Mixed formulations for fluid-poroelastic structure interaction

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This thesis focuses on the development of mixed finite element methods for the coupled problem arising in the interaction between free fluid flow and flow in a deformable poroelastic medium. We adopt the Stokes or the Navier-Stokes equations to model the free fluid region, and the Biot system to describe the poroelastic medium. On the interface, mass conservation, balance of stresses and the slip with friction conditions are imposed via the Lagrange multiplier method.

We first develop a new mixed elasticity formulation for the Stokes-Biot problem. We establish the existence and uniqueness of a solution for the continuous weak formulation and perform stability and error analyses for the semi-discrete continuous-in-time mixed finite element approximation. We present numerical experiments that verify the theoretical results and illustrate the robustness of the method with respect to the physical parameters.

We then extend the previous results for the Stokes-Biot problem by considering dual-mixed formulations in both the fluid and structure regions. Well-posedness and stability results are established for the continuous weak formulation, as well as a semi-discrete continuous-in-time formulation with non-matching grids. In addition, we develop a new multipoint stress-flux mixed finite element method by involving the vertex quadrature rule. Well-posedness and error analysis with corresponding rates of convergences for the fully-discrete scheme are complemented by several numerical experiments.

Next, we propose an augmented fully mixed formulation for the coupled quasi-static Navier-Stokes – Biot model by introducing a "nonlinear-pseudostress" tensor linking the pseudostress tensor with the convective term in the Navier-Stokes equations and augmenting the variational formulation with suitable Galerkin redundant terms. We show well-posedness, derive stability and error analysis results for the associated mixed finite element approximation and conduct several numerical experiments.

Finally, we derive a fully mixed formulation with weakly symmetric stresses for the

Navier-Stokes – Biot model. We develop an extension of the multipoint stress-flux mixed finite element method that allows for local elimination of the fluid and poroelastic stresses, vorticity, and rotation, resulting in a positive definite finite volume scheme. A numerical convergence study is presented for the fully discrete scheme.

Keywords: numerical analysis, mixed finite element methods, FPSI, Stokes-Biot model, Navier-Stokes – Biot model, multipoint stress-flux, augmented formulation, finite volume method.

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Preface

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1.0 Introduction

1.1 Motivation and overview

The interaction of a free fluid with a deformable porous medium, referred to as fluid-poroelastic structure interaction (FPSI), is a challenging multiphysics problem. There has been an increased interest in this problem in recent years, due to its wide range of applications in petroleum engineering, hydrology, environmental sciences, and biomedical engineering, such as predicting and controlling processes arising in gas and oil extraction from naturally or hydraulically fractured reservoirs, cleanup of groundwater flow in deformable aquifers, designing industrial filters, and modeling blood-vessel interactions in blood flows. For this physical phenomenon, the free fluid region can be modeled by the Stokes or Navier–Stokes equations, while the flow through the deformable porous medium is modeled by the Biot system of poroelasticity [19]. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation that describes the average velocity of the fluid in the pores. The two regions are coupled via dynamic and kinematic interface conditions, including balance of forces, continuity of normal flux, continuity of normal stress and a no slip or slip with friction tangential velocity condition. The FPSI system exhibits features of both coupled Stokes–Darcy flows [42, 43, 47, 53, 62, 71, 78] and fluid–structure interaction (FSI) [17, 29, 46, 70], both of which have been extensively studied.

To our knowledge, one of the first works in analyzing the Stokes-Biot coupled problem is [75], where a fully dynamic system is considered and well-posedness is established by rewriting the problem as a parabolic system and using semigroup methods. One of the first numerical studies is presented in [16], using the Navier-Stokes equations to model the free fluid flow. The authors develop a variational multiscale finite element method and propose both monolithic and iterative partitioned methods for the solution of the coupled system. A non-iterative operator splitting scheme is developed in [27] for an arterial flow model that includes a thin elastic membrane separating the two regions, using a non-mixed pressure formulation for the flow in the poroelastic region. In [38], the fully dynamic coupled

Navier-Stokes/Biot system with a pressure-based Darcy formulation is analyzed. Finite element methods for mixed Darcy formulations, where the continuity of normal flux condition becomes essential, are considered in [25, 26] using Nitsche's interior penalty method and in [9, 10] using a pressure Lagrange multiplier formulation. More recently, a nonlinear quasi-static Stokes–Biot model for non-Newtonian fluids is studied in [4]. The authors establish well-posedness of the weak formulation in Banach space setting, along with stability and convergence of the finite element approximation. Additional works include optimization-based decoupling method [37], a second order in time split scheme [61], various discretization methods [18,36,79], dimensionally reduced model for flow through fractures [28], and coupling with transport [5].

To the best of our knowledge, all of the previous works consider displacement-based discretizations of the elasticity equation in the Biot system. In this thesis we develop a mixed finite element discretization of the quasi-static Stokes–Biot system using a mixed elasticity formulation with a weakly symmetric poroelastic stress. The advantages of mixed finite element methods for elasticity include locking-free behavior, robustness with respect to the physical parameters, local momentum conservation, and accurate stress approximations with continuous normal components across element edges or faces. Here we consider a three-field stress–displacement–rotation elasticity formulation. This formulation allows for mixed finite element methods with reduced number of degrees of freedom, see e.g. [11, 13]. It is also the basis for the multipoint stress mixed finite element method [6, 7], where stress and rotation can be locally eliminated, resulting in a positive definite cell-centered scheme for the displacement. We consider a mixed velocity–pressure Darcy formulation, resulting in a five-field Biot formulation, which was proposed in [63] and studied further in [8], where a multipoint stress-flux mixed finite element method is developed. We note that our analysis can be easily extended to the strongly symmetric mixed elasticity formulation, which leads to the four-field mixed Biot formulation developed in [82]. Finally, for the Stokes equations we consider the classical velocity–pressure formulation. The weak formulation for the resulting Stokes–Biot system has not been studied in the literature. One main difference from the previous works with displacement-based elasticity formulations [4, 10] is that the normal component of the poroelastic stress appears explicitly in the interface terms. Correspondingly, we introduce a Lagrange multiplier with a physical meaning of structure velocity that is used to impose weakly the balance of force and the BJS condition. In addition, a Darcy pressure Lagrange multiplier is used to impose weakly the continuity of normal flux.

Since the weak formulation of the Stokes–Biot system considered in this thesis is new, we first show that it has a unique solution. This is done by casting it in the form of a degenerate evolution saddle point system and employing results from classical semigroup theory for differential equations with monotone operators [74]. We then present a semi-discrete continuous-in-time formulation, which is based on employing stable mixed finite element spaces for the Stokes, Darcy, and elasticity equations on grids that may be non-matching along the interface, as well as suitable choices for the Lagrange multiplier finite element spaces. Well-posedness of the semi-discrete formulation is established with a similar argument to the continuous case, using discrete inf-sup conditions for the divergence and interface bilinear forms. Stability and optimal order error estimates are then derived for all variables in their natural space-time norms. We emphasize that the estimates hold uniformly in the limit of the storativity coefficient s_0 going to zero, which is a locking regime for non-mixed elasticity discretizations for the Biot system. In addition, our results are robust with respect to $a_{\min}$, the lower bound for the compliance tensor A, which relates to another locking phenomena in poroelasticity called Poisson locking [83]. Furthermore, we do not use Gronwall's inequality in the stability bound, thus obtaining long-time stability for our method. We present several computational experiments for a fully discrete finite element method designed to verify the convergence theory, illustrate the behavior of the method for a problem modeling an interaction between surface and subsurface hydrological systems, and study the robustness of the method with respect to the physical parameters. In particular, the numerical experiments illustrate the locking-free properties of the mixed finite element method for the Stokes–Biot system.

We discuss the mixed elasticity finite element method in details in Chapter 2, which is organized as follows. In Section 2.1, we present the model problem and derive its continuous weak formulation. Well-posedness of the continuous formulation is proved in Section 2.2, where existence and uniqueness of solution are established. The semi-discrete continuous-in-time approximation is introduced in Section 2.3. There the well-posedness, as well as its stability and error analyses are performed. Finally, numerical experiments are presented in Section 2.4.

Motivated by the advantages of mixed finite element methods for elasticity, we then develop a new fully mixed formulation of the quasi-static Stokes-Biot model, which is based on dual mixed formulations for all three components - Darcy, elasticity, and Stokes. In particular, we use a velocity-pressure Darcy formulation, a weakly symmetric stress-displacement-rotation elasticity formulation, and a weakly symmetric stress-velocity-vorticity Stokes formulation. This formulation exhibits multiple advantages, including local conservation of mass for the Darcy fluid, local poroelastic and Stokes momentum conservation, and accurate approximations with continuous normal components across element edges or faces for the Darcy velocity, the poroelastic stress, and the free fluid stress. In addition, dual mixed formulations are known for their locking-free properties and robustness with respect to the physical parameters, as discussed previously.

Our five-field dual mixed Biot formulation is the same as the one considered in Chapter 2. Our three-field dual mixed Stokes formulation is based on the models developed in [50,51]. In particular, we introduce the stress tensor and subsequently eliminate the pressure unknown, by utilizing the deviatoric stress. In order to impose the symmetry of the Stokes stress and poroelastic stress tensors, the vorticity and structure rotation, respectively, are introduced as additional unknowns. The transmission conditions consisting of mass conservation, conservation of momentum, and the Beavers–Joseph–Saffman slip with friction condition are imposed weakly via the incorporation of additional Lagrange multipliers: the traces of the fluid velocity, structure velocity and the poroelastic media pressure on the interface. The resulting variational system of equations is then ordered so that it shows a twofold saddle point structure. The well-posedness and uniqueness of both the continuous and semidiscrete continuous-in-time formulations are proved by employing classical results for parabolic problems [74, 76] and monotone operators, and an abstract theory for twofold saddle point problems [1, 49]. In the discrete problem, for the three components of the model we consider suitable stable mixed finite element spaces on non-matching grids across the interface, coupled through either conforming or non-conforming Lagrange multiplier discretizations. We develop stability and error analysis, establishing rates of convergence to the true solution. The estimates we establish are uniform in the limit of the storativity coefficient going to zero.

Another main contribution related to this formulation is the development of a new mixed finite element method for the Stokes-Biot model that can be reduced to a positive definite cell-centered pressure-velocities-traces system. We recall the multipoint flux mixed finite element (MFMFE) method for Darcy flow developed in [24,57,80,81], where the lowest order Brezzi-Douglas-Marini $\mathbb{BDM}_1$ velocity spaces [22,23,66] and piecewise constant pressure are utilized. An alternative formulation based on a broken Raviart-Thomas velocity space is developed in [60]. The use of the vertex quadrature rule for the velocity bilinear form localizes the interaction between velocity degrees of freedom around mesh vertices and leads to a block-diagonal mass matrix. Consequently, the velocity can be locally eliminated, resulting in a cell-centered pressure system. In turn, the multipoint stress mixed finite element (MSMFE) method for elasticity is developed in [6, 7]. It utilizes stable weakly symmetric elasticity finite element triples with $\mathbb{BDM}_1$ stress spaces [7,13,15,21,44,64]. Similarly to the MFMFE method, an application of the vertex quadrature rule for the stress and rotation bilinear forms allows for local stress and rotation elimination, resulting in a cell-centered displacement system. We also refer the reader to the related finite volume multipoint stress approximation (MPSA) method for elasticity [58, 67, 68]. Recently, combining the MSMFE and MFMFE methods, a multipoint stress-flux mixed finite element (MSFMFE) method for the Biot poroelasticity model is developed in [8]. There, the dual mixed finite element system is reduced to a cell-centered displacement-pressure system. The reduced system is comparable in cost to the finite volume method developed in [69].

In this thesis we note for the first time that the MSMFE method for elasticity can be applied to the weakly symmetric stress-velocity-vorticity Stokes formulation from [50, 51] when $\mathbb{BDM}_1$-based stable finite element triples are utilized. With the application of the vertex quadrature rule, the fluid stress and vorticity can be locally eliminated, resulting in a positive definite cell-centered velocity system. To the best of our knowledge, this is the first such scheme for Stokes in the literature.

Finally, we combine the MFMFE method for Darcy with the MSMFE methods for elasticity and Stokes to develop a multipoint stress-flux mixed finite element for the Stokes-Biot system. We analyze the stability and convergence of the semidiscrete formulation. We further consider the fully discrete system with backward Euler time discretization and show that the algebraic system on each time step can be reduced to a positive definite cell-centered pressure-velocities-traces system.

The discussion on the fully mixed formulation of the Stokes-Biot model together with the multipoint stress-flux mixed finite element method are presented in Chapter 3. In Section 3.1, we derive a fully-mixed variational formulation for the Stokes-Biot model, which is written as a degenerate evolution problem with a twofold saddle point structure. Next, existence, uniqueness and stability of the solution of the weak formulation are obtained in Section 3.2. The corresponding semi-discrete continuous-in-time approximation is introduced and analyzed in Section 3.3, where the discrete analogue of the theory used in the continuous case is employed to prove its well-posedness. Error estimates and rates of convergence are also derived there. In Section 3.4, the multipoint stress-flux mixed finite element method is presented and the corresponding rates of convergence are provided, along with the analysis of the reduced cell-centered system. Finally, numerical experiments illustrating the accuracy of our mixed finite element method and its applications to coupling surface and subsurface flows and flow through poroelastic medium with a cavity are reported in Section 3.5.

While the Stokes model describes the motion of creeping flow, the Navier-Stokes equations could be used to model fast flows of scientific and engineering interests. The coupled Navier-Stokes – Biot model is of importance due to its applications to problems such as blood flow and industrial filters. In [16], the authors design residual-based stabilization techniques for the Biot system, motivated by the variational multiscale approach, and propose both a semi-implicit monolithic method and an extension of domain decomposition techniques for the Navier-Stokes – Biot system, where the main variables are fluid velocity, fluid pressure, structure velocity, filtration velocity and Darcy pressure. Theoretical analysis including well-posedness and a priori error estimates for the fully dynamic coupled Navier-Stokes – Biot model is established in [38] using velocity-pressure Navier-Stokes formulation, a pressure Darcy formulation and a displacement formulation for elasticity. To the best of our knowledge, dual mixed formulations for Navier-Stokes – Biot model have not been studied in the literature. Thus another topic of our interest is to extend the work to study a fully-mixed formulation of the quasi-static Navier-Stokes – Biot model, which is based on dual mixed formulations for all three components - Darcy, elasticity and Navier Stokes. The problem becomes much harder since it is nonlinear, due to a convective term in the Navier-Stokes equations. For this, we consider pseudostress-based formulations for the Navier-Stokes problems. These kinds of formulations allow for a unified analysis for Newtonian and non-Newtonian flows. Moreover, they yield direct approximations of several other quantities of physical interest such as the fluid stress tensor, the fluid pressure and the fluid vorticity. Here, similarly to [33], we introduce a nonlinear pseudostress tensor linking the pseudostress tensor with the convective term, which together with the fluid velocity, yield a pseudostress-velocity Navier-Stokes formulation. Furthermore, in order to relax the hypotheses on the finite element spaces, we augment the mixed formulation with some redundant Garlerkin-type terms arising from the equilibrium and constitutive equations. Our five-field dual mixed Biot formulation is still the same as the one considered in Chapter 2. Also, similar as the fully-mixed formulation for the Stokes-Biot model, the transmission conditions are imposed weakly through the introduction of three Lagrange multipliers: the traces of the fluid velocity, structure velocity and the Darcy pressure on the interface.

We present the analysis of the augmented fully-mixed formulation for the quasi-static Navier-Stokes – Biot model in Chapter 4. We state the model problem, together with its continuous formulation in Section 4.1. Since the problem is nonlinear, for the well-posedness we apply a fixed point approach as well as rewrite the problem into a parabolic system to fit in classical semigroup theory for differential equations with monotone operators [74]. The details are discussed in Section 4.2. We then present a semi-discrete continuous-in-time formulation based on employing stable mixed finite element spaces for the Navier-Stokes, Darcy and elasticity equations on non-matching grids along the interface, together with suitable choices for the Lagrange multiplier finite element spaces in Section 4.3. Well-posedness and stability analysis results are established using a similar argument to the continuous case. Also, we develop error analysis and establish rates of convergence for all variables in their natural norms. Finally in Section 4.4, we present several numerical experiments for a fully discrete finite element method to validate the theoretical rates of convergence and illustrate the behavior of the method for modelling blood flow in an artery bifurcation as well as industrial filters.

For the last part of this thesis, we discuss a fully-mixed formulation for the Navier-Stokes – Biot model. The problem we consider involves the time derivative of the fluid velocity, together with suitable Banach spaces for the nonlinear fluid stress tensor and the fluid velocity. We adopt the nonstandard pseudostress-velocity-vorticity formulation for the Navier-Stokes equations and the five-field dual mixed formulation for the Biot system including a stress-displacement-rotation formulation of elasticity with a velocity-pressure formulation for Darcy flow. Based on the fully-mixed formulation, we present a cell-centered finite volume method, where the multipoint stress-flux mixed finite element method is employed for the Navier-Stokes and elasticity equations, and the multipoint flux mixed finite element method is used for Darcy's flow. The formulation and the method together with a convergence numerical test are discussed in Chapter 5.

1.2 Preliminaries

In this section we introduce some definitions and fix some notations. Let $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{N}$ denote the sets of $n \times n$ matrices, $n \times n$ symmetric matrices and $n \times n$ skew-symmetric matrices, respectively. Let $\mathcal{O} \subset \mathrm{R}^n$, $n \in \{2,3\}$, denote a domain with Lipschitz boundary. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$, we denote by $\mathrm{L}^{\mathrm{p}}(\mathcal{O})$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^{\mathrm{p}}(\mathcal{O})}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\mathcal{O})}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\mathcal{O}) = \mathrm{L}^{\mathrm{p}}(\mathcal{O})$. If $\mathrm{p} = 2$ we write $\mathrm{H}^{\mathrm{s}}(\mathcal{O})$ in place of $\mathrm{W}^{\mathrm{s},2}(\mathcal{O})$, and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^{\mathrm{s}}(\mathcal{O})}$. Similar notation is used for a section Γ of the boundary of $\mathcal{O}$. By $\mathbf{Z}$ and $\mathbb{Z}$ we will denote the corresponding vectorial and tensorial counterparts of a generic scalar functional space Z. The $\mathrm{L}^2(\mathcal{O})$ inner product for scalar, vector, or tensor valued functions is denoted by $(\cdot,\cdot)_{\mathcal{O}}$. The $\mathrm{L}^2(\Gamma)$ inner product or duality pairing is denoted by $\langle\cdot,\cdot\rangle_{\Gamma}$. For any vector field $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$,, we set the gradient, divergence operators and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}$$

For any tensor fields $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} := (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n}\tau_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n}\tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix in $\mathrm{R}^{n\times n}$. In addition, we recall the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \Big\{\mathbf{v}\in\mathbf{L}^2(\mathcal{O}): \quad \mathrm{div}(\mathbf{v})\in\mathrm{L}^2(\mathcal{O})\Big\},$$

equipped with the norm $\|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{div};\mathcal{O})} := \|\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\mathrm{div}(\mathbf{v})\|^2_{\mathrm{L}^2(\mathcal{O})}$. The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div};\mathcal{O})$ and endowed with the norm $\|\boldsymbol{\tau}\|^2_{\mathbb{H}(\mathbf{div};\mathcal{O})} := \|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\mathcal{O})} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{\mathbf{L}^2(\mathcal{O})}$. Finally, given a separable Banach space V endowed with the norm $\|\cdot\|_{\mathrm{V}}$, we let $\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})$ be the space of classes of functions $f:(0,T)\to\mathrm{V}$ that are Bochner measurable and such that $\|f\|_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})}<\infty$, with

$$\|f\|^{\mathrm{p}}_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^{\mathrm{p}}_{\mathrm{V}}\,dt, \quad \|f\|_{\mathrm{L}^{\infty}(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]}\|f(t)\|_{\mathrm{V}}.$$

We employ $\mathbf{0}$ to denote the null vector or tensor, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

We end this section by describing briefly some finite element spaces, including Taylor-Hood and the MINI elements which are stable Stokes finite element pairs, and the Raviart-Thomas (RT) and the Brezzi-Douglas-Marini (BDM) elements which are stable Darcy mixed finite element pairs [23]. In the generalised Taylor-Hood elements, on triangles or tetrahedra, velocities are approximated by a standard $\mathbf{P}_k$ element and pressures by a standard continuous P_{k-1}, where P_k denotes the polynomials of total degree $k\geq 1$. This choice has an analogue on rectangles or cubes using a $\mathbf{Q}_k$ element for velocities and a Q_{k-1} element for pressures, where Q_k stands for polynomials of degree k in each variable. The MINI elements adopts $\mathbf{P}_1^b$, the space of continuous piecewise linear polynomials enriched elementwise by cubic bubble functions, for velocities, and P_1 for pressures. On the other hand, RT space and BDM space are built for approximations of H(div) to preserve the continuity of the normal traces. In particular, on triangles or tetrahedra elements E, we have

$$\mathbf{RT}_k(E) = \mathrm{V}_h^k(E) \times \mathrm{W}_h^k(E) \quad \text{where} \quad \mathrm{V}_h^k(E) = \mathbf{P}_k(E) + \mathbf{x}\,\mathrm{P}_k(E),\ \mathrm{W}_h^k(E) = \mathrm{P}_k(E);$$

$$\mathbf{BDM}_k(E) = \mathrm{V}_h^k(E) \times \mathrm{W}_h^k(E) \quad \text{where} \quad \mathrm{V}_h^k(E) = \mathbf{P}_k(E),\ \mathrm{W}_h^k(E) = \mathrm{P}_{k-1}(E).$$

2.0 A mixed elasticity formulation for the Stokes-Biot model

2.1 The model problem and weak formulation

Let $\Omega \subset \mathrm{R}^n$, $n \in \{2,3\}$, be a Lipschitz domain, which is subdivided into two non-overlapping and possibly non-connected regions: fluid region Ω_f and poroelastic region Ω_p. Let $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$ denote the (nonempty) interface between these regions and let $\Gamma_f = \partial\Omega_f \setminus \Gamma_{fp}$ and $\Gamma_p = \partial\Omega_p \setminus \Gamma_{fp}$ denote the external parts on the boundary $\partial\Omega$. We denote by $\mathbf{n}_f$ and $\mathbf{n}_p$ the unit normal vectors that point outward from $\partial\Omega_f$ and $\partial\Omega_p$, respectively, noting that $\mathbf{n}_f = -\mathbf{n}_p$ on Γ_{fp}. Let $(\mathbf{u}_\star, p_\star)$ be the velocity-pressure pair in $\Omega_\star$ with $\star \in \{f, p\}$, and let $\boldsymbol{\eta}_p$ be the displacement in Ω_p. Let $\mu > 0$ be the fluid viscosity, let $\mathbf{f}_\star$ be the body force terms, and let $q_\star$ be external source or sink terms.

We assume that the flow in Ω_f is governed by the Stokes equations, which are written in the following stress-velocity-pressure formulation:

$$\boldsymbol{\sigma}_f = -p_f\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_f), \quad -\mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f, \quad \mathrm{div}(\mathbf{u}_f) = q_f \quad \text{in} \quad \Omega_f \times (0,T], \tag{2.1.1a}$$

$$\mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f \times (0,T], \tag{2.1.1b}$$

where $\boldsymbol{\sigma}_f$ is the stress tensor, $\mathbf{e}(\mathbf{u}_f) := \dfrac{1}{2}\,(\nabla\mathbf{u}_f + (\nabla\mathbf{u}_f)^{\mathrm{t}})$ stands for the deformation rate tensor, and $T > 0$ is the final time.

In turn, let $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_p$ be the elastic and poroelastic stress tensors, respectively, satisfying

$$A\,\boldsymbol{\sigma}_e = \mathbf{e}(\boldsymbol{\eta}_p) \quad \text{and} \quad \boldsymbol{\sigma}_p := \boldsymbol{\sigma}_e - \alpha_p\,p_p\,\mathbf{I} \quad \text{in} \quad \Omega_p \times (0,T], \tag{2.1.2}$$

where $0 < \alpha_p \leq 1$ is the Biot–Willis constant, and $A : \mathbb{S} \to \mathbb{M}$ is the symmetric and positive definite compliance tensor, which in the isotropic case has the form, for all tensors $\boldsymbol{\tau} \in \mathbb{S}$,

$$A(\boldsymbol{\tau}) := \frac{1}{2\,\mu_p}\left(\boldsymbol{\tau} - \frac{\lambda_p}{2\,\mu_p + n\,\lambda_p}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}\right), \quad \text{with} \quad A^{-1}(\boldsymbol{\tau}) = 2\,\mu_p\,\boldsymbol{\tau} + \lambda_p\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}, \tag{2.1.3}$$

satisfying

$$\forall\,\boldsymbol{\tau} \in \mathrm{R}^{n\times n}, \quad a_{\min}\,\boldsymbol{\tau} : \boldsymbol{\tau} \,\leq\, A(\boldsymbol{\tau}) : \boldsymbol{\tau} \,\leq\, a_{\max}\,\boldsymbol{\tau} : \boldsymbol{\tau} \quad \forall\,\mathbf{x} \in \Omega_p, \tag{2.1.4}$$

with $a_{\min} = 1/(2\mu_{\max} + n\,\lambda_{\max})$ and $a_{\max} = 1/2\mu_{\min}$. In this case, $\boldsymbol{\sigma}_e := \lambda_p \operatorname{div}(\boldsymbol{\eta}_p)\,\mathbf{I} + 2\,\mu_p\,\mathbf{e}(\boldsymbol{\eta}_p)$, and $0 < \lambda_{\min} \le \lambda_p(\mathbf{x}) \le \lambda_{\max}$ and $0 < \mu_{\min} \le \mu_p(\mathbf{x}) \le \mu_{\max}$ are the Lamé parameters. We extend the definition of A on $\mathbb{M}$ such that it is a positive constant multiple of the identity map on $\mathbb{N}$ as in [63]. The poroelasticity region Ω_p is governed by the quasi-static Biot system [19]:

$$-\,\mathbf{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p, \quad \mu\,\mathbf{K}^{-1}\mathbf{u}_p + \nabla\,p_p = \mathbf{0},$$

$$\frac{\partial}{\partial t}\big(s_0\,p_p + \alpha_p \operatorname{div}(\boldsymbol{\eta}_p)\big) + \operatorname{div}(\mathbf{u}_p) = q_p \quad \text{in} \quad \Omega_p \times (0,T], \tag{2.1.5a}$$

$$\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}} \times (0,T], \quad p_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}} \times (0,T], \tag{2.1.5b}$$

$$\boldsymbol{\sigma}_p\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{N}} \times (0,T], \quad \boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{D}} \times (0,T], \tag{2.1.5c}$$

where $\Gamma_p = \Gamma_p^{\mathrm{N}} \cup \Gamma_p^{\mathrm{D}} = \tilde{\Gamma}_p^{\mathrm{N}} \cup \tilde{\Gamma}_p^{\mathrm{D}}$, $s_0 > 0$ is a storativity coefficient and $\mathbf{K}(\mathbf{x})$ is the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \le k_{\max}$,

$$\forall\,\mathbf{w} \in \mathrm{R}^n, \quad k_{\min}\,\mathbf{w}\cdot\mathbf{w} \,\le\, (\mathbf{K}\mathbf{w})\cdot\mathbf{w} \,\le\, k_{\max}\,\mathbf{w}\cdot\mathbf{w} \quad \forall\,\mathbf{x} \in \Omega_p. \tag{2.1.6}$$

To avoid the issue with restricting the mean value of the pressure, we assume that $|\Gamma_p^{\mathrm{D}}| > 0$. We also assume that Γ_f^{D}, Γ_p^{D}, and $\tilde{\Gamma}_p^{\mathrm{D}}$ are not adjacent to the interface Γ_{fp}, i.e., $\exists\, s > 0$ such that $\operatorname{dist}(\Gamma_f^{\mathrm{D}}, \Gamma_{fp}) \ge s$, $\operatorname{dist}(\Gamma_p^{\mathrm{D}}, \Gamma_{fp}) \ge s$, and $\operatorname{dist}(\tilde{\Gamma}_p^{\mathrm{D}}, \Gamma_{fp}) \ge s$. This assumption is used to simplify the characterization of the normal trace spaces on Γ_{fp}.

Next, we introduce the following transmission conditions on the interface Γ_{fp} [10,16,26,75]:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \left(\frac{\partial\,\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot \mathbf{n}_p \,=\, 0, \quad \boldsymbol{\sigma}_f\mathbf{n}_f + \boldsymbol{\sigma}_p\mathbf{n}_p \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_{fp} \times (0,T], \tag{2.1.7a}$$

$$\boldsymbol{\sigma}_f\mathbf{n}_f + \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \sqrt{\mathbf{K}_j^{-1}} \left\{\left(\mathbf{u}_f - \frac{\partial\,\boldsymbol{\eta}_p}{\partial t}\right)\cdot \mathbf{t}_{f,j}\right\} \mathbf{t}_{f,j} \,=\, -\,p_p\mathbf{n}_f \quad \text{on} \quad \Gamma_{fp} \times (0,T], \tag{2.1.7b}$$

where $\mathbf{t}_{f,j}$, $1 \le j \le n-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $\mathbf{K}_j = (\mathbf{K}\,\mathbf{t}_{f,j})\cdot\mathbf{t}_{f,j}$, and $\alpha_{\mathtt{BJS}} \ge 0$ is an experimentally determined friction coefficient. The equations in (2.1.7a) correspond to mass conservation and conservation of momentum on Γ_{fp}, respectively, whereas the equation (2.1.7b) can be decomposed into its normal and tangential components, as follows:

$$(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f \;=\; -\,p_p, \quad (\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{t}_{f,j} \;=\; -\,\mu\,\alpha_{\mathtt{BJS}} \sqrt{\mathbf{K}_j^{-1}} \left(\mathbf{u}_f - \frac{\partial\, \boldsymbol{\eta}_p}{\partial t} \right) \cdot \mathbf{t}_{f,j} \quad \text{on} \quad \Gamma_{fp} \times (0,T],$$

representing balance of normal stress and the Beaver–Joseph–Saffman (BJS) slip with friction condition, respectively.

Finally, the above system of equations is complemented by the initial condition $p_p(\mathbf{x},0) = p_{p,0}(\mathbf{x})$ in Ω_p. We stress that, similarly to [65], compatible initial data for the rest of the variables can be constructed from $p_{p,0}$ in a way that all equations in the system (2.1.1)–(2.1.7), except for the unsteady conservation of mass equation in the first row of (2.1.5a), hold at $t = 0$. This will be established in Lemma 2.2.10 below. We will consider a weak formulation with a time-differentiated elasticity equation and compatible initial data $(\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

We next derive a weak formulation of the Stokes-Biot model given by (2.1.1)–(2.1.7). Throughout Chapter 2, we define the fluid velocity space and fluid pressure space as the Hilbert spaces

$$\mathbf{V}_f := \Big\{ \mathbf{v}_f \in \mathbf{H}^1(\Omega_f) : \mathbf{v}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f \Big\}, \qquad \mathrm{W}_f := \mathrm{L}^2(\Omega_f),$$

respectively, endowed with the corresponding standard norms

$$\|\mathbf{v}_f\|_{\mathbf{V}_f} := \|\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)}, \qquad \|w_f\|_{\mathrm{W}_f} := \|w_f\|_{\mathrm{L}^2(\Omega_f)}.$$

For the structure region, we introduce a new variable, the structure velocity $\mathbf{u}_s := \partial_t \boldsymbol{\eta}_p$, using the notation $\partial_t := \dfrac{\partial}{\partial t}$. We will develop a formulation that uses $\mathbf{u}_s$ instead of $\boldsymbol{\eta}_p$, which is better suitable for analysis. To impose the symmetry condition on $\boldsymbol{\sigma}_p$ weakly, we introduce the rotation operator $\boldsymbol{\rho}_p := \dfrac{1}{2}(\nabla \boldsymbol{\eta}_p - \nabla \boldsymbol{\eta}_p^{\mathrm{t}})$. In the weak formulation we will use its time derivative $\boldsymbol{\gamma}_p := \partial_t \boldsymbol{\rho}_p = \dfrac{1}{2}(\nabla \mathbf{u}_s - \nabla \mathbf{u}_s^{\mathrm{t}})$. We introduce the Hilbert spaces

$$\mathbf{V}_p := \Big\{ \mathbf{v}_p \in \mathbf{H}(\mathrm{div};\Omega_p) : \; \mathbf{v}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}} \Big\}, \qquad \mathrm{W}_p := \mathrm{L}^2(\Omega_p),$$

$$\mathbb{X}_p := \Big\{ \boldsymbol{\tau}_p \in \mathbb{H}(\mathbf{div};\Omega_p) : \; \boldsymbol{\tau}_p \mathbf{n}_p = 0 \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{N}} \Big\},$$

$$\mathbf{V}_s := \mathbf{L}^2(\Omega_p), \qquad \mathbb{Q}_p := \Big\{ \boldsymbol{\chi}_p \in \mathbb{L}^2(\Omega_p) : \boldsymbol{\chi}_p^{\mathrm{t}} = -\,\boldsymbol{\chi}_p \Big\},$$

endowed with the standard norms, respectively,

$$\|\mathbf{v}_p\|_{\mathbf{V}_p} := \|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)}, \qquad \|w_p\|_{\mathrm{W}_p} := \|w_p\|_{\mathrm{L}^2(\Omega_p)},$$

$$\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} := \|\boldsymbol{\tau}_p\|_{\mathbb{H}(\mathrm{div};\Omega_p)}, \qquad \|\mathbf{v}_s\|_{\mathbf{V}_s} := \|\mathbf{v}_s\|_{\mathbf{L}^2(\Omega_p)}, \qquad \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p} := \|\boldsymbol{\chi}_p\|_{\mathbb{L}^2(\Omega_p)}.$$

We further introduce two Lagrange multipliers:

$$\lambda := -(\boldsymbol{\sigma}_f \mathbf{n}_f)\cdot\mathbf{n}_f = p_p, \quad \text{and} \quad \boldsymbol{\theta} := \mathbf{u}_s \quad \text{on} \quad \Gamma_{fp}.$$

The first one is standard in Stokes–Darcy and Stokes–Biot models with a mixed Darcy formulation and it is used to impose weakly continuity of flux, cf. the first equation in (2.1.7a). The second one is needed in the mixed elasticity formulation, since the trace of $\mathbf{u}_s$ on Γ_{fp} is not well defined for $\mathbf{u}_s \in \mathbf{L}^2(\Omega_p)$. It will be used to impose weakly the continuity of normal stress condition $\boldsymbol{\sigma}_f\mathbf{n}_f\cdot\mathbf{n}_f = \boldsymbol{\sigma}_p\mathbf{n}_p\cdot\mathbf{n}_p$ and the BJS condition, cf. (2.1.7b). For the Lagrange multiplier spaces we need $\Lambda_p = (\mathbf{V}_p\cdot\mathbf{n}_p)'$ and $\boldsymbol{\Lambda}_s = (\mathbb{X}_p\mathbf{n}_p)'$. According to the normal trace theorem, since $\mathbf{v}_p \in \mathbf{V}_p \subset \mathbf{H}(\mathrm{div};\Omega_p)$, then $\mathbf{v}_p\cdot\mathbf{n}_p \in \mathrm{H}^{-1/2}(\partial\Omega_p)$. It is shown in [47] that if $\mathbf{v}_p\cdot\mathbf{n}_p = 0$ on $\partial\Omega_p\backslash\Gamma_{fp}$, then $\mathbf{v}_p\cdot\mathbf{n}_p \in \mathrm{H}^{-1/2}(\Gamma_{fp})$. In our case, since $\mathbf{v}_p\cdot\mathbf{n}_p = 0$ on Γ_p^{N} and $\mathrm{dist}\,(\Gamma_p^{\mathrm{D}},\Gamma_{fp}) \geq s > 0$, the argument can be modified as follows. For any $\xi \in \mathrm{H}^{1/2}(\Gamma_{fp})$, let $E_1\xi$ be a continuous extension to $\mathrm{H}^{1/2}(\Gamma_{fp}\cup\Gamma_p^{\mathrm{N}})$ such that $E_1\xi = 0$ on $\partial(\Gamma_{fp}\cup\Gamma_p^{\mathrm{N}})$, then let $E_2(E_1\xi) \in \mathrm{H}^{1/2}(\partial\Omega)$ be a continuous extension of $E_1\xi$ such that $E_2(E_1\xi) = 0$ on Γ_p^{D}. We then have

$$\langle \mathbf{v}_p\cdot\mathbf{n}_p, \xi\rangle_{\Gamma_{fp}} = \langle \mathbf{v}_p\cdot\mathbf{n}_p, E_1\xi\rangle_{\Gamma_{fp}\cup\Gamma_p^{\mathrm{N}}} = \langle \mathbf{v}_p\cdot\mathbf{n}_p, E_2(E_1\xi)\rangle_{\partial\Omega_p}$$

and

$$\langle \mathbf{v}_p\cdot\mathbf{n}_p, \xi\rangle_{\Gamma_{fp}} \leq \|\mathbf{v}_p\cdot\mathbf{n}_p\|_{\mathrm{H}^{-1/2}(\partial\Omega_p)}\|E_2(E_1\xi)\|_{\mathrm{H}^{1/2}(\partial\Omega_p)} \leq C\|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)}\|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}. \quad (2.1.8)$$

Similarly, for any $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{fp})$,

$$\langle \boldsymbol{\sigma}_p\mathbf{n}_p, \boldsymbol{\phi}\rangle_{\Gamma_{fp}} \leq C\|\boldsymbol{\sigma}_p\|_{\mathbb{H}(\mathrm{div};\Omega_p)}\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}. \quad (2.1.9)$$

Thus we can take

$$\Lambda_p := \mathrm{H}^{1/2}(\Gamma_{fp}), \quad \boldsymbol{\Lambda}_s := \mathbf{H}^{1/2}(\Gamma_{fp})$$

with norms

$$\|\xi\|_{\Lambda_p} := \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \quad \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} := \|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}. \tag{2.1.10}$$

We now proceed with the derivation of the variational formulation of (2.1.1)–(2.1.7). We test the first equation in (2.1.1a) with an arbitrary $\mathbf{v}_f \in \mathbf{V}_f$, integrate by parts, and combine with the BJS interface condition in (2.1.7b). We test the third equation in (2.1.5a) by $w_p \in \mathrm{W}_p$ and make use of (2.1.2) and the fact that

$$\operatorname{div}(\boldsymbol{\eta}_p) = \operatorname{tr}(\mathbf{e}(\boldsymbol{\eta}_p)) = \operatorname{tr}(A\boldsymbol{\sigma}_e) = \operatorname{tr}A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}),$$

as well as $\operatorname{tr}(\boldsymbol{\tau})w = \boldsymbol{\tau} : (w\mathbf{I})\ \forall\, \boldsymbol{\tau} \in \mathbb{M}, w \in \mathbb{R}$. In addition, (2.1.2) gives

$$A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}) = \nabla \boldsymbol{\eta}_p - \boldsymbol{\rho}_p.$$

In the weak formulation we will use its time differentiated version

$$\partial_t A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}) = \nabla \mathbf{u}_s - \boldsymbol{\gamma}_p,$$

which is tested by $\boldsymbol{\tau}_p \in \mathbb{X}_p$. Finally, we impose the remaining equations weakly, as well as the symmetry of $\boldsymbol{\sigma}_p$ and the interface conditions (2.1.7), obtaining the following mixed variational formulation: Given

$$\mathbf{f}_f : [0,T] \to \mathbf{V}_f', \quad \mathbf{f}_p : [0,T] \to \mathbf{V}_s', \quad q_f : [0,T] \to \mathrm{W}_f', \quad q_p : [0,T] \to \mathrm{W}_p'$$

and $(\boldsymbol{\sigma}_{p,0}, p_{p,0}) \in \mathbb{X}_p \times \mathrm{W}_p$, find $(\mathbf{u}_f, p_f, \boldsymbol{\sigma}_p, \mathbf{u}_s, \boldsymbol{\gamma}_p, \mathbf{u}_p, p_p, \lambda, \boldsymbol{\theta}) : [0,T] \to \mathbf{V}_f \times \mathrm{W}_f \times \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \mathbf{V}_p \times \mathrm{W}_p \times \Lambda_p \times \boldsymbol{\Lambda}_s$ such that $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$ and, for a.e. $t \in (0,T)$ and for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in \mathrm{W}_f$, $\boldsymbol{\tau}_p \in \mathbb{X}_p$, $\mathbf{v}_s \in \mathbf{V}_s$, $\boldsymbol{\chi}_p \in \mathbb{Q}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in \mathrm{W}_p$, $\xi \in \Lambda_p$, and $\boldsymbol{\phi} \in \boldsymbol{\Lambda}_s$,

$$\begin{aligned}&(2\mu\mathbf{e}(\mathbf{u}_f), \mathbf{e}(\mathbf{v}_f))_{\Omega_f} - (\operatorname{div}(\mathbf{v}_f), p_f)_{\Omega_f} + \langle \mathbf{v}_f \cdot \mathbf{n}_f, \lambda \rangle_{\Gamma_{fp}} \\ &\qquad + \sum_{j=1}^{n-1} \langle \mu\alpha_{\mathtt{BJS}} \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_f - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, \mathbf{v}_f \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f},\end{aligned} \tag{2.1.11a}$$

$$(\operatorname{div}(\mathbf{u}_f), w_f)_{\Omega_f} = (q_f, w_f)_{\Omega_f}, \tag{2.1.11b}$$

$$(\partial_t A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\mathbf{div}(\boldsymbol{\tau}_p), \mathbf{u}_s)_{\Omega_p} + (\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p)_{\Omega_p} - \langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\theta} \rangle_{\Gamma_{fp}} = 0, \tag{2.1.11c}$$

$$(\mathbf{div}(\boldsymbol{\sigma}_p), \mathbf{v}_s)_{\Omega_p} = -(\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}, \tag{2.1.11d}$$

$$(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p)_{\Omega_p} = 0, \tag{2.1.11e}$$

$$(\mu \mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p} - (\mathrm{div}(\mathbf{v}_p), p_p)_{\Omega_p} + \langle \mathbf{v}_p \cdot \mathbf{n}_p, \lambda \rangle_{\Gamma_{fp}} = 0, \tag{2.1.11f}$$

$$(s_0 \partial_t p_p, w_p)_{\Omega_p} + \alpha_p(\partial_t A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), w_p \mathbf{I})_{\Omega_p} + (\mathrm{div}(\mathbf{u}_p), w_p)_{\Omega_p} = (q_p, w_p)_{\Omega_p}, \tag{2.1.11g}$$

$$\langle \mathbf{u}_f \cdot \mathbf{n}_f + \boldsymbol{\theta} \cdot \mathbf{n}_p + \mathbf{u}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} = 0, \tag{2.1.11h}$$

$$\langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \lambda \rangle_{\Gamma_{fp}} - \sum_{j=1}^{n-1} \langle \mu \alpha_{\mathtt{BJS}} \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_f - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi} \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\phi} \rangle_{\Gamma_{fp}} = 0. \tag{2.1.11i}$$

In the above, (2.1.11a)–(2.1.11b) are the Stokes equations, (2.1.11c)–(2.1.11e) are the elasticity equations, (2.1.11f)–(2.1.11g) are the Darcy equations, and (2.1.11h)–(2.1.11i) enforce weakly the interface conditions.

Remark 2.1.1. *The time differentiated equation* (2.1.11c) *allows us to eliminate the displacement variable* $\boldsymbol{\eta}_p$ *and obtain a formulation that uses only* $\mathbf{u}_s$. *As part of the analysis we will construct suitable initial data such that, by integrating* (2.1.11c) *in time, we can recover the original equation*

$$(A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\mathbf{div}(\boldsymbol{\tau}_p), \boldsymbol{\eta}_p)_{\Omega_p} + (\boldsymbol{\tau}_p, \boldsymbol{\rho}_p)_{\Omega_p} - \langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\omega} \rangle_{\Gamma_{fp}} = 0, \tag{2.1.12}$$

where $\boldsymbol{\omega} := \boldsymbol{\eta}_p|_{\Gamma_{fp}}$.

In order to obtain a structure suitable for analysis, we combine the equations for the variables with coercive bilinear forms, $\mathbf{u}_f$, $\mathbf{u}_p$, $\boldsymbol{\sigma}_p$, and p_p, together with $\boldsymbol{\theta}$, which is coupled with them via the continuity of flux and BJS conditions. We further combine the rest of the equations. Introducing the bilinear forms

$$a_f(\mathbf{u}_f, \mathbf{v}_f) := (2\,\mu\,\mathbf{e}(\mathbf{u}_f), \mathbf{e}(\mathbf{v}_f))_{\Omega_f},$$

$$a_p(\mathbf{u}_p, \mathbf{v}_p) := (\mu \mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}, \quad a_p^p(p_p, w_p) := (s_0 p_p, w_p)_{\Omega_p},$$

$$b_\star(\mathbf{v}_\star, w_\star) := -(\mathrm{div}(\mathbf{v}_\star), w_\star)_{\Omega_\star},\ \star \in \{f, p\}, \quad b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) := (\mathbf{div}(\boldsymbol{\tau}_p), \mathbf{v}_s)_{\Omega_p},$$

$$b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi}) := -\langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\phi} \rangle_{\Gamma_{fp}}, \quad b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\chi}_p) := (\boldsymbol{\tau}_p, \boldsymbol{\chi}_p)_{\Omega_p},$$

$$a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) := (A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p w_p \mathbf{I})_{\Omega_p},$$

$$a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) := \sum_{j=1}^{n-1} \langle \mu \alpha_{\mathtt{BJS}} \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_f - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, (\mathbf{v}_f - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}},$$

$$b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\phi}; \xi) := \langle \mathbf{v}_f \cdot \mathbf{n}_f + \boldsymbol{\phi} \cdot \mathbf{n}_p + \mathbf{v}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}},$$

the system (2.1.11) can be written as follows:

$$\begin{aligned}
& a_f(\mathbf{u}_f, \mathbf{v}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) + b_p(\mathbf{v}_p, p_p) + b_f(\mathbf{v}_f, p_f) \\
& \quad + b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\phi}; \lambda) + a_p^p(\partial_t p_p, w_p) + a_e(\partial_t \boldsymbol{\sigma}_p, \partial_t p_p; \boldsymbol{\tau}_p, w_p) \\
& \quad + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) - b_p(\mathbf{u}_p, w_p) = (\mathbf{f}_f, \mathbf{v}_f) + (q_p, w_p)_{\Omega_p}, \\
& -b_f(\mathbf{u}_f, w_f) - b_s(\boldsymbol{\sigma}_p, \mathbf{v}_s) - b_{\mathrm{sk}}(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p) - b_\Gamma(\mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\theta}; \xi) = (q_f, w_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{v}_s).
\end{aligned} \tag{2.1.13}$$

We group the spaces and test functions as:

$$\mathbf{Q} := \mathbf{V}_f \times \boldsymbol{\Lambda}_s \times \mathbf{V}_p \times \mathbb{X}_p \times \mathrm{W}_p, \quad \mathbf{S} := \mathrm{W}_f \times \mathbf{V}_s \times \mathbb{Q}_p \times \Lambda_p,$$

$$\mathbf{p} := (\mathbf{u}_f, \boldsymbol{\theta}, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p) \in \mathbf{Q}, \quad \mathbf{r} := (p_f, \mathbf{u}_s, \boldsymbol{\gamma}_p, \lambda) \in \mathbf{S},$$

$$\mathbf{q} := (\mathbf{v}_f, \boldsymbol{\phi}, \mathbf{v}_p, \boldsymbol{\tau}_p, w_p) \in \mathbf{Q}, \quad \mathbf{s} := (w_f, \mathbf{v}_s, \boldsymbol{\chi}_p, \xi) \in \mathbf{S},$$

where the spaces $\mathbf{Q}$ and $\mathbf{S}$ are endowed with the norms, respectively,

$$\|\mathbf{q}\|_{\mathbf{Q}} = \|\mathbf{v}_f\|_{\mathbf{V}_f} + \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} + \|\mathbf{v}_p\|_{\mathbf{V}_p} + \|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} + \|w_p\|_{\mathrm{W}_p},$$

$$\|\mathbf{s}\|_{\mathbf{S}} = \|w_f\|_{\mathrm{W}_f} + \|\mathbf{v}_s\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p} + \|\xi\|_{\Lambda_p}.$$

Hence, we can write (2.1.13) in an operator notation as a degenerate evolution problem in a mixed form:

$$\begin{aligned}
\partial_t \mathcal{E}_1 \mathbf{p}(t) + \mathcal{A} \mathbf{p}(t) + \mathcal{B}' \mathbf{r}(t) &= \mathbf{F}(t) \quad \text{in} \quad \mathbf{Q}', \\
-\mathcal{B} \mathbf{p}(t) &= \mathbf{G}(t) \quad \text{in} \quad \mathbf{S}'.
\end{aligned} \tag{2.1.14}$$

The operators $\mathcal{A} : \mathbf{Q} \to \mathbf{Q}'$, $\mathcal{B} : \mathbf{Q} \to \mathbf{S}'$ and the functionals $\mathbf{F}(t) \in \mathbf{Q}'$, $\mathbf{G}(t) \in \mathbf{S}'$ are defined as follows:

$$\mathcal{A} = \begin{pmatrix} A_f + A^f_{\mathtt{BJS}} & (A^{fs}_{\mathtt{BJS}})' & 0 & 0 & 0 \\ A^{fs}_{\mathtt{BJS}} & A^s_{\mathtt{BJS}} & 0 & (B^p_n)' & 0 \\ 0 & 0 & A_p & 0 & B'_p \\ 0 & -B^p_n & 0 & 0 & 0 \\ 0 & 0 & -B_p & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B_f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_s & 0 \\ 0 & 0 & 0 & B_{\mathrm{sk}} & 0 \\ B^f_\Gamma & B^s_\Gamma & B^p_\Gamma & 0 & 0 \end{pmatrix}, \tag{2.1.15}$$

$$\mathbf{F}(t) = \begin{pmatrix} \mathbf{f}_f \\ 0 \\ 0 \\ \mathbf{0} \\ q_p \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} q_f \\ \mathbf{f}_p \\ 0 \\ 0 \end{pmatrix},$$

where

$$(A_f \mathbf{u}_f, \mathbf{v}_f) = a_f(\mathbf{u}_f, \mathbf{v}_f), \qquad (A_p \mathbf{u}_p, \mathbf{v}_p) = a_p(\mathbf{u}_p, \mathbf{v}_p),$$

$$(B_p \mathbf{u}_p, w_p) = b_p(\mathbf{u}_p, w_p), \qquad (B^p_n \boldsymbol{\sigma}_p, \boldsymbol{\phi}) = -b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}),$$

$$(A^f_{\mathtt{BJS}} \mathbf{u}_f, \mathbf{v}_f) = a_{\mathtt{BJS}}(\mathbf{u}_f, \mathbf{0}; \mathbf{v}_f, \mathbf{0}), \quad (A^{fs}_{\mathtt{BJS}} \mathbf{u}_f, \boldsymbol{\phi}) = a_{\mathtt{BJS}}(\mathbf{u}_f, \mathbf{0}; \mathbf{0}, \boldsymbol{\phi}),$$

$$(A^s_{\mathtt{BJS}} \boldsymbol{\theta}, \boldsymbol{\phi}) = a_{\mathtt{BJS}}(\mathbf{0}, \boldsymbol{\theta}; \mathbf{0}, \boldsymbol{\phi}),$$

$$(B_f \mathbf{u}_f, w_f) = b_f(\mathbf{u}_f, w_f), \quad (B_s \boldsymbol{\sigma}_p, \mathbf{v}_s) = b_s(\boldsymbol{\sigma}_p, \mathbf{v}_s), \quad (B_{\mathrm{sk}} \boldsymbol{\sigma}_p, \boldsymbol{\chi}_p) = b_{\mathrm{sk}}(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_s),$$

$$(B^f_\Gamma \mathbf{u}_f, \xi) = b_\Gamma(\mathbf{u}_f, \mathbf{0}, \mathbf{0}; \xi), \quad (B^s_\Gamma \boldsymbol{\theta}, \xi) = b_\Gamma(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}; \xi), \quad (B^p_\Gamma \mathbf{u}_p, \xi) = b_\Gamma(\mathbf{0}, \mathbf{u}_p, \mathbf{0}; \xi).$$

The operator $\mathcal{E}_1 : \mathbf{Q} \to \mathbf{Q}'$ is given by:

$$\mathcal{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A^s_e & A^{sp}_e \\ 0 & 0 & 0 & (A^{sp}_e)' & A^p_p + A^p_e \end{pmatrix},$$

where

$$(A_e^s \boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) = a_e(\boldsymbol{\sigma}_p, 0; \boldsymbol{\tau}_p, 0), \qquad (A_e^{sp} \boldsymbol{\sigma}_p, w_p) = a_e(\boldsymbol{\sigma}_p, 0; \mathbf{0}, w_p),$$

$$(A_e^p p_p, w_p) = a_e(\mathbf{0}, p_p; \mathbf{0}, w_p), \qquad (A_p^p p_p, w_p) = a_p^p(p_p, w_p).$$

2.2 Well-posedness of the weak formulation

2.2.1 Preliminaries

We start with exploring important properties of the operators introduced in the previous section.

Lemma 2.2.1. *The linear operators $\mathcal{A}$ and $\mathcal{E}_1$ are continuous and monotone.*

Proof. Continuity follows from the Cauchy-Schwarz inequality and the trace inequalities (2.1.8)–(2.1.9). In particular,

$$a_f(\mathbf{u}_f, \mathbf{v}_f) \le 2\mu \|\mathbf{u}_f\|_{\mathbf{V}_f} \|\mathbf{v}_f\|_{\mathbf{V}_f}, \qquad a_p(\mathbf{u}_p, \mathbf{v}_p) \le \mu k_{\min}^{-1} \|\mathbf{u}_p\|_{\mathbf{L}^2(\Omega_p)} \|\mathbf{v}_p\|_{\mathbf{L}^2(\Omega_p)},$$

$$\begin{aligned} a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) &\le \mu \alpha_{\mathtt{BJS}} k_{\min}^{-1/2} |\mathbf{u}_f - \boldsymbol{\theta}|_{a_{\mathtt{BJS}}} |\mathbf{v}_f - \boldsymbol{\phi}|_{a_{\mathtt{BJS}}} \\ &\le C(\|\mathbf{u}_f\|_{\mathbf{V}_f} + \|\boldsymbol{\theta}\|_{\mathbf{L}^2(\Gamma_{fp})})(\|\mathbf{v}_f\|_{\mathbf{V}_f} + \|\boldsymbol{\phi}\|_{\mathbf{L}^2(\Gamma_{fp})}), \end{aligned} \tag{2.2.1}$$

$$b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi}) \le C \|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s}, \qquad b_p(\mathbf{v}_p, w_p) \le \|\mathbf{v}_p\|_{\mathbf{V}_p} \|w_p\|_{\mathrm{W}_p},$$

where, for $\mathbf{v}_f \in \mathbf{V}_f$, $\boldsymbol{\phi} \in \boldsymbol{\Lambda}_f$, $|\mathbf{v}_f - \boldsymbol{\phi}|^2_{a_{\mathtt{BJS}}} := \sum_{j=1}^{n-1} \langle (\mathbf{v}_f - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j}, (\mathbf{v}_f - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}}$, and we have used the trace inequality, for a domain $\mathcal{O}$ and $S \subset \partial\mathcal{O}$,

$$\|\varphi\|_{\mathrm{H}^{1/2}(S)} \le C \|\varphi\|_{\mathrm{H}^1(\mathcal{O})} \quad \forall\, \varphi \in \mathrm{H}^1(\mathcal{O}). \tag{2.2.2}$$

Thus we have

$$\begin{aligned} (\mathcal{A}\mathbf{p}, \mathbf{q}) &= a_f(\mathbf{u}_f, \mathbf{v}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \theta) \\ &\quad + b_p(\mathbf{v}_p, p_p) - b_p(\mathbf{u}_p, w_p) \\ &\le C \|\mathbf{p}\|_{\mathbf{Q}} \|\mathbf{q}\|_{\mathbf{Q}} \end{aligned} \tag{2.2.3}$$

and

$$(\mathcal{E}_1\mathbf{p},\mathbf{q}) = (s_0 p_p, w_p)_{\Omega_p} + (A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p w_p \mathbf{I})_{\Omega_p} \leq C\|\mathbf{p}\|_{\mathbf{Q}}\|\mathbf{q}\|_{\mathbf{Q}}. \tag{2.2.4}$$

Therefore $\mathcal{A}$ and $\mathcal{E}_1$ are continuous. The monotonicity of $\mathcal{A}$ follows from

$$\begin{aligned}
a_f(\mathbf{v}_f,\mathbf{v}_f) &= 2\mu\|\mathbf{e}(\mathbf{v}_f)\|^2_{\mathbb{L}^2(\Omega_f)} \geq 2\mu C_K^2\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)},\\
a_p(\mathbf{v}_p,\mathbf{v}_p) &= \mu\|\mathbf{K}^{-1/2}\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} \geq \mu k_{\max}^{-1}\|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)},\\
a_{\mathtt{BJS}}(\mathbf{v}_f,\boldsymbol{\phi};\mathbf{v}_f,\boldsymbol{\phi}) &\geq \mu\alpha_{\mathtt{BJS}}k_{\max}^{-1/2}|\mathbf{v}_f-\boldsymbol{\phi}|^2_{a_{\mathtt{BJS}}},
\end{aligned} \tag{2.2.5}$$

where we used Korn's inequality $\|\mathbf{e}(\mathbf{v}_f)\| \geq C_K\|\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)}$ in the first bound. The monotonicity of $\mathcal{E}_1$ follows from

$$(\mathcal{E}_1\mathbf{q},\mathbf{q}) \;=\; s_0\|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}\,(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}. \tag{2.2.6}$$

□

Lemma 2.2.2. *The linear operator $\mathcal{B}$ is continuous. Furthermore, there exist positive constants β_1, β_2, and β_3 such that*

$$\beta_1(\|\mathbf{v}_s\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p}) \leq \sup_{\boldsymbol{\tau}_p\in\mathbb{X}_p\ s.t.\ \boldsymbol{\tau}_p\mathbf{n}_p=\mathbf{0}\ on\,\Gamma_{fp}} \frac{b_s(\boldsymbol{\tau}_p,\mathbf{v}_s) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p,\boldsymbol{\chi}_p)}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}}, \qquad \forall\,\mathbf{v}_s\in\mathbf{V}_s,\ \boldsymbol{\chi}_p\in\mathbb{Q}_p, \tag{2.2.7}$$

$$\beta_2(\|w_f\|_{\mathrm{W}_f} + \|w_p\|_{\mathrm{W}_p} + \|\xi\|_{\Lambda_p}) \leq \sup_{(\mathbf{v}_f,\mathbf{v}_p)\in\mathbf{V}_f\times\mathbf{V}_p} \frac{b_f(\mathbf{v}_f,w_f) + b_p(\mathbf{v}_p,w_p) + b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0};\xi)}{\|(\mathbf{v}_f,\mathbf{v}_p)\|_{\mathbf{V}_f\times\mathbf{V}_p}},$$

$$\forall\,w_f\in\mathrm{W}_f,\ w_p\in\mathrm{W}_p,\ \ and\ \xi\in\Lambda_p, \tag{2.2.8}$$

$$\beta_3\|\boldsymbol{\phi}\|_{\mathbf{\Lambda}_s} \leq \sup_{\boldsymbol{\tau}_p\in\mathbb{X}_p\ s.t.\,\mathbf{div}(\boldsymbol{\tau}_p)=0} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_p,\boldsymbol{\phi})}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}}, \qquad \forall\,\boldsymbol{\phi}\in\mathbf{\Lambda}_s. \tag{2.2.9}$$

Proof. The definition (2.1.15) of $\mathcal{B}$ implies

$$
\begin{aligned}
(\mathcal{B}\mathbf{q}, \mathbf{s}) &= b_f(\mathbf{v}_f, w_f) + b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\chi}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\phi}; \xi) \\
&\le \|\mathrm{div}(\mathbf{v}_f)\|_{\mathrm{L}^2(\Omega_f)} \|w_f\|_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{div}(\boldsymbol{\tau}_p)\|_{\mathbf{L}^2(\Omega_p)} \|\mathbf{v}_s\|_{\mathbf{L}^2(\Omega_p)} + \|\boldsymbol{\tau}_p\|_{\mathbb{L}^2(\Omega_p)} \|\boldsymbol{\chi}_p\|_{\mathbb{L}^2(\Omega_p)} \\
&\quad + C\|\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)} \|\xi\|_{\mathrm{L}^2(\Gamma_{fp})} + C\|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)} \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})} + \|\boldsymbol{\phi}\|_{\mathbf{L}^2(\Gamma_{fp})} \|\xi\|_{\mathrm{L}^2(\Gamma_{fp})} \\
&\le C\|\mathbf{q}\|_{\mathbf{Q}} \|\mathbf{s}\|_{\mathbf{S}}, \qquad (2.2.10)
\end{aligned}
$$

so $\mathcal{B}$ is continuous. Next, inf-sup condition (2.2.7) follows from [50, Section 2.4.3]. We note that the restriction $\boldsymbol{\tau}_p \mathbf{n}_p = \mathbf{0}$ on Γ_{fp} allows us to eliminate the term $b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta})$ when applying this inf-sup condition, see (2.2.26) below. Inf-sup condition (2.2.8) follows from a modification of the argument in Lemmas 3.1 and 3.2 in [43] to account for $|\Gamma_p^D| > 0$. Finally, (2.2.9) can be proved using the argument in [50, Lemma 4.2]. □

2.2.2 Existence and uniqueness of a solution

We will establish existence of a solution to the weak formulation (2.1.14) using the following key result.

Theorem 2.2.3. [74, Theorem IV.6.1(b)] *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given for the real vector space E to its algebraic dual E^*, and let E_b' be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = (\mathcal{N}x(x))^{1/2} \qquad x \in E.$$

Let $\mathcal{M} \subset E \times E_b'$ be a relation with domain $\mathcal{D} = \{x \in E : \mathcal{M}(x) \neq \emptyset\}$. Assume that $\mathcal{M}$ is monotone and $Rg(\mathcal{N} + \mathcal{M}) = E_b'$. Then, for each $u_0 \in \mathcal{D}$ and for each $f \in \mathrm{W}^{1,1}(0,T;E_b')$, there is a solution u of

$$\frac{d}{dt}\big(\mathcal{N}\,u(t)\big) + \mathcal{M}\big(u(t)\big) \ni f(t) \quad \textit{a.e. } 0 < t < T, \qquad (2.2.11)$$

with

$$\mathcal{N}\,u \in \mathrm{W}^{1,\infty}(0,T;E_b'), \quad u(t) \in \mathcal{D}, \quad \textit{for a.e. } 0 \le t \le T, \quad \textit{and} \quad \mathcal{N}\,u(0) = \mathcal{N}\,u_0.$$

We cast (2.1.14) in the form (2.2.11) by setting

$$E = \mathbf{Q} \times \mathbf{S}, \quad u = \begin{pmatrix} \mathbf{p} \\ \mathbf{r} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} \mathcal{E}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix}, \quad f = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}. \tag{2.2.12}$$

The seminorm induced by the operator $\mathcal{E}_1$ is $|\mathbf{q}|^2_{\mathcal{E}_1} := s_0 \|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$, cf. (2.2.6). Since $s_0 > 0$, it is equivalent to $\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{L}^2(\Omega_p)}$. We denote by $\mathbb{X}_{p,2}$ and $\mathrm{W}_{p,2}$ the closures of the spaces $\mathbb{X}_p$ and W_p, respectively, with respect to the norms $\|\boldsymbol{\tau}_p\|_{\mathbb{X}_{p,2}} := \|\boldsymbol{\tau}_p\|_{\mathbb{L}^2(\Omega_p)}$ and $\|w_p\|_{\mathrm{W}_{p,2}} := \|w_p\|_{\mathrm{L}^2(\Omega_p)}$. Then the Hilbert space E'_b in Theorem 2.2.3 in our case is

$$E'_b := \mathbf{Q}'_{2,0} \times \mathbf{S}'_{2,0}, \quad \text{where} \quad \mathbf{Q}'_{2,0} := \mathbf{0} \times \mathbf{0} \times \mathbf{0} \times \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}, \quad \mathbf{S}'_{2,0} := \mathbf{0} \times \mathbf{0} \times \mathbf{0} \times \mathbf{0}. \tag{2.2.13}$$

We further define $\mathcal{D} := \{(\mathbf{p}, \mathbf{r}) \in \mathbf{Q} \times \mathbf{S} : \ \mathcal{M}(\mathbf{p}, \mathbf{r}) \in E'_b\}$.

Remark 2.2.1. *The above definition of the space E'_b and the corresponding domain $\mathcal{D}$ implies that, in order to apply Theorem 2.2.3 for our problem* (2.1.14)*, we need to restrict $\mathbf{f}_f = \mathbf{0}$, $q_f = 0$, and $\mathbf{f}_p = \mathbf{0}$. To avoid this restriction we will employ a translation argument [76] to reduce the existence for* (2.1.14) *to existence for the following initial-value problem: Given initial data $(\widehat{\mathbf{p}}_0, \widehat{\mathbf{r}}_0) \in \mathcal{D}$ and source terms $(\widehat{g}_{\boldsymbol{\tau}_p}, \widehat{g}_{w_p}) : (0,T) \to \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$, find $(\mathbf{p}, \mathbf{r}) : [0,T] \to \mathbf{Q} \times \mathbf{S}$ such that $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\widehat{\boldsymbol{\sigma}}_{p,0}, \widehat{p}_{p,0})$ and, for a.e. $t \in (0,T)$,*

$$\begin{aligned} \partial_t\, \mathcal{E}_1\, \mathbf{p}(t) + \mathcal{A}\, \mathbf{p}(t) + \mathcal{B}'\, \mathbf{r}(t) &= \widehat{\mathbf{F}}(t) \quad \text{in} \quad \mathbf{Q}'_{2,0}, \\ -\mathcal{B}\, \mathbf{p}(t) &= \mathbf{0} \qquad \text{in} \quad \mathbf{S}'_{2,0}, \end{aligned} \tag{2.2.14}$$

where $\widehat{\mathbf{F}}(t) = (\mathbf{0}, 0, 0, \widehat{g}_{\boldsymbol{\tau}_p}, \widehat{g}_{w_p})^{\mathrm{t}}$.

In order to apply Theorem 2.2.3 for problem (2.2.14), we need to 1) establish the required properties of the operators $\mathcal{N}$ and $\mathcal{M}$, 2) prove the range condition $Rg(\mathcal{N} + \mathcal{M}) = E'_b$, and 3) construct compatible initial data $(\widehat{\mathbf{p}}_0, \widehat{\mathbf{r}}_0) \in \mathcal{D}$. We proceed with a sequence of lemmas establishing these results.

Lemma 2.2.4. *The linear operator $\mathcal{N}$ defined in* (2.2.12) *is continuous, symmetric, and monotone. The linear operator $\mathcal{M}$ defined in* (2.2.12) *is continuous and monotone.*

Proof. The stated properties follow easily from the properties of the operators $\mathcal{E}_1$, $\mathcal{A}$, and $\mathcal{B}$ established in Lemma 2.2.1 and Lemma 2.2.2. $\square$

Next, we establish the range condition $Rg(\mathcal{N} + \mathcal{M}) = E_b'$, which is done by solving the related resolvent system. In fact, we will show a stronger result by considering a resolvent system where all source terms may be non-zero. This stronger result will be used in the translation argument for proving existence of the original problem (2.1.14). In particular, consider the following resolvent system: Given $\widehat{g}_{\mathbf{v}_f} \in \mathbf{V}_f'$, $\widehat{g}_{w_f} \in \mathrm{W}_f'$, $\widehat{g}_{\boldsymbol{\tau}_p} \in \mathbb{X}_{p,2}'$, $\widehat{g}_{\mathbf{v}_s} \in \mathbf{V}_s'$, $\widehat{g}_{\boldsymbol{\chi}_p} \in \mathbb{Q}_p'$, $\widehat{g}_{\mathbf{v}_p} \in \mathbf{V}_p'$, $\widehat{g}_{w_p} \in \mathrm{W}_{p,2}'$, $\widehat{g}_{\xi} \in \Lambda_p'$, and $\widehat{g}_{\boldsymbol{\phi}} \in \boldsymbol{\Lambda}_s'$, find $(\mathbf{u}_f, p_f, \boldsymbol{\sigma}_p, \mathbf{u}_s, \boldsymbol{\gamma}_p, \mathbf{u}_p, p_p, \lambda, \boldsymbol{\theta}) \in \mathbf{V}_f \times \mathrm{W}_f \times \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \mathbf{V}_p \times \mathrm{W}_p \times \Lambda_p \times \boldsymbol{\Lambda}_s$ such that for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in \mathrm{W}_f$, $\boldsymbol{\tau}_p \in \mathbb{X}_p$, $\mathbf{v}_s \in \mathbf{V}_s$, $\boldsymbol{\chi}_p \in \mathbb{Q}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in \mathrm{W}_p$, $\xi \in \Lambda_p$, and $\boldsymbol{\phi} \in \boldsymbol{\Lambda}_s$,

$$
\begin{aligned}
& a_f(\mathbf{u}_f, \mathbf{v}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) + b_p(\mathbf{v}_p, p_p) + b_f(\mathbf{v}_f, p_f) \\
& \quad + b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p) + b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\phi}; \lambda) + a_p^p(p_p, w_p) + a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) \\
& \quad + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) - b_p(\mathbf{u}_p, w_p) \\
& \qquad = (\widehat{g}_{\mathbf{v}_f}, \mathbf{v}_f)_{\Omega_f} + (\widehat{g}_{\boldsymbol{\phi}}, \boldsymbol{\phi})_{\Omega_p} + (\widehat{g}_{\mathbf{v}_p}, \mathbf{v}_p)_{\Omega_p} + (\widehat{g}_{\boldsymbol{\tau}_p}, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{g}_{w_p}, w_p)_{\Omega_p}, \\
& -b_f(\mathbf{u}_f, w_f) - b_s(\boldsymbol{\sigma}_p, \mathbf{v}_s) - b_{\mathrm{sk}}(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p) - b_\Gamma(\mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\theta}; \xi) \\
& \qquad = (\widehat{g}_{w_f}, w_f)_{\Omega_f} + (\widehat{g}_{\mathbf{v}_s}, \mathbf{v}_s)_{\Omega_p} + (\widehat{g}_{\boldsymbol{\chi}_p}, \boldsymbol{\chi}_p)_{\Omega_p} + (\widehat{g}_{\xi}, \xi)_{\Omega_p}.
\end{aligned}
\tag{2.2.15}
$$

Letting

$$
\mathbf{Q}_2 = \mathbf{V}_f \times \boldsymbol{\Lambda}_s \times \mathbf{V}_p \times \mathbb{X}_{p,2} \times \mathrm{W}_{p,2},
$$

the resolvent system (2.2.15) can be written in an operator form as

$$
\begin{aligned}
(\mathcal{E}_1 + \mathcal{A})\mathbf{p} + \mathcal{B}'\mathbf{r} = \widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{Q}_2', \\
-\mathcal{B}\mathbf{p} = \widehat{\mathbf{G}} \quad \text{in} \quad \mathbf{S}'.
\end{aligned}
\tag{2.2.16}
$$

where $\widehat{\mathbf{F}} \in \mathbf{Q}_2'$ and $\widehat{\mathbf{G}} \in \mathbf{S}'$ are the functionals on the right hand side of (2.2.15).

To prove the solvability of this resolvent system, we use a regularization technique, following the approach in [4, 76]. To that end, we introduce operators that will be used to regularize the problem. Let $R_{\mathbf{u}_p} : \mathbf{V}_p \to \mathbf{V}'_p$, $R_{\boldsymbol{\sigma}_p} : \mathbb{X}_p \to \mathbb{X}'_p$, $R_{p_p} : \mathrm{W}_p \to \mathrm{W}'_p$, $L_{p_f} : \mathrm{W}_f \to \mathrm{W}'_f$, $L_{\mathbf{u}_s} : \mathbf{V}_s \to \mathbf{V}'_s$, and $L_{\boldsymbol{\gamma}_p} : \mathbb{Q}_p \to \mathbb{Q}'_p$ be defined as follows:

$$(R_{\mathbf{u}_p}\mathbf{u}_p, \mathbf{v}_p) = r_{\mathbf{u}_p}(\mathbf{u}_p, \mathbf{v}_p) := (\mathrm{div}(\mathbf{u}_p), \mathrm{div}(\mathbf{v}_p))_{\Omega_p},$$

$$(R_{\boldsymbol{\sigma}_p}\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) = r_{\boldsymbol{\sigma}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) := (\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\mathbf{div}(\boldsymbol{\sigma}_p), \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p},$$

$$(R_{p_p}p_p, w_p) = r_{p_p}(p_p, w_p) := (p_p, w_p)_{\Omega_p}, \quad (L_{p_f}p_f, w_f) = l_{p_f}(p_f, w_f) := (p_f, w_f)_{\Omega_f},$$

$$(L_{\mathbf{u}_s}\mathbf{u}_s, \mathbf{v}_s) = l_{\mathbf{u}_s}(\mathbf{u}_s, \mathbf{v}_s) := (\mathbf{u}_s, \mathbf{v}_s)_{\Omega_p}, \quad (L_{\boldsymbol{\gamma}_p}\boldsymbol{\gamma}_p, \boldsymbol{\chi}_P) = l_{\boldsymbol{\gamma}_p}(\boldsymbol{\gamma}_p, \boldsymbol{\chi}_p) := (\boldsymbol{\gamma}_p, \boldsymbol{\chi}_p)_{\Omega_p}.$$

The following operator properties follow immediately from the above definitions.

