1}{n} < 1$ for $H \to \infty$. The sign of the acceleration depends on the value of n. For $n < \frac{3}{2}$ it follows that $U > \frac{2}{3}$ everywhere, and hence $\ddot{a} < 0$; there is no accelerating phase. For $n > 2$ there is $U < \frac{1}{2}$ for $H \to \infty$; all trajectories pass from an accelerating to a decelerating phase. In the intermediate parameter range $\frac{3}{2} \leq n \leq 2$ the accelerating phase does not cover the whole edge $H \to \infty$ of the phase diagram, but only the part $X < 2n - 3$. This means in particular that the big bang singularity at $H \to \infty, X \to 1$ is located in the decelerating phase. Trajectories starting from the big bang singularity may either bypass the accelerating phase completely or experience a transient positive acceleration. In all of these cases we have $w_{\mathrm{DE}} > -1$.

(iv) In the case $\alpha > 0$ and $n < 0$ shown in Fig. 1(c) we need to discuss two regions of the phase diagram separately. For $H < H^\diamond$ there is $V < 0$, and hence also $U < 0$, $\ddot{a} > 0$, and $w_{\mathrm{DE}} > -1$. For $H > H^\diamond$ we find $W > V > 0$; hence $U > 1$, $\ddot{a} < 0$, and

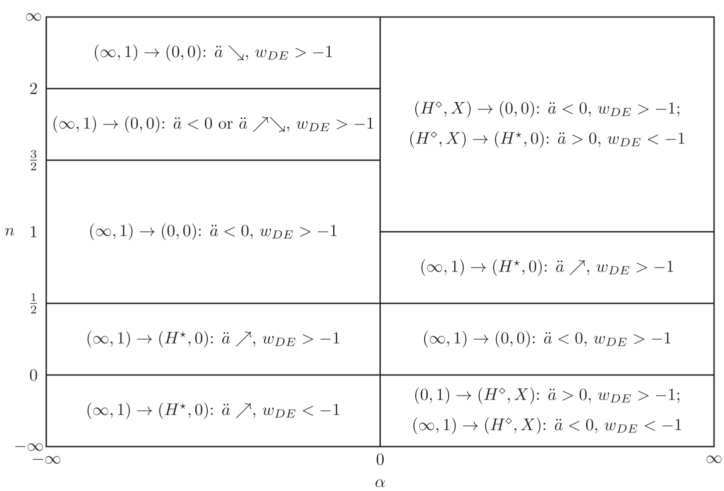

FIG. 2. Physical trajectories $(H_i, X_i) \to (H_f, X_f)$ in the power law model with $H \geq 0$, classified by their asymptotic initial and final states (H_i, X_i) and (H_f, X_f). Due to the symmetry of the phase diagrams, each of these has a corresponding trajectory with $H \leq 0$, which can be obtained by replacing $H \mapsto -H$ and reversing the direction of the arrows. Here $\ddot{a}\nearrow$ indicates a transition from deceleration to acceleration, while $\ddot{a}\searrow$ indicates a transition in the opposite direction.

$w_{\rm DE} < -1$. However, since these regions are separated by a singularity, there are no transitions between accelerating and decelerating phases or between phantom and nonphantom dark energy.

(v) For $\alpha > 0$ and $0 < n < \frac{1}{2}$ as shown in Fig. 1(d) we have the limiting cases $U \to \frac{1}{n} > 2$ for $H \to 0$ and $U \to 1$ for $H \to \infty$. We thus have $U > 1$ everywhere, and therefore $\ddot{a} < 0$; there is no accelerating phase. There is also no phantom dark energy since $w_{\rm DE} > -1$.

(vi) When $\alpha > 0$ and $\frac{1}{2} < n < 1$ as shown in Fig. 1(e), there is a de Sitter fixed point with $U \to 0$ for $H \to H^\star$; hence $\ddot{a} > 0$, while $U \to 1$ for $H \to \infty$, and thus $\ddot{a} < 0$. It follows that there is a transition from a decelerating to an accelerating phase. We still find $w_{\rm DE} > -1$ also in this case. Such dark energy behavior was noted in Ref. [36].