Lemma 2.2.5. *The operators $R_{\mathbf{u}_p}$, $R_{\boldsymbol{\sigma}_p}$, R_{p_p}, L_{p_f}, $L_{\mathbf{u}_s}$, and $L_{\boldsymbol{\gamma}_p}$ are continuous and monotone.*

For the regularization of the Lagrange multipliers, let $\psi(\lambda) \in \mathrm{H}^1(\Omega_p)$ be the weak solution of

$$-\mathrm{div}(\nabla\psi(\lambda)) = 0 \quad \text{in} \quad \Omega_p,$$

$$\psi(\lambda) = \lambda \quad \text{on} \quad \Gamma_{fp}, \quad \nabla\psi(\lambda)\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p.$$

Elliptic regularity and the trace inequality (2.2.2) imply that there exist positive constants c and C such that

$$c\|\psi(\lambda)\|_{\mathrm{H}^1(\Omega_p)} \le \|\lambda\|_{\mathrm{H}^{1/2}(\Gamma_{fp})} \le C\|\psi(\lambda)\|_{\mathrm{H}^1(\Omega_p)}. \tag{2.2.17}$$

We define $L_\lambda : \Lambda_p \to \Lambda'_p$ as

$$(L_\lambda\lambda, \xi) = l_\lambda(\lambda, \xi) := (\nabla\psi(\lambda), \nabla\psi(\xi))_{\Omega_p}. \tag{2.2.18}$$

Similarly, let $\boldsymbol{\varphi}(\boldsymbol{\theta}) \in \mathbf{H}^1(\Omega_p)$ be the weak solution of

$$-\mathbf{div}(\nabla\boldsymbol{\varphi}(\boldsymbol{\theta})) = \mathbf{0} \quad \text{in} \quad \Omega_p,$$

$$\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\theta} \quad \text{on} \quad \Gamma_{fp}, \quad \nabla\boldsymbol{\varphi}(\boldsymbol{\theta})\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p,$$

satisfying

$$c\|\boldsymbol{\varphi}(\boldsymbol{\theta})\|_{\mathrm{H}^1(\Omega_p)} \le \|\boldsymbol{\theta}\|_{\mathrm{H}^{1/2}(\Gamma_{fp})} \le C\|\boldsymbol{\varphi}(\boldsymbol{\theta})\|_{\mathrm{H}^1(\Omega_p)}. \tag{2.2.19}$$

Let $R_{\boldsymbol{\theta}} : \boldsymbol{\Lambda}_s \to \boldsymbol{\Lambda}_s'$ be defined as

$$(R_{\boldsymbol{\theta}}\boldsymbol{\theta}, \boldsymbol{\phi}) = r_{\boldsymbol{\theta}}(\boldsymbol{\theta}, \boldsymbol{\phi}) := (\nabla\boldsymbol{\varphi}(\boldsymbol{\theta}), \nabla\boldsymbol{\varphi}(\boldsymbol{\phi}))_{\Omega_p}. \tag{2.2.20}$$

Lemma 2.2.6. *The operators L_λ and $R_{\boldsymbol{\theta}}$ are continuous and coercive.*

Proof. It follows from (2.2.17) and (2.2.19) that there exist positive constants c and C such that

$$\begin{aligned}
&(L_\lambda\lambda, \xi) \leq C\|\lambda\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}\|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, && (L_\lambda\lambda, \lambda) \geq c\|\lambda\|^2_{\mathrm{H}^{1/2}(\Gamma_{fp})}, && \forall\, \lambda, \xi \in \Lambda_p,\\
&(R_{\boldsymbol{\theta}}\boldsymbol{\theta}, \boldsymbol{\phi}) \leq C\|\boldsymbol{\theta}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}, && (R_{\boldsymbol{\theta}}\boldsymbol{\theta}, \boldsymbol{\theta}) \geq c\|\boldsymbol{\theta}\|^2_{\mathbf{H}^{1/2}(\Gamma_{fp})}, && \forall\, \boldsymbol{\theta}, \boldsymbol{\phi} \in \boldsymbol{\Lambda}_s.
\end{aligned} \tag{2.2.21}$$

□

Lemma 2.2.7. *For every $\widehat{\mathbf{F}} \in \mathbf{Q}_2'$ and $\widehat{\mathbf{G}} \in \mathbf{S}'$, there exists a solution of the resolvent system (2.2.16).*

Proof. Define the operators $\mathcal{R} : \mathbf{Q} \to \mathbf{Q}_2'$ and $\mathcal{L} : \mathbf{S} \to \mathbf{S}'$ such that, for any $\mathbf{p} = (\mathbf{u}_f, \boldsymbol{\theta}, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p)$, $\mathbf{q} = (\mathbf{v}_f, \boldsymbol{\phi}, \mathbf{v}_p, \boldsymbol{\tau}_p, w_p) \in \mathbf{Q}$ and $\mathbf{r} = (p_f, \mathbf{u}_s, \boldsymbol{\gamma}_p, \lambda)$, $\mathbf{s} = (w_f, \mathbf{v}_s, \boldsymbol{\chi}_p, \xi) \in \mathbf{S}$,

$$(\mathcal{R}\mathbf{p}, \mathbf{q}) := (R_{\mathbf{u}_p}\mathbf{u}_p, \mathbf{v}_p) + (R_{\boldsymbol{\sigma}_p}\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) + (R_{p_p}p_p, w_p) + (R_{\boldsymbol{\theta}}\boldsymbol{\theta}, \boldsymbol{\phi}),$$

$$(\mathcal{L}\mathbf{r}, \mathbf{s}) := (L_{p_f}p_f, w_f) + (L_{\mathbf{u}_s}\mathbf{u}_s, \mathbf{v}_s) + (L_{\boldsymbol{\gamma}_p}\boldsymbol{\gamma}_p, \boldsymbol{\chi}_p) + (L_\lambda\lambda, \xi).$$

For $\epsilon > 0$, consider a regularization of (2.2.15): Given $\widehat{\mathbf{F}} = (\widehat{g}_{\mathbf{v}_f}, \widehat{g}_{\boldsymbol{\phi}}, \widehat{g}_{\mathbf{v}_p}, \widehat{g}_{\boldsymbol{\tau}_p}, \widehat{g}_{w_p}) \in \mathbf{Q}_2'$ and $\widehat{\mathbf{G}} = (\widehat{g}_{w_f}, \widehat{g}_{\mathbf{v}_s}, \widehat{g}_{\boldsymbol{\chi}_p}, \widehat{g}_{\xi}) \in \mathbf{S}'$, find $\mathbf{p}_\epsilon = (\mathbf{u}_{f,\epsilon}, \boldsymbol{\theta}_\epsilon, \mathbf{u}_{p,\epsilon}, \boldsymbol{\sigma}_{p,\epsilon}, p_{p,\epsilon}) \in \mathbf{Q}$ and $\mathbf{r}_\epsilon = (p_{f,\epsilon}, \mathbf{u}_{s,\epsilon}, \boldsymbol{\gamma}_{p,\epsilon}, \lambda_\epsilon)$ $\in \mathbf{S}$ such that

$$\begin{aligned}
(\epsilon\mathcal{R} + \mathcal{E}_1 + \mathcal{A})\mathbf{p}_\epsilon + \mathcal{B}'\mathbf{r}_\epsilon = \widehat{\mathbf{F}} \quad &\text{in} \quad \mathbf{Q}_2',\\
-\mathcal{B}\mathbf{p}_\epsilon + \epsilon\mathcal{L}\mathbf{r}_\epsilon = \widehat{\mathbf{G}} \quad &\text{in} \quad \mathbf{S}'.
\end{aligned} \tag{2.2.22}$$

Let the operator $\mathcal{O} : \mathbf{Q} \times \mathbf{S} \to \mathbf{Q}_2' \times \mathbf{S}'$ be defined as

$$\mathcal{O}\begin{pmatrix} \mathbf{q} \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} \epsilon\mathcal{R} + \mathcal{E}_1 + \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \epsilon\mathcal{L} \end{pmatrix}\begin{pmatrix} \mathbf{q} \\ \mathbf{s} \end{pmatrix}.$$

We have

$$\left(\mathcal{O}\begin{pmatrix}\mathbf{p}\\ \mathbf{r}\end{pmatrix},\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix}\right) = ((\epsilon\mathcal{R}+\mathcal{E}_1+\mathcal{A})\mathbf{p},\mathbf{q}) + (\mathcal{B}'\mathbf{r},\mathbf{q}) - (\mathcal{B}\mathbf{p},\mathbf{s}) + \epsilon(\mathcal{L}\mathbf{r},\mathbf{s}).$$

Lemmas 2.2.1–2.2.6 imply that $\mathcal{O}$ is continuous. Moreover, using the coercivity and monotonicity bounds (2.2.5), (2.2.6), and (2.2.21), we have

$$\begin{aligned}
\left(\mathcal{O}\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix},\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix}\right) &= ((\epsilon\mathcal{R}+\mathcal{E}_1+\mathcal{A})\mathbf{q},\mathbf{q}) + (\epsilon\mathcal{L}\mathbf{s},\mathbf{s})\\
&= \epsilon r_{\mathbf{u}_p}(\mathbf{v}_p,\mathbf{v}_p) + \epsilon r_{\boldsymbol{\sigma}_p}(\boldsymbol{\tau}_p,\boldsymbol{\tau}_p) + \epsilon r_{\boldsymbol{\theta}}(\boldsymbol{\phi},\boldsymbol{\phi}) + \epsilon r_{p_p}(w_p,w_p) + a_p(\mathbf{v}_p,\mathbf{v}_p)\\
&\quad + (A(\boldsymbol{\tau}_p+\alpha_p w_p\mathbf{I}),\boldsymbol{\tau}_p+\alpha_p w_p\mathbf{I}) + (s_0 w_p,w_p) + a_f(\mathbf{v}_f,\mathbf{v}_f) + a_{\mathtt{BJS}}(\mathbf{v}_f,\boldsymbol{\phi};\mathbf{v}_f,\boldsymbol{\phi})\\
&\quad + \epsilon l_{p_f}(w_f,w_f) + \epsilon l_{\mathbf{u}_s}(\mathbf{v}_s,\mathbf{v}_s) + \epsilon l_{\boldsymbol{\gamma}_p}(\boldsymbol{\chi}_p,\boldsymbol{\chi}_p) + \epsilon l_\lambda(\xi,\xi)\\
&\geq C(\epsilon\|\mathrm{div}(\mathbf{v}_p)\|^2_{\mathrm{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\mathbf{div}(\boldsymbol{\tau}_p)\|^2_{\mathbf{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\phi}\|^2_{\mathbf{H}^{1/2}(\Gamma_{fp})} + \epsilon\|w_p\|^2_{\mathrm{L}^2(\Omega_p)}\\
&\quad + \|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\tau}_p+\alpha_p w_p\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{e}(\mathbf{v}_f)\|^2_{\mathbb{L}^2(\Omega_f)}\\
&\quad + |\mathbf{v}_f-\boldsymbol{\phi}|^2_{a_{\mathtt{BJS}}} + \epsilon\|w_f\|^2_{\mathrm{L}^2(\Omega_p)} + \epsilon\|\mathbf{v}_s\|^2_{\mathbf{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\chi}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\xi\|^2_{\mathrm{H}^{1/2}(\Gamma_{fp})}),
\end{aligned} \tag{2.2.23}$$

which implies that $\mathcal{O}$ is coercive. Thus, an application of the Lax-Milgram theorem establishes the existence of a unique solution $(\mathbf{p}_\epsilon,\mathbf{r}_\epsilon)\in\mathbf{Q}\times\mathbf{S}$ of (2.2.22). Now, from (2.2.22) and (2.2.23) we obtain

$$\begin{aligned}
&\epsilon\|\mathrm{div}(\mathbf{u}_{p,\epsilon})\|^2_{\mathrm{L}^2(\Omega_p)} + \epsilon\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|^2_{\mathbf{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\theta}_\epsilon\|^2_{\mathbf{H}^{1/2}(\Gamma_{fp})} + \epsilon\|\boldsymbol{\sigma}_{p,\epsilon}\|^2_{\mathbb{L}^2(\Omega_p)} + \epsilon\|p_{p,\epsilon}\|^2_{\mathrm{L}^2(\Omega_p)}\\
&\quad + \|\mathbf{u}_{p,\epsilon}\|^2_{\mathbf{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\sigma}_{p,\epsilon}+\alpha_p p_{p,\epsilon}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|p_{p,\epsilon}\|^2_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{u}_{f,\epsilon}\|^2_{\mathbf{H}^1(\Omega_f)}\\
&\quad + |\mathbf{u}_{f,\epsilon}-\boldsymbol{\theta}_\epsilon|^2_{a_{\mathtt{BJS}}} + \epsilon\|p_{f,\epsilon}\|^2_{\mathrm{L}^2(\Omega_p)} + \epsilon\|\mathbf{u}_{s,\epsilon}\|^2_{\mathbf{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\gamma}_{p,\epsilon}\|^2_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\lambda_\epsilon\|^2_{\mathrm{H}^{1/2}(\Gamma_{fp})}\\
&\leq C(\|\widehat{g}_{\mathbf{v}_f}\|_{\mathbf{L}^2(\Omega_f)}\|\mathbf{u}_{f,\epsilon}\|_{\mathbf{L}^2(\Omega_f)} + \|\widehat{g}_{\boldsymbol{\phi}}\|_{\mathbf{L}^2(\Omega_p)}\|\boldsymbol{\theta}_\epsilon\|_{\mathbf{L}^2(\Omega_p)} + \|\widehat{g}_{\mathbf{v}_p}\|_{\mathbf{L}^2(\Omega_p)}\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}\\
&\quad + \|\widehat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)}\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{g}_{w_p}\|_{\mathrm{L}^2(\Omega_p)}\|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)} + \|\widehat{g}_{w_f}\|_{\mathrm{L}^2(\Omega_f)}\|p_{f,\epsilon}\|_{\mathrm{L}^2(\Omega_f)}\\
&\quad + \|\widehat{g}_{\mathbf{v}_s}\|_{\mathbf{L}^2(\Omega_p)}\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{L}^2(\Omega_p)} + \|\widehat{g}_{\boldsymbol{\chi}_p}\|_{\mathbb{L}^2(\Omega_p)}\|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{g}_\xi\|_{\mathrm{L}^2(\Omega_p)}\|\lambda_\epsilon\|_{\mathrm{L}^2(\Omega_p)}),
\end{aligned} \tag{2.2.24}$$

which implies that $\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}$, $\|A^{1/2}(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}$ and $\|\mathbf{u}_{f,\epsilon}\|_{\mathbf{H}^1(\Omega_f)}$ are bounded independently of ϵ. Next, from (2.2.22) we have

$$(A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + \epsilon(\boldsymbol{\sigma}_{p,\epsilon}, \boldsymbol{\tau}_p)_{\Omega_p} + \epsilon(\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon}), \mathbf{div}(\boldsymbol{\tau}_p)_{\Omega_p})$$
$$+ b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_\epsilon) + b_s(\boldsymbol{\tau}_p, \mathbf{u}_{s,\epsilon}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_{p,\epsilon}) = (\widehat{g}_{\tau_p}, \boldsymbol{\tau}_p)_{\Omega_p}. \tag{2.2.25}$$

Applying the inf-sup condition (2.2.7) results in

$$\begin{aligned}
&\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{L}^2(\Omega_p)} + \|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} \leq C \sup_{\boldsymbol{\tau}_p \in \mathbb{X}_p \text{ s.t. } \boldsymbol{\tau}_p \mathbf{n}_p = \mathbf{0} \text{ on } \Gamma_{fp}} \frac{b_s(\boldsymbol{\tau}_p, \mathbf{u}_{s,\epsilon}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_{p,\epsilon})}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} \\
&\quad = C \sup_{\boldsymbol{\tau}_p \in \mathbb{X}_p \text{ s.t. } \boldsymbol{\tau}_p \mathbf{n}_p = \mathbf{0} \text{ on } \Gamma_{fp}} \Bigg(\frac{-(A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} - \epsilon(\boldsymbol{\sigma}_{p,\epsilon}, \boldsymbol{\tau}_p)_{\Omega_p}}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} \\
&\qquad + \frac{-\epsilon(\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon}), \mathbf{div}(\boldsymbol{\tau}_p)_{\Omega_p}))_{\Omega_p} - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_\epsilon) + (\widehat{g}_{\boldsymbol{\tau}_p}, \boldsymbol{\tau}_p)_{\Omega_p}}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} \Bigg) \\
&\quad \leq C(\|A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)} + \|\widehat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)}),
\end{aligned} \tag{2.2.26}$$

where the term $b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_\epsilon)$ vanishes due to the restriction $\boldsymbol{\tau}_p \mathbf{n}_p = \mathbf{0}$ on Γ_{fp}. Also, applying the inf-sup condition (2.2.9) and using (2.2.25), we obtain

$$\begin{aligned}
&\|\boldsymbol{\theta}_\epsilon\|_{\mathbf{H}^{1/2}(\Gamma_{fp})} \leq C \sup_{\boldsymbol{\tau}_p \in \mathbb{X}_p \text{ s.t. } \mathbf{div}(\boldsymbol{\tau}_p = 0)} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_\epsilon)}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} \\
&= C \sup_{\boldsymbol{\tau}_p \in \mathbb{X}_p \text{ s.t. } \mathbf{div}(\boldsymbol{\tau}_p = 0)} \frac{(-A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} - \epsilon(\boldsymbol{\sigma}_{p,\epsilon}, \boldsymbol{\tau}_p)_{\Omega_p} - b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_{p,\epsilon}) + (\widehat{g}_{\boldsymbol{\tau}_p}, \boldsymbol{\tau}_p)_{\Omega_p}}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} \\
&\leq C(\|A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + \|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_f)} + \|\widehat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)}).
\end{aligned} \tag{2.2.27}$$

Bounds (2.2.26) and (2.2.27) imply that $\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}$ $\|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)}$, and $\|\boldsymbol{\theta}_\epsilon\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}$ are bounded independently of ϵ. In addition, (2.2.22) gives

$$a_p(\mathbf{u}_{p,\epsilon}, \mathbf{v}_p) + \epsilon(\mathrm{div}(\mathbf{u}_{p,\epsilon}), \mathrm{div}(\mathbf{v}_p))_{\Omega_p} + b_p(\mathbf{v}_p, p_{p,\epsilon}) + \langle \mathbf{v}_p \cdot \mathbf{n}_p, \lambda_\epsilon \rangle_{\Gamma_{fp}} + a_f(\mathbf{u}_{f,\epsilon}, \mathbf{v}_f)$$
$$+ a_{\mathtt{BJS}}(\mathbf{u}_{f,\epsilon}, \boldsymbol{\theta}_\epsilon; \mathbf{v}_f, \mathbf{0}) + b_f(\mathbf{v}_f, p_{f,\epsilon}) + \langle \mathbf{v}_f \cdot \mathbf{n}_f, \lambda_\epsilon \rangle_{\Gamma_{fp}} = 0, \tag{2.2.28}$$

so applying the inf-sup condition (2.2.8), we obtain

$$\|p_{f,\epsilon}\|_{\mathrm{L}^2(\Omega_f)} + \|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)} + \|\lambda_\epsilon\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}$$

$$\leq C \sup_{(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0})\in\mathbf{V}_f\times\mathbf{V}_p\times\mathbf{\Lambda}_s} \frac{b_f(\mathbf{v}_f,p_{f,\epsilon})+b_p(\mathbf{v}_p,p_{p,\epsilon})+b_\Gamma(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0};\lambda_\epsilon)}{\|(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0})\|_{\mathbf{V}_f\times\mathbf{V}_p\times\mathbf{\Lambda}_s}}$$

$$= C \sup_{(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0})\in\mathbf{V}_f\times\mathbf{V}_p\times\mathbf{\Lambda}_s} \left(\frac{-a_p(\mathbf{u}_{p,\epsilon},\mathbf{v}_p)-\epsilon(\operatorname{div}(\mathbf{u}_{p,\epsilon}),\operatorname{div}(\mathbf{v}_p))}{\|(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0})\|_{\mathbf{V}_f\times\mathbf{V}_p\times\mathbf{\Lambda}_s}} \right.$$

$$\left. + \frac{-a_f(\mathbf{u}_{f,\epsilon},\mathbf{v}_f)-a_{\mathtt{BJS}}(\mathbf{u}_{f,\epsilon},\boldsymbol{\theta}_\epsilon;\mathbf{v}_f,\mathbf{0})}{\|(\mathbf{v}_f,\mathbf{v}_p,\mathbf{0})\|_{\mathbf{V}_f\times\mathbf{V}_p\times\mathbf{\Lambda}_s}} \right)$$

$$\leq C(\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}+\epsilon\|\operatorname{div}(\mathbf{u}_{p,\epsilon})\|_{\mathrm{L}^2(\Omega_p)}+\|\mathbf{u}_{f,\epsilon}\|_{\mathbf{H}^1(\Omega_f)}+|\mathbf{u}_{f,\epsilon}-\boldsymbol{\theta}_\epsilon|_{a_{\mathtt{BJS}}}). \tag{2.2.29}$$

Therefore we have that $\|p_{f,\epsilon}\|_{\mathrm{L}^2(\Omega_f)}$, $\|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)}$ and $\|\lambda_\epsilon\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}$ are also bounded independently of ϵ.

Since $\mathbf{div}(\mathbb{X}_p)=\mathbf{V}_s$, by taking $\mathbf{v}_s=\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})$ in (2.2.22), we have

$$\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)} \leq \epsilon\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}+\|\widehat{g}_{\mathbf{v}_s}\|_{\mathbf{L}^2(\Omega_p)}, \tag{2.2.30}$$

which implies that $\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)}$ is bounded independently of ϵ. Since $\|A^{1/2}(\boldsymbol{\sigma}_{p,\epsilon}+\alpha_p p_{p,\epsilon}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}$, $\|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)}$ and $\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)}$ are all bounded independently of ϵ, the same holds for $\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{H}(\mathrm{div},\Omega_p)}$. Finally, since $\operatorname{div}(\mathbf{V}_p)=\mathrm{W}_p$, by taking $w_p=\operatorname{div}(\mathbf{u}_{p,\epsilon})$ in (2.2.22), we have

$$\|\operatorname{div}(\mathbf{u}_{p,\epsilon})\|_{\mathrm{L}^2(\Omega_p)} \leq C(\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)}+(s_0+\epsilon)\|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)}+\|\widehat{g}_{w_p}\|_{\mathrm{L}^2(\Omega_p)}), \tag{2.2.31}$$

so $\|\operatorname{div}(\mathbf{u}_{p,\epsilon})\|_{\mathrm{L}^2(\Omega_p)}$, and therefore $\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{V}_p}$ is bounded independently of ϵ. Thus we conclude that all the variables are bounded independently of ϵ.

Since $\mathbf{Q}$ and $\mathbf{S}$ are reflexive Banach spaces, as $\epsilon\to 0$ we can extract weakly convergent subsequences $\{\mathbf{p}_{\epsilon,n}\}_{n=1}^\infty$ and $\{\mathbf{r}_{\epsilon,n}\}_{n=1}^\infty$ such that $\mathbf{p}_{\epsilon,n}\to\mathbf{p}$ in $\mathbf{Q}$, $\mathbf{r}_{\epsilon,n}\to\mathbf{r}$ in $\mathbf{S}$. Taking the limit in (2.2.22), we obtain that $(\mathbf{p},\mathbf{r})$ is a solution to (2.2.16). □

Lemma 2.2.8. *For $\mathcal{N}$, $\mathcal{M}$ and E_b' defined in* (2.2.12) *and* (2.2.13)*, it holds that $Rg(\mathcal{N}+\mathcal{M})=E_b'$, that is, given $f\in E_b'$, there exists $v\in\mathcal{D}$ such that $(\mathcal{N}+\mathcal{M})v=f$.*

Proof. Given any $\widehat{g}_{\boldsymbol{\tau}_p} \in \mathbb{X}'_{p,2}$ and $\widehat{g}_{w_p} \in \mathrm{W}'_{p,2}$, according to Lemma 2.2.7, there exist $(\mathbf{p},\mathbf{r}) \in \mathbf{Q}\times\mathbf{S}$ such that

$$(\mathcal{E}_1+\mathcal{A})\mathbf{p}+\mathcal{B}'\mathbf{r}=\widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{Q}'_{2,0},$$

$$-\mathcal{B}\mathbf{p}=\mathbf{0} \quad \text{in} \quad \mathbf{S}'_{2,0},$$

where $\widehat{\mathbf{F}}=(\mathbf{0},0,0,\widehat{g}_{\boldsymbol{\tau}_p},\widehat{g}_{w_p})^{\mathrm{t}} \in \mathbf{Q}'_{2,0}$, implying the range condition. □

We are now ready to establish existence for the auxiliary initial value problem (2.2.14), assuming compatible initial data.

Theorem 2.2.9. *For each compatible initial data* $(\widehat{\mathbf{p}}_0,\widehat{\mathbf{r}}_0) \in \mathcal{D}$ *and each* $(\widehat{g}_{\boldsymbol{\tau}_p},\widehat{g}_{w_p}) \in \mathrm{W}^{1,1}(0,T;\mathbb{X}'_{p,2})\times \mathrm{W}^{1,1}(0,T;\mathrm{W}'_{p,2})$*, there exists a solution to* (2.2.14) *with* $(\boldsymbol{\sigma}_p(0),p_p(0)) = (\widehat{\boldsymbol{\sigma}}_{p,0},\widehat{p}_{p,0})$ *and* $(\mathbf{u}_f,p_f,\boldsymbol{\sigma}_p,\mathbf{u}_s,\boldsymbol{\gamma}_p,\mathbf{u}_p,p_p,\lambda,\boldsymbol{\theta}) : [0,T] \to \mathbf{V}_f\times \mathrm{W}_f\times\mathbb{X}_p\times\mathbf{V}_s\times\mathbb{Q}_p\times\mathbf{V}_p\times \mathrm{W}_p\times\Lambda_p\times\boldsymbol{\Lambda}_s$ *such that* $(\boldsymbol{\sigma}_p,p_p)\in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p))\times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$*.*

Proof. Using Lemma 2.2.4 and Lemma 2.2.8, we apply Theorem 2.2.3 with E, $\mathcal{N}$ and $\mathcal{M}$ defined in (2.2.12) to obtain existence of a solution to (2.2.14) with $\boldsymbol{\sigma}_p \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p))$ and $p_p \in \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$. □

We will employ Theorem 2.2.9 to obtain existence of a solution to our problem (2.1.13). To that end, we first construct compatible initial data $(\mathbf{p}_0,\mathbf{r}_0)$.

Lemma 2.2.10. *Assume that the initial data* $p_{p,0}\in \mathrm{W}_p\cap \mathrm{H}$*, where*

$$\mathrm{H} := \big\{ w_p \in \mathrm{H}^1(\Omega_p) : \mathbf{K}\nabla w_p \in \mathbf{H}^1(\Omega_p),\ \mathbf{K}\nabla w_p\cdot\mathbf{n}_p = 0 \quad on \quad \Gamma_p^{\mathrm{N}},\ w_p = 0 \quad on \quad \Gamma_p^{\mathrm{D}} \big\}. \tag{2.2.32}$$

Then, there exist $\mathbf{p}_0 := (\mathbf{u}_{f,0},\boldsymbol{\theta}_0,\mathbf{u}_{p,0},\boldsymbol{\sigma}_{p,0},p_{p,0}) \in \mathbf{Q}$ *and* $\mathbf{r}_0 := (p_{f,0},\mathbf{u}_{s,0},\boldsymbol{\gamma}_{p,0},\lambda_0)\in\mathbf{S}$ *such that*

$$\begin{aligned} \mathcal{A}\mathbf{p}_0+\mathcal{B}'\mathbf{r}_0 &= \widehat{\mathbf{F}}_0 \quad in \quad \mathbf{Q}'_2, \\ -\mathcal{B}\mathbf{p}_0 &= \mathbf{G}(0) \quad in \quad \mathbf{S}', \end{aligned} \tag{2.2.33}$$

where $\widehat{\mathbf{F}}_0 = (\mathbf{f}_f(0),0,0,\widehat{g}_{\boldsymbol{\tau}_p},\widehat{g}_{w_p})^{\mathrm{t}} \in \mathbf{Q}'_2$*, with suitable* $\widehat{g}_{\boldsymbol{\tau}_p}\in\mathbb{X}'_{p,2}$ *and* $\widehat{g}_{w_p}\in \mathrm{W}'_{p,2}$*.*

Proof. Our approach is to solve a sequence of well-defined subproblems, using the previously obtained solutions as data to guarantee that we obtain a solution of the coupled problem (2.2.33). We proceed as follows.

1. Define $\mathbf{u}_{p,0} := -\mu^{-1}\mathbf{K}\nabla p_{p,0} \in \mathbf{H}^1(\Omega_p)$, with $p_{p,0} \in \mathrm{W}_p \cap \mathrm{H}$, cf. (2.2.32). It follows that

$$\mu\mathbf{K}^{-1}\mathbf{u}_{p,0} = -\nabla p_{p,0}, \quad \mathrm{div}(\mathbf{u}_{p,0}) = -\mu^{-1}\mathrm{div}(\mathbf{K}\nabla p_{p,0}) \quad \text{in} \quad \Omega_p, \quad \mathbf{u}_{p,0}\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}.$$

Next, define $\lambda_0 = p_{p,0}|_{\Gamma_{fp}} \in \Lambda_p$. Testing the first two equations above with $\mathbf{v}_p \in \mathbf{V}_p$ and $w_p \in \mathrm{W}_p$, respectively, we obtain

$$\begin{aligned} a_p(\mathbf{u}_{p,0},\mathbf{v}_p) + b_p(\mathbf{v}_p, p_{p,0}) + \langle \mathbf{v}_p\cdot\mathbf{n}_p, \lambda_0\rangle_{\Gamma_{fp}} &= 0, && \forall\, \mathbf{v}_p \in \mathbf{V}_p, \\ -b_p(\mathbf{u}_{p,0}, w_p) &= -\mu^{-1}(\mathrm{div}(\mathbf{K}\nabla p_{p,0}), w_p)_{\Omega_p}, && \forall\, w_p \in \mathrm{W}_p. \end{aligned} \tag{2.2.34}$$

2. Define $(\mathbf{u}_{f,0}, p_{f,0}) \in \mathbf{V}_f \times \mathrm{W}_f$ such that

$$\begin{aligned} & a_f(\mathbf{u}_{f,0},\mathbf{v}_f) + b_f(\mathbf{v}_f, p_{f,0}) \\ & = -\sum_{j=1}^{n-1}\langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}\mathbf{u}_{p,0}\cdot\mathbf{t}_{f,j}, \mathbf{v}_f\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}} - \langle \mathbf{v}_f\cdot\mathbf{n}_f, \lambda_0\rangle_{\Gamma_{fp}} + (\mathbf{f}_f(0), \mathbf{v}_f)_{\Omega_f}, \quad \forall\, \mathbf{v}_f \in \mathbf{V}_f, \\ & -b_f(\mathbf{u}_{f,0}, w_f) = (q_f(0), w_f), \qquad \forall\, w_f \in \mathrm{W}_f. \end{aligned} \tag{2.2.35}$$

This is a well-posed problem, since it corresponds to the weak solution of the Stokes system with mixed boundary conditions on Γ_{fp}. Note that λ_0 and $\mathbf{u}_{p,0}$ are data for this problem.

3. Define $(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\eta}_{p,0}, \boldsymbol{\rho}_{p,0}, \boldsymbol{\omega}_0) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \boldsymbol{\Lambda}_s$ such that

$$\begin{aligned} & (A\boldsymbol{\sigma}_{p,0}, \boldsymbol{\tau}_p)_{\Omega_p} + b_s(\boldsymbol{\tau}_p, \boldsymbol{\eta}_{p,0}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\rho}_{p,0}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\omega}_0) = -(A\alpha_p p_{p,0}\mathbf{I}, \boldsymbol{\tau}_p)_{\Omega_p}, && \forall\, \boldsymbol{\tau}_p \in \mathbb{X}_p, \\ & -b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}) = \sum_{j=1}^{n-1}\langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}\mathbf{u}_{p,0}\cdot\mathbf{t}_{f,j}, \boldsymbol{\phi}\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\phi}\cdot\mathbf{n}_p, \lambda_0\rangle_{\Gamma_{fp}}, && \forall\, \boldsymbol{\phi} \in \boldsymbol{\Lambda}_s, \\ & -b_s(\boldsymbol{\sigma}_{p,0}, \mathbf{v}_s) = (\mathbf{f}_p(0), \mathbf{v}_s)_{\Omega_p}, && \forall\, \mathbf{v}_s \in \mathbf{V}_s, \\ & -b_{\mathrm{sk}}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi}_p) = 0, && \forall\, \boldsymbol{\chi}_p \in \mathbb{Q}_p. \end{aligned} \tag{2.2.36}$$

This is a well-posed problem corresponding to the weak solution of the mixed elasticity system with mixed boundary conditions on Γ_{fp}. Note that $p_{p,0}$, $\mathbf{u}_{p,0}$ and λ_0 are data for this problem. Here $\boldsymbol{\eta}_{p,0}$, $\boldsymbol{\rho}_{p,0}$, and $\boldsymbol{\omega}_0$ are auxiliary variables that are not part of the constructed initial data. However, they can be used to recover the variables $\boldsymbol{\eta}_p$, $\boldsymbol{\rho}_p$, and $\boldsymbol{\omega}$ that satisfy the non-differentiated equation (2.1.12).

4. Define $\boldsymbol{\theta}_0 \in \boldsymbol{\Lambda}_s$ as

$$\boldsymbol{\theta}_0 = \mathbf{u}_{f,0} - \mathbf{u}_{p,0} \quad \text{on} \quad \Gamma_{fp}, \tag{2.2.37}$$

where $\mathbf{u}_{f,0}$ and $\mathbf{u}_{p,0}$ are data obtained in the previous steps. Note that (2.2.37) implies that the BJS terms in (2.2.35) and (2.2.36) can be rewritten with $\mathbf{u}_{p,0} \cdot \mathbf{t}_{f,j}$ replaced by $(\mathbf{u}_{f,0} - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}$ and that (2.1.11h) holds for the initial data.

5. Define $(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$ such that

$$\begin{aligned}
&(A\widehat{\boldsymbol{\sigma}}_{p,0}, \boldsymbol{\tau}_p)_{\Omega_p} + b_s(\boldsymbol{\tau}_p, \mathbf{u}_{s,0}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_{p,0}) = -b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_0), && \forall\, \boldsymbol{\tau}_p \in \mathbb{X}_p, \\
&-b_s(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{v}_s) = 0, && \forall\, \mathbf{v}_s \in \mathbf{V}_s, \\
&-b_{\mathrm{sk}}(\widehat{\boldsymbol{\sigma}}_{p,0}, \boldsymbol{\chi}_p) = 0, && \forall\, \boldsymbol{\chi}_p \in \mathbb{Q}_p.
\end{aligned} \tag{2.2.38}$$

This is a well-posed problem, since it corresponds to the weak solution of the mixed elasticity system with Dirichlet data $\boldsymbol{\theta}_0$ on Γ_{fp}. We note that $\widehat{\boldsymbol{\sigma}}_{p,0}$ is an auxiliary variable not used in the initial data.

Combining (2.2.34)–(2.2.38), we obtain $(\mathbf{u}_{f,0}, \boldsymbol{\theta}_0, \mathbf{u}_{p,0}, \boldsymbol{\sigma}_{p,0}, p_{p,0}) \in \mathbf{Q}$ and $(p_{f,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{p,0}, \lambda_0) \in \mathbf{S}$ satisfying (2.2.33) with

$$(\widehat{g}_{\boldsymbol{\tau}_p}, \boldsymbol{\tau}_p)_{\Omega_p} = -(A(\widehat{\boldsymbol{\sigma}}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p}, \quad (\widehat{g}_{w_p}, w_p)_{\Omega_p} = -b_p(\mathbf{u}_{p,0}, w_p).$$

The above equations imply

$$\|\widehat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{g}_{w_p}\|_{\mathrm{L}^2(\Omega_p)} \leq C(\|\widehat{\boldsymbol{\sigma}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\mathrm{div}(\mathbf{u}_{p,0})\|_{\mathrm{L}^2(\Omega_p)}),$$

hence $(\widehat{g}_{\boldsymbol{\tau}_p}, \widehat{g}_{w_p}) \in \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$, completing the proof. □

We are now ready to prove the main result of this section.

Theorem 2.2.11. *For each compatible initial data* $(\mathbf{p}_0, \mathbf{r}_0) \in \mathcal{D}$ *constructed in Lemma 2.2.10 and each*

$$\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_f'), \quad q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p'),$$

there exists a unique solution of (2.1.11) $(\mathbf{u}_f, p_f, \boldsymbol{\sigma}_p, \mathbf{u}_s, \boldsymbol{\gamma}_p, \mathbf{u}_p, p_p, \lambda, \boldsymbol{\theta}) : [0,T] \to \mathbf{V}_f \times \mathrm{W}_f \times \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \mathbf{V}_p \times \mathrm{W}_p \times \Lambda_p \times \mathbf{\Lambda}_s$ *such that* $(\boldsymbol{\sigma}_p, p_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

Proof. For each fixed time $t \in [0,T]$, Lemma 2.2.7 implies that there exists a solution to the resolvent system (2.2.16) with $\widehat{\mathbf{F}} = \mathbf{F}(t)$ and $\widehat{\mathbf{G}} = \mathbf{G}(t)$ defined in (2.1.14). In other words, there exist $(\widetilde{\mathbf{p}}(t), \widetilde{\mathbf{r}}(t))$ such that

$$\begin{aligned} (\mathcal{E}_1 + \mathcal{A})\,\widetilde{\mathbf{p}}(t) + \mathcal{B}'\,\widetilde{\mathbf{r}}(t) = \mathbf{F}(t) \quad \text{in} \quad \mathbf{Q}_2', \\ -\mathcal{B}\,\widetilde{\mathbf{p}}(t) = \mathbf{G}(t) \quad \text{in} \quad \mathbf{S}'. \end{aligned} \tag{2.2.39}$$

We look for a solution to (2.1.14) in the form $\mathbf{p}(t) = \widetilde{\mathbf{p}}(t) + \widehat{\mathbf{p}}(t)$, $\mathbf{r}(t) = \widetilde{\mathbf{r}}(t) + \widehat{\mathbf{r}}(t)$. Subtracting (2.2.39) from (2.1.14) leads to the reduced evolution problem

$$\begin{aligned} \partial_t\,\mathcal{E}_1\,\widehat{\mathbf{p}}(t) + \mathcal{A}\,\widehat{\mathbf{p}}(t) + \mathcal{B}'\,\widehat{\mathbf{r}}(t) = \mathcal{E}_1\,\widetilde{\mathbf{p}}(t) - \partial_t\,\mathcal{E}_1\,\widetilde{\mathbf{p}}(t) \quad \text{in} \quad \mathbf{Q}_{2,0}', \\ -\mathcal{B}\,\widehat{\mathbf{p}}(t) = \mathbf{0} \quad \text{in} \quad \mathbf{S}_{2,0}', \end{aligned} \tag{2.2.40}$$

with initial condition $\widehat{\mathbf{p}}(0) = \mathbf{p}_0 - \widetilde{\mathbf{p}}(0)$ and $\widehat{\mathbf{r}}(0) = \mathbf{r}_0 - \widetilde{\mathbf{r}}(0)$. Subtracting (2.2.39) at $t = 0$ from (2.2.33) gives

$$\begin{aligned} \mathcal{A}\,\widehat{\mathbf{p}}(0) + \mathcal{B}'\,\widehat{\mathbf{r}}(0) = \mathcal{E}_1\widetilde{\mathbf{p}}(0) + \widehat{\mathbf{F}}_0 - \mathbf{F}(0) \quad \text{in} \quad \mathbf{Q}_{2,0}', \\ -\mathcal{B}\,\widehat{\mathbf{p}}(0) = \mathbf{0} \quad \text{in} \quad \mathbf{S}_{2,0}', \end{aligned}$$

We emphasize that in the above, $\widehat{\mathbf{F}}_0 - \mathbf{F}(0) = (\mathbf{0}, 0, 0, \widehat{g}_{\boldsymbol{\tau}_p}, \widehat{g}_{w_p} - q_p(0))^{\mathrm{t}} \in \mathbf{Q}_{2,0}'$. Therefore, $\mathcal{M}\begin{pmatrix} \widehat{\mathbf{p}}(0) \\ \widehat{\mathbf{r}}(0) \end{pmatrix} \in E_b'$, i.e., $(\widehat{\mathbf{p}}(0), \widehat{\mathbf{r}}(0)) \in \mathcal{D}$. Thus, the reduced evolution problem (2.2.40) is in the form of (2.2.14). According to Theorem 2.2.9, it has a solution, which establishes the existence of a solution to (2.1.11) with the stated regularity satisfying $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

We next show that the solution is unique. Since the problem is linear, it is sufficient to prove that the problem with zero data has only the zero solution. Taking $\mathbf{F} = \mathbf{G} = \mathbf{0}$ in (2.1.14) and testing it with the solution $(\mathbf{p}, \mathbf{r})$ yields

$$\frac{1}{2}\partial_t \left(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|p_p\|^2_{\mathrm{L}^2(\Omega_p)} \right)$$
$$+ a_p(\mathbf{u}_p, \mathbf{u}_p) + a_f(\mathbf{u}_f, \mathbf{u}_f) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{u}_f, \boldsymbol{\theta}) = 0.$$

Integrating in time from 0 to $t \in (0, T]$ and using that the initial data is zero, as well as the coercivity of a_p and a_f and monotonicity of $a_{\mathtt{BJS}}$, cf. (2.2.5), we conclude that $\boldsymbol{\sigma}_p = \mathbf{0}$, $p_p = 0$, $\mathbf{u}_p = \mathbf{0}$, and $\mathbf{u}_f = 0$. Then the inf-sup conditions (2.2.7)–(2.2.9) imply that $\mathbf{u}_s = \mathbf{0}$, $\boldsymbol{\gamma}_p = \mathbf{0}$, $\boldsymbol{\theta} = \mathbf{0}$, $p_f = 0$, and $\lambda = 0$, using arguments similar to (2.2.26)–(2.2.29). Therefore the solution of (2.1.13) is unique. □

Corollary 2.2.12. *The solution of* (2.1.13) *satisfies* $\mathbf{u}_f(0) = \mathbf{u}_{f,0}$, $p_f(0) = p_{f,0}$, $\mathbf{u}_p(0) = \mathbf{u}_{p,0}$, $\lambda(0) = \lambda_0$, *and* $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$.

Proof. Let $\overline{\mathbf{u}}_f := \mathbf{u}_f(0) - \mathbf{u}_{f,0}$, with a similar definition and notation for the rest of the variables. Since Theorem 2.2.3 implies that $\mathcal{M}(u) \in \mathrm{L}^\infty(0, T; E_b')$, we can take $t \to 0^+$ in all equations without time derivatives in (2.1.13) and using that the initial data $(\mathbf{p}_0, \mathbf{r}_0)$ satisfies the same equations at $t = 0$, cf. (2.2.33), and that $\overline{\boldsymbol{\sigma}}_p = \mathbf{0}$ and $\overline{p}_p = 0$, we obtain

$$(2\mu \mathbf{e}(\overline{\mathbf{u}}_f), \mathbf{e}(\mathbf{v}_f))_{\Omega_f} - (\mathrm{div}(\mathbf{v}_f), \overline{p}_f)_{\Omega_f} + \langle \mathbf{v}_f \cdot \mathbf{n}_f, \overline{\lambda} \rangle_{\Gamma_{fp}}$$
$$+ \sum_{j=1}^{n-1} \langle \mu \alpha_{\mathtt{BJS}} \sqrt{\mathbf{K}_j^{-1}} (\overline{\mathbf{u}}_f - \overline{\boldsymbol{\theta}}) \cdot \mathbf{t}_{f,j}, \mathbf{v}_f \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} = 0, \tag{2.2.41a}$$

$$(\mathrm{div}(\overline{\mathbf{u}}_f), w_f)_{\Omega_f} = 0, \tag{2.2.41b}$$

$$(\mu \mathbf{K}^{-1} \overline{\mathbf{u}}_p, \mathbf{v}_p)_{\Omega_p} + \langle \mathbf{v}_p \cdot \mathbf{n}_p, \overline{\lambda} \rangle_{\Gamma_{fp}} = 0, \tag{2.2.41c}$$

$$\langle \overline{\mathbf{u}}_f \cdot \mathbf{n}_f + \overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p + \overline{\mathbf{u}}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} = 0, \tag{2.2.41d}$$

$$\langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \overline{\lambda} \rangle_{\Gamma_{fp}} - \sum_{j=1}^{n-1} \langle \mu \alpha_{\mathtt{BJS}} \sqrt{\mathbf{K}_j^{-1}} (\overline{\mathbf{u}}_f - \overline{\boldsymbol{\theta}}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi} \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} = 0. \tag{2.2.41e}$$

Taking $(\mathbf{v}_f, w_f, \mathbf{v}_p, \xi, \boldsymbol{\phi}) = (\overline{\mathbf{u}}_f, \overline{p}_f, \overline{\mathbf{u}}_p, \overline{\lambda}, \overline{\boldsymbol{\theta}})$ and combining the equations results in

$$\|\overline{\mathbf{u}}_f\|^2_{\mathbf{H}^1(\Omega_f)} + \|\overline{\mathbf{u}}_p\|^2_{\mathrm{L}^2(\Omega_p)} + |\overline{\mathbf{u}}_f - \overline{\boldsymbol{\theta}}|^2_{a_{\mathrm{BJS}}} \leq 0,$$

which implies $\overline{\mathbf{u}}_f = \mathbf{0}$, $\overline{\mathbf{u}}_p = \mathbf{0}$ and $\overline{\boldsymbol{\theta}} \cdot \mathbf{t}_{f,j} = 0$. Then (2.2.41d) implies that $\langle \overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} = 0$ for all $\xi \in \mathrm{H}^{1/2}(\Gamma_{fp})$. We note that $\mathbf{n}_p$ may be discontinuous on Γ_{fp}, resulting in $\overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p \in \mathrm{L}^2(\Gamma_{fp})$. However, since $\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, we obtain $\overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p = 0$, thus $\overline{\boldsymbol{\theta}} = \mathbf{0}$. Using the inf-sup condition (2.2.8), together with (2.2.41a) and (2.2.41c), we conclude that $\overline{p}_f = 0$ and $\overline{\lambda} = 0$. □

Remark 2.2.2. *As we noted in Remark 2.1.1, the time differentiated equation* (2.1.11c) *can be used to recover the non-differentiated equation* (2.1.12)*. In particular, recalling the initial data construction* (2.2.36)*, let*

$$\forall\, t \in [0, T], \quad \boldsymbol{\eta}_p(t) = \boldsymbol{\eta}_{p,0} + \int_0^t \mathbf{u}_s(s)\, ds,\ \boldsymbol{\rho}_p(t) = \boldsymbol{\rho}_{p,0} + \int_0^t \boldsymbol{\gamma}_p(s)\, ds,\ \boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \int_0^t \boldsymbol{\theta}(s)\, ds.$$

Then (2.1.12) *follows from integrating* (2.1.11c) *from* 0 *to* $t \in (0, T]$ *and using the first equation in* (2.2.36)*.*

2.3 Semi-discrete formulation

2.3.1 Semi-discrete continuous-in-time formulation

In this section we introduce the semi-discrete continuous-in-time approximation of (2.1.14). We assume for simplicity that Ω_f and Ω_p are polygonal domains. Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular [39] affine finite element partitions of Ω_f and Ω_p, respectively, which may be non-matching along the interface Γ_{fp}. Here h is the maximum element diameter. Let $(\mathbf{V}_{fh}, \mathrm{W}_{fh}) \subset (\mathbf{V}_f, \mathrm{W}_f)$ be any stable Stokes finite element pair, such as Taylor-Hood or the MINI elements [23], and let $(\mathbf{V}_{ph}, \mathrm{W}_{ph}) \subset (\mathbf{V}_p, \mathrm{W}_p)$ be any stable Darcy mixed finite element pair, such as the Raviart-Thomas (RT) or the Brezzi-Douglas-Marini (BDM) elements [23]. Let $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph}) \subset (\mathbb{X}_p, \mathbf{V}_s, \mathbb{Q}_p)$ by any stable finite element triple for mixed elasticity with weak stress symmetry, such as the spaces developed in [11, 13, 20]. We note that these spaces satisfy

$$\mathrm{div}(\mathbf{V}_{ph}) = \mathrm{W}_{ph}, \quad \mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}. \tag{2.3.1}$$

For the Lagrange multipliers, we choose non-conforming approximations:

$$\Lambda_{ph} := \mathbf{V}_{ph}\cdot\mathbf{n}_p\,|_{\Gamma_{fp}}, \quad \mathbf{\Lambda}_{sh} := \mathbb{X}_{ph}\mathbf{n}_p\,|_{\Gamma_{fp}} \tag{2.3.2}$$

with norms $\|\xi\|_{\Lambda_{ph}} := \|\xi\|_{\mathrm{L}^2(\Gamma_{fp})}, \quad \|\boldsymbol{\phi}\|_{\mathbf{\Lambda}_{sh}} := \|\boldsymbol{\phi}\|_{\mathbf{L}^2(\Gamma_{fp})}$.

The semi-discrete continuous-in-time problem is: Given $\mathbf{f}_f : [0,T] \to \mathbf{V}'_f$, $\mathbf{f}_p : [0,T] \to \mathbf{V}'_s$, $q_f : [0,T] \to \mathrm{W}'_f$, $q_p : [0,T] \to \mathrm{W}'_p$, and $(\boldsymbol{\sigma}_{ph,0}, p_{ph,0}) \in \mathbb{X}_{ph}\times\mathrm{W}_{ph}$, find $(\mathbf{u}_{fh}, p_{fh}, \boldsymbol{\sigma}_{ph}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h, \boldsymbol{\theta}_h) : [0,T] \to \mathbf{V}_{fh}\times\mathrm{W}_{fh}\times\mathbb{X}_{ph}\times\mathbf{V}_{sh}\times\mathbb{Q}_{ph}\times\mathbf{V}_{ph}\times\mathrm{W}_{ph}\times\Lambda_{ph}\times\mathbf{\Lambda}_{sh}$ such that $(\boldsymbol{\sigma}_{ph}(0), p_{ph}(0)) = (\boldsymbol{\sigma}_{ph,0}, p_{ph,0})$ and, for a.e. $t\in(0,T)$ and for all $\mathbf{v}_{fh}\in\mathbf{V}_{fh}$, $w_{fh}\in\mathrm{W}_{fh}$, $\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph}$, $\mathbf{v}_{sh}\in\mathbf{V}_{sh}$, $\boldsymbol{\chi}_{ph}\in\mathbb{Q}_{ph}$, $\mathbf{v}_{ph}\in\mathbf{V}_{ph}$, $w_{ph}\in\mathrm{W}_{ph}$, $\xi_h\in\Lambda_{ph}$, and $\boldsymbol{\phi}_h\in\mathbf{\Lambda}_{sh}$,

$$\begin{aligned}&(2\,\mu\,\mathbf{e}(\mathbf{u}_{fh}),\mathbf{e}(\mathbf{v}_{fh}))_{\Omega_f} - (\mathrm{div}(\mathbf{v}_{fh}), p_{fh})_{\Omega_f} + \langle \mathbf{v}_{fh}\cdot\mathbf{n}_f, \lambda_h\rangle_{\Gamma_{fp}}\\ &\qquad + \sum_{j=1}^{n-1}\langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{fh}-\boldsymbol{\theta}_h)\cdot\mathbf{t}_{f,j}, \mathbf{v}_{fh}\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}} = (\mathbf{f}_f, \mathbf{v}_{fh})_{\Omega_f},\end{aligned} \tag{2.3.3a}$$

$$(\mathrm{div}(\mathbf{u}_{fh}), w_{fh})_{\Omega_f} = (q_f, w_{fh})_{\Omega_f}, \tag{2.3.3b}$$

$$(\partial_t A(\boldsymbol{\sigma}_{ph}+\alpha_p p_{ph}\mathbf{I}), \boldsymbol{\tau}_{ph})_{\Omega_p} + (\mathbf{div}(\boldsymbol{\tau}_{ph}), \mathbf{u}_{sh})_{\Omega_p} + (\boldsymbol{\tau}_{ph}, \boldsymbol{\gamma}_{ph})_{\Omega_p} - \langle \boldsymbol{\tau}_{ph}\mathbf{n}_p, \boldsymbol{\theta}_h\rangle_{\Gamma_{fp}} = 0, \tag{2.3.3c}$$

$$(\mathbf{div}(\boldsymbol{\sigma}_{ph}), \mathbf{v}_{sh})_{\Omega_p} = -(\mathbf{f}_p, \mathbf{v}_{sh})_{\Omega_p}, \tag{2.3.3d}$$

$$(\boldsymbol{\sigma}_{ph}, \boldsymbol{\chi}_{ph})_{\Omega_p} = 0, \tag{2.3.3e}$$

$$(\mu\mathbf{K}^{-1}\mathbf{u}_{ph}, \mathbf{v}_{ph})_{\Omega_p} - (\mathrm{div}(\mathbf{v}_{ph}), p_{ph})_{\Omega_p} + \langle \mathbf{v}_{ph}\cdot\mathbf{n}_p, \lambda_h\rangle_{\Gamma_{fp}} = 0, \tag{2.3.3f}$$

$$(s_0\partial_t p_{ph}, w_{ph})_{\Omega_p} + \alpha_p(\partial_t A(\boldsymbol{\sigma}_{ph}+\alpha_p p_{ph}\mathbf{I}), w_{ph}\mathbf{I})_{\Omega_p} + (\mathrm{div}(\mathbf{u}_{ph}), w_{ph})_{\Omega_p} = (q_p, w_{ph})_{\Omega_p}, \tag{2.3.3g}$$

$$\langle \mathbf{u}_{fh}\cdot\mathbf{n}_f + \boldsymbol{\theta}_h\cdot\mathbf{n}_p + \mathbf{u}_{ph}\cdot\mathbf{n}_p, \xi_h\rangle_{\Gamma_{fp}} = 0, \tag{2.3.3h}$$

$$\langle \boldsymbol{\phi}_h\cdot\mathbf{n}_p, \lambda_h\rangle_{\Gamma_{fp}} - \sum_{j=1}^{n-1}\langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{fh}-\boldsymbol{\theta}_h)\cdot\mathbf{t}_{f,j}, \boldsymbol{\phi}_h\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}} + \langle \boldsymbol{\sigma}_{ph}\mathbf{n}_p, \boldsymbol{\phi}_h\rangle_{\Gamma_{fp}} = 0. \tag{2.3.3i}$$

Remark 2.3.1. *We note that, since* $\mathrm{H}^{1/2}(\Gamma_{fp})$ *is dense in* $\mathrm{L}^2(\Gamma_{fp})$*, the continuous variational equations* (2.1.11h) *and* (2.1.11i) *hold for test functions in* $\mathrm{L}^2(\Gamma_{fp})$*, assuming that the solution is smooth enough. In particular, they hold for* $\xi_h \in \Lambda_{ph}$ *and* $\boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}$*, respectively.*

The formulation (2.3.3) can be equivalently written as

$$
\begin{aligned}
& a_f(\mathbf{u}_{fh}, \mathbf{v}_{fh}) + a_p(\mathbf{u}_{ph}, \mathbf{v}_{ph}) + a_{\mathtt{BJS}}(\mathbf{u}_{fh}, \boldsymbol{\theta}_h; \mathbf{v}_{fh}, \boldsymbol{\phi}_h) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{ph}, \boldsymbol{\phi}_h) + b_p(\mathbf{v}_{ph}, p_{ph}) \\
& \quad + b_f(\mathbf{v}_{fh}, p_{fh}) + b_s(\boldsymbol{\tau}_{ph}, \mathbf{u}_{sh}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, \boldsymbol{\gamma}_{ph}) + b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\phi}_h; \lambda_h) + a_p^p(\partial_t p_{ph}, w_{ph}) \\
& \quad + a_e(\partial_t \boldsymbol{\sigma}_{ph}, \partial_t p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\theta}_h) - b_p(\mathbf{u}_{ph}, w_{ph}) = (\mathbf{f}_f, \mathbf{v}_{fh}) + (q_p, w_{ph})_{\Omega_p}, \\
& -b_f(\mathbf{u}_{fh}, w_{fh}) - b_s(\boldsymbol{\sigma}_{ph}, \mathbf{v}_{sh}) - b_{\mathrm{sk}}(\boldsymbol{\sigma}_{ph}, \boldsymbol{\chi}_{ph}) - b_\Gamma(\mathbf{u}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\theta}_h; \xi_h) \\
& \quad = (q_f, w_{fh})_{\Omega_f} + (\mathbf{f}_p, \mathbf{v}_{sh})_{\Omega_p}.
\end{aligned}
\tag{2.3.4}
$$

We group the spaces and test functions as in the continuous case:

$$
\begin{gathered}
\mathbf{Q}_h := \mathbf{V}_{fh} \times \boldsymbol{\Lambda}_{sh} \times \mathbf{V}_{ph} \times \mathbb{X}_{ph} \times \mathrm{W}_{ph}, \quad \mathbf{S}_h := \mathrm{W}_{fh} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \times \Lambda_{ph}, \\
\mathbf{p}_h := (\mathbf{u}_{fh}, \boldsymbol{\theta}_h, \mathbf{u}_{ph}, \boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathbf{Q}_h, \quad \mathbf{r}_h := (p_{fh}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph}, \lambda_h) \in \mathbf{S}_h, \\
\mathbf{q}_h := (\mathbf{v}_{fh}, \boldsymbol{\phi}_h, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, w_{ph}) \in \mathbf{Q}_h, \quad \mathbf{s}_h := (w_{fh}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}, \xi_h) \in \mathbf{S}_h,
\end{gathered}
$$

where the spaces $\mathbf{Q}_h$ and $\mathbf{S}_h$ are endowed with the norms, respectively,

$$
\begin{gathered}
\|\mathbf{q}_h\|_{\mathbf{Q}_h} = \|\mathbf{v}_{fh}\|_{\mathbf{V}_f} + \|\boldsymbol{\phi}_h\|_{\boldsymbol{\Lambda}_{sh}} + \|\mathbf{v}_{ph}\|_{\mathbf{V}_p} + \|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p} + \|w_{ph}\|_{\mathrm{W}_p}, \\
\|\mathbf{s}_h\|_{\mathbf{S}_h} = \|w_{fh}\|_{\mathrm{W}_f} + \|\mathbf{v}_{sh}\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_{ph}\|_{\mathbb{Q}_p} + \|\xi_h\|_{\Lambda_{ph}}.
\end{gathered}
$$

Hence, we can write (2.3.4) in an operator notation as a degenerate evolution problem in a mixed form:

$$
\begin{aligned}
\partial_t\, \mathcal{E}_1\, \mathbf{p}_h(t) + \mathcal{A}\, \mathbf{p}_h(t) + \mathcal{B}'\, \mathbf{r}_h(t) = \mathbf{F}(t) \quad & \text{in} \quad \mathbf{Q}_h', \\
-\mathcal{B}\, \mathbf{p}_h(t) = \mathbf{G}(t) \quad & \text{in} \quad \mathbf{S}_h'.
\end{aligned}
\tag{2.3.5}
$$

Next, we state the discrete inf-sup conditions.

Lemma 2.3.1. *There exist positive constants $\beta_{h,1}$, $\beta_{h,2}$, and $\beta_{h,3}$ independent of h such that*

$$\beta_{h,1}(\|\mathbf{v}_{sh}\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_{ph}\|_{\mathbb{Q}_p}) \leq \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph} \text{ s.t. } \boldsymbol{\tau}_{ph}\mathbf{n}_p=\mathbf{0} \text{ on } \Gamma_{fp}} \frac{b_s(\boldsymbol{\tau}_{ph},\mathbf{v}_{sh}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph},\boldsymbol{\chi}_{ph})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}},$$

$$\forall\, \mathbf{v}_{sh} \in \mathbf{V}_{sh},\, \boldsymbol{\chi} \in \mathbb{Q}_{ph}, \tag{2.3.6}$$

$$\beta_{h,2}(\|w_{fh}\|_{\mathrm{W}_f} + \|w_{ph}\|_{\mathrm{W}_p} + \|\xi_h\|_{\Lambda_{ph}})$$
$$\leq \sup_{(\mathbf{v}_{fh},\mathbf{v}_{ph})\in\mathbf{V}_{fh}\times\mathbf{V}_{ph}} \frac{b_f(\mathbf{v}_{fh},w_{fh}) + b_p(\mathbf{v}_{ph},w_{ph}) + b_\Gamma(\mathbf{v}_{fh},\mathbf{v}_{ph},\mathbf{0};\xi_h)}{\|(\mathbf{v}_{fh},\mathbf{v}_{ph})\|_{\mathbf{V}_f\times\mathbf{V}_p}},$$

$$\forall\, w_{fh} \in \mathrm{W}_{fh}, w_{ph} \in \mathrm{W}_{ph}, \xi_h \in \Lambda_{ph}, \tag{2.3.7}$$

$$\beta_{h,3}\|\boldsymbol{\phi}_h\|_{\boldsymbol{\Lambda}_{sh}} \leq \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph} \text{ s.t. } \mathbf{div}(\boldsymbol{\tau}_{ph})=\mathbf{0}} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph},\boldsymbol{\phi}_p)}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}}, \qquad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}. \tag{2.3.8}$$

Proof. Inequality (2.3.6) can be shown using the argument in [6, Theorem 4.1]. Inequality (2.3.7) is proved in [4, Theorem 5.2]. Inequality (2.3.8) can be derived as in [4, Lemma 5.1]. □

We next discuss the construction of compatible discrete initial data $(\mathbf{p}_{h,0}, \mathbf{r}_{h,0})$ based on a modification of the step-by-step procedure for the continuous initial data.

1. Let $P_h^{\boldsymbol{\Lambda}_s} : \boldsymbol{\Lambda}_s \to \boldsymbol{\Lambda}_{sh}$ be the L^2-projection operator, satisfying, for all $\boldsymbol{\phi} \in \mathbf{L}^2(\Gamma_{fp})$,

$$\langle \boldsymbol{\phi} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\phi}, \boldsymbol{\phi}_h\rangle_{\Gamma_{fp}} = 0 \qquad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}. \tag{2.3.9}$$

Define

$$\boldsymbol{\theta}_{h,0} = P_h^{\boldsymbol{\Lambda}_s}\,\boldsymbol{\theta}_0. \tag{2.3.10}$$

2. Define $(\mathbf{u}_{fh,0}, p_{fh,0}) \in \mathbf{V}_{fh} \times \mathrm{W}_{fh}$ and $(\mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{ph} \times \mathrm{W}_{ph} \times \Lambda_{ph}$ by solving a coupled Stokes-Darcy problem: for all $\mathbf{v}_{fh} \in \mathbf{V}_{fh}$, $w_{fh} \in \mathrm{W}_{fh}$, $\mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $w_{ph} \in \mathrm{W}_{ph}$, $\xi_h \in \Lambda_{ph}$,

$$a_f(\mathbf{u}_{fh,0},\mathbf{v}_{fh}) + b_f(\mathbf{v}_{fh},p_{fh,0}) + \sum_{j=1}^{n-1}\langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{fh,0} - \boldsymbol{\theta}_{h,0})\cdot\mathbf{t}_{f,j}, \mathbf{v}_{fh}\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}}$$
$$+ \langle \mathbf{v}_{fh}\cdot\mathbf{n}_f, \lambda_{h,0}\rangle_{\Gamma_{fp}}$$
$$= a_f(\mathbf{u}_{f,0},\mathbf{v}_{fh}) + b_f(\mathbf{v}_{fh},p_{f,0}) + \sum_{j=1}^{n-1}\langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{f,0} - \boldsymbol{\theta}_0)\cdot\mathbf{t}_{f,j}, \mathbf{v}_{fh}\cdot\mathbf{t}_{f,j}\rangle_{\Gamma_{fp}}$$

$$+ \langle \mathbf{v}_{fh} \cdot \mathbf{n}_f, \lambda_0 \rangle_{\Gamma_{fp}} = (\mathbf{f}_f(0), \mathbf{v}_{fh})_{\Omega_f},$$

$$- b_f(\mathbf{u}_{fh,0}, w_{fh}) = -b_f(\mathbf{u}_{f,0}, w_{fh}) = (q_f(0), w_{fh}),$$

$$a_p(\mathbf{u}_{ph,0}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, p_{ph,0}) + \langle \mathbf{v}_{ph} \cdot \mathbf{n}_p, \lambda_{h,0} \rangle_{\Gamma_{fp}}$$

$$= a_p(\mathbf{u}_{p,0}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, p_{p,0}) + \langle \mathbf{v}_{ph} \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0,$$

$$- b_p(\mathbf{u}_{ph,0}, w_{ph}) = -b_p(\mathbf{u}_{p,0}, w_{ph}) = -\mu^{-1}(\mathrm{div}(\mathbf{K}\nabla p_{p,0}), w_{ph})_{\Omega_p},$$

$$- \langle \mathbf{u}_{ph,0} \cdot \mathbf{n}_p + \mathbf{u}_{fh,0} \cdot \mathbf{n}_f + \boldsymbol{\theta}_{h,0} \cdot \mathbf{n}_p, \xi_h \rangle_{\Gamma_{fp}} = -\langle \mathbf{u}_{p,0} \cdot \mathbf{n}_p + \mathbf{u}_{f,0} \cdot \mathbf{n}_f + \boldsymbol{\theta}_0 \cdot \mathbf{n}_p, \xi_h \rangle_{\Gamma_{fp}} = 0. \tag{2.3.11}$$

This is a well-posed problem due to the inf-sup condition (2.3.8), using the theory of saddle point problems [23], see [43, 62].

3. Define $(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}, \boldsymbol{\omega}_{h,0}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \times \boldsymbol{\Lambda}_{sh}$ such that, for all $\boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $\mathbf{v}_{sh} \in \mathbf{V}_{sh}$, $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$, $\boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}$,

$$(A\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\tau}_{ph}, \boldsymbol{\eta}_{ph,0}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, \boldsymbol{\rho}_{ph,0}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\omega}_{h,0}) + (A\alpha_p p_{ph,0}\mathbf{I}, \boldsymbol{\tau}_{ph})_{\Omega_p}$$

$$= (A\boldsymbol{\sigma}_{p,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\tau}_{ph}, \boldsymbol{\eta}_{p,0}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, \boldsymbol{\rho}_{p,0}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\omega}_0) + (A\alpha_p p_{p,0}\mathbf{I}, \boldsymbol{\tau}_{ph})_{\Omega_p} = 0,$$

$$- b_s(\boldsymbol{\sigma}_{ph,0}, \mathbf{v}_{sh}) = -b_s(\boldsymbol{\sigma}_{p,0}, \mathbf{v}_{sh}) = (\mathbf{f}_p(0), \mathbf{v}_{sh})_{\Omega_p},$$

$$- b_{\mathrm{sk}}(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\chi}_{ph}) = -b_{\mathrm{sk}}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi}_{ph}) = 0,$$

$$- b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\phi}_h) - \sum_{j=1}^{n-1} \langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{fh,0} - \boldsymbol{\theta}_{h,0}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}_h \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \boldsymbol{\phi}_h \cdot \mathbf{n}_p, \lambda_{h,0} \rangle_{\Gamma_{fp}}$$

$$= -b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}_h) - \sum_{j=1}^{n-1} \langle \mu\alpha_{\mathtt{BJS}}\sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{f,0} - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}_h \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \boldsymbol{\phi}_h \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0. \tag{2.3.12}$$

It can be shown that the above problem is well-posed using the finite element theory for elasticity with weak stress symmetry [11,13] and the inf-sup condition (2.3.8) for the Lagrange multiplier $\boldsymbol{\omega}_{h,0}$.

4. Define $(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$ such that, for all $\boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $\mathbf{v}_{sh} \in \mathbf{V}_{sh}$, $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$,

$$(A\widehat{\boldsymbol{\sigma}}_{ph,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\tau}_{ph}, \mathbf{u}_{sh,0}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, \boldsymbol{\gamma}_{ph,0}) = -b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\theta}_{h,0}),$$

$$
\begin{aligned}
&- b_s(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{v}_{sh}) = 0, \\
&- b_{\mathrm{sk}}(\widehat{\boldsymbol{\sigma}}_{ph,0}, \boldsymbol{\chi}_{ph}) = 0.
\end{aligned} \tag{2.3.13}
$$

This is a well posed discrete mixed elasticity problem [11, 13].

We then define $\mathbf{p}_{h,0} = (\mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0}, \mathbf{u}_{p,0}, \boldsymbol{\sigma}_{ph,0}, p_{p,0})$ and $\mathbf{r}_{h,0} = (p_{fh,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0}, \lambda_{h,0})$. This construction guarantees that the discrete initial data is compatible in the sense of Lemma 2.2.10:

$$
\begin{aligned}
\mathcal{A}\mathbf{p}_{h,0} + \mathcal{B}'\mathbf{r}_{h,0} &= \overline{\mathbf{F}}_0 \quad \text{in} \quad \mathbf{Q}'_h, \\
-\mathcal{B}\mathbf{p}_{h,0} &= \mathbf{G}(0) \quad \text{in} \quad \mathbf{S}'_h,
\end{aligned} \tag{2.3.14}
$$

where $\overline{\mathbf{F}}_0 = (\mathbf{f}_f(0), 0, 0, \overline{g}_{\boldsymbol{\tau}_p}, \overline{g}_{w_p})^{\mathrm{t}} \in \mathbf{Q}'_2$, with suitable $\overline{g}_{\boldsymbol{\tau}_p} \in \mathbb{X}'_{p,2}$ and $\overline{g}_{w_p} \in \mathrm{W}'_{p,2}$. Furthermore, it provides compatible initial data for the non-differentiated elasticity variables $(\boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}, \boldsymbol{\omega}_{h,0})$ in the sense of the first equation in (2.2.36).

The well-posedness of the problem (2.3.5) follows from similar arguments to the proof of Theorem 2.2.11.

Theorem 2.3.2. *For each* $\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}'_f)$, $\mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}'_s)$, $q_f \in \mathrm{W}^{1,1}(0,T;\mathrm{W}'_f)$, *and* $q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}'_p)$, *and initial data* $(\mathbf{p}_{h,0}, \mathbf{r}_{h,0})$ *satisfying* (2.3.14), *there exists a unique solution of* (2.3.3) $(\mathbf{u}_{fh}, p_{fh}, \boldsymbol{\sigma}_{ph}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h, \boldsymbol{\theta}_h) : [0,T] \to \mathbf{V}_{fh} \times \mathrm{W}_{fh} \times \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \times \mathbf{V}_{ph} \times \mathrm{W}_{ph} \times \Lambda_{ph} \times \boldsymbol{\Lambda}_{sh}$ *such that* $(\boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\mathbf{u}_{fh}(0), p_{fh}(0), \boldsymbol{\sigma}_{ph}(0), \mathbf{u}_{ph}(0), p_{ph}(0), \lambda_h(0), \boldsymbol{\theta}_h(0)) = (\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\sigma}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}, \boldsymbol{\theta}_{h,0})$.

Proof. With the discrete inf-sup conditions (2.3.6)–(2.3.8) and the discrete initial data construction described in (2.3.9)–(2.3.12), the proof is similar to the proofs of Theorem 2.2.11 and Corollary 2.2.12, with two differences due to non-conforming choices of the Lagrange multiplier spaces equipped with L^2-norms. The first is in the continuity of the bilinear forms $b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\phi}_h)$, cf. (2.2.1), and $b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\phi}_h; \xi_h)$, cf. (2.2.10). In particular, using the discrete trace-inverse inequality for piecewise polynomial functions, $\|\varphi\|_{L^2(\Gamma_{fp})} \le C h^{-1/2} \|\varphi\|_{L^2(\Omega_p)}$, we have

$$
b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\phi}_h) \le C h^{-1/2} \|\boldsymbol{\tau}_{ph}\|_{\mathbb{L}^2(\Omega_p)} \|\boldsymbol{\phi}_h\|_{\mathbf{L}^2(\Gamma_{fp})}
$$

and

$$b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\phi}_h; \xi_h) \leq C(\|\mathbf{v}_{fh}\|_{\mathbf{H}^1(\Omega_f)} + h^{-1/2}\|\mathbf{v}_{ph}\|_{\mathbf{L}^2(\Omega_p)} + \|\boldsymbol{\phi}_h\|_{\mathbf{L}^2(\Gamma_{fp})})\|\xi_h\|_{\mathrm{L}^2(\Gamma_{fp})}.$$

Therefore these bilinear forms are continuous for any given mesh. Second, the operators L_λ and $R_{\boldsymbol{\theta}}$ from Lemma 2.2.6 are now defined as $L_\lambda : \Lambda_{ph} \to \Lambda'_{ph},\ (L_\lambda\, \lambda_h, \xi_h) := \langle \lambda_h, \xi_h \rangle_{\Gamma_{fp}}$ and $R_{\boldsymbol{\theta}} : \boldsymbol{\Lambda}_{sh} \to \boldsymbol{\Lambda}'_{sh},\ (R_{\boldsymbol{\theta}}\, \boldsymbol{\theta}_h, \boldsymbol{\phi}_h) := \langle \boldsymbol{\theta}_h, \boldsymbol{\phi}_h \rangle_{\Gamma_{fp}}$. The fact that L_λ and $R_{\boldsymbol{\theta}}$ are continuous and coercive follows immediately from their definitions, since $(L_\lambda\, \xi_h, \xi_h) = \|\xi\|^2_{\Lambda_{ph}}$ and $(R_{\boldsymbol{\theta}}\, \boldsymbol{\phi}_h, \boldsymbol{\phi}_h) = \|\boldsymbol{\phi}_h\|^2_{\boldsymbol{\Lambda}_{sh}}$. We note that the proof of Corollary 2.2.12 works in the discrete case due to the choice of the discrete initial data as the elliptic projection of the continuous initial data, cf. (2.3.11) and (2.3.12). □

Remark 2.3.2. *As in the continuous case, we can recover the non-differentiated elasticity variables with*

$$\forall\, t \in [0,T], \quad \boldsymbol{\eta}_{ph}(t) = \boldsymbol{\eta}_{ph,0} + \int_0^t \mathbf{u}_{sh}(s)\, ds,$$

$$\boldsymbol{\rho}_{ph}(t) = \boldsymbol{\rho}_{ph,0} + \int_0^t \boldsymbol{\gamma}_{ph}(s)\, ds, \quad \boldsymbol{\omega}_h(t) = \boldsymbol{\omega}_{h,0} + \int_0^t \boldsymbol{\theta}_h(s)\, ds.$$

Then (2.1.12) *holds discretely, which follows from integrating the third equation in* (2.3.3) *from* 0 *to* $t \in (0,T]$ *and using the discrete version of the first equation in* (2.2.36).

2.3.2 Stability analysis

In this section we establish a stability bound for the solution of semi-discrete continuous-in-time formulation (2.3.5). We emphasize that the stability constant is independent of s_0 and $a_{\min}$, indicating robustness of the method in the limits of small storativity and almost incompressible media, which are known to cause locking in numerical methods for the Biot system [83]. Furthermore, since we do not utilize Gronwall's inequality, we obtain long-time stability for our method.

Theorem 2.3.3. *Assuming sufficient regularity of the data, for the solution to the semi-discrete problem* (2.3.3)*, there exists a constant* C *independent of* h*,* s_0 *and* $a_{\min}$ *such that*

$$
\begin{aligned}
&\|\mathbf{u}_{fh}\|_{\mathrm{L}^\infty(0,T;\mathbf{V}_f)} + \|\mathbf{u}_{fh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_f)} + |\mathbf{u}_{fh} - \boldsymbol{\theta}_h|_{\mathrm{L}^\infty(0,T;a_{\mathtt{BJS}})} + |\mathbf{u}_{fh} - \boldsymbol{\theta}_h|_{\mathrm{L}^2(0,T;a_{\mathtt{BJS}})} \\
&\quad + \|p_{fh}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_f)} + \|p_{fh}\|_{\mathrm{L}^2(0,T;\mathrm{W}_f)} + \|A^{1/2}\boldsymbol{\sigma}_{ph}\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} \\
&\quad + \|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{u}_{sh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_s)} \\
&\quad + \|\boldsymbol{\gamma}_{ph}\|_{\mathrm{L}^2(0,T;\mathbb{Q}_p)} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} + \|p_{ph}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\quad + \sqrt{s_0}\|\partial_t p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} + \|\lambda_h\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} + \|\lambda_h\|_{\mathrm{L}^2(0,T;\Lambda_{ph})} + \|\boldsymbol{\theta}_h\|_{\mathrm{L}^2(0,T;\boldsymbol{\Lambda}_{sh})} \\
&\leq C\Big(\|\mathbf{f}_f\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\Omega_f))} + \|\mathbf{f}_p\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\Omega_p))} + \|q_f\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_f))} + \|q_p\|_{\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_p))} \\
&\qquad + \|p_{p,0}\|_{\mathrm{H}^1(\Omega_p)} + \|\mathrm{div}(\mathbf{K}\nabla p_{p,0})\|_{\mathrm{L}^2(\Omega_p)}\Big). \qquad (2.3.15)
\end{aligned}
$$

Proof. By taking $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \xi_h, \boldsymbol{\phi}_h) = (\mathbf{u}_{fh}, p_{fh}, \boldsymbol{\sigma}_{ph}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph}, \mathbf{u}_{ph}, p_{ph}, \lambda_h, \boldsymbol{\theta}_h)$ in (2.3.3) and adding up all the equations, we get

$$
\begin{aligned}
&a_f(\mathbf{u}_{fh}, \mathbf{u}_{fh}) + a_{\mathtt{BJS}}(\mathbf{u}_{fh}, \boldsymbol{\theta}_h; \mathbf{u}_{fh}, \boldsymbol{\theta}_h) + a_e(\partial_t\boldsymbol{\sigma}_{ph}, \partial_t p_{ph}; \boldsymbol{\sigma}_{ph}, p_{ph}) + a_p(\mathbf{u}_{ph}, \mathbf{u}_{ph}) \\
&\quad + a_p^p(\partial_t p_{ph}, p_{ph}) = (\mathbf{f}_f, \mathbf{u}_{fh})_{\Omega_f} + (q_f, p_{fh})_{\Omega_f} + (\mathbf{f}_p, \mathbf{u}_{sh})_{\Omega_p} + (q_p, p_{ph})_{\Omega_p}. \qquad (2.3.16)
\end{aligned}
$$

Using the algebraic identity $\int_S v\,\partial_t v = \frac{1}{2}\partial_t\|v\|^2_{\mathrm{L}^2(S)}$, and employing the coercivity properties of a_f and a_p, and the semi-positive definiteness of $a_{\mathtt{BJS}}$, cf. (2.2.5), we obtain

$$
\begin{aligned}
&2\mu C_K^2\|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \mu\alpha_{\mathtt{BJS}}k_{\max}^{-1/2}|\mathbf{u}_{fh} - \boldsymbol{\theta}_h|^2_{a_{\mathtt{BJS}}} + \frac{1}{2}\partial_t\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \mu k_{\max}^{-1}\|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + \frac{1}{2}s_0\partial_t\|p_{ph}\|^2_{\mathrm{W}_p} \leq (\mathbf{f}_f, \mathbf{u}_{fh})_{\Omega_f} + (q_f, p_{fh})_{\Omega_f} + (\mathbf{f}_p, \mathbf{u}_{sh})_{\Omega_p} + (q_p, p_{ph})_{\Omega_p}.
\end{aligned}
$$

Integrating from 0 to any $t \in (0, T]$ and applying the Cauchy-Schwarz and Young's inequalities, we get

$$
\begin{aligned}
&\int_0^t \big(2\mu C_K^2\|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + \mu\alpha_{\mathtt{BJS}}k_{\max}^{-1/2}|\mathbf{u}_{fh} - \boldsymbol{\theta}_h|^2_{a_{\mathtt{BJS}}} + \mu k_{\max}^{-1}\|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)}\big)\,ds \\
&+ \frac{1}{2}\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} - \frac{1}{2}\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&+ \frac{1}{2}s_0\|p_{ph}(t)\|^2_{\mathrm{W}_p} - \frac{1}{2}s_0\|p_{ph}(0)\|^2_{\mathrm{W}_p}
\end{aligned}
$$

$$\leq \frac{\epsilon}{2}\int_0^t (\|\mathbf{u}_{fh}\|^2_{\mathbf{L}^2(\Omega_f)} + \|p_{fh}\|^2_{\mathrm{W}_f} + \|\mathbf{u}_{sh}\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_{ph}\|^2_{\mathrm{W}_p})\, ds$$
$$+ \frac{1}{2\epsilon}\int_0^t (\|\mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|q_p\|^2_{\mathrm{L}^2(\Omega_p)})\, ds. \tag{2.3.17}$$

From the discrete inf-sup conditions (2.3.6)–(2.3.8) and (2.3.3a), (2.3.3c), and (2.3.3f), we have

$$\begin{aligned}
&\|p_{fh}\|_{\mathrm{W}_f} + \|p_{ph}\|_{\mathrm{W}_p} + \|\lambda_h\|_{\Lambda_{ph}} \\
&\quad \leq C \sup_{(\mathbf{v}_{fh},\mathbf{v}_{ph})\in \mathbf{V}_{fh}\times\mathbf{V}_{ph}} \frac{b_f(\mathbf{v}_{fh},p_{fh}) + b_p(\mathbf{v}_{ph},p_{ph}) + b_\Gamma(\mathbf{v}_{fh},\mathbf{v}_{ph},\mathbf{0};\lambda_h)}{\|(\mathbf{v}_{fh},\mathbf{v}_{ph})\|_{\mathbf{V}_f\times\mathbf{V}_p}} \\
&\quad = C \sup_{(\mathbf{v}_{fh},\mathbf{v}_{ph})\in \mathbf{V}_{fh}\times\mathbf{V}_{ph}} \frac{-a_f(\mathbf{u}_{fh},\mathbf{v}_{fh}) - a_{\mathtt{BJS}}(\mathbf{u}_{fh},\boldsymbol{\theta}_h;\mathbf{v}_{fh},\mathbf{0}) + (\mathbf{f}_f,\mathbf{v}_{fh})_{\Omega_f} - a_p(\mathbf{u}_{ph},\mathbf{v}_{ph})}{\|\mathbf{v}_{fh}\|_{\mathbf{V}_f} + \|\mathbf{v}_{ph}\|_{\mathbf{V}_p}} \\
&\quad \leq C(\|\mathbf{u}_{fh}\|_{\mathbf{V}_f} + |\mathbf{u}_{fh} - \boldsymbol{\theta}_h|_{a_{\mathtt{BJS}}} + \|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\Omega_p)}),
\end{aligned} \tag{2.3.18}$$

$$\begin{aligned}
\|\mathbf{u}_{sh}\|_{\mathbf{V}_s} + \|\boldsymbol{\gamma}_{ph}\|_{\mathbb{Q}_p} &\leq C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph}\ \text{s.t.}\ \boldsymbol{\tau}_{ph}\mathbf{n}_p=\mathbf{0}\ \text{on}\ \Gamma_{fp}} \frac{b_s(\boldsymbol{\tau}_{ph},\mathbf{u}_{sh}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph},\boldsymbol{\gamma}_{ph})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&= C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph}\ \text{s.t.}\ \boldsymbol{\tau}_{ph}\mathbf{n}_p=\mathbf{0}\ \text{on}\ \Gamma_{fp}} \frac{-(A\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I}),\boldsymbol{\tau}_{ph}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph},\boldsymbol{\theta}_h)}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&\leq C\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)},
\end{aligned} \tag{2.3.19}$$

$$\begin{aligned}
\|\boldsymbol{\theta}_h\|_{\boldsymbol{\Lambda}_{sh}} &\leq C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph}\ \text{s.t.}\ \mathbf{div}(\boldsymbol{\tau}_{ph})=\mathbf{0}} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph},\boldsymbol{\theta}_h)}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&= C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph}\ \text{s.t.}\ \mathbf{div}(\boldsymbol{\tau}_{ph})=\mathbf{0}} \frac{-(A\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I}),\boldsymbol{\tau}_{ph}) - b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph},\boldsymbol{\gamma}_{ph}) - b_s(\boldsymbol{\tau}_{ph},\mathbf{u}_{sh})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&\leq C(\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|\boldsymbol{\gamma}_{ph}\|_{\mathbb{Q}_p}).
\end{aligned} \tag{2.3.20}$$

Combining (2.3.17) with (2.3.18)–(2.3.20), and choosing ϵ small enough, results in

$$\begin{aligned}
&\int_0^t \Big(\|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + |\mathbf{u}_{fh} - \boldsymbol{\theta}_h|^2_{a_{\mathtt{BJS}}} + \|p_{fh}\|^2_{\mathrm{W}_f} + \|\mathbf{u}_{sh}\|^2_{\mathbf{V}_s} + \|\boldsymbol{\gamma}_{ph}\|^2_{\mathbb{Q}_p} + \|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_{ph}\|^2_{\mathrm{W}_p} \\
&\qquad + \|\lambda_h\|^2_{\Lambda_{ph}} + \|\boldsymbol{\theta}_h\|^2_{\boldsymbol{\Lambda}_{sh}}\Big) ds + \|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|p_{ph}(t)\|^2_{\mathrm{W}_p} \\
&\quad \leq C\Big(\int_0^t (\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)}
\end{aligned}$$

$$+ \|q_p\|^2_{\mathrm{L}^2(\Omega_p)}\Big) ds + \|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|p_{ph}(0)\|^2_{\mathrm{L}^2(\Omega_p)}\Bigg). \tag{2.3.21}$$

To get a bound for $\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))}$, we differentiate in time (2.3.3a), (2.3.3d), (2.3.3e), (2.3.3f), and (2.3.3i), take $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \xi_h, \boldsymbol{\phi}_h) = (\mathbf{u}_{fh}, \partial_t p_{fh}, \partial_t\boldsymbol{\sigma}_{ph}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph}, \mathbf{u}_{ph}, \partial_t p_{ph}, \partial_t\lambda_h, \boldsymbol{\theta}_h)$ in (2.3.3), and add all equations, to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t a_f(\mathbf{u}_{fh}, \mathbf{u}_{fh}) + \frac{1}{2}\partial_t a_{\mathtt{BJS}}(\mathbf{u}_{fh}, \boldsymbol{\theta}_h; \mathbf{u}_{fh}, \boldsymbol{\theta}_h) + \|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&+ \frac{1}{2}\partial_t a_p(\mathbf{u}_{ph}, \mathbf{u}_{ph}) + s_0\|\partial_t p_{ph}\|^2_{\mathrm{W}_p} \\
&\quad = (\partial_t\mathbf{f}_f, \mathbf{u}_{fh})_{\Omega_f} + (q_f, \partial_t p_{fh})_{\Omega_f} + (\partial_t\mathbf{f}_p, \mathbf{u}_{sh})_{\Omega_p} + (q_p, \partial_t p_{ph})_{\Omega_p}.
\end{aligned} \tag{2.3.22}$$

We next integrate (2.3.22) in time from 0 to an arbitrary $t \in (0, T]$ and use integration by parts in time for the last two terms:

$$\begin{aligned}
\int_0^t (q_f, \partial_t p_{fh})_{\Omega_f} ds + \int_0^t (q_p, \partial_t p_{ph})_{\Omega_p} ds = (q_f, p_{fh})_{\Omega_f}\Big|_0^t - \int_0^t (\partial_t q_f, p_{fh})_{\Omega_f}\, ds \\
+ (q_p, p_{ph})_{\Omega_p}\Big|_0^t - \int_0^t (\partial_t q_p, p_{ph})_{\Omega_p}\, ds.
\end{aligned}$$

Making use of the continuity of a_f, a_p and $a_{\mathtt{BJS}}$, cf. (2.2.1), the coercivity of a_f and a_p the and semi-positive definiteness of $a_{\mathtt{BJS}}$, cf. (2.2.5), and the Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}
&\mu C_K^2\|\mathbf{u}_{fh}(t)\|^2_{\mathbf{V}_f} + \frac{1}{2}\mu\alpha_{\mathtt{BJS}}k_{\max}^{-1/2}|(\mathbf{u}_{fh} - \boldsymbol{\theta}_h)(t)|^2_{a_{\mathtt{BJS}}} + \frac{1}{2}\mu k_{\max}^{-1}\|\mathbf{u}_{ph}(t)\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\qquad + \int_0^t (\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t p_{ph}\|^2_{\mathrm{W}_p})\, ds \\
&\leq \frac{\epsilon}{2}\Big(\int_0^t (\|\mathbf{u}_{fh}\|^2_{\mathbf{L}^2(\Omega_f)} + \|p_{fh}\|^2_{\mathrm{W}_f} + \|\mathbf{u}_{sh}\|^2_{\mathbf{V}_s} + \|p_{ph}\|^2_{\mathrm{W}_p})\, ds + \|p_{fh}(t)\|^2_{\mathrm{W}_f} + \|p_{ph}(t)\|^2_{\mathrm{W}_p}\Big) \\
&\quad + \frac{1}{2\epsilon}\Big(\int_0^t (\|\partial_t\mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|\partial_t q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\partial_t\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t q_p\|^2_{\mathrm{L}^2(\Omega_p)})\, ds + \|q_f(t)\|^2_{\mathrm{L}^2(\Omega_f)} \\
&\quad + \|q_p(t)\|^2_{\mathrm{L}^2(\Omega_p)}\Big) + \mu\|\mathbf{u}_{fh}(0)\|^2_{\mathbf{H}^1(\Omega_f)} + \frac{1}{2}\mu\alpha_{\mathtt{BJS}}k_{\min}^{-1/2}|(\mathbf{u}_{fh} - \boldsymbol{\theta}_h)(0)|^2_{a_{\mathtt{BJS}}} + \frac{1}{2}\|p_{fh}(0)\|^2_{\mathrm{W}_f} \\
&\quad + \frac{1}{2}\mu k_{\min}^{-1}\|\mathbf{u}_{ph}(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \frac{1}{2}\|p_{ph}(0)\|^2_{\mathrm{W}_p} + \frac{1}{2}\|q_f(0)\|^2_{\mathrm{L}^2(\Omega_f)} + \frac{1}{2}\|q_p(0)\|^2_{\mathrm{L}^2(\Omega_p)}.
\end{aligned} \tag{2.3.23}$$

We note that the terms on the first four terms in the first line on the right hand side are controlled in (2.3.21), while the terms $\|p_{fh}(t)\|_{\mathrm{W}_f}$ and $\|p_{ph}(t)\|_{\mathrm{W}_p}$ are controlled in the inf-sup bound (2.3.18). Thus, combining (2.3.18), (2.3.21) and (2.3.23), and taking ϵ small enough, we obtain

$$\int_0^t \Big(\|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f} + |\mathbf{u}_{fh} - \boldsymbol{\theta}_h|^2_{a_{\mathtt{BJS}}} + \|p_{fh}\|^2_{\mathrm{W}_f} + \|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{u}_{sh}\|^2_{\mathbf{V}_s}$$
$$+ \|\boldsymbol{\gamma}_{ph}\|^2_{\mathbb{Q}_p} + \|\mathbf{u}_{ph}\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_{ph}\|^2_{\mathrm{W}_p} + s_0\|\partial_t p_{ph}\|^2_{\mathrm{W}_p} + \|\lambda_h\|^2_{\Lambda_{ph}} + \|\boldsymbol{\theta}_h\|^2_{\boldsymbol{\Lambda}_{sh}} \Big) ds$$
$$+ \|\mathbf{u}_{fh}(t)\|^2_{\mathbf{V}_f} + |(\mathbf{u}_{fh} - \boldsymbol{\theta}_h)(t)|^2_{a_{\mathtt{BJS}}} + \|p_{fh}(t)\|^2_{\mathrm{W}_f} + \|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)}$$
$$+ \|\mathbf{u}_{ph}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_{ph}(t)\|^2_{\mathrm{W}_p} + \|\lambda_h(t)\|^2_{\Lambda_{ph}}$$
$$\leq C\bigg(\int_0^t \big(\|\mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|q_p\|^2_{\mathrm{L}^2(\Omega_p)}\big) ds + \|\mathbf{f}_f(t)\|^2_{\mathrm{L}^2(\Omega_f)}$$
$$+ \int_0^t (\|\partial_t \mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|\partial_t \mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|\partial_t q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\partial_t q_p\|^2_{\mathrm{L}^2(\Omega_p)})\, ds$$
$$+ \|q_f(t)\|^2_{\mathrm{L}^2(\Omega_f)} + \|q_p(t)\|^2_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{u}_{fh}(0)\|^2_{\mathbf{V}_f} + |(\mathbf{u}_{fh} - \boldsymbol{\theta}_h)(0)|^2_{a_{\mathtt{BJS}}} + \|p_{fh}(0)\|^2_{\mathrm{W}_f}$$
$$+ \|A^{1/2}\boldsymbol{\sigma}_{ph}(0)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{u}_{ph}(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_{ph}(0)\|^2_{\mathrm{W}_p} + \|q_f(0)\|^2_{\mathrm{L}^2(\Omega_f)} + \|q_p(0)\|^2_{\mathrm{L}^2(\Omega_p)} \bigg). \tag{2.3.24}$$

We remark that in the above bound we have obtained control on $\|p_{ph}(t)\|_{\mathrm{L}^2(\Omega_p)}$ independent of s_0. To bound the initial data terms above, we recall that $(\mathbf{u}_{fh}(0), p_{fh}(0), \boldsymbol{\sigma}_{ph}(0), \mathbf{u}_{ph}(0), p_{ph}(0), \lambda_h(0), \boldsymbol{\theta}_h(0)) = (\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\sigma}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}, \boldsymbol{\theta}_{h,0})$ and the construction of the discrete initial data (2.3.11)–(2.3.12). Combining the two systems and using the steady-state version of the arguments presented in (2.3.16)–(2.3.18), we obtain

$$\|\mathbf{u}_{fh}(0)\|_{\mathbf{V}_f} + \|p_{fh}(0)\|_{\mathrm{W}_f} + \|A^{1/2}\boldsymbol{\sigma}_{ph}(0)\|_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{u}_{ph}(0)\|_{\mathbf{L}^2(\Omega_p)}$$
$$+ \|p_{ph}(0)\|_{\mathrm{W}_p} + |(\mathbf{u}_{fh} - \boldsymbol{\theta}_h)(0)|_{a_{\mathtt{BJS}}}$$
$$\leq C(\|\mathrm{div}(\mathbf{K}\nabla p_{p,0})\|_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{f}_f(0)\|_{\mathbf{L}^2(\Omega_f)} + \|q_f(0)\|_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{f}_p(0)\|_{\mathbf{L}^2(\Omega_p)}. \tag{2.3.25}$$

We complete the argument by deriving bounds for $\|\mathrm{div}(\mathbf{u}_{ph})\|_{\mathrm{L}^2(\Omega_p)}$ and $\|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathbf{L}^2(\Omega_p)}$. Due to (2.3.1), we can choose $w_{ph} = \mathrm{div}(\mathbf{u}_{ph})$ in (2.3.3g), obtaining

$$\|\mathrm{div}(\mathbf{u}_{ph})\|^2_{\mathrm{L}^2(\Omega_p)}$$

$$= -(A\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I}), \operatorname{div}(\mathbf{u}_{ph}))_{\Omega_p} - (s_0\partial_t p_{ph}, \operatorname{div}(\mathbf{u}_{ph}))_{\Omega_p} + (q_p, \operatorname{div}(\mathbf{u}_{ph}))_{\Omega_p}$$

$$\leq (a_{\max}^{1/2}\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t p_{ph}\|_{\mathrm{L}^2(\Omega_p)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)})\|\operatorname{div}(\mathbf{u}_{ph})\|_{\mathrm{L}^2(\Omega_p)},$$

therefore

$$\int_0^t \|\operatorname{div}(\mathbf{u}_{ph})\|^2_{\mathrm{L}^2(\Omega_p)} ds \leq C\int_0^t (\|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t p_{ph}\|^2_{\mathrm{L}^2(\Omega_p)} + \|q_p\|^2_{\mathrm{L}^2(\Omega_p)})ds. \tag{2.3.26}$$

Similarly, the choice of $\mathbf{v}_{sh} = \mathbf{div}(\boldsymbol{\sigma}_{ph})$ in (2.3.3d) gives

$$\|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathbf{L}^2(\Omega_p)} \leq \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} \quad \text{and} \quad \int_0^t \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|^2_{\mathbf{L}^2(\Omega_p)} ds \leq \int_0^t \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)} ds. \tag{2.3.27}$$

Combining (2.3.24)–(2.3.27), we conclude (2.3.15), where we also use

$$\|A^{1/2}\boldsymbol{\sigma}_{ph}(t)\|_{\mathbb{L}^2(\Omega_p)} \leq C(\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})(t)\|_{\mathbb{L}^2(\Omega_p)} + \|p_{ph}(t)\|_{\mathrm{L}^2(\Omega_p)}).$$

□

2.3.3 Error analysis

In this section we derive a priori error estimate for the semi-discrete formulation (2.3.3). We assume that the finite element spaces contain polynomials of degrees $s_{\mathbf{u}_f}$ and s_{p_f} for $\mathbf{V}_{fh}$ and W_{fh}, $s_{\mathbf{u}_p}$ and s_{p_p} for $\mathbf{V}_{ph}$ and W_{ph}, $s_{\boldsymbol{\sigma}_p}$, $s_{\mathbf{u}_s}$, and $s_{\boldsymbol{\gamma}_p}$ for $\mathbb{X}_{ph}$, $\mathbf{V}_{sh}$, and $\mathbb{Q}_{ph}$, $s_{\boldsymbol{\theta}}$ and s_λ for $\boldsymbol{\Lambda}_{sh}$ and Λ_{ph}. Next, we define interpolation operators into the finite elements spaces that will be used in the error analysis.

We recall that $P_h^{\boldsymbol{\Lambda}_s} : \boldsymbol{\Lambda}_s \to \boldsymbol{\Lambda}_{sh}$ is the L^2-projection operator, cf. (2.3.9), and define $P_h^{\Lambda_p} : \Lambda_p \to \Lambda_{ph}$ as the L^2-projection operator, satisfying, for any $\xi \in \mathrm{L}^2(\Gamma_{fp})$, $\langle \xi - P_h^{\Lambda_p}\xi, \xi_h\rangle_{\Gamma_{fp}} = 0$ $\forall\, \xi_h \in \Lambda_{ph}$. Since the discrete Lagrange multiplier spaces are chosen as $\boldsymbol{\Lambda}_{sh} = \mathbb{X}_{ph}\,\mathbf{n}_p|_{\Gamma_{fp}}$ and $\Lambda_{ph} = \mathbf{X}_{ph} \cdot \mathbf{n}_p|_{\Gamma_{fp}}$, respectively, we have

$$\langle \boldsymbol{\phi} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\phi}, \boldsymbol{\tau}_{ph}\,\mathbf{n}_p\rangle_{\Gamma_{fp}} = 0, \quad \forall\, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}, \qquad \langle \xi - P_h^{\Lambda_p}\xi, \mathbf{v}_{ph}\cdot\mathbf{n}_p\rangle_{\Gamma_{fp}} = 0, \quad \forall\, \mathbf{v}_{ph} \in \mathbf{V}_{ph}. \tag{2.3.28}$$

These operators have approximation properties [39],

$$\|\boldsymbol{\phi} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\phi}\|_{\mathbf{L}^2(\Gamma_{fp})} \leq C h^{s_{\boldsymbol{\theta}}+1}\|\boldsymbol{\phi}\|_{\mathbf{H}^{s_{\boldsymbol{\theta}}+1}(\Gamma_{fp})}, \quad \|\xi - P_h^{\Lambda_p}\xi\|_{\mathrm{L}^2(\Gamma_{fp})} \leq C h^{s_\lambda+1}\|\xi\|_{\mathrm{H}^{s_\lambda+1}(\Gamma_{fp})}. \tag{2.3.29}$$

Similarly, we introduce $P_h^{\mathrm{W}_f} : \mathrm{W}_f \to \mathrm{W}_{fh}$, $P_h^{\mathrm{W}_p} : \mathrm{W}_p \to \mathrm{W}_{ph}$, $P_h^{\mathbf{V}_s} : \mathbf{V}_s \to \mathbf{V}_{sh}$ and $P_h^{\mathbb{Q}_p} : \mathbb{Q}_p \to \mathbb{Q}_{ph}$ as L^2-projection operators, satisfying

$$\begin{aligned} &(w_f - P_h^{\mathrm{W}_f} w_f, w_{fh})_{\Omega_f} = 0, \quad \forall\, w_{fh} \in \mathrm{W}_{fh}, && (w_p - P_h^{\mathrm{W}_p} w_p, w_{ph})_{\Omega_p} = 0, \quad \forall\, w_{ph} \in \mathrm{W}_{ph}, \\ &(\mathbf{v}_s - P_h^{\mathbf{V}_s}\mathbf{v}_s, \mathbf{v}_{sh})_{\Omega_p} = 0, \quad \forall\, \mathbf{v}_{sh} \in \mathbf{V}_{sh}, && (\boldsymbol{\chi}_p - P_h^{\mathbb{Q}_p}\boldsymbol{\chi}_p, \boldsymbol{\chi}_{ph})_{\Omega_p} = 0, \quad \forall\, \boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}, \end{aligned} \tag{2.3.30}$$

with approximation properties [39],

$$\begin{aligned} \|w_f - P_h^{\mathrm{W}_f} w_f\|_{\mathrm{L}^2(\Omega_f)} &\leq C h^{s_{p_f}+1}\|w_f\|_{\mathrm{H}^{s_{p_f}+1}(\Omega_f)}, \\ \|w_p - P_h^{\mathrm{W}_p} w_p\|_{\mathrm{L}^2(\Omega_p)} &\leq C h^{s_{p_p}+1}\|w_p\|_{\mathrm{H}^{s_{p_p}+1}(\Omega_p)}, \\ \|\mathbf{v}_s - P_h^{\mathbf{V}_s}\mathbf{v}_s\|_{\mathbf{L}^2(\Omega_p)} &\leq C h^{s_{\mathbf{u}_s}+1}\|\mathbf{v}_s\|_{\mathbf{H}^{s_{\mathbf{u}_s}+1}(\Omega_p)}, \\ \|\boldsymbol{\chi}_p - P_h^{\mathbb{Q}_p}\boldsymbol{\chi}_p\|_{\mathbb{L}^2(\Omega_p)} &\leq C h^{s_{\boldsymbol{\gamma}_p}+1}\|\boldsymbol{\chi}_p\|_{\mathbb{H}^{s_{\boldsymbol{\gamma}_p}+1}(\Omega_p)}. \end{aligned} \tag{2.3.31}$$

Next, we consider a Stokes-like projection operator $I_h^{\mathbf{V}_f} : \mathbf{V}_f \to \mathbf{V}_{fh}$, defined by solving the problem: find $I_h^{\mathbf{V}_f}\mathbf{v}_f$ and $\widetilde{p}_{fh} \in \mathrm{W}_{fh}$ such that

$$\begin{aligned} a_f(I_h^{\mathbf{V}_f}\mathbf{v}_f, \mathbf{v}_{fh}) - b_f(\mathbf{v}_{fh}, \widetilde{p}_{fh}) &= a_f(\mathbf{v}_f, \mathbf{v}_{fh}), && \forall\, \mathbf{v}_{fh} \in \mathbf{V}_{fh}, \\ b_f(I_h^{\mathbf{V}_f}\mathbf{v}_f, w_{fh}) &= b_f(\mathbf{v}_f, w_{fh}), && \forall\, w_{fh} \in \mathrm{W}_{fh}. \end{aligned} \tag{2.3.32}$$

The operator $I_h^{\mathbf{V}_f}$ satisfies the approximation property [45]:

$$\|\mathbf{v}_f - I_h^{\mathbf{V}_f}\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)} \leq C h^{s_{\mathbf{u}_f}}\|\mathbf{v}_f\|_{\mathbf{H}^{s_{\mathbf{u}_f}+1}(\Omega_f)}. \tag{2.3.33}$$

Let $I_h^{\mathbf{V}_p}$ be the mixed finite element interpolant onto $\mathbf{V}_{ph}$, which satisfies for all $\mathbf{v}_p \in \mathbf{V}_p \cap \mathbf{H}^1(\Omega_p)$,

$$\begin{aligned} &(\mathrm{div}(I_h^{\mathbf{V}_p}\mathbf{v}_p), w_{ph})_{\Omega_p} = (\mathrm{div}(\mathbf{v}_p), w_{ph})_{\Omega_p}, && \forall\, w_{ph} \in \mathrm{W}_{ph}, \\ &\langle I_h^{\mathbf{V}_p}\mathbf{v}_p \cdot \mathbf{n}_p, \mathbf{v}_{ph} \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}} = \langle \mathbf{v}_p \cdot \mathbf{n}_p, \mathbf{v}_{ph} \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}}, && \forall\, \mathbf{v}_{ph} \in \mathbf{V}_{ph}, \end{aligned} \tag{2.3.34}$$

and

$$\begin{aligned} \|\mathbf{v}_p - I_h^{\mathbf{V}_p}\mathbf{v}_p\|_{\mathbf{L}^2(\Omega_p)} &\le Ch^{s_{\mathbf{u}_p}+1}\|\mathbf{v}_p\|_{\mathbf{H}^{s_{\mathbf{u}_p}+1}(\Omega_p)}, \\ \|\operatorname{div}(\mathbf{v}_p - I_h^{\mathbf{V}_p}\mathbf{v}_p)\|_{\mathrm{L}^2(\Omega_p)} &\le Ch^{s_{\mathbf{u}_p}+1}\|\operatorname{div}(\mathbf{v}_p)\|_{\mathrm{H}^{s_{\mathbf{u}_p}+1}(\Omega_p)}. \end{aligned} \tag{2.3.35}$$

For $\mathbb{X}_{ph}$, we consider the weakly symmetric elliptic projection introduced in [14] and extended in [59] to the case of Neumann boundary condition: given $\boldsymbol{\sigma}_p \in \mathbb{X}_p \cap \mathbb{H}^1(\Omega_p)$, find $(\widetilde{\boldsymbol{\sigma}}_{ph}, \widetilde{\boldsymbol{\eta}}_{ph}, \widetilde{\boldsymbol{\rho}}_{ph}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$ such that

$$\begin{aligned} &(\widetilde{\boldsymbol{\sigma}}_{ph}, \boldsymbol{\tau}_{ph}) + (\widetilde{\boldsymbol{\eta}}_{ph}, \mathbf{div}(\boldsymbol{\tau}_{ph})) + (\widetilde{\boldsymbol{\rho}}_{ph}, \boldsymbol{\tau}_{ph}) = (\boldsymbol{\sigma}_p, \boldsymbol{\tau}_{ph}), && \forall\, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}^0, \\ &(\mathbf{div}(\widetilde{\boldsymbol{\sigma}}_{ph}), \mathbf{v}_{sh}) = (\mathbf{div}(\boldsymbol{\sigma}_p), \mathbf{v}_{sh}), && \forall\, \mathbf{v}_{sh} \in \mathbf{V}_{sh}, \\ &(\widetilde{\boldsymbol{\sigma}}_{ph}, \boldsymbol{\chi}_{ph}) = (\boldsymbol{\sigma}_p, \boldsymbol{\chi}_{ph}), && \forall\, \boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}, \\ &\langle \widetilde{\boldsymbol{\sigma}}_{ph}\mathbf{n}_p, \boldsymbol{\tau}_{ph}\mathbf{n}_p \rangle_{\Gamma_{fp}} = \langle \boldsymbol{\sigma}_p\mathbf{n}_p, \boldsymbol{\tau}_{ph}\mathbf{n}_p \rangle_{\Gamma_{fp}}, && \forall\, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}^{\Gamma_{fp}}, \end{aligned} \tag{2.3.36}$$

where $\mathbb{X}_{ph}^0 = \{\boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph} : \boldsymbol{\tau}_{ph}\mathbf{n}_p = \mathbf{0} \text{ on } \Gamma_{fp}\}$, and $\mathbb{X}_{ph}^{\Gamma_{fp}}$ is the complement of $\mathbb{X}_{ph}^0$ in $\mathbb{X}_{ph}$, which spans the degrees of freedoms on Γ_{fp}. We define $I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p := \widetilde{\boldsymbol{\sigma}}_{ph}$, which satisfies

$$\begin{aligned} \|\boldsymbol{\sigma}_p - I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p\|_{\mathbb{L}^2(\Omega_p)} &\le h^{s_{\boldsymbol{\sigma}_p}+1}\|\boldsymbol{\sigma}_p\|_{\mathbb{H}^{s_{\boldsymbol{\sigma}_p}+1}(\Omega_p)}, \\ \|\mathbf{div}(\boldsymbol{\sigma}_p - I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p)\|_{\mathbf{L}^2(\Omega_p)} &\le Ch^{s_{\boldsymbol{\sigma}_p}+1}\|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathbf{H}^{s_{\boldsymbol{\sigma}_p}+1}(\Omega_p)}. \end{aligned} \tag{2.3.37}$$

We now establish the main result of this section.

Theorem 2.3.4. *Assuming sufficient regularity of the solution to the continuous problem* (2.1.11)*, for the solution of the semi-discrete problem* (2.3.3)*, there exists a constant C independent of h, s_0, and $a_{\min}$ such that*

$$\begin{aligned} &\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{\mathrm{L}^\infty(0,T;\mathbf{V}_f)} + \|\mathbf{u}_f - \mathbf{u}_{fh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_f)} + |(\mathbf{u}_f - \boldsymbol{\theta}) - (\mathbf{u}_{fh} - \boldsymbol{\theta}_h)|_{\mathrm{L}^\infty(0,T;a_{\mathtt{BJS}})} \\ &\quad + |(\mathbf{u}_f - \boldsymbol{\theta}) - (\mathbf{u}_{fh} - \boldsymbol{\theta}_h)|_{\mathrm{L}^2(0,T;a_{\mathtt{BJS}})} + \|p_f - p_{fh}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_f)} + \|p_f - p_{fh}\|_{\mathrm{L}^2(0,T;\mathrm{W}_f)} \\ &\quad + \|A^{1/2}(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} \\ &\quad + \|\mathbf{div}(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} + \|A^{1/2}\partial_t((\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}) - (\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I}))\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} \\ &\quad + \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_s)} + \|\boldsymbol{\gamma}_p - \boldsymbol{\gamma}_{ph}\|_{\mathrm{L}^2(0,T;\mathbb{Q}_p)} + \|\mathbf{u}_p - \mathbf{u}_{ph}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} \\ &\quad + \|\mathbf{u}_p - \mathbf{u}_{ph}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} + \|p_p - p_{ph}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_p - p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{s_0}\|\partial_t(p_p - p_{ph})\|_{\mathrm{L}^2(0,T;\mathrm{W}_p))} + \|\lambda - \lambda_h\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} + \|\lambda - \lambda_h\|_{\mathrm{L}^2(0,T;\Lambda_{ph})} \\
&+ \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{\mathrm{L}^2(0,T;\boldsymbol{\Lambda}_{sh})} \\
\leq\ & C\sqrt{\exp(T)}\Big(h^{s_{\mathbf{u}_f}}\|\mathbf{u}_f\|_{\mathrm{H}^1(0,T;\mathbf{H}^{s_{\mathbf{u}_f}+1}(\Omega_f))} + h^{s_{p_f}+1}\|p_f\|_{\mathrm{H}^1(0,T;\mathrm{H}^{s_{p_f}+1}(\Omega_f))} \\
&+ h^{s_{\boldsymbol{\sigma}_p}+1}(\|\boldsymbol{\sigma}_p\|_{\mathrm{H}^1(0,T;\mathbb{H}^{s_{\boldsymbol{\sigma}_p}+1}(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^{s_{\boldsymbol{\sigma}_p}+1}(\Omega_p))}) \\
&+ h^{s_{\mathbf{u}_s}+1}\|\mathbf{u}_s\|_{\mathrm{L}^2(0,T;\mathbf{H}^{s_{\mathbf{u}_s}+1}(\Omega_p))} + h^{s_{\boldsymbol{\gamma}_p}+1}\|\gamma_p\|_{\mathrm{H}^1(0,T;\mathbb{H}^{s_{\boldsymbol{\gamma}_p}+1}(\Omega_p))} \\
&+ h^{s_{\mathbf{u}_p}+1}(\|\mathbf{u}_p\|_{\mathrm{H}^1(0,T;\mathbf{H}^{s_{\mathbf{u}_p}+1}(\Omega_p))} + \|\mathrm{div}(\mathbf{u}_p)\|_{\mathrm{L}^2(0,T;\mathrm{H}^{s_{\mathbf{u}_p}+1}(\Omega_p))}) + h^{s_{p_p}+1}\|p_p\|_{\mathrm{H}^1(0,T;\mathrm{H}^{s_{p_p}+1}(\Omega_p))} \\
&+ h^{s_\lambda+1}\|\lambda\|_{\mathrm{H}^1(0,T;\mathrm{H}^{s_\lambda+1}(\Gamma_{fp}))} + h^{s_{\boldsymbol{\theta}}+1}\|\boldsymbol{\theta}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{s_{\boldsymbol{\theta}}+1}(\Gamma_{fp}))} + h^{s_{\boldsymbol{\gamma}_p}+1}\|\boldsymbol{\rho}_p(0)\|_{\mathbb{H}^{s_{\boldsymbol{\gamma}_p}+1}(\Omega_p)}\Big).
\end{aligned}
\tag{2.3.38}
$$

Proof. We introduce the error terms as the differences of the solutions to (2.1.11) and (2.3.3) and decompose them into approximation and discretization errors using the interpolation operators:

$$
\begin{aligned}
&e_{\mathbf{u}_f} := \mathbf{u}_f - \mathbf{u}_{fh} = (\mathbf{u}_f - I_h^{\mathbf{V}_f}\mathbf{u}_f) + (I_h^{\mathbf{V}_f}\mathbf{u}_f - \mathbf{u}_{fh}) := e_{\mathbf{u}_f}^I + e_{\mathbf{u}_f}^h, \\
&e_{p_f} := p_f - p_{fh} = (p_f - P_h^{\mathrm{W}_f}p_f) + (P_h^{\mathrm{W}_f}p_f - p_{fh}) := e_{p_f}^I + e_{p_f}^h, \\
&e_{\mathbf{u}_p} := \mathbf{u}_p - \mathbf{u}_{ph} = (\mathbf{u}_p - I_h^{\mathbf{V}_p}\mathbf{u}_p) + (I_h^{\mathbf{V}_p}\mathbf{u}_p - \mathbf{u}_{ph}) := e_{\mathbf{u}_p}^I + e_{\mathbf{u}_p}^h, \\
&e_{p_p} := p_p - p_{ph} = (p_p - P_h^{\mathrm{W}_p}p_p) + (P_h^{\mathrm{W}_p}p_p - p_{ph}) := e_{p_p}^I + e_{p_p}^h, \\
&e_{\boldsymbol{\sigma}_p} := \boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph} = (\boldsymbol{\sigma}_p - I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p) + (I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph}) := e_{\boldsymbol{\sigma}_p}^I + e_{\boldsymbol{\sigma}_p}^h, \\
&e_{\mathbf{u}_s} := \mathbf{u}_s - \mathbf{u}_{sh} = (\mathbf{u}_s - P_h^{\mathbf{V}_s}\mathbf{u}_s) + (P_h^{\mathbf{V}_s}\mathbf{u}_s - \mathbf{u}_{sh}) := e_{\mathbf{u}_s}^I + e_{\mathbf{u}_s}^h, \\
&e_{\boldsymbol{\gamma}_p} := \boldsymbol{\gamma}_p - \boldsymbol{\gamma}_{ph} = (\boldsymbol{\gamma}_p - P_h^{\mathbb{Q}_p}\boldsymbol{\gamma}_p) + (P_h^{\mathbb{Q}_p}\boldsymbol{\gamma}_p - \boldsymbol{\gamma}_{ph}) := e_{\boldsymbol{\gamma}_p}^I + e_{\boldsymbol{\gamma}_p}^h, \\
&e_{\boldsymbol{\theta}} := \boldsymbol{\theta} - \boldsymbol{\theta}_h = (\boldsymbol{\theta} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\theta}) + (P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\theta} - \boldsymbol{\theta}_h) := e_{\boldsymbol{\theta}}^I + e_{\boldsymbol{\theta}}^h, \\
&e_\lambda := \lambda - \lambda_h = (\lambda - P_h^{\boldsymbol{\Lambda}_p}\lambda) + (P_h^{\boldsymbol{\Lambda}_p}\lambda - \lambda_h) := e_\lambda^I + e_\lambda^h.
\end{aligned}
\tag{2.3.39}
$$

We also define the approximation errors for non-differentiated variables:

$$
e_{\boldsymbol{\eta}_p}^I = \boldsymbol{\eta}_p - P_h^{\mathbf{V}_s}\boldsymbol{\eta}_p, \quad e_{\boldsymbol{\rho}_p}^I = \boldsymbol{\rho}_p - P_h^{\mathbb{Q}_p}\boldsymbol{\rho}_p, \quad e_{\boldsymbol{\omega}}^I = \boldsymbol{\omega} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\omega}.
$$

We form the error equations by subtracting the semi-discrete equations (2.3.3) from the continuous equations (2.1.11):

$$a_f(e_{\mathbf{u}_f}, \mathbf{v}_{fh}) + b_f(\mathbf{v}_{fh}, e_{p_f}) + b_\Gamma(\mathbf{v}_{fh}, \mathbf{0}, \mathbf{0}; e_\lambda) + a_{\mathtt{BJS}}(e_{\mathbf{u}_f}, e_{\boldsymbol{\theta}}; \mathbf{v}_{fh}, \mathbf{0}) = 0, \tag{2.3.40a}$$

$$- b_f(e_{\mathbf{u}_f}, w_{fh}) = 0, \tag{2.3.40b}$$

$$a_e(\partial_t e_{\boldsymbol{\sigma}_p}, \partial_t e_{p_p}; \boldsymbol{\tau}_{ph}, 0) + b_s(\boldsymbol{\tau}_{ph}, e_{\mathbf{u}_s}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, e_{\boldsymbol{\gamma}_p}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e_{\boldsymbol{\theta}}) = 0, \tag{2.3.40c}$$

$$- b_s(e_{\boldsymbol{\sigma}_p}, \mathbf{v}_{sh}) = 0, \tag{2.3.40d}$$

$$- b_{\mathrm{sk}}(e_{\boldsymbol{\sigma}_p}, \boldsymbol{\chi}_{ph}) = 0, \tag{2.3.40e}$$

$$a_p(e_{\mathbf{u}_p}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, e_{p_p}) + b_\Gamma(\mathbf{0}, \mathbf{v}_{ph}, \mathbf{0}; e_\lambda) = 0, \tag{2.3.40f}$$

$$a_p^p(\partial_t e_{p_p}, w_{ph}) + a_e(\partial_t e_{\boldsymbol{\sigma}_p}, \partial_t e_{p_p}; \mathbf{0}, w_{ph}) - b_p(e_{\mathbf{u}_p}, w_{ph}) = 0, \tag{2.3.40g}$$

$$- b_\Gamma(e_{\mathbf{u}_f}, e_{\mathbf{u}_p}, e_{\boldsymbol{\theta}}; \xi_h) = 0, \tag{2.3.40h}$$

$$b_\Gamma(\mathbf{0}, \mathbf{0}, \boldsymbol{\phi}_h; e_\lambda) + a_{\mathtt{BJS}}(e_{\mathbf{u}_f}, e_{\boldsymbol{\theta}}; \mathbf{0}, \boldsymbol{\phi}_h) - b_{\mathbf{n}_p}(e_{\boldsymbol{\sigma}_p}, \boldsymbol{\phi}_h) = 0. \tag{2.3.40i}$$

Setting $\mathbf{v}_{fh} = e^h_{\mathbf{u}_f}, w_{fh} = e^h_{p_f}, \boldsymbol{\tau}_{ph} = e^h_{\boldsymbol{\sigma}_p}, \mathbf{v}_{sh} = e^h_{\mathbf{u}_s}, \boldsymbol{\chi}_{ph} = e^h_{\boldsymbol{\gamma}_p}, \mathbf{v}_{ph} = e^h_{\mathbf{u}_p}, w_{ph} = e^h_{p_p}, \xi_h = e^h_\lambda, \boldsymbol{\phi}_h = e^h_{\boldsymbol{\theta}}$, and summing the equations, we obtain

$$\begin{aligned}
&a_f(e^I_{\mathbf{u}_f}, e^h_{\mathbf{u}_f}) + a_f(e^h_{\mathbf{u}_f}, e^h_{\mathbf{u}_f}) + a_{\mathtt{BJS}}(e^I_{\mathbf{u}_f}, e^I_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) + a_{\mathtt{BJS}}(e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) \\
&+ a_e(\partial_t e^I_{\boldsymbol{\sigma}_p}, \partial_t e^I_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) + a_e(\partial_t e^h_{\boldsymbol{\sigma}_p}, \partial_t e^h_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) + a_p(e^I_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) + a_p(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) \\
&+ a_p^p(\partial_t e^I_{p_p}, e^h_{p_p}) + a_p^p(\partial_t e^h_{p_p}, e^h_{p_p}) - b_{\mathbf{n}_p}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\theta}}) + b_p(e^h_{\mathbf{u}_p}, e^I_{p_p}) + b_f(e^h_{\mathbf{u}_f}, e^I_{p_f}) \\
&+ b_s(e^h_{\boldsymbol{\sigma}_p}, e^I_{\mathbf{u}_s}) + b_{\mathrm{sk}}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p}) + b_\Gamma(e^h_{\mathbf{u}_f}, e^h_{\mathbf{u}_p}, e^h_{\boldsymbol{\theta}}; e^I_\lambda) + b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}}) - b_p(e^I_{\mathbf{u}_p}, e^h_{p_p}) \\
&- b_f(e^I_{\mathbf{u}_f}, e^h_{p_f}) - b_s(e^I_{\boldsymbol{\sigma}_p}, e^h_{\mathbf{u}_s}) - b_{\mathrm{sk}}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\gamma}_p}) - b_\Gamma(e^I_{\mathbf{u}_f}, e^I_{\mathbf{u}_p}, e^I_{\boldsymbol{\theta}}; e^h_\lambda) = 0.
\end{aligned}$$

Due to (2.3.1) and the properties of the projection operators (2.3.28), (2.3.30), (2.3.32), (2.3.34) and (2.3.36), we have

$$b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}}) = 0, \quad \langle e^h_{\mathbf{u}_p} \cdot \mathbf{n}_p, e^I_\lambda \rangle_{\Gamma_{fp}} = 0,$$

$$a_p^p(\partial_t e^I_{p_p}, e^h_{p_p})_{\Omega_p} = 0, \quad b_p(e^h_{\mathbf{u}_p}, e^I_{p_p}) = 0, \quad b_s(e^h_{\boldsymbol{\sigma}_p}, e^I_{\mathbf{u}_s}) = 0,$$

$$b_f(e^I_{\mathbf{u}_f}, e^h_{p_f}) = 0, \qquad b_p(e^I_{\mathbf{u}_p}, e^h_{p_p}) = 0, \quad \langle e^I_{\mathbf{u}_p} \cdot \mathbf{n}_p, e^h_\lambda \rangle_{\Gamma_{fp}} = 0,$$

$$b_s(e^I_{\boldsymbol{\sigma}_p}, e^h_{\mathbf{u}_s}) = 0, \quad b_{\mathrm{sk}}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\gamma}_p}) = 0, \quad b_{\mathbf{n}_p}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\theta}}) = 0.$$

With the use of the algebraic identity $\int_S v\,\partial_t v = \frac{1}{2}\partial_t \|v\|^2_{\mathrm{L}^2(S)}$, the error equation (2.3.3) becomes

$$\begin{aligned}
& a_f(e^h_{\mathbf{u}_f}, e^h_{\mathbf{u}_f}) + a_{\mathtt{BJS}}(e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) + \frac{1}{2}\partial_t \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + a_p(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) + \frac{1}{2} s_0 \partial_t \|e^h_{p_p}\|^2_{\mathrm{W}_p} \\
& = -a_f(e^I_{\mathbf{u}_f}, e^h_{\mathbf{u}_f}) - a_{\mathtt{BJS}}(e^I_{\mathbf{u}_f}, e^I_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) - a_e(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, \partial_t\, e^I_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) - a_p(e^I_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) \\
&\quad - b_f(e^h_{\mathbf{u}_f}, e^I_{p_f}) - b_{\mathrm{sk}}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p}) - b_\Gamma(e^h_{\mathbf{u}_f}, \mathbf{0}, e^h_{\boldsymbol{\theta}}; e^I_\lambda) + b_\Gamma(e^I_{\mathbf{u}_f}, \mathbf{0}, e^I_{\boldsymbol{\theta}}; e^h_\lambda).
\end{aligned} \tag{2.3.41}$$

We proceed by integrating (2.3.41) from 0 to $t \in (0, T]$, applying the coercivity properties of a_f and a_p, the semi-positive definiteness of $a_{\mathtt{BJS}}$ (2.2.5), the Cauchy-Schwarz inequality, the trace inequality (2.2.2), and Young's inequality, to get

$$\begin{aligned}
& \|e^h_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \|e^h_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{L}^2(\Omega_p))} + s_0 \|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} \\
& \leq \epsilon \Big(\|e^h_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\,\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|A^{1/2} e^h_{\boldsymbol{\sigma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|e^h_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|e^h_\lambda\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|e^h_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} \Big) \\
&\quad + \frac{C}{\epsilon} \Big(\|e^I_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + \|e^I_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)} \\
&\quad + \|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} + \|e^I_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} \\
&\quad + \|e^I_\lambda\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|e^I_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} \Big) + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|e^h_{p_p}(0)\|^2_{\mathrm{W}_p},
\end{aligned} \tag{2.3.42}$$

Here we also used that the extension of A from $\mathbb{S}$ to $\mathbb{M}$ can be chosen as the identity operator, therefore, cf. [63], there exists $c > 0$ such that

$$b_{\mathrm{sk}}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p}) = \frac{1}{c}(e^h_{\boldsymbol{\sigma}_p}, A e^I_{\boldsymbol{\gamma}_p})_{\Omega_p} = \frac{1}{c}(A^{1/2} e^h_{\boldsymbol{\sigma}_p}, A^{1/2} e^I_{\boldsymbol{\gamma}_p})_{\Omega_p} \leq \frac{a_{\max}^{1/2}}{c} \|A^{1/2} e^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega_p)} \|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}. \tag{2.3.43}$$

On the other hand, from the discrete inf-sup condition (2.3.6), and using (2.3.40a) and (2.3.40f), we have

$$
\begin{aligned}
&\|e^h_{p_f}\|_{\mathrm{W}_f} + \|e^h_{p_p}\|_{\mathrm{W}_p} + \|e^h_\lambda\|_{\Lambda_{ph}} \\
&\leq C \sup_{(\mathbf{v}_{fh},\mathbf{v}_{ph})\in\mathbf{V}_{fh}\times\mathbf{V}_{ph}} \frac{b_f(\mathbf{v}_{fh}, e^h_{p_f}) + b_p(\mathbf{v}_{ph}, e^h_{p_p}) + b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \mathbf{0}; e^h_\lambda)}{\|(\mathbf{v}_{fh}, \mathbf{v}_{ph})\|_{\mathbf{V}_f\times\mathbf{V}_p}} \\
&= C \sup_{(\mathbf{v}_{fh},\mathbf{v}_{ph})\in\mathbf{V}_{fh}\times\mathbf{V}_{ph}} \left(\frac{-a_f(e^h_{\mathbf{u}_f}, \mathbf{v}_{fh}) - a_{\mathtt{BJS}}(e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; \mathbf{v}_{fh}, \mathbf{0}) - a_f(e^I_{\mathbf{u}_f}, \mathbf{v}_{fh})}{\|\mathbf{v}_{fh}\|_{\mathbf{V}_f} + \|\mathbf{v}_{ph}\|_{\mathbf{V}_p}} \right. \\
&\quad \left. + \frac{-a_{\mathtt{BJS}}(e^I_{\mathbf{u}_f}, e^I_{\boldsymbol{\theta}}; \mathbf{v}_{fh}, \mathbf{0}) - a_p(e^h_{\mathbf{u}_p}, \mathbf{v}_{ph}) - a_p(e^I_{\mathbf{u}_p}, \mathbf{v}_{ph}) - b_f(\mathbf{v}_{fh}, e^I_{p_f}) - b_\Gamma(\mathbf{v}_{fh}, \mathbf{0}, \mathbf{0}; e^I_\lambda)}{\|\mathbf{v}_{fh}\|_{\mathbf{V}_f} + \|\mathbf{v}_{ph}\|_{\mathbf{V}_p}} \right) \\
&\leq C(\|e^h_{\mathbf{u}_f}\|_{\mathbf{V}_f} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|_{a_{\mathtt{BJS}}} + \|e^I_{\mathbf{u}_f}\|_{\mathbf{V}_f} + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|_{a_{\mathtt{BJS}}} + \|e^h_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} + \|e^I_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \|e^I_{p_f}\|_{\mathrm{W}_f} + \|e^I_\lambda\|_{\Lambda_{ph}}), \qquad (2.3.44)
\end{aligned}
$$

where we also used (2.3.1), (2.3.28) and (2.3.30). Similarly, the inf-sup condition (2.3.7) and (2.3.40c) give

$$
\begin{aligned}
&\|e^h_{\mathbf{u}_s}\|_{\mathbf{V}_s} + \|e^h_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p} \leq C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph} \text{ s.t. } \boldsymbol{\tau}_{ph}\mathbf{n}_p=\mathbf{0} \text{ on } \Gamma_{fp}} \frac{b_s(\boldsymbol{\tau}_{ph}, e^h_{\mathbf{u}_s}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\gamma}_p})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&= C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph} \text{ s.t. } \boldsymbol{\tau}_{ph}\mathbf{n}_p=\mathbf{0} \text{ on } \Gamma_{fp}} \left(\frac{-a_e(\partial_t e^h_{\boldsymbol{\sigma}_p}, \partial_t e^h_{p_p}; \boldsymbol{\tau}_{ph}, 0) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\theta}})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \right. \\
&\quad \left. + \frac{-a_e(\partial_t e^I_{\boldsymbol{\sigma}_p}, \partial_t e^I_{p_p}; \boldsymbol{\tau}_{ph}, 0) - b^p_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, e^I_{\gamma_p})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \right) \\
&\leq C(\|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}), \qquad (2.3.45)
\end{aligned}
$$

where we also used (2.3.1) and (2.3.30). Finally, using the inf-sup condition (2.3.8) and (2.3.40c), we obtain

$$
\begin{aligned}
&\|e^h_{\boldsymbol{\theta}}\|_{\mathbf{\Lambda}_{sh}} \leq C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph} \text{ s.t. } \mathbf{div}(\boldsymbol{\tau}_{ph})=\mathbf{0}} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\theta}})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&= C \sup_{\boldsymbol{\tau}_{ph}\in\mathbb{X}_{ph} \text{ s.t. } \mathbf{div}(\boldsymbol{\tau}_{ph})=\mathbf{0}} \left(\frac{-a_e(\partial_t e^h_{\boldsymbol{\sigma}_p}, \partial_t e^h_{p_p}; \boldsymbol{\tau}_{ph}, 0) - b_s(\boldsymbol{\tau}_{ph}, e^h_{\mathbf{u}_s}) - b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\gamma}_p})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \right.
\end{aligned}
$$

$$+ \frac{-a_e(\partial_t e^I_{\boldsymbol{\sigma}_p}, \partial_t e^I_{p_p}; \boldsymbol{\tau}_{ph}, 0) - b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, e^I_{\boldsymbol{\gamma}_p})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}}\Bigg)$$

$$\leq C(\|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p} + \|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}), \tag{2.3.46}$$

where we also used (2.3.28).

We next derive bounds for $\|\mathrm{div}(e^h_{\mathbf{u}_p})\|_{\mathrm{L}^2(\Omega_p)}$ and $\|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega_p)}$. Due to (2.3.1), we can choose $w_{ph} = \mathrm{div}(e^h_{\mathbf{u}_p})$ in (2.3.40g), obtaining

$$\begin{aligned}
&\|\mathrm{div}(e^h_{\mathbf{u}_p})\|^2_{\mathrm{L}^2(\Omega_p)} \\
&= -(s_0\partial_t e^h_{p_p}, \mathrm{div}(e^h_{\mathbf{u}_p}))_{\Omega_p} - (A\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I}), \mathrm{div}(e^h_{\mathbf{u}_p}))_{\Omega_p} - (A\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I}), \mathrm{div}(e^h_{\mathbf{u}_p}))_{\Omega_p} \\
&\leq (s_0\|\partial_t e^h_{p_p}\|_{\mathrm{W}_p} + a^{1/2}_{\max}\|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \\
&\quad + a^{1/2}_{\max}\|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)})\|\mathrm{div}(e^h_{\mathbf{u}_p})\|_{\mathrm{L}^2(\Omega_p)}.
\end{aligned} \tag{2.3.47}$$

Similarly, the choice of $\mathbf{v}_{sh} = \mathbf{div}(e^h_{\boldsymbol{\sigma}_p})$ in (2.3.40d) gives

$$\|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})(t)\|_{\mathbf{L}^2(\Omega_p)} = 0 \quad \text{and} \quad \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,t;\mathbf{L}^2(\Omega_p))} = 0. \tag{2.3.48}$$

Combining (2.3.42) with (2.3.44)–(2.3.48) and choosing ϵ small enough, results in

$$\begin{aligned}
&\|e^h_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathrm{BJS}})} + \|e^h_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathrm{L}^2(0,t;\mathbf{L}^2(\Omega_p))} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_s}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_s)} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} \\
&\quad + \|e^h_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|e^h_{p_p}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_p)} + s_0\|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + \|e^h_{\lambda}\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|e^h_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} \\
&\leq C(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \\
&\quad + s_0\|\partial_t e^h_{p_p}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_p)} + \|e^I_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathrm{BJS}})} + \|e^I_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)} \\
&\quad + \|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} + \|e^I_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|e^I_{\lambda}\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} \\
&\quad + \|e^I_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|e^h_{p_p}(0)\|^2_{\mathrm{L}^2(\Omega_p)}),
\end{aligned} \tag{2.3.49}$$

where we also used

$$\|A^{1/2}e^h_{\boldsymbol{\sigma}_p}\|_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \leq C(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|e^h_{p_p}\|_{\mathrm{L}^2(0,t;\mathrm{W}_p)}). \tag{2.3.50}$$

In order to bound $\|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p}+\alpha_p e^h_{p_p}\mathbf{I})\|_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))}$ and $s_0\|\partial_t e^h_{p_p}\|_{\mathrm{L}^2(0,t;\mathrm{W}_p)}$, we differentiate in time (2.1.11a), (2.1.11d), (2.1.11e), (2.1.11f), and (2.1.11i) in the continuous equations and (2.3.3a), (2.3.3d), (2.3.3e), (2.3.3f), and (2.3.3i) in the semi-discrete equations, subtract the two systems, take $(\mathbf{v}_{fh}, w_{fh}, \boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}, \mathbf{v}_{ph}, w_{ph}, \xi_h, \boldsymbol{\phi}_h) = (e^h_{\mathbf{u}_f}, \partial_t e^h_{p_f}, \partial_t e^h_{\boldsymbol{\sigma}_p}, e^h_{\mathbf{u}_s}, e^h_{\boldsymbol{\gamma}_p}, e^h_{\mathbf{u}_p}, \partial_t e^h_{p_p}, \partial_t e^h_\lambda, e^h_{\boldsymbol{\theta}})$, and add all the equations together to obtain, in a way similar to (2.3.41),

$$
\begin{aligned}
&\frac{1}{2}\partial_t a_f(e^h_{\mathbf{u}_f}, e^h_{\mathbf{u}_f}) + \frac{1}{2}\partial_t a_{\mathtt{BJS}}(e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) + \|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \frac{1}{2}\partial_t a_p(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) + s_0\|\partial_t e^h_{p_p}\|^2_{\mathrm{W}_p} \\
&= -a_f(\partial_t e^I_{\mathbf{u}_f}, e^h_{\mathbf{u}_f}) - a_{\mathtt{BJS}}(\partial_t e^I_{\mathbf{u}_f}, \partial_t e^I_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) - a_e(\partial_t e^I_{\boldsymbol{\sigma}_p}, \partial_t e^I_{p_p}; \partial_t e^h_{\boldsymbol{\sigma}_p}, \partial_t e^h_{p_p}) - a_p(\partial_t e^I_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) \\
&\quad - b_f(e^h_{\mathbf{u}_f}, \partial_t e^I_{p_f}) - b_{\mathrm{sk}}(\partial_t e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p}) - b_\Gamma(e^h_{\mathbf{u}_f}, \mathbf{0}, e^h_{\boldsymbol{\theta}}; \partial_t e^I_\lambda) + b_\Gamma(e^I_{\mathbf{u}_f}, \mathbf{0}, e^I_{\boldsymbol{\theta}}; \partial_t e^h_\lambda). \qquad (2.3.51)
\end{aligned}
$$

Using integration by parts in time, we obtain

$$
\begin{aligned}
&\int_0^t b_{\mathrm{sk}}(\partial_t e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p})ds = b_{\mathrm{sk}}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p})\Big|_0^t - \int_0^t b_{\mathrm{sk}}(e^h_{\boldsymbol{\sigma}_p}, \partial_t e^I_{\boldsymbol{\gamma}_p})ds, \\
&\int_0^t \langle e^I_{\mathbf{u}_f}\cdot\mathbf{n}_f, \partial_t e^h_\lambda\rangle_{\Gamma_{fp}} ds = \langle e^I_{\mathbf{u}_f}\cdot\mathbf{n}_f, e^h_\lambda\rangle_{\Gamma_{fp}}\Big|_0^t - \int_0^t \langle \partial_t e^I_{\mathbf{u}_f}\cdot\mathbf{n}_f, e^h_\lambda\rangle_{\Gamma_{fp}} ds, \\
&\int_0^t \langle e^I_{\boldsymbol{\theta}}\cdot\mathbf{n}_p, \partial_t e^h_\lambda\rangle_{\Gamma_{fp}} ds = \langle e^I_{\boldsymbol{\theta}}\cdot\mathbf{n}_p, e^h_\lambda\rangle_{\Gamma_{fp}}\Big|_0^t - \int_0^t \langle \partial_t e^I_{\boldsymbol{\theta}}\cdot\mathbf{n}_p, e^h_\lambda\rangle_{\Gamma_{fp}} ds.
\end{aligned}
$$

We integrate (2.3.51) over $(0,t)$ and apply the coercivity properties of a_f and a_p, the semi-positive definiteness of $a_{\mathtt{BJS}}$ (2.2.5), the Cauchy-Schwarz inequality, the trace inequality (2.2.2), and Young's inequality, to obtain

$$
\begin{aligned}
&\|e^h_{\mathbf{u}_f}(t)\|^2_{\mathbf{V}_f} + |(e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}})(t)|^2_{a_{\mathtt{BJS}}} + \|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|e^h_{\mathbf{u}_p}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + s_0\|\partial_t e^h_{p_p}\|^2_{\mathrm{L}^2(0,t;W_p)} \\
&\le \epsilon\Big(\|e^h_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + \|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} \\
&\quad + \|e^h_{p_p}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_p)} + \|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + \|e^h_\lambda(t)\|^2_{\Lambda_{ph}} + \|e^h_\lambda\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|e^h_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})}\Big) \\
&\quad + \frac{C}{\epsilon}\Big(\|\partial_t e^I_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + |\partial_t(e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}})|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + \|\partial_t e^I_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|\partial_t e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} + \|e^I_{\boldsymbol{\gamma}_p}(t)\|^2_{\mathbb{Q}_p} \\
&+ \|\partial_t e^I_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|\partial_t e^I_{\lambda}\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|\partial_t e^I_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} + \|e^I_{\mathbf{u}_f}(t)\|^2_{\mathbf{V}_f} + \|e^I_{\boldsymbol{\theta}}(t)\|^2_{\boldsymbol{\Lambda}_s}\Big) \\
&+ \|e^h_{\mathbf{u}_f}(0)\|^2_{\mathbf{V}_f} + |(e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}})(0)|^2_{a_{\mathtt{BJS}}} + \|A^{1/2}e^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{\lambda}(0)\|^2_{\Lambda_p} \\
&+ \|e^I_{\mathbf{u}_f}(0)\|^2_{\mathbf{V}_f} + \|e^I_{\boldsymbol{\gamma}_p}(0)\|^2_{\mathbb{Q}_p} + \|e^I_{\boldsymbol{\theta}}(0)\|^2_{\boldsymbol{\Lambda}_{sh}}, \qquad (2.3.52)
\end{aligned}
$$

where we also used $b_{\mathrm{sk}}(e^h_{\boldsymbol{\sigma}_p}, \partial_t e^I_{\boldsymbol{\gamma}_p}) \le C\|A^{1/2}e^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega_p)}\|\partial_t e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}$, cf. (2.3.43), and

$$
\|A^{1/2}e^h_{\boldsymbol{\sigma}_p}(t)\|_{\mathbb{L}^2(\Omega_p)} \le C(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|_{\mathbb{L}^2(\Omega_p)} + \|e^h_{p_p}(t)\|_{\mathrm{W}_p}).
$$

In addition, the choice of $\mathbf{v}_{sh} = \mathbf{div}(\partial_t e^h_{\boldsymbol{\sigma}_p})$ in the time differentiated version of (2.3.40) gives

$$
\|\mathbf{div}(\partial_t e^h_{\boldsymbol{\sigma}_p})(t)\|_{\mathbf{L}^2(\Omega_p)} = 0 \quad \text{and} \quad \|\mathbf{div}(\partial_t e^h_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,t;\mathbf{L}^2(\Omega_p))} = 0. \qquad (2.3.53)
$$

Thus, combining (2.3.52) with (2.3.44), (2.3.49) and (2.3.53), and taking ϵ small enough, we obtain

$$
\begin{aligned}
&\|e^h_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + \|e^h_{\mathbf{u}_f}(t)\|^2_{\mathbf{V}_f} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + |(e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}})(t)|^2_{a_{\mathtt{BJS}}} + \|e^h_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)} \\
&+ \|e^h_{p_f}(t)\|_{\mathrm{W}_f} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{div}(\partial_t e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \\
&+ \|\mathbf{div}(\partial_t e^h_{\boldsymbol{\sigma}_p})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} \\
&+ \|e^h_{\mathbf{u}_s}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_s)} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} + \|e^h_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|e^h_{\mathbf{u}_p}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{p_p}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_p)} \\
&+ \|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + s_0\|\partial_t e^h_{p_p}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_p)} + \|e^h_{\lambda}\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|e^h_{\lambda}(t)\|_{\Lambda_{ph}} + \|e^h_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} \\
&\le C(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|e^I_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + \|\partial_t e^I_{\mathbf{u}_f}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_f)} + \|e^I_{\mathbf{u}_f}(t)\|^2_{\mathbf{V}_f} \\
&\quad + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + |\partial_t(e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}})|^2_{\mathrm{L}^2(0,t;a_{\mathtt{BJS}})} + |(e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}})(t)|_{a_{\mathtt{BJS}}} + \|e^I_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)} \\
&\quad + \|\partial_t e^I_{p_f}\|^2_{\mathrm{L}^2(0,t;\mathrm{W}_f)} + \|e^I_{p_f}(t)\|^2_{\mathrm{W}_f} + \|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} \\
&\quad + \|\partial_t e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathrm{L}^2(0,t;\mathbb{Q}_p)} + \|e^I_{\boldsymbol{\gamma}_p}(t)\|^2_{\mathbb{Q}_p} + \|e^I_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|\partial_t e^I_{\mathbf{u}_p}\|^2_{\mathrm{L}^2(0,t;\mathbf{V}_p)} + \|e^I_{\mathbf{u}_p}(t)\|_{\mathbf{V}_p} \\
&\quad + \|e^I_{\lambda}\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|\partial_t e^I_{\lambda}\|^2_{\mathrm{L}^2(0,t;\Lambda_{ph})} + \|e^I_{\lambda}(t)\|_{\Lambda_{ph}} + \|e^I_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} + \|\partial_t e^I_{\boldsymbol{\theta}}\|^2_{\mathrm{L}^2(0,t;\boldsymbol{\Lambda}_{sh})} \\
&\quad + \|e^I_{\boldsymbol{\theta}}(t)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^h_{\mathbf{u}_f}(0)\|^2_{\mathbf{V}_f} + |(e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}})(0)|^2_{a_{\mathtt{BJS}}} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)}
\end{aligned}
$$

$$+ s_0\|e^h_{p_p}(0)\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}e^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_\lambda(0)\|^2_{\Lambda_{ph}} + \|e^I_{\mathbf{u}_f}(0)\|^2_{\mathbf{V}_f}$$

$$+ \|e^I_{\boldsymbol{\gamma}_p}(0)\|^2_{\mathbb{Q}_p} + \|e^I_{\boldsymbol{\theta}}(0)\|^2_{\boldsymbol{\Lambda}_{sh}}). \tag{2.3.54}$$

We remark that in the above bound we have obtained control on $\|e^h_{p_p}(t)\|_{\mathrm{W}_p}$ independent of s_0.

We next establish a bound on the initial data terms above. We recall that $(\mathbf{u}_f(0), p_f(0), \boldsymbol{\sigma}_p(0), \mathbf{u}_p(0), p_p(0), \lambda(0), \boldsymbol{\theta}(0)) = (\mathbf{u}_{f,0}, p_{f,0}, \boldsymbol{\sigma}_{p,0}, \mathbf{u}_{p,0}, p_{p,0}, \lambda_0, \boldsymbol{\theta}_0)$, cf. Corollary 2.2.12, and $(\mathbf{u}_{fh}(0), p_{fh}(0), \boldsymbol{\sigma}_{ph}(0), \mathbf{u}_{ph}(0), p_{ph}(0), \lambda_h(0), \boldsymbol{\theta}_h(0)) = (\mathbf{u}_{fh,0}, p_{fh,0}, \boldsymbol{\sigma}_{ph,0}, \mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}, \boldsymbol{\theta}_{h,0})$, cf. Theorem 2.3.2. We first note that, since $\boldsymbol{\theta}_{h,0} = P^{\boldsymbol{\Lambda}_s}_h \boldsymbol{\theta}_0$,

$$e^h_{\boldsymbol{\theta}}(0) = \mathbf{0}. \tag{2.3.55}$$

Next, similarly to (2.3.25), we obtain

$$\|e^h_{\mathbf{u}_f}(0)\|^2_{\mathbf{V}_f} + |(e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}})(0)|^2_{a_{\mathtt{BJS}}} + \|A^{1/2}e^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)}$$

$$+ \|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} + \|e^h_\lambda(0)\|^2_{\Lambda_{ph}}$$

$$\le C\big(\|e^I_{\mathbf{u}_f}(0)\|_{\mathbf{V}_f} + |e^I_{\mathbf{u}_f}(0) - e^I_{\boldsymbol{\theta}}(0)|^2_{a_{\mathtt{BJS}}} + \|e^I_{p_f}(0)\|_{\mathrm{W}_f} + \|e^I_{\boldsymbol{\sigma}_p}(0)\|_{\mathbb{X}_p} + \|e^I_{\boldsymbol{\rho}_p}(0)\|_{\mathbb{Q}_p}$$

$$+ \|e^I_{\mathbf{u}_p}(0)\|_{\mathbf{V}_p} + \|e^I_{p_p}(0)\|_{\mathrm{W}_p} + \|e^I_\lambda(0)\|_{\Lambda_p} + \|e^I_{\boldsymbol{\theta}}(0)\|_{\boldsymbol{\Lambda}_{sh}}\big). \tag{2.3.56}$$

Combining (2.3.54)-(2.3.56), using Gronwall's inequality for $\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathrm{L}^2(0,t;\mathbb{L}^2(\Omega_p))}$, the triangle inequality, and the approximation properties (2.3.29), (2.3.31), (2.3.33), and (2.3.35), we obtain (2.3.38). □

2.4 Numerical results

In this section we present the results from a series of numerical tests illustrating the performance of the proposed method. We employ the backward Euler method for the time discretization. Let $\Delta t = T/N$ be the time step, $t_n = n\Delta t$, $n = 0, \cdots, N$. Let $d_t\, u^n := (u^n - u^{n-1})/\Delta t$, where $u^n := u(t_n)$. The fully discrete method reads: given $(\mathbf{p}_h^0, \mathbf{r}_h^0) = (\mathbf{p}_h(0), \mathbf{r}_h(0))$ satisfying (2.3.14), find $(\mathbf{p}_h^n, \mathbf{r}_h^n) \in \mathbf{Q}_h \times \mathbf{S}_h$, $n = 1, \cdots, N$, such that for all $(\mathbf{q}_h, \mathbf{s}_h) \in \mathbf{Q}_h \times \mathbf{S}_h$,

$$\begin{aligned} d_t\, \mathcal{E}_1\, (\mathbf{p}_h^n)(\mathbf{q}_h) + \mathcal{A}\, (\mathbf{p}_h^n)(\mathbf{q}_h) + \mathcal{B}'\, (\mathbf{r}_h^n)(\mathbf{q}_h) &= \mathbf{F}(\mathbf{q}_h), \\ -\mathcal{B}\, (\mathbf{p}_h^n)(\mathbf{s}_h) &= \mathbf{G}(\mathbf{s}_h). \end{aligned} \tag{2.4.1}$$

Our implementation is on triangular grids and it is based on the `FreeFem++` finite element package [55]. For spatial discretization we use the MINI elements $\mathbf{P}_1^b - \mathrm{P}_1$ for the Stokes spaces $(\mathbf{V}_{fh}, \mathrm{W}_{fh})$, where P_1^b stands for the space of continuous piecewise linear polynomials enhanced elementwise by cubic bubbles, the lowest order Raviart-Thomas elements $\mathbf{RT}_0 - \mathrm{P}_0$ for the Darcy spaces $(\mathbf{V}_{ph}, \mathrm{W}_{ph})$, and the $\mathbb{BDM}_1 - \mathbf{P}_0 - \mathrm{P}_1$ elements [20] for the elasticity spaces $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph})$. According to (2.3.2), for the Lagrange multiplier spaces we choose piecewise constants for Λ_{ph} and discontinuous piecewise linears for $\mathbf{\Lambda}_{sh}$. We present two examples. Example 1 is used to corroborate the rates of convergence. In Example 2 we present simulations of the coupling of surface and subsurface hydrological systems, focusing on the qualitative behavior of the solution.

2.4.1 Example 1: convergence test

For the convergence study we consider a test case with domain $\Omega = (0, 1) \times (-1, 1)$ and a known analytical solution. We associate the upper half with the Stokes flow, while the lower half represents the flow in the poroelastic structure governed by the Biot system. The physical parameters are $\mathbf{K} = \mathbf{I}$, $\mu = 1$, $\alpha_p = 1$, $\alpha_{\mathtt{BJS}} = 1$, $s_0 = 1$, $\lambda_p = 1$, and $\mu_p = 1$. The solution in the Stokes region is

$$\mathbf{u}_f = \pi \cos(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}, \quad p_f = e^t \sin(\pi x) \cos(\frac{\pi y}{2}) + 2\pi \cos(\pi t).$$

The Biot solution is chosen accordingly to satisfy the interface conditions at $y = 0$:

$$\mathbf{u}_p = \pi e^t \begin{pmatrix} -\cos(\pi x)\cos(\frac{\pi y}{2}) \\ \frac{1}{2}\sin(\pi x)\sin(\frac{\pi y}{2}) \end{pmatrix}, \ p_p = e^t \sin(\pi x)\cos(\frac{\pi y}{2}), \ \boldsymbol{\eta}_p = \sin(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}.$$

The right hand side functions $\mathbf{f}_f$, q_f, $\mathbf{f}_p$, and q_p are computed using the above solution. The model problem is complemented with Dirichlet boundary conditions and initial data obtained from the true solution. The total simulation time for this test case is $T = 0.01$ and the time step is $\Delta t = 10^{-3}$. The time step is sufficiently small, so that the time discretization error does not affect the spatial convergence rates.

In Table 2.4.1, we report errors on a sequence of refined meshes, which are matching along the interface. We use the notation $\|\cdot\|_{l^\infty(V)}$ and $\|\cdot\|_{l^2(V)}$ to denote the time-discrete space-time errors. For all errors we report the $\|\cdot\|_{l^2(V)}$ norms with the exception of the error $e_{\boldsymbol{\sigma}_p}$, for which we have a bound only in l^∞ in time. We observe at least $O(h)$ convergence for all norms, which is consistent with the theoretical results stated in Theorem 2.3.4. The observed $O(h^2)$ convergence for $\|e_{\boldsymbol{\sigma}_p}\|_{l^\infty(\mathbb{L}^2(\Omega_p))}$, $\|e_{\boldsymbol{\gamma}_p}\|_{l^2(\mathbb{Q}_p)}\|$, and $\|e_{\boldsymbol{\theta}}\|_{l^2(\boldsymbol{\Lambda}_{sh})}$ corresponds to the second order of approximation in the spaces $\mathbb{X}_{ph}$, $\mathbb{Q}_{ph}$, and $\boldsymbol{\Lambda}_{sh}$, respectively, and indicates that the convergence rates for these variables are not affected by the lower rate for the rest of the variables. Next, noting that the analysis in Theorem 2.3.4 is not restricted to the case of matching grids, we provide the convergence results obtained with non-matching grids along the interface. The results in Table 2.4.2 are obtained by setting the ratio between the characteristic mesh sizes to be $h_{\text{Stokes}} = \frac{5}{8} h_{\text{Biot}}$. The results in Table 2.4.3 are with $h_{\text{Biot}} = \frac{5}{8} h_{\text{Stokes}}$. The convergence rates in both tables agree with the statement of Theorem 2.3.4.

n	$\|e_{\mathbf{u}_f}\|_{l^2(\mathbf{V}_f)}$	rate	$\|e_{p_f}\|_{l^2(\mathrm{W}_f)}$	rate	$\|e_{\boldsymbol{\sigma}_p}\|_{l^\infty(\mathbb{L}^2(\Omega_p))}$	rate
8	7.731e-03	0.0	2.601e-03	0.0	7.454e-02	0.0
16	3.860e-03	1.0	8.319e-04	1.6	2.572e-02	1.5
32	1.929e-03	1.0	2.759e-04	1.6	8.775e-03	1.6
64	9.640e-04	1.0	9.419e-05	1.6	2.784e-03	1.7
128	4.819e-04	1.0	3.270e-05	1.5	8.224e-04	1.8

n	$\|e_{\mathbf{div}(\boldsymbol{\sigma}_p)}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate	$\|e_{\mathbf{u}_s}\|_{l^2(\mathbf{V}_s)}$	rate	$\|e_{\boldsymbol{\gamma}_p}\|_{l^2(\mathbb{Q}_p)}$	rate	$\|e_{\mathbf{u}_p}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate
8	1.032e-01	0.0	7.141e-02	0.0	1.926e-01	0.0	1.046e-01	0.0
16	5.169e-02	1.0	3.550e-02	1.0	5.171e-02	1.9	5.224e-02	1.0
32	2.586e-02	1.0	1.773e-02	1.0	1.372e-02	1.9	2.612e-02	1.0
64	1.293e-02	1.0	8.862e-03	1.0	3.633e-03	1.9	1.306e-02	1.0
128	6.465e-03	1.0	4.431e-03	1.0	9.497e-04	1.9	6.532e-03	1.0

n	$\|e_{\mathrm{div}(\mathbf{u}_p)}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate	$\|e_{p_p}\|_{l^2(\mathrm{W}_p)}$	rate	$\|e_\lambda\|_{l^2(\Lambda_{ph})}$	rate	$\|e_{\boldsymbol{\theta}}\|_{l^2(\boldsymbol{\Lambda}_{sh})}$	rate
8	1.223e-01	0.0	1.033e-01	0.0	1.140e-01	0.0	3.232e-02	0.0
16	5.457e-02	1.2	5.172e-02	1.0	5.675e-02	1.0	6.446e-03	2.3
32	2.693e-02	1.0	2.587e-02	1.0	2.835e-02	1.0	1.238e-03	2.4
64	1.442e-02	0.9	1.294e-02	1.0	1.417e-02	1.0	2.328e-04	2.4
128	9.001e-03	0.7	6.468e-03	1.0	7.085e-03	1.0	4.442e-05	2.4

Table 2.4.1: Example 1, Mesh sizes, errors and rates of convergences in matching grids.

n	$\|e_{\mathbf{u}_f}\|_{l^2(\mathbf{V}_f)}$	rate	$\|e_{p_f}\|_{l^2(\mathrm{W}_f)}$	rate	$\|e_{\boldsymbol{\sigma}_p}\|_{l^\infty(\mathbb{L}^2(\Omega_p))}$	rate
8	1.171e-02	0.0	8.326e-03	0.0	8.800e-02	0.0
16	5.725e-03	1.0	2.616e-03	1.7	3.220e-02	1.5
32	2.835e-03	1.0	9.239e-04	1.5	1.084e-02	1.6
64	1.411e-03	1.0	3.256e-04	1.5	3.262e-03	1.7
128	7.037e-04	1.0	1.152e-04	1.5	9.161e-04	1.8

n	$\|e_{\mathbf{div}(\boldsymbol{\sigma}_p)}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate	$\|e_{\mathbf{u}_s}\|_{l^2(\mathbf{V}_s)}$	rate	$\|e_{\boldsymbol{\gamma}_p}\|_{l^2(\mathbb{Q}_p)}$	rate	$\|e_{\mathbf{u}_p}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate
8	1.032e-01	0.0	7.632e-02	0.0	2.255e-01	0.0	1.049e-01	0.0
16	5.170e-02	1.0	3.810e-02	1.0	6.617e-02	1.8	5.226e-02	1.0
32	2.587e-02	1.0	1.905e-02	1.0	1.955e-02	1.8	2.613e-02	1.0
64	1.293e-02	1.0	9.524e-03	1.0	5.773e-03	1.8	1.306e-02	1.0
128	6.467e-03	1.0	4.762e-03	1.0	1.638e-03	1.8	6.532e-03	1.0

n	$\|e_{\mathrm{div}(\mathbf{u}_p)}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate	$\|e_{p_p}\|_{l^2(\mathrm{W}_p)}$	rate	$\|e_{\lambda}\|_{l^2(\Lambda_{ph})}$	rate	$\|e_{\boldsymbol{\theta}}\|_{l^2(\mathbf{\Lambda}_{sh})}$	rate
8	1.323e-01	0.0	1.033e-01	0.0	1.141e-01	0.0	3.272e-02	0.0
16	5.742e-02	1.2	5.172e-02	1.0	5.675e-02	1.0	6.733e-03	2.3
32	2.738e-02	1.1	2.587e-02	1.0	2.835e-02	1.0	1.314e-03	2.4
64	1.448e-02	0.9	1.294e-02	1.0	1.417e-02	1.0	2.502e-04	2.4
128	9.007e-03	0.7	6.468e-03	1.0	7.085e-03	1.0	4.820e-05	2.4

Table 2.4.2: Example 1, Mesh sizes, errors and rates of convergences in nonmatching grids.

n	$\|e_{\mathbf{u}_f}\|_{l^2(\mathbf{V}_f)}$	rate	$\|e_{p_f}\|_{l^2(\mathrm{W}_f)}$	rate	$\|e_{\boldsymbol{\sigma}_p}\|_{l^\infty(\mathbb{L}^2(\Omega_p))}$	rate
8	7.203e-03	0.0	5.066e-03	0.0	1.661e-01	0.0
16	3.561e-03	1.0	1.404e-03	1.9	6.387e-02	1.4
32	1.768e-03	1.0	4.843e-04	1.5	2.298e-02	1.5
64	8.807e-04	1.0	1.697e-04	1.5	7.441e-03	1.6
128	4.396e-04	1.0	5.977e-05	1.5	2.178e-03	1.8

n	$\|e_{\mathbf{div}(\boldsymbol{\sigma}_p)}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate	$\|e_{\mathbf{u}_s}\|_{l^2(\mathbf{V}_s)}$	rate	$\|e_{\boldsymbol{\gamma}_p}\|_{l^2(\mathbb{Q}_p)}$	rate	$\|e_{\mathbf{u}_p}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate
8	1.644e-01	0.0	1.230e-01	0.0	4.521e-01	0.0	1.698e-01	0.0
16	8.264e-02	1.0	6.100e-02	1.0	1.504e-01	1.6	8.374e-02	1.0
32	4.137e-02	1.0	3.048e-02	1.0	4.373e-02	1.8	4.180e-02	1.0
64	2.069e-02	1.0	1.524e-02	1.0	1.293e-02	1.8	2.090e-02	1.0
128	1.035e-02	1.0	7.619e-03	1.0	3.798e-03	1.8	1.045e-02	1.0

n	$\|e_{\mathrm{div}(\mathbf{u}_p)}\|_{l^2(\mathbf{L}^2(\Omega_p))}$	rate	$\|e_{p_p}\|_{l^2(\mathrm{W}_p)}$	rate	$\|e_\lambda\|_{l^2(\Lambda_{ph})}$	rate	$\|e_{\boldsymbol{\theta}}\|_{l^2(\boldsymbol{\Lambda}_{sh})}$	rate
8	2.430e-01	0.0	1.649e-01	0.0	1.849e-01	0.0	9.021e-02	0.0
16	1.004e-01	1.3	8.270e-02	1.0	9.101e-02	1.0	1.977e-02	2.2
32	4.474e-02	1.2	4.138e-02	1.0	4.538e-02	1.0	3.990e-03	2.3
64	2.203e-02	1.0	2.070e-02	1.0	2.268e-02	1.0	7.683e-04	2.4
128	1.215e-02	0.9	1.035e-02	1.0	1.134e-02	1.0	1.461e-04	2.4

Table 2.4.3: Example 1, Mesh sizes, errors and rates of convergences in nonmatching grids.