(vii) Finally, in the case $\alpha > 0$ and $n > 1$ the physical phase space $W \geq 0$ splits into two regions divided by a singularity at $H = H^\diamond$, as shown in Fig. 1(f), which we discuss separately. For $H < H^\diamond$ we have $W > V > 0$, and thus $U > 1$; hence $\ddot{a} < 0$ and $w_{\rm DE} > -1$. In contrast, for $H > H^\diamond$ we find $V < 0$, which yields $U < 0$; thus $\ddot{a} > 0$ and $w_{\rm DE} < -1$. Since accelerating and decelerating phases are separated by a singularity, there is no transition. The same holds for phantom and nonphantom dark energy.

H. Observational properties

We finally discuss how to derive a number of observational parameters for the power law model. For the density parameters (76) we find the expressions

$$\Omega_m = (1 - X)[1 + (1 - 2n)\alpha(6H^2)^{n-1}], \tag{96a}$$

$$\Omega_r = X[1 + (1 - 2n)\alpha(6H^2)^{n-1}], \tag{96b}$$

$$\Omega_{\rm DE} = -(1 - 2n)\alpha(6H^2)^{n-1}. \tag{96c}$$

We further calculate the deceleration parameter, which can be found from the acceleration (91) and is given by

$$q = -1 + \frac{X + 3}{2}\frac{6H^2 + (1 - 2n)\alpha(6H^2)^n}{6H^2 + n(1 - 2n)\alpha(6H^2)^n}. \tag{97}$$

We do not attempt to fit the parameters α and n of the power law model based on the observational properties derived generically in this article, since this particular class of models has already been extensively studied and numerous numerical fits have been obtained [8,36,38–40,42–49]. Instead, we only give a qualitative estimate based on the phase diagrams shown in Fig. 1, again with the purpose of illustrating the generic formalism developed in this article. If one assumes that the qualitative behavior of the Hubble parameter is described by a big bang singularity $H \to \infty$ at a finite time in the past, followed by an expansion, that finally leads to an accelerated expansion at a de Sitter fixed point, one is led to the conclusion that the expansion history of the universe is best described by either of the phase diagrams 1(a) or 1(e). Hence, one concludes that the model parameters must satisfy either $\alpha < 0$ and $n < \frac{1}{2}$ or $\alpha > 0$ and $\frac{1}{2} < n < 1$. Remarkably, both of these possibilities are consistent with a positive density parameter $\Omega_{\rm DE}$, as can be seen from equation (96c).

This concludes our discussion of the power law model. We have seen that our general formalism reproduces a large number of results which have been previously obtained in individual studies. These findings demonstrate the validity and usefulness of our formalism.

VII. CONCLUSION

In this article we have derived a two-dimensional dynamical system from the flat FLRW cosmological field equations of a generic $f(T)$ gravity theory, where the matter content is given by a combination of dust and radiation. We have shown that the full cosmological dynamics of this model depend only on a single function $W(H)$ of the Hubble parameter H, which is derived from the function $f(T)$ defining the particular theory under consideration. Instead of choosing a particular form of $f(T)$, we have kept the function fully generic and derived a number of physically relevant properties of the whole family of $f(T)$ theories.

Our main result is comprised of numerous conditions on the Friedmann function $W(H)$, which determine the existence and stability of fixed points in the cosmological dynamics, the possibility of a bounce or turnaround, the existence and severity of finite time singularities, the existence of accelerating and decelerating phases of the expansion of the universe, and transitions between them as well as the possibility of crossing the phantom divide. As a fully generic result, we have shown that there exist no periodic orbits in the phase space, and no oscillating universe solutions, independent of the choice of the function $f(T)$. Further, we have shown how points on the phase space and the shape of the Friedmann function $W(H)$ at these points can be related to observational cosmological parameters. Note that our chosen matter content manifestly satisfies all energy conditions, and that all features we discussed are direct consequences of the modified gravitational dynamics.

To illustrate our results and the general formalism, we have applied it to a generic power law model $f(T) = T + \alpha(-T)^n$. We have shown how the dynamics on the physical phase space depend on the constant parameters α and n of the model and displayed the phase diagrams for all qualitatively different values of these parameters. We have further characterized all possible trajectories in these phase spaces and their acceleration and effective dark energy. In particular, we have shown that it is not possible to dynamically cross the phantom divide $w_{\mathrm{DE}} = -1$ in these models. We have finally shown that there are no trajectories that start from an initial accelerating period (which would be interpreted as inflation), become decelerating, and finally transition back to an accelerating de Sitter phase.

The formalism and generic results derived in this article can now be applied to any particular $f(T)$ gravity theory or class of such theories in order to get a systematic overview of its cosmological behavior. It is left for future works to scrutinize other models in a similar manner, finally arriving at a catalog of $f(T)$ theories, classified by the dynamical properties of their cosmologies. Our results further hint towards the possibility to reverse the line of investigation and to construct heuristic $f(T)$ models based on a set of desired cosmological features. Once a class of models or a parameter range with viable dynamical behaviors has been confirmed, it can be subjected to further studies by other methods, e.g., the evolution of perturbations, local gravitational constraints, etc.

Finally, one may also consider a more general class of modified teleparallel theories augmented with a nonminimally coupled scalar field [60–63], a Gauss-Bonnet term [64,65], a boundary term [66], combinations of those [67,68], or higher derivatives of the torsion scalar [69]. Additionally, one may consider actions which are not a function of the torsion scalar (2), but of different contractions of the torsion tensor [70–72]. It should be straightforward to generalize our formalism to such theories, and thus to use our results to determine their cosmological dynamics.

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APPENDIX: CLASSIFICATION OF PHASE SPACE POINTS

In order to provide a better overview of all conditions on the Friedmann function W listed in the main part of the

<table>
<tr><th rowspan="2"></th><th rowspan="2">$W = 0$</th><th rowspan="2">$W \to \infty$</th><th colspan="4">$W > 0$</th></tr>
<tr><th colspan="2">$W_H = 0$</th><th>$W_H \neq 0$</th><th>$W_H \to \pm\infty$</th></tr>
<tr><td>$H > 0$</td><td>expan. dS FP;
1. i | 1. iii</td><td>expan. sing. FP;
2. ii | 2. iii</td><td colspan="2" rowspan="2">type II sing.;
6. i</td><td rowspan="2">$\dot{H} \neq 0$</td><td rowspan="2">pseudo FP,
type IV sing.;
2. iva & 7. i</td></tr>
<tr><td>$H < 0$</td><td>contr. dS FP;
1. ii | 1. iii</td><td>contr. sing. FP;
2. i | 2. iii</td></tr>
<tr><td rowspan="3">$H = 0$</td><td rowspan="3">static FP;
1. via</td><td rowspan="3">static sing. FP;
2. ivb</td><td>$W_{HH} = 0$</td><td>type II sing.;
6. ii</td><td rowspan="3">static FP;
1. iv | 1. v</td><td rowspan="3">static FP;
1. vib</td></tr>
<tr><td>$W_{HH} \neq 0$</td><td>bounce / turnaround;
3</td></tr>
<tr><td>$W_{HH} \to \pm\infty$</td><td>static FP, type IV sing.;
1. vic & 7. ii</td></tr>
<tr><td>$H \to \pm\infty$</td><td colspan="3">$(\ln W)_H \to 0$
type I or type III sing.; 5</td><td colspan="3">$(\ln W)_H \nrightarrow 0$
inf. time sing.</td></tr>
</table>

FIG. 3. Classification of all points in the compactified physical phase space $\bar{\mathcal{P}}$. Gray fields indicate conditions, while white fields show the physical consequence if all conditions in the same row and column are satisfied. Numbers and codes refer to the statements and cases in the main part of the article. FP = fixed point, dS = de Sitter.

article, we provide a graphical ordering scheme of all values H that belong to the compactified phase space $\bar{\mathcal{P}}$ in Fig. 3. Table entries refer to the corresponding general statements detailed in Secs. III to V. If several statements apply simultaneously to the same phase space point (H, X), they are listed with an ampersand (&) character. If several statements apply to the same value of H, but different values of X, they are separated with a pipe (|) character.