2.4.2 Example 2: coupling of surface and subsurface hydrological systems

In this example, we illustrate the behavior of the method for a problem motivated by the coupling of surface and subsurface hydrological systems and test its robustness with respect to physical parameters. On the domain $\Omega = (0,2) \times (-1,1)$, we associate the upper half with surface flow, such as lake or river, modeled by the Stokes equations while the lower half represents subsurface flow in a poroelastic aquifer, governed by the Biot system. The appropriate interface conditions are enforced along the interface $y = 0$. We consider three cases with different values of $\mathbf{K}$, s_0, λ_p and μ_p, as described in Table 2.4.4, while we set the

	$\mathbf{K}$	s_0	λ_p	μ_p
Case 1	$\mathbf{I}$	1	1	1
Case 2	$10^{-4} \times \mathbf{I}$	10^{-4}	10^6	1
Case 3	$10^{-4} \times \mathbf{I}$	10^{-4}	10^6	10^6

Table 2.4.4: Set of parameters for the sensitivity analysis

rest of the physical parameters to be $\mu = 1$, $\alpha_p = 1$, and $\alpha_{\mathtt{BJS}} = 1$. In the discussion we will also refer to the Young's modulus E and the Poisson's ratio ν, which are related to the Lamé coefficients via

$$\nu = \frac{\lambda_p}{2(\lambda_p + \mu_p)}, \quad E = \frac{(3\lambda_p + 2\mu_p)\mu_p}{\lambda_p + \mu_p}.$$

The body forces and external source are zero, as well as the initial conditions. The flow is driven by a parabolic fluid velocity on the left boundary of fluid region. The boundary conditions are as follows:

$$\mathbf{u}_f = (-40y(y-1) \;\; 0)^{\mathrm{t}} \quad \text{on} \quad \Gamma_{f,\text{left}}, \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_{f,\text{top}} \cup \Gamma_{f,\text{right}},$$

$$p_p = 0 \quad \text{and} \quad \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{p,\text{bottom}},$$

$$\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{and} \quad \mathbf{u}_s = \mathbf{0} \quad \text{on} \quad \Gamma_{p,\text{left}} \cup \Gamma_{p,right},$$

The simulation is run for a total time $T = 3$ with a time step $\Delta t = 0.06$.

For each case, we present the plots of computed velocities, first and second columns of stresses (top plots), first column components of poroelastic stress (middle plots), displacement and Darcy pressure (bottom plots) at final time $T = 3$.

Case 1 focuses on the qualitative behavior of the solution. The computed solution at the final time $T = 3$ is shown in Figure 2.4.1. On the top left, the arrows represent the velocity vectors $\mathbf{u}_f$ and $\mathbf{u}_p + \partial_t \boldsymbol{\eta}_p$ in the two regions, while the color shows the vertical components of these vectors. The other two plots on the top show the computed stress. The arrows in both plots represent the second columns of the negative stresses $-(\boldsymbol{\sigma}_{f,12}, \boldsymbol{\sigma}_{f,22})^{\mathrm{t}}$ and $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$. The colors show $-\boldsymbol{\sigma}_{f,12}$ and $-\boldsymbol{\sigma}_{p,12}$ in the middle plot and $-\boldsymbol{\sigma}_{f,22}$ and $-\boldsymbol{\sigma}_{p,22}$ in the right plot. Since the Stokes stress is much larger than the poroelastic stress, the arrows in the fluid region are scaled by a factor 1/5 for visualization purpose and the color scale is more suitable for the Stokes region. The poroelastic stresses are presented separately in the middle row with their own color range. The bottom plots show the displacement vector and its magnitude on the left and the poroelastic pressure on the right.

From the velocity plot we observe that the fluid is driven into the poroelastic medium due to zero pressure at the bottom, which simulates gravity. The mass conservation $\mathbf{u}_f \cdot \mathbf{n}_f + (\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p) \cdot \mathbf{n}_p = 0$ on the interface with $\mathbf{n}_p = (0,1)^{\mathrm{t}}$ indicates continuity of second components of these two velocity vectors, which is observed from the color plot of the velocity. In addition, the conservation of momentum $\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0$ implies that $-\boldsymbol{\sigma}_{f,12} = -\boldsymbol{\sigma}_{p,12}$ and $-\boldsymbol{\sigma}_{f,22} = -\boldsymbol{\sigma}_{p,22}$ on the interface. These conditions are verified from the two stress color plots on the top row. We observe large fluid stress near the top boundary, which is due to the no slip condition there, as well as large fluid stress along the interface, which is due to the slip with friction interface condition. A singularity in the left lower corner appears due to the mismatch in inflow boundary conditions between the fluid and poroelastic regions. The bottom plots show that the infiltration of fluid from the Stokes region into the poroelastic region causes deformation of the medium and larger Darcy pressure. Furthermore, comparing the right middle and bottom plots, we note the match along the interface between $-\boldsymbol{\sigma}_{p,22}$ and p_p, which is consistent with the balance of force and momentum conservation conditions $-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p$ and $\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0$, respectively.

In Case 2 we test the model for a problem that exhibits both locking regimes for poroelasticity: 1) small permeability and storativity and 2) almost incompressible material [83]. In particular, we take $\mathbf{K} = 10^{-4} \times \mathbf{I}$ and $s_0 = 10^{-4}$. Furthermore, the choice $\lambda_p = 10^6$, $\mu_p = 1$ results in Poisson's ratio $\nu = 0.4999995$. The computed solution does not exhibit locking or oscillations. The behavior is qualitatively similar to Case 1, with larger fluid and poroelastic stresses and a Darcy pressure gradient.

In Case 3, the Lamé coefficient μ_p is increased from 1 to 10^6, resulting in a much stiffer poroelastic medium, which is typical in subsurface flow applications. The solution is again

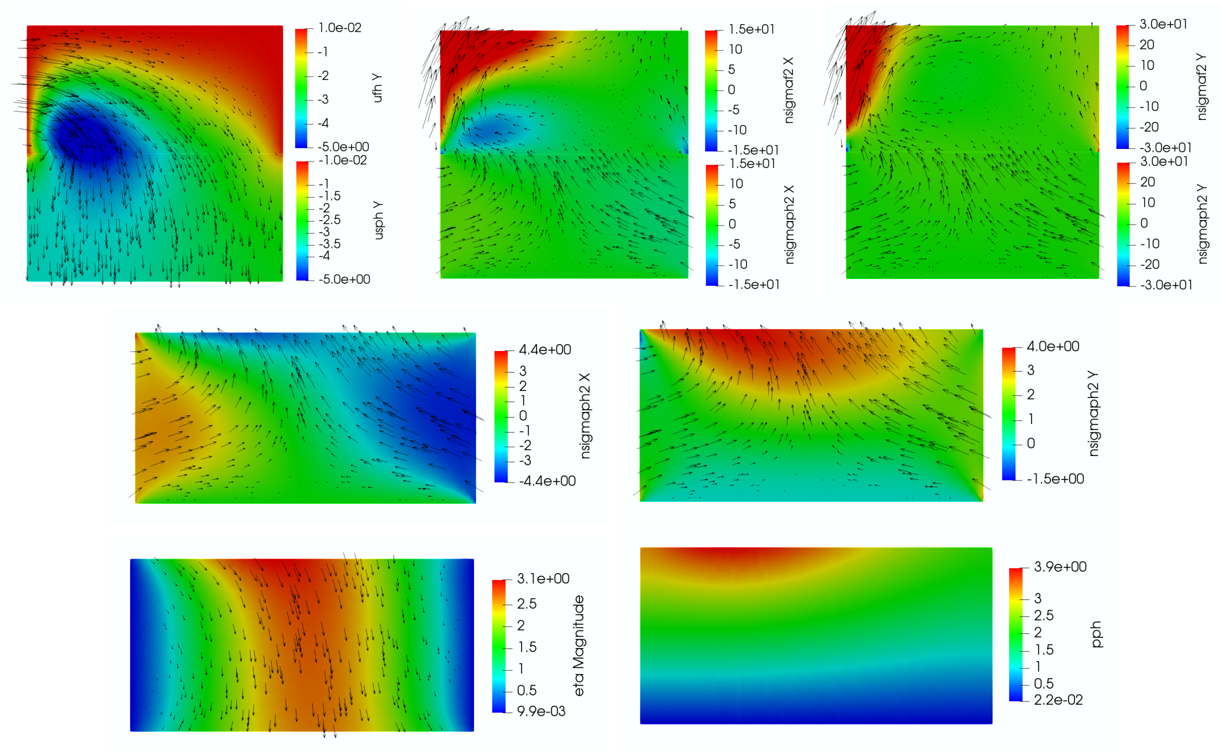


Figure 2.4.1: Example 2, Case 1.

$\mathbf{K} = \mathbf{I}$, $s_0 = 1$, $\lambda_p = 1$, $\mu_p = 1$. Computed solution at final time $T = 3$. Top left: velocities $\mathbf{u}_f$ and $\mathbf{u}_p + \partial_t \boldsymbol{\eta}_p$ (arrows), $\mathbf{u}_{f,2}$ and $\mathbf{u}_{p,2} + \partial_t \boldsymbol{\eta}_{p,2}$ (color). Top middle and right: stresses $-(\boldsymbol{\sigma}_{f,12}, \boldsymbol{\sigma}_{f,22})^{\mathrm{t}}$ and $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$ (arrows); top middle: $-\boldsymbol{\sigma}_{f,12}$ and $-\boldsymbol{\sigma}_{p,12}$ (color); top right: $-\boldsymbol{\sigma}_{f,22}$ and $-\boldsymbol{\sigma}_{p,22}$ (color). Middle: poroelastic stress $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$ (arrows); middle left: $-\boldsymbol{\sigma}_{p,12}$ (color); middle right: $-\boldsymbol{\sigma}_{p,22}$ (color). Bottom left: displacement $\boldsymbol{\eta}_p$ (arrows), $|\boldsymbol{\eta}_p|$ (color). Bottom right: Darcy pressure p_p.

free of locking effects or oscillations, but it differs significantly from Case 2, including three orders of magnitude larger stresses and Darcy pressure, as well as smaller displacement and Darcy velocity.

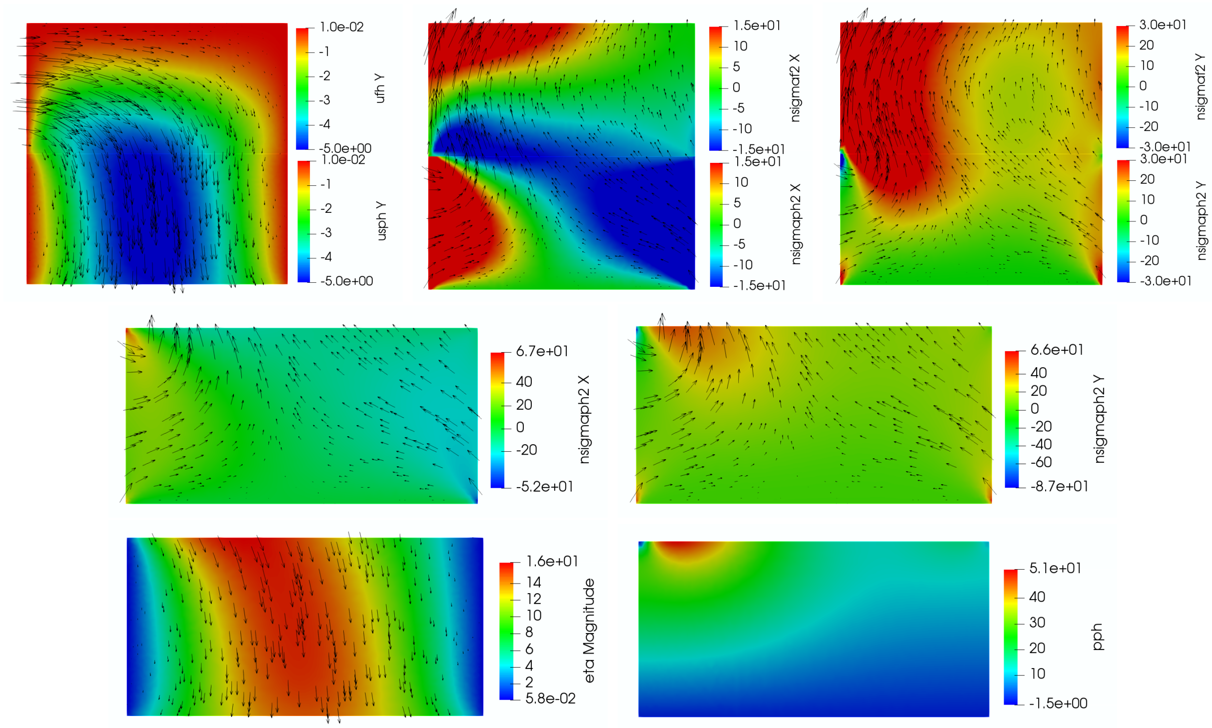


Figure 2.4.2: Example 2, Case 2.

$\mathbf{K} = 10^{-4} \times \mathbf{I}$, $s_0 = 10^{-4}$, $\lambda_p = 10^6$, $\mu_p = 1$. Computed solution at final time $T = 3$. Top left: velocities $\mathbf{u}_f$ and $\mathbf{u}_p + \partial_t \boldsymbol{\eta}_p$ (arrows), $\mathbf{u}_{f,2}$ and $\mathbf{u}_{p,2} + \partial_t \boldsymbol{\eta}_{p,2}$ (color). Top middle and right: stresses $-(\boldsymbol{\sigma}_{f,12}, \boldsymbol{\sigma}_{f,22})^{\mathrm{t}}$ and $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$ (arrows); top middle: $-\boldsymbol{\sigma}_{f,12}$ and $-\boldsymbol{\sigma}_{p,12}$ (color); top right: $-\boldsymbol{\sigma}_{f,22}$ and $-\boldsymbol{\sigma}_{p,22}$ (color). Middle: poroelastic stress $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$ (arrows); middle left: $-\boldsymbol{\sigma}_{p,12}$ (color); middle right: $-\boldsymbol{\sigma}_{p,22}$ (color). Bottom left: displacement $\boldsymbol{\eta}_p$ (arrows), $|\boldsymbol{\eta}_p|$ (color). Bottom right: Darcy pressure p_p.

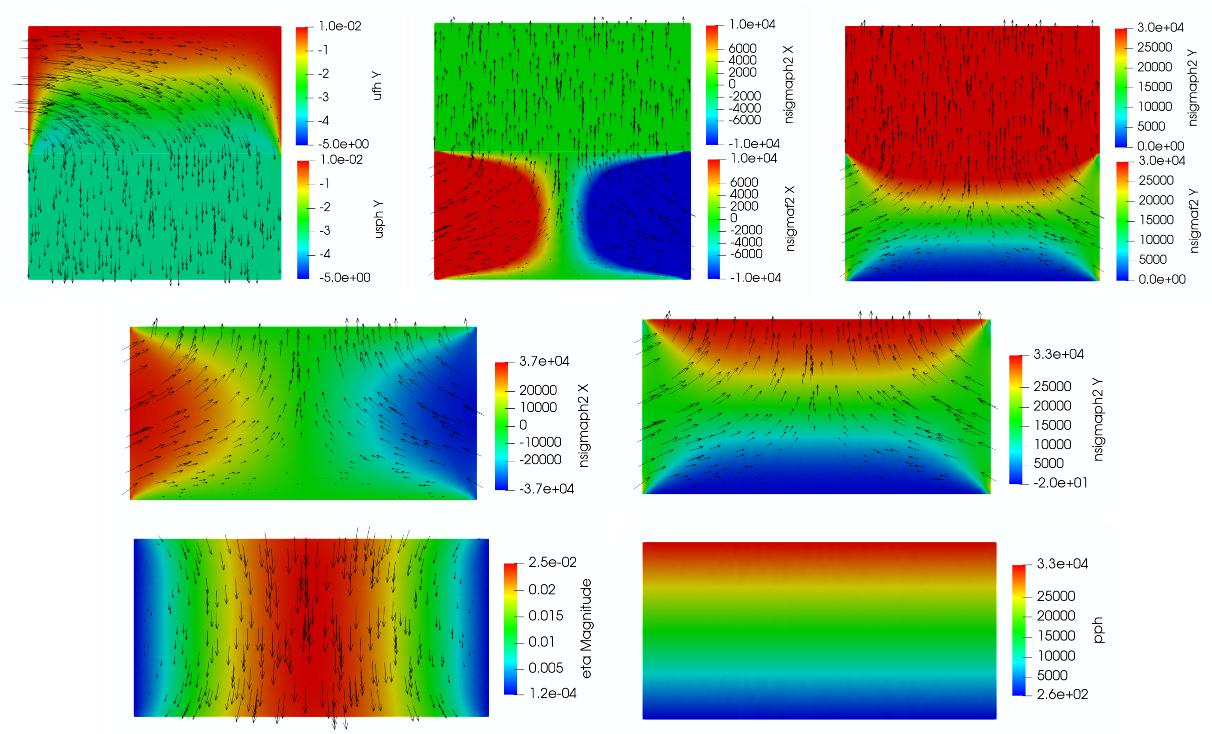


Figure 2.4.3: Example 2, Case 3.

$\mathbf{K} = 10^{-4} \times \mathbf{I}$, $s_0 = 10^{-4}$, $\lambda_p = 10^6$, $\mu_p = 10^6$. Computed solution at final time $T = 3$. Top left: velocities $\mathbf{u}_f$ and $\mathbf{u}_p + \partial_t \boldsymbol{\eta}_p$ (arrows), $\mathbf{u}_{f,2}$ and $\mathbf{u}_{p,2} + \partial_t \boldsymbol{\eta}_{p,2}$ (color). Top middle and right: stresses $-(\boldsymbol{\sigma}_{f,12}, \boldsymbol{\sigma}_{f,22})^{\mathrm{t}}$ and $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$ (arrows); top middle: $-\boldsymbol{\sigma}_{f,12}$ and $-\boldsymbol{\sigma}_{p,12}$ (color); top right: $-\boldsymbol{\sigma}_{f,22}$ and $-\boldsymbol{\sigma}_{p,22}$ (color). Middle: poroelastic stress $-(\boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{p,22})^{\mathrm{t}}$ (arrows); middle left: $-\boldsymbol{\sigma}_{p,12}$ (color); middle right: $-\boldsymbol{\sigma}_{p,22}$ (color). Bottom left: displacement $\boldsymbol{\eta}_p$ (arrows), $|\boldsymbol{\eta}_p|$ (color). Bottom right: Darcy pressure p_p.

3.0 A multipoint stress-flux mixed finite element method for the Stokes-Biot model

3.1 The model problem and weak formulation

The model problem we study in this Chapter is similar to the Stokes-Biot model in Chapter 2. The only difference lies in the fluid region, where we consider a dual mixed formulation. In particular, the flow in Ω_f is governed by the Stokes equations, which are written in the following stress-velocity-pressure formulation:

$$\begin{gathered}
\boldsymbol{\sigma}_f = -p_f \mathbf{I} + 2\mu \mathbf{e}(\mathbf{u}_f), \quad -\mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f, \quad \operatorname{div}(\mathbf{u}_f) = q_f \quad \text{in} \quad \Omega_f \times (0,T], \\
\boldsymbol{\sigma}_f \mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}} \times (0,T], \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{D}} \times (0,T],
\end{gathered} \tag{3.1.1}$$

where $\Gamma_f = \Gamma_f^{\mathrm{N}} \cup \Gamma_f^{\mathrm{D}}$. Since we would like to derive a dual-mixed formulation for the Stokes-Biot model, we adopt the approach from [1, 50], and include as a new variable the vorticity tensor $\boldsymbol{\gamma}_f$,

$$\boldsymbol{\gamma}_f := \frac{1}{2}\left(\nabla \mathbf{u}_f - (\nabla \mathbf{u}_f)^{\mathrm{t}}\right).$$

In this way, owing to the fact that $\operatorname{tr}(\mathbf{e}(\mathbf{u}_f)) = \operatorname{div}(\mathbf{u}_f) = q_f$, we find that (3.1.1) can be rewritten, equivalently, as the set of equations with unknowns $\boldsymbol{\sigma}_f, \boldsymbol{\gamma}_f$ and $\mathbf{u}_f$, given by

$$\begin{gathered}
\frac{1}{2\mu}\boldsymbol{\sigma}_f^{\mathrm{d}} = \nabla \mathbf{u}_f - \boldsymbol{\gamma}_f - \frac{1}{n} q_f \mathbf{I}, \quad -\mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f \quad \text{in} \quad \Omega_f \times (0,T], \\
\boldsymbol{\sigma}_f = \boldsymbol{\sigma}_f^{\mathrm{t}}, \quad p_f = -\frac{1}{n}\left(\operatorname{tr}(\boldsymbol{\sigma}_f) - 2\mu q_f\right) \quad \text{in} \quad \Omega_f \times (0,T], \\
\boldsymbol{\sigma}_f \mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}} \times (0,T], \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{D}} \times (0,T].
\end{gathered} \tag{3.1.2}$$

Notice that the fourth equation in (3.1.2) has allowed us to eliminate the pressure p_f from the system and provides a formula for its approximation through a post-processing procedure. For simplicity we assume that $|\Gamma_f^{\mathrm{N}}| > 0$, which will allow us to control $\boldsymbol{\sigma}_f$ by $\boldsymbol{\sigma}_f^{\mathrm{d}}$. The case $|\Gamma_f^{\mathrm{N}}| = 0$ can be handled as in [50–52] by introducing an additional variable corresponding to the mean value of $\operatorname{tr}(\boldsymbol{\sigma}_f)$.

The Biot system and the interface conditions are exactly the same as the one in Section 2.1 of Chapter 2. We present them here for completeness.

$$
\begin{aligned}
&-\mathbf{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p, \quad \mu\,\mathbf{K}^{-1}\mathbf{u}_p + \nabla\,p_p = \mathbf{0},\\
&\frac{\partial}{\partial t}\big(s_0\,p_p + \alpha_p\,\mathrm{div}(\boldsymbol{\eta}_p)\big) + \mathrm{div}(\mathbf{u}_p) = q_p \quad \text{in} \quad \Omega_p\times(0,T],
\end{aligned} \tag{3.1.3a}
$$

$$
\mathbf{u}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}\times(0,T], \quad p_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}}\times(0,T], \tag{3.1.3b}
$$

$$
\boldsymbol{\sigma}_p\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{N}}\times(0,T], \quad \boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{D}}\times(0,T], \tag{3.1.3c}
$$

$$
\mathbf{u}_f\cdot\mathbf{n}_f + \left(\frac{\partial\,\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot\mathbf{n}_p \;=\; 0, \quad \boldsymbol{\sigma}_f\mathbf{n}_f + \boldsymbol{\sigma}_p\mathbf{n}_p \;=\; \mathbf{0} \quad \text{on} \quad \Gamma_{fp}\times(0,T], \tag{3.1.3d}
$$

$$
\boldsymbol{\sigma}_f\mathbf{n}_f + \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\sqrt{\mathbf{K}_j^{-1}}\left\{\left(\mathbf{u}_f - \frac{\partial\,\boldsymbol{\eta}_p}{\partial t}\right)\cdot\mathbf{t}_{f,j}\right\}\mathbf{t}_{f,j} \;=\; -\,p_p\mathbf{n}_f \quad \text{on} \quad \Gamma_{fp}\times(0,T]. \tag{3.1.3e}
$$

Finally, the above system of equations is complemented by the initial condition $p_p(\mathbf{x},0) = p_{p,0}(\mathbf{x})$ in Ω_p. We stress that, similarly to [65], compatible initial data for the rest of the variables can be constructed from $p_{p,0}$ in a way that all equations in the Stokes-Biot system, except for the unsteady conservation of mass equation in the second row of (3.1.3a), hold at $t = 0$. This will be established in Lemma 3.2.8 below. We will consider a weak formulation with a time-differentiated elasticity equation and compatible initial data $(\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

We then proceed analogously to [4, Section 3] (see also [50]) and derive a weak formulation of the coupled Stokes-Biot problem. For the stress tensor, velocity, and vorticity in the Stokes region, we use the Hilbert spaces, respectively,

$$
\begin{gathered}
\mathbb{X}_f := \Big\{\boldsymbol{\tau}_f \in \mathbb{H}(\mathbf{div};\Omega_f) : \boldsymbol{\tau}_f\mathbf{n}_f = \mathbf{0} \;\text{ on }\; \Gamma_f^{\mathrm{N}}\Big\},\\
\mathbf{V}_f := \mathbf{L}^2(\Omega_f), \quad \mathbb{Q}_f := \Big\{\boldsymbol{\chi}_f \in \mathbb{L}^2(\Omega_f) : \boldsymbol{\chi}_f^{\mathrm{t}} = -\,\boldsymbol{\chi}_f\Big\},
\end{gathered}
$$

endowed with the corresponding norms

$$
\|\boldsymbol{\tau}_f\|_{\mathbb{X}_f} := \|\boldsymbol{\tau}_f\|_{\mathbb{H}(\mathbf{div};\Omega_f)}, \quad \|\mathbf{v}_f\|_{\mathbf{V}_f} := \|\mathbf{v}_f\|_{\mathbf{L}^2(\Omega_f)}, \quad \|\boldsymbol{\chi}_f\|_{\mathbb{Q}_f} := \|\boldsymbol{\chi}_f\|_{\mathbb{L}^2(\Omega_f)}.
$$

In the Biot region, we introduce the structure velocity $\mathbf{u}_s := \partial_t\,\boldsymbol{\eta}_p \in \mathbf{V}_s$ satisfying $\mathbf{u}_s = \mathbf{0}$ on $\tilde{\Gamma}_p^{\mathrm{D}}\times(0,T]$ and the rotation operator $\boldsymbol{\rho}_p := \dfrac{1}{2}(\nabla\boldsymbol{\eta}_p - \nabla\boldsymbol{\eta}_p^{\mathrm{t}})$. Notice that in the weak formulation we will use its time derivative, that is, the structure rotation velocity $\boldsymbol{\gamma}_p := \partial_t \boldsymbol{\rho}_p = \frac{1}{2}\left(\nabla \mathbf{u}_s - (\nabla \mathbf{u}_s)^{\mathrm{t}}\right)$. We introduce the Hilbert spaces:

$$\mathbb{X}_p := \Big\{ \boldsymbol{\tau}_p \in \mathbb{H}(\mathbf{div};\Omega_p) : \boldsymbol{\tau}_p \mathbf{n}_p = \mathbf{0} \ \text{ on } \ \tilde{\Gamma}_p^{\mathrm{N}} \Big\},$$

$$\mathbf{V}_s := \mathbf{L}^2(\Omega_p), \quad \mathbb{Q}_p := \Big\{ \boldsymbol{\chi}_p \in \mathbb{L}^2(\Omega_p) : \boldsymbol{\chi}_p^{\mathrm{t}} = -\,\boldsymbol{\chi}_p \Big\},$$

$$\mathbf{V}_p := \Big\{ \mathbf{v}_p \in \mathbf{H}(\mathrm{div};\Omega_p) : \ \mathbf{v}_p \cdot \mathbf{n}_p = 0 \ \text{ on } \ \Gamma_p^{\mathrm{N}} \Big\}, \quad \mathrm{W}_p := \mathrm{L}^2(\Omega_p),$$

endowed with the standard norms

$$\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} := \|\boldsymbol{\tau}_p\|_{\mathbb{H}(\mathbf{div};\Omega_p)}, \quad \|\mathbf{v}_s\|_{\mathbf{V}_s} := \|\mathbf{v}_s\|_{\mathbf{L}^2(\Omega_p)}, \quad \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p} := \|\boldsymbol{\chi}_p\|_{\mathbb{L}^2(\Omega_p)},$$

$$\|\mathbf{v}_p\|_{\mathbf{V}_p} := \|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)}, \quad \|w_p\|_{\mathrm{W}_p} := \|w_p\|_{\mathrm{L}^2(\Omega_p)}.$$

Finally, analogously to [4, 10, 47, 50, 65] we need to introduce three Lagrange multipliers modeling the Stokes velocity, structure velocity and Darcy pressure on the interface, respectively,

$$\boldsymbol{\varphi} := \mathbf{u}_f|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_f, \quad \boldsymbol{\theta} := \mathbf{u}_s|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_s, \quad \text{and} \quad \lambda := p_p|_{\Gamma_{fp}} \in \Lambda_p.$$

The reason for introducing these Lagrange multipliers is twofold. First, $\mathbf{u}_f$, $\mathbf{u}_s$, and p_p are all modeled in the L^2 space, thus they do not have sufficient regularity for their traces on Γ_{fp} to be well defined. Second, the Lagrange multipliers are utilized to impose weakly the transmission conditions (3.1.3d)–(3.1.3e). For the Lagrange multiplier spaces we need $\Lambda_p := (\mathbf{V}_p \cdot \mathbf{n}_p|_{\Gamma_{fp}})'$, $\boldsymbol{\Lambda}_f := (\mathbb{X}_f\, \mathbf{n}_f|_{\Gamma_{fp}})'$, and $\boldsymbol{\Lambda}_s := (\mathbb{X}_p\, \mathbf{n}_p|_{\Gamma_{fp}})'$. According to the normal trace theorem, it holds that

$$\langle \mathbf{v}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} \leq C \|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)} \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \quad \forall\, \mathbf{v}_p \in \mathbf{V}_p,\ \xi \in \mathrm{H}^{1/2}(\Gamma_{fp}), \tag{3.1.4}$$

and

$$\langle \boldsymbol{\tau}_\star\, \mathbf{n}_\star, \boldsymbol{\psi} \rangle_{\Gamma_{fp}} \leq C \|\boldsymbol{\tau}_\star\|_{\mathbb{H}(\mathrm{div};\Omega_\star)} \|\boldsymbol{\psi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}, \quad \forall\, \boldsymbol{\tau}_\star \in \mathbb{X}_\star,\ \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma_{fp}),\ \star \in \{f, p\}. \tag{3.1.5}$$

Therefore we can take $\Lambda_p := \mathrm{H}^{1/2}(\Gamma_{fp})$, $\boldsymbol{\Lambda}_f := \mathbf{H}^{1/2}(\Gamma_{fp})$, and $\boldsymbol{\Lambda}_s := \mathbf{H}^{1/2}(\Gamma_{fp})$, endowed with the norms

$$\|\xi\|_{\Lambda_p} := \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \quad \|\boldsymbol{\psi}\|_{\boldsymbol{\Lambda}_f} := \|\boldsymbol{\psi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}, \quad \text{and} \quad \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} := \|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}. \tag{3.1.6}$$

We now proceed with the derivation of our Lagrange multiplier variational formulation for the coupling of the Stokes – Biot problems. We adopt the same derivation process in Section 2.1 for the Biot system. Then, similarly to [4,10,50,51], we test the first equation of (3.1.2) with arbitrary $\boldsymbol{\tau}_f \in \mathbb{X}_f$, integrate by parts, utilize the fact that $\boldsymbol{\sigma}_f^{\mathrm{d}} : \boldsymbol{\tau}_f = \boldsymbol{\sigma}_f^{\mathrm{d}} : \boldsymbol{\tau}_f^{\mathrm{d}}$, impose the remaining equations weakly, and utilize the transmission conditions in (3.1.3d)–(3.1.3e) to obtain the variational problem,

$$
\begin{aligned}
&\frac{1}{2\mu}\,(\boldsymbol{\sigma}_f^{\mathrm{d}},\boldsymbol{\tau}_f^{\mathrm{d}})_{\Omega_f} + (\mathbf{u}_f,\mathbf{div}(\boldsymbol{\tau}_f))_{\Omega_f} + (\boldsymbol{\gamma}_f,\boldsymbol{\tau}_f)_{\Omega_f} - \langle \boldsymbol{\tau}_f\mathbf{n}_f,\boldsymbol{\varphi}\rangle_{\Gamma_{fp}} = -\frac{1}{n}\,(q_f\,\mathbf{I},\boldsymbol{\tau}_f)_{\Omega_f},\\
&-\ (\mathbf{v}_f,\mathbf{div}(\boldsymbol{\sigma}_f))_{\Omega_f} = (\mathbf{f}_f,\mathbf{v}_f)_{\Omega_f},\\
&-\ (\boldsymbol{\sigma}_f,\boldsymbol{\chi}_f)_{\Omega_f} = 0,\\
&(\partial_t\,A(\boldsymbol{\sigma}_p + \alpha_p\,p_p\,\mathbf{I}),\boldsymbol{\tau}_p)_{\Omega_p} +\ (\mathbf{u}_s,\mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p} + (\boldsymbol{\gamma}_p,\boldsymbol{\tau}_p)_{\Omega_p} - \langle \boldsymbol{\tau}_p\mathbf{n}_p,\boldsymbol{\theta}\rangle_{\Gamma_{fp}} = 0,\\
&-\ (\mathbf{v}_s,\mathbf{div}(\boldsymbol{\sigma}_p))_{\Omega_p} = (\mathbf{f}_p,\mathbf{v}_s)_{\Omega_p},\\
&-\ (\boldsymbol{\sigma}_p,\boldsymbol{\chi}_p)_{\Omega_p} = 0,\\
&\mu\,(\mathbf{K}^{-1}\mathbf{u}_p,\mathbf{v}_p)_{\Omega_p} - (p_p,\mathrm{div}(\mathbf{v}_p))_{\Omega_p} + \langle \mathbf{v}_p\cdot\mathbf{n}_p,\lambda\rangle_{\Gamma_{fp}} = 0,\\
&(s_0\,\partial_t\,p_p,w_p)_{\Omega_p} + \alpha_p\,(\partial_t\,A(\boldsymbol{\sigma}_p + \alpha_p\,p_p\,\mathbf{I}),w_p\,\mathbf{I})_{\Omega_p} +\ (w_p,\mathrm{div}(\mathbf{u}_p))_{\Omega_p} = (q_p,w_p)_{\Omega_p},\\
&-\ \langle \boldsymbol{\varphi}\cdot\mathbf{n}_f + (\boldsymbol{\theta}+\mathbf{u}_p)\cdot\mathbf{n}_p,\xi\rangle_{\Gamma_{fp}} = 0,\\
&\langle \boldsymbol{\sigma}_f\mathbf{n}_f,\boldsymbol{\psi}\rangle_{\Gamma_{fp}} + \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\left\langle \sqrt{\mathbf{K}_j^{-1}}\,(\boldsymbol{\varphi}-\boldsymbol{\theta})\cdot\mathbf{t}_{f,j},\boldsymbol{\psi}\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} +\ \langle \boldsymbol{\psi}\cdot\mathbf{n}_f,\lambda\rangle_{\Gamma_{fp}} = 0,\\
&\langle \boldsymbol{\sigma}_p\mathbf{n}_p,\boldsymbol{\phi}\rangle_{\Gamma_{fp}} - \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\left\langle \sqrt{\mathbf{K}_j^{-1}}\,(\boldsymbol{\varphi}-\boldsymbol{\theta})\cdot\mathbf{t}_{f,j},\boldsymbol{\phi}\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} +\ \langle \boldsymbol{\phi}\cdot\mathbf{n}_p,\lambda\rangle_{\Gamma_{fp}} = 0.
\end{aligned}
\tag{3.1.7}
$$

The last three equations impose weakly the transmission conditions (3.1.3d)–(3.1.3e). In particular, the equation with test function ξ imposes the mass conservation, the equation with $\boldsymbol{\psi}$ imposes (3.1.3e), which is a combination of balance of normal stress and the BJS condition, while the equation with $\boldsymbol{\phi}$ imposes the conservation of momentum. We emphasize that this is a new formulation. To our knowledge, this is the first fully dual-mixed formulation for the Stokes-Biot problem.

Remark 3.1.1. *The time differentiated equation in the fourth row of* (3.1.7) *allows us to eliminate the displacement variable* $\boldsymbol{\eta}_p$ *and obtain a formulation that uses only* $\mathbf{u}_s$*. As part* *of the analysis we will construct suitable initial data such that, by integrating in time the fourth equation of* (3.1.7)*, we can recover the original equation*

$$(A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\boldsymbol{\eta}_p, \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p} + (\boldsymbol{\rho}_p, \boldsymbol{\tau}_p)_{\Omega_p} - \langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\omega} \rangle_{\Gamma_{fp}} = 0, \tag{3.1.8}$$

where $\boldsymbol{\omega} := \boldsymbol{\eta}_p|_{\Gamma_{fp}}$.

To simplify the notation, we set the following bilinear forms:

$$\begin{gathered}
a_f(\boldsymbol{\sigma}_f, \boldsymbol{\tau}_f) \,:=\, \frac{1}{2\,\mu}\,(\boldsymbol{\sigma}_f^{\mathrm{d}}, \boldsymbol{\tau}_f^{\mathrm{d}})_{\Omega_f}, \quad a_p(\mathbf{u}_p, \mathbf{v}_p) \,:=\, \mu\,(\mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}, \\
a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) \,:=\, (A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})_{\Omega_p}, \\
b_f(\boldsymbol{\tau}_f, \mathbf{v}_f) \,:=\, (\mathbf{div}(\boldsymbol{\tau}_f), \mathbf{v}_f)_{\Omega_f}, \quad b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) \,:=\, (\mathbf{div}(\boldsymbol{\tau}_p), \mathbf{v}_s)_{\Omega_p}, \\
b_p(\mathbf{v}_p, w_p) \,:=\, -\,(\mathrm{div}(\mathbf{v}_p), w_p)_{\Omega_p}, \quad b_\Gamma(\mathbf{v}_p, \xi) \,:=\, \langle \mathbf{v}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}}, \\
b_{\mathrm{sk},\star}(\boldsymbol{\tau}_\star, \boldsymbol{\chi}_\star) \,:=\, (\boldsymbol{\tau}_\star, \boldsymbol{\chi}_\star)_{\Omega_\star}, \quad b_{\mathbf{n}_\star}(\boldsymbol{\tau}_\star, \boldsymbol{\psi}) \,:=\, -\,\langle \boldsymbol{\tau}_\star \mathbf{n}_\star, \boldsymbol{\psi} \rangle_{\Gamma_{fp}}, \text{ with } \star \in \big\{f, p\big\},
\end{gathered} \tag{3.1.9}$$

and

$$\begin{gathered}
c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\psi}, \boldsymbol{\phi}) \,:=\, \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi} - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, (\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}}, \\
c_\Gamma(\boldsymbol{\psi}, \boldsymbol{\phi}; \xi) \,:=\, \langle \boldsymbol{\psi} \cdot \mathbf{n}_f, \xi \rangle_{\Gamma_{fp}} + \langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}}.
\end{gathered} \tag{3.1.10}$$

There are many different ways of ordering the variables in (3.1.7). For the sake of the subsequent analysis, we proceed as in [50] and [4], and adopt one leading to an evolution problem in a doubly-mixed form. Hence, the variational formulation for the system (3.1.7) reads: Given

$$\mathbf{f}_f : [0,T] \to \mathbf{V}_f',\ \mathbf{f}_p : [0,T] \to \mathbf{V}_s',\ q_f : [0,T] \to \mathbb{X}_f',\ q_p : [0,T] \to \mathrm{W}_p', \quad p_{p,0} \in \mathrm{W}_p,\ \boldsymbol{\sigma}_{p,0} \in \mathbb{X}_p,$$

find $(\boldsymbol{\sigma}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p, \boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) : [0,T] \to \mathbb{X}_f \times \mathbf{V}_p \times \mathbb{X}_p \times \mathrm{W}_p \times \boldsymbol{\Lambda}_f \times \boldsymbol{\Lambda}_s \times \Lambda_p \times \mathbf{V}_f \times \mathbf{V}_s \times \mathbb{Q}_f \times \mathbb{Q}_p$, such that $p_p(0) = p_{p,0}$, $\boldsymbol{\sigma}_p(0) = \boldsymbol{\sigma}_{p,0}$ and for a.e. $t \in (0,T)$:

$$\begin{aligned}
& a_f(\boldsymbol{\sigma}_f, \boldsymbol{\tau}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_e(\partial_t\, \boldsymbol{\sigma}_p, \partial_t\, p_p; \boldsymbol{\tau}_p, w_p) + (s_0\, \partial_t\, p_p, w_p)_{\Omega_p} \\
& \quad +\ b_p(\mathbf{v}_p, p_p) - b_p(\mathbf{u}_p, w_p) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\varphi}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_\Gamma(\mathbf{v}_p, \lambda)
\end{aligned}$$

$$
\begin{aligned}
&+\ b_f(\boldsymbol{\tau}_f, \mathbf{u}_f) + b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_f, \boldsymbol{\gamma}_f) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p) \ = \ -\frac{1}{n}\,(q_f\,\mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f} + (q_p, w_p)_{\Omega_p}, \\
&-\ b_{\mathbf{n}_f}(\boldsymbol{\sigma}_f, \boldsymbol{\psi}) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) - b_\Gamma(\mathbf{u}_p, \xi) + c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\psi}, \boldsymbol{\phi}) + c_\Gamma(\boldsymbol{\psi}, \boldsymbol{\phi}; \lambda) - c_\Gamma(\boldsymbol{\varphi}, \boldsymbol{\theta}; \xi) = 0, \\
&-\ b_f(\boldsymbol{\sigma}_f, \mathbf{v}_f) - b_s(\boldsymbol{\sigma}_p, \mathbf{v}_s) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_f, \boldsymbol{\chi}_f) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p) \ = \ (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}, \quad (3.1.11)
\end{aligned}
$$

$\forall\ \boldsymbol{\tau}_f \in \mathbb{X}_f, \mathbf{v}_p \in \mathbf{V}_p, \boldsymbol{\tau}_p \in \mathbb{X}_p, w_p \in \mathrm{W}_p, \boldsymbol{\psi} \in \boldsymbol{\Lambda}_f, \boldsymbol{\phi} \in \boldsymbol{\Lambda}_s, \xi \in \Lambda_p, \mathbf{v}_f \in \mathbf{V}_f, \mathbf{v}_s \in \mathbf{V}_s, \boldsymbol{\chi}_f \in \mathbb{Q}_f, \boldsymbol{\chi}_p \in \mathbb{Q}_p$.

Now, we group the spaces and test functions as follows:

$$
\mathbf{X} := \mathbb{X}_f \times \mathbf{V}_p \times \mathbb{X}_p \times \mathrm{W}_p, \quad \mathbf{Y} := \boldsymbol{\Lambda}_f \times \boldsymbol{\Lambda}_s \times \Lambda_p, \quad \mathbf{Z} := \mathbf{V}_f \times \mathbf{V}_s \times \mathbb{Q}_f \times \mathbb{Q}_p,
$$

$$
\begin{aligned}
\underline{\boldsymbol{\sigma}} &:= (\boldsymbol{\sigma}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda) \in \mathbf{Y}, \quad \underline{\mathbf{u}} := (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}, \\
\underline{\boldsymbol{\tau}} &:= (\boldsymbol{\tau}_f, \mathbf{v}_p, \boldsymbol{\tau}_p, w_p) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \boldsymbol{\phi}, \xi) \in \mathbf{Y}, \quad \underline{\mathbf{v}} := (\mathbf{v}_f, \mathbf{v}_s, \boldsymbol{\chi}_f, \boldsymbol{\chi}_p) \in \mathbf{Z},
\end{aligned}
$$

where the spaces $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are endowed with the norms, respectively,

$$
\begin{aligned}
&\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}} := \|\boldsymbol{\tau}_f\|_{\mathbb{X}_f} + \|\mathbf{v}_p\|_{\mathbf{V}_p} + \|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} + \|w_p\|_{\mathrm{W}_p}, \quad \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} := \|\boldsymbol{\psi}\|_{\boldsymbol{\Lambda}_f} + \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} + \|\xi\|_{\Lambda_p}, \\
&\|\underline{\mathbf{v}}\|_{\mathbf{Z}} := \|\mathbf{v}_f\|_{\mathbf{V}_f} + \|\mathbf{v}_s\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_f\|_{\mathbb{Q}_f} + \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p}.
\end{aligned}
$$

Hence, we can write (3.1.11) in an operator notation as a degenerate evolution problem in a doubly-mixed form:

$$
\begin{aligned}
\frac{\partial}{\partial t}\,\mathcal{E}(\underline{\boldsymbol{\sigma}}(t)) + \mathcal{A}(\underline{\boldsymbol{\sigma}}(t)) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}(t)) + \mathcal{B}'(\underline{\mathbf{u}}(t)) &= \mathbf{F}(t) \quad \text{in} \quad \mathbf{X}', \\
-\mathcal{B}_1(\underline{\boldsymbol{\sigma}}(t)) + \mathcal{C}(\underline{\boldsymbol{\varphi}}(t)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Y}', \qquad (3.1.12) \\
-\mathcal{B}\,(\underline{\boldsymbol{\sigma}}(t)) &= \mathbf{G}(t) \quad \text{in} \quad \mathbf{Z}',
\end{aligned}
$$

where, according to (3.1.9)–(3.1.10), the operators $\mathcal{A} : \mathbf{X} \to \mathbf{X}', \mathcal{B}_1 : \mathbf{X} \to \mathbf{Y}', \mathcal{C} : \mathbf{Y} \to \mathbf{Y}'$, and $\mathcal{B} : \mathbf{X} \to \mathbf{Z}'$, are defined by

$$
\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) &:= a_f(\boldsymbol{\sigma}_f, \boldsymbol{\tau}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) - b_p(\mathbf{u}_p, w_p), \\
\mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\psi}}) &:= b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\psi}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi}) + b_\Gamma(\mathbf{v}_p, \xi), \qquad (3.1.13) \\
\mathcal{C}(\underline{\boldsymbol{\varphi}})(\underline{\boldsymbol{\psi}}) &:= c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\psi}, \boldsymbol{\phi}) + c_\Gamma(\boldsymbol{\psi}, \boldsymbol{\phi}; \lambda) - c_\Gamma(\boldsymbol{\varphi}, \boldsymbol{\theta}; \xi),
\end{aligned}
$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{v}}) \,:=\, b_f(\boldsymbol{\tau}_f, \mathbf{v}_f) + b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_f, \boldsymbol{\chi}_f) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\chi}_p), \tag{3.1.14}$$

whereas the operator $\mathcal{E} : \mathbf{X} \to \mathbf{X}'$ is given by

$$\mathcal{E}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) \,:=\, a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) + (s_0\, p_p, w_p)_{\Omega_p}, \tag{3.1.15}$$

and the functionals $\mathbf{F} \in \mathbf{X}'$, $\mathbf{G} \in \mathbf{Z}'$ are defined as

$$\mathbf{F}(\underline{\boldsymbol{\tau}}) \,:=\, -\frac{1}{n}\,(q_f\,\mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f} + (q_p, w_p)_{\Omega_p} \quad \text{and} \quad \mathbf{G}(\underline{\mathbf{v}}) \,:=\, (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}. \tag{3.1.16}$$

3.2 Well-posedness of the weak formulation

In this section we establish the solvability of (3.1.12) (equivalently (3.1.11)). To that end we first collect some previous results that will be used in the forthcoming analysis.

3.2.1 Preliminaries

We begin by recalling the key result 2.2.3 given in [74, Theorem IV.6.1(b)] that will be used to establish the existence of a solution to (3.1.12). In addition, in order to show the range condition of Theorem 2.2.3 in our context, we will require the following theorem whose proof can be derived similarly to [49, Theorem 2.2] (see also [1, Theorem 3.13] for a generalized nonlinear Banach version).

Theorem 3.2.1. *Let X, Y, and Z be Hilbert spaces, and let X', Y', Z' be their respective duals. Let $A : X \to X'$, $S : Y \to Y'$, $B_1 : X \to Y'$, and $B : X \to Z'$ be linear bounded operators. We also let $B_1' : Y \to X'$ and $B' : Z \to X'$ be the corresponding adjoints. Finally, we let V be the kernel of B, that is*

$$V \,:=\, \Big\{ \boldsymbol{\tau} \in X : \quad B(\boldsymbol{\tau})(\mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in Z \Big\}.$$

Assume that

(i) $A|_V : V \to V'$ *is elliptic, that is, there exists a constant* $\alpha > 0$ *such that*

$$A(\boldsymbol{\tau})(\boldsymbol{\tau}) \,\geq\, \alpha\, \|\boldsymbol{\tau}\|_X^2 \quad \forall\, \boldsymbol{\tau} \in V.$$

(ii) S *is positive semi-definite on* Y*, that is,*

$$S(\boldsymbol{\psi})(\boldsymbol{\psi}) \,\geq\, 0 \quad \forall\, \boldsymbol{\psi} \in Y.$$

(iii) B_1 *satisfies an inf-sup condition on* $V \times Y$*, that is, there exists* $\beta_1 > 0$ *such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau} \in V} \frac{B_1(\boldsymbol{\tau})(\boldsymbol{\psi})}{\|\boldsymbol{\tau}\|_X} \,\geq\, \beta_1\, \|\boldsymbol{\psi}\|_Y \quad \forall\, \boldsymbol{\psi} \in Y.$$

(iv) B *satisfies an inf-sup condition on* $X \times Z$*, that is, there exists* $\beta > 0$ *such that*

$$\sup_{\mathbf{0} \neq \boldsymbol{\tau} \in X} \frac{B(\boldsymbol{\tau})(\mathbf{v})}{\|\boldsymbol{\tau}\|_X} \,\geq\, \beta\, \|\mathbf{v}\|_Z \quad \forall\, \mathbf{v} \in Z.$$

Then, for each $(F_1, F_2, G) \in X' \times Y' \times Z'$ *there exists a unique* $(\boldsymbol{\sigma}, \boldsymbol{\varphi}, \mathbf{u}) \in X \times Y \times Z$*, such that*

$$\begin{aligned}
A(\boldsymbol{\sigma})(\boldsymbol{\tau}) + B_1'(\boldsymbol{\varphi})(\boldsymbol{\tau}) + B'(\mathbf{u})(\boldsymbol{\tau}) \;&=\; F_1(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in X,\\
B_1(\boldsymbol{\sigma})(\boldsymbol{\psi}) - S(\boldsymbol{\varphi})(\boldsymbol{\psi}) \;&=\; F_2(\boldsymbol{\psi}) \quad \forall\, \boldsymbol{\psi} \in Y,\\
B(\boldsymbol{\sigma})(\mathbf{v}) \;&=\; G(\mathbf{v}) \quad\;\; \forall\, \mathbf{v} \in Z.
\end{aligned}$$

Moreover, there exists $C > 0$*, depending only on* $\alpha, \beta_1, \beta, \|A\|, \|S\|$*, and* $\|B_1\|$ *such that*

$$\|(\boldsymbol{\sigma}, \boldsymbol{\varphi}, \mathbf{u})\|_{X \times Y \times Z} \,\leq\, C \Big\{ \|F_1\|_{X'} + \|F_2\|_{Y'} + \|G\|_{Z'} \Big\}.$$

At this point we recall, for later use, that there exist positive constants $c_1(\Omega_f)$ and $c_2(\Omega_f)$, such that (see, [23, Proposition IV.3.1] and [48, Lemma 2.5], respectively)

$$c_1(\Omega_f)\, \|\boldsymbol{\tau}_{f,0}\|_{\mathbb{L}^2(\Omega_f)}^2 \,\leq\, \|\boldsymbol{\tau}_f^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega_f)}^2 + \|\mathbf{div}(\boldsymbol{\tau}_f)\|_{\mathbf{L}^2(\Omega_f)}^2 \quad \forall\, \boldsymbol{\tau}_f = \boldsymbol{\tau}_{f,0} + \ell\, \mathbf{I} \in \mathbb{H}(\mathbf{div}; \Omega_f) \tag{3.2.1}$$

and

$$c_2(\Omega_f)\, \|\boldsymbol{\tau}_f\|_{\mathbb{X}_f}^2 \,\leq\, \|\boldsymbol{\tau}_{f,0}\|_{\mathbb{X}_f}^2 \quad \forall\, \boldsymbol{\tau}_f = \boldsymbol{\tau}_{f,0} + \ell\, \mathbf{I} \in \mathbb{X}_f, \tag{3.2.2}$$

where $\boldsymbol{\tau}_{f,0} \in \mathbb{H}_0(\mathbf{div}; \Omega_f) := \Big\{ \boldsymbol{\tau}_f \in \mathbb{H}(\mathbf{div}; \Omega_f) : \quad (\mathrm{tr}(\boldsymbol{\tau}_f), 1)_{\Omega_f} = 0 \Big\}$ and $\ell \in \mathrm{R}$. We emphasize that (3.2.2) holds since each $\boldsymbol{\tau}_f \in \mathbb{X}_f$ satisfies the boundary condition $\boldsymbol{\tau}_f \mathbf{n}_f = \mathbf{0}$ on Γ_f^{N} with $|\Gamma_f^{\mathrm{N}}| > 0$.

3.2.2 The resolvent system

Now, we proceed to analyze the solvability of (3.1.12) (equivalently (3.1.11)). First, recalling the definition of the operators $\mathcal{A}, \mathcal{B}_1, \mathcal{B}, \mathcal{C}$, and $\mathcal{E}$ (cf. (3.1.13), (3.1.14) and (3.1.15)), we note that problem (3.1.12) can be written in the form of (2.2.11) with

$$
\begin{gathered}
E = \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}, \quad u = \begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{\varphi}} \\ \underline{\mathbf{u}} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} \mathcal{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \\
\mathcal{M} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_1' & \mathcal{B}' \\ -\mathcal{B}_1 & \mathcal{C} & \mathbf{0} \\ -\mathcal{B} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad f = \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{G} \end{pmatrix}.
\end{gathered} \tag{3.2.3}
$$

In addition, the norm induced by the operator $\mathcal{E}$ is $|\underline{\boldsymbol{\tau}}|_{\mathcal{E}}^2 := s_0 \|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$, which is equivalent to $\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{L}^2(\Omega_p)}$ since $s_0 > 0$. We denote by $\mathbb{X}_{p,2}$ and $\mathrm{W}_{p,2}$ the closures of the spaces $\mathbb{X}_p$ and W_p, respectively, with respect to the norms $\|\boldsymbol{\tau}_p\|_{\mathbb{X}_{p,2}} := \|\boldsymbol{\tau}_p\|_{\mathbb{L}^2(\Omega_p)}$ and $\|w_p\|_{\mathrm{W}_{p,2}} := \|w_p\|_{\mathrm{L}^2(\Omega_p)}$. Note that $\mathbb{X}_{p,2}' = \mathbb{L}^2(\Omega_p)$ and $\mathrm{W}_{p,2}' = \mathrm{W}_p'$. Next, denoting $\mathbf{X}_{2,0}' := \mathbf{0} \times \mathbf{0} \times \mathbb{X}_{p,2}' \times \mathrm{W}_{p,2}'$, $\mathbf{Y}_{2,0}' := \mathbf{0} \times \mathbf{0} \times \mathbf{0}$, and $\mathbf{Z}_{2,0}' := \mathbf{0} \times \mathbf{0} \times \mathbf{0} \times \mathbf{0}$, the Hilbert space E_b' and domain $\mathcal{D}$ in Theorem 2.2.3 for our context are

$$
E_b' := \mathbf{X}_{2,0}' \times \mathbf{Y}_{2,0}' \times \mathbf{Z}_{2,0}', \quad \mathcal{D} := \Big\{ (\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} : \quad \mathcal{M}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in E_b' \Big\}. \tag{3.2.4}
$$

Remark 3.2.1. *The above definition of the space E_b' and the corresponding domain $\mathcal{D}$ implies that, in order to apply Theorem 2.2.3 for our problem* (3.1.12)*, we need to restrict $\mathbf{f}_f = \mathbf{0}, q_f = 0$, and $\mathbf{f}_p = \mathbf{0}$. To avoid this restriction we will employ a translation argument [76] to reduce the existence for* (3.1.12) *to existence for the following initial-value problem: Given initial data $(\widehat{\underline{\boldsymbol{\sigma}}}_0, \widehat{\underline{\boldsymbol{\varphi}}}_0, \widehat{\underline{\mathbf{u}}}_0) \in \mathcal{D}$ and source terms $(\widehat{\mathbf{f}}_p, \widehat{q}_p) : [0,T] \to \mathbb{X}_{p,2}' \times \mathrm{W}_{p,2}'$, find $(\widehat{\underline{\boldsymbol{\sigma}}}, \widehat{\underline{\boldsymbol{\varphi}}}, \widehat{\underline{\mathbf{u}}}) \in [0,T] \to \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ such that $(\widehat{\boldsymbol{\sigma}}_p(0), \widehat{p}_p(0)) = (\widehat{\boldsymbol{\sigma}}_{p,0}, \widehat{p}_{p,0})$ and, for a.e. $t \in (0,T)$,*

$$
\begin{aligned}
\frac{\partial}{\partial t}\, \mathcal{E}(\widehat{\underline{\boldsymbol{\sigma}}}(t)) + \mathcal{A}(\widehat{\underline{\boldsymbol{\sigma}}}(t)) + \mathcal{B}_1'(\widehat{\underline{\boldsymbol{\varphi}}}(t)) + \mathcal{B}'(\widehat{\underline{\mathbf{u}}}(t)) &= \widehat{\mathbf{F}}(t) \quad \text{in} \quad \mathbf{X}_{2,0}', \\
-\,\mathcal{B}_1(\widehat{\underline{\boldsymbol{\sigma}}}(t)) + \mathcal{C}(\widehat{\underline{\boldsymbol{\varphi}}}(t)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Y}_{2,0}', \\
-\,\mathcal{B}\,(\widehat{\underline{\boldsymbol{\sigma}}}(t)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Z}_{2,0}',
\end{aligned} \tag{3.2.5}
$$

where $\widehat{\mathbf{F}} = (\mathbf{0}, \mathbf{0}, \widehat{\mathbf{f}}_p, \widehat{q}_p)^{\mathrm{t}}$.

In order to apply Theorem 2.2.3 for problem (3.2.5), we need to: (1) establish the required properties of the operators $\mathcal{N}$ and $\mathcal{M}$, (2) prove the range condition $Rg(\mathcal{N}+\mathcal{M}) = E_b'$, and (3) construct compatible initial data $(\underline{\widehat{\boldsymbol{\sigma}}}_0, \underline{\widehat{\boldsymbol{\varphi}}}_0, \underline{\widehat{\mathbf{u}}}_0) \in \mathcal{D}$. We proceed with a sequence of lemmas establishing these results.

Lemma 3.2.2. *The linear operators $\mathcal{N}$ and $\mathcal{M}$ defined in* (3.2.3) *are continuous and monotone. In addition, $\mathcal{N}$ is symmetric.*

Proof. First, from the definition of the operators $\mathcal{E}, \mathcal{A}, \mathcal{B}_1, \mathcal{C}$ and $\mathcal{B}$ (cf. (3.1.13), (3.1.14), (3.1.15)) it is clear that both $\mathcal{N}$ and $\mathcal{M}$ (cf. (3.2.3)) are linear and continuous, using the trace inequalities (3.1.4)–(3.1.5) for the continuity of $\mathcal{B}_1$. In turn, $\mathcal{N}$ is symmetric since $\mathcal{E}$ is. Finally, using (2.1.6), we have

$$
\begin{aligned}
\mathcal{E}(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) &= s_0 \|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p w_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}, \\
\mathcal{A}(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) &\geq \frac{1}{2\,\mu} \|\boldsymbol{\tau}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \mu\, k_{\max}^{-1} \|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} \quad \forall\, \underline{\boldsymbol{\tau}} \in \mathbf{X},
\end{aligned}
\tag{3.2.6}
$$

and recalling the definition of the operator $\mathcal{C}$ (cf. (3.1.10), (3.1.13)), we obtain

$$
\mathcal{C}(\underline{\boldsymbol{\psi}})(\underline{\boldsymbol{\psi}}) = \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} (\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j}, (\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \geq \frac{\mu\, \alpha_{\mathtt{BJS}}}{\sqrt{k_{\max}}} |\boldsymbol{\psi} - \boldsymbol{\phi}|^2_{\mathtt{BJS}}\,,
\tag{3.2.7}
$$

for all $\underline{\boldsymbol{\psi}} = (\boldsymbol{\psi}, \boldsymbol{\phi}, \xi) \in \mathbf{Y}$, where $|\boldsymbol{\psi} - \boldsymbol{\phi}|^2_{\mathtt{BJS}} := \sum_{j=1}^{n-1} \|(\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j}\|^2_{\mathrm{L}^2(\Gamma_{fp})}$. Thus, combining (3.2.6) and (3.2.7), and the fact that the operators $\mathcal{E}, \mathcal{A}, \mathcal{C}$ are linear, we deduce the monotonicity of the operators $\mathcal{N}$ and $\mathcal{M}$ completing the proof. □

Next, we establish the range condition $Rg(\mathcal{N}+\mathcal{M}) = E_b'$, which is done by solving the related resolvent system. In fact, we will show a stronger result by considering a resolvent system where all source terms in $\mathbf{F}$ and $\mathbf{G}$ may be non-zero. This stronger result will be used in the translation argument for proving existence of the original problem (3.1.12). More precisely, let

$$
\mathbf{X}_2 := \mathbb{X}_f \times \mathbf{V}_p \times \mathbb{X}_{p,2} \times \mathrm{W}_{p,2} \supset \mathbf{X}
$$

and note that $\mathbf{X}_2' = \mathbb{X}_f' \times \mathbf{V}_p' \times \mathbb{X}_{p,2}' \times \mathrm{W}_{p,2}' \subset \mathbf{X}'$. We consider the following resolvent system:

$$
\begin{aligned}
(\mathcal{E}+\mathcal{A})(\underline{\boldsymbol{\sigma}}) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}) + \mathcal{B}'(\underline{\mathbf{u}}) &= \widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{X}_2', \\
-\,\mathcal{B}_1(\underline{\boldsymbol{\sigma}}) + \mathcal{C}(\underline{\boldsymbol{\varphi}}) &= \mathbf{0} \quad \text{in} \quad \mathbf{Y}', \\
-\,\mathcal{B}\,(\underline{\boldsymbol{\sigma}}) &= \widehat{\mathbf{G}} \quad \text{in} \quad \mathbf{Z}',
\end{aligned} \tag{3.2.8}
$$

where $\widehat{\mathbf{F}} \in \mathbf{X}_2'$ and $\widehat{\mathbf{G}} \in \mathbf{Z}'$ are such that

$$
\widehat{\mathbf{F}}(\underline{\boldsymbol{\tau}}) := (\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_f}, \boldsymbol{\tau}_f)_{\Omega_f} + (\widehat{\mathbf{f}}_{\mathbf{u}_p}, \mathbf{v}_p)_{\Omega_p} + (\widehat{\mathbf{f}}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{q}_p, w_p)_{\Omega_p}\,,
$$

$$
\widehat{\mathbf{G}}(\underline{\mathbf{v}}) := (\widehat{\mathbf{f}}_{\mathbf{u}_f}, \mathbf{v}_f)_{\Omega_f} + (\widehat{\mathbf{f}}_{\mathbf{u}_s}, \mathbf{v}_s)_{\Omega_p} + (\widehat{\mathbf{f}}_{\boldsymbol{\gamma}_f}, \boldsymbol{\chi}_f)_{\Omega_f} + (\widehat{\mathbf{f}}_{\boldsymbol{\gamma}_p}, \boldsymbol{\chi}_p)_{\Omega_p}\,.
$$

We next focus on proving that the resolvent system (3.2.8) is well-posed. We start with the following preliminary lemma.

Lemma 3.2.3. *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ *be a solution to* (3.2.8)*. Then, for any positive constant* κ*, it satisfies*

$$
\begin{aligned}
(\mathcal{E}+\widetilde{\mathcal{A}})(\underline{\boldsymbol{\sigma}}) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}) + \mathcal{B}'(\underline{\mathbf{u}}) &= \widetilde{\mathbf{F}} \quad \textit{in} \quad \mathbf{X}_2', \\
\mathcal{B}_1(\underline{\boldsymbol{\sigma}}) - \mathcal{C}(\underline{\boldsymbol{\varphi}}) &= \mathbf{0} \quad \textit{in} \quad \mathbf{Y}', \\
\mathcal{B}\,(\underline{\boldsymbol{\sigma}}) &= -\,\widehat{\mathbf{G}} \quad \textit{in} \quad \mathbf{Z}',
\end{aligned} \tag{3.2.9}
$$

where

$$
\widetilde{\mathcal{A}}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) := \mathcal{A}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) + \kappa \Big\{ (\mathrm{div}(\mathbf{u}_p), \mathrm{div}(\mathbf{v}_p))_{\Omega_p} + \big(s_0\, p_p + \alpha_p \,\mathrm{tr}\big(A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})\big), \mathrm{div}(\mathbf{v}_p)\big)_{\Omega_p} \Big\}, \tag{3.2.10}
$$

and

$$
\widetilde{\mathbf{F}}(\underline{\boldsymbol{\tau}}) := \widehat{\mathbf{F}}(\underline{\boldsymbol{\tau}}) + \kappa\,\big(\widehat{q}_p, \mathrm{div}(\mathbf{v}_p)\big)_{\Omega_p}.
$$

Conversely, if $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ *is a solution to* (3.2.9)*, then it is also a solution to* (3.2.8)*.*

Proof. Let $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ be a solution to (3.2.8). Using that $\mathrm{div}\,\mathbf{V}_p = \mathrm{W}_p$, we take $\underline{\boldsymbol{\tau}} = (\mathbf{0}, w_p) = (\mathbf{0}, \mathrm{div}(\mathbf{v}_p)) \in \mathbf{X}$ in the first row of (3.2.8), multiply by a positive constant κ and add that term to (3.2.8), to obtain (3.2.9). Conversely, if $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ satisfies (3.2.9) we employ similar arguments, but now subtracting, to recover (3.2.8). □

Problem (3.2.9) has the same structure as the one in Theorem 3.2.1. Therefore, in what follows we apply this result to establish the well-posedness of (3.2.9). To that end, we first observe that the kernel of the operator $\mathcal{B}$, cf. (3.1.14), can be written as

$$\mathbf{V} := \Big\{ \underline{\boldsymbol{\tau}} \in \mathbf{X} : \quad \mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{v}}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{Z} \Big\} = \widetilde{\mathbb{X}}_f \times \mathbf{V}_p \times \widetilde{\mathbb{X}}_p \times \mathrm{W}_p \tag{3.2.11}$$

where

$$\widetilde{\mathbb{X}}_\star := \Big\{ \boldsymbol{\tau}_\star \in \mathbb{X}_\star : \quad \boldsymbol{\tau}_\star = \boldsymbol{\tau}_\star^{\mathrm{t}} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}_\star) = \mathbf{0} \quad \text{in} \quad \Omega_\star \Big\}, \quad \star \in \{f, p\}.$$

We next verify the hypotheses of Theorem 3.2.1. We begin by noting that the operators $\widetilde{\mathcal{A}}, \mathcal{B}_1, \mathcal{C}, \mathcal{B}$, and $\mathcal{E}$ are linear and continuous. Next, we proceed with the ellipticity of the operator $\mathcal{E} + \widetilde{\mathcal{A}}$ on $\mathbf{V}$.

Lemma 3.2.4. *Assume that*

$$\kappa \in \left(0, 2 \min\left\{\delta_1, \frac{\delta_2}{\alpha_p}\right\}\right) \quad \textit{with} \quad \delta_1 \in \left(0, \frac{2}{s_0}\right) \quad \textit{and} \quad \delta_2 \in \left(0, \frac{4\mu_{min}}{n\,\alpha_p}\left(1 - \frac{s_0}{2}\,\delta_1\right)\right).$$

Then, the operator $\mathcal{E} + \widetilde{\mathcal{A}}$ is elliptic on $\mathbf{V}$.

Proof. From the definition of $\widetilde{\mathcal{A}}$, cf. (3.2.10), and considering $\underline{\boldsymbol{\tau}} \in \mathbf{V}$ we get

$$\begin{aligned}(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) \;=\; & \frac{1}{2\,\mu}\,\|\boldsymbol{\tau}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \mu\|\mathbf{K}^{-1/2}\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} + s_0\,\|w_p\|^2_{\mathrm{W}_p} \\ & + \|A^{1/2}(\boldsymbol{\tau}_p\alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \kappa\,\|\mathrm{div}(\mathbf{v}_p)\|^2_{\mathrm{L}^2(\Omega_p)} + s_0\,\kappa\,(w_p, \mathrm{div}(\mathbf{v}_p))_{\Omega_p} \\ & + \alpha_p\,\kappa\,(A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I}), A^{1/2}(\mathrm{div}(\mathbf{v}_p)\,\mathbf{I}))_{\Omega_p}.\end{aligned}$$

Hence, using the Cauchy–Schwarz and Young's inequalities, (2.1.6), (2.1.4), and (3.2.1)–(3.2.2), we obtain

$$\begin{aligned}&(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) \\ &\geq \frac{C_{\mathrm{d}}}{2\,\mu}\|\boldsymbol{\tau}_f\|^2_{\mathbb{X}_f} + \mu\,k_{\max}^{-1}\|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \kappa\left(\left(1 - \frac{s_0}{2}\delta_1\right) - \frac{n\,\alpha_p}{4\mu_{\min}}\delta_2\right)\|\mathrm{div}(\mathbf{v}_p)\|^2_{\mathrm{L}^2(\Omega_p)} \\ &\quad + \left(1 - \frac{\alpha_p}{2\,\delta_2}\,\kappa\right)\|A^{1/2}\,(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\left(1 - \frac{\kappa}{2\,\delta_1}\right)\|w_p\|^2_{\mathrm{W}_p},\end{aligned}$$

where $C_{\rm d} := C_1(\Omega_f)\, C_2(\Omega_f)$. Then, using the stipulated hypotheses on δ_1, δ_2 and κ, we can define the positive constants

$$\alpha_1(\Omega_f) := \frac{C_{\rm d}}{2\,\mu}, \quad \alpha_2(\Omega_p) := \min\left\{ \mu\, k_{\max}^{-1}, \kappa\, \left(\left(1 - \frac{s_0}{2}\,\delta_1\right) - \frac{n\,\alpha_p}{4\mu_{\min}}\,\delta_2 \right) \right\},$$

$$\alpha_3(\Omega_p) := \frac{s_0}{2}\,\left(1 - \frac{\kappa}{2\,\delta_1}\right), \quad \alpha_4(\Omega_p) := \min\left\{ \left(1 - \frac{\alpha_p}{2\,\delta_2}\,\kappa\right), \alpha_3(\Omega_p) \right\}$$

which allow us to obtain

$$\begin{aligned}(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) \,\geq\, & \alpha_1(\Omega_f)\, \|\boldsymbol{\tau}_f\|^2_{\mathbb{X}_f} + \alpha_2(\Omega_p)\, \|\mathbf{v}_p\|^2_{\mathbf{V}_p} + \alpha_3(\Omega_p)\, \|w_p\|^2_{\mathrm{W}_p} \\ & + \alpha_4(\Omega_p)\, \left(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{W}_p} \right).\end{aligned} \tag{3.2.12}$$

In turn, from (2.1.4) and using the triangle inequality, we deduce

$$\begin{aligned}\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} \,\leq\, & (2\,\mu_{\max} + n\,\lambda_{\max})\, \left(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}(\alpha_p\, w_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \right) \\ \leq\, & C_p\, \left(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{W}_p} \right),\end{aligned} \tag{3.2.13}$$

where $C_p := (2\,\mu_{\max} + n\,\lambda_{\max})\max\left\{1, \frac{n\,\alpha_p^2}{2\mu_{\min}}\right\}$. A combination of (3.2.12) and (3.2.13), and the fact that $\mathbf{div}(\boldsymbol{\tau}_p) = \mathbf{0}$ in Ω_p, implies

$$(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) \,\geq\, \alpha(\Omega_f, \Omega_p)\, \|\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}} \quad \forall\, \underline{\boldsymbol{\tau}} \in \mathbf{V},$$

with $\alpha(\Omega_f, \Omega_p) := \min\left\{\alpha_1(\Omega_f), \alpha_2(\Omega_p), \alpha_3(\Omega_p), \alpha_4(\Omega_p)/C_p\right\}$, hence $\mathcal{E} + \widetilde{\mathcal{A}}$ is elliptic on $\mathbf{V}$. □

Remark 3.2.2. *To maximize the ellipticity constant $\alpha(\Omega_f, \Omega_p)$, we can choose explicitly the parameter κ by taking the parameters δ_1 and δ_2 as the middle points of their feasible ranges. More precisely, we can simply take*

$$\delta_1 = \frac{1}{s_0}, \quad \delta_2 = \frac{\mu_{min}}{n\,\alpha_p}, \quad \kappa = \min\left\{ \frac{1}{s_0}, \frac{\mu_{\min}}{n\,\alpha_p^2} \right\}.$$

We continue with the verification of the hypotheses of Theorem 3.2.1.

Lemma 3.2.5. *There exist positive constants* β_1 *and* β*, such that*

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{V}} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\psi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \geq \beta_1 \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} \quad \forall\, \underline{\boldsymbol{\psi}} \in \mathbf{Y}, \tag{3.2.14}$$

and

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{X}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{v}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \geq \beta \|\underline{\mathbf{v}}\|_{\mathbf{Z}} \quad \forall\, \underline{\mathbf{v}} \in \mathbf{Z}. \tag{3.2.15}$$

Proof. We begin with the proof of (3.2.14). Due the diagonal character of operator $\mathcal{B}_1$, cf. (3.1.13), we need to show individual inf-sup conditions for $b_{\mathbf{n}_f}$, $b_{\mathbf{n}_p}$, and b_Γ. The inf-sup condition for b_Γ follows from a slight adaptation of the argument in [43, Lemma 3.2] to account for the presence of Dirichlet boundary Γ_p^{D}, using that $\mathrm{dist}\,(\Gamma_p^{\mathrm{D}}, \Gamma_{fp}) \geq s > 0$. The inf-sup conditions for $b_{\mathbf{n}_f}$ and $b_{\mathbf{n}_p}$ follow in a similar way. Since the kernel space $\mathbf{V}$ consists of symmetric and divergence-free tensors, the argument in [43, Lemma 3.2] must be modified to account for that. For example, in Ω_f we solve a problem

$$\mathbf{div}(\mathbf{e}(\mathbf{v}_f)) = \mathbf{0} \ \text{ in } \ \Omega_f, \quad \mathbf{e}(\mathbf{v}_f)\,\mathbf{n}_f = \boldsymbol{\xi} \ \text{ on } \ \Gamma_{fp}\cup\Gamma_f^{\mathrm{N}}, \quad \mathbf{v}_f = \mathbf{0} \ \text{ on } \ \Gamma_f^{\mathrm{D}}, \tag{3.2.16}$$

for given data $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_{fp}\cup\Gamma_f^{\mathrm{N}})$ such that $\boldsymbol{\xi} = \mathbf{0}$ on Γ_f^{N}. We recall that Γ_f^{N} is adjacent to Γ_{fp}. Furthermore, $|\Gamma_f^{\mathrm{D}}| > 0$, which guarantees the solvability of the problem. We refer to [43, Lemma 3.2] for further details.

Finally, proceeding as above, using the diagonal character of operator $\mathcal{B}$, cf. (3.1.14), and employing the theory developed in [48, Section 2.4.3] to our context, we can deduce (3.2.15). □

Now, we are in a position to establish that the resolvent system associated to (3.2.5) is well-posed.

Lemma 3.2.6. *For* $\mathcal{N}, \mathcal{M}$ *and* E_b' *defined in* (3.2.3)–(3.2.4)*, it holds that* $Rg(\mathcal{N}+\mathcal{M}) = E_b'$*, that is, given* $f \in E_b'$*, there exists* $v \in \mathcal{D}$ *such that* $(\mathcal{N}+\mathcal{M})(v) = f$*.*

Proof. Let us consider $\widehat{\mathbf{F}} = (\mathbf{0}, \mathbf{0}, \widehat{\mathbf{f}}_p, \widehat{q}_p)^{\mathrm{t}}$ and $\widehat{\mathbf{G}} = \mathbf{0}$ in (3.2.8)–(3.2.9) and κ as in Lemma 3.2.4. The well-posedness of (3.2.9) follows from (3.2.7), Lemmas 3.2.4 and 3.2.5, and a straightforward application of Theorem 3.2.1 with $A = \mathcal{E} + \widetilde{\mathcal{A}}, B_1 = \mathcal{B}_1, S = \mathcal{C}$, and $B = \mathcal{B}$. Then, employing Lemma 3.2.3 we conclude that there exists a unique solution of the resolvent system of (3.2.5), implying the range condition. □

We are now ready to establish existence for the auxiliary initial value problem (3.2.5), assuming compatible initial data.

Lemma 3.2.7. *For each compatible initial data* $(\widehat{\underline{\boldsymbol{\sigma}}}_0, \widehat{\underline{\boldsymbol{\varphi}}}_0, \widehat{\underline{\mathbf{u}}}_0) \in \mathcal{D}$ *and each* $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathrm{W}^{1,1}(0,T;\mathbb{X}'_{p,2}) \times \mathrm{W}^{1,1}(0,T;\mathrm{W}'_{p,2})$*, the problem* (3.2.5) *has a solution* $(\widehat{\underline{\boldsymbol{\sigma}}}, \widehat{\underline{\boldsymbol{\varphi}}}, \widehat{\underline{\mathbf{u}}}) : [0,T] \to \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ *such that* $(\widehat{\boldsymbol{\sigma}}_p, \widehat{p}_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\widehat{\boldsymbol{\sigma}}_p(0), \widehat{p}_p(0)) = (\widehat{\boldsymbol{\sigma}}_{p,0}, \widehat{p}_{p,0})$.

Proof. The assertion of the lemma follows by applying Theorem 2.2.3 with $E, \mathcal{N}, \mathcal{M}$ defined in (3.2.3), using Lemmas 3.2.2 and 3.2.6. □

We will employ Lemma 3.2.7 to obtain existence of a solution to our problem (3.1.12). To that end, we first construct compatible initial data $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$.

Lemma 3.2.8. *Assume that the initial data* $p_{p,0} \in \mathrm{W}_p \cap \mathrm{H}$*, where*

$$\mathrm{H} := \Big\{ w_p \in \mathrm{H}^1(\Omega_p) : \quad \mathbf{K}\nabla w_p \in \mathbf{H}^1(\Omega_p),\ \mathbf{K}\nabla w_p \cdot \mathbf{n}_p = 0 \ \text{on}\ \Gamma_p^{\mathrm{N}}, \quad w_p = 0 \ \text{on}\ \Gamma_p^{\mathrm{D}} \Big\}. \tag{3.2.17}$$

Then, there exist $\underline{\boldsymbol{\sigma}}_0 := (\boldsymbol{\sigma}_{f,0}, \mathbf{u}_{p,0}, \boldsymbol{\sigma}_{p,0}, p_{p,0}) \in \mathbf{X}$, $\underline{\boldsymbol{\varphi}}_0 := (\boldsymbol{\varphi}_0, \boldsymbol{\theta}_0, \lambda_0) \in \mathbf{Y}$, *and* $\underline{\mathbf{u}}_0 := (\mathbf{u}_{f,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{f,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbf{Z}$ *such that*

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}_0) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}_0) + \mathcal{B}'(\underline{\mathbf{u}}_0) &= \widehat{\mathbf{F}}_0 && \text{in} \quad \mathbf{X}_2', \\ -\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_0) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_0) &= \mathbf{0} && \text{in} \quad \mathbf{Y}', \\ -\mathcal{B}(\underline{\boldsymbol{\sigma}}_0) &= \mathbf{G}(0) && \text{in} \quad \mathbf{Z}', \end{aligned} \tag{3.2.18}$$

where $\widehat{\mathbf{F}}_0 = (q_f(0), \mathbf{0}, \widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0})^{\mathrm{t}} \in \mathbf{X}_2'$*, with suitable* $(\widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0}) \in \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$.

Proof. Following the approach from [4, Lemma 4.15], the initial data is constructed by solving a sequence of well-defined subproblems. We take the following steps.

1. Define $\mathbf{u}_{p,0} := -\dfrac{1}{\mu}\mathbf{K}\nabla p_{p,0}$, with $p_{p,0} \in \mathrm{H}$, cf. (3.2.17). It follows that $\mathbf{u}_{p,0} \in \mathbf{H}(\mathrm{div};\Omega_p)$ and

$$\mu\,\mathbf{K}^{-1}\mathbf{u}_{p,0} = -\nabla p_{p,0}, \quad \mathrm{div}(\mathbf{u}_{p,0}) = -\frac{1}{\mu}\,\mathrm{div}(\mathbf{K}\nabla p_{p,0}) \quad \text{in} \quad \Omega_p, \quad \mathbf{u}_{p,0}\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}. \tag{3.2.19}$$

Next, defining $\lambda_0 := p_{p,0}|_{\Gamma_{fp}} \in \Lambda_p$, (3.2.19) implies

$$a_p(\mathbf{u}_{p,0}, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_{p,0}) + b_\Gamma(\mathbf{v}_p, \lambda_0) = 0 \quad \forall\, \mathbf{v}_p \in \mathbf{V}_p. \tag{3.2.20}$$

2. Define $(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\varphi}_0, \mathbf{u}_{f,0}, \boldsymbol{\gamma}_{f,0}) \in \mathbb{X}_f \times \boldsymbol{\Lambda}_f \times \mathbf{V}_f \times \mathbb{Q}_f$ as the unique solution of the problem

$$\begin{aligned}
&a_f(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\tau}_f) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\varphi}_0) + b_f(\boldsymbol{\tau}_f, \mathbf{u}_{f,0}) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_f, \boldsymbol{\gamma}_{f,0}) = -\frac{1}{n}\,(q_f(0)\,\mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f},\\
&-b_{\mathbf{n}_f}(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\psi}) = -\mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\mathbf{u}_{p,0}\cdot \mathbf{t}_{f,j}, \boldsymbol{\psi}\cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\psi}\cdot\mathbf{n}_f, \lambda_0\rangle_{\Gamma_{fp}},\\
&-b_f(\boldsymbol{\sigma}_{f,0}, \mathbf{v}_f) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\chi}_f) = (\mathbf{f}_f(0), \mathbf{v}_f)_{\Omega_f}
\end{aligned} \tag{3.2.21}$$

for all $(\boldsymbol{\tau}_f, \boldsymbol{\psi}, \mathbf{v}_f, \boldsymbol{\chi}_f) \in \mathbb{X}_f \times \boldsymbol{\Lambda}_f \times \mathbf{V}_f \times \mathbb{Q}_f$. Note that (3.2.21) is well-posed, since it corresponds to the weak solution of the Stokes problem in a mixed formulation and its solvability can be shown using classical Babuška-Brezzi theory. Note also that $\mathbf{u}_{p,0}$ and λ_0 are data for this problem.

3. Define $(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\omega}_0, \boldsymbol{\eta}_{p,0}, \boldsymbol{\rho}_{p,0}) \in \mathbb{X}_p \times \boldsymbol{\Lambda}_s \times \mathbf{V}_s \times \mathbb{Q}_p$, as the unique solution of the problem

$$\begin{aligned}
&(A(\boldsymbol{\sigma}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p} + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\omega}_0) + b_s(\boldsymbol{\tau}_p, \boldsymbol{\eta}_{p,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\rho}_{p,0}) \;=\; -(A(\alpha_p\, p_{p,0}\,\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p}\\
&-b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}) \;=\; \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\mathbf{u}_{p,0}\cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}\cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\phi}\cdot\mathbf{n}_p, \lambda_0\rangle_{\Gamma_{fp}}\\
&-b_s(\boldsymbol{\sigma}_{p,0}, \mathbf{v}_s) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi}_p) \;=\; (\mathbf{f}_p(0), \mathbf{v}_s)_{\Omega_p},
\end{aligned} \tag{3.2.22}$$

for all $(\boldsymbol{\tau}_p, \boldsymbol{\phi}, \mathbf{v}_s, \boldsymbol{\chi}_p) \in \mathbb{X}_p \times \boldsymbol{\Lambda}_s \times \mathbf{V}_s \times \mathbb{Q}_p$. Problem (3.2.22) corresponds to the weak solution of the elasticity problem in a mixed formulation and its solvability can be shown using classical Babuška-Brezzi theory. Note that $p_{p,0}, \mathbf{u}_{p,0}$, and λ_0 are data for this problem. Here $\boldsymbol{\eta}_{p,0}, \boldsymbol{\rho}_{p,0}$, and $\boldsymbol{\omega}_0$ are auxiliary variables that are not part of the constructed initial data. However, they can be used to recover the variables $\boldsymbol{\eta}_p, \boldsymbol{\rho}_p$, and $\boldsymbol{\omega}$ that satisfy the non-differentiated equation (3.1.8).

4. Define $\boldsymbol{\theta}_0 \in \boldsymbol{\Lambda}_s$ as

$$\boldsymbol{\theta}_0 \;:=\; \boldsymbol{\varphi}_0 - \mathbf{u}_{p,0} \quad \text{on} \quad \Gamma_{fp}, \tag{3.2.23}$$

where $\boldsymbol{\varphi}_0$ and $\mathbf{u}_{p,0}$ are data obtained in the previous steps. Note that (3.2.23) implies that the BJS terms in (3.2.21) and (3.2.22) can be rewritten with $\mathbf{u}_{p,0} \cdot \mathbf{t}_{f,j} = (\boldsymbol{\varphi}_0 - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}$ and that the ninth equation in (3.1.7) holds for the initial data, that is,

$$-\left\langle \boldsymbol{\varphi}_0 \cdot \mathbf{n}_f + (\boldsymbol{\theta}_0 + \mathbf{u}_{p,0}) \cdot \mathbf{n}_p, \xi \right\rangle_{\Gamma_{fp}} \;=\; 0 \quad \forall\, \xi \in \Lambda_p. \tag{3.2.24}$$

5. Finally, define $(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$, as the unique solution of the problem

$$\begin{aligned} (A(\widehat{\boldsymbol{\sigma}}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p} + b_s(\boldsymbol{\tau}_p, \mathbf{u}_{s,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_{p,0}) \;&=\; -b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_0) \\ -b_s(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{v}_s) - b_{\mathrm{sk},p}(\widehat{\boldsymbol{\sigma}}_{p,0}, \boldsymbol{\chi}_p) \;&=\; 0, \end{aligned} \tag{3.2.25}$$

for all $(\boldsymbol{\tau}_p, \mathbf{v}_s, \boldsymbol{\chi}_p) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$. Problem (3.2.25) corresponds to the weak solution of the elasticity problem in Ω_p with Dirichlet datum $\boldsymbol{\theta}_0$ on Γ_{fp}.

Combining (3.2.20), (3.2.21), the second and third equations in (3.2.22), (3.2.24), and the first equation in (3.2.25), we obtain $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ satisfying (3.2.18) with

$$(\widehat{\mathbf{f}}_{p,0}, \boldsymbol{\tau}_p)_{\Omega_p} \;=\; -(A(\widehat{\boldsymbol{\sigma}}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p} \quad \text{and} \quad (\widehat{q}_{p,0}, w_p)_{\Omega_p} \;=\; -b_p(\mathbf{u}_{p,0}, w_p). \tag{3.2.26}$$

The above equations imply

$$\|\widehat{\mathbf{f}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{q}_{p,0}\|_{\mathrm{L}^2(\Omega_p)} \leq C\, \left(\|\widehat{\boldsymbol{\sigma}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\mathrm{div}(\mathbf{u}_{p,0})\|_{\mathrm{L}^2(\Omega_p)}\right),$$

hence $(\widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0}) \in \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$, completing the proof. □

3.2.3 The main result

We are now ready to prove the main result of this section.

Theorem 3.2.9. *For each compatible initial data* $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$ *constructed in Lemma 3.2.8 and each*

$$\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}_f'), \quad q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p'),$$

there exists a unique solution of (3.1.12), $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) : [0,T] \to \mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$, *such that* $(\boldsymbol{\sigma}_p, p_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

Proof. For each fixed time $t \in [0,T]$, Lemma 3.2.6 implies that there exists a solution to the resolvent system (3.2.8) with $\widehat{\mathbf{F}} = \mathbf{F}(t)$ and $\widehat{\mathbf{G}} = \mathbf{G}(t)$ defined in (3.1.16). More precisely, there exist $(\underline{\widetilde{\boldsymbol{\sigma}}}(t), \underline{\widetilde{\boldsymbol{\varphi}}}(t), \underline{\widetilde{\mathbf{u}}}(t))$ such that

$$\begin{aligned}
\big(\mathcal{E}+\mathcal{A}\big)(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) + \mathcal{B}_1'(\underline{\widetilde{\boldsymbol{\varphi}}}(t)) + \mathcal{B}'(\underline{\widetilde{\mathbf{u}}}(t)) &= \mathbf{F}(t) \quad \text{in} \quad \mathbf{X}_2', \\
-\mathcal{B}_1(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) + \mathcal{C}(\underline{\widetilde{\boldsymbol{\varphi}}}(t)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Y}', \\
-\mathcal{B}\,(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) &= \mathbf{G}(t) \quad \text{in} \quad \mathbf{Z}'.
\end{aligned} \tag{3.2.27}$$

We look for a solution to (3.1.12) in the form $\underline{\boldsymbol{\sigma}}(t) = \underline{\widetilde{\boldsymbol{\sigma}}}(t) + \underline{\widehat{\boldsymbol{\sigma}}}(t)$, $\underline{\boldsymbol{\varphi}}(t) = \underline{\widetilde{\boldsymbol{\varphi}}}(t) + \underline{\widehat{\boldsymbol{\varphi}}}(t)$, and $\underline{\mathbf{u}}(t) = \underline{\widetilde{\mathbf{u}}}(t) + \underline{\widehat{\mathbf{u}}}(t)$. Subtracting (3.2.27) from (3.1.12) leads to the reduced evolution problem

$$\begin{aligned}
\partial_t\mathcal{E}(\underline{\widehat{\boldsymbol{\sigma}}}(t)) + \mathcal{A}(\underline{\widehat{\boldsymbol{\sigma}}}(t)) + \mathcal{B}_1'(\underline{\widehat{\boldsymbol{\varphi}}}(t)) + \mathcal{B}'(\underline{\widehat{\mathbf{u}}}(t)) &= \mathcal{E}(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) - \partial_t\mathcal{E}(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) \quad \text{in} \quad \mathbf{X}_{2,0}', \\
-\mathcal{B}_1(\underline{\widehat{\boldsymbol{\sigma}}}(t)) + \mathcal{C}(\underline{\widehat{\boldsymbol{\varphi}}}(t)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Y}_{2,0}', \\
-\mathcal{B}\,(\underline{\widehat{\boldsymbol{\sigma}}}(t)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Z}_{2,0}',
\end{aligned} \tag{3.2.28}$$

with initial condition $\underline{\widehat{\boldsymbol{\sigma}}}(0) = \underline{\boldsymbol{\sigma}}_0 - \underline{\widetilde{\boldsymbol{\sigma}}}(0)$, $\underline{\widehat{\boldsymbol{\varphi}}}(0) = \underline{\boldsymbol{\varphi}}_0 - \underline{\widetilde{\boldsymbol{\varphi}}}(0)$, and $\underline{\widehat{\mathbf{u}}}(0) = \underline{\mathbf{u}}_0 - \underline{\widetilde{\mathbf{u}}}(0)$. Subtracting (3.2.27) at $t = 0$ from (3.2.18) gives

$$\begin{aligned}
\mathcal{A}(\underline{\widehat{\boldsymbol{\sigma}}}(0)) + \mathcal{B}_1'(\underline{\widehat{\boldsymbol{\varphi}}}(0)) + \mathcal{B}'(\underline{\widehat{\mathbf{u}}}(0)) &= \mathcal{E}(\underline{\widetilde{\boldsymbol{\sigma}}}(0)) + \widehat{\mathbf{F}}_0 - \mathbf{F}(0) \quad \text{in} \quad \mathbf{X}_{2,0}', \\
-\mathcal{B}_1(\underline{\widehat{\boldsymbol{\sigma}}}(0)) + \mathcal{C}(\underline{\widehat{\boldsymbol{\varphi}}}(0)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Y}_{2,0}', \\
-\mathcal{B}\,(\underline{\widehat{\boldsymbol{\sigma}}}(0)) &= \mathbf{0} \quad \text{in} \quad \mathbf{Z}_{2,0}'.
\end{aligned} \tag{3.2.29}$$

We emphasize that in (3.2.29), $\widehat{\mathbf{F}}_0 - \mathbf{F}(0) = (\mathbf{0}, \mathbf{0}, \widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0} - q_p(0))^{\mathrm{t}} \in \mathbf{X}'_{2,0}$. Thus, $\mathcal{M}(\widehat{\underline{\boldsymbol{\sigma}}}(0), \widehat{\underline{\boldsymbol{\varphi}}}(0), \widehat{\underline{\mathbf{u}}}(0)) \in E'_b$, i.e., $(\widehat{\underline{\boldsymbol{\sigma}}}(0), \widehat{\underline{\boldsymbol{\varphi}}}(0), \widehat{\underline{\mathbf{u}}}(0)) \in \mathcal{D}$ (cf. (3.2.4)). Thus, the reduced evolution problem (3.2.28) is in the form of (3.2.5). According to Lemma 3.2.7, it has a solution, which establishes the existence of a solution to (3.1.12) with the stated regularity satisfying $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

We next show that the solution of (3.1.12) is unique. Since the problem is linear, it is sufficient to prove that the problem with zero data has only the zero solution. Taking $\mathbf{F} = \mathbf{G} = \mathbf{0}$ in (3.1.12) and testing it with the solution $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}})$ yields

$$\begin{aligned} &\frac{1}{2}\,\partial_t \left(\|A^{1/2}\,(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|p_p\|^2_{\mathrm{W}_p} \right) \\ &\quad + \frac{1}{2\,\mu}\,\|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(\mathbf{u}_p, \mathbf{u}_p) + \mathcal{C}(\underline{\boldsymbol{\varphi}})(\underline{\boldsymbol{\varphi}}) \;=\; 0, \end{aligned}$$

which together with (3.2.13), (2.1.6) to bound a_p (cf. (3.1.9)), the semi-definite positive property of $\mathcal{C}$ (cf. (3.2.7)), integrating in time from 0 to $t \in (0, T]$, and using that the initial data is zero, implies

$$\|\boldsymbol{\sigma}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \|p_p\|^2_{\mathrm{W}_p} + \int_0^t \left(\|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{u}_p\|^2_{\mathbf{L}^2(\Omega_p)} \right) ds \leq 0. \tag{3.2.30}$$

It follows from (3.2.30) that $\boldsymbol{\sigma}^{\mathrm{d}}_f(t) = \mathbf{0}, \mathbf{u}_p(t) = \mathbf{0}, \boldsymbol{\sigma}_p(t) = \mathbf{0}$, and $p_p(t) = 0$ for all $t \in (0, T]$.

Now, taking $\underline{\boldsymbol{\tau}} \in \mathbf{V}$ (cf. (3.2.11)) in the first equation of (3.1.12) and employing the inf-sup condition of $\mathcal{B}_1$ (cf. (3.2.14)), with $\underline{\boldsymbol{\psi}} = \underline{\boldsymbol{\varphi}} = (\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda) \in \mathbf{Y}$, yields

$$\widetilde{\beta}\,\|\underline{\boldsymbol{\varphi}}\|_{\mathbf{Y}} \;\leq\; \sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{V}} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\varphi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} = -\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{V}} \frac{(\partial_t\,\mathcal{E} + \mathcal{A})(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} = 0.$$

Thus, $\boldsymbol{\varphi}(t) = \mathbf{0}, \boldsymbol{\theta}(t) = \mathbf{0}$, and $\lambda(t) = 0$ for all $t \in (0, T]$. In turn, from the inf-sup condition of $\mathcal{B}$ (cf. (3.2.15)), with $\underline{\mathbf{v}} = \underline{\mathbf{u}} = (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}$, we get

$$\beta\,\|\underline{\mathbf{u}}\|_{\mathbf{Z}} \leq \sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{X}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{u}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} = -\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{X}} \frac{(\partial_t\,\mathcal{E} + \mathcal{A})(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) + \mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\varphi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \;=\; 0.$$

Therefore, $\mathbf{u}_f(t) = \mathbf{0}, \mathbf{u}_s(t) = \mathbf{0}, \boldsymbol{\gamma}_f(t) = \mathbf{0}$, and $\boldsymbol{\gamma}_p(t) = \mathbf{0}$ for all $t \in (0, T]$. Finally, from the third row in (3.1.11), we have the identity

$$b_f(\boldsymbol{\sigma}_f, \mathbf{v}_f) = 0 \quad \forall\, \mathbf{v}_f \in \mathbf{V}_f.$$

Taking $\mathbf{v}_f = \mathbf{div}(\boldsymbol{\sigma}_f) \in \mathbf{V}_f$, we deduce that $\mathbf{div}(\boldsymbol{\sigma}_f(t)) = \mathbf{0}$ for all $t \in (0, T]$, which combined with the fact that $\boldsymbol{\sigma}_f^{\mathrm{d}}(t) = \mathbf{0}$ for all $t \in (0, T]$, and estimates (3.2.1)–(3.2.2) yields $\boldsymbol{\sigma}_f(t) = \mathbf{0}$ for all $t \in (0, T]$. Then, (3.1.12) has a unique solution. □

Corollary 3.2.10. *The solution of* (3.1.12) *satisfies* $\boldsymbol{\sigma}_f(0) = \boldsymbol{\sigma}_{f,0}, \mathbf{u}_f(0) = \mathbf{u}_{f,0}, \boldsymbol{\gamma}_f(0) = \boldsymbol{\gamma}_{f,0}, \mathbf{u}_p(0) = \mathbf{u}_{p,0}, \boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0$*,* $\lambda(0) = \lambda_0$*, and* $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$*.*

Proof. Let $\overline{\boldsymbol{\sigma}}_f := \boldsymbol{\sigma}_f(0) - \boldsymbol{\sigma}_{f,0}$, with a similar definition and notation for the rest of the variables. Since Theorem 2.2.3 implies that $\mathcal{M}(u) \in \mathrm{L}^\infty(0, T; E_b')$, we can take $t \to 0$ in all equations without time derivatives in (3.2.28), and therefore also in (3.1.12). Using that the initial data $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$ satisfies the same equations at $t = 0$ (cf. (3.2.18)), and that $\overline{\boldsymbol{\sigma}}_p = \mathbf{0}$ and $\overline{p}_p = 0$, we obtain

$$
\begin{aligned}
& \frac{1}{2\mu} (\overline{\boldsymbol{\sigma}}_f^{\mathrm{d}}, \boldsymbol{\tau}_f^{\mathrm{d}})_{\Omega_f} + (\overline{\mathbf{u}}_f, \mathbf{div}(\boldsymbol{\tau}_f))_{\Omega_f} + (\overline{\boldsymbol{\gamma}}_f, \boldsymbol{\tau}_f)_{\Omega_f} - \langle \boldsymbol{\tau}_f \mathbf{n}_f, \overline{\boldsymbol{\varphi}} \rangle_{\Gamma_{fp}} = 0, \\
& \mu \, (\mathbf{K}^{-1} \overline{\mathbf{u}}_p, \mathbf{v}_p)_{\Omega_p} + \left\langle \mathbf{v}_p \cdot \mathbf{n}_p, \overline{\lambda} \right\rangle_{\Gamma_{fp}} = 0, \\
& -\, (\mathbf{v}_f, \mathbf{div}(\overline{\boldsymbol{\sigma}}_f))_{\Omega_f} = 0, \\
& -\, (\overline{\boldsymbol{\sigma}}_f, \boldsymbol{\chi}_f)_{\Omega_f} = 0, \\
& -\, \left\langle \overline{\boldsymbol{\varphi}} \cdot \mathbf{n}_f + \left(\overline{\boldsymbol{\theta}} + \overline{\mathbf{u}}_p\right) \cdot \mathbf{n}_p, \xi \right\rangle_{\Gamma_{fp}} = 0, \\
& \langle \overline{\boldsymbol{\sigma}}_f \mathbf{n}_f, \boldsymbol{\psi} \rangle_{\Gamma_{fp}} + \mu \, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} \left(\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}\right) \cdot \mathbf{t}_{f,j}, \boldsymbol{\psi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\psi} \cdot \mathbf{n}_f, \overline{\lambda} \right\rangle_{\Gamma_{fp}} = 0, \\
& -\, \mu \, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} \left(\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}\right) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \overline{\lambda} \right\rangle_{\Gamma_{fp}} = 0.
\end{aligned}
\tag{3.2.31}
$$

Taking $(\boldsymbol{\tau}_f, \mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\chi}_f, \xi, \boldsymbol{\psi}, \boldsymbol{\phi}) = (\overline{\boldsymbol{\sigma}}_f, \overline{\mathbf{u}}_p, \overline{\mathbf{u}}_f, \overline{\boldsymbol{\gamma}}_f, \overline{\lambda}, \overline{\boldsymbol{\varphi}}, \overline{\boldsymbol{\theta}})$ and combining the equations results in

$$
\|\overline{\boldsymbol{\sigma}}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \|\overline{\mathbf{u}}_p\|^2_{\mathbf{L}^2(\Omega_p)} + |\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \le 0 \,, \tag{3.2.32}
$$

implying $\overline{\boldsymbol{\sigma}}_f^{\mathrm{d}} = \mathbf{0}, \overline{\mathbf{u}}_p = \mathbf{0}$, and $(\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}) \cdot \mathbf{t}_{f,j} = 0$. The inf-sup conditions (3.2.14)–(3.2.15), together with (3.2.31), imply that $\overline{\mathbf{u}}_f = \mathbf{0}, \overline{\boldsymbol{\gamma}}_f = 0, \overline{\boldsymbol{\varphi}} = \mathbf{0}$, and $\overline{\lambda} = 0$. Then (3.2.32) yields $\overline{\boldsymbol{\theta}} \cdot \mathbf{t}_{f,j} = 0$. In turn, the fifth equation in (3.2.31) implies that $\left\langle \overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p, \xi \right\rangle_{\Gamma_{fp}} = 0$ for all $\xi \in \mathrm{H}^{1/2}(\Gamma_{fp})$. Note that $\mathbf{n}_p$ may be discontinuous on Γ_{fp}, thus $\overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p \in \mathrm{L}^2(\Gamma_{fp})$. Since

$\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, then $\overline{\boldsymbol{\theta}}\cdot\mathbf{n}_p = 0$, and we conclude that $\overline{\boldsymbol{\theta}} = \mathbf{0}$. In addition, taking $\mathbf{v}_f = \mathbf{div}(\overline{\boldsymbol{\sigma}}_f) \in \mathbf{V}_f$ in the third equation of (3.2.31) we deduce that $\mathbf{div}(\overline{\boldsymbol{\sigma}}_f) = \mathbf{0}$, which, combined with (3.2.1)–(3.2.2), yields $\overline{\boldsymbol{\sigma}}_f = \mathbf{0}$, completing the proof. □

Remark 3.2.3. *As we noted in Remark 3.1.1, the fourth equation in* (3.1.7) *can be used to recover the non-differentiated equation* (3.1.8)*. In particular, recalling the initial data construction* (3.2.22)*, let*

$$\forall\, t \in [0,T], \quad \boldsymbol{\eta}_p(t) = \boldsymbol{\eta}_{p,0} + \int_0^t \mathbf{u}_s(s)\,ds,\ \boldsymbol{\rho}_p(t) = \boldsymbol{\rho}_{p,0} + \int_0^t \boldsymbol{\gamma}_p(s)\,ds,\ \boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \int_0^t \boldsymbol{\theta}(s)\,ds.$$

Then (3.1.8) *follows from integrating the fourth equation in* (3.1.7) *from* 0 *to* $t \in (0,T]$ *and using the first equation in* (3.2.22)*.*

We end this section with a stability bound for the solution of (3.1.12). We will use the inf-sup condition

$$\|p_p\|_{\mathrm{W}_p} + \|\lambda\|_{\Lambda_p} \,\leq\, c \sup_{\mathbf{0}\neq\mathbf{v}_p\in\mathbf{V}_p} \frac{b_p(\mathbf{v}_p,p_p) + b_\Gamma(\mathbf{v}_p,\lambda)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}}, \tag{3.2.33}$$

which follows from a slight adaptation of [52, Lemma 3.3].

Theorem 3.2.11. *For the solution of* (3.1.12)*, assuming sufficient regularity of the data, there exists a positive constant C independent of s_0 such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma}_f\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|\boldsymbol{\sigma}_f\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_p\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{u}_p\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} + |\boldsymbol{\varphi}-\boldsymbol{\theta}|_{\mathrm{L}^\infty(0,T;\mathtt{BJS})} \\
&\quad + |\boldsymbol{\varphi}-\boldsymbol{\theta}|_{\mathrm{L}^2(0,T;\mathtt{BJS})} + \|\lambda\|_{\mathrm{L}^\infty(0,T;\Lambda_p)} + \|\underline{\boldsymbol{\varphi}}\|_{\mathrm{L}^2(0,T;\mathbf{Y})} + \|\underline{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{Z})} + \|A^{1/2}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} + \|p_p\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_p\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\quad + \|\partial_t\, A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \sqrt{s_0}\|\partial_t\, p_p\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\leq C\Big(\|\mathbf{f}_f\|_{\mathrm{H}^1(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{\mathrm{H}^1(0,T;\mathbf{V}_s')} + \|q_f\|_{\mathrm{H}^1(0,T;\mathbb{X}_f')} + \|q_p\|_{\mathrm{H}^1(0,T;\mathrm{W}_p')} \\
&\qquad + (1+\sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \|\mathbf{K}\nabla p_{p,0}\|_{\mathrm{H}^1(\Omega_p)}\Big).
\end{aligned} \tag{3.2.34}$$

Proof. We begin by choosing $(\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}, \underline{\mathbf{v}}) = (\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}})$ in (3.1.11) to get

$$\begin{aligned}
&\frac{1}{2}\,\partial_t \Big(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\, p_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|p_p\|^2_{\mathrm{W}_p} \Big) + \frac{1}{2\,\mu}\,\|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} \\
&\quad + a_p(\mathbf{u}_p, \mathbf{u}_p) + c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\varphi}, \boldsymbol{\theta}) \\
&\quad = -\frac{1}{n}\,(q_f\,\mathbf{I}, \boldsymbol{\sigma}_f)_{\Omega_f} + (q_p, p_p)_{\Omega_p} + (\mathbf{f}_f, \mathbf{u}_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{u}_s)_{\Omega_p}. \qquad (3.2.35)
\end{aligned}$$

Next, we integrate (3.2.35) from 0 to $t \in (0, T]$, use the coercivity bounds (3.2.6)–(3.2.7), and apply the Cauchy–Schwarz and Young's inequalities, to find

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\, p_p\,\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|p_p(t)\|^2_{\mathrm{W}_p} + \int_0^t \Big(\|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{u}_p\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\varphi} - \boldsymbol{\theta}|^2_{\mathtt{BJS}} \Big)\, ds \\
&\quad \le C \Bigg(\int_0^t \Big(\|\mathbf{f}_f\|^2_{\mathbf{V}'_f} + \|\mathbf{f}_p\|^2_{\mathbf{V}'_s} + \|q_f\|^2_{\mathbb{X}'_f} + \|q_p\|^2_{\mathrm{W}'_p} \Big)\, ds + \|A^{1/2}(\boldsymbol{\sigma}_p(0) + \alpha_p\, p_p(0)\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\qquad + s_0\,\|p_p(0)\|^2_{\mathrm{W}_p} \Bigg) + \delta \int_0^t \Big(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{X}_f} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s} \Big)\, ds, \qquad (3.2.36)
\end{aligned}$$

where $\delta > 0$ will be suitably chosen. In addition, (3.2.33) and the first equation in (3.1.11), yields

$$\|p_p\|_{\mathrm{W}_p} + \|\lambda\|_{\Lambda_p} \le c \sup_{\mathbf{0} \neq \mathbf{v}_p \in \mathbf{V}_p} \frac{b_p(\mathbf{v}_p, p_p) + b_\Gamma(\mathbf{v}_p, \lambda)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}} = -c \sup_{\mathbf{0} \neq \mathbf{v}_p \in \mathbf{V}_p} \frac{a_p(\mathbf{u}_p, \mathbf{v}_p)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}} \le C\,\|\mathbf{u}_p\|_{\mathbf{L}^2(\Omega_p)}. \qquad (3.2.37)$$

Taking $\underline{\boldsymbol{\tau}} \in \mathbf{V}$ (cf. (3.2.11)) in the first equation of (3.1.12), using the continuity of the operators $\mathcal{E}$ and $\mathcal{A}$ in Lemma 3.2.2, and the inf-sup condition of $\mathcal{B}_1$ for $\underline{\boldsymbol{\varphi}} \in \mathbf{Y}$ (cf. (3.2.14)), we deduce

$$\begin{aligned}
&\beta_1\,\|\underline{\boldsymbol{\varphi}}\|_{\mathbf{Y}} \le \sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{V}} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\varphi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} = -\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{V}} \frac{(\partial_t\,\mathcal{E} + \mathcal{A})(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) - \mathbf{F}(\underline{\boldsymbol{\tau}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \\
&\le C\big(\|\boldsymbol{\sigma}_f\|_{\mathbb{X}_f} + \|\mathbf{u}_p\|_{\mathbf{V}_p} + \|\partial_t\, A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \qquad (3.2.38) \\
&\quad + \sqrt{s_0}\|\partial_t\, p_p\|_{\mathrm{W}_p} + \|q_f\|_{\mathbb{X}'_f} + \|q_p\|_{\mathrm{W}'_p} \big).
\end{aligned}$$

In turn, from the first equation in (3.1.12), applying the inf-sup condition of $\mathcal{B}$ (cf. (3.2.15)) for $\underline{\mathbf{u}} = (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}$, and (3.2.38), we obtain

$$\begin{aligned}
\beta\,\|\underline{\mathbf{u}}\|_{\mathbf{Z}} \,&\leq\, \sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{X}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{u}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \,=\, -\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{X}} \frac{(\partial_t\,\mathcal{E} + \mathcal{A})(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) + \mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\varphi}}) - \mathbf{F}(\underline{\boldsymbol{\tau}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \\
&\leq C\big(\|\boldsymbol{\sigma}_f\|_{\mathbb{X}_f} + \|\mathbf{u}_p\|_{\mathbf{V}_p} + \|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \sqrt{s_0}\|\partial_t\,p_p\|_{\mathrm{W}_p} + \|q_f\|_{\mathbb{X}_f'} + \|q_p\|_{\mathrm{W}_p'}\big).
\end{aligned} \tag{3.2.39}$$

In addition, taking $w_p = \mathrm{div}(\mathbf{u}_p)$, $\mathbf{v}_f = \mathbf{div}(\boldsymbol{\sigma}_f)$, and $\mathbf{v}_s = \mathbf{div}(\boldsymbol{\sigma}_p)$ in the first and third equations of (3.1.11), we get

$$\begin{aligned}
\|\mathbf{div}(\boldsymbol{\sigma}_f)\|_{\mathbf{L}^2(\Omega_f)} \,&\leq\, \|\mathbf{f}_f\|_{\mathbf{V}_f'}, \quad \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathbf{L}^2(\Omega_p)} \,\leq\, \|\mathbf{f}_p\|_{\mathbf{V}_s'}\,, \\
\|\mathrm{div}(\mathbf{u}_p)\|_{\mathrm{L}^2(\Omega_p)} \,&\leq\, C\left(\|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \sqrt{s_0}\|\partial_t\,p_p\|_{\mathrm{W}_p} + \|q_p\|_{\mathrm{W}_p'}\right).
\end{aligned} \tag{3.2.40}$$

Then, combining (3.2.36)–(3.2.40), using (3.2.1)–(3.2.2), and choosing δ small enough, we obtain

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|p_p(t)\|^2_{\mathrm{W}_p} \\
&\quad + \int_0^t \Big(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p\|^2_{\mathbf{V}_p} + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_p\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi} - \boldsymbol{\theta}|^2_{\mathtt{BJS}} + \|\underline{\boldsymbol{\varphi}}\|^2_{\mathbf{Y}} + \|\underline{\mathbf{u}}\|^2_{\mathbf{Z}}\Big)\,ds \\
&\leq C\Bigg(\int_0^t \Big(\|\mathbf{f}_f\|^2_{\mathbf{V}_f'} + \|\mathbf{f}_p\|^2_{\mathbf{V}_s'} + \|q_f\|^2_{\mathbb{X}_f'} + \|q_p\|^2_{\mathrm{W}_p'}\Big)\,ds + \|A^{1/2}(\boldsymbol{\sigma}_p(0) + \alpha_p\,p_p(0)\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + s_0\,\|p_p(0)\|^2_{\mathrm{W}_p} + \int_0^t \Big(\|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\,p_p\|^2_{\mathrm{W}_p}\Big)\,ds\Bigg).
\end{aligned} \tag{3.2.41}$$

Finally, in order to bound the last two terms in (3.2.41), we test (3.1.11) with $\underline{\boldsymbol{\tau}} = (\partial_t\,\boldsymbol{\sigma}_f, \mathbf{u}_p, \partial_t\,\boldsymbol{\sigma}_p, \partial_t\,p_p) \in \mathbf{X}$, $\underline{\boldsymbol{\psi}} = (\boldsymbol{\varphi}, \boldsymbol{\theta}, \partial_t\,\lambda) \in \mathbf{Y}$, $\underline{\mathbf{v}} = (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}$ and differentiate in time the rows in (3.1.11) associated to $\mathbf{v}_p, \boldsymbol{\psi}, \boldsymbol{\phi}, \mathbf{v}_f, \mathbf{v}_s, \boldsymbol{\chi}_f$ and $\boldsymbol{\chi}_p$, to deduce

$$\begin{aligned}
&\frac{1}{2}\,\partial_t\left(\frac{1}{2\,\mu}\,\|\boldsymbol{\sigma}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(\mathbf{u}_p, \mathbf{u}_p) + c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\varphi}, \boldsymbol{\theta})\right) + \|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\,p_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + s_0\,\|\partial_t\,p_p\|^2_{\mathrm{W}_p} \,=\, \frac{1}{n}\,(q_f\,\mathbf{I}, \partial_t\,\boldsymbol{\sigma}_f)_{\Omega_f} + (q_p, \partial_t\,p_p)_{\Omega_p} + (\partial_t\,\mathbf{f}_f, \mathbf{u}_f)_{\Omega_f} + (\partial_t\,\mathbf{f}_p, \mathbf{u}_s)_{\Omega_p},
\end{aligned}$$

which together with the identities

$$\int_0^t (q_f\,\mathbf{I}, \partial_t\,\boldsymbol{\sigma}_f)_{\Omega_f} = (q_f\,\mathbf{I}, \boldsymbol{\sigma}_f)_{\Omega_f}\Big|_0^t - \int_0^t (\partial_t\,q_f\,\mathbf{I}, \boldsymbol{\sigma}_f)_{\Omega_f},$$
$$\int_0^t (q_p, \partial_t\,p_p)_{\Omega_p} = (q_p, p_p)_{\Omega_p}\Big|_0^t - \int_0^t (\partial_t\,q_p, p_p)_{\Omega_p},$$

and the positive semi-definite property of $\mathcal{C}$ (cf. (3.2.7)), yields

$$\begin{aligned}
&\|\boldsymbol{\sigma}_f^{\mathrm{d}}(t)\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{u}_p(t)\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\varphi}(t) - \boldsymbol{\theta}(t)|^2_{\mathtt{BJS}} \\
&\quad + \int_0^t \left(\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|\partial_t p_p\|^2_{\mathrm{W}_p}\right) ds \\
&\le C \Bigg(\int_0^t \left(\|\partial_t\,\mathbf{f}_f\|^2_{\mathbf{V}_f'} + \|\partial_t\,\mathbf{f}_p\|^2_{\mathbf{V}_s'} + \|\partial_t\,q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\partial_t\,q_p\|^2_{\mathrm{W}_p'}\right) ds + \|q_f(t)\|^2_{\mathbb{X}_f'} + \|q_p(t)\|^2_{\mathrm{W}_p'} \\
&\quad + \|q_f(0)\|^2_{\mathbb{X}_f'} + \|q_p(0)\|^2_{\mathrm{W}_p'} + \|\boldsymbol{\sigma}_f(0)\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_p(0)\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}(0) - \boldsymbol{\theta}(0)|^2_{\mathtt{BJS}} \Bigg) \\
&\quad + \delta_1 \left(\|\boldsymbol{\sigma}_f(t)\|^2_{\mathbb{X}_f} + \|p_p(t)\|^2_{\mathrm{W}_p}\right) + \delta_2 \int_0^t \left(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s}\right) ds.
\end{aligned} \tag{3.2.42}$$

Using (3.2.37) and the first two inequalities in (3.2.40), and choosing δ_1 small enough, we derive from (3.2.42) and (3.2.1)–(3.2.2) that

$$\begin{aligned}
&\|\boldsymbol{\sigma}_f(t)\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{div}(\boldsymbol{\sigma}_p(t))\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\varphi}(t) - \boldsymbol{\theta}(t)|^2_{\mathtt{BJS}} + \|p_p(t)\|^2_{\mathrm{W}_p} + \|\lambda(t)\|^2_{\Lambda_p} \\
&\quad + \int_0^t \left(\|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|\partial_t\,p_p\|^2_{\mathrm{W}_p}\right) ds \\
&\le C \Bigg(\int_0^t \left(\|\partial_t\,\mathbf{f}_f\|^2_{\mathbf{V}_f'} + \|\partial_t\,\mathbf{f}_p\|^2_{\mathbf{V}_s'} + \|\partial_t\,q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\partial_t q_p\|^2_{\mathrm{W}_p'}\right) ds + \|\mathbf{f}_f(t)\|^2_{\mathbf{V}_f'} + \|\mathbf{f}_p(t)\|^2_{\mathbf{V}_s'} \\
&\quad + \|q_f(t)\|^2_{\mathbb{X}_f'} + \|q_p(t)\|^2_{\mathrm{W}_p'} + \|q_f(0)\|^2_{\mathbb{X}_f'} + \|q_p(0)\|^2_{\mathrm{W}_p'} + \|\boldsymbol{\sigma}_f(0)\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p(0)\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \|p_p(0)\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}(0) - \boldsymbol{\theta}(0)|^2_{\mathtt{BJS}} \Bigg) + \delta_2 \int_0^t \left(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{X}_f} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s}\right) ds.
\end{aligned} \tag{3.2.43}$$

We next bound the initial data terms in (3.2.41) and (3.2.43). Recalling from Corollary 3.2.10 that $(\underline{\boldsymbol{\sigma}}(0), \boldsymbol{\varphi}(0), \boldsymbol{\theta}(0)) = (\underline{\boldsymbol{\sigma}}_0, \boldsymbol{\varphi}_0, \boldsymbol{\theta}_0)$, using the stability of the continuous initial data problems (3.2.19)–(3.2.22) and the steady-state version of the arguments leading to (3.2.41), we obtain

$$\begin{aligned} &\|\boldsymbol{\sigma}_f(0)\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\sigma}_p(0))\|^2_{\mathbb{L}^2(\Omega_p)} + \|p_p(0)\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}(0) - \boldsymbol{\theta}(0)|^2_{\mathtt{BJS}} \\ &\quad \leq C \left(\|p_{p,0}\|^2_{\mathrm{W}_p} + \|\mathbf{K}\nabla p_{p,0}\|^2_{\mathrm{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|^2_{\mathbf{V}_f'} + \|\mathbf{f}_p(0)\|^2_{\mathbf{V}_s'} + \|q_f(0)\|^2_{\mathbb{X}_f'} \right), \end{aligned} \tag{3.2.44}$$

Therefore, combining (3.2.41) with (3.2.43) and (3.2.44), choosing δ_2 small enough, and using the estimate (cf. (3.2.13)):

$$\|A^{1/2}(\boldsymbol{\sigma}_p(t))\|_{\mathbb{L}^2(\Omega_p)} \leq C \left(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p \, p_p \, \mathbf{I})(t)\|_{\mathbb{L}^2(\Omega_p)} + \|p_p(t)\|_{\mathrm{W}_p} \right), \tag{3.2.45}$$

and the Sobolev embedding of $\mathrm{H}^1(0,T)$ into $\mathrm{L}^\infty(0,T)$, we conclude (3.2.34). □

3.3 Semi-discrete formulation

In this section we introduce and analyze the semidiscrete continuous-in-time approximation of (3.1.12). We analyze its solvability by employing the strategy developed in Section 3.2. In addition, we derive error estimates with rates of convergence.

3.3.1 Semi-discrete continuous-in-time formulation

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular and quasi-uniform affine finite element partitions of Ω_f and Ω_p, respectively. The two partitions may be non-matching along the interface Γ_{fp}. For the discretization, we consider the following conforming finite element spaces:

$$\mathbb{X}_{fh} \times \mathbf{V}_{fh} \times \mathbb{Q}_{fh} \subset \mathbb{X}_f \times \mathbf{V}_f \times \mathbb{Q}_f, \quad \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \subset \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p, \quad \mathbf{V}_{ph} \times \mathrm{W}_{ph} \subset \mathbf{V}_p \times \mathrm{W}_p.$$

We take $(\mathbb{X}_{fh}, \mathbf{V}_{fh}, \mathbb{Q}_{fh})$ and $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph})$ to be any stable finite element spaces for mixed elasticity with weakly imposed stress symmetry, such as the Amara–Thomas [3], PEERS [12], Stenberg [77], Arnold–Falk–Winther [13, 15], or Cockburn–Gopalakrishnan–Guzman

[40] families of spaces. We choose $(\mathbf{V}_{ph}, \mathrm{W}_{ph})$ to be any stable mixed finite element Darcy spaces, such as the Raviart–Thomas or Brezzi-Douglas-Marini spaces [23]. For the Lagrange multipliers $(\mathbf{\Lambda}_{fh}, \mathbf{\Lambda}_{sh}, \Lambda_{ph})$ we consider the following two options of discrete spaces.

(S1) Conforming spaces:

$$\mathbf{\Lambda}_{fh} \subset \mathbf{\Lambda}_f, \quad \mathbf{\Lambda}_{sh} \subset \mathbf{\Lambda}_s, \quad \Lambda_{ph} \subset \Lambda_p\,, \tag{3.3.1}$$

equipped with $\mathrm{H}^{1/2}$-norms as in (3.1.6). If the normal traces of the spaces $\mathbb{X}_{fh}$, $\mathbb{X}_{ph}$, or $\mathbf{V}_{ph}$ contain piecewise polynomials in P_k on simplices or Q_k on cubes with $k \geq 1$, where P_k denotes polynomials of total degree k and Q_k stands for polynomials of degree k in each variable, we take the Lagrange multiplier spaces to be continuous piecewise polynomials in P_k or Q_k on the traces of the corresponding subdomain grids. In the case of $k = 0$, we take the Lagrange multiplier spaces to be continuous piecewise polynomials in P_1 or Q_1 on grids obtained by coarsening by two the traces of the subdomain grids.

(S2) Non-conforming spaces:

$$\mathbf{\Lambda}_{fh} := \mathbb{X}_{fh}\mathbf{n}_f|_{\Gamma_{fp}}, \quad \mathbf{\Lambda}_{sh} := \mathbb{X}_{ph}\mathbf{n}_p|_{\Gamma_{fp}}, \quad \Lambda_{ph} := \mathbf{V}_{ph}\cdot\mathbf{n}_p|_{\Gamma_{fp}}\,, \tag{3.3.2}$$

which consist of discontinuous piecewise polynomials and are equipped with L^2-norms.

It is also possible to mix conforming and non-conforming choices, but we will focus on **(S1)** and **(S2)** for simplicity of the presentation.

Remark 3.3.1. *We note that, since $\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, the last three equations in the continuous weak formulation* (3.1.7) *hold for test functions in $\mathrm{L}^2(\Gamma_{fp})$, assuming that the solution is smooth enough. In particular, these equations hold for $\xi_h \in \Lambda_{ph}$, $\boldsymbol{\psi}_h \in \mathbf{\Lambda}_{fh}$, and $\boldsymbol{\phi}_h \in \mathbf{\Lambda}_{sh}$ in both the conforming case* **(S1)** *and the non-conforming case* **(S2)**.

Now, we group the spaces similarly to the continuous case:

$$\mathbf{X}_h := \mathbb{X}_{fh} \times \mathbf{V}_{ph} \times \mathbb{X}_{ph} \times \mathrm{W}_{ph},\ \mathbf{Y}_h := \mathbf{\Lambda}_{fh} \times \mathbf{\Lambda}_{sh} \times \Lambda_{ph},\ \mathbf{Z}_h := \mathbf{V}_{fh} \times \mathbf{V}_{sh} \times \mathbb{Q}_{fh} \times \mathbb{Q}_{ph},$$

$$\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathbf{X}_h,\ \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \boldsymbol{\theta}_h, \lambda_h) \in \mathbf{Y}_h,\ \underline{\mathbf{u}}_h := (\mathbf{u}_{fh}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{fh}, \boldsymbol{\gamma}_{ph}) \in \mathbf{Z}_h,$$

$$\underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, w_{ph}) \in \mathbf{X}_h,\ \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \boldsymbol{\phi}_h, \xi_h) \in \mathbf{Y}_h,\ \underline{\mathbf{v}}_h := (\mathbf{v}_{fh}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{fh}, \boldsymbol{\chi}_{ph}) \in \mathbf{Z}_h.$$

The spaces $\mathbf{X}_h$ and $\mathbf{Z}_h$ are endowed with the same norms as their continuous counterparts. For $\mathbf{Y}_h$ we consider the norm $\|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{Y}_h} := \|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}_{fh}} + \|\boldsymbol{\phi}_h\|_{\boldsymbol{\Lambda}_{sh}} + \|\xi_h\|_{\Lambda_{ph}}$, where

$$\|\xi_h\|_{\Lambda_{ph}} := \begin{cases} \|\xi_h\|_{\Lambda_p} \text{ for conforming subspaces } (\mathbf{S1}) \text{ (cf. (3.1.6))}, \\ \\ \|\xi_h\|_{\mathrm{L}^2(\Gamma_{fp})} \text{ for non-conforming subspaces } (\mathbf{S2}). \end{cases} \tag{3.3.3}$$

Analogous notation is used for $\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}_{fh}}$ and $\|\boldsymbol{\phi}_h\|_{\boldsymbol{\Lambda}_{sh}}$.

The continuity of all operators in the discrete case follows from their continuity in the continuous case (cf. Lemma 3.2.2), with the exception of $\mathcal{B}_1$ (cf. (3.1.13)) in the case of non-conforming Lagrange multipliers (**S2**). In this case it follows for each fixed h from the discrete trace-inverse inequality for piecewise polynomial functions, $\|\varphi\|_{\mathrm{L}^2(\Gamma)} \leq C h^{-1/2} \|\varphi\|_{\mathrm{L}^2(\mathcal{O})}$, where $\Gamma \subset \partial\mathcal{O}$. In particular,

$$b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\psi}) \leq C \|\boldsymbol{\tau}_f\|_{\mathbb{L}^2(\Gamma_{fp})} \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Gamma_{fp})} \leq C h^{-1/2} \|\boldsymbol{\tau}_f\|_{\mathbb{L}^2(\Omega_f)} \|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Gamma_{fp})}, \tag{3.3.4}$$

with similar bounds for $b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi})$ and $b_\Gamma(\mathbf{v}_p, \xi)$.

We next discuss the discrete inf-sup conditions that are satisfied by the finite element spaces. Let

$$\widetilde{\mathbf{X}}_h := \Big\{ \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h : \quad \boldsymbol{\tau}_{fh}\mathbf{n}_f = \mathbf{0} \quad \text{and} \quad \boldsymbol{\tau}_{ph}\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{fp} \Big\}. \tag{3.3.5}$$

In addition, define the discrete kernel of the operator $\mathcal{B}$ as

$$\mathbf{V}_h := \Big\{ \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h : \quad \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Z}_h \Big\} = \widetilde{\mathbb{X}}_{fh} \times \mathbf{V}_{ph} \times \widetilde{\mathbb{X}}_{ph} \times \mathrm{W}_{ph}, \tag{3.3.6}$$

where

$$\widetilde{\mathbb{X}}_{\star h} := \Big\{ \boldsymbol{\tau}_{\star h} \in \mathbb{X}_{\star h} : (\boldsymbol{\tau}_{\star h}, \boldsymbol{\xi}_{\star h})_{\Omega_\star} = 0 \;\; \forall\, \boldsymbol{\xi}_{\star h} \in \mathbb{Q}_{\star h} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}_{\star h}) = \mathbf{0} \quad \text{in} \quad \Omega_\star \Big\},$$

for $\star \in \{f, p\}$. In the above, $\mathbf{div}(\boldsymbol{\tau}_{\star h}) = \mathbf{0}$ follows from $\mathbf{div}(\mathbb{X}_{fh}) = \mathbf{V}_{fh}$ and $\mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}$, which is true for all stable elasticity spaces.

Lemma 3.3.1. *There exist positive constants $\widetilde{\beta}$ and $\widetilde{\beta}_1$ such that*

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}_h\in\widetilde{\mathbf{X}}_h} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \geq \widetilde{\beta}\,\|\underline{\mathbf{v}}_h\|_{\mathbf{Z}} \quad \forall\,\underline{\mathbf{v}}_h \in \mathbf{Z}_h, \tag{3.3.7}$$

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}_h\in\mathbf{V}_h} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\psi}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \geq \widetilde{\beta}_1\,\|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{Y}_h} \quad \forall\,\underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h. \tag{3.3.8}$$

Proof. We begin with the proof of (3.3.7). We recall that the space $\mathbf{X}_h$ consists of stresses and velocities with zero normal traces on the Neumann boundaries, while the space $\widetilde{\mathbf{X}}_h$ involves further restriction on Γ_{fp}. The inf-sup condition (3.3.7) without restricting the normal stress or velocity on the subdomain boundary follows from the stability of the elasticity and Darcy finite element spaces. The restricted inf-sup condition (3.3.7) can be shown using the argument in [6, Theorem 4.2].

We continue with the proof of (3.3.8). Similarly to the continuous case, due the diagonal character of operator $\mathcal{B}_1$ (cf. (3.1.13)), we need to show individual inf-sup conditions for $b_{\mathbf{n}_f}$, $b_{\mathbf{n}_p}$, and b_Γ. We first focus on b_Γ. For the conforming case **(S1)** (cf. (3.3.1)), the proof of (3.3.8) can be derived from a slight adaptation of [43, Lemma 4.4] (see also [50, Section 5.3] for the case $k = 0$), whereas from [4, Section 5.1] we obtain the proof for the non-conforming version **(S2)** (cf. (3.3.2)). We next consider the inf-sup condition (3.3.8) for $b_{\mathbf{n}_f}$, with argument for $b_{\mathbf{n}_p}$ being similar. The proof utilizes a suitable interpolant of $\boldsymbol{\tau}_f := \mathbf{e}(\mathbf{v}_f)$, the solution to the auxiliary problem (3.2.16). Due to the stability of the spaces $(\mathbb{X}_{fh}, \mathbf{V}_{fh}, \mathbb{Q}_{fh})$ (cf. (3.3.7)), there exists an interpolant $\tilde{\Pi}_h^f : \mathbb{H}^1(\Omega_f) \to \mathbb{X}_{fh}$ satisfying

$$\begin{gathered} b_f(\tilde{\Pi}_h^f\boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \mathbf{v}_{fh}) = 0 \quad \forall\,\mathbf{v}_{fh} \in \mathbf{V}_{fh}, \quad b_{\mathrm{sk},f}(\tilde{\Pi}_h^f\boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \boldsymbol{\chi}_{fh}) = 0 \quad \forall\,\boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}, \\ \langle(\tilde{\Pi}_h^f\boldsymbol{\tau}_f - \boldsymbol{\tau}_f)\mathbf{n}_f, \boldsymbol{\tau}_{fh}\mathbf{n}_f\rangle_{\Gamma_{fp}\cup\Gamma_f^N} = 0 \quad \forall\,\boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh}. \end{gathered} \tag{3.3.9}$$

The interpolant $\tilde{\Pi}_h^f\boldsymbol{\tau}_f$ is defined as the elliptic projection of $\boldsymbol{\tau}_f$ satisfying Neumann boundary condition on $\Gamma_{fp} \cup \Gamma_f^N$ [59, (3.11)–(3.15)]. Due to (3.3.9), it holds that $\tilde{\Pi}_h^f\boldsymbol{\tau}_f \in \widetilde{\mathbb{X}}_{fh}$. With this interpolant, the proof of (3.3.8) for b_Γ discussed above can be easily modified for $b_{\mathbf{n}_f}$, see [43, Lemma 4.4] and [50, Section 5.3] for **(S1)** and [4, Section 5.1] for **(S2)**. $\square$

Remark 3.3.2. *The stability analysis requires only a discrete inf-sup condition for $\mathcal{B}$ in $\mathbf{X}_h \times \mathbf{Z}_h$. The more restrictive inf-sup condition* (3.3.7) *is used in the error analysis in order to simplify the proof.*

Finally, we will utilize the following inf-sup condition: there exists a constant $c > 0$ such that

$$\|p_{ph}\|_{\mathrm{W}_p} + \|\lambda_h\|_{\Lambda_{ph}} \le c \sup_{\mathbf{0}\neq\mathbf{v}_{ph}\in\mathbf{V}_{ph}} \frac{b_p(\mathbf{v}_{ph}, p_{ph}) + b_\Gamma(\mathbf{v}_{ph}, \lambda_h)}{\|\mathbf{v}_{ph}\|_{\mathbf{V}_p}} \tag{3.3.10}$$

whose proof for the conforming case (3.3.1) follows from a slight adaptation of the one in [52, Lemma 5.1], whereas the non-conforming case (3.3.2) can be found in [4, Section 5.1].

The semidiscrete continuous-in-time approximation to (3.1.12) reads: find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ such that for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, and for a.e. $t \in (0,T)$,

$$\begin{aligned}
\frac{\partial}{\partial t}\,\mathcal{E}(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{A}(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h) &= \mathbf{F}(\underline{\boldsymbol{\tau}}_h), \\
-\,\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_h)(\underline{\boldsymbol{\psi}}_h) &= 0, \\
-\,\mathcal{B}\,(\underline{\boldsymbol{\sigma}}_h)(\underline{\mathbf{v}}_h) &= \mathbf{G}(\underline{\mathbf{v}}_h).
\end{aligned} \tag{3.3.11}$$

We next discuss the choice of compatible discrete initial data $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$, whose construction is based on a modification of the step-by-step procedure for the continuous initial data.

1. Define $\boldsymbol{\theta}_{h,0} := P_h^{\boldsymbol{\Lambda}_s}(\boldsymbol{\theta}_0)$, where $P_h^{\boldsymbol{\Lambda}_s} : \boldsymbol{\Lambda}_s \to \boldsymbol{\Lambda}_{sh}$ is the classical L^2-projection operator, satisfying, for all $\boldsymbol{\phi} \in \mathbf{L}^2(\Gamma_{fp})$,

$$\left\langle \boldsymbol{\phi} - P_h^{\boldsymbol{\Lambda}_s}(\boldsymbol{\phi}), \boldsymbol{\phi}_h \right\rangle_{\Gamma_{fp}} = 0 \quad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}\,.$$

2. Define $(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\varphi}_{h,0}, \mathbf{u}_{fh,0}, \boldsymbol{\gamma}_{fh,0}) \in \mathbb{X}_{fh} \times \boldsymbol{\Lambda}_{fh} \times \mathbf{V}_{fh} \times \mathbb{Q}_{fh}$ and $(\mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{ph} \times \mathrm{W}_{ph} \times \Lambda_{ph}$ by solving a coupled Stokes-Darcy problem:

$$\begin{aligned}
& a_f(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\tau}_{fh}) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\varphi}_{h,0}) + b_f(\boldsymbol{\tau}_{fh}, \mathbf{u}_{fh,0}) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\gamma}_{fh,0}) \\
&\quad = a_f(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\tau}_{fh}) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\varphi}_0) + b_f(\boldsymbol{\tau}_{fh}, \mathbf{u}_{f,0}) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\gamma}_{f,0}) = -\frac{1}{n}\,(q_f(0)\,\mathbf{I}, \boldsymbol{\tau}_{fh})_{\Omega_f}, \\
& - b_{\mathbf{n}_f}(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\psi}_h) + \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi}_{h,0} - \boldsymbol{\theta}_{h,0}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\psi}_h \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \langle \boldsymbol{\psi}_h \cdot \mathbf{n}_f, \lambda_{h,0} \rangle_{\Gamma_{fp}}
\end{aligned}$$

$$= \; -b_{\mathbf{n}_f}(\boldsymbol{\sigma}_{f,0},\boldsymbol{\psi}_h) + \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\left\langle\sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi}_0 - \boldsymbol{\theta}_0)\cdot\mathbf{t}_{f,j},\boldsymbol{\psi}_h\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} + \langle\boldsymbol{\psi}_h\cdot\mathbf{n}_f,\lambda_0\rangle_{\Gamma_{fp}}$$

$$= \; 0,$$

$$-\,b_f(\boldsymbol{\sigma}_{fh,0},\mathbf{v}_{fh}) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_{fh,0},\boldsymbol{\chi}_{fh}) = -b_f(\boldsymbol{\sigma}_{f,0},\mathbf{v}_{fh}) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_{f,0},\boldsymbol{\chi}_{fh}) = (\mathbf{f}_f(0),\mathbf{v}_{fh})_{\Omega_f},$$

$$a_p(\mathbf{u}_{ph,0},\mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph},p_{ph,0}) + b_\Gamma(\mathbf{v}_{ph},\lambda_{h,0}) = a_p(\mathbf{u}_{p,0},\mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph},p_{p,0}) + b_\Gamma(\mathbf{v}_{ph},\lambda_0) = 0\,,$$

$$-\,b_p(\mathbf{u}_{ph,0},w_{ph}) = -b_p(\mathbf{u}_{p,0},w_{ph}) = -\mu^{-1}(\mathrm{div}(\mathbf{K}\nabla p_{p,0}),w_{ph})_{\Omega_p},$$

$$-\left\langle\boldsymbol{\varphi}_{h,0}\cdot\mathbf{n}_f + (\boldsymbol{\theta}_{h,0}+\mathbf{u}_{ph,0})\cdot\mathbf{n}_p,\xi_h\right\rangle_{\Gamma_{fp}} = -\left\langle\boldsymbol{\varphi}_0\cdot\mathbf{n}_f + (\boldsymbol{\theta}_0+\mathbf{u}_{p,0})\cdot\mathbf{n}_p,\xi_h\right\rangle_{\Gamma_{fp}} = 0, \quad (3.3.12)$$

for all $(\boldsymbol{\tau}_{fh},\boldsymbol{\psi}_h,\mathbf{v}_{fh},\boldsymbol{\chi}_{fh}) \in \mathbb{X}_{fh}\times\boldsymbol{\Lambda}_{fh}\times\mathbf{V}_{fh}\times\mathbb{Q}_{fh}$ and $(\mathbf{v}_{ph},w_{ph},\xi_h)\in\mathbf{V}_{ph}\times\mathrm{W}_{ph}\times\Lambda_{ph}$. Note that (3.3.12) is well-posed as a direct application of Theorem 3.2.1. Note also that $\boldsymbol{\theta}_{h,0}$ is data for this problem.

3. Define $(\boldsymbol{\sigma}_{ph,0},\boldsymbol{\omega}_{h,0},\boldsymbol{\eta}_{ph,0},\boldsymbol{\rho}_{ph,0})\in\mathbb{X}_{ph}\times\boldsymbol{\Lambda}_{sh}\times\mathbf{V}_{sh}\times\mathbb{Q}_{ph}$, as the unique solution of the problem

$$(A(\boldsymbol{\sigma}_{ph,0}),\boldsymbol{\tau}_{ph})_{\Omega_p} + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph},\boldsymbol{\omega}_{h,0}) + b_s(\boldsymbol{\tau}_{ph},\boldsymbol{\eta}_{ph,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph},\boldsymbol{\rho}_{ph,0}) + (A(\alpha_p\,p_{ph,0}\,\mathbf{I}),\boldsymbol{\tau}_{ph})_{\Omega_p}$$

$$= \; (A(\boldsymbol{\sigma}_{p,0}),\boldsymbol{\tau}_{ph})_{\Omega_p} + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph},\boldsymbol{\omega}_0) + b_s(\boldsymbol{\tau}_{ph},\boldsymbol{\eta}_{p,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph},\boldsymbol{\rho}_{p,0}) + (A(\alpha_p\,p_{p,0}\,\mathbf{I}),\boldsymbol{\tau}_{ph})_{\Omega_p}$$

$$= 0,$$

$$-\,b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{ph,0},\boldsymbol{\phi}_h) + \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\left\langle\sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi}_{h,0} - \boldsymbol{\theta}_{h,0})\cdot\mathbf{t}_{f,j},\boldsymbol{\phi}_h\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} + \langle\boldsymbol{\phi}_h\cdot\mathbf{n}_p,\lambda_{h,0}\rangle_{\Gamma_{fp}}$$

$$= -b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0},\boldsymbol{\phi}_h) + \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\left\langle\sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi}_0 - \boldsymbol{\theta}_0)\cdot\mathbf{t}_{f,j},\boldsymbol{\phi}_h\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} + \langle\boldsymbol{\phi}_h\cdot\mathbf{n}_p,\lambda_0\rangle_{\Gamma_{fp}} = 0,$$

$$-\,b_s(\boldsymbol{\sigma}_{ph,0},\mathbf{v}_{sh}) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_{ph,0},\boldsymbol{\chi}_{ph}) = -b_s(\boldsymbol{\sigma}_{p,0},\mathbf{v}_{sh}) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_{p,0},\boldsymbol{\chi}_{ph}) = (\mathbf{f}_p(0),\mathbf{v}_{sh})_{\Omega_p}, \quad (3.3.13)$$

for all $(\boldsymbol{\tau}_{ph},\boldsymbol{\phi}_h,\mathbf{v}_{sh},\boldsymbol{\chi}_{ph})\in\mathbb{X}_{ph}\times\boldsymbol{\Lambda}_{sh}\times\mathbf{V}_{sh}\times\mathbb{Q}_{ph}$. Note that the well-posedness of (3.3.13) follows from the classical Babuška-Brezzi theory. Note also that $p_{ph,0},\boldsymbol{\varphi}_{h,0},\boldsymbol{\theta}_{h,0}$, and $\lambda_{h,0}$ are data for this problem.

4. Finally, define $(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$, as the unique solution of the problem

$$
\begin{aligned}
&(A(\widehat{\boldsymbol{\sigma}}_{ph,0}), \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\tau}_{ph}, \mathbf{u}_{sh,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\gamma}_{ph,0}) \;=\; -b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\theta}_{h,0})\,, \\
&-b_s(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{v}_{sh}) - b_{\mathrm{sk},p}(\widehat{\boldsymbol{\sigma}}_{ph,0}, \boldsymbol{\chi}_{ph}) \;=\; 0\,,
\end{aligned}
\tag{3.3.14}
$$

for all $(\boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$. Problem (3.3.14) is well-posed as a direct application of the classical Babuška-Brezzi theory. Note that $\boldsymbol{\theta}_{h,0}$ is data for this problem.

We then define $\underline{\boldsymbol{\sigma}}_{h,0} = (\boldsymbol{\sigma}_{fh,0}, \mathbf{u}_{ph,0}, \boldsymbol{\sigma}_{ph,0}, p_{ph,0}) \in \mathbf{X}_h$, $\underline{\boldsymbol{\varphi}}_{h,0} = (\boldsymbol{\varphi}_{h,0}, \boldsymbol{\theta}_{h,0}, \lambda_{h,0}) \in \mathbf{Y}_h$, and $\underline{\mathbf{u}}_{h,0} = (\mathbf{u}_{fh,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{fh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbf{Z}_h$. This construction guarantees that the discrete initial data is compatible in the sense of Lemma 3.2.8:

$$
\begin{aligned}
\mathcal{A}(\underline{\boldsymbol{\sigma}}_{h,0})(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_{h,0}) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_{h,0}) &= \widehat{\mathbf{F}}_{h,0}(\underline{\boldsymbol{\tau}}_h) \quad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h, \\
-\,\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_{h,0})(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_{h,0})(\underline{\boldsymbol{\psi}}_h) &= 0 \qquad\qquad\; \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h, \\
-\,\mathcal{B}\,(\underline{\boldsymbol{\sigma}}_{h,0})(\underline{\mathbf{v}}_h) &= \mathbf{G}_0(\underline{\mathbf{v}}_h) \quad\;\; \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h,
\end{aligned}
\tag{3.3.15}
$$

where $\widehat{\mathbf{F}}_{h,0} = (q_f(0), \mathbf{0}, \widehat{\mathbf{f}}_{ph,0}, \widehat{q}_{ph,0})^{\mathrm{t}} \in \mathbf{X}'_2$ and $\mathbf{G}_0 = \mathbf{G}(0) \in \mathbf{Z}'$, with $\widehat{\mathbf{f}}_{ph,0} \in \mathbb{X}'_{p,2}$ and $\widehat{q}_{ph,0} \in \mathrm{W}'_{p,2}$ suitable data. Furthermore, it provides compatible initial data for the non-differentiated elasticity variables $(\boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}, \boldsymbol{\omega}_{h,0})$ in the sense of the first equation in (3.2.22) (cf. (3.3.13)).

3.3.2 Existence and uniqueness of a solution

Now, we establish the well-posedness of problem (3.3.11) and the corresponding stability bound.

Theorem 3.3.2. *For each compatible initial data* $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ *satisfying* (3.3.15) *and*

$$
\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}'_f), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}'_s), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}'_f), \quad q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}'_p)\,,
$$

there exists a unique solution of (3.3.11), $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ *such that* $(\boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{X}_{ph}) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_{ph})$, *and* $(\underline{\boldsymbol{\sigma}}_h(0), \underline{\boldsymbol{\varphi}}_h(0), \mathbf{u}_{fh}(0), \boldsymbol{\gamma}_{fh}(0)) =$

$(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \mathbf{u}_{fh,0}, \boldsymbol{\gamma}_{fh,0})$. *Moreover, assuming sufficient regularity of the data, there exists a positive constant* C *independent of* h *and* s_0*, such that*

$$
\begin{aligned}
&\|\boldsymbol{\sigma}_{fh}\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|\boldsymbol{\sigma}_{fh}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} \\
&\quad + |\boldsymbol{\varphi}_h - \boldsymbol{\theta}_h|_{\mathrm{L}^\infty(0,T;\mathtt{BJS})} + |\boldsymbol{\varphi}_h - \boldsymbol{\theta}_h|_{\mathrm{L}^2(0,T;\mathtt{BJS})} + \|\lambda_h\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} + \|\underline{\boldsymbol{\varphi}}_h\|_{\mathrm{L}^2(0,T;\mathbf{Y}_h)} \\
&\quad + \|\underline{\mathbf{u}}_h\|_{\mathrm{L}^2(0,T;\mathbf{Z})} + \|A^{1/2}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} \\
&\quad + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} + \|p_{ph}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\quad + \|\partial_t\, A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \sqrt{s_0}\|\partial_t\, p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\leq C\Big(\|\mathbf{f}_f\|_{\mathrm{H}^1(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{\mathrm{H}^1(0,T;\mathbf{V}_s')} + \|q_f\|_{\mathrm{H}^1(0,T;\mathbb{X}_f')} + \|q_p\|_{\mathrm{H}^1(0,T;\mathrm{W}_p')} \\
&\qquad + (1+\sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \|\mathbf{K}\nabla p_{p,0}\|_{\mathrm{H}^1(\Omega_p)}\Big).
\end{aligned} \tag{3.3.16}
$$

Proof. From the fact that $\mathbf{X}_h \subset \mathbf{X}$, $\mathbf{Z}_h \subset \mathbf{Z}$, and $\mathbf{div}(\mathbb{X}_{fh}) = \mathbf{V}_{fh}$, $\mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}$, $\mathrm{div}(\mathbf{V}_{ph}) = \mathrm{W}_{ph}$, considering $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ satisfying (3.3.15), and employing the continuity and monotonicity properties of the operators $\mathcal{N}$ and $\mathcal{M}$ (cf. Lemma 3.2.2 and (3.3.4)), as well as the discrete inf-sup conditions (3.3.7), (3.3.8), and (3.3.10), the proof is identical to the proofs of Theorems 3.2.9 and 3.2.11, and Corollary 3.2.10. We note that the proof of Corollary 3.2.10 works in the discrete case due to the choice of the discrete initial data as the elliptic projection of the continuous initial data (cf. (3.3.12)–(3.3.14)). □

Remark 3.3.3. *As in the continuous case, we can recover the non-differentiated elasticity variables*

$$
\boldsymbol{\eta}_{ph}(t) = \boldsymbol{\eta}_{ph,0} + \int_0^t \mathbf{u}_{sh}(s)\,ds, \quad \boldsymbol{\rho}_{ph}(t) = \boldsymbol{\rho}_{ph,0} + \int_0^t \boldsymbol{\gamma}_{ph}(s)\,ds, \quad \boldsymbol{\omega}_h(t) = \boldsymbol{\omega}_{h,0} + \int_0^t \boldsymbol{\theta}_h(s)\,ds\,,
$$

for each $t \in [0,T]$*. Then* (3.1.8) *holds discretely, which follows from integrating the equation associated to* $\boldsymbol{\tau}_{ph}$ *in* (3.3.11) *from* 0 *to* $t \in (0,T]$ *and using the first equation in* (3.3.13) *(cf.* (3.2.22)*).*

3.3.3 Error analysis

We proceed with establishing rates of convergence. To that end, let us set $\mathrm{V} \in \big\{\mathrm{W}_p, \mathbf{V}_f, \mathbf{V}_s, \mathbb{Q}_f, \mathbb{Q}_p\big\}$, $\Lambda \in \big\{\mathbf{\Lambda}_f, \mathbf{\Lambda}_s, \Lambda_p\big\}$ and let V_h, Λ_h be the discrete counterparts. Let $P_h^{\mathrm{V}} : \mathrm{V} \to \mathrm{V}_h$ and $P_h^{\Lambda} : \Lambda \to \Lambda_h$ be the L^2-projection operators, satisfying

$$\begin{aligned} (u - P_h^{\mathrm{V}}(u), v_h)_{\Omega_\star} \;&=\; 0 \quad \forall\, v_h \in \mathrm{V}_h, \\ \langle \varphi - P_h^{\Lambda}(\varphi), \psi_h \rangle_{\Gamma_{fp}} \;&=\; 0 \quad \forall\, \psi_h \in \Lambda_h, \end{aligned} \tag{3.3.17}$$

where $\star \in \{f, p\}$, $u \in \big\{p_p, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p\big\}$, $\varphi \in \big\{\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda\big\}$, and v_h, ψ_h are the corresponding discrete test functions. We have the approximation properties [39]:

$$\begin{aligned} \|u - P_h^{\mathrm{V}}(u)\|_{\mathrm{L}^2(\Omega_\star)} \;&\leq\; C h^{s_u+1} \, \|u\|_{\mathrm{H}^{s_u+1}(\Omega_\star)}, \\ \|\varphi - P_h^{\Lambda}(\varphi)\|_{\Lambda_h} \;&\leq\; C h^{s_\varphi+r} \, \|\varphi\|_{\mathrm{H}^{s_\varphi+1}(\Gamma_{fp})}, \end{aligned} \tag{3.3.18}$$

where $s_u \in \big\{s_{p_p}, s_{\mathbf{u}_f}, s_{\mathbf{u}_s}, s_{\boldsymbol{\gamma}_f}, s_{\boldsymbol{\gamma}_p}\big\}$ and $s_\varphi \in \big\{s_{\boldsymbol{\varphi}}, s_{\boldsymbol{\theta}}, s_\lambda\big\}$ are the degrees of polynomials in the spaces V_h and Λ_h, respectively, and (cf. (3.3.3)),

$$\|\varphi\|_{\Lambda_h} := \begin{cases} \|\varphi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \text{ with } r = 1/2 \text{ in (3.3.18) for conforming spaces } \textbf{(S1)}, \\ \|\varphi\|_{\mathrm{L}^2(\Gamma_{fp})}, \text{ with } r = 1 \text{ in (3.3.18) for non-conforming spaces } \textbf{(S2)}. \end{cases}$$

Next, denote $\mathrm{X} \in \big\{\mathbb{X}_f, \mathbb{X}_p, \mathbf{V}_p\big\}$, $\sigma \in \big\{\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_p, \mathbf{u}_p\big\} \in \mathrm{X}$ and let X_h and τ_h be their discrete counterparts. For the case **(S2)** when the discrete Lagrange multiplier spaces are chosen as in (3.3.2), (3.3.17) implies

$$\langle \varphi - P_h^{\Lambda}(\varphi), \tau_h \mathbf{n}_\star \rangle_{\Gamma_{fp}} \;=\; 0 \quad \forall\, \tau_h \in \mathrm{X}_h, \tag{3.3.19}$$

where $\star \in \{f, p\}$. We note that (3.3.19) does not hold for the case **(S1)**.

Let $I_h^{\mathrm{X}} : \mathrm{X} \cap \mathrm{H}^1(\Omega_\star) \to \mathrm{X}_h$ be the mixed finite element projection operator [23] satisfying

$$\begin{aligned} (\mathrm{div}(I_h^{\mathrm{X}}(\sigma)), w_h)_{\Omega_\star} &= (\mathrm{div}(\sigma), w_h)_{\Omega_\star} \quad \forall\, w_h \in \mathrm{W}_h, \\ \big\langle I_h^{\mathrm{X}}(\sigma)\mathbf{n}_\star, \tau_h \mathbf{n}_\star \big\rangle_{\Gamma_{fp}} &= \langle \sigma \mathbf{n}_\star, \tau_h \mathbf{n}_\star \rangle_{\Gamma_{fp}} \quad \forall\, \tau_h \in \mathrm{X}_h, \end{aligned} \tag{3.3.20}$$

and

$$\|\sigma - I_h^{\mathrm{X}}(\sigma)\|_{\mathrm{L}^2(\Omega_\star)} \,\leq\, C\, h^{s_\sigma+1}\|\sigma\|_{\mathrm{H}^{s_\sigma+1}(\Omega_\star)},$$
$$\|\mathrm{div}(\sigma - I_h^{\mathrm{X}}(\sigma))\|_{\mathrm{L}^2(\Omega_\star)} \,\leq\, C\, h^{s_\sigma+1}\|\mathrm{div}(\sigma)\|_{\mathrm{H}^{s_\sigma+1}(\Omega_\star)}, \tag{3.3.21}$$

where $w_h \in \big\{\mathbf{v}_{fh}, \mathbf{v}_{sh}, w_{ph}\big\}$, $\mathrm{W}_h \in \big\{\mathbf{V}_f, \mathbf{V}_s, \mathrm{W}_p\big\}$, and $s_\sigma \in \big\{s_{\boldsymbol{\sigma}_f}, s_{\boldsymbol{\sigma}_p}, s_{\mathbf{u}_p}\big\}$ – the degrees of polynomials in the spaces X_h.

Now, let $(\boldsymbol{\sigma}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p, \boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p)$ and $(\boldsymbol{\sigma}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\sigma}_{ph}, p_{ph}, \boldsymbol{\varphi}_h, \boldsymbol{\theta}_h, \lambda_h, \mathbf{u}_{fh}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{fh}, \boldsymbol{\gamma}_{ph})$ be the solutions of (3.1.12) and (3.3.11), respectively. We introduce the error terms as the differences of these two solutions and decompose them into approximation and discretization errors using the interpolation operators:

$$\begin{aligned}
e_\sigma &:= \sigma - \sigma_h \,=\, (\sigma - I_h^{\mathrm{X}}(\sigma)) + (I_h^{\mathrm{X}}(\sigma) - \sigma_h) \,:=\, e_\sigma^I + e_\sigma^h, \quad \sigma \in \big\{\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_p, \mathbf{u}_p\big\},\\
e_\varphi &:= \varphi - \varphi_h \,=\, (\varphi - P_h^{\Lambda}(\varphi)) + (P_h^{\Lambda}(\varphi) - \varphi_h) \,:=\, e_\varphi^I + e_\varphi^h, \quad \varphi \in \big\{\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda\big\},\\
e_u &:= u - u_h \,=\, (u - P_h^{\mathrm{V}}(u)) + (P_h^{\mathrm{V}}(u) - u_h) \,:=\, e_u^I + e_u^h, \quad u \in \big\{p_p, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p\big\}.
\end{aligned} \tag{3.3.22}$$

Then, we set the errors

$$e_{\underline{\boldsymbol{\sigma}}} := (e_{\boldsymbol{\sigma}_f}, e_{\mathbf{u}_p}, e_{\boldsymbol{\sigma}_p}, e_{p_p}), \quad e_{\underline{\boldsymbol{\varphi}}} := (e_{\boldsymbol{\varphi}}, e_{\boldsymbol{\theta}}, e_\lambda), \quad \text{and} \quad e_{\underline{\mathbf{u}}} := (e_{\mathbf{u}_f}, e_{\mathbf{u}_s}, e_{\boldsymbol{\gamma}_f}, e_{\boldsymbol{\gamma}_p}).$$

We next form the error system by subtracting the discrete problem (3.3.11) from the continuous one (3.1.12). Using that $\mathbf{X}_h \subset \mathbf{X}$ and $\mathbf{Z}_h \subset \mathbf{Z}$, as well as Remark 3.3.1, we obtain

$$\begin{aligned}
(\partial_t\, \mathcal{E} + \mathcal{A})(e_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(e_{\underline{\boldsymbol{\varphi}}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(e_{\underline{\mathbf{u}}}) &= 0 \quad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h,\\
-\,\mathcal{B}_1(e_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(e_{\underline{\boldsymbol{\varphi}}})(\underline{\boldsymbol{\psi}}_h) &= 0 \quad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h,\\
-\,\mathcal{B}(e_{\underline{\boldsymbol{\sigma}}})(\underline{\mathbf{v}}_h) &= 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h.
\end{aligned} \tag{3.3.23}$$

We now establish the main result of this section.

Theorem 3.3.3. *For the solutions of the continuous and discrete problems* (3.1.12) *and* (3.3.11), *respectively, assuming sufficient regularity of the true solution according to* (3.3.18) *and* (3.3.21), *there exists a positive constant C independent of h and s_0, such that*

$$\|e_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|e_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|e_{\mathbf{u}_p}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|e_{\mathbf{u}_p}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} + |e_{\boldsymbol{\varphi}} - e_{\boldsymbol{\theta}}|_{\mathrm{L}^\infty(0,T;\mathtt{BJS})}$$

$$
\begin{aligned}
&+ |e_{\boldsymbol{\varphi}} - e_{\boldsymbol{\theta}}|_{\mathrm{L}^2(0,T;\mathtt{BJS})} + \|e_\lambda\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} + \|e_{\underline{\boldsymbol{\varphi}}}\|_{\mathrm{L}^2(0,T;\mathbf{Y}_h)} + \|e_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{Z})} \\
&+ \|A^{1/2}(e_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(e_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{div}(e_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} \\
&+ \|e_{p_p}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|e_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} + \|\partial_t\, A^{1/2}(e_{\boldsymbol{\sigma}_p} + \alpha_p e_{p_p}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} \\
&+ \sqrt{s_0}\|\partial_t\, e_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
\leq\ & C\sqrt{\exp(T)}\left(h^{s_{\underline{\boldsymbol{\sigma}}}+1} + h^{s_{\underline{\boldsymbol{\varphi}}}+r} + h^{s_{\underline{\mathbf{u}}}+1}\right),
\end{aligned} \tag{3.3.24}
$$

where $s_{\underline{\boldsymbol{\sigma}}} = \min\{s_{\boldsymbol{\sigma}_f}, s_{\mathbf{u}_p}, s_{\boldsymbol{\sigma}_p}, s_{p_p}\}$, $s_{\underline{\boldsymbol{\varphi}}} = \min\{s_{\boldsymbol{\varphi}}, s_{\boldsymbol{\theta}}, s_\lambda\}$, $s_{\underline{\mathbf{u}}} = \min\{s_{\mathbf{u}_f}, s_{\mathbf{u}_s}, s_{\boldsymbol{\gamma}_f}, s_{\boldsymbol{\gamma}_p}\}$, *and* r *is defined in* (3.3.18)*.*

Proof. We present in detail the proof for the conforming case **(S1)**. The proof in the non-conforming case **(S2)** is simpler, since several error terms are zero. We explain the differences at the end of the proof.

We proceed as in Theorem 3.2.11. Taking $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) = (e^h_{\underline{\boldsymbol{\sigma}}}, e^h_{\underline{\boldsymbol{\varphi}}}, e^h_{\underline{\mathbf{u}}})$ in (3.3.23), we obtain

$$
\begin{aligned}
&\frac{1}{2}\,\partial_t\left(a_e(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) + s_0\,(e^h_{p_p}, e^h_{p_p})_{\Omega_p}\right) + a_f(e^h_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\sigma}_f}) + a_p(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) + c_{\mathtt{BJS}}(e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}; e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}) \\
&= -\,a_f(e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\sigma}_f}) - a_p(e^I_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) - a_e(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, \partial_t\, e^I_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) - \mathcal{C}(e^I_{\underline{\boldsymbol{\varphi}}})(e^h_{\underline{\boldsymbol{\varphi}}}) \\
&\quad -\, b_{\mathbf{n}_f}(e^h_{\boldsymbol{\sigma}_f}, e^I_{\boldsymbol{\varphi}}) - b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}}) - b_\Gamma(e^h_{\mathbf{u}_p}, e^I_\lambda) + b_{\mathbf{n}_f}(e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\varphi}}) + b_{\mathbf{n}_p}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\theta}}) + b_\Gamma(e^I_{\mathbf{u}_p}, e^h_\lambda) \\
&\quad -\, b_{\mathrm{sk},f}(e^h_{\boldsymbol{\sigma}_f}, e^I_{\boldsymbol{\gamma}_f}) - b_{\mathrm{sk},p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p}) + b_{\mathrm{sk},f}(e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\gamma}_f}) + b_{\mathrm{sk},p}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\gamma}_p}),
\end{aligned} \tag{3.3.25}
$$

where, the right-hand side of (3.3.25) has been simplified, since the projection properties (3.3.17) and (3.3.20), and the fact that $\mathrm{div}(e^h_{\mathbf{u}_p}) \in \mathrm{W}_{ph}$, $\mathbf{div}(e^h_{\boldsymbol{\sigma}_f}) \in \mathbf{V}_{fh}$, and $\mathbf{div}(e^h_{\boldsymbol{\sigma}_p}) \in \mathbf{V}_{sh}$, imply that the following terms are zero:

$$
s_0(\partial_t\, e^I_{p_p}, e^h_{p_p}),\ b_p(e^h_{\mathbf{u}_p}, e^I_{p_p}),\ b_p(e^I_{\mathbf{u}_p}, e^h_{p_p}),\ b_f(e^h_{\boldsymbol{\sigma}_f}, e^I_{\mathbf{u}_f}),\ b_f(e^I_{\boldsymbol{\sigma}_f}, e^h_{\mathbf{u}_f}),\ b_s(e^h_{\boldsymbol{\sigma}_p}, e^I_{\mathbf{u}_s}),\ b_s(e^I_{\boldsymbol{\sigma}_p}, e^h_{\mathbf{u}_s}). \tag{3.3.26}
$$

In turn, from the equations in (3.3.23) corresponding to test functions $\mathbf{v}_{fh}$, $\mathbf{v}_{sh}$, and w_{ph}, using the projection properties (3.3.20), we find that

$$b_f(e^h_{\boldsymbol{\sigma}_f}, \mathbf{v}_{fh}) = 0 \quad \forall\, \mathbf{v}_{fh} \in \mathbf{V}_{fh}, \quad b_s(e^h_{\boldsymbol{\sigma}_p}, \mathbf{v}_{sh}) = 0 \quad \forall\, \mathbf{v}_{sh} \in \mathbf{V}_{sh},$$

$$b_p(e^h_{\mathbf{u}_p}, w_{ph}) = a_e(\partial_t\, e^h_{\boldsymbol{\sigma}_p}, \partial_t\, e^h_{p_p}; \mathbf{0}, w_{ph}) + a_e(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, \partial_t\, e^I_{p_p}; \mathbf{0}, w_{ph})$$

$$+(s_0\, \partial_t\, e^h_{p_p}, w_{ph})_{\Omega_p} \quad \forall\, w_{ph} \in \mathrm{W}_{ph}.$$

Therefore $\mathbf{div}(e^h_{\boldsymbol{\sigma}_\star}) = \mathbf{0}$ in $\Omega_\star$, with $\star \in \{f,p\}$, and using (3.2.1)–(3.2.2) we deduce

$$\begin{aligned}
&\|(e^h_{\boldsymbol{\sigma}_f})^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} \,\geq\, C\, \|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f}, \quad \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega_p)} \,=\, 0\,, \\
&\|\mathrm{div}(e^h_{\mathbf{u}_p})\|_{\mathrm{L}^2(\Omega_p)} \,\leq\, C\, \Big(\|\partial_t\, A^{1/2}(e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \\
&\qquad + \|\partial_t\, A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \sqrt{s_0}\, \|\partial_t\, e^h_{p_p}\|_{\mathrm{W}_p} \Big)\,.
\end{aligned} \tag{3.3.27}$$

Then, applying the ellipticity and continuity bounds of the bilinear forms involved in (3.3.25) (cf. Lemma 3.2.2) and the Cauchy–Schwarz and Young's inequalities, in combination with (3.3.27), we get

$$\begin{aligned}
&\partial_t \Big(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|e^h_{p_p}\|^2_{\mathrm{W}_p} \Big) + \|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} \\
&\quad + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + |e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \\
&\leq\, C \Big(\|e^I_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^I_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + \|e^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{X}_p} + |e^I_{\boldsymbol{\varphi}} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \\
&\quad + \|\partial_t\, A^{1/2}\, (e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\, (e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \|\partial_t\, A^{1/2}\, (e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|\partial_t\, e^h_{p_p}\|^2_{\mathrm{W}_p} \Big) \\
&\quad + \delta_1 \Big(\|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \Big) \\
&\quad + \delta_2 \Big(\|e^h_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^h_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \Big),
\end{aligned}$$

where for the bound on $b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}})$ we used the trace inequality (3.1.5) and the fact that $\mathbf{div}(e^h_{\boldsymbol{\sigma}_p}) = \mathbf{0}$. Next, integrating from 0 to $t \in (0,T]$, using (3.2.13) to control the term $\|e^h_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)}$, and choosing δ_1 small enough, we find that

$$\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|e^h_{p_p}(t)\|^2_{\mathrm{W}_p}$$

$$
\begin{aligned}
&+ \int_0^t \left(\|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + |e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \right) ds \\
&\leq C \Bigg(\int_0^t \left(\|e^I_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^I_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |e^I_{\boldsymbol{\varphi}} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} + \|e^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{X}_p} \right) ds \\
&+ \int_0^t \left(\|\partial_t\, A^{1/2}\, (e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\, (e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \right) ds \\
&+ \int_0^t \left(\|\partial_t\, A^{1/2}\, (e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, e^h_{p_p}\|^2_{\mathrm{W}_p} \right) ds + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\,\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&+ s_0\|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} \Bigg) + \delta_2 \int_0^t \left(\|e^h_{p_p}\|^2_{\mathrm{W}_p} + \|e^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^h_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \right) ds \,. \qquad (3.3.28)
\end{aligned}
$$

On the other hand, taking $\underline{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, 0) \in \mathbf{V}_h$ (cf. (3.3.6)) in the first equation of (3.3.23), we obtain

$$
\mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(e^h_{\underline{\boldsymbol{\varphi}}}) \,=\, -\,(\partial_t\, \mathcal{E} + \mathcal{A})(e_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) - \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(e^I_{\underline{\boldsymbol{\varphi}}})\,,
$$

In the above, thanks to the projection properties (3.3.17), the following terms are zero: $b_p(\mathbf{v}_{ph}, e^I_{p_p})$, $b_f(\boldsymbol{\tau}_{fh}, e^I_{\mathbf{u}_f})$, and $b_s(\boldsymbol{\tau}_{ph}, e^I_{\mathbf{u}_s})$. Then the discrete inf-sup condition of $\mathcal{B}_1$ (cf. (3.3.8)) for $e^h_{\underline{\boldsymbol{\varphi}}} = (e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}, e^h_{\lambda}) \in \mathbf{Y}_h$ gives

$$
\begin{aligned}
\|e^h_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}_h} \,\leq\, C\Big(&\|e^I_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|e^I_{\mathbf{u}_p}\|_{\mathbf{V}_p} + \|e^I_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}_h} + \|e^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \\
&+ \|\partial_t\, A^{1/2}\, (e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|_{\mathbf{V}_p} + \|e^h_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} \\
&+ \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} + \|\partial_t\, A^{1/2}\, (e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^h_{p_p}\|_{\mathrm{W}_p} \Big)\,. \qquad (3.3.29)
\end{aligned}
$$

In turn, to bound $\|e^h_{\underline{\mathbf{u}}}\|_{\mathbf{Z}}$, we test (3.3.23) with $\underline{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_{fh}, \mathbf{0}, \boldsymbol{\tau}_{ph}, 0) \in \widetilde{\mathbf{X}}_h$ (cf. (3.3.5)), to find that

$$
\mathcal{B}(\underline{\boldsymbol{\tau}}_h)(e^h_{\underline{\mathbf{u}}}) \,=\, -\Big(a_f(e_{\boldsymbol{\sigma}_f}, \boldsymbol{\tau}_{fh}) + a_e(\partial_t\, e_{\boldsymbol{\sigma}_p}, \partial_t\, e_{p_p}; \boldsymbol{\tau}_{ph}, 0) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(e^I_{\underline{\mathbf{u}}}) \Big).
$$

In the above, the terms $b_f(\boldsymbol{\tau}_{fh}, e^I_{\mathbf{u}_f})$ and $b_s(\boldsymbol{\tau}_{ph}, e^I_{\mathbf{u}_s})$ are zero, due to the projection property (3.3.17). Then, the discrete inf-sup condition of $\mathcal{B}$ (cf. (3.3.7)) for $e^h_{\underline{\mathbf{u}}} \in \mathbf{Z}_h$, yields

$$
\begin{aligned}
\|e^h_{\underline{\mathbf{u}}}\|_{\mathbf{Z}} \,\leq\, C\, \Big(&\|e^I_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|\partial_t\, A^{1/2}\, (e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\gamma}_f}\|_{\mathbb{Q}_f} + \|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p} \\
&+ \|e^h_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|\partial_t\, A^{1/2}\, (e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \Big)\,.
\end{aligned} \qquad (3.3.30)
$$

Finally, to bound $\|e^h_{p_p}\|_{\mathrm{W}_p}$, we test (3.3.23) with $\underline{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, 0) \in \mathbf{X}_h$ to get

$$b_p(\mathbf{v}_{ph}, e^h_{p_p}) + b_\Gamma(\mathbf{v}_{ph}, e^h_\lambda) \;=\; -\Big(a_p(e_{\mathbf{u}_p}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, e^I_{p_p}) + b_\Gamma(\mathbf{v}_{ph}, e^I_\lambda)\Big).$$

Note that $b_p(\mathbf{v}_{ph}, e^I_{p_p}) = 0$ due to the projection property (3.3.17), thus the discrete inf-sup condition (3.3.10) gives

$$\|e^h_{p_p}\|_{\mathrm{W}_p} + \|e^h_\lambda\|_{\Lambda_{ph}} \le C\Big(\|e^I_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} + \|e^I_\lambda\|_{\Lambda_{ph}} + \|e^h_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)}\Big). \tag{3.3.31}$$

Combining (3.3.28) with (3.3.29), (3.3.30), and (3.3.31), choosing δ_2 small enough, and employing the Gronwall's inequality to deal with the term $\int_0^t \|A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\, ds$, we obtain

$$\begin{aligned}
&\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + \int_0^t \Big(\|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} \\
&\quad + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{p_p}\|^2_{\mathrm{W}_p} + |e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^h_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}}\Big)\, ds \\
&\le C\,\exp(T)\Bigg(\int_0^t \Big(\|e^I_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}} + \|e^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^I_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} + |e^I_{\boldsymbol{\varphi}} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \\
&\quad + \|\partial_t\, A^{1/2}\,(e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\Big)\, ds + \int_0^t \Big(\|\partial_t\, A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad +\; s_0\|\partial_t\, e^h_{p_p}\|^2_{\mathrm{W}_p}\Big)\, ds + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|e^h_{p_p}(0)\|^2_{\mathrm{W}_p}\Bigg).
\end{aligned} \tag{3.3.32}$$

Now, in order to bound $\int_0^t \Big(\|\partial_t\, A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, e^h_{p_p}\|^2_{\mathrm{W}_p}\Big)\, ds$ on the right-hand side of (3.3.32), we test (3.3.23) with $\underline{\boldsymbol{\tau}}_h = (\partial_t e^h_{\boldsymbol{\sigma}_f}, e^h_{\mathbf{u}_p}, \partial_t e^h_{\boldsymbol{\sigma}_p}, \partial_t e^h_{p_p})$, $\underline{\boldsymbol{\psi}}_h = (e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}, \partial_t e^h_\lambda)$, and $\underline{\mathbf{v}}_h = (e^h_{\mathbf{u}_f}, e^h_{\mathbf{u}_s}, e^h_{\boldsymbol{\gamma}_f}, e^h_{\boldsymbol{\gamma}_p})$, differentiate in time the rows in (3.3.23) associated to $\mathbf{v}_{ph}, \boldsymbol{\psi}_h$, $\boldsymbol{\phi}_h, \mathbf{v}_{fh}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{fh}, \boldsymbol{\chi}_{ph}$, and employ the projections properties (3.3.17)–(3.3.20) to eliminate some of the terms (cf. (3.3.26)), obtaining

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big(\frac{1}{2\mu}\|(e^h_{\boldsymbol{\sigma}_f})^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) + c_{\mathtt{BJS}}(e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}; e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}})\Big) \\
&\quad + \|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t e^h_{p_p}\|^2_{\mathrm{W}_p} \\
&= -\,a_f(e^I_{\boldsymbol{\sigma}_f}, \partial_t\, e^h_{\boldsymbol{\sigma}_f}) - a_p(\partial_t\, e^I_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) - a_e(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, \partial_t\, e^I_{p_p}; \partial_t\, e^h_{\boldsymbol{\sigma}_p}, \partial_t\, e^h_{p_p}) \\
&\quad - c_{\mathtt{BJS}}(\partial_t\, e^I_{\boldsymbol{\varphi}}, \partial_t\, e^I_{\boldsymbol{\theta}}; e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}) + c_\Gamma(e^h_{\boldsymbol{\varphi}}, e^h_{\boldsymbol{\theta}}; \partial_t\, e^I_\lambda) - c_\Gamma(e^I_{\boldsymbol{\varphi}}, e^I_{\boldsymbol{\theta}}; \partial_t\, e^h_\lambda)
\end{aligned}$$

$$
\begin{aligned}
&- \, b_{\mathbf{n}_f}(\partial_t\, e^h_{\boldsymbol{\sigma}_f}, e^I_{\boldsymbol{\varphi}}) - b_{\mathbf{n}_p}(\partial_t\, e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}}) - b_\Gamma(e^h_{\mathbf{u}_p}, \partial_t\, e^I_\lambda) + b_{\mathbf{n}_f}(\partial_t\, e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\varphi}}) \\
&+ b_{\mathbf{n}_p}(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\theta}}) + b_\Gamma(e^I_{\mathbf{u}_p}, \partial_t\, e^h_\lambda) - b_{\mathrm{sk},f}(\partial_t\, e^h_{\boldsymbol{\sigma}_f}, e^I_{\boldsymbol{\gamma}_f}) - b_{\mathrm{sk},p}(\partial_t\, e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\gamma}_p}) \\
&+ \, b_{\mathrm{sk},f}(\partial_t\, e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\gamma}_f}) + b_{\mathrm{sk},p}(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\gamma}_p})\,. \qquad (3.3.33)
\end{aligned}
$$

Then, integrating (3.3.33) from 0 to $t \in (0,T]$, using the identities

$$
\begin{aligned}
&\int_0^t a_f(e^I_{\boldsymbol{\sigma}_f}, \partial_t\, e^h_{\boldsymbol{\sigma}_f})\, ds \;=\; a_f(e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\sigma}_f})\Big|_0^t - \int_0^t a_f(\partial_t\, e^I_{\boldsymbol{\sigma}_f}, e^h_{\boldsymbol{\sigma}_f})\, ds\,, \\
&\int_0^t b_{\mathbf{n}_\star}(\partial_t\, e^h_{\boldsymbol{\sigma}_\star}, e^I_\circ)\, ds \;=\; b_{\mathbf{n}_\star}(e^h_{\boldsymbol{\sigma}_\star}, e^I_\circ)\Big|_0^t - \int_0^t b_{\mathbf{n}_\star}(e^h_{\boldsymbol{\sigma}_\star}, \partial_t\, e^I_\circ)\, ds\,,\ \star \in \{f,p\},\ \circ \in \{\boldsymbol{\varphi}, \boldsymbol{\theta}\}\,, \\
&\int_0^t b_{\mathrm{sk},\star}(\partial_t\, e^h_{\boldsymbol{\sigma}_\star}, e^I_{\boldsymbol{\gamma}_\star})\, ds \;=\; b_{\mathrm{sk},\star}(e^h_{\boldsymbol{\sigma}_\star}, e^I_{\boldsymbol{\gamma}_\star})\Big|_0^t - \int_0^t b_{\mathrm{sk},\star}(e^h_{\boldsymbol{\sigma}_\star}, \partial_t\, e^I_{\boldsymbol{\gamma}_\star})\, ds\,, \\
&\int_0^t \big\langle e^I_\diamond \cdot \mathbf{n}_f, \partial_t\, e^h_\lambda \big\rangle_{\Gamma_{fp}}\, ds \;=\; \big\langle e^I_\diamond \cdot \mathbf{n}_f, e^h_\lambda \big\rangle_{\Gamma_{fp}}\Big|_0^t - \int_0^t \big\langle \partial_t\, e^I_\diamond \cdot \mathbf{n}_f, e^h_\lambda \big\rangle_{\Gamma_{fp}}\, ds\,,\ \diamond \in \{\boldsymbol{\varphi}, \boldsymbol{\theta}, \mathbf{u}_p\},
\end{aligned}
\qquad (3.3.34)
$$

and applying the ellipticity and continuity bounds of the bilinear forms involved (cf. Lemma 3.2.2), the Cauchy-Schwarz and Young's inequalities, and the fact that $\mathbf{div}(e^h_{\boldsymbol{\sigma}_\star}) = \mathbf{0}$ in $\Omega_\star$ with $\star \in \{f,p\}$ (cf. (3.3.27)), we obtain

$$
\begin{aligned}
&\|e^h_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p}(t))\|^2_{\mathbf{L}^2(\Omega_p)} + |(e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}})(t)|^2_{\mathtt{BJS}} \\
&\quad + \int_0^t \Big(\|\partial_t\, A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|\partial_t\, e^h_{p_p}\|^2_{\mathrm{W}_p} \Big)\, ds \\
&\le\, C \Bigg(\|e^I_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{L}^2(\Omega_f)} + \|e^I_{\mathbf{u}_p}(t)\|^2_{\mathbf{V}_p} + \|e^I_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\varphi}}(t)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|e^I_{\boldsymbol{\theta}}(t)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^I_{\boldsymbol{\gamma}_f}(t)\|^2_{\mathbb{Q}_f} \\
&\quad + \|e^I_{\boldsymbol{\gamma}_p}(t)\|^2_{\mathbb{Q}_p} + \int_0^t \Big(\|\partial_t\, e^I_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\partial_t\, e^I_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |\partial_t\,(e^I_{\boldsymbol{\varphi}} - e^I_{\boldsymbol{\theta}})|^2_{\mathtt{BJS}} + \|e^I_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\partial_t\, e^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} \\
&\quad + \|\partial_t\, e^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\partial_t\, e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} + \|\partial_t\, A^{1/2}\,(e^I_{\boldsymbol{\sigma}_p} + \alpha_p\, e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|\partial_t\, e^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{X}_p} \Big)\, ds \\
&\quad + \|e^I_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{L}^2(\Omega_f)} + \|e^I_{\mathbf{u}_p}(0)\|^2_{\mathbf{V}_p} + \|e^I_{\boldsymbol{\varphi}}(0)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|e^I_{\boldsymbol{\theta}}(0)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^I_{\boldsymbol{\gamma}_f}(0)\|^2_{\mathbb{Q}_f} \Bigg) \\
&+\, \delta_3 \Bigg(\|e^h_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{X}_f} + \|e^h_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_\lambda(t)\|^2_{\Lambda_{ph}} + \int_0^t \Big(\|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \Big)\, ds \\
&\quad + \int_0^t \Big(\|e^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^h_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} \Big)\, ds \Bigg) + \frac{1}{2} \int_0^t \|\partial_t\, A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\, ds
\end{aligned}
$$

$$+ \, C \left(\|e^h_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{X}_p} + |(e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}})(0)|^2_{\mathtt{BJS}} + \|e^h_{\lambda}(0)\|^2_{\Lambda_{ph}} \right). \tag{3.3.35}$$

We note that $\|e^h_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\lambda}(t)\|^2_{\Lambda_{ph}}$ can be bounded by using (3.2.13) and (3.3.31), whereas all the other terms with δ_3 can be bounded by the left hand side of (3.3.32). Thus, combining (3.3.32) with (3.3.31) and (3.3.35), using algebraic manipulations, and choosing δ_3 small enough, we get

$$\begin{aligned}
&\|e^h_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + |(e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}})(t)|^2_{\mathtt{BJS}} + \|e^h_{\lambda}(t)\|^2_{\Lambda_{ph}} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p}(t))\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + \int_0^t \Big(\|e^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} \\
&\quad + \|e^h_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{p_p}\|^2_{\mathrm{W}_p} + \|\partial_t\, A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p} + \alpha_p\, e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, e^h_{p_p}\|^2_{\mathrm{W}_p} \Big)\, ds \\
&\leq C\, \exp(T) \Bigg(\|e^I_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{L}^2(\Omega_f)} + \|e^I_{\mathbf{u}_p}(t)\|^2_{\mathbf{V}_p} + \|e^I_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\varphi}}(t)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|e^I_{\boldsymbol{\theta}}(t)\|^2_{\boldsymbol{\Lambda}_{sh}} \\
&\quad + \|e^I_{\boldsymbol{\gamma}_f}(t)\|^2_{\mathbb{Q}_f} + \|e^I_{\boldsymbol{\gamma}_p}(t)\|^2_{\mathbb{Q}_p} + \int_0^t \Big(\|e^I_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}} + \|e^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|e^I_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} + |e^I_{\boldsymbol{\varphi}} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|\partial_t\, e^I_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}} \Big)\, ds \\
&\quad + \int_0^t \Big(\|\partial_t\, e^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + |\partial_t\,(e^I_{\boldsymbol{\varphi}} - e^I_{\boldsymbol{\theta}})|^2_{\mathtt{BJS}} + \|\partial_t\, e^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\partial_t\, e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \Big)\, ds + \|e^I_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{L}^2(\Omega_f)} \\
&\quad + \|e^I_{\mathbf{u}_p}(0)\|^2_{\mathbf{V}_p} + \|e^I_{\boldsymbol{\varphi}}(0)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|e^I_{\boldsymbol{\theta}}(0)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^I_{\boldsymbol{\gamma}_f}(0)\|^2_{\mathbb{Q}_f} + \|e^h_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \|e^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{X}_p} + (1+s_0)\|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} + |(e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}})(0)|^2_{\mathtt{BJS}} + \|e^h_{\lambda}(0)\|^2_{\Lambda_{ph}} \Bigg).
\end{aligned} \tag{3.3.36}$$

Finally, we establish a bound on the initial data terms above. In fact, proceeding as in (2.3.25), recalling from Corollary 3.2.10 and Theorem 3.3.2 that $(\underline{\boldsymbol{\sigma}}(0), \underline{\boldsymbol{\varphi}}(0)) = (\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0)$ and $(\underline{\boldsymbol{\sigma}}_h(0), \underline{\boldsymbol{\varphi}}_h(0)) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0})$, using similar arguments to (3.3.32) in combination with the error system derived from (3.3.12)–(3.3.13), we deduce

$$\begin{aligned}
&\|e^h_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{V}_p} + \|A^{1/2}\,(e^h_{\boldsymbol{\sigma}_p}(0))\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p}(0))\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} \\
&\quad + |(e^h_{\boldsymbol{\varphi}} - e^h_{\boldsymbol{\theta}})(0)|^2_{\mathtt{BJS}} + \|e^h_{\lambda}(0)\|^2_{\Lambda_{ph}} \,\leq\, C\left(\|e^I_{\underline{\boldsymbol{\sigma}}_0}\|^2_{\mathbf{X}} + \|e^I_{\underline{\widetilde{\boldsymbol{\varphi}}}_0}\|^2_{\mathbf{Y}_h} + \|e^I_{\underline{\widetilde{\mathbf{u}}}_0}\|^2_{\mathbf{Z}} \right),
\end{aligned} \tag{3.3.37}$$

where $\underline{\boldsymbol{\sigma}}_0 = (\boldsymbol{\sigma}_{f,0}, \mathbf{u}_{p,0}, \boldsymbol{\sigma}_{p,0}, p_{p,0})$, $\widetilde{\underline{\boldsymbol{\varphi}}}_0 = (\boldsymbol{\varphi}_0, \boldsymbol{\omega}_0, \lambda_0)$ and $\widetilde{\underline{\mathbf{u}}}_0 = (\mathbf{u}_{f,0}, \boldsymbol{\eta}_{p,0}, \boldsymbol{\gamma}_{f,0}, \boldsymbol{\rho}_{p,0})$, and $e^I_{\underline{\boldsymbol{\sigma}}_0}, e^I_{\widetilde{\underline{\boldsymbol{\varphi}}}_0}, e^I_{\widetilde{\underline{\mathbf{u}}}_0}$ denote their corresponding approximation errors. Thus, using the error decomposition (3.3.22) in combination with (3.3.36)–(3.3.37), the triangle inequality, (3.2.13) and the approximation properties (3.3.18) and (3.3.21), we obtain (3.3.24) with a positive constant C depending on parameters $\mu, \lambda_p, \mu_p, \alpha_p, k_{\min}, k_{\max}, \alpha_{\mathtt{BJS}}$, and the extra regularity assumptions for $\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}$, and $\underline{\mathbf{u}}$ whose expressions are obtained from the right-hands side of (3.3.18) and (3.3.21). This completes the proof in the conforming case **(S1)**.

The proof in the non-conforming case **(S2)** follows by using similar arguments. We exploit the projection property (3.3.19) to conclude that some terms in (3.3.25) are zero, namely $b_{\mathbf{n}_f}(e^h_{\boldsymbol{\sigma}_f}, e^I_{\boldsymbol{\varphi}})$, $b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}})$, and $b_\Gamma(e^h_{\mathbf{u}_p}, e^I_\lambda)$, as well as terms appearing in the operator $\mathcal{C}$ (cf. (3.1.10)): $\left\langle e^h_{\boldsymbol{\varphi}} \cdot \mathbf{n}_f, e^I_\lambda \right\rangle_{\Gamma_{fp}}$, $\left\langle e^I_{\boldsymbol{\varphi}} \cdot \mathbf{n}_f, e^h_\lambda \right\rangle_{\Gamma_{fp}}$, $\left\langle e^h_{\boldsymbol{\theta}} \cdot \mathbf{n}_p, e^I_\lambda \right\rangle_{\Gamma_{fp}}$, and $\left\langle e^I_{\boldsymbol{\theta}} \cdot \mathbf{n}_p, e^h_\lambda \right\rangle_{\Gamma_{fp}}$. In addition, in the non-conforming version of (3.3.29) the terms $\|e^I_\lambda\|_{\Lambda_{ph}}$, $\|e^I_{\boldsymbol{\varphi}}\|_{\boldsymbol{\Lambda}_{fh}}$, and $\|e^I_{\boldsymbol{\theta}}\|_{\boldsymbol{\Lambda}_{sh}}$ do not appear, since the bilinear forms $b_\Gamma(\mathbf{v}_{ph}, e^I_\lambda)$, $b_{\mathbf{n}_f}(\boldsymbol{\tau}_{fh}, e^I_{\boldsymbol{\varphi}})$, and $b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^I_{\boldsymbol{\theta}})$ are zero by a direct application of the projection property (3.3.19). □

3.4 A multipoint stress-flux mixed finite element method

In this section, inspired by previous works on the multipoint flux mixed finite element method for Darcy flow [24, 57, 80, 81] and the multipoint stress mixed finite element method for elasticity [6–8], we present a vertex quadrature rule that allows for local elimination of the stresses, rotations, and Darcy fluxes, leading to a positive-definite cell-centered pressure-velocities-traces system. We emphasize that, to the best of our knowledge, this is the first time such method is developed for the Stokes equations. To that end, the finite element spaces to be considered for both $(\mathbb{X}_{fh}, \mathbf{V}_{fh}, \mathbb{Q}_{fh})$ and $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph})$ are the triple $\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1$, which have been shown to be stable for mixed elasticity with weak stress symmetry in [20, 21, 44], whereas $(\mathbf{V}_{ph}, \mathrm{W}_{ph})$ is chosen to be $\mathbf{BDM}_1 - \mathrm{P}_0$ [22], and the Lagrange multiplier spaces $(\boldsymbol{\Lambda}_{fh}, \boldsymbol{\Lambda}_{sh}, \Lambda_{ph})$ are either $\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1$ or $\mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$ satisfying **(S1)** or **(S2)** (cf. (3.3.1), (3.3.2)), respectively, where $\mathrm{P}_1^{\mathrm{dc}}$ denotes the piecewise linear discontinuous finite element space and $\mathbf{P}_1^{\mathrm{dc}}$ is its corresponding vector version.

3.4.1 A quadrature rule setting

Let $S_\star$ denote the space of elementwise continuous functions on $\mathcal{T}_h^\star$. For any pair of tensor or vector valued functions φ and ψ with elements in $S_\star$, we define the vertex quadrature rule as in [81] (see also [6, 8]):

$$(\varphi,\psi)_{Q,\Omega_\star} := \sum_{E\in\mathcal{T}_h^\star} (\varphi,\psi)_{Q,E} = \sum_{E\in\mathcal{T}_h^\star} \frac{|E|}{s} \sum_{i=1}^{s} \varphi(\mathbf{r}_i)\cdot\psi(\mathbf{r}_i), \tag{3.4.1}$$

where $\star \in \{f,p\}$, $s=3$ on triangles and $s=4$ on tetrahedra, $\mathbf{r}_i$, $i=1,\ldots,s$, are the vertices of the element E, and $\cdot$ denotes the inner product for both vectors and tensors.

We will apply the quadrature rule for the bilinear forms a_f, a_p, a_e and $b_{\mathrm{sk},\star}$, which will be denoted by a_f^h, a_p^h, a_e^h and $b_{\mathrm{sk},\star}^h$, respectively. These bilinear forms involve the stress spaces $\mathbb{X}_{fh}$ and $\mathbb{X}_{ph}$, the vorticity space $\mathbb{Q}_{fh}$ and rotation space $\mathbb{Q}_{ph}$, and the Darcy velocity space $\mathbf{V}_{ph}$. The $\mathbf{BDM}_1$ spaces have for degrees of freedom $s-1$ normal components on each element edge (face), which can be associated with the vertices of the edge (face). At any element vertex $\mathbf{r}_i$, the value of a tensor or vector function is uniquely determined by its normal components at the associated two edges or three faces. Also, the vorticity space $\mathbb{Q}_{fh}$ and the rotation space $\mathbb{Q}_{ph}$ are vertex-based. Therefore the application of the vertex quadrature rule (3.4.1) for the bilinear forms involving the above spaces results in coupling only the degrees of freedom associated with a mesh vertex, which allows for local elimination of these variables. Next, we state a preliminary lemma to be used later on, which has been proved in [8, Lemma 3.1] and [6, Lemma 2.2].

Lemma 3.4.1. *There exist positive constants C_0 and C_1 independent of h, such that for any linear uniformly bounded and positive-definite operator L, there hold*

$$(L(\varphi),\varphi)_{Q,\Omega_\star} \geq C_0\,\|\varphi\|^2_{\Omega_\star}, \quad (L(\varphi),\psi)_{Q,\Omega_\star} \leq C_1\,\|\varphi\|_{\Omega_\star}\|\psi\|_{\Omega_\star}, \quad \forall\,\varphi,\psi\in S_\star, \quad \star\in\{f,p\}.$$

Consequently, the bilinear form $(L(\varphi),\varphi)_{Q,\Omega_\star}$ is an inner product in $\mathrm{L}^2(\Omega_\star)$ and $(L(\varphi),\varphi)^{1/2}_{Q,\Omega_\star}$ is a norm equivalent to $\|\varphi\|_{\Omega_\star}$.

The semidiscrete coupled multipoint stress-flux mixed finite element method for (3.1.12) reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ such that for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, and for a.e. $t \in (0,T)$,

$$
\begin{aligned}
\frac{\partial}{\partial t}\,\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{A}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h) &= \mathbf{F}(\underline{\boldsymbol{\tau}}_h), \\
- \mathcal{B}_1(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_h)(\underline{\boldsymbol{\psi}}_h) &= 0, \\
- \mathcal{B}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\mathbf{v}}_h) &= \mathbf{G}(\underline{\mathbf{v}}_h),
\end{aligned}
\tag{3.4.2}
$$

where

$$
\begin{aligned}
\mathcal{A}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) &:= a_f^h(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) + a_p^h(\mathbf{u}_{ph}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, p_{ph}) - b_p(\mathbf{u}_{ph}, w_{ph}), \\
\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) &:= a_e^h(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) + (s_0\, p_{ph}, w_{ph})_{\Omega_p}, \\
\mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) &:= b_f(\boldsymbol{\tau}_{fh}, \mathbf{v}_{fh}) + b_s(\boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}) + b^h_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\chi}_{fh}) + b^h_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\chi}_{ph}).
\end{aligned}
$$

We next discuss the discrete inf-sup conditions. We recall the space $\widetilde{\mathbf{X}}_h$ defined in (3.3.5). We also define the discrete kernel of the operator $\mathcal{B}_h$ as

$$
\widehat{\mathbf{V}}_h := \Big\{ \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h : \quad \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) = 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h \Big\} = \widehat{\mathbb{X}}_{fh} \times \mathbf{V}_{ph} \times \widehat{\mathbb{X}}_{ph} \times \mathrm{W}_{ph}, \tag{3.4.3}
$$

where

$$
\widehat{\mathbb{X}}_{\star h} := \Big\{ \boldsymbol{\tau}_{\star h} \in \mathbb{X}_{\star h} : \, (\boldsymbol{\tau}_{\star h}, \boldsymbol{\xi}_{\star h})_{Q,\Omega_\star} = 0 \ \ \forall\, \boldsymbol{\xi}_{\star h} \in \mathbb{Q}_{\star h} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}_{\star h}) = \mathbf{0} \quad \text{in} \quad \Omega_\star \Big\},
$$

for $\star \in \{f, p\}$, emphasizing the difference from the discrete kernel of $\mathcal{B}$ defined in (3.3.6).

Lemma 3.4.2. *There exist positive constants $\widehat{\beta}$ and $\widehat{\beta}_1$, such that*

$$
\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}}_h \in \widetilde{\mathbf{X}}_h} \frac{\mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \ \geq\ \widehat{\beta}\, \|\underline{\mathbf{v}}_h\|_{\mathbf{Z}} \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h, \tag{3.4.4}
$$

$$
\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}}_h \in \widehat{\mathbf{V}}_h} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\psi}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \ \geq\ \widehat{\beta}_1\, \|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{Y}_h} \quad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h. \tag{3.4.5}
$$

Proof. The proof of (3.4.4) follows from a slight adaptation of the argument in [6, Theorem 4.2]. The proof of (3.4.5) is similar to the proof of (3.3.8). The main difference is replacing the interpolant satisfying (3.3.9) by an interpolant $\hat{\Pi}_h^f : \mathbb{H}^1(\Omega_f) \to \mathbb{X}_{fh}$ satisfying

$$b_f(\hat{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \mathbf{v}_{fh}) = 0 \quad \forall\, \mathbf{v}_{fh} \in \mathbf{V}_{fh}, \quad b_{\mathrm{sk},f}^h(\hat{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \boldsymbol{\chi}_{fh}) = 0 \quad \forall\, \boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh},$$
$$\langle (\hat{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f)\mathbf{n}_f, \boldsymbol{\tau}_{fh}\mathbf{n}_f \rangle_{\Gamma_{fp} \cup \Gamma_f^N} = 0 \quad \forall\, \boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh},$$

whose existence follows from the inf-sup condition for $\mathcal{B}_h$ (3.4.4). □

We can establish the following well-posedness result.

Theorem 3.4.3. *For each compatible initial data* $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ *satisfying* (3.3.15) *and*

$$\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}_f'), \quad q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p'),$$

there exists a unique solution of (3.4.2), $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ *such that* $(\boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{X}_{ph}) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_{ph})$, *and* $(\underline{\boldsymbol{\sigma}}_h(0), \underline{\boldsymbol{\varphi}}_h(0), \mathbf{u}_{fh}(0), \boldsymbol{\gamma}_{fh}(0)) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \mathbf{u}_{fh,0}, \boldsymbol{\gamma}_{fh,0})$. *Moreover, assuming sufficient regularity of the data, a stability bound as in* (3.3.16) *also holds.*

Proof. The theorem follows from similar arguments to the proof of Theorem 3.3.2, in conjunction with Lemmas 3.4.1 and 3.4.2. □

3.4.2 Error analysis

Now, we obtain the error estimates and theoretical rates of convergence for the multipoint stress-flux mixed scheme (3.4.2). To that end, for each $\boldsymbol{\sigma}_{fh}$, $\boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh}$, $\mathbf{u}_{ph}$, $\mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $\boldsymbol{\sigma}_{ph}$, $\boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, p_{ph}, $w_{ph} \in \mathrm{W}_{ph}$, $\boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}$, and $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$, we denote the quadrature errors by

$$\begin{aligned}
\delta_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) &= a_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) - a_f^h(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}), \\
\delta_p(\mathbf{u}_{ph}, \mathbf{v}_{ph}) &= a_p(\mathbf{u}_{ph}, \mathbf{v}_{ph}) - a_p^h(\mathbf{u}_{ph}, \mathbf{v}_{ph}), \\
\delta_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) &= a_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) - a_e^h(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}), \\
\delta_{\mathrm{sk},\star}(\boldsymbol{\chi}_{\star h}, \boldsymbol{\tau}_{\star h}) &= b_{\mathrm{sk},\star}(\boldsymbol{\chi}_{\star h}, \boldsymbol{\tau}_{\star h}) - b_{\mathrm{sk},\star}^h(\boldsymbol{\chi}_{\star h}, \boldsymbol{\tau}_{\star h}), \quad \star \in \{f, p\}.
\end{aligned} \tag{3.4.6}$$

Next, for the operator A (cf. (2.1.3)) we will say that $A \in \mathbb{W}^{1,\infty}_{\mathcal{T}_h^p}$ if $A \in \mathbb{W}^{1,\infty}(E)$ for all $E \in \mathcal{T}_h^p$ and $\|A\|_{\mathbb{W}^{1,\infty}(E)}$ is uniformly bounded independently of h. Similar notation holds for $\mathbf{K}^{-1}$. In the next lemma we establish bounds on the quadrature errors. The proof follows from a slight adaptation of [6, Lemma 5.2] to our context (see also [8, 81]).

Lemma 3.4.4. *If $\mathbf{K}^{-1} \in \mathbb{W}^{1,\infty}_{\mathcal{T}_h^p}$ and $A \in \mathbb{W}^{1,\infty}_{\mathcal{T}_h^p}$, then there is a constant $C > 0$ independent of h such that*

$$\begin{aligned}
|\delta_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh})| &\leq C \sum_{E \in \mathcal{T}_h^f} h \,\|\boldsymbol{\sigma}_{fh}\|_{\mathbb{H}^1(E)} \,\|\boldsymbol{\tau}_{fh}\|_{\mathbb{L}^2(E)}, \\
|\delta_p(\mathbf{u}_{ph}, \mathbf{v}_{ph})| &\leq C \sum_{E \in \mathcal{T}_h^p} h \,\|\mathbf{K}^{-1}\|_{\mathbb{W}^{1,\infty}(E)} \,\|\mathbf{u}_{ph}\|_{\mathbf{H}^1(E)} \,\|\mathbf{v}_{ph}\|_{\mathbf{L}^2(E)}, \\
|\delta_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph})| &\leq C \sum_{E \in \mathcal{T}_h^p} h \,\|A\|_{\mathbb{W}^{1,\infty}(E)} \|(\boldsymbol{\sigma}_{ph}, p_{ph})\|_{\mathbb{H}^1(E)\times \mathrm{L}^2(E)} \|(\boldsymbol{\tau}_{ph}, w_{ph})\|_{\mathbb{L}^2(E)\times \mathrm{L}^2(E)}, \\
|\delta_{\mathrm{sk},\star}(\boldsymbol{\tau}_{\star h}, \boldsymbol{\chi}_{\star h})| &\leq C \sum_{E \in \mathcal{T}_h^\star} h \,\|\boldsymbol{\tau}_{\star h}\|_{\mathbb{L}^2(E)} \,\|\boldsymbol{\chi}_{\star h}\|_{\mathbb{H}^1(E)}, \quad \star \in \{f, p\}, \\
|\delta_{\mathrm{sk},\star}(\boldsymbol{\tau}_{\star h}, \boldsymbol{\chi}_{\star h})| &\leq C \sum_{E \in \mathcal{T}_h^\star} h \,\|\boldsymbol{\tau}_{\star h}\|_{\mathbb{H}^1(E)} \,\|\boldsymbol{\chi}_{\star h}\|_{\mathbb{L}^2(E)}, \quad \star \in \{f, p\},
\end{aligned}$$

for all $\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh}$, $\mathbf{u}_{ph}, \mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $\boldsymbol{\sigma}_{ph}, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $p_{ph}, w_{ph} \in \mathrm{W}_{ph}$, $\boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}$, $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$.

We are ready to establish the convergence of the multipoint stress-flux mixed finite element method.

Theorem 3.4.5. *For the solutions of the continuous and semidiscrete problems* (3.1.12) *and* (3.4.2)*, respectively, assuming sufficient regularity of the true solution according to* (3.3.18) *and* (3.3.21)*, there exists a positive constant C independent of h and s_0, such that*

$$\begin{aligned}
&\|e_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|e_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|e_{\mathbf{u}_p}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|e_{\mathbf{u}_p}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} + |e_{\boldsymbol{\varphi}} - e_{\boldsymbol{\theta}}|_{\mathrm{L}^\infty(0,T;\mathtt{BJS})} \\
&\quad + |e_{\boldsymbol{\varphi}} - e_{\boldsymbol{\theta}}|_{\mathrm{L}^2(0,T;\mathtt{BJS})} + \|e_\lambda\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} + \|e_{\underline{\boldsymbol{\varphi}}}\|_{\mathrm{L}^2(0,T;\mathbf{Y}_h)} + \|e_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{Z})} \\
&\quad + \|A^{1/2}(e_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(e_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|e_{p_p}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} \\
&\quad + \|\mathbf{div}(e_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} + \|e_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} + \|\partial_t\, A^{1/2}(e_{\boldsymbol{\sigma}_p} + \alpha_p e_{p_p}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} \\
&\quad + \sqrt{s_0}\|\partial_t\, e_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \leq C\left(h + h^{1+r}\right),
\end{aligned} \tag{3.4.7}$$

where r is defined in (3.3.18).

Proof. To obtain the error equations, we subtract the multipoint stress-flux mixed finite element formulation (3.4.2) from the continuous one (3.1.12). Using the error decomposition (3.3.22) and applying some algebraic manipulations, we obtain the error system:

$$
\begin{aligned}
&\big(\partial_t\,\mathcal{E}_h + \mathcal{A}_h\big)(e^h_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(e^h_{\underline{\boldsymbol{\varphi}}}) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(e^h_{\underline{\mathbf{u}}}) \\
&\quad = -\big(\partial_t\,\mathcal{E} + \mathcal{A}\big)(e^I_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) - \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(e^I_{\underline{\boldsymbol{\varphi}}}) - \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(e^I_{\underline{\mathbf{u}}}) - \boldsymbol{\delta}_{fep}(I_h(\underline{\boldsymbol{\sigma}}), P_h(\underline{\mathbf{u}}))(\underline{\boldsymbol{\tau}}_h), \\
&-\mathcal{B}_1(e^h_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(e^h_{\underline{\boldsymbol{\varphi}}})(\underline{\boldsymbol{\psi}}_h) \;=\; \mathcal{B}_1(e^I_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\psi}}_h) - \mathcal{C}(e^I_{\underline{\boldsymbol{\varphi}}})(\underline{\boldsymbol{\psi}}_h) \\
&-\mathcal{B}_h(e^h_{\underline{\boldsymbol{\sigma}}})(\underline{\mathbf{v}}_h) \;=\; \mathcal{B}(e^I_{\underline{\boldsymbol{\sigma}}})(\underline{\mathbf{v}}_h) + \boldsymbol{\delta}_{fp}(I_h(\underline{\boldsymbol{\sigma}}))(\underline{\mathbf{v}}_h)\,,
\end{aligned}
\tag{3.4.8}
$$

for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, where

$$
\begin{aligned}
\boldsymbol{\delta}_{fep}(I_h(\underline{\boldsymbol{\sigma}}), P_h(\underline{\mathbf{u}}))(\underline{\boldsymbol{\tau}}_h) \;:=\; &-\delta_f(I_h^{\mathbb{X}_f}(\boldsymbol{\sigma}_f), \boldsymbol{\tau}_{fh}) - \delta_e(I_h^{\mathbb{X}_p}(\boldsymbol{\sigma}_p), p_p; \boldsymbol{\tau}_{ph}, w_{ph}) \\
&- \delta_p(I_h^{\mathbf{V}_p}(\mathbf{u}_p), \mathbf{v}_{ph}) - \delta_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, P_h^{\mathbb{Q}_f}(\boldsymbol{\gamma}_f)) - \delta_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, P_h^{\mathbb{Q}_p}(\boldsymbol{\gamma}_p))
\end{aligned}
$$

and

$$
\boldsymbol{\delta}_{fp}(I_h(\underline{\boldsymbol{\sigma}}))(\underline{\mathbf{v}}_h) \;:=\; \delta_{\mathrm{sk},f}(I_h^{\mathbb{X}_f}(\boldsymbol{\sigma}_f), \boldsymbol{\chi}_{fh}) + \delta_{\mathrm{sk},p}(I_h^{\mathbb{X}_p}(\boldsymbol{\sigma}_p), \boldsymbol{\chi}_{ph})\,.
$$

Notice that the error system (3.4.8) is similar to (3.3.23), except for the additional quadrature error terms. The rest of the proof follows from the arguments in the proof of (3.3.24), using Lemmas 3.4.1, 3.4.2 and 3.4.4, and utilizing the continuity bounds of the interpolation operators $I_h^{\mathbf{X}_\star}, I_h^{\mathbf{v}_p}, P_h^{\mathbb{Q}_\star}$ [6, Lemma 5.1]:

$$
\begin{aligned}
\|I_h^{\mathbb{X}_\star}(\boldsymbol{\tau}_{\star h})\|_{\mathbb{H}^1(E)} \;&\leq\; C\,\|\boldsymbol{\tau}_{\star h}\|_{\mathbb{H}^1(E)} \quad \forall\,\boldsymbol{\tau}_{\star h} \in \mathbb{H}^1(E)\,, \quad \star \in \{f, p\}\,, \\
\|P_h^{\mathbb{Q}_\star}(\boldsymbol{\chi}_{\star h})\|_{\mathbb{H}^1(E)} \;&\leq\; C\,\|\boldsymbol{\chi}_{\star h}\|_{\mathbb{H}^1(E)} \quad \forall\,\boldsymbol{\chi}_{\star h} \in \mathbb{H}^1(E)\,, \\
\|I_h^{\mathbf{V}_p}(\mathbf{v}_{ph})\|_{\mathbf{H}^1(E)} \;&\leq\; C\,\|\mathbf{v}_{ph}\|_{\mathbf{H}^1(E)} \quad \forall\,\mathbf{v}_{ph} \in \mathbf{H}^1(E)\,.
\end{aligned}
$$

We omit further details, and refer to [6, 8, 81] for more details on the error analysis of the multipoint flux and multipoint stress mixed finite element methods on simplicial grids. □

3.4.3 Reduction to a cell-centered pressure-velocities-traces system

In this section we focus on the fully discrete problem associated to (3.4.2) (cf. (3.1.12), (3.3.11)), and describe how to obtain a reduced cell-centered system for the algebraic problem at each time step. For the time discretization we employ the backward Euler method. Let Δt be the time step, $T = M\,\Delta t$, $t_m = m\,\Delta t$, $m = 0,\dots,M$. Let $d_t\, u^m := (\Delta t)^{-1}(u^m - u^{m-1})$ be the first order (backward) discrete time derivative, where $u^m := u(t_m)$. Then the fully discrete model reads: given $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\boldsymbol{\varphi}}_h^0, \underline{\mathbf{u}}_h^0) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ satisfying (3.3.15), find $(\underline{\boldsymbol{\sigma}}_h^m, \underline{\boldsymbol{\varphi}}_h^m, \underline{\mathbf{u}}_h^m) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, $m = 1,\dots,M$, such that for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$,

$$
\begin{aligned}
d_t\, \mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\tau}}_h) + \mathcal{A}_h(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h^m) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h^m) &= \mathbf{F}(\underline{\boldsymbol{\tau}}_h)\,, \\
-\,\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_h^m)(\underline{\boldsymbol{\psi}}_h) &= 0\,, \\
-\,\mathcal{B}_h(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\mathbf{v}}_h) &= \mathbf{G}(\underline{\mathbf{v}}_h)\,.
\end{aligned}
\tag{3.4.9}
$$

Remark 3.4.1. *The well-posedness and error estimate associated to the fully discrete problem* (3.4.9) *can be derived employing similar arguments to Theorems 3.4.3 and 3.4.5 in combination with the theory developed in [10, Sections 6 and 9]. In particular, we note that at each time step the well-posedness of the fully discrete problem* (3.4.9)*, with* $m = 1,\dots,M$*, follows from similar arguments to the proof of Lemma 3.2.6.*

Notice that the first row in (3.4.9) can be rewritten equivalently as

$$
\big((\Delta t)^{-1}\mathcal{E}_h + \mathcal{A}_h\big)\,(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h^m) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h^m) \;=\; \mathbf{F}(\underline{\boldsymbol{\tau}}_h) + (\Delta t)^{-1}\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h^{m-1})(\underline{\boldsymbol{\tau}}_h)\,.
\tag{3.4.10}
$$

Let us associate with the operators in (3.4.9)–(3.4.10) matrices denoted in the same way. We then have

$$
\big((\Delta t)^{-1}\,\mathcal{E}_h + \mathcal{A}_h\big) = \begin{pmatrix} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f} & 0 & 0 & 0 \\ 0 & A_{\mathbf{u}_p\mathbf{u}_p} & 0 & A^{\mathrm{t}}_{\mathbf{u}_p p_p} \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} & A^{\mathrm{t}}_{\boldsymbol{\sigma}_p p_p} \\ 0 & -A_{\mathbf{u}_p p_p} & A_{\boldsymbol{\sigma}_p p_p} & A_{p_p p_p} \end{pmatrix}, \quad \mathcal{B}_h = \begin{pmatrix} A_{\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 \\ A_{\boldsymbol{\sigma}_f\gamma_f} & 0 & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\gamma_p} & 0 \end{pmatrix},
$$

$$\mathcal{B}_1 = \begin{pmatrix} A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 \\ 0 & A_{\mathbf{u}_p\lambda} & 0 & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\lambda} \\ A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\theta}\lambda} \\ -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & 0 \end{pmatrix},$$

with

$$A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f} \sim a_f^h(\cdot,\cdot),\ A_{\mathbf{u}_p\mathbf{u}_p} \sim a_p^h(\cdot,\cdot),\ A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} \sim (\Delta t)^{-1}\, a_e^h(\cdot,0;\cdot,0),\ A_{\boldsymbol{\sigma}_p p_p} \sim (\Delta t)^{-1} a_e^h(\cdot,0;\mathbf{0},\cdot),$$

$$A_{p_p p_p} \sim (\Delta t)^{-1} a_e^h(\mathbf{0},\cdot;\mathbf{0},\cdot) + (\Delta t)^{-1}(s_0\,\cdot,\cdot)_{\Omega_p}, A_{\mathbf{u}_p p_p} \sim b_p(\cdot,\cdot),\ A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} \sim b_{\mathbf{n}_f}(\cdot,\cdot),$$

$$A_{\mathbf{u}_p\lambda} \sim b_\Gamma(\cdot,\cdot),\ A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} \sim b_{\mathbf{n}_p}(\cdot,\cdot),\ A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} \sim c_{\mathtt{BJS}}(\cdot,\mathbf{0};\cdot,\mathbf{0}),\ A_{\boldsymbol{\varphi}\boldsymbol{\theta}} \sim c_{\mathtt{BJS}}(\cdot,\mathbf{0};\mathbf{0},\cdot),$$

$$A_{\boldsymbol{\theta}\boldsymbol{\theta}} \sim c_{\mathtt{BJS}}(\mathbf{0},\cdot;\mathbf{0},\cdot),\ A_{\boldsymbol{\varphi}\lambda} \sim c_\Gamma(\cdot,\mathbf{0};\cdot),\ A_{\boldsymbol{\theta}\lambda} \sim c_\Gamma(\mathbf{0},\cdot;\cdot),\ A_{\boldsymbol{\sigma}_f\mathbf{u}_f} \sim b_f(\cdot,\cdot),$$

$$A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} \sim b^h_{\mathrm{sk},f}(\cdot,\cdot),\ A_{\boldsymbol{\sigma}_p\mathbf{u}_s} \sim b_s(\cdot,\cdot),\ A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} \sim b^h_{\mathrm{sk},p}(\cdot,\cdot),$$

where the notation $A \sim a$ means that the matrix A is associated with the bilinear form a. Denoting the algebraic vectors corresponding to the variables $\underline{\boldsymbol{\sigma}}_h^m$, $\underline{\boldsymbol{\varphi}}_h^m$, and $\underline{\mathbf{u}}_h^m$ in the same way, we can then write the system (3.4.9) in a matrix-vector form as

$$\begin{pmatrix} (\Delta t)^{-1}\,\mathcal{E}_h + \mathcal{A}_h & \mathcal{B}_1^{\mathrm{t}} & \mathcal{B}_h^{\mathrm{t}} \\ -\mathcal{B}_1 & \mathcal{C} & 0 \\ -\mathcal{B}_h & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\sigma}}_h^m \\ \underline{\boldsymbol{\varphi}}_h^m \\ \underline{\mathbf{u}}_h^m \end{pmatrix} = \begin{pmatrix} \mathbf{F} + (\Delta t)^{-1}\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h^{m-1}) \\ 0 \\ \mathbf{G} \end{pmatrix}. \tag{3.4.11}$$

As we noted in Section 3.4.1, due to the the use of the vertex quadrature rule, the degrees of freedom (DOFs) of the Stokes stress $\boldsymbol{\sigma}_{fh}^m$, Darcy velocity $\mathbf{u}_{ph}^m$ and poroelastic stress tensor $\boldsymbol{\sigma}_{ph}^m$ associated with a mesh vertex become decoupled from the rest of the DOFs. As a result, the assembled mass matrices have a block-diagonal structure with one block per mesh vertex. The dimension of each block equals the number of DOFs associated with the vertex. These matrices can then be easily inverted with local computations. Inverting each local block in $A_{\mathbf{u}_p\mathbf{u}_p}$ allows for expressing the Darcy velocity DOFs associated with a vertex in terms of the Darcy pressure p_{ph}^m at the centers of the elements that share the vertex, as well as the trace unknown λ_h^m on neighboring edges (faces) for vertices on Γ_{fp}. Similarly, inverting each local block in $A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}$ allows for expressing the Stokes stress DOFs associated with a vertex in terms of neighboring Stokes velocity $\mathbf{u}_{fh}^m$, vorticity $\boldsymbol{\gamma}_{fh}^m$, and trace $\boldsymbol{\varphi}_h^m$. Finally, inverting each local block in $A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}$ allows for expressing the poroelastic stress DOFs associated with a vertex in terms of neighboring Darcy pressure p_{ph}^m, structure velocity $\mathbf{u}_{sh}^m$, structure rotation $\boldsymbol{\gamma}_{ph}^m$, and trace $\boldsymbol{\theta}_h^m$. Then we have

$$
\begin{aligned}
\mathbf{u}_{ph}^m &= -A_{\mathbf{u}_p\mathbf{u}_p}^{-1}A_{\mathbf{u}_p p_p}^{\mathrm{t}}\, p_{ph}^m - A_{\mathbf{u}_p\mathbf{u}_p}^{-1}A_{\mathbf{u}_p\lambda}^{\mathrm{t}}\,\lambda_h^m,\\
\boldsymbol{\sigma}_{fh}^m &= -A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}}\,\boldsymbol{\varphi}_h^m - A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}\,\mathbf{u}_{fh}^m - A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}}\,\boldsymbol{\gamma}_{fh}^m,\\
\boldsymbol{\sigma}_{ph}^m &= -A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}}\, p_{ph}^m - A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}\,\boldsymbol{\theta}_h^m - A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}\,\mathbf{u}_{sh}^m - A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}\,\boldsymbol{\gamma}_{ph}^m.
\end{aligned}
\tag{3.4.12}
$$

The reduced matrix associated to (3.4.11) in terms of $(p_{ph}^m,\boldsymbol{\varphi}_h^m,\boldsymbol{\theta}_h^m,\lambda_h^m,\mathbf{u}_{fh}^m,\mathbf{u}_{sh}^m,\boldsymbol{\gamma}_{fh}^m,\boldsymbol{\gamma}_{ph}^m)$ is given by

$$
\begin{pmatrix}
A_{p_p\boldsymbol{\sigma}_p p_p}+A_{p_p\mathbf{u}_p p_p} & 0 & -A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{p_p\mathbf{u}_p\lambda} & 0 & -A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & -A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}\\
0 & A_{\boldsymbol{\varphi}\boldsymbol{\varphi}}+A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}}^{\mathrm{t}} & A_{\boldsymbol{\varphi}\lambda}^{\mathrm{t}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0\\
A_{p_p\boldsymbol{\sigma}_p\theta}^{\mathrm{t}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}}+A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\lambda}^{\mathrm{t}} & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}\\
A_{p_p\mathbf{u}_p\lambda}^{\mathrm{t}} & -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & A_{\lambda\mathbf{u}_p\lambda} & 0 & 0 & 0 & 0\\
0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0\\
A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}} & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}\\
0 & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}} & 0 & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0\\
A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}} & 0 & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}} & 0 & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}
\end{pmatrix}
\tag{3.4.13}
$$

where

$A_{p_p\boldsymbol{\sigma}_p p_p} = A_{p_p p_p} - A_{\boldsymbol{\sigma}_p p_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}}$, $A_{p_p\mathbf{u}_p p_p} = A_{\mathbf{u}_p p_p}A_{\mathbf{u}_p\mathbf{u}_p}^{-1}A_{\mathbf{u}_p p_p}^{\mathrm{t}}$, $A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p p_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}$,

$A_{p_p\mathbf{u}_p\lambda} = A_{\mathbf{u}_p p_p}A_{\mathbf{u}_p\mathbf{u}_p}^{-1}A_{\mathbf{u}_p\lambda}^{\mathrm{t}}$, $A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} = A_{\boldsymbol{\sigma}_p p_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}$, $A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} = A_{\boldsymbol{\sigma}_p p_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}$,

$A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}}$, $A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}$,

$A_{\lambda\mathbf{u}_p\lambda} = A_{\mathbf{u}_p\lambda}A_{\mathbf{u}_p\mathbf{u}_p}^{-1}A_{\mathbf{u}_p\lambda}^{\mathrm{t}}$, $A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}$, $A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} = A_{\boldsymbol{\sigma}_f\mathbf{u}_f}A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}$,

$A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} = A_{\boldsymbol{\sigma}_f\mathbf{u}_f}A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1}A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}}$, $A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}$, $A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} = A_{\boldsymbol{\sigma}_p\mathbf{u}_s}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}$,

$A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} = A_{\boldsymbol{\sigma}_p\mathbf{u}_s}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}$, $A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} = A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}$, $A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}$,

$$A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} = A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f},\; A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}. \tag{3.4.14}$$

Furthermore, due to the vertex quadrature rule, the vorticity and structure rotation DOFs corresponding to each vertex of the grid become decoupled from the rest of the DOFs, leading to block-diagonal matrices $A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}$ and $A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}$. Recalling the matrix definitions in (3.4.14), each block is symmetric and positive definite and thus locally invertible, due the positive definiteness of $A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}$ and $A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}$ and the inf-sup condition (3.3.7). We then have

$$\begin{aligned}
\boldsymbol{\gamma}^m_{fh} &= -A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}\,\boldsymbol{\varphi}^m_h - A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}\,\mathbf{u}^m_{fh},\\
\boldsymbol{\gamma}^m_{ph} &= -A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}\,p^m_{ph} - A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}\,\boldsymbol{\theta}^m_h - A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}\,\mathbf{u}^m_{sh},
\end{aligned} \tag{3.4.15}$$

and using some algebraic manipulation, we obtain the reduced problem $\mathbf{A}\vec{\mathbf{p}}^m_h = \vec{\mathbf{F}}$, with vector solution $\vec{\mathbf{p}}^m_h := (p^m_{ph}, \boldsymbol{\varphi}^m_h, \boldsymbol{\theta}^m_h, \lambda^m_h, \mathbf{u}^m_{fh}, \mathbf{u}^m_{sh})$ and matrix

$$\mathbf{A} = \begin{pmatrix}
\widetilde{A}_{p_p\boldsymbol{\sigma}_p p_p} + A_{p_p\mathbf{u}_p p_p} & 0 & -\widetilde{A}_{p_p\boldsymbol{\sigma}_p\theta} & A_{p_p\mathbf{u}_p\lambda} & 0 & -\widetilde{A}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s}\\
0 & \widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} + A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\lambda} & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0\\
\widetilde{A}^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & \widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} + A_{\boldsymbol{\theta}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\theta}\lambda} & 0 & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}\\
A^{\mathrm{t}}_{p_p\mathbf{u}_p\lambda} & -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & A_{\lambda\mathbf{u}_p\lambda} & 0 & 0\\
0 & \widetilde{A}^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & 0\\
\widetilde{A}^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & \widetilde{A}^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 & 0 & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s}
\end{pmatrix} \tag{3.4.16}$$

where

$$\begin{aligned}
&\widetilde{A}_{p_p\boldsymbol{\sigma}_p p_p} = A_{p_p\boldsymbol{\sigma}_p p_p} + A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\; \widetilde{A}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} - A_{p\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}},\\
&\widetilde{A}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} = A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} - A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\; \widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} - A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}},\\
&\widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} - A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f},\; \widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} - A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}},\\
&\widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} - A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\; \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} = A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} - A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f},\\
&\widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} = A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} - A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},
\end{aligned} \tag{3.4.17}$$

and the right hand side vector $\vec{\mathbf{F}}$ has been obtained by transforming the right-hand side in (3.4.9) accordingly to the procedure above. Note that, after solving the problem with matrix (3.4.16), we can recover $\mathbf{u}_{ph}^m, \boldsymbol{\sigma}_{fh}^m, \boldsymbol{\sigma}_{ph}^m$ and $\boldsymbol{\gamma}_{fh}^m, \boldsymbol{\gamma}_{ph}^m$ through the formulae (3.4.12) and (3.4.15), respectively, thus obtaining the full solution to (3.4.9).

Lemma 3.4.6. *The cell-centered finite difference system for the pressure-velocities-traces problem* (3.4.16) *is positive definite.*

Proof. Consider a vector $\vec{\mathbf{q}}^{\mathrm{t}} = (w_{ph}^{\mathrm{t}}\ \boldsymbol{\psi}_h^{\mathrm{t}}\ \boldsymbol{\phi}_h^{\mathrm{t}}\ \xi_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}) \neq \vec{\mathbf{0}}$. Employing the matrices in (3.4.14) and (3.4.17) and some algebraic manipulations, we obtain

$$
\begin{aligned}
\vec{\mathbf{q}}^{\mathrm{t}}\,\mathbf{A}\,\vec{\mathbf{q}} \;=\;& w_{ph}^{\mathrm{t}}\big(A_{p_pp_p} - A_{\boldsymbol{\sigma}_pp_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_pp_p}^{\mathrm{t}}\big)w_{ph} + w_{ph}^{\mathrm{t}}A_{p_p\boldsymbol{\sigma}_p\gamma_p}A_{\gamma_p\boldsymbol{\sigma}_p\gamma_p}^{-1}A_{p_p\boldsymbol{\sigma}_p\gamma_p}^{\mathrm{t}}w_{ph} \\
&+ \big(A_{\mathbf{u}_pp_p}^{\mathrm{t}}\,w_{ph} + A_{\mathbf{u}_p\lambda}^{\mathrm{t}}\,\xi_h\big)^{\mathrm{t}}A_{\mathbf{u}_p\mathbf{u}_p}^{-1}\big(A_{\mathbf{u}_pp_p}^{\mathrm{t}}\,w_{ph} + A_{\mathbf{u}_p\lambda}^{\mathrm{t}}\,\xi_h\big) + (\boldsymbol{\psi}_h^{\mathrm{t}}\ \boldsymbol{\phi}_h^{\mathrm{t}})\begin{pmatrix} A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}}^{\mathrm{t}} \\ A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}} \end{pmatrix}\begin{pmatrix} \boldsymbol{\psi}_h \\ \boldsymbol{\phi}_h \end{pmatrix} \\
&+ (\boldsymbol{\psi}_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}})\begin{pmatrix} \widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} \\ \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} \end{pmatrix}\begin{pmatrix} \boldsymbol{\psi}_h \\ \mathbf{v}_{fh} \end{pmatrix} + (\boldsymbol{\phi}_h^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}})\begin{pmatrix} \widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} \\ \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} \end{pmatrix}\begin{pmatrix} \boldsymbol{\phi}_h \\ \mathbf{v}_{sh} \end{pmatrix}.
\end{aligned}
\tag{3.4.18}
$$

Now, we focus on analyzing the six terms in the right-hand side of (3.4.18). The first term is non-negative due to [56, Theorem 7.7.6] and the fact that the matrix $A_{p_pp_p} - A_{\boldsymbol{\sigma}_pp_p}A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1}A_{\boldsymbol{\sigma}_pp_p}^{\mathrm{t}}$ is a Schur complement of the matrix

$$
\begin{pmatrix} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} & A_{\boldsymbol{\sigma}_pp_p}^{\mathrm{t}} \\ A_{\boldsymbol{\sigma}_pp_p} & A_{p_pp_p} \end{pmatrix},
$$

which is positive semi-definite as a consequence of the ellipticity property of the operator a_e (cf. (3.1.9) and (3.2.6)). The second term is nonnegative, since the matrix $A_{\gamma_p\boldsymbol{\sigma}_p\gamma_p}$ is positive definite, as noted in (3.4.15). The third term is positive for $(w_{ph}^{\mathrm{t}}\ \xi_h^{\mathrm{t}}) \neq \vec{\mathbf{0}}$, due to the positive-definiteness of $A_{\mathbf{u}_p\mathbf{u}_p}^{-1}$ and the inf-sup condition (3.3.10). The fourth term is non-negative since the operator $\mathcal{C}$ (cf. (3.2.7)) is positive semi-definite. The matrices in the last two terms are Schur complements of the matrices

$$
A_f := \begin{pmatrix} A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A_{\gamma_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} \\ A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\gamma_f} \\ A_{\gamma_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\gamma_f}^{\mathrm{t}} & A_{\gamma_f\boldsymbol{\sigma}_f\gamma_f} \end{pmatrix} \quad \text{and} \quad A_p := \begin{pmatrix} A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\gamma_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} \\ A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\gamma_p} \\ A_{\gamma_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\gamma_p}^{\mathrm{t}} & A_{\gamma_p\boldsymbol{\sigma}_p\gamma_p} \end{pmatrix},
$$

respectively, which are positive definite. In particular, for $\vec{\mathbf{v}}_f^{\mathrm{t}} = (\boldsymbol{\psi}_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}\ \boldsymbol{\chi}_{fh}^{\mathrm{t}}) \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{v}}_p^{\mathrm{t}} = (\boldsymbol{\phi}_h^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}\ \boldsymbol{\chi}_{ph}^{\mathrm{t}}) \neq \vec{\mathbf{0}}$, we have

$$\begin{aligned}
\vec{\mathbf{v}}_f^{\mathrm{t}} A_f \vec{\mathbf{v}}_f &= \big(A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}\,\boldsymbol{\psi}_h + A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\mathbf{u}_f}\,\mathbf{v}_{fh} + A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\gamma_f}\,\boldsymbol{\chi}_{fh}\big)^{\mathrm{t}} A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}\big(A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}\,\boldsymbol{\psi}_h + A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\mathbf{u}_f}\,\mathbf{v}_{fh} + A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\gamma_f}\,\boldsymbol{\chi}_{fh}\big)\\
&> 0,\\
\vec{\mathbf{v}}_p^{\mathrm{t}} A_p \vec{\mathbf{v}}_p &= \big(A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}\,\boldsymbol{\phi}_h + A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\mathbf{u}_s}\,\mathbf{v}_{sh} + A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\gamma_p}\,\boldsymbol{\chi}_{ph}\big)^{\mathrm{t}} A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}\big(A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}\,\boldsymbol{\phi}_h + A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\mathbf{u}_s}\,\mathbf{v}_{sh} + A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\gamma_p}\,\boldsymbol{\chi}_{ph}\big) > 0,
\end{aligned}$$

due to the positive-definiteness of $A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}$ and $A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}$, along with the combined inf-sup condition for $\mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\psi}}_h)$. The latter follows from the inf-sup conditions (3.4.4) and (3.4.5), using that (3.4.5) holds in the kernel of $\mathcal{B}_h$. Then, applying again [56, Theorem 7.7.6], we conclude that the last two terms in (3.4.18) are positive for $(\boldsymbol{\psi}_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}) \neq \vec{\mathbf{0}}$ and $(\boldsymbol{\phi}_h^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}) \neq \vec{\mathbf{0}}$. Therefore $\vec{\mathbf{q}}^{\mathrm{t}}\,\mathbf{A}\,\vec{\mathbf{q}} > 0$ for all $\vec{\mathbf{q}} \neq \vec{\mathbf{0}}$, implying that the matrix $\mathbf{A}$ from (3.4.16) is positive definite. □

Remark 3.4.2. *The solution of the reduced system with the matrix* $\mathbf{A}$ *from* (3.4.16) *results in significant computational savings compared to the original system* (3.4.11)*. In particular, five of the eleven variables have been eliminated. Three of the remaining variables are Lagrange multipliers that appear only on the interface* Γ_{fp}*. The other three are the cell-centered velocities and Darcy pressure, with only* n *DOFs per element in the Stokes region and* $n+1$ *DOFs per element in the Biot region, which are the smallest possible number of DOFs for the sub-problems. Furthermore, since the reduced system is positive definite, efficient iterative solvers such as GMRES can be utilized for its solution.*

3.5 Numerical results

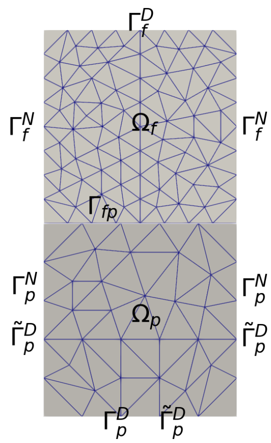
In this section we present numerical results that illustrate the behavior of the fully discrete multipoint stress-flux mixed finite element method (3.4.9). Our implementation is in two dimensions and it is based on `FreeFem++` [55], in conjunction with the direct linear solver `UMFPACK` [41]. For spatial discretization, we use the $(\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1)$ spaces for Stokes, the $(\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1) - (\mathbf{BDM}_1 - \mathrm{P}_0)$ spaces for Biot, and either $(\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1)$ or

$\mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$ for the Lagrange multipliers. We present three examples. Example 1 is used to corroborate the rates of convergence. Example 2 is a simulation of the coupling of surface and subsurface hydrological systems, focusing on the qualitative behavior of the solution. Example 3 illustrates an application to flow in a poroelastic medium with an irregularly shaped cavity, using physically realistic parameters.

3.5.1 Example 1: convergence test

In this test we study the convergence rates for the space discretization using an analytical solution. The domain is $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, where $\Omega_f = (0,1) \times (0,1)$ and $\Omega_p = (0,1) \times (-1,0)$. In particular, the upper half is associated with the Stokes flow, while the lower half represents the flow in the poroelastic structure governed by the Biot system, see Figure 3.5.1 (left). The interface conditions are enforced along the interface Γ_{fp}. The parameters and analytical solution are given in Figure 3.5.1 (right). The solution is designed to satisfy the interface conditions (3.1.3d)–(3.1.3e). The right hand side functions $\mathbf{f}_f, q_f, \mathbf{f}_p$ and q_p are computed from (3.1.1)–(3.1.3) using the true solution. The model problem is then complemented with the appropriate boundary conditions, which are described in Figure 3.5.1 (left), and initial data. Notice that the boundary conditions for $\boldsymbol{\sigma}_f, \mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p$, and $\boldsymbol{\eta}_p$ (cf. (3.1.2) and (3.1.3)) are not homogeneous and therefore the right-hand side of the resulting system must be modified accordingly. The total simulation time for this example is $T = 0.01$ and the time step is $\Delta t = 10^{-3}$. The time step is sufficiently small, so that the time discretization error does not affect the convergence rates.

Tables 3.5.1 and 4.4.1 show the convergence history for a sequence of quasi-uniform mesh refinements with non-matching grids along the interface employing conforming and non-conforming spaces for the Lagrange multipliers (cf. (3.3.1)–(3.3.2)), respectively. In the tables, h_f and h_p denote the mesh sizes in Ω_f and Ω_p, respectively, while the mesh sizes for their traces on Γ_{fp} are h_{tf} and h_{tp}, satisfying $h_{tf} = \frac{5}{8} h_{tp}$. We note that the Stokes pressure and the displacement at time t_m are recovered by the post-processed formulae $p_f^m = -\frac{1}{n}(\mathrm{tr}(\boldsymbol{\sigma}_f^m) - 2\,\mu\, q_f^m)$ (cf. (3.1.2)) and $\boldsymbol{\eta}_p^m = \boldsymbol{\eta}_p^{m-1} + \Delta t\, \mathbf{u}_s^m$ (cf. Remark 3.3.3), respectively. The results illustrate that spatial rates of convergence $\mathcal{O}(h)$, as provided by



$$\mu = 1, \quad \alpha_p = 1, \quad \lambda_p = 1, \quad \mu_p = 1,$$

$$s_0 = 1, \quad \mathbf{K} = \mathbf{I}, \quad \alpha_{\mathtt{BJS}} = 1,$$

$$\mathbf{u}_f = \pi\,\cos(\pi\,t)\begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix},$$

$$p_f = \exp(t)\,\sin(\pi\,x)\,\cos\left(\frac{\pi\,y}{2}\right) + 2\,\pi\,\cos(\pi\,t),$$

$$p_p = \exp(t)\,\sin(\pi\,x)\,\cos\left(\frac{\pi\,y}{2}\right),$$

$$\mathbf{u}_p = -\frac{1}{\mu}\,\mathbf{K}\,\nabla p_p, \quad \boldsymbol{\eta}_p = \sin(\pi\,t)\begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}.$$

Figure 3.5.1: Example 1, domain and coarsest mesh level (left), parameters and analytical solution (right).

Theorem 3.4.5, are attained for all subdomain variables in their natural norms. The Lagrange multiplier variables, which are approximated in $\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1$ and $\mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$, exhibit rates of convergence $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h^2)$ in the $\mathrm{H}^{1/2}$ and L^2-norms on Γ_{fp}, respectively, which is consistent with the order of approximation.

3.5.2 Example 2: coupled surface and subsurface flows

In this example, we simulate coupling of surface and subsurface flows, which could be used to describe the interaction between a river and an aquifer. We consider the domain $\Omega = (0,2) \times (-1,1)$. We associate the upper half with the river flow modeled by Stokes equations, while the lower half represents the flow in the aquifer governed by the Biot system. The appropriate interface conditions are enforced along the interface $y = 0$. In this example we focus on the qualitative behavior of the solution and use unit physical parameters:

$$\mu = 1, \quad \alpha_p = 1, \quad \lambda_p = 1, \quad \mu_p = 1, \quad s_0 = 1, \quad \mathbf{K} = \mathbf{I}, \quad \alpha_{\mathtt{BJS}} = 1.$$

The body forces terms and external source are set to zero, as well as the initial conditions. The flow is driven through a parabolic fluid velocity on the left boundary of the fluid region with boundary conditions specified as follows:

$$
\begin{aligned}
&\mathbf{u}_f = (-40y(y-1)\;\; 0)^t && \text{on} \quad \Gamma_{f,left},\\
&\mathbf{u}_f = \mathbf{0} && \text{on} \quad \Gamma_{f,top},\\
&\boldsymbol{\sigma}_f \mathbf{n}_f = \mathbf{0} && \text{on} \quad \Gamma_{f,right},\\
&p_p = 0 \quad \text{and} \quad \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0} && \text{on} \quad \Gamma_{p,bottom},\\
&\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{and} \quad \mathbf{u}_s = \mathbf{0} && \text{on} \quad \Gamma_{p,left} \cup \Gamma_{p,right}.
\end{aligned}
$$

The simulation is run for a total time $T = 3$ with a time step $\Delta t = 0.06$. The computed solution is presented in Figure 3.5.2. From the velocity plot (top left), we see that the flow in the Stokes region is moving primarily from left to right, driven by the parabolic inflow condition, with some of the fluid percolating downward into the poroelastic medium due to the zero pressure at the bottom, which simulates gravity. The mass conservation $\mathbf{u}_f \cdot \mathbf{n}_f + \big(\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p\big) \cdot \mathbf{n}_p = 0$ on the interface with $\mathbf{n}_p = (0,1)^{\mathrm{t}}$ indicates the continuity of the second components of the fluid velocity and Darcy velocity when the displacement becomes steady, which is observed from the color plot of the vertical velocity. The stress plots (top middle and right) illustrate the ability of our fully mixed formulation to compute accurate $\mathbb{H}(\mathbf{div})$ stresses in both the fluid and poroelastic regions, without the need for numerical differentiation. In addition, the conservation of momentum $\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0}$ and balance of normal stress $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -p_p$ imply that $\boldsymbol{\sigma}_{f,12} = \boldsymbol{\sigma}_{p,12}$, $\boldsymbol{\sigma}_{f,22} = \boldsymbol{\sigma}_{p,22}$ and $-\boldsymbol{\sigma}_{f,22} = p_p$ on the interface. These conditions are verified from the top middle and right color plots, as well as the bottom left plot. Furthermore, the arrows in the stress plots are formed by the second columns of the stresses, whose traces on the interface are $\boldsymbol{\sigma}_f \mathbf{n}_f$ and $-\boldsymbol{\sigma}_p \mathbf{n}_p$, respectively. For visualization purpose, the Stokes stress is scaled by a factor of 1/5 compared to the poroelastic stress, due to large difference in their magnitudes away from the interface. Nevertheless, the continuity of the vector field across the interface is evident, consistent with the conservation of momentum condition $\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0}$. The overall qualitative behavior of the computed stresses is consistent with the specified boundary and interface conditions. In particular, we observe large fluid stress along the top boundary due to the no slip condition, as well as along the interface due to the slip with friction condition. The singularity near the lower left corner of the Stokes region is due to the mismatch in boundary conditions between the fluid and poroelastic regions. Finally, the last plot shows that the inflow from the Stokes region causes deformation of the poroelastic medium.

3.5.3 Example 3: irregularly shaped fluid-filled cavity

This example features highly irregularly shaped cavity motivated by modeling flow through vuggy or naturally fractured reservoirs or aquifers. It uses physical units and realistic parameter values taken from the reservoir engineering literature [54]:

$$\mu = 10^{-6} \text{ kPa s}, \quad \alpha_p = 1, \quad \lambda_p = 5/18 \times 10^7 \text{ kPa}, \quad \mu_p = 5/12 \times 10^7 \text{ kPa},$$

$$s_0 = 6.89 \times 10^{-2} \text{ kPa}^{-1}, \quad \mathbf{K} = 10^{-8} \times \mathbf{I} \text{ m}^2, \quad \alpha_{\texttt{BJS}} = 1.$$

We emphasize that the problem features very small permeability and storativity, as well as large Lamé parameters. These are parameter regimes that are known to lead locking in modeling of the Biot system of poroelasticity [63, 83]. The domain is $\Omega = (0,1) \times (0,1)$, with a large fluid-filled cavity in the interior. The body forces and external sources are set to zero. The flow is driven from left to right via a pressure drop of 1 kPa, with boundary conditions specified as follows:

$$\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f = 1000, \quad \mathbf{u}_f \cdot \mathbf{t}_f = 0 \quad \text{on} \quad \Gamma_{f,right},$$

$$p_p = 1001 \quad \text{on} \quad \Gamma_{p,left}, \; p_p = 1000 \quad \text{on} \quad \Gamma_{p,right} \quad \text{and} \quad \mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_{p,top} \cup \Gamma_{p,bottom},$$

$$\boldsymbol{\sigma}_p \, \mathbf{n}_p = -\alpha_p \, p_p \, \mathbf{n}_p \quad \text{on} \quad \Gamma_{p,left} \cup \Gamma_{p,right} \quad \text{and} \quad \mathbf{u}_s = \mathbf{0} \quad \text{on} \quad \Gamma_{p,top} \cup \Gamma_{p,bottom}.$$

The total simulation time is $T = 10\,\text{s}$ with a time step of size $\Delta t = 0.05\,\text{s}$. To avoid inconsistency between the initial and boundary conditions for p_p, we start with $p_p = 1000$ on $\Gamma_{p,left}$ and gradually increase it to reach $p_p = 1001$ at $t = 0.5\,\text{s}$. Similar adjustment is done for $\boldsymbol{\sigma}_p \mathbf{n}_p$.

The simulation results at the final time $T = 10\,\text{s}$ are shown in Figure 3.5.3. In the top plots, we present the Darcy pressure and Darcy velocity vector, the displacement vector with its magnitude, and the first row of the poroelastic stress with its magnitude. Since the pressure variation is small relative to its value, for visualization purpose we plot its difference from the reference pressure, $p_p - 1000$. The Darcy velocity and the pressure drop are largest in the region between the left inflow boundary and the cavity. The displacement is largest around the cavity, due to the large fluid velocity within the cavity and the slip with friction interface condition. The poroelastic stress exhibits singularities near some of the sharp tips of the cavity. The bottom plots show the fluid pressure and velocity vector, the velocity vector with its magnitude, and the first row of the fluid stress with its magnitude. Similarly to the Darcy pressure, we plot $p_f - 1000$. A channel-like flow profile is clearly visible within the cavity, with the largest velocity along a central path away from the cavity walls. The fluid pressure is decreasing from left to right along the central path of the cavity. Consistent with the poroelastic stress, the fluid stress near the tips of the cavity is relatively larger. We emphasize that, despite the locking regime of the parameters, the computed solution is free of locking and spurious oscillations. This example illustrates the ability of our method to handle computationally challenging problems with physically realistic parameters in poroelastic locking regimes.

h_f	$\|e_{\boldsymbol{\sigma}_f}\|_{\ell^2(0,T;\mathbb{X}_f)}$ error	rate	$\|e_{\mathbf{u}_f}\|_{\ell^2(0,T;\mathbf{V}_f)}$ error	rate	$\|e_{\boldsymbol{\gamma}_f}\|_{\ell^2(0,T;\mathbb{Q}_f)}$ error	rate	$\|e_{p_f}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega_f))}$ error	rate
0.1964	2.2E-02	–	2.7E-02	–	2.4E-03	–	6.3E-03	–
0.0997	1.2E-02	0.95	1.4E-02	1.00	9.3E-04	1.41	3.1E-03	1.05
0.0487	5.7E-03	0.99	6.8E-03	0.99	4.2E-04	1.11	1.6E-03	0.93
0.0250	2.9E-03	1.04	3.4E-03	1.04	2.0E-04	1.13	7.8E-04	1.07
0.0136	1.4E-03	1.14	1.7E-03	1.15	9.4E-05	1.23	3.9E-04	1.15
0.0072	7.1E-04	1.08	8.4E-04	1.10	4.7E-05	1.09	2.0E-04	1.02

h_p	$\|e_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{X}_p)}$ error	rate	$\|e_{\mathbf{u}_s}\|_{\ell^2(0,T;\mathbf{V}_s)}$ error	rate	$\|e_{\boldsymbol{\gamma}_p}\|_{\ell^2(0,T;\mathbb{Q}_p)}$ error	rate	$\|e_{\mathbf{u}_p}\|_{\ell^2(0,T;\mathbf{V}_p)}$ error	rate	$\|e_{p_p}\|_{\ell^\infty(0,T;\mathrm{W}_p)}$ error	rate
0.2828	2.7E-01	–	4.3E-02	–	3.4E-02	–	1.0E-01	–	7.5E-02	–
0.1646	1.4E-01	1.27	2.2E-02	1.23	9.4E-03	2.38	5.2E-02	1.27	3.8E-02	1.25
0.0779	6.7E-02	0.97	1.1E-02	0.96	2.2E-03	1.96	2.5E-02	1.00	1.9E-02	0.93
0.0434	3.4E-02	1.17	5.4E-03	1.19	5.8E-04	2.25	1.2E-02	1.24	9.4E-03	1.22
0.0227	1.7E-02	1.06	2.7E-03	1.07	2.0E-04	1.68	5.9E-03	1.08	4.7E-03	1.07
0.0124	8.4E-03	1.15	1.4E-03	1.15	8.1E-05	1.48	2.9E-03	1.15	2.4E-03	1.14

$\|e_{\boldsymbol{\eta}_p}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega_p))}$ error	rate	h_{tf}	$\|e_{\boldsymbol{\varphi}}\|_{\ell^2(0,T;\boldsymbol{\Lambda}_f)}$ error	rate	h_{tp}	$\|e_{\boldsymbol{\theta}}\|_{\ell^2(0,T;\boldsymbol{\Lambda}_s)}$ error	rate	$\|e_{\lambda}\|_{\ell^2(0,T;\Lambda_p)}$ error	rate
2.7E-04	–	1/8	1.6E-03	–	1/5	1.6E-02	–	6.9E-03	–
1.4E-04	1.23	1/16	3.7E-04	2.11	1/10	5.7E-03	1.49	2.5E-03	1.49
6.7E-05	0.96	1/32	1.3E-04	1.45	1/20	1.2E-03	2.31	8.5E-04	1.52
3.4E-05	1.19	1/64	4.6E-05	1.54	1/40	3.4E-04	1.76	3.0E-04	1.50
1.7E-05	1.07	1/128	1.2E-05	1.96	1/80	1.1E-04	1.62	1.1E-04	1.50
8.4E-06	1.15	1/256	3.6E-06	1.70	1/160	2.2E-05	2.34	3.7E-05	1.54

Table 3.5.1: Example 1, errors and convergence rates with piecewise linear Lagrange multipliers.

h_f	$\|e_{\boldsymbol{\sigma}_f}\|_{\ell^2(0,T;\mathbb{X}_f)}$ error	rate	$\|e_{\mathbf{u}_f}\|_{\ell^2(0,T;\mathbf{V}_f)}$ error	rate	$\|e_{\boldsymbol{\gamma}_f}\|_{\ell^2(0,T;\mathbb{Q}_f)}$ error	rate	$\|e_{p_f}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega_f))}$ error	rate
0.1964	2.2E-02	–	2.7E-02	–	2.4E-03	–	6.1E-03	–
0.0997	1.2E-02	0.94	1.4E-02	1.00	9.7E-04	1.31	3.1E-03	1.02
0.0487	5.7E-03	0.99	6.8E-03	0.99	4.2E-04	1.16	1.6E-03	0.92
0.0250	2.8E-03	1.04	3.4E-03	1.04	2.0E-04	1.13	7.8E-04	1.07
0.0136	1.4E-03	1.14	1.7E-03	1.15	9.4E-05	1.23	3.9E-04	1.15
0.0072	7.1E-04	1.08	8.4E-04	1.09	4.7E-05	1.09	2.0E-04	1.02

h_p	$\|e_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{X}_p)}$ error	rate	$\|e_{\mathbf{u}_s}\|_{\ell^2(0,T;\mathbf{V}_s)}$ error	rate	$\|e_{\boldsymbol{\gamma}_p}\|_{\ell^2(0,T;\mathbb{Q}_p)}$ error	rate	$\|e_{\mathbf{u}_p}\|_{\ell^2(0,T;\mathbf{V}_p)}$ error	rate	$\|e_{p_p}\|_{\ell^\infty(0,T;\mathrm{W}_p)}$ error	rate
0.2828	2.7E-01	–	4.3E-02	–	3.4E-02	–	1.0E-01	–	7.5E-02	–
0.1646	1.4E-01	1.27	2.2E-02	1.23	9.4E-03	2.39	5.2E-02	1.26	3.8E-02	1.25
0.0779	6.7E-02	0.97	1.1E-02	0.96	2.2E-03	1.96	2.5E-02	1.00	1.9E-02	0.93
0.0434	3.4E-02	1.17	5.4E-03	1.19	5.8E-04	2.25	1.2E-02	1.24	9.4E-03	1.22
0.0227	1.7E-02	1.06	2.7E-03	1.07	2.0E-04	1.67	5.9E-03	1.08	4.7E-03	1.07
0.0124	8.4E-03	1.15	1.4E-03	1.15	8.1E-05	1.48	2.9E-03	1.15	2.4E-03	1.14

$\|e_{\boldsymbol{\eta}_p}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega_p))}$ error	rate	h_{tf}	$\|e_{\boldsymbol{\varphi}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	h_{tp}	$\|e_{\boldsymbol{\theta}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	$\|e_{\lambda}\|_{\ell^2(0,T;\mathrm{L}^2(\Gamma_{fp}))}$ error	rate
2.7E-04	–	1/8	4.1E-04	–	1/5	7.9E-03	–	1.1E-03	–
1.4E-04	1.23	1/16	2.0E-04	1.04	1/10	2.9E-03	1.46	3.1E-04	1.87
6.7E-05	0.96	1/32	2.4E-05	3.07	1/20	5.7E-04	2.34	7.7E-05	2.01
3.4E-05	1.19	1/64	6.4E-06	1.89	1/40	1.5E-04	1.89	1.9E-05	2.00
1.7E-05	1.07	1/128	1.6E-06	1.97	1/80	3.8E-05	2.01	4.9E-06	1.98
8.4E-06	1.15	1/256	4.0E-07	2.02	1/160	9.0E-06	2.09	1.2E-06	2.09

Table 3.5.2: Example 1, errors and convergence rates with discontinuous piecewise linear Lagrange multipliers.

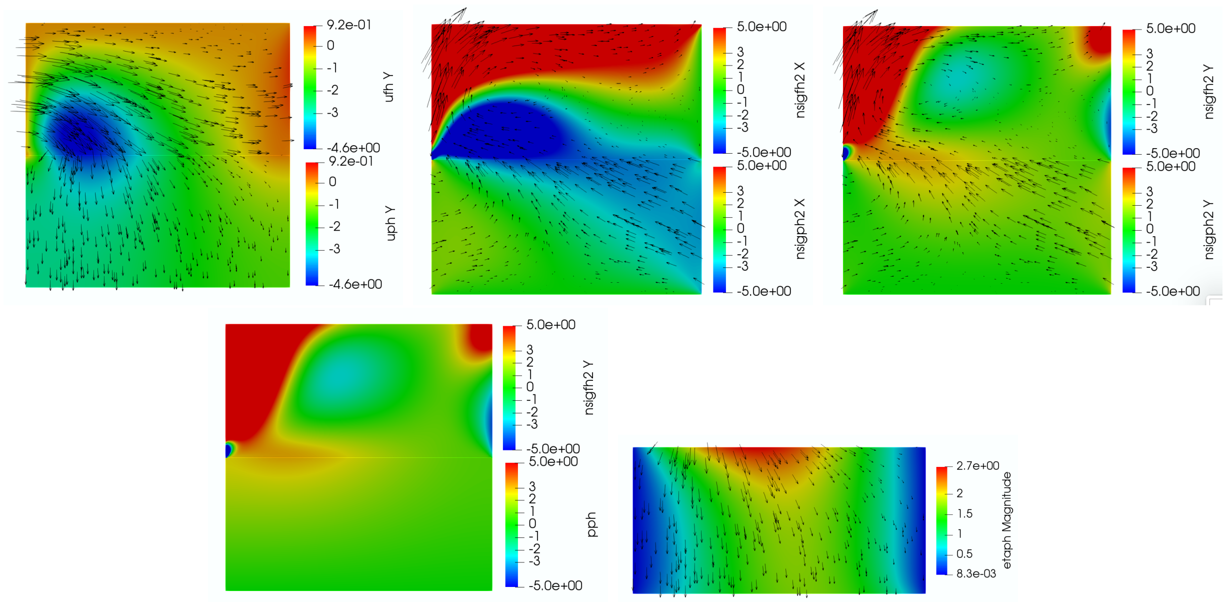


Figure 3.5.2: Example 2, computed solution at T = 3.

Top left: velocities $\mathbf{u}_{fh}$ and $\mathbf{u}_{ph}$ (arrows), $\mathbf{u}_{fh,2}$ and $\mathbf{u}_{ph,2}$ (color). Top middle and right: negative stresses $-(\boldsymbol{\sigma}_{fh,12}, \boldsymbol{\sigma}_{fh,22})^{\mathrm{t}}$ and $-(\boldsymbol{\sigma}_{ph,12}, \boldsymbol{\sigma}_{ph,22})^{\mathrm{t}}$ (arrows); middle: $-\boldsymbol{\sigma}_{fh,12}$ and $-\boldsymbol{\sigma}_{ph,12}$ (color); right: $-\boldsymbol{\sigma}_{fh,22}$ and $-\boldsymbol{\sigma}_{ph,22}$ (color). Bottom left: negative Stokes stress $-\boldsymbol{\sigma}_{fh,22}$ and Darcy pressure p_{ph}. Bottom right: displacement $\boldsymbol{\eta}_{ph}$ (arrows) and its magnitude (color).

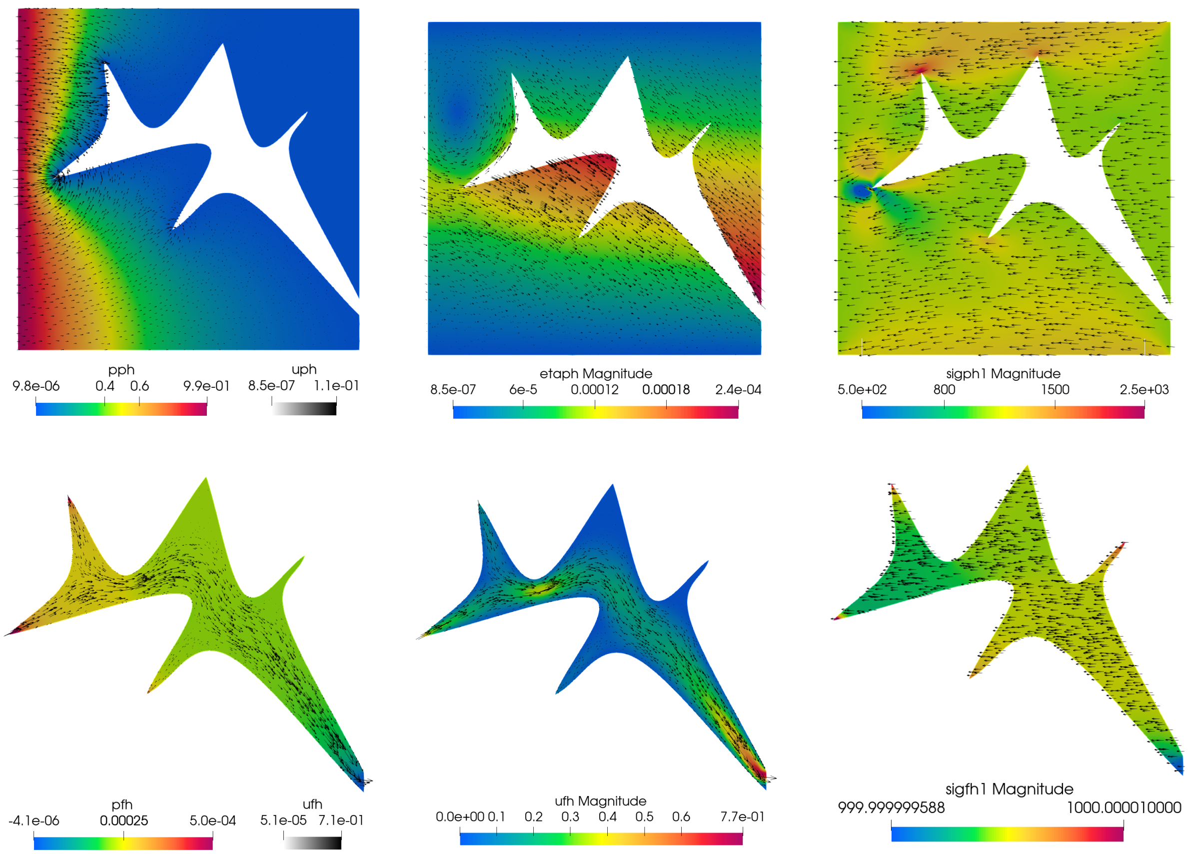


Figure 3.5.3: Example 3, computed solution at T = 10 s.

Top left: Darcy velocity (arrows) and pressure (color). Top middle: displacement (arrows) and its magnitude (color). Top right: first row of the poroelastic stress tensor (arrows) and its magnitude (color). Bottom left: Stokes velocity (arrows) and pressure (color). Bottom middle: Stokes velocity (arrows) and its magnitude (color). Bottom right: first row of the Stokes stress (arrows) and its magnitude (color).

4.0 An augmented fully-mixed formulation for the quasi-static Navier-Stokes – Biot model

4.1 The model problem and weak formulation

We consider the same Lipschitz domain consisted of fluid region Ω_f and poroelastic region Ω_p. Let ρ_f be the density, with other terms defined as in Section 2.1. We assume that the flow in Ω_f is governed by the Navier–Stokes equations:

$$\rho_f\,(\nabla\mathbf{u}_f)\,\mathbf{u}_f - \mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f, \quad \mathrm{div}(\mathbf{u}_f) = q_f \quad \text{in} \quad \Omega_f\times(0,T], \tag{4.1.1a}$$

$$(\boldsymbol{\sigma}_f - \rho_f\,(\mathbf{u}_f\otimes\mathbf{u}_f))\,\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}}\times(0,T], \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{D}}\times(0,T], \tag{4.1.1b}$$

where $\Gamma_f = \Gamma_f^{\mathrm{N}}\cup\Gamma_f^{\mathrm{D}}$, $\mathbf{e}(\mathbf{u}_f)$ and $\boldsymbol{\sigma}_f$ denote the deformation and the stress tensors, respectively:

$$\mathbf{e}(\mathbf{u}_f) := \frac{1}{2}\,\left(\nabla\mathbf{u}_f + (\nabla\mathbf{u}_f)^{\mathrm{t}}\right), \quad \boldsymbol{\sigma}_f := -\,p_f\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_f).$$

While the standard strong Navier–Stokes equations are presented above to describe the behaviour of the fluid in Ω_f, in this thesis we make use of an equivalent version of (4.1.1) based on the introduction of a pseudostress tensor relating the stress tensor $\boldsymbol{\sigma}_f$ with the convective term. More precisely, analogously to [30, 32, 34], we introduce the nonlinear-pseudostress tensor

$$\mathbf{T}_f := \boldsymbol{\sigma}_f - \rho_f\,(\mathbf{u}_f\otimes\mathbf{u}_f) = -\,p_f\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_f) - \rho_f\,(\mathbf{u}_f\otimes\mathbf{u}_f) \quad \text{in} \quad \Omega_f\times(0,T].$$

In this way, owing to the fact that $\mathrm{tr}(\mathbf{e}(\mathbf{u}_f)) = \mathrm{div}(\mathbf{u}_f) = q_f$, we find that (4.1.1) can be rewritten, equivalently, as the set of equations with unknowns $\mathbf{T}_f$ and $\mathbf{u}_f$, given by

$$\frac{1}{2\,\mu}\,\mathbf{T}_f^{\mathrm{d}} = \nabla\mathbf{u}_f - \boldsymbol{\gamma}_f(\mathbf{u}_f) - \frac{\rho_f}{2\,\mu}\,(\mathbf{u}_f\otimes\mathbf{u}_f)^{\mathrm{d}} - \frac{1}{n}\,q_f\,\mathbf{I} \quad \text{in} \quad \Omega_f\times(0,T], \tag{4.1.2a}$$

$$-\,\rho_f\,q_f\,\mathbf{u}_f - \mathbf{div}(\mathbf{T}_f) = \mathbf{f}_f, \quad \mathbf{T}_f = \mathbf{T}_f^{\mathrm{t}} \quad \text{in} \quad \Omega_f\times(0,T], \tag{4.1.2b}$$

$$\mathbf{T}_f\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{N}\times(0,T], \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{D}\times(0,T], \tag{4.1.2c}$$

$$p_f = -\frac{1}{n}\,(\mathrm{tr}(\mathbf{T}_f) + \rho_f\,\mathrm{tr}(\mathbf{u}_f\otimes\mathbf{u}_f) - 2\,\mu\,q_f) \quad \text{in} \quad \Omega_f\times(0,T], \tag{4.1.2d}$$

where $\boldsymbol{\gamma}_f(\mathbf{u}_f) := \frac{1}{2}\,(\nabla\mathbf{u}_f - (\nabla\mathbf{u}_f)^{\mathrm{t}})$ is the vorticity (or skew-symmetric part of the velocity gradient tensor $\nabla\mathbf{u}_f$). Notice that (4.1.2d) allows us to eliminate the pressure p_f from the system (which anyway can be approximated later on through a post-processing procedure). For simplicity we assume that $|\Gamma_f^{\mathrm{N}}| > 0$, which will allow us to control $\mathbf{T}_f$ by $\mathbf{T}_f^{\mathrm{d}}$. The case $|\Gamma_f^{\mathrm{N}}| = 0$ can be handled as in [50–52] by introducing an additional variable corresponding to the mean value of $\mathrm{tr}(\mathbf{T}_f)$.

The Biot system is similar as in Section 2.1, but with different boundary conditions for simplicity:

$$-\mathrm{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p \quad \text{in} \quad \Omega_p\times(0,T], \quad \mu\,\mathbf{K}^{-1}\,\mathbf{u}_p + \nabla\,p_p = \mathbf{0} \quad \text{in} \quad \Omega_p\times(0,T], \tag{4.1.3a}$$

$$\frac{\partial}{\partial t}\big(s_0\,p_p + \alpha_p\,\mathrm{div}(\boldsymbol{\eta}_p)\big) + \mathrm{div}(\mathbf{u}_p) = q_p \quad \text{in} \quad \Omega_p\times(0,T], \tag{4.1.3b}$$

$$\mathbf{u}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}\times(0,T], \quad p_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}}\times(0,T], \quad \boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \Gamma_p\times(0,T]. \tag{4.1.3c}$$

The transmission conditions are the same as the one in Section 2.1 of Chapter 2. We present them here for completeness.

$$\mathbf{u}_f\cdot\mathbf{n}_f + \left(\frac{\partial\,\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot\mathbf{n}_p \;=\; 0 \quad \text{on} \quad \Gamma_{fp}\times(0,T], \tag{4.1.4a}$$

$$\boldsymbol{\sigma}_f\mathbf{n}_f + \boldsymbol{\sigma}_p\mathbf{n}_p \;=\; \mathbf{0} \quad \text{on} \quad \Gamma_{fp}\times(0,T], \tag{4.1.4b}$$

$$\boldsymbol{\sigma}_f\mathbf{n}_f + \mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\sqrt{\mathbf{K}_j^{-1}}\left\{\left(\mathbf{u}_f - \frac{\partial\,\boldsymbol{\eta}_p}{\partial t}\right)\cdot\mathbf{t}_{f,j}\right\}\mathbf{t}_{f,j} \;=\; -\,p_p\mathbf{n}_f \quad \text{on} \quad \Gamma_{fp}\times(0,T]. \tag{4.1.4c}$$

We remark here that (4.1.4b)–(4.1.4c) can be rewritten in terms of tensor $\mathbf{T}_f$ as follows:

$$\begin{gathered}\mathbf{T}_f\mathbf{n}_f + \rho_f\,(\mathbf{u}_f\otimes\mathbf{u}_f)\mathbf{n}_f + \boldsymbol{\sigma}_p\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{fp}\times(0,T],\\ \mathbf{T}_f\mathbf{n}_f + \rho_f\,(\mathbf{u}_f\otimes\mathbf{u}_f)\mathbf{n}_f\\ = -\,\mu\,\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\sqrt{\mathbf{K}_j^{-1}}\left\{\left(\mathbf{u}_f - \frac{\partial\boldsymbol{\eta}_p}{\partial t}\right)\cdot\mathbf{t}_{f,j}\right\}\mathbf{t}_{f,j} - p_p\mathbf{n}_f \quad \text{on} \quad \Gamma_{fp}\times(0,T],\end{gathered} \tag{4.1.5}$$

Finally, the above system of equations is complemented by the initial condition $p_p(\mathbf{x},0) = p_{p,0}(\mathbf{x})$ in Ω_p. We stress that, similarly to [65], compatible initial data for the rest of the variables can be constructed from $p_{p,0}$ in a way that all equations in the system (4.1.2), (4.1.3), (4.1.4a) and (4.1.5), except for the unsteady conservation of mass equation (4.1.3b), hold at $t = 0$. This will be established in Lemma 4.2.10 below. We will consider a weak formulation with a time-differentiated elasticity equation and compatible initial data $(\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

We then proceed analogously to [4, Section 3] (see also [34, 50]) and derive a weak formulation of the coupled problem given by (4.1.2), (4.1.3), (4.1.4a) and (4.1.5). Similarly to [32,34], in the sequel we will employ the following Hilbert spaces to deal with the nonlinear pseudostress tensor and velocity of the Navier–Stokes equation, respectively, that is

$$\mathbb{X}_f := \Big\{ \mathbf{R}_f \in \mathbb{H}(\mathbf{div};\Omega_f) : \quad \mathbf{R}_f\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}} \Big\},$$
$$\mathbf{V}_f := \Big\{ \mathbf{v}_f \in \mathbf{H}^1(\Omega_f) : \quad \mathbf{v}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{D}} \Big\},$$

endowed with the corresponding norms

$$\|\mathbf{R}_f\|_{\mathbb{X}_f} := \|\mathbf{R}_f\|_{\mathbb{H}(\mathbf{div};\Omega_f)}, \quad \|\mathbf{v}_f\|_{\mathbf{V}_f} := \|\mathbf{v}_f\|_{\mathbf{H}^1(\Omega_f)}.$$

For the Biot region, we begin by introducing the structure velocity $\mathbf{u}_s := \partial_t\,\boldsymbol{\eta}_p \in \mathbf{V}_s$ satisfying $\mathbf{u}_s = \mathbf{0}$ on Γ_p, cf. (4.1.3c), the rotation operator $\boldsymbol{\rho}_p := \dfrac{1}{2}(\nabla\boldsymbol{\eta}_p - \nabla\boldsymbol{\eta}_p^{\mathrm{t}})$ and its time derivative, that is, the structure rotation operator $\boldsymbol{\gamma}_p := \partial_t\boldsymbol{\rho}_p = \dfrac{1}{2}\left(\nabla\mathbf{u}_s - (\nabla\mathbf{u}_s)^{\mathrm{t}}\right)$ which will be used in the weak formulation. In turn, we set the spaces $\mathbb{X}_p := \mathbb{H}(\mathbf{div};\Omega_p)$, $\mathbf{V}_s := \mathbf{L}^2(\Omega_p)$, $\mathrm{W}_p := \mathrm{L}^2(\Omega_p)$ and introduce the following subspaces of $\mathbb{L}^2(\Omega_p)$ and $\mathbf{H}(\mathrm{div};\Omega_p)$, respectively

$$\mathbb{Q}_p := \Big\{ \boldsymbol{\chi}_p \in \mathbb{L}^2(\Omega_p) : \quad \boldsymbol{\chi}_p^{\mathrm{t}} = -\,\boldsymbol{\chi}_p \Big\},$$
$$\mathbf{V}_p := \Big\{ \mathbf{v}_p \in \mathbf{H}(\mathrm{div};\Omega_p) : \quad \mathbf{v}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}} \Big\},$$

endowed with the standard norms. In addition, we need to introduce two Lagrange multipliers which has a meaning of the structure velocity and Darcy pressure on the interface, respectively,

$$\boldsymbol{\theta} := \mathbf{u}_s|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_s \quad \text{and} \quad \lambda := p_p|_{\Gamma_{fp}} \in \Lambda_p,$$

together with their spaces $\Lambda_p := (\mathbf{V}_p \cdot \mathbf{n}_p)'$ and $\boldsymbol{\Lambda}_s := (\mathbb{X}_p\,\mathbf{n}_p)'$. We take $\Lambda_p := \mathrm{H}^{1/2}(\Gamma_{fp})$ as in Section 2.1 and recall that it holds that

$$\langle \mathbf{v}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} \leq C \|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)} \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \quad \forall \mathbf{v}_p \in \mathbf{V}_p,\ \xi \in \mathrm{H}^{1/2}(\Gamma_{fp}). \tag{4.1.6}$$

Now for $\boldsymbol{\Lambda}_s$, observe that, if $\mathbf{E}_{0,p} : \mathbf{H}^{1/2}(\Gamma_{fp}) \to \mathbf{L}^2(\partial\Omega_p)$ is the extension operator defined by

$$\mathbf{E}_{0,p}(\boldsymbol{\phi}) := \begin{cases} \boldsymbol{\phi} & \text{on} \quad \Gamma_{fp} \\ \mathbf{0} & \text{on} \quad \Gamma_p \end{cases} \qquad \forall\, \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{fp}),$$

then it holds that $\forall \boldsymbol{\tau}_p \in \mathbb{X}_p,\ \boldsymbol{\phi} \in \mathbf{H}^{1/2}_{p,0}(\Gamma_{fp})$,

$$\langle \boldsymbol{\tau}_p\,\mathbf{n}_p, \boldsymbol{\phi} \rangle_{\Gamma_{fp}} = \langle \boldsymbol{\tau}_p\,\mathbf{n}_p, \mathbf{E}_{0,p}(\boldsymbol{\phi}) \rangle_{\partial\Omega_p} \leq C \|\boldsymbol{\tau}_p\|_{\mathbb{H}(\mathbf{div};\Omega_p)} \|\mathbf{E}_{0,p}(\boldsymbol{\phi})\|_{\mathbf{H}^{1/2}(\partial\Omega_p)}, \tag{4.1.7}$$

where

$$\begin{aligned} \mathbf{H}^{1/2}_{p,0}(\Gamma_{fp}) := & \ \Big\{ \mathbf{v}|_{\Gamma_{fp}} : \quad \mathbf{v} \in \mathbf{H}^1(\Omega_p) \quad \text{and} \quad \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_p \Big\} \\ = & \ \Big\{ \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{fp}) : \quad \mathbf{E}_{0,p}(\boldsymbol{\phi}) \in \mathbf{H}^{1/2}(\partial\Omega_p) \Big\}. \end{aligned}$$

Thus analogously to [34, 50] we take $\boldsymbol{\Lambda}_s := \mathbf{H}^{1/2}_{p,0}(\Gamma_{fp})$. In this way, the spaces Λ_p and $\boldsymbol{\Lambda}_s$ are endowed with the norms

$$\|\xi\|_{\Lambda_p} := \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})} \quad \text{and} \quad \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} := \|\mathbf{E}_{0,p}(\boldsymbol{\phi})\|_{\mathbf{H}^{1/2}(\partial\Omega_p)}.$$

We now proceed with the derivation of our Lagrange multiplier variational formulation for the coupling of the Navier–Stokes – Biot problems. Similarly to [4, 34], we test (4.1.2a) with arbitrary $\mathbf{R}_f \in \mathbb{X}_f$, integrate by parts and utilize the fact that $\mathbf{T}^{\mathrm{d}}_f : \mathbf{R}_f = \mathbf{T}^{\mathrm{d}}_f : \mathbf{R}^{\mathrm{d}}_f$. We apply the same derivation process as in Section 2.1 for the Biot model, then impose the remaining equations weakly, as well as the symmetry of $\mathbf{T}_f$ and $\boldsymbol{\sigma}_p$, and the transmission conditions in (4.1.4a) and (4.1.5) to obtain the variational problem,

$$\begin{aligned} & \frac{1}{2\mu}(\mathbf{T}^{\mathrm{d}}_f, \mathbf{R}^{\mathrm{d}}_f)_{\Omega_f} + (\mathbf{u}_f, \mathbf{div}(\mathbf{R}_f))_{\Omega_f} - \langle \mathbf{R}_f \mathbf{n}_f, \mathbf{u}_f \rangle_{\Gamma_{fp}} + (\boldsymbol{\gamma}_f(\mathbf{u}_f), \mathbf{R}_f)_{\Omega_f} \\ & \quad + \frac{\rho_f}{2\mu}((\mathbf{u}_f \otimes \mathbf{u}_f)^{\mathrm{d}}, \mathbf{R}_f)_{\Omega_f} = -\frac{1}{n}\,(q_f\,\mathbf{I}, \mathbf{R}_f)_{\Omega_f}, \end{aligned} \tag{4.1.8a}$$

$$-\rho_f\,(q_f\,\mathbf{u}_f, \mathbf{v}_f)_{\Omega_f} - (\mathbf{v}_f, \mathbf{div}(\mathbf{T}_f))_{\Omega_f} = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}, \tag{4.1.8b}$$

$$-\,(\mathbf{T}_f, \boldsymbol{\gamma}_f(\mathbf{v}_f))_{\Omega_f} = 0, \tag{4.1.8c}$$

$$(\partial_t\, A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\,\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\mathbf{u}_s, \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p} - \langle \boldsymbol{\tau}_p\mathbf{n}_p, \boldsymbol{\theta}\rangle_{\Gamma_{fp}} = 0, \tag{4.1.8d}$$

$$-\,(\mathbf{v}_s, \mathbf{div}(\boldsymbol{\sigma}_p))_{\Omega_p} = (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}, \tag{4.1.8e}$$

$$-\,(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p)_{\Omega_p} = 0, \tag{4.1.8f}$$

$$\mu(\mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p} - (p_p, \mathrm{div}(\mathbf{v}_p))_{\Omega_p} + \langle \mathbf{v}_p\cdot\mathbf{n}_p, \lambda\rangle_{\Gamma_{fp}} = 0, \tag{4.1.8g}$$

$$s_0\,(\partial_t\, p_p, w_p)_{\Omega_p} + (\partial_t\, A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\,\mathbf{I}), \alpha_p\, w_p\,\mathbf{I})_{\Omega_p} + (w_p, \mathrm{div}(\mathbf{u}_p))_{\Omega_p} = (q_p, w_p)_{\Omega_p}, \tag{4.1.8h}$$

$$-\,\langle \mathbf{u}_f\cdot\mathbf{n}_f + (\boldsymbol{\theta} + \mathbf{u}_p)\cdot\mathbf{n}_p, \xi\rangle_{\Gamma_{fp}} = 0, \tag{4.1.8i}$$

$$\langle \boldsymbol{\phi}\cdot\mathbf{n}_p, \lambda\rangle_{\Gamma_{fp}} - \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\,(\mathbf{u}_f - \boldsymbol{\theta})\cdot\mathbf{t}_{f,j}, \boldsymbol{\phi}\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} + \langle \boldsymbol{\sigma}_p\mathbf{n}_p, \boldsymbol{\phi}\rangle_{\Gamma_{fp}} = 0, \tag{4.1.8j}$$

$$\begin{aligned}&\langle \mathbf{T}_f\mathbf{n}_f, \mathbf{v}_f\rangle_{\Gamma_{fp}} + \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\,(\mathbf{u}_f - \boldsymbol{\theta})\cdot\mathbf{t}_{f,j}, \mathbf{v}_f\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} \\ &\quad + \rho_f\,\langle \mathbf{u}_f\cdot\mathbf{n}_f, \mathbf{u}_f\cdot\mathbf{v}_f\rangle_{\Gamma_{fp}} + \langle \mathbf{v}_f\cdot\mathbf{n}_f, \lambda\rangle_{\Gamma_{fp}} = 0.\end{aligned} \tag{4.1.8k}$$

In the above, (4.1.8a)–(4.1.8c) are the Navier-Stokes equations, (4.1.8d)–(4.1.8f) are the elasticity equations, (4.1.8g)–(4.1.8h) are the Darcy equations, and (4.1.8i)–(4.1.8k) enforce weakly the interface conditions. Notice that, similarly to [2, eq. (3.5)] and since $\Big\{\boldsymbol{\gamma}_f(\mathbf{v}_f) : \mathbf{v}_f \in \mathbf{H}^1(\Omega_f)\Big\}$ is a proper-subspace of the skew-symmetric tensor space, (4.1.8c) imposes the symmetry of $\mathbf{T}_f$ in an ultra-weak sense. Notice also that the fifth term in (4.1.8a) and the third term in (4.1.8k) require $\mathbf{u}_f$ to live in a smaller space than $\mathbf{L}^2(\Omega_f)$. In fact, by applying the Cauchy–Schwarz and Hölder inequalities, the continuous injection $\mathbf{i}_c$ of $\mathbf{H}^1(\Omega_f)$ into $\mathbf{L}^4(\Omega_f)$ and $\mathbf{i}_\Gamma$ of $\mathbf{H}^{1/2}(\partial\Omega_f)$ into $\mathbf{L}^4(\partial\Omega_f)$, and the continuous trace operator $\gamma_0 : \mathbf{H}^1(\Omega_f) \to \mathbf{L}^2(\partial\Omega_f)$, we find that there holds

$$\begin{aligned}\big|((\mathbf{u}_f\otimes\mathbf{w}_f)^{\mathrm{d}}, \mathbf{R}_f)_{\Omega_f}\big| &\le \|\mathbf{u}_f\|_{\mathbf{L}^4(\Omega_f)}\|\mathbf{w}_f\|_{\mathbf{L}^4(\Omega_f)}\|\mathbf{R}_f\|_{\mathbb{L}^2(\Omega_f)} \\ &\le \|\mathbf{i}_c\|^2\|\mathbf{u}_f\|_{\mathbf{H}^1(\Omega_f)}\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)}\|\mathbf{R}_f\|_{\mathbb{X}_f},\end{aligned} \tag{4.1.9}$$

$$|\langle \mathbf{w}_f\cdot\mathbf{n}_f, \mathbf{u}_f\cdot\mathbf{v}_f\rangle_{\Gamma_{fp}}| \le \|\mathbf{i}_\Gamma\|^2\|\gamma_0\|\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)}\|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\|(\mathbf{R}_f, \mathbf{v}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f},$$

for all $\mathbf{u}_f, \mathbf{v}_f, \mathbf{w}_f \in \mathbf{H}^1(\Omega_f)$ and $\mathbf{T}_f, \mathbf{R}_f \in \mathbb{X}_f$. According to this, we propose to look for the unknown $\mathbf{u}_f$ in $\mathbf{V}_f$ and to restrict the set of corresponding test functions $\mathbf{v}_f$ to the same space. Finally, we augment the resulting system through the incorporation of the following redundant Galerkin-type terms:

$$\kappa_1 \left(\rho_f\, q_f\, \mathbf{u}_f + \mathbf{div}(\mathbf{T}_f), \mathbf{div}(\mathbf{R}_f)\right)_{\Omega_f} = -\,\kappa_1 \left(\mathbf{f}_f, \mathbf{div}(\mathbf{R}_f)\right)_{\Omega_f} \quad \forall\, \mathbf{R}_f \in \mathbb{X}_f, \tag{4.1.10a}$$

$$\kappa_2 \left(\mathbf{e}(\mathbf{u}_f) - \frac{\rho_f}{2\,\mu}\,(\mathbf{u}_f \otimes \mathbf{u}_f)^{\mathrm{d}} - \frac{1}{2\,\mu}\,\mathbf{T}_f^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f)\right)_{\Omega_f} = \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{v}_f))_{\Omega_f} \quad \forall\, \mathbf{v}_f \in \mathbf{V}_f, \tag{4.1.10b}$$

where κ_1 and κ_2 are positive parameters to be specified later. Notice that the foregoing terms are nothing but consistent expressions, arising from the equilibrium and constitutive equations. It is easy to see that each solution of the original system is also a solution of the resulting augmented one, and hence by solving the latter we find all the solutions of the former.

Remark 4.1.1. *The time differentiated equation* (4.1.8d) *allows us to eliminate the displacement variable* $\boldsymbol{\eta}_p$ *and obtain a formulation that uses only* $\mathbf{u}_s$. *As part of the analysis we will construct suitable initial data such that, by integrating* (4.1.8d) *in time, we can recover the original equation*

$$(A(\boldsymbol{\sigma}_p + \alpha p_p \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\boldsymbol{\rho}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\boldsymbol{\eta}_p, \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p} - \langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\omega}\rangle_{\Gamma_{fp}} = 0, \tag{4.1.11}$$

where $\boldsymbol{\omega} := \boldsymbol{\eta}_p|_{\Gamma_{fp}}$.

Now, it is clear that there are many different way of ordering the Lagrange multiplier formulation described above, but for the sake of the subsequent analysis, we proceed as in [4], and adopt one leading to an evolution problem in a mixed form. For this purpose, given $\mathbf{w}_f \in \mathbf{V}_f$, we set the following bilinear forms:

$$\begin{aligned} a_f(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) &:= \frac{1}{2\,\mu}\,(\mathbf{T}_f^{\mathrm{d}}, \mathbf{R}_f^{\mathrm{d}})_{\Omega_f} + \kappa_1\,(\mathbf{div}(\mathbf{T}_f), \mathbf{div}(\mathbf{R}_f))_{\Omega_f} \\ &+ \rho_f\,(q_f\,\mathbf{u}_f, \kappa_1\,\mathbf{div}(\mathbf{R}_f) - \mathbf{v}_f)_{\Omega_f} + (\mathbf{u}_f, \mathbf{div}(\mathbf{R}_f))_{\Omega_f} - (\mathbf{v}_f, \mathbf{div}(\mathbf{T}_f))_{\Omega_f} \\ &+ (\boldsymbol{\gamma}_f(\mathbf{u}_f), \mathbf{R}_f)_{\Omega_f} - (\mathbf{T}_f, \boldsymbol{\gamma}_f(\mathbf{v}_f))_{\Omega_f} + \langle \mathbf{T}_f \mathbf{n}_f, \mathbf{v}_f\rangle_{\Gamma_{fp}} - \langle \mathbf{R}_f \mathbf{n}_f, \mathbf{u}_f\rangle_{\Gamma_{fp}} \end{aligned}$$

$$+ \kappa_2 \left(\mathbf{e}(\mathbf{u}_f) - \frac{1}{2\,\mu}\, \mathbf{T}_f^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f) \right)_{\Omega_f},$$

$$\kappa_{\mathbf{w}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) := \frac{\rho_f}{2\,\mu} \left((\mathbf{u}_f \otimes \mathbf{w}_f)^{\mathrm{d}}, \mathbf{R}_f - \kappa_2\, \mathbf{e}(\mathbf{v}_f) \right)_{\Omega_f} + \rho_f\, \langle \mathbf{w}_f \cdot \mathbf{n}_f, \mathbf{u}_f \cdot \mathbf{v}_f \rangle_{\Gamma_{fp}},$$

$$a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) := (A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p w_p \mathbf{I})_{\Omega_p},$$

$$a_p(\mathbf{u}_p, \mathbf{v}_p) := \mu\, (\mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}, \quad b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi}) := \langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\phi} \rangle_{\Gamma_{fp}},$$

$$b_p(w_p, \mathbf{v}_p) := -\,(w_p, \mathrm{div}(\mathbf{v}_p))_{\Omega_p}, \quad b_s(\mathbf{v}_s, \boldsymbol{\tau}_p) := (\mathbf{v}_s, \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p}, \quad b_{\mathrm{sk}}(\boldsymbol{\chi}_p, \boldsymbol{\tau}_p) := (\boldsymbol{\chi}_p, \boldsymbol{\tau}_p)_{\Omega_p},$$

and the interface terms

$$a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) := \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_f - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, (\mathbf{v}_f - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}},$$

$$b_\Gamma(\mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\phi}; \xi) := \langle \mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\phi} + \mathbf{v}_p) \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}},$$

Hence, the Lagrange variational formulation for the system (4.1.8) and (4.1.10), reads: Given,

$$\mathbf{f}_f : [0,T] \to \mathbf{V}_f', \quad \mathbf{f}_p : [0,T] \to \mathbf{V}_s', \quad q_f : [0,T] \to \mathbb{X}_f', \quad q_p : [0,T] \to \mathrm{W}_p'$$

and $(\boldsymbol{\sigma}_{p,0}, p_{p,0}) \in \mathbb{X}_p \times \mathrm{W}_p$, find $(\boldsymbol{\sigma}_p, p_p, \mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}, \lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p) : [0,T] \to \mathbb{X}_p \times \mathrm{W}_p \times \mathbf{V}_p \times \mathbb{X}_f \times \mathbf{V}_f \times \boldsymbol{\Lambda}_s \times \Lambda_p \times \mathbf{V}_s \times \mathbb{Q}_p$, such that $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$, for a.e. $t \in (0,T)$ and for all $\boldsymbol{\tau}_p \in \mathbb{X}_p, w_p \in \mathrm{W}_p, \mathbf{v}_p \in \mathbf{V}_p, \mathbf{R}_f \in \mathbb{X}_f, \mathbf{v}_f \in \mathbf{V}_f, \boldsymbol{\phi} \in \boldsymbol{\Lambda}_s, \xi \in \Lambda_p, \mathbf{v}_s \in \mathbf{V}_s, \boldsymbol{\chi}_p \in \mathbb{Q}_p$,

$$\begin{aligned}
& s_0\,(\partial_t\, p_p, w_p)_{\Omega_p} + a_e(\partial_t\, \boldsymbol{\sigma}_p, \partial_t\, p_p; \boldsymbol{\tau}_p, w_p) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_f(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) \\
& \quad +\, \kappa_{\mathbf{u}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) + b_p(p_p, \mathbf{v}_p) - b_p(w_p, \mathbf{u}_p) \\
& \quad +\, b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_s(\mathbf{u}_s, \boldsymbol{\tau}_p) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p) + b_\Gamma(\mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\phi}; \lambda) \\
& \quad =\, -\,(\mathbf{f}_f, \kappa_1\, \mathbf{div}(\mathbf{R}_f) - \mathbf{v}_f)_{\Omega_f} - \frac{1}{n}\,(q_f\, \mathbf{I}, \mathbf{R}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{v}_f))_{\Omega_f} + (q_p, w_p)_{\Omega_p}, \\
& -b_s(\mathbf{v}_s, \boldsymbol{\sigma}_p) - b_{\mathrm{sk}}(\boldsymbol{\chi}_p, \boldsymbol{\sigma}_p) - b_\Gamma(\mathbf{u}_p, \mathbf{u}_f, \boldsymbol{\theta}; \xi) \,=\, (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p},
\end{aligned} \tag{4.1.12}$$

Now, we group the spaces, unknowns and test functions as follows:

$$\mathbf{Q} := \mathbb{X}_p \times \mathrm{W}_p \times \mathbf{V}_p \times \mathbb{X}_f \times \mathbf{V}_f \times \mathbf{\Lambda}_s, \quad \mathbf{S} := \Lambda_p \times \mathbf{V}_s \times \mathbb{Q}_p,$$

$$\mathbf{p} := (\boldsymbol{\sigma}_p, p_p, \mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}) \in \mathbf{Q}, \quad \mathbf{r} := (\lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p) \in \mathbf{S},$$

$$\mathbf{q} := (\boldsymbol{\tau}_p, w_p, \mathbf{v}_p, \mathbf{R}_f, \mathbf{v}_f, \boldsymbol{\phi}) \in \mathbf{Q}, \quad \mathbf{s} := (\xi, \mathbf{v}_s, \boldsymbol{\chi}_p) \in \mathbf{S},$$

where the spaces $\mathbf{Q}$ and $\mathbf{S}$ are respectively endowed with the norms

$$\begin{aligned} \|\mathbf{q}\|_{\mathbf{Q}} &= \|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} + \|w_p\|_{\mathrm{W}_p} + \|\mathbf{v}_p\|_{\mathbf{V}_p} + \|\mathbf{R}_f\|_{\mathbb{X}_f} + \|\mathbf{v}_f\|_{\mathbf{V}_f} + \|\boldsymbol{\phi}\|_{\mathbf{\Lambda}_s}, \\ \|\mathbf{s}\|_{\mathbf{S}} &= \|\xi\|_{\Lambda_p} + \|\mathbf{v}_s\|_{\mathbf{V}_p} + \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p}. \end{aligned}$$

Hence, we can write (4.1.12) in an operator notation as a degenerate evolution problem in a mixed form:

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{E}\,\mathbf{p}(t) + (\mathcal{A} + \mathcal{K}_{\mathbf{u}_f})\,\mathbf{p}(t) + \mathcal{B}'\,\mathbf{r}(t) &= \mathbf{F}(t) \quad \text{in} \quad \mathbf{Q}', \\ -\mathcal{B}\,\mathbf{p}(t) &= \mathbf{G}(t) \quad \text{in} \quad \mathbf{S}', \end{aligned} \tag{4.1.13}$$

where, the operators $\mathcal{A} : \mathbf{Q} \to \mathbf{Q}'$, $\mathcal{K}_{\mathbf{w}_f} : \mathbf{Q} \to \mathbf{Q}'$, $\mathcal{B} : \mathbf{Q} \to \mathbf{S}'$, and the functionals $\mathbf{F} \in \mathbf{Q}'$, $\mathbf{G} \in \mathbf{S}'$ are defined as follows:

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & B'_{\mathbf{n}_p} \\ 0 & 0 & B'_p & 0 & 0 & 0 \\ 0 & -B_p & A_p & 0 & 0 & 0 \\ 0 & 0 & 0 & A^e_f + A^{r_1}_f & B'_f + A^{r_2}_f & 0 \\ 0 & 0 & 0 & -B_f + A^{r_3}_f & A^f_f + A^{r_4}_f + A^f_{\mathtt{BJS}} & (A^{fs}_{\mathtt{BJS}})' \\ -B_{\mathbf{n}_p} & 0 & 0 & 0 & A^{fs}_{\mathtt{BJS}} & A^f_{\mathtt{BJS}} \end{pmatrix},$$

$$\mathcal{K}_{\mathbf{w}_f} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K^{fe}_{\mathbf{w}_f} & 0 \\ 0 & 0 & 0 & 0 & K^f_{\mathbf{w}_f} + K^{\Gamma}_{\mathbf{w}_f} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 & B^p_\Gamma & 0 & B^f_\Gamma & B^s_\Gamma \\ B_s & 0 & 0 & 0 & 0 & 0 \\ B_{\mathrm{sk}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
\mathbf{F} = \begin{pmatrix} 0 \\ q_p \\ 0 \\ -\dfrac{1}{n}\, q_f\, \mathrm{tr} - \kappa_1\, \mathbf{f}_f \cdot \mathbf{div} \\ \dfrac{\kappa_2}{n}\, q_f\, \mathrm{div} + \mathbf{f}_f \\ 0 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 \\ \mathbf{f}_p \\ 0 \end{pmatrix},
$$

where

$$
(A_p\, \mathbf{u}_p, \mathbf{v}_p) = a_p(\mathbf{u}_p, \mathbf{v}_p), \quad (A_f^e\, \mathbf{T}_f, \mathbf{R}_f) = \frac{1}{2\,\mu}\, (\mathbf{T}_f^{\mathrm{d}}, \mathbf{R}_f^{\mathrm{d}})_{\Omega_f}, \quad (A_f^f\, \mathbf{u}_f, \mathbf{v}_f) = -\rho_f\, (q_f\, \mathbf{u}_f, \mathbf{v}_f),
$$

$$
(A_f^{r_1}\, \mathbf{T}_f, \mathbf{R}_f) = \kappa_1\, (\mathbf{div}(\mathbf{T}_f), \mathbf{div}(\mathbf{R}_f))_{\Omega_f}, \quad (A_f^{r_2}\, \mathbf{u}_f, \mathbf{R}_f) = \kappa_1\, \rho_f\, (q_f\, \mathbf{u}_f, \mathbf{div}(\mathbf{R}_f))_{\Omega_f},
$$

$$
(A_f^{r_3}\, \mathbf{T}_f, \mathbf{v}_f) = -\kappa_2\, \frac{1}{2\,\mu}\, \big(\mathbf{T}_f^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f)\big)_{\Omega_f}, \quad (A_f^{r_4}\, \mathbf{u}_f, \mathbf{v}_f) = \kappa_2 (\mathbf{e}(\mathbf{u}_f), \mathbf{e}(\mathbf{v}_f))_{\Omega_f}
$$

$$
(A_{\mathtt{BJS}}^f\, \mathbf{u}_f, \mathbf{v}_f) = a_{\mathtt{BJS}}(\mathbf{u}_f, \mathbf{0}; \mathbf{v}_f, \mathbf{0}), \quad (A_{\mathtt{BJS}}^{fs}\, \mathbf{u}_f, \boldsymbol{\phi}) = a_{\mathtt{BJS}}(\mathbf{u}_f, \mathbf{0}; \mathbf{0}, \boldsymbol{\phi}),
$$

$$
(A_{\mathtt{BJS}}^s\, \boldsymbol{\theta}, \boldsymbol{\phi}) = a_{\mathtt{BJS}}(\mathbf{0}, \boldsymbol{\theta}; \mathbf{0}, \boldsymbol{\phi}),
$$

$$
(B_p\, p_p, \mathbf{v}_p) = -b_p(p_p, \mathbf{v}_p), \quad (B_{\mathbf{n}_p}\, \boldsymbol{\sigma}_p, \boldsymbol{\phi}) = -b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}),
$$

$$
(B_f\, \mathbf{T}_f, \mathbf{v}_f) = (\mathbf{v}_f, \mathbf{div}(\mathbf{T}_f))_{\Omega_f} + (\mathbf{T}_f, \boldsymbol{\gamma}_f(\mathbf{v}_f))_{\Omega_f} - \langle \mathbf{T}_f \mathbf{n}_f, \mathbf{v}_f \rangle_{\Gamma_{fp}},
$$

$$
(K_{\mathbf{w}_f}^{fe}\, \mathbf{u}_f, \mathbf{R}_f) = \frac{\rho_f}{2\,\mu}\, ((\mathbf{u}_f \otimes \mathbf{w}_f)^{\mathrm{d}}, \mathbf{R}_f)_{\Omega_f}, \quad (K_{\mathbf{w}_f}^f\, \mathbf{u}_f, \mathbf{v}_f) = -\kappa_2\, \frac{\rho_f}{2\,\mu}\, ((\mathbf{u}_f \otimes \mathbf{w}_f)^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f))_{\Omega_f},
$$

$$
(K_{\mathbf{w}_f}^{\Gamma}\, \mathbf{u}_f, \mathbf{v}_f) = \rho_f\, \langle \mathbf{w}_f \cdot \mathbf{n}_f, \mathbf{u}_f \cdot \mathbf{v}_f \rangle_{\Gamma_{fp}}, \quad (B_s\, \mathbf{v}_s, \boldsymbol{\sigma}_p) = b_s(\mathbf{v}_s, \boldsymbol{\sigma}_p),
$$

$$
(B_{\mathrm{sk}}\, \boldsymbol{\chi}_p, \boldsymbol{\sigma}_p) = b_{\mathrm{sk}}(\boldsymbol{\chi}_p, \boldsymbol{\sigma}_p), \quad (B_\Gamma^p\, \mathbf{u}_p, \xi) = b_\Gamma(\mathbf{u}_p, \mathbf{0}, \mathbf{0}; \xi),
$$

$$
(B_\Gamma^f\, \mathbf{u}_f, \xi) = b_\Gamma(\mathbf{0}, \mathbf{u}_f, \mathbf{0}; \xi), \quad (B_\Gamma^s\, \boldsymbol{\theta}, \xi) = b_\Gamma(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}; \xi).
$$

The operator $\mathcal{E} : \mathbf{Q} \to \mathbf{Q}'$ is given by:

$$\mathcal{E} = \begin{pmatrix} A_e^s & A_e^{sp} & 0 & 0 & 0 & 0 \\ (A_e^{sp})' & A_p^p + A_e^p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$(A_e^s \boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) = a_e(\boldsymbol{\sigma}_p, 0; \boldsymbol{\tau}_p, 0), \quad (A_e^{sp} \boldsymbol{\sigma}_p, w_p) = a_e(\boldsymbol{\sigma}_p, 0; \mathbf{0}, w_p),$$

$$(A_e^p p_p, w_p) = a_e(\mathbf{0}, p_p; \mathbf{0}, w_p), \quad (A_p^p p_p, w_p) = (s_0 \, p_p, w_p)_{\Omega_p}.$$

4.2 Well-posedness of the weak formulation

4.2.1 Stability properties

We start by establishing the stability properties of the operators $\mathcal{A}$, $\mathcal{K}_{\mathbf{w}_f}$, $\mathcal{B}$ and $\mathcal{E}$. In the sequel, we make use of the following well-known estimates: there exist positive constants $c_1(\Omega_f)$ and $c_2(\Omega_f)$, such that (see, [23, Proposition IV.3.1] and [48, Lemma 2.5], respectively)

$$c_1(\Omega_f) \, \|\mathbf{R}_{f,0}\|^2_{\mathbb{L}^2(\Omega_f)} \, \leq \, \|\mathbf{R}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{div}(\mathbf{R}_f)\|^2_{\mathbf{L}^2(\Omega_f)} \quad \forall \, \mathbf{R}_f = \mathbf{R}_{f,0} + \ell \, \mathbf{I} \in \mathbb{H}(\mathbf{div}; \Omega_f) \tag{4.2.1}$$

and

$$c_2(\Omega_f) \, \|\mathbf{R}_f\|^2_{\mathbb{X}_f} \, \leq \, \|\mathbf{R}_{f,0}\|^2_{\mathbb{X}_f} \quad \forall \, \mathbf{R}_f = \mathbf{R}_{f,0} + \ell \, \mathbf{I} \in \mathbb{X}_f, \tag{4.2.2}$$

where $\mathbf{R}_{f,0} \in \mathbb{H}_0(\mathbf{div}; \Omega_f) := \Big\{ \mathbf{R}_f \in \mathbb{H}(\mathbf{div}; \Omega_f) : \quad (\mathrm{tr}(\mathbf{R}_f), 1)_{\Omega_f} = 0 \Big\}$ and $\ell \in \mathrm{R}$. We emphasize that (4.2.2) holds since each $\mathbf{R}_f \in \mathbb{X}_f$ satisfies the boundary condition $\mathbf{R}_f \mathbf{n}_f = \mathbf{0}$

on Γ_f^{N} with $|\Gamma_f^{\mathrm{N}}| > 0$. In addition, we recall Korn inequality, that is there exists positive constants $c_3(\Omega_f)$ such that

$$c_3(\Omega_f)\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)} \le \|\mathbf{e}(\mathbf{v}_f)\|^2_{\mathbb{L}^2(\Omega_f)} \le \|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)} \quad \forall\, \mathbf{v}_f \in \mathbf{H}^1(\Omega_f) \tag{4.2.3}$$

and also notice that

$$\|\boldsymbol{\gamma}_f(\mathbf{v}_f)\|^2_{\mathbb{L}^2(\Omega_f)} \le \|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)} \quad \forall\, \mathbf{v}_f \in \mathbf{H}^1(\Omega_f) \tag{4.2.4}$$

Lemma 4.2.1. *Given $q_f \in \mathrm{L}^4(\Omega_f)$ and $\mathbf{w}_f \in \mathbf{V}_f$, the operators $\mathcal{A}$, $\mathcal{K}_{\mathbf{w}_f}$, $\mathcal{B}$ and $\mathcal{E}$ are linear and bounded as follows,*

$$\begin{gathered}\mathcal{A}(\mathbf{p})(\mathbf{q}) \le C_{\mathcal{A}}\|\mathbf{p}\|_{\mathbf{Q}}\|\mathbf{q}\|_{\mathbf{Q}}, \quad \mathcal{K}_{\mathbf{w}_f}(\mathbf{p})(\mathbf{q}) \le C_{\mathcal{K}}\|\mathbf{w}_f\|_{\mathbf{V}_f}\|\mathbf{p}\|_{\mathbf{Q}}\|\mathbf{q}\|_{\mathbf{Q}},\\ \mathcal{B}(\mathbf{q})(\mathbf{s}) \le C_{\mathcal{B}}\|\mathbf{q}\|_{\mathbf{Q}}\|\mathbf{s}\|_{\mathbf{S}}, \quad \mathcal{E}(\mathbf{p})(\mathbf{q}) \le C_{\mathcal{E}}\|\mathbf{p}\|_{\mathbf{Q}}\|\mathbf{q}\|_{\mathbf{Q}},\end{gathered} \tag{4.2.5}$$

where $C_{\mathcal{A}}$, $C_{\mathcal{K}}$, $C_{\mathcal{B}}$ and $C_{\mathcal{E}}$ are positive constants depending on μ, $\mathbf{K}$, ρ_f, $\alpha_{\mathtt{BJS}}$, q_f, s_0, κ_1 and κ_2.

Proof. We begin noting that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}$ are clearly linear and bounded, using the trace inequalities (4.1.6)–(4.1.7) for continuity of b_Γ and $b_{\mathbf{n}_p}$. As for $\mathcal{K}_{\mathbf{w}_f}$, we make use of (4.2.3), combining with the continuity of the embedding $\mathbf{i}_c : \mathbf{H}^1(\Omega_f) \to \mathbf{L}^4(\Omega_f)$ and $\mathbf{i}_\Gamma : \mathbf{H}^{1/2}(\partial\Omega_f) \to \mathbf{L}^4(\partial\Omega_f)$, and the continuity of the trace operator $\gamma_0 : \mathbf{H}^1(\Omega_f) \to \mathbf{L}^2(\partial\Omega_f)$, to deduce that given $q_f \in \mathrm{L}^4(\Omega_f)$ and $\mathbf{w}_f \in \mathbf{V}_f$, $\mathcal{K}_{\mathbf{w}_f}$ is linear and bounded. In particular, we have

$$\begin{gathered}\frac{\rho_f}{2\,\mu}|((\mathbf{u}_f \otimes \mathbf{w}_f)^{\mathrm{d}}, \mathbf{R}_f - \kappa_2\mathbf{e}(\mathbf{v}_f))_{\Omega_f}|\\ \le c_4\frac{\rho_f}{2\,\mu}\|\mathbf{i}_c\|^2\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)}\|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\|(\mathbf{R}_f, \mathbf{v}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f},\\ \rho_f|\langle \mathbf{w}_f \cdot \mathbf{n}_f, \mathbf{v}_f \cdot \mathbf{v}_f\rangle_{\Gamma_{fp}}| \le \rho_f\|\mathbf{i}_\Gamma\|^2\|\gamma_0\|\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)}\|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\|(\mathbf{R}_f, \mathbf{v}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\end{gathered}$$

where $c_4 = \max\{1, \kappa_2\}$, and $\|(\mathbf{R}_f, \mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} = \|\mathbf{R}_f\|^2_{\mathbb{X}_f} + \|\mathbf{v}_f\|^2_{\mathbf{V}_f}$, so indeed we have that $C_{\mathcal{K}} = \max\left\{c_4\dfrac{\rho_f}{2\,\mu}\|\mathbf{i}_c\|^2, \rho_f\|\mathbf{i}_\Gamma\|^2\|\gamma_0\|\right\}$. □

Next, we establish the monotonicity of the operators $\mathcal{A} + \mathcal{K}_{\mathbf{w}_f}$ and $\mathcal{E}$, respectively.

Lemma 4.2.2. *Assume $\kappa_1 > 0$, $0 < \kappa_2 < 2\mu\, c_3(\Omega_f)$,*

$$\|q_f\|_{\mathrm{L}^4(\Omega_f)} \le \min\Big\{1, \frac{\kappa_2 c_3(\Omega_f)}{4\rho_f \|\mathbf{i}_c\|^2 (1+\kappa_2\rho_f/2)}\Big\}, \tag{4.2.6}$$

and $\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)} \le r_0$, where

$$r_0 := \frac{\alpha_f}{2C_{\mathcal{K}}}, \quad \alpha_f = \min\Big\{c_1(\Omega_f)\min\{\frac{1}{4\mu}, \frac{\kappa_1}{4}\}, \frac{\kappa_1}{4}, \frac{\kappa_2 c_3(\Omega_f)}{4}\Big\}, \tag{4.2.7}$$

then $\mathcal{A} + \mathcal{K}_{\mathbf{w}_f}$ and $\mathcal{E}$ are monotone as follows,

$$(\mathcal{A} + \mathcal{K}_{\mathbf{w}_f})(\mathbf{q})(\mathbf{q}) \ge \alpha_{\mathcal{AK}} \|\mathbf{q}\|^2_{\mathbf{Q}}, \quad \mathcal{E}(\mathbf{q})(\mathbf{q}) \ge \alpha_{\mathcal{E}} \|\mathbf{q}\|^2_{\mathbf{Q}}, \tag{4.2.8}$$

where $\alpha_{\mathcal{AK}}$ is a positive constant depending on μ, $\mathbf{K}$, $\alpha_{\mathtt{BJS}}$, κ_1, κ_2, $c_1(\Omega_f)$ and $c_3(\Omega_f)$, and $\alpha_{\mathcal{E}}$ is a nonnegative constant depending on s_0. In particular,

$$\begin{gathered} a_f(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f) \ge \alpha_f \|(\mathbf{R}_f, \mathbf{v}_f)\|^2_{\mathbb{X}_f \times \mathbf{V}_f}, \\ a_f(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\mathbf{w}_f}(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f) \ge \frac{\alpha_f}{2} \|(\mathbf{R}_f, \mathbf{v}_f)\|^2_{\mathbb{X}_f \times \mathbf{V}_f}, \\ a_p(\mathbf{v}_p, \mathbf{v}_p) \ge \mu k_{max}^{-1} \|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)}, \qquad a_{\mathtt{BJS}}(\mathbf{v}_f, \boldsymbol{\phi}; \mathbf{v}_f, \boldsymbol{\phi}) \ge c_{\mathtt{BJS}} |\mathbf{v}_f - \boldsymbol{\phi}|^2_{\mathtt{BJS}}. \end{gathered} \tag{4.2.9}$$

Proof. From the definition of the operator A (c.f. (2.1.3)), using triangle inequality, we deduce that

$$\begin{aligned} \|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} &\le 2\,(2\,\mu_p + n\,\lambda_p)\,\Big(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}(\alpha_p\, w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\Big) \\ &\le C_p\,\Big(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\, w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{W}_p}\Big), \end{aligned}$$

where $C_p := 2\,\max\big\{2\,\mu_p + n\,\lambda_p, n\,\alpha_p^2\big\}$. Thus combining with the definition of $\mathcal{E}$, we get

$$\begin{aligned} (\mathcal{E})(\mathbf{q})(\mathbf{q}) &= s_0 \|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}\,(\boldsymbol{\tau}_p + \alpha_p\, w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\ &\ge \frac{s_0}{2}\|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \alpha_1(\Omega_p)(\|A^{1/2}\,(\boldsymbol{\tau}_p + \alpha_p\, w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{L}^2(\Omega_p)}) \\ &\ge \alpha_{\mathcal{E}}(\Omega_p)(\|w_p\|^2_{W_p} + \|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)}), \end{aligned}$$

with $\alpha_1(\Omega_p) = \min\{s_0/2,\, 1\}$ and $\alpha_{\mathcal{E}}(\Omega_p) = \alpha_1(\Omega_p)$.

In turn, utilizing Young's inequality, (4.1.9) and (4.2.3), we have

$$|\kappa_1\,\rho_f\,(q_f\,\mathbf{v}_f, \mathbf{div}(\mathbf{R}_f))_{\Omega_f}| \le \frac{\kappa_1}{2}\|\mathbf{div}(\mathbf{R}_f)\|^2_{\mathbf{L}^2(\Omega_f)} + \frac{\kappa_1}{2}\rho_f^2\,\|\mathbf{i}_c\|^2\|q_f\|^2_{\mathrm{L}^4(\Omega_f)}\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)},$$

$$|\rho_f\,(q_f\,\mathbf{v}_f,\mathbf{v}_f)_{\Omega_f}| \le \rho_f\,\|\mathbf{i}_c\|^2\|q_f\|_{\mathrm{L}^4(\Omega_f)}\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)},$$

$$|\frac{1}{2\mu}\kappa_2(\mathbf{R}_f^{\mathrm{d}},\mathbf{e}(\mathbf{v}_f))_{\Omega_f}| \le \frac{1}{4\mu}\|\mathbf{R}_f^{\mathrm{d}}\|^2_{\mathbf{L}^2(\Omega_f)} + \frac{1}{4\mu}\kappa_2^2\,\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)},$$

$$\kappa_2\,(\mathbf{e}(\mathbf{v}_f),\mathbf{e}(\mathbf{v}_f))_{\Omega_f} \ge \kappa_2\,c_3(\Omega_f)\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)},$$

thus we could get that

$$\begin{aligned} a_f(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) &\ge \frac{1}{4\mu}\|\mathbf{R}_f^{\mathrm{d}}\|^2_{\mathbf{L}^2(\Omega_f)} + \frac{\kappa_1}{2}\|\mathbf{div}(\mathbf{R}_f)\|^2_{\mathbf{L}^2(\Omega_f)} \\ &+ \Big\{\kappa_2\big(c_3(\Omega_f) - \frac{1}{4\mu}\kappa_2\big) - \rho_f\|\mathbf{i}_c\|^2\|q_f\|_{\mathrm{L}^4(\Omega_f)}\big(1+\frac{\kappa_1}{2}\rho_f\|q_f\|_{\mathrm{L}^4(\Omega_f)}\big)\Big\}\|\mathbf{v}_f\|^2_{\mathbf{H}^1(\Omega_f)} \\ &\ge \alpha_2\|\mathbf{R}_f\|^2_{\mathbb{X}_f} + \alpha_3\|\mathbf{v}_f\|^2_{\mathbf{V}_f} \ge \alpha_f\|(\mathbf{R}_f,\mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f}, \end{aligned}$$

if $\kappa_2 \le 2\,\mu\,c_3(\Omega_f)$, and $\|q_f\|_{L^4(\Omega_f)} \le \min\Big\{1, \dfrac{\kappa_2 c_3(\Omega_f)}{4\,\rho_f\,\|\mathbf{i}_c\|^2\,(1+\kappa_2\rho_f/2)}\Big\}$, where $\alpha_2 = \min\Big\{c_1(\Omega_f)\min\{\dfrac{1}{4\mu},\dfrac{\kappa_1}{4}\},\dfrac{\kappa_1}{4}\Big\}$, $\alpha_3 = \kappa_2 c_3(\Omega_f)/4$, and $\alpha_f = \min\{\alpha_2,\alpha_3\}$. Furthermore, there holds

$$\begin{aligned} a_f(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) + \kappa_{\mathbf{w}_f}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) &\ge a_f(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) - |\kappa_{\mathbf{w}_f}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f)| \\ &\ge (\alpha_f - C_{\mathcal{K}}\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)})\|(\mathbf{R}_f,\mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} \ge \frac{\alpha_f}{2}\|(\mathbf{R}_f,\mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f}, \end{aligned} \tag{4.2.10}$$

where we used $\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)} \le \dfrac{\alpha_f}{2C_{\mathcal{K}}}$ in the last inequality.

Finally, from the definition of a_p and $a_{\mathtt{BJS}}$, we have

$$a_p(\mathbf{v}_p,\mathbf{v}_p) \ge \mu k_{max}^{-1}\|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)},$$

$$a_{\mathtt{BJS}}(\mathbf{v}_f,\boldsymbol{\phi};\mathbf{v}_f,\boldsymbol{\phi}) = \nu\alpha_{\mathtt{BJS}}\sum_{j=1}^{n-1}\left\langle\sqrt{\mathbf{K}_j^{-1}}(\mathbf{v}_f-\boldsymbol{\phi})\cdot\mathbf{t}_{f,j},(\mathbf{v}_f-\boldsymbol{\phi})\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} \ge c_{\mathtt{BJS}}|\mathbf{v}_f-\boldsymbol{\phi}|^2_{\mathtt{BJS}}, \tag{4.2.11}$$

where $c_{\mathtt{BJS}}$ is a positive constant that only depends on μ, $\alpha_{\mathtt{BJS}}$ and $\mathbf{K}$, and we define for $\mathbf{v}_f\in\mathbf{V}_f$, $\boldsymbol{\phi}\in\boldsymbol{\Lambda}_f$,

$$|\mathbf{v}_f-\boldsymbol{\phi}|^2_{\mathtt{BJS}} := \sum_{j=1}^{n-1}\|(\mathbf{v}_f-\boldsymbol{\phi})\cdot\mathbf{t}_{f,j}\|^2_{\mathrm{L}^2(\Gamma_{fp})}.$$

The monotonicity of $\mathcal{A}+\mathcal{K}_{\mathbf{w}_f}$ follows from (4.2.10) and (4.2.11). $\square$

Next we define

$$\widetilde{\mathbb{X}}_p := \Big\{\boldsymbol{\tau}_p \in \mathbb{X}_p : \quad \mathbf{div}(\boldsymbol{\tau}_p) = \mathbf{0} \quad \text{in} \quad \Omega_p\Big\}, \quad \widehat{\mathbb{X}}_p := \Big\{\boldsymbol{\tau}_p \in \mathbb{X}_p : \quad \boldsymbol{\tau}_p\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{fp}\Big\},$$

then the inf-sup conditions are given by the following lemma.

Lemma 4.2.3. *There exist constants β_1, β_2, $\beta_3 > 0$ such that*

$$\beta_1(\|\mathbf{v}_s\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p}) \leq \sup_{\mathbf{0}\neq\boldsymbol{\tau}_p\in\widehat{\mathbb{X}}_p} \frac{b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) + b_{sk}(\boldsymbol{\tau}_p, \boldsymbol{\chi}_p)}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}}, \qquad \forall \mathbf{v}_s \in \mathbf{V}_s, \boldsymbol{\chi}_p \in \mathbb{Q}_p, \quad (4.2.12)$$

$$\beta_2(\|w_p\|_{W_p} + \|\xi\|_{\Lambda_p}) \leq \sup_{\mathbf{0}\neq\mathbf{v}_p\in\mathbf{V}_p} \frac{b_p(\mathbf{v}_p, w_p) + b_\Gamma(\mathbf{0}, \mathbf{v}_p, \mathbf{0}; \xi)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}}, \qquad \forall w_p \in \mathrm{W}_p, \xi \in \Lambda_p, \quad (4.2.13)$$

$$\beta_3\|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} \leq \sup_{\mathbf{0}\neq\boldsymbol{\tau}_p\in\widetilde{\mathbb{X}}_p} \frac{b_n^p(\boldsymbol{\tau}_p, \boldsymbol{\phi})}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}}, \qquad \forall \phi \in \boldsymbol{\Lambda}_s. \quad (4.2.14)$$

Proof. The inf-sup condition (4.2.12) is a result from [13], and inf-sup condition (4.2.13) follows from a modification of the argument in Lemmas 3.1 and 3.2 in [43] to account for $|\Gamma_p^D| > 0$. Finally, (4.2.14) can be proved from using the argument in [50, Lemma 4.2]. □

We now establish the well-posedness of (4.1.13) (equivalently (4.1.12)). We start with some preliminary results that will serve for the forthcoming analysis.

4.2.2 Well-posedness analysis

We begin by recalling Theorem 2.2.3 to establish the existence of a solution to (4.1.13) (see [74, Theorem IV.6.1(b)] for details).

Remark 4.2.1. *The problem* (4.1.13) *is a degenerate evolution problem in a mixed form, which fits the structure of the problem studied in the theorem above. However, note that in the theorem, f is restricted in the space $\mathrm{W}^{1,1}(0,T;E_b')$ arising from $\mathcal{N}$. If we would like $u(t)$ in the theorem to cover for all the variables in our case, we will have to restrict data as $\mathbf{f}_f = \mathbf{f}_p = \mathbf{0}$ and $q_f = 0$. To avoid this restriction, we will reformulate the problem as a parabolic problem for $\boldsymbol{\sigma}_p$ and p_p as in [4].*

We denote by the $\mathbf{E}_2$ the closure of the space $\mathbf{E} := \mathbb{X}_p \times \mathrm{W}_p$ with respect to the norm and inner product induced by the operator $\mathcal{E}$, that is,

$$\begin{gathered} \|(\boldsymbol{\tau}_p, w_p)\|_{\mathbf{E}_2} := ((\boldsymbol{\tau}_p, w_p), (\boldsymbol{\tau}_p, w_p))^{1/2}_{\mathbf{E}_2}, \\ ((\boldsymbol{\tau}_1, w_1), (\boldsymbol{\tau}_2, w_2))_{\mathbf{E}_2} := a_e(\boldsymbol{\tau}_1, w_1; \boldsymbol{\tau}_2, w_2) + (s_0 w_1, w_2)_{\Omega_p}. \end{gathered} \tag{4.2.15}$$

From the definition of the operator A, cf. (2.1.3), and the fact that $s_0 > 0$, we could see that the norm $\|\cdot\|_{\mathbf{E}_2}$ in (4.2.15) is equivalent to the standard product norm

$$\|(\boldsymbol{\tau}_p, w_p)\|_{\widehat{\mathbf{E}}_2} := \left(\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{W}_p}\right)^{1/2}, \tag{4.2.16}$$

which implies that $\mathbf{E}_2 = \mathbb{L}^2(\Omega_p) \times \mathrm{W}_p \supset \mathbb{X}_p \times \mathrm{W}_p$. Now let us set $\mathbf{Q}_2 = \mathbb{L}^2(\Omega_p) \times \mathrm{W}_p \times \mathbf{V}_p \times \mathbb{X}_f \times \mathbf{V}_f \times \boldsymbol{\Lambda}_s$, then $\mathbf{Q}_2' = \mathbb{L}^2(\Omega_p) \times \mathrm{W}_p' \times \mathbf{V}_p' \times \mathbb{X}_f' \times \mathbf{V}_f' \times \boldsymbol{\Lambda}_s' \subset \mathbf{Q}'$. Next, we define the domain associated to the resolvent system of (4.1.12) similar to [4, Section 4.1],

$\mathcal{D} := \Big\{(\boldsymbol{\sigma}_p, p_p) \in \mathbb{X}_p \times \mathrm{W}_p : \quad$ for given $(q_f, \mathbf{f}_f, \mathbf{f}_p) \in \mathbb{X}_f' \times \mathbf{V}_f' \times \mathbf{V}_s'$,
there exists $((\mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}), (\lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p)) \in (\mathbf{V}_p \times \mathbb{X}_f \times \mathbf{V}_f \times \boldsymbol{\Lambda}_s) \times \mathbf{S}$ such that $\forall\, (\mathbf{q}, \mathbf{s}) \in \mathbf{Q} \times \mathbf{S}$:

$$\begin{aligned}
& s_0\,(p_p, w_p)_{\Omega_p} + a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_f(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) \\
& + \kappa_{\mathbf{u}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) + b_p(p_p, \mathbf{v}_p) - b_p(w_p, \mathbf{u}_p) \\
& + b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_s(\mathbf{u}_s, \boldsymbol{\tau}_p) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p) + b_\Gamma(\mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\phi}; \lambda) \\
& = -\,(\mathbf{f}_f, \kappa_1\,\mathbf{div}(\mathbf{R}_f) - \mathbf{v}_f)_{\Omega_f} - \frac{1}{n}\,(q_f\,\mathbf{I}, \mathbf{R}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{v}_f))_{\Omega_f} \\
& \quad + (\widehat{\mathbf{f}}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{q}_p, w_p)_{\Omega_p}, \\
& - b_s(\mathbf{v}_s, \boldsymbol{\sigma}_p) - b_{\mathrm{sk}}(\boldsymbol{\chi}_p, \boldsymbol{\sigma}_p) - b_\Gamma(\mathbf{u}_p, \mathbf{u}_f, \boldsymbol{\theta}; \xi) \;=\; (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p},
\end{aligned} \tag{4.2.17}$$

and for some $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathbf{E}_2'$ satisfying

$$\|\widehat{\mathbf{f}}_p\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{q}_p\|_{\mathrm{L}^2(\Omega_p)} \le \widehat{C}_{ep}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big) \tag{4.2.18}$$

for some constant $\widehat{C}_{ep}\Big\} \subset \mathbf{E}_2$.

Note that the resolvent system (4.2.17) can be written in an operator form as

$$
\begin{aligned}
(\mathcal{E} + \mathcal{A} + \mathcal{K}_{\mathbf{u}_f})\,\mathbf{p} + \mathcal{B}'\,\mathbf{r} &= \widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{Q}_2', \\
-\mathcal{B}\,\mathbf{p} &= \mathbf{G} \quad \text{in} \quad \mathbf{S}',
\end{aligned}
\tag{4.2.19}
$$

where $\widehat{\mathbf{F}} \in \mathbf{Q}'$ is the functional on the right hand side of (4.2.17).

Note that there may be more than one $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathbf{E}_2'$ that generate the same $(\boldsymbol{\sigma}_p, p_p) \in \mathcal{D}$. In view of this, we introduce the multivalued operator $\mathcal{M}(\cdot)$ with domain $\mathcal{D}$ defined by

$$
\mathcal{M}(\boldsymbol{\sigma}_p, p_p) \,:=\, \Big\{ (\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_p, p_p) : \ (\boldsymbol{\sigma}_p, p_p) \ \text{satisfies} \ (4.2.17) \ \text{for} \ (\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathbb{L}^2(\Omega_p) \times \mathrm{W}_p' \Big\},
\tag{4.2.20}
$$

where $\widehat{\mathcal{E}}$ is the top left 2×2 block of $\mathcal{E}$. Associated with $\mathcal{M}(\cdot)$ we have the relation $\mathcal{M} \subset \mathbf{E}\times\mathbf{E}_2'$ with domain $\mathcal{D}$, where $[\mathbf{v}, \mathbf{f}] \in \mathcal{M}$ if $\mathbf{v} \in \mathcal{D}$ and $\mathbf{f} \in \mathcal{M}(\mathbf{v})$.

Next we consider the following parabolic problem: Given $(h_{\boldsymbol{\sigma}_p}, h_{p_p}) \in \mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,1}(0,T;\mathrm{W}_p')$, find $(\boldsymbol{\sigma}_p, p_p) \in \mathcal{D}$ satisfying

$$
\frac{d}{dt}\,\widehat{\mathcal{E}} \begin{pmatrix} \boldsymbol{\sigma}_p(t) \\ p_p(t) \end{pmatrix} + \mathcal{M} \begin{pmatrix} \boldsymbol{\sigma}_p(t) \\ p_p(t) \end{pmatrix} \ni \begin{pmatrix} h_{\boldsymbol{\sigma}_p}(t) \\ h_{p_p}(t) \end{pmatrix}, \quad \textit{a.e. } t \in (0,T).
\tag{4.2.21}
$$

Using Theorem 2.2.3, we can show that the problem (4.1.13) is well-posed. To that end, we proceed in the following manner.

Step 1. Introduce a fixed-point $\mathcal{J}$ associated to problem (4.2.17).
Step 2. Prove $\mathcal{J}$ is a contraction mapping and conclude that the domain $\mathcal{D}$, cf. (4.2.17), is nonempty.
Step 3. Show the solvability of the parabolic problem (4.2.21).
Step 4. Show that the original problem (4.1.13) is a special case of problem (4.2.21).

4.2.2.1 Step 1: A fixed-point approach

We begin the solvability analysis of (4.2.17) or equivalently that the domain $\mathcal{D}$ is nonempty by defining the operator $\mathcal{J} : \mathbf{V}_f \to \mathbf{V}_f$ by

$$\mathcal{J}(\mathbf{w}_f) := \mathbf{u}_f \quad \forall\, \mathbf{w}_f \in \mathbf{V}_f, \tag{4.2.22}$$

where $\mathbf{p} := (\boldsymbol{\sigma}_p, p_p, \mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}) \in \mathbf{Q}$ is the first component of the unique solution (to be confirmed below) of the problem: Find $(\mathbf{p}, \mathbf{r}) \in \mathbf{Q} \times \mathbf{S}$, such that

$$\begin{aligned} (\mathcal{E} + \mathcal{A} + \mathcal{K}_{\mathbf{w}_f})\, \mathbf{p} + \mathcal{B}'\, \mathbf{r} = \widehat{\mathbf{F}} \quad &\text{in} \quad \mathbf{Q}_2', \\ -\mathcal{B}\, \mathbf{p} = \mathbf{G} \quad &\text{in} \quad \mathbf{S}'. \end{aligned} \tag{4.2.23}$$

Thus it is not hard to see that $(\mathbf{p}, \mathbf{r}) \in \mathbf{Q} \times \mathbf{S}$ is a solution of (4.2.17) if and only if $\mathbf{u}_f \in \mathbf{V}_f$ is a fixed-point of $\mathcal{J}$, that is,

$$\mathcal{J}(\mathbf{u}_f) = \mathbf{u}_f. \tag{4.2.24}$$

In this way, in what follows we focus on proving that $\mathcal{J}$ possesses a unique fixed-point. However, we remark in advance that the definition of $\mathcal{J}$ will make sense only in a closed ball of $\mathbf{V}_f$.

Before continuing with the solvability analysis of (4.2.24), we provided the hypotheses under which $\mathcal{J}$ is well-defined. To that end, we introduce operators that will be used to regularize the problem (4.2.23). Let $R_{\boldsymbol{\sigma}_p} : \mathbb{X}_p \to \mathbb{X}_p'$, $R_{p_p} : \mathrm{W}_p \to \mathrm{W}_p'$, $R_{\mathbf{u}_p} : \mathbf{V}_p \to \mathbf{V}_p'$, $L_{\mathbf{u}_s} : \mathbf{V}_s \to \mathbf{V}_s'$, and $L_{\boldsymbol{\gamma}_p} : \mathbb{Q}_p \to \mathbb{Q}_p'$ be defined as follows:

$$\begin{aligned}
&(R_{\boldsymbol{\sigma}_p}\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) = r_{\boldsymbol{\sigma}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p) := (\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\mathbf{div}(\boldsymbol{\sigma}_p), \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p}, \\
&(R_{p_p} p_p, w_p) = r_{p_p}(p_p, w_p) := (p_p, w_p)_{\Omega_p}, \\
&(R_{\mathbf{u}_p}\mathbf{u}_p, \mathbf{v}_p) = r_{\mathbf{u}_p}(\mathbf{u}_p, \mathbf{v}_p) := (\mathrm{div}(\mathbf{u}_p), \mathrm{div}(\mathbf{v}_p))_{\Omega_p}, \\
&(L_{\mathbf{u}_s}\mathbf{u}_s, \mathbf{v}_s) = l_{\mathbf{u}_s}(\mathbf{u}_s, \mathbf{v}_s) := (\mathbf{u}_s, \mathbf{v}_s)_{\Omega_p}, \\
&(L_{\boldsymbol{\gamma}_p}\boldsymbol{\gamma}_p, \boldsymbol{\chi}_P) = l_{\boldsymbol{\gamma}_p}(\boldsymbol{\gamma}_p, \boldsymbol{\chi}_p) := (\boldsymbol{\gamma}_p, \boldsymbol{\chi}_p)_{\Omega_p}.
\end{aligned}$$

The following operator properties follow immediately from the above definitions.

Lemma 4.2.4. *The operators $R_{\boldsymbol{\sigma}_p}$, R_{p_p}, R_{u_p}, $L_{\mathbf{u}_s}$, and $L_{\boldsymbol{\gamma}_p}$ are bounded, continuous, coercive and monotone.*

It was shown in [43] that there is a bounded extension of λ from $\mathrm{H}^{1/2}(\Gamma_{fp})$ to $\mathrm{H}^{1/2}(\partial\Omega_p)$ defined as $E_\Gamma\lambda := \gamma_1\psi(\lambda)$, where $\gamma_1 : \mathrm{H}^1(\Omega_p) \to \mathrm{H}^{1/2}(\partial\Omega_p)$ is the trace operator and $\psi(\lambda) \in \mathrm{H}^1(\Omega_p)$ is the weak solution of

$$-\mathrm{div}(\nabla\psi(\lambda)) = 0 \quad \text{in} \quad \Omega_p,$$
$$\psi(\lambda) = \lambda \quad \text{on} \quad \Gamma_{fp}, \quad \nabla\psi(\lambda)\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}, \quad \psi(\lambda) = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}}.$$

In addition, according to [4], there exists generic constants c_4, $c_5 > 0$ such that

$$c_4\|\psi(\lambda)\|_{\mathrm{H}^1(\Omega_p)} \le \|\lambda\|_{\mathrm{H}^{1/2}(\Gamma_{fp})} \le c_5\|\psi(\lambda)\|_{\mathrm{H}^1(\Omega_p)}.$$

Then we define $L_\lambda : \Lambda_p \to \Lambda_p'$ as

$$(L_\lambda\lambda, \xi) = l_\lambda(\lambda, \xi) := (\nabla\psi(\lambda), \nabla\psi(\xi))_{\Omega_p}. \tag{4.2.25}$$

Similarly, there is a bounded extension of $\boldsymbol{\theta}$ from $\mathbf{H}^{1/2}(\Gamma_{fp})$ to $\mathbf{H}^{1/2}(\partial\Omega_p)$ defined as $E_\Gamma\boldsymbol{\theta} := \gamma_2\boldsymbol{\varphi}(\boldsymbol{\theta})$, where $\gamma_2 : \mathbf{H}^1(\Omega_p) \to \mathbf{H}^{1/2}(\partial\Omega_p)$ is defined similarly as before and $\boldsymbol{\varphi}(\boldsymbol{\theta}) \in \mathbf{H}^1(\Omega_p)$ is the weak solution of

$$-\mathbf{div}\,(\nabla\boldsymbol{\varphi}(\boldsymbol{\theta})) = \mathbf{0} \quad \text{in} \quad \Omega_p,$$
$$\boldsymbol{\varphi}(\boldsymbol{\theta}) = \boldsymbol{\theta} \quad \text{on} \quad \Gamma_{fp}, \quad \boldsymbol{\varphi}(\boldsymbol{\theta}) = \mathbf{0} \quad \text{on} \quad \Gamma_p.$$

Elliptic regularity and trace inequality imply that $\|\boldsymbol{\theta}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}$ and $\|\boldsymbol{\varphi}(\boldsymbol{\theta})\|_{\mathbf{H}^1(\Omega_p)}$ are equivalent norms, so $R_{\boldsymbol{\theta}} : \mathbf{\Lambda}_s \to \mathbf{\Lambda}_s'$ is defined as

$$(R_{\boldsymbol{\theta}}\boldsymbol{\theta}, \boldsymbol{\phi}) = r_{\boldsymbol{\theta}}(\boldsymbol{\theta}, \boldsymbol{\phi}) := (\nabla\boldsymbol{\varphi}(\boldsymbol{\theta}), \nabla\boldsymbol{\varphi}(\boldsymbol{\phi}))_{\Omega_p}. \tag{4.2.26}$$

Lemma 4.2.5. *The operators L_λ and $R_{\boldsymbol{\theta}}$ are bounded, continuous, coercive and monotone.*

Proof. The result can be obtained similarly as the proof of Lemma 4.2.4, using the equivalence of norms mentioned before. In particular, there exists generic constants c_Γ and C_Γ such that

$$(L_\lambda\lambda,\xi) \le C_\Gamma\|\lambda\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}\|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \qquad (L_\lambda\lambda,\lambda) \ge c_\Gamma\|\lambda\|^2_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \qquad \forall\,\lambda,\xi\in\Lambda_p,$$
$$(R_{\boldsymbol{\theta}}\boldsymbol{\theta},\boldsymbol{\phi}) \le C_\Gamma\|\boldsymbol{\theta}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}, \qquad (R_{\boldsymbol{\theta}}\boldsymbol{\theta},\boldsymbol{\phi}) \ge c_\Gamma\|\boldsymbol{\theta}\|^2_{\mathbf{H}^{1/2}(\Gamma_{fp})}, \qquad \forall\,\boldsymbol{\theta},\boldsymbol{\phi}\in\mathbf{\Lambda}_s.$$

□

Theorem 4.2.6. *Let $r\in(0,r_0)$ with r_0 given by (4.2.7) and let $\mathbf{f}_f\in\mathbf{L}^2(\Omega_f)$, $\mathbf{f}_p\in\mathbf{L}^2(\Omega_p)$, $q_f\in\mathrm{L}^2(\Omega_f)$, and $q_p\in\mathrm{L}^2(\Omega_p)$. Assume conditions in Lemma 4.2.2, then for each $\mathbf{w}_f$ such that $\|\mathbf{w}_f\|_{\mathbf{H}^1(\Omega_f)}\le r$ and for each $(\widehat{\mathbf{f}}_p,\widehat{q}_p)$ satisfying (4.2.18), there exists a unique solution of the resolvent system (4.2.23). Moreover, there exists a constant $C_{\mathcal{J}}>0$, independent of $\mathbf{w}_f$ and the data $\mathbf{f}_f$, $\mathbf{f}_p$, q_f, and q_p, such that*

$$\|\mathcal{J}(\mathbf{w}_f)\|_{\mathbf{V}_f} \le \|(\mathbf{p},\mathbf{r})\|_{\mathbf{Q}\times\mathbf{S}} \le C_{\mathcal{J}}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)}+\|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)}+\|q_f\|_{\mathrm{L}^2(\Omega_f)}+\|q_p\|_{\mathrm{L}^2(\Omega_p)}\big). \quad (4.2.27)$$

Proof. For $\mathbf{p}=(\boldsymbol{\sigma}_p,p_p,\mathbf{u}_p,\mathbf{T}_f,\mathbf{u}_f,\boldsymbol{\theta})$, $\mathbf{q}=(\boldsymbol{\tau}_p,w_p,\mathbf{v}_p,\mathbf{R}_f,\mathbf{v}_f,\boldsymbol{\phi})\in\mathbf{Q}$ and $\mathbf{r}=(\lambda,\mathbf{u}_s,\boldsymbol{\gamma}_p)$, $\mathbf{s}=(\xi,\mathbf{v}_s,\boldsymbol{\chi}_p)\in\mathbf{S}$, define the operators $\mathcal{R}:\mathbf{Q}\to\mathbf{Q}'$ and $\mathcal{L}:\mathbf{S}\to\mathbf{S}'$ as

$$\begin{aligned}(\mathcal{R}\,\mathbf{p},\mathbf{q}) &:= (R_{\boldsymbol{\sigma}_p}\boldsymbol{\sigma}_p,\boldsymbol{\tau}_p)+(R_{p_p}p_p,w_p)+(R_{\mathbf{u}_p}\mathbf{u}_p,\mathbf{v}_p)+(R_{\boldsymbol{\theta}}\boldsymbol{\theta},\boldsymbol{\phi}),\\ (\mathcal{L}\,\mathbf{r},\mathbf{s}) &:= (L_\lambda\lambda,\xi)+(L_{\mathbf{u}_s}\mathbf{u}_s,\mathbf{v}_s)+(L_{\boldsymbol{\gamma}_p}\boldsymbol{\gamma}_p,\boldsymbol{\chi}_p).\end{aligned} \quad (4.2.28)$$

For $\epsilon>0$, consider a regularization of (4.2.23) defined by: Given $\widehat{\mathbf{F}}\in\mathbf{Q}_2'$ and $\mathbf{G}\in\mathbf{S}'$, find $\mathbf{p}_\epsilon=(\boldsymbol{\sigma}_{p,\epsilon},p_{p,\epsilon},\mathbf{u}_{p,\epsilon},\mathbf{T}_{f,\epsilon},\mathbf{u}_{f,\epsilon},\boldsymbol{\theta}_\epsilon)\in\mathbf{Q}$ and $\mathbf{r}_\epsilon=(\lambda_\epsilon,\mathbf{u}_{s,\epsilon},\boldsymbol{\gamma}_{p,\epsilon})\in\mathbf{S}$ such that

$$\begin{aligned}(\epsilon\mathcal{R}+\mathcal{E}+\mathcal{A}+\mathcal{K}_{\mathbf{w}_f})\,\mathbf{p}_\epsilon+\mathcal{B}'\,\mathbf{r}_\epsilon &= \widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{Q}_2',\\ -\mathcal{B}\,\mathbf{p}_\epsilon+\epsilon\mathcal{L}\,\mathbf{r}_\epsilon &= \mathbf{G} \quad \text{in} \quad \mathbf{S}'.\end{aligned} \quad (4.2.29)$$

Let the operator $\mathcal{O}:\mathbf{Q}\times\mathbf{S}\to\mathbf{Q}'\times\mathbf{S}'$ be defined as

$$\mathcal{O}\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix} = \begin{pmatrix}\epsilon\mathcal{R}+\mathcal{E}+\mathcal{A}+\mathcal{K}_{\mathbf{w}_f} & \mathcal{B}'\\ -\mathcal{B} & \epsilon\mathcal{L}\end{pmatrix}\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix}.$$

Note that

$$\left(\mathcal{O}\begin{pmatrix}\mathbf{p}\\ \mathbf{r}\end{pmatrix},\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix}\right) = \big((\epsilon\mathcal{R}+\mathcal{E}+\mathcal{A}+\mathcal{K}_{\mathbf{w}_f})\,\mathbf{p},\mathbf{q}\big) + (\mathcal{B}'\,\mathbf{r},\mathbf{q}) - (\mathcal{B}\,\mathbf{p},\mathbf{s}) + \epsilon(\mathcal{L}\,\mathbf{r},\mathbf{s}),$$

thus we could conclude that $\mathcal{O}$ is bounded and continuous from Lemma 4.2.1 and Lemma 4.2.4–4.2.5. Moreover, using coercivity bounds from Lemma 4.2.2 and Lemma 4.2.4–4.2.5, we also have

$$\begin{aligned}
&\left(\mathcal{O}\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix},\begin{pmatrix}\mathbf{q}\\ \mathbf{s}\end{pmatrix}\right) = ((\epsilon\mathcal{R}+\mathcal{E}+\mathcal{A}+\mathcal{K}_{\mathbf{w}_f})\,\mathbf{q},\mathbf{q}) + (\epsilon\mathcal{L}\,\mathbf{s},\mathbf{s})\\
&= \epsilon r_{\boldsymbol{\sigma}_p}(\boldsymbol{\tau}_p,\boldsymbol{\tau}_p) + \epsilon r_{p_p}(w_p,w_p) + \epsilon r_{\mathbf{u}_p}(\mathbf{v}_p,\mathbf{v}_p) + \epsilon r_{\boldsymbol{\theta}}(\boldsymbol{\phi},\boldsymbol{\phi}) + (s_0 w_p,w_p) + a_e(\boldsymbol{\tau}_p,w_p;\boldsymbol{\tau}_p,w_p)\\
&\quad + a_p(\mathbf{v}_p,\mathbf{v}_p) + a_f(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) + \kappa_{\mathbf{w}_f}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) + a_{\mathtt{BJS}}(\mathbf{v}_f,\boldsymbol{\phi};\mathbf{v}_f,\boldsymbol{\phi})\\
&\quad + \epsilon l_\lambda(\xi,\xi) + \epsilon l_{\mathbf{u}_s}(\mathbf{v}_s,\mathbf{v}_s) + \epsilon l_{\boldsymbol{\gamma}_p}(\boldsymbol{\chi}_p,\boldsymbol{\chi}_p)\\
&\geq C\big(\epsilon\|\boldsymbol{\tau}_p\|^2_{\mathbb{X}_p} + \epsilon\|w_p\|^2_{\mathrm{W}_p} + \epsilon\|\mathrm{div}(\mathbf{v}_p)\|^2_{\mathrm{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\phi}\|^2_{\boldsymbol{\Lambda}_s} + s_0\|w_p\|^2_{\mathrm{W}_p}\\
&\quad + \|A^{1/2}(\boldsymbol{\tau}_p+\alpha_p w_p\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{v}_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{R}_f\|^2_{\mathbb{X}_f} + \|\mathbf{v}_f\|^2_{\mathbf{V}_f}\\
&\quad + |\mathbf{v}_f-\boldsymbol{\phi}|^2_{\mathtt{BJS}} + \epsilon\|\xi\|^2_{\Lambda_p} + \epsilon\|\mathbf{v}_s\|^2_{\mathbf{V}_s} + \epsilon\|\boldsymbol{\chi}_p\|^2_{\mathbb{Q}_p}\big),
\end{aligned} \tag{4.2.30}$$

which implies that $\mathcal{O}$ is coercive. Thus, an application of the Lax-Milgram theorem establishes the existence of a solution $(\mathbf{p}_\epsilon,\mathbf{r}_\epsilon)\in\mathbf{Q}\times\mathbf{S}$ of (4.2.29). Now, from (4.2.29) and (4.2.30), we have

$$\begin{aligned}
&\epsilon\|\boldsymbol{\sigma}_{p,\epsilon}\|^2_{\mathbb{X}_p} + \epsilon\|p_{p,\epsilon}\|^2_{\mathrm{W}_p} + \epsilon\|\mathrm{div}(\mathbf{u}_{p,\epsilon})\|^2_{\mathrm{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\theta}_\epsilon\|^2_{\boldsymbol{\Lambda}_s} + s_0\|p_{p,\epsilon}\|^2_{\mathrm{W}_p}\\
&\quad + \|A^{1/2}(\boldsymbol{\sigma}_{p,\epsilon}+\alpha_p p_{p,\epsilon}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{u}_{p,\epsilon}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{T}_{f,\epsilon}\|^2_{\mathbb{X}_f} + \|\mathbf{u}_{f,\epsilon}\|^2_{\mathbf{V}_f}\\
&\quad + |\mathbf{u}_{f,\epsilon}-\boldsymbol{\theta}_\epsilon|^2_{\mathtt{BJS}} + \epsilon\|\lambda_\epsilon\|^2_{\Lambda_p} + \epsilon\|\mathbf{u}_{s,\epsilon}\|^2_{\mathbf{V}_s} + \epsilon\|\boldsymbol{\gamma}_{p,\epsilon}\|^2_{\mathbb{Q}_p}\\
&\leq C\big(\|\hat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)}\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + \|\hat{g}_{w_p}\|_{\mathrm{L}^2(\Omega_p)}\|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)} + \|\hat{g}_{\mathbf{R}_f}\|_{\mathbb{L}^2(\Omega_f)}\|\mathbf{T}_{f,\epsilon}\|_{\mathbb{L}^2(\Omega_f)}\\
&\quad + \|\hat{g}_{\mathbf{v}_f}\|_{\mathbf{L}^2(\Omega_f)}\|\mathbf{u}_{f,\epsilon}\|_{\mathbf{L}^2(\Omega_f)} + \|\hat{g}_{\mathbf{v}_s}\|_{\mathbf{L}^2(\Omega_p)}\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}\big),
\end{aligned} \tag{4.2.31}$$

which implies that $\|A^{1/2}(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$, $\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{L}^2(\Omega_p)}$, $\|\mathbf{T}_{f,\epsilon}\|_{\mathbb{X}_f}$ and $\|\mathbf{u}_{f,\epsilon}\|_{\mathbf{V}_f}$ are bounded independently of ϵ. Next, we apply the inf-sup conditions in Lemma 4.2.3 and using (4.2.29) to get

$$\begin{aligned}
\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{V}_s} + \|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{Q}_p} &\le C\big(\|A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \epsilon\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbb{L}^2(\Omega_p)} + \|\hat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)}\big), \\
\|p_{p,\epsilon}\|_{\mathrm{W}_p} + \|\lambda_\epsilon\|_{\Lambda_p} &\le C\big(\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{L}^2(\Omega_p)} + \epsilon\|\mathrm{div}(\mathbf{u}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)}\big), \\
\|\boldsymbol{\theta}_\epsilon\|_{\boldsymbol{\Lambda}_s} &\le C\big(\|A(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \epsilon\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + \|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_f)} + \|\hat{g}_{\boldsymbol{\tau}_p}\|_{\mathbb{L}^2(\Omega_p)}\big), \quad (4.2.32)
\end{aligned}$$

which implies that $\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{V}_s}$, $\|\boldsymbol{\gamma}_{p,\epsilon}\|_{\mathbb{Q}_p}$, $\|p_{p,\epsilon}\|_{\mathrm{W}_p}$, $\|\lambda_\epsilon\|_{\Lambda_p}$ and $\|\boldsymbol{\theta}_\epsilon\|_{\boldsymbol{\Lambda}_s}$ are bounded independently of ϵ.

Since $\mathbf{div}(\mathbb{X}_p) = \mathbf{V}_s$, by taking $\mathbf{v}_s = \mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})$ in (4.2.29), we have

$$\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)} \le \epsilon\|\mathbf{u}_{s,\epsilon}\|_{\mathbf{L}^2(\Omega_p)} + \|\hat{g}_{\mathbf{v}_s}\|_{\mathbf{L}^2(\Omega_p)}, \quad (4.2.33)$$

which implies that $\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)}$ is bounded independently of ϵ. Since $\|A^{1/2}(\boldsymbol{\sigma}_{p,\epsilon} + \alpha_p p_{p,\epsilon}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$, $\|p_{p,\epsilon}\|_{\mathrm{W}_p}$ and $\|\mathbf{div}(\boldsymbol{\sigma}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)}$ are all bounded independently of ϵ, the same holds for $\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{X}_p}$. Finally, since $\mathrm{div}(\mathbf{V}_p) = \mathrm{W}_p$, by taking $w_p = \mathrm{div}(\mathbf{u}_{p,\epsilon})$ in (4.2.29), we have

$$\|\mathrm{div}(\mathbf{u}_{p,\epsilon})\|_{\mathrm{L}^2(\Omega_p)} \le C\big(\|\boldsymbol{\sigma}_{p,\epsilon}\|_{\mathbb{L}^2(\Omega_p)} + (s_0 + \epsilon)\|p_{p,\epsilon}\|_{\mathrm{L}^2(\Omega_p)} + \|\hat{g}_{w_p}\|_{\mathrm{L}^2(\Omega_p)}\big), \quad (4.2.34)$$

so $\|\mathrm{div}(\mathbf{u}_{p,\epsilon})\|_{\mathbf{L}^2(\Omega_p)}$, and therefore $\|\mathbf{u}_{p,\epsilon}\|_{\mathbf{V}_p}$ is bounded independently of ϵ. Therefore, we conclude that all the variables are bounded independently of ϵ. In addition, from (4.2.31)–(4.2.34) with (4.2.18), we conclude there exists $C_{\mathcal{J}} > 0$ independent of ϵ, such that

$$\|(\mathbf{p}_\epsilon, \mathbf{r}_\epsilon)\|_{\mathbf{Q}\times\mathbf{S}} \le C_{\mathcal{J}}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big). \quad (4.2.35)$$

Since $\mathbf{Q}$ and $\mathbf{S}$ are reflexive Banach spaces, and $\mathcal{E}$, $\mathcal{A}$, $\mathcal{K}_{\mathbf{w}_f}$, $\mathcal{B}$, $\widehat{\mathbf{F}}$ and $\mathbf{G}$ are continuous, as $\epsilon \to 0$ we can extract weakly convergent subsequences $\{\mathbf{p}_{\epsilon,n}\}_{n=1}^\infty$ and $\{\mathbf{r}_{\epsilon,n}\}_{n=1}^\infty$ such that $\mathbf{p}_{\epsilon,n} \to \mathbf{p}$ in $\mathbf{Q}$, $\mathbf{r}_{\epsilon,n} \to \mathbf{r}$ in $\mathbf{S}$, and $(\mathbf{p}, \mathbf{r})$ is a solution to (4.2.23). Moreover, proceeding analogously to (4.2.35) we derive (4.2.27).

Finally, we prove that the solution is unique. Let $(\mathbf{p},\mathbf{r})$ and $(\widetilde{\mathbf{p}},\widetilde{\mathbf{r}})$ be two solutions corresponding to the same data, we deduce that for all $(\mathbf{q},\mathbf{s}) \in \mathbf{Q}\times\mathbf{S}$:

$$\begin{aligned}(\mathcal{E}+\mathcal{A}+\mathcal{K}_{\mathbf{w}_f})(\mathbf{p}-\widetilde{\mathbf{p}})(\mathbf{q}) + \mathbf{B}'(\mathbf{r}-\widetilde{\mathbf{r}})(\mathbf{q}) &= \mathbf{0}, \\ -\mathcal{B}(\mathbf{p}-\widetilde{\mathbf{p}})(\mathbf{s}) &= \mathbf{0}\end{aligned} \tag{4.2.36}$$

Taking (4.2.36) with $\mathbf{q}=\mathbf{p}-\widetilde{\mathbf{p}}$ and $\mathbf{s}=\mathbf{r}-\widetilde{\mathbf{r}}$, combining with the monotonicity and coercivity results in Lemma 4.2.2 yields

$$\begin{aligned}&\alpha_{\mathcal{E}}(\Omega_p)\big(\|p_p-\widetilde{p}_p\|^2_{\mathrm{W}_p} + \|\boldsymbol{\sigma}_p-\widetilde{\boldsymbol{\sigma}}_p\|^2_{\mathbb{L}^2(\Omega_p)}\big) + \mu k_{\max}^{-1}\|\mathbf{u}_p-\widetilde{\mathbf{u}}_p\|^2_{\mathbf{L}^2(\Omega_p)} \\ &+c_{\mathtt{BJS}}|(\mathbf{u}_f-\widetilde{\mathbf{u}}_f)-(\boldsymbol{\phi}-\widetilde{\boldsymbol{\phi}})|^2_{\mathtt{BJS}} + \frac{\alpha_f}{2}\|(\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} \,\le\, \mathbf{0},\end{aligned} \tag{4.2.37}$$

so it follows that $p_p=\widetilde{p}_p$, $\boldsymbol{\sigma}_p=\widetilde{\boldsymbol{\sigma}}_p$, $\mathbf{u}_p=\widetilde{\mathbf{u}}_p$, $\mathbf{T}_f=\widetilde{\mathbf{T}}_f$, and $\mathbf{u}_f=\widetilde{\mathbf{u}}_f$. Next, employing the inf-sup conditions in Lemma 4.2.3, one can deduce easily that the rest variables are unique too. □

4.2.2.2 Step 2: The domain D is nonempty

In this section we proceed analogously to [34] by means of the well-known Banach fixed-point theorem to show that $\mathcal{D}$, cf. (4.2.17), is nonempty.

Lemma 4.2.7. *Let $r\in(0,r_0)$ with r_0 given by (4.2.7) and let $\mathbf{W}_r$ be the closed ball defined by*

$$\mathbf{W}_r := \big\{\mathbf{w}_f\in\mathbf{V}_f:\ \|\mathbf{w}_f\|_{\mathbf{V}_f}\le r\big\}, \tag{4.2.38}$$

and assume conditions in Lemma 4.2.2 are satisfied. Then, for all $\mathbf{w}_f$, $\widetilde{\mathbf{w}}_f\in\mathbf{W}_r$ there holds

$$\|\mathcal{J}(\mathbf{w}_f)-\mathcal{J}(\widetilde{\mathbf{w}}_f)\|_{\mathbf{V}_f} \le \frac{C_{\mathcal{J}}}{r_0}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big)\|\mathbf{w}_f-\widetilde{\mathbf{w}}_f\|_{\mathbf{V}_f}, \tag{4.2.39}$$

where $C_{\mathcal{J}}$ is the constant given by (4.2.27).

Proof. Given $\mathbf{w}_f$, $\widetilde{\mathbf{w}}_f \in \mathbf{W}_r$, we let $\mathbf{u}_f := \mathcal{J}(\mathbf{w}_f)$ and $\widetilde{\mathbf{u}}_f := \mathcal{J}(\widetilde{\mathbf{w}}_f)$. According to the definition of $\mathcal{J}$, cf. (4.2.22)–(4.2.23), it follows that

$$\begin{aligned} (\mathcal{E} + \mathcal{A} + \mathcal{K}_{\mathbf{w}_f})\,\mathbf{p} + \mathcal{B}'\,\mathbf{r} &= \widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{Q}_2', \\ -\mathcal{B}\,\mathbf{p} &= \mathbf{G} \quad \text{in} \quad \mathbf{S}'. \end{aligned}$$

and

$$\begin{aligned} (\mathcal{E} + \mathcal{A} + \mathcal{K}_{\widetilde{\mathbf{w}}_f})\,\widetilde{\mathbf{p}} + \mathcal{B}'\,\widetilde{\mathbf{r}} &= \widehat{\mathbf{F}} \quad \text{in} \quad \mathbf{Q}_2', \\ -\mathcal{B}\,\widetilde{\mathbf{p}} &= \mathbf{G} \quad \text{in} \quad \mathbf{S}'. \end{aligned}$$

Subtracting the second rows of both problems, we obtain that

$$-\mathcal{B}\,(\mathbf{p} - \widetilde{\mathbf{p}}) = \mathbf{0} \quad \text{in} \quad \mathbf{S}',$$

which implies that $(\mathbf{p} - \widetilde{\mathbf{p}}) \in \ker(\mathcal{B})$. So we then subtract the first rows of both problems and test with $\mathbf{q} = \mathbf{p} - \widetilde{\mathbf{p}}$, we obtain

$$(\mathcal{E} + \mathcal{A} + \mathcal{K}_{\mathbf{w}_f})(\mathbf{p} - \widetilde{\mathbf{p}})(\mathbf{p} - \widetilde{\mathbf{p}}) = -\mathcal{K}_{\mathbf{w}_f - \widetilde{\mathbf{w}}_f}(\widetilde{\mathbf{p}})(\mathbf{p} - \widetilde{\mathbf{p}}),$$

which together with the continuity of $\mathcal{K}_{\mathbf{w}_f}$ with $\mathbf{w}_f \in \mathbf{W}_r$, cf. Lemma 4.2.1, and the monotonicity of $\mathcal{A} + \mathcal{K}_{\mathbf{w}_f}$ and $\mathcal{E}$, cf. Lemma 4.2.2, implies that

$$\frac{\alpha_f}{2}\|\mathbf{u}_f - \widetilde{\mathbf{u}}_f\|_{\mathbf{V}_f} \le C_{\mathcal{K}}\|\widetilde{\mathbf{u}}_f\|_{\mathbf{V}_f}\|\mathbf{w}_f - \widetilde{\mathbf{w}}_f\|_{\mathbf{V}_f}.$$

Therefore, combining with the definition of r_0, cf. (4.2.7), and the bound of $\|\widetilde{\mathbf{u}}_f\|_{\mathbf{V}_f}$, cf. (4.2.27), we get

$$\begin{aligned} \|\mathbf{u} - \widetilde{\mathbf{u}}_f\|_{\mathbf{V}_f} &\le C_{\mathcal{J}}\frac{2C_{\mathcal{K}}}{\alpha_f}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big)\|\mathbf{w}_f - \widetilde{\mathbf{w}}_f\|_{\mathbf{V}_f} \\ &= \frac{C_{\mathcal{J}}}{r_0}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big)\|\mathbf{w}_f - \widetilde{\mathbf{w}}_f\|_{\mathbf{V}_f}. \end{aligned}$$

□

We are now in position of establishing the main result of this section.

Theorem 4.2.8. *Given $r \in (0, r_0)$, with r_0 given by* (4.2.7)*, we let $\mathbf{W}_r$ be as in* (4.2.38)*, assume conditions in Lemma 4.2.2, and in addition, assume that the data satisfy*

$$C_{\mathcal{J}}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big) \leq r. \tag{4.2.40}$$

Then, the problem (4.2.19) *has a unique solution $(\mathbf{p}, \mathbf{r}) \in \mathbf{Q} \times \mathbf{S}$ with $\mathbf{u}_f \in \mathbf{W}_r$, and there holds*

$$\|(\mathbf{p}, \mathbf{r})\|_{\mathbf{Q}\times\mathbf{S}} \leq C_{\mathcal{J}}\big(\|\mathbf{f}_f\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)} + \|q_f\|_{\mathrm{L}^2(\Omega_f)} + \|q_p\|_{\mathrm{L}^2(\Omega_p)}\big). \tag{4.2.41}$$

In addition, for $\mathcal{M}$ defined by (4.2.20) *we have $Rg(\widehat{\mathcal{E}} + \mathcal{M}) = \mathbf{E}_2'$.*

Proof. We start by noticing that (4.2.40) implies that $\mathcal{J} : \mathbf{W}_r \to \mathbf{W}_r$ is well-defined. Combining the result (4.2.39) and assumption (4.2.40), we have that

$$\|\mathcal{J}(\mathbf{w}_f) - \mathcal{J}(\widetilde{\mathbf{w}}_f)\|_{\mathbf{V}_f} \leq \frac{r}{r_0}\|\mathbf{w}_f - \widetilde{\mathbf{w}}_f\|_{\mathbf{V}_f}, \tag{4.2.42}$$

so $\mathcal{J}$ is a contraction mapping. Thus by the classical Banach fixed-point theorem, we conclude that $\mathcal{J}$ has a unique fixed-point $\mathbf{u}_f \in \mathbf{W}_r$, or equivalently, (4.2.19) is well-posed and then the domain $\mathcal{D}$, cf. (4.2.17), is nonempty. And (4.2.41) follows directly from (4.2.27).

On the other hand, to show $Rg(\widehat{\mathcal{E}} + \mathcal{M}) = \mathbf{E}_2'$, we need to show that for $\mathbf{f} \in \mathbf{E}_2'$ there is a $\mathbf{v} \in \mathcal{D}$ such that $\mathbf{f} \in (\widehat{\mathcal{E}} + \mathcal{M})(\mathbf{v})$. In fact, given $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathbf{E}_2'$, Theorem 4.2.6 with $\mathbf{w}_f = \mathbf{u}_f$ implies that there exists $(\widehat{\boldsymbol{\sigma}}_p, \widehat{p}_p) \in \mathcal{D}$ such that (4.2.19) is satisfied. Hence $(\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\widehat{\boldsymbol{\sigma}}_p, \widehat{p}_p) \in \mathcal{M}(\widehat{\boldsymbol{\sigma}}_p, \widehat{p}_p)$ and therefore it follows that $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in (\widehat{\mathcal{E}} + \mathcal{M})(\widehat{\boldsymbol{\sigma}}_p, \widehat{p}_p)$. □

4.2.2.3 Step 3: Solvability of the parabolic problem

In this section we establish the existence of a solution to (4.2.21). We begin by showing that $\mathcal{M}$ defined by (4.2.20) is a monotone operator.

Lemma 4.2.9. *Let* $r \in (0, r_0)$ *with* r_0 *defined by* (4.2.7)*, assume conditions in Lemma 4.2.2, and assume that the data satisfy* (4.2.40)*. Then, the operator* $\mathcal{M}$ *defined by* (4.2.20) *is monotone.*

Proof. To show that $\mathcal{M}$ is monotone, we need to show for $\mathbf{f} \in \mathcal{M}(\mathbf{v})$, $\widetilde{\mathbf{f}} \in \mathcal{M}(\widetilde{\mathbf{v}})$ that $(\mathbf{f} - \widetilde{\mathbf{f}}, \mathbf{v} - \widetilde{\mathbf{v}})_{\Omega_p} \geq 0$. For $(\boldsymbol{\sigma}_p, p_p) \in \mathcal{D}$, $(\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_p, p_p) \in \mathcal{M}(\boldsymbol{\sigma}_p, p_p)$ with $(\widehat{\mathbf{f}}_p, \widehat{q}_p)$ satisfying condition in (4.2.17), and $(\boldsymbol{\tau}_p, w_p) \in \mathbf{E}$, we have

$$\begin{aligned}
&\big((\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_p, p_p), (\boldsymbol{\tau}_p, w_p)\big)_{\Omega_p} \\
&\quad = (\widehat{\mathbf{f}}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{q}_p, w_p)_{\Omega_p} - (A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})_{\Omega_p} - (s_0\, p_p, w_p)_{\Omega_p} \\
&\quad = -b_p(w_p, \mathbf{u}_p) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_s(\mathbf{u}_s, \boldsymbol{\tau}_p) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p).
\end{aligned} \tag{4.2.43}$$

Also from (4.2.17), $((\mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}), \lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p)$ satisfy

$$\begin{aligned}
&s_0\,(p_p, w_p)_{\Omega_p} + a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_f(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\mathbf{u}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) \\
&\quad + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{v}_f, \boldsymbol{\phi}) + b_p(p_p, \mathbf{v}_p) - b_p(w_p, \mathbf{u}_p) + b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_s(\mathbf{u}_s, \boldsymbol{\tau}_p) \\
&\quad + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p) + b_\Gamma(\mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\phi}; \lambda) \\
&\quad = -(\mathbf{f}_f, \kappa_1\, \mathbf{div}(\mathbf{R}_f) - \mathbf{v}_f)_{\Omega_f} - \frac{1}{n}\,(q_f\, \mathbf{I}, \mathbf{R}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{v}_f))_{\Omega_f} \\
&\qquad + (\widehat{\mathbf{f}}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{q}_p, w_p)_{\Omega_p}, \\
&-b_s(\mathbf{v}_s, \boldsymbol{\sigma}_p) - b_{\mathrm{sk}}(\boldsymbol{\chi}_p, \boldsymbol{\sigma}_p) - b_\Gamma(\mathbf{u}_p, \mathbf{u}_f, \boldsymbol{\theta}; \xi) \;=\; (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}.
\end{aligned} \tag{4.2.44}$$

Similarly, for $(\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p) \in \mathcal{D}$, $(\widetilde{\mathbf{f}}_p, \widetilde{q}_p) - \widehat{\mathcal{E}}(\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p) \in \mathcal{M}(\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p)$ with $(\widetilde{\mathbf{f}}_p, \widetilde{q}_p)$ satisfying condition in (4.2.17), and $(\boldsymbol{\tau}_p, w_p) \in \mathbf{E}$,

$$\big((\widetilde{\mathbf{f}}_p, \widetilde{q}_p) - \widehat{\mathcal{E}}(\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p), (\boldsymbol{\tau}_p, w_p)\big)_{\Omega_p} = -b_p(w_p, \widetilde{\mathbf{u}}_p) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \widetilde{\boldsymbol{\theta}}) + b_s(\widetilde{\mathbf{u}}_s, \boldsymbol{\tau}_p) + b_{\mathrm{sk}}(\widetilde{\gamma}_p, \boldsymbol{\tau}_p), \tag{4.2.45}$$

and the corresponding $((\widetilde{\mathbf{u}}_p, \widetilde{\mathbf{T}}_f, \widetilde{\mathbf{u}}_f, \widetilde{\boldsymbol{\theta}}), \widetilde{\lambda}, \widetilde{\mathbf{u}}_s, \widetilde{\boldsymbol{\gamma}}_p)$ satisfy

$$
\begin{aligned}
& s_0\,(\widetilde{p}_p, w_p)_{\Omega_p} + a_e(\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p; \boldsymbol{\tau}_p, w_p) + a_p(\widetilde{\mathbf{u}}_p, \mathbf{v}_p) + a_f(\widetilde{\mathbf{T}}_f, \widetilde{\mathbf{u}}_f; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\widetilde{\mathbf{u}}_f}(\widetilde{\mathbf{T}}_f, \widetilde{\mathbf{u}}_f; \mathbf{R}_f, \mathbf{v}_f) \\
&\quad + a_{\mathtt{BJS}}(\widetilde{\mathbf{u}}_f, \widetilde{\boldsymbol{\theta}}; \mathbf{v}_f, \boldsymbol{\phi}) + b_p(\widetilde{p}_p, \mathbf{v}_p) - b_p(w_p, \widetilde{\mathbf{u}}_p) + b_{\mathbf{n}_p}(\widetilde{\boldsymbol{\sigma}}_p, \boldsymbol{\phi}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \widetilde{\boldsymbol{\theta}}) + b_s(\widetilde{\mathbf{u}}_s, \boldsymbol{\tau}_p) \\
&\quad + b_{\mathrm{sk}}(\widetilde{\boldsymbol{\gamma}}_p, \boldsymbol{\tau}_p) + b_\Gamma(\mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\phi}; \widetilde{\lambda}) \\
&\quad = -\,(\mathbf{f}_f, \kappa_1\,\mathbf{div}(\mathbf{R}_f) - \mathbf{v}_f)_{\Omega_f} - \frac{1}{n}\,(q_f\,\mathbf{I}, \mathbf{R}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{v}_f))_{\Omega_f} \\
&\qquad + (\widetilde{\mathbf{f}}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\widetilde{q}_p, w_p)_{\Omega_p}, \\
& -b_s(\mathbf{v}_s, \widetilde{\boldsymbol{\sigma}}_p) - b_{\mathrm{sk}}(\boldsymbol{\chi}_p, \widetilde{\boldsymbol{\sigma}}_p) - b_\Gamma(\widetilde{\mathbf{u}}_p, \widetilde{\mathbf{u}}_f, \widetilde{\boldsymbol{\theta}}; \xi) \;=\; (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}.
\end{aligned}
\tag{4.2.46}
$$

With the association $\mathbf{v} = (\boldsymbol{\sigma}_p, p_p)$, $\widetilde{\mathbf{v}} = (\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p)$, $\mathbf{f} = (\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_p, p_p)$, and $\widetilde{\mathbf{f}} = (\widetilde{\mathbf{f}}_p, \widetilde{q}_p) - \widehat{\mathcal{E}}(\widetilde{\boldsymbol{\sigma}}_p, \widetilde{p}_p)$, we deduce that

$$
\begin{aligned}
(\mathbf{f} - \widetilde{\mathbf{f}}, \mathbf{v} - \widetilde{\mathbf{v}})_{\Omega_p} &= -b_p(p_p - \widetilde{p}_p, \mathbf{u}_p - \widetilde{\mathbf{u}}_p) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p - \widetilde{\boldsymbol{\sigma}}_p, \boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}}) + b_s(\mathbf{u}_s - \widetilde{\mathbf{u}}_s, \boldsymbol{\sigma}_p - \widetilde{\boldsymbol{\sigma}}_p) \\
&\quad + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p - \widetilde{\boldsymbol{\gamma}}_p, \boldsymbol{\sigma}_p - \widetilde{\boldsymbol{\sigma}}_p).
\end{aligned}
\tag{4.2.47}
$$

Testing the first equation in (4.2.44) with $(\boldsymbol{\tau}_p, w_p, \mathbf{v}_p, \mathbf{R}_f, \mathbf{v}_f, \boldsymbol{\phi}) = (\mathbf{0}, 0, \mathbf{u}_p - \widetilde{\mathbf{u}}_p, \mathbf{T}_f - \widetilde{\mathbf{T}}_f, \mathbf{u}_f - \widetilde{\mathbf{u}}_f, \boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}})$ and the second equation in (4.2.44) and (4.2.46) with $(\xi, \mathbf{v}_s, \boldsymbol{\chi}_p) = (\lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p)$, we obtain

$$
\begin{aligned}
& a_p(\mathbf{u}_p, \mathbf{u}_p - \widetilde{\mathbf{u}}_p) + a_f(\mathbf{T}_f, \mathbf{u}_f; \mathbf{T}_f - \widetilde{\mathbf{T}}_f, \mathbf{u}_f - \widetilde{\mathbf{u}}_f) + \kappa_{\mathbf{u}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{T}_f - \widetilde{\mathbf{T}}_f, \mathbf{u}_f - \widetilde{\mathbf{u}}_f) \\
& + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{u}_f - \widetilde{\mathbf{u}}_f, \boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}}) + b_p(p_p, \mathbf{u}_p - \widetilde{\mathbf{u}}_p) + b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}}) \\
& - b_s(\mathbf{u}_s, \boldsymbol{\sigma}_p - \widetilde{\boldsymbol{\sigma}}_p) - b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\sigma}_p - \widetilde{\boldsymbol{\sigma}}_p) \\
& = -(\mathbf{f}_f, \kappa_1 \mathbf{div}(\mathbf{T}_f - \widetilde{\mathbf{T}}_f) - (\mathbf{u}_f - \widetilde{\mathbf{u}}_f))_{\Omega_f} - \frac{1}{n}\,(q_f\,\mathbf{I}, \mathbf{T}_f - \widetilde{\mathbf{T}}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{u}_f - \widetilde{\mathbf{u}}_f))_{\Omega_f}.
\end{aligned}
\tag{4.2.48}
$$

Repeating the same argument for the problem of $((\widetilde{\mathbf{u}}_p, \widetilde{\mathbf{T}}_f, \widetilde{\mathbf{u}}_f, \widetilde{\boldsymbol{\theta}}), \widetilde{\lambda}, \widetilde{\mathbf{u}}_s, \widetilde{\boldsymbol{\gamma}}_p)$, we deduce a similar identity as (4.2.48). Subtracting these two identities to get an expression for the right hand side of (4.2.47), and then replace back into (4.2.47), we have

$$
\begin{aligned}
(\mathbf{f}-\widetilde{\mathbf{f}},\mathbf{v}-\widetilde{\mathbf{v}})_{\Omega_p} &= a_p(\mathbf{u}_p-\widetilde{\mathbf{u}}_p,\mathbf{u}_p-\widetilde{\mathbf{u}}_p) + a_f(\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f;\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\\
&\quad + \kappa_{\mathbf{u}_f}(\mathbf{T}_f,\mathbf{u}_f;\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f) - \kappa_{\widetilde{\mathbf{u}}_f}(\widetilde{\mathbf{T}}_f,\widetilde{\mathbf{u}}_f;\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\\
&\quad + a_{\mathtt{BJS}}(\mathbf{u}_f-\widetilde{\mathbf{u}}_f,\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}};\mathbf{u}_f-\widetilde{\mathbf{u}}_f,\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\\
&= a_p(\mathbf{u}_p-\widetilde{\mathbf{u}}_p,\mathbf{u}_p-\widetilde{\mathbf{u}}_p) + a_f(\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f;\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\\
&\quad + \kappa_{\mathbf{u}_f-\widetilde{\mathbf{u}}_f}(\mathbf{T}_f,\mathbf{u}_f;\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f) + \kappa_{\widetilde{\mathbf{u}}_f}(\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f;\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\\
&\quad + a_{\mathtt{BJS}}(\mathbf{u}_f-\widetilde{\mathbf{u}}_f,\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}};\mathbf{u}_f-\widetilde{\mathbf{u}}_f,\boldsymbol{\theta}-\widetilde{\boldsymbol{\theta}})\\
&\geq \big(\alpha_f - C_{\mathcal{K}}(\|(\mathbf{T}_f,\mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f} + \|(\widetilde{\mathbf{T}}_f,\widetilde{\mathbf{u}}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f})\big)\|(\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f},
\end{aligned}
\tag{4.2.49}
$$

where we have employed the monotonicity of a_p, a_f and $a_{\mathtt{BJS}}$, cf. Lemma 4.2.2, and the continuity of $\kappa_{\mathbf{u}_f}$, cf. Lemma 4.2.1. Finally, recalling that both $\|(\mathbf{T}_f,\mathbf{v}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}$ and $\|(\widetilde{\mathbf{T}}_f,\widetilde{\mathbf{v}}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}$ are bounded by data, cf. (4.2.41), with the assumption on data (4.2.40), we obtain

$$
(\mathbf{f}-\widetilde{\mathbf{f}},\mathbf{v}-\widetilde{\mathbf{v}})_{\Omega_p} \geq (\alpha_f - 2\,r_0\,C_{\mathcal{K}})\|(\mathbf{T}_f-\widetilde{\mathbf{T}}_f,\mathbf{u}_f-\widetilde{\mathbf{u}}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} = 0, \tag{4.2.50}
$$

which implies the monotonicity of $\mathcal{M}$ and conclude the proof. □

Next, in order to prove that (4.2.21) has a solution in $\mathcal{D}$, we need to show that $(\boldsymbol{\sigma}_{p,0},p_{p,0})$ live in $\mathcal{D}$.

Lemma 4.2.10. *Let $(q_f(0),\mathbf{f}_f(0),\mathbf{f}_p(0)) \in \mathbb{X}_f' \times \mathbf{V}_f' \times \mathbf{V}_s'$. Assume the initial condition $p_{p,0} \in \mathrm{W}_p \cap \mathrm{H}$, where*

$$
\mathrm{H} := \Big\{ w_p \in \mathrm{H}^1(\Omega_p) : \quad \mathbf{K}\nabla w_p \in \mathbf{H}^1(\Omega_p),\ \mathbf{K}\nabla w_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}},\ w_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}} \Big\}. \tag{4.2.51}
$$

In addition, assume $\mathbf{f}_f(0)$, $q_f(0)$ and $p_{p,0}$ satisfy a small data condition

$$
C_{\mathcal{J},0}(\|\mathbf{f}_f(0)\|_{\mathbf{L}^2(\Omega_f)} + \|q_f(0)\|_{\mathrm{L}^2(\Omega_f)} + \|p_{p,0}\|_{\mathrm{H}^1(\Omega_p)}) \leq r_{f,0} \tag{4.2.52}
$$

where $C_{\mathcal{J},0}$ *and* $r_{f,0}$ *are defined in a similar manner as in* (4.2.40). *Then, there exists* $\mathbf{p}_0 := (\boldsymbol{\sigma}_{p,0}, p_{p,0}, \mathbf{u}_{p,0}, \mathbf{T}_{f,0}, \mathbf{u}_{f,0}, \boldsymbol{\theta}_0) \in \mathbf{Q}$, *and* $\mathbf{r}_0 := (\boldsymbol{\lambda}_0, \mathbf{u}_{s,0}\boldsymbol{\gamma}_{p,0}) \in \mathbf{S}$ *such that* (4.2.17) *holds for suitable* $(\widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0}) \in \mathbf{E}_2'$.

Proof. We proceed as in [4, Lemma 4.10]. In fact, we solve a sequence of well-defined subproblems, using the previously obtained solutions as data to guarantee that we obtain a solution of the coupled problem. We take the following steps.

1. Define $\mathbf{u}_{p,0} := -\dfrac{1}{\mu}\mathbf{K}\nabla p_{p,0}$, with $p_{p,0} \in \mathrm{W}_p \cap \mathrm{H}$, cf. (4.2.51), it follows that

$$\mu\,\mathbf{K}^{-1}\mathbf{u}_{p,0} = -\nabla p_{p,0}, \quad \mathrm{div}(\mathbf{u}_{p,0}) = -\frac{1}{\mu}\,\mathrm{div}(\mathbf{K}\nabla p_{p,0}) \quad \text{in} \quad \Omega_p, \quad \mathbf{u}_{p,0}\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}. \tag{4.2.53}$$

Next, defining $\lambda_0 := p_{p,0}|_{\Gamma_{fp}} \in \Lambda_p$, integrating by parts the first equation in (4.2.53) and impose in a weak sense the second equation of (4.2.53), we obtain

$$\begin{aligned} a_p(\mathbf{u}_{p,0}, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_{p,0}) + b_\Gamma(\mathbf{v}_p, \mathbf{0}, \mathbf{0}; \lambda_0) &= 0, \quad \forall\, \mathbf{v}_p \in \mathbf{V}_p, \\ -b_p(\mathbf{u}_{p,0}, w_p) = -\frac{1}{\mu}\,(\mathrm{div}(\mathbf{K}\nabla p_{p,0}), w_p)_{\Omega_p}, & \qquad \forall\, w_p \in \mathrm{W}_p. \end{aligned} \tag{4.2.54}$$

2. Define $(\mathbf{T}_{f,0}, \mathbf{u}_{f,0}) \in \mathbb{X}_f \times \mathbf{V}_f$ associated to the problem

$$\begin{aligned} & a_f(\mathbf{T}_{f,0}, \mathbf{u}_{f,0}; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\mathbf{u}_{f,0}}(\mathbf{T}_{f,0}, \mathbf{u}_{f,0}; \mathbf{R}_f, \mathbf{v}_f) \\ & = \; -a_{\mathtt{BJS}}(\mathbf{u}_{p,0}, \mathbf{0}; \mathbf{v}_f, \mathbf{0}) - \langle \mathbf{v}_f\cdot\mathbf{n}_f, \lambda_0\rangle_{\Gamma_{fp}} - (\mathbf{f}_f(0), \kappa_1\,\mathbf{div}(\mathbf{R}_f) - \mathbf{v}_f)_{\Omega_f} \\ & \quad -\frac{1}{n}\,(q_f(0)\,\mathbf{I}, \mathbf{R}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f(0), \mathrm{div}(\mathbf{v}_f))_{\Omega_f}, \qquad \forall (\mathbf{R}_f, \mathbf{v}_f) \in \mathbb{X}_f \times \mathbf{V}_f. \end{aligned} \tag{4.2.55}$$

Notice that (4.2.55) is well-posed, since it corresponds to the weak solution of the augmented mixed formulation for the Navier-Stokes problem with mixed boundary conditions. We would like to point out that to show the well-posedness, a fixed point approach needs to be adopted with a small data assumption (4.2.52). We refer to [33] for more details. Notice also that $\mathbf{u}_{p,0}$ and λ_0 are data for this problem.

3. Define $(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\eta}_{p,0}, \boldsymbol{\rho}_{p,0}, \boldsymbol{\psi}_0) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \boldsymbol{\Lambda}_s$ such that

$$
\begin{aligned}
&(A\boldsymbol{\sigma}_{p,0}, \boldsymbol{\tau}_p)_{\Omega_p} + b_s(\boldsymbol{\eta}_{p,0}, \boldsymbol{\tau}_p) + b_{\mathrm{sk}}(\boldsymbol{\rho}_{p,0}, \boldsymbol{\tau}_p) - b_{\mathbf{n}_p}(\boldsymbol{\psi}_0, \boldsymbol{\tau}_p) = -(A\alpha p_{p,0}\mathbf{I}, \boldsymbol{\tau}_p)_{\Omega_p}, && \forall\, \boldsymbol{\tau}_p \in \mathbb{X}_p, \\
&b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}) = -a_{\mathtt{BJS}}(\mathbf{u}_{p,0}, \mathbf{0}; \mathbf{0}, \boldsymbol{\phi}) - \langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}}, && \forall\, \boldsymbol{\phi} \in \boldsymbol{\Lambda}_s, \\
&-b_s(\boldsymbol{\sigma}_{p,0}, \mathbf{v}_s) = (\mathbf{f}_p(0), \mathbf{v}_s)_{\Omega_p}, && \forall\, \mathbf{v}_s \in \mathbf{V}_s, \\
&-b_{\mathrm{sk}}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi}_p) = 0, && \forall\, \boldsymbol{\chi}_p \in \mathbb{Q}_p.
\end{aligned}
\tag{4.2.56}
$$

This is a well-posed problem corresponding to the weak solution of the mixed elasticity system with mixed boundary conditions on Γ_{fp}. Note that $p_{p,0}$, $\mathbf{u}_{p,0}$ and λ_0 are data for this problem. Here $\boldsymbol{\eta}_{p,0}$, $\boldsymbol{\rho}_{p,0}$, and $\boldsymbol{\psi}_0$ are auxiliary variables that are not part of the constructed initial data. However, they can be used to recover the variables $\boldsymbol{\eta}_p$, $\boldsymbol{\rho}_p$, and $\boldsymbol{\psi}$ that satisfy the non-differentiated equation (2.1.12).

4. Define $\boldsymbol{\theta}_0 \in \boldsymbol{\Lambda}_s$ as

$$
\boldsymbol{\theta}_0 = \mathbf{u}_{f,0} - \mathbf{u}_{p,0} \quad \text{on} \quad \Gamma_{fp}, \tag{4.2.57}
$$

where $\mathbf{u}_{f,0}$ and $\mathbf{u}_{p,0}$ are data obtained in the previous steps. Note that (4.2.57) implies that the BJS terms in (4.2.55) and (4.2.56) can be rewritten with $\mathbf{u}_{p,0} \cdot \mathbf{t}_{f,j} = (\mathbf{u}_{f,0} - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}$ and that (4.1.8i) holds for the initial data.

5. Define $(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$ such that

$$
\begin{aligned}
&(A\widehat{\boldsymbol{\sigma}}_{p,0}, \boldsymbol{\tau}_p)_{\Omega_p} + b_s(\mathbf{u}_{s,0}, \boldsymbol{\tau}_p) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_{p,0}, \boldsymbol{\tau}_p) = b_{\mathbf{n}_p}(\boldsymbol{\theta}_0, \boldsymbol{\tau}_p), && \forall\, \boldsymbol{\tau}_p \in \mathbb{X}_p, \\
&-b_s(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{v}_s) = 0, && \forall\, \mathbf{v}_s \in \mathbf{V}_s, \\
&-b_{\mathrm{sk}}(\widehat{\boldsymbol{\sigma}}_{p,0}, \boldsymbol{\chi}_p) = 0, && \forall\, \boldsymbol{\chi}_p \in \mathbb{Q}_p.
\end{aligned}
\tag{4.2.58}
$$

This is a well-posed problem, since it corresponds to the weak solution of the mixed elasticity system with Dirichlet data $\boldsymbol{\theta}_0$ on Γ_{fp}. We note that $\widehat{\boldsymbol{\sigma}}_{p,0}$ is an auxilliary variable not used in the initial data.

Combining (4.2.53)–(4.2.58), we obtain $(\boldsymbol{\sigma}_{p,0}, p_{p,0}, \mathbf{u}_{p,0}, \mathbf{T}_{f,0}, \mathbf{u}_{f,0}, \boldsymbol{\theta}_0) \in \mathbf{Q}$ and $(\boldsymbol{\lambda}_0, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbf{S}$ satisfying (4.2.17) with $\widehat{\mathbf{f}}_{p,0}$ and $\widehat{q}_{p,0}$ such that

$$\begin{aligned}(\widehat{\mathbf{f}}_{p,0}, \boldsymbol{\tau}_p)_{\Omega_p} &= a_e(\boldsymbol{\sigma}_{p,0}, p_{p,0}; \boldsymbol{\tau}_p, 0) - (A(\widehat{\boldsymbol{\sigma}}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p}, \\ (\widehat{q}_{p,0}, w_p)_{\Omega_p} &= (s_0\, p_{p,0}, w_p)_{\Omega_p} + a_e(\boldsymbol{\sigma}_{p,0}, p_{p,0}; \mathbf{0}, w_p) - b_p(\mathbf{u}_{p,0}, w_p),\end{aligned} \tag{4.2.59}$$

resulting in

$$\|\widehat{\mathbf{f}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{q}_{p,0}\|_{\mathrm{L}^2(\Omega_p)} \le C(\|p_{p,0}\|_{\mathrm{W}_p} + \|\boldsymbol{\sigma}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{\boldsymbol{\sigma}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\mathrm{div}(\mathbf{u}_{p,0})\|_{\mathrm{L}^2(\Omega_p)}), \tag{4.2.60}$$

thus $(\widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0}) \in \mathbf{E}_2'$. Then, from the construction of the initial data (4.2.53)–(4.2.58), we could deduce that there exists a constant