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Curriculum Vitae
In English

Personal information:

Name	Ulbossyn Ualikhanova
Date of birth	01.12.1990
Address	W. Ostwaldi Str 1, 50411 Tartu, Estonia
Telephone	+372 5673 3970
E-mail	ulbossyn.ualikhanova@gmail.com

Education:

2016 – 2021	doctoral studies (physics 80342), University of Tartu
2012 – 2014	master's studies (physics), ENU, Astana, KZ
2008 – 2012	bachelor's studies (physics), ENU, Astana, KZ
2002 – 2008	school-lyceum №7, Shymkent, KZ
2000 – 2002	secondary school №19, Shymkent, KZ
1997 – 2000	secondary school №34, Shymkent, KZ

Work experience:

2014 – 2016	teacher (1.5), Eurasian National University (ENU), Astana, Kazakhstan (KZ)

Awards and stipends:

Dora Plus scholarship for participation in the

1. The 27th Workshop on General Relativity and Gravitation in Japan (Higashi Hiroshima, 2017).
2. Gravity Malta: Gravitational waves, black holes and fundamental physics (Malta, 2018).
3. The 29th Workshop on General Relativity and Gravitation in Japan (Kobe, 2019).

Kristjan Jaak's program stipend for participation in the

1. Fifteenth Marcel Grossmann Meeting (Rome, 2018).
2. 10th Alexander Friedmann International Seminar (St. Petersburg, 2019).

Mobility funding from the European Union Framework Programme Horizon 2020 through COST Action: CA15117 for a research stay at

1. Physics Department of Federico II University (Naples, 2019).

Publications:

1. U. Ualikhanova and M. Hohmann "Parameterized post-Newtonian limit of general teleparallel gravity theories" *Phys. Rev.* D **100**, 104011 (2019)
2. M. Hohmann, C. Pfeifer, U. Ualikhanova, J.L. Said "Propagation of gravitational waves in symmetric teleparallel gravity theories" *Phys. Rev.* D **99**, 024009 (2019)
3. M. Hohmann, M. Kršsšák, C. Pfeifer, U. Ualikhanova "Propagation of gravitational waves in teleparallel gravity theories" *Phys. Rev.* D **98**, 124004 (2018)
4. M. Hohmann, L. Järv, U. Ualikhanova "Covariant formulation of scalar-torsion gravity" *Phys. Rev.* D **97**, 104011 (2018)
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Curriculum Vitae
eesti keeles

Isiklik informatsioon:

Nimi	Ulbossyn Ualikhanova
Sünniaeg	01.12.1990
Aadress	W. Ostwaldi tn 1, 50411 Tartu
Telefon	+372 5673 3970
E-mail	ulbossyn.ualikhanova@gmail.com

Hariduskäik:

2016 – 2021	doktoriõpe (füüsika 80342), Tartu Ülikool
2012 – 2014	magistriõpe (füüsika), ENU, Astana, KZ
2008 – 2012	bakalaureuseõpe (füüsika), ENU, Astana, KZ
2002 – 2008	kool-lütseum №7, Shymkent, KZ
2000 – 2002	põhikool №19, Shymkent, KZ
1997 – 2000	põhikool №34, Shymkent, KZ

Töökogemus:

2014 – 2016	õpetaja (1.5), Euraasia Rahvuslik Ülikool (ENU), Astana, Kasahstan (KZ)

Uurimistoetused ja stipendiumid:

DoRa stipendium osalemiseks:
1. The 27th Workshop on General Relativity and Gravitation in Japan (Higashi Hiroshima, 2017).
2. Gravity Malta: Gravitational waves, black holes and fundamental physics (Malta, 2018).
3. The 29th Workshop on General Relativity and Gravitation in Japan (Kobe, 2019).

Kristjan Jaagu välissõidu stipendium osalemiseks:
1. Fifteenth Marcel Grossmann Meeting (Rooma, 2018).
2. 10th Alexander Friedmann International Seminar (Peterburi, 2019).

Mobiilsustoetus Euroopa Liidu Horisont 2020 raamprogrammi COST projektist CA15117 uurimistööks:
1. Physics Department of Federico II University (Naples, 2019).

Publikatsioonid:

1. U. Ualikhanova and M. Hohmann "Parameterized post-Newtonian limit of general teleparallel gravity theories" *Phys. Rev.* D **100**, 104011 (2019)
2. M. Hohmann, C. Pfeifer, U. Ualikhanova, J.L. Said "Propagation of gravitational waves in symmetric teleparallel gravity theories" *Phys. Rev.* D **99**, 024009 (2019)
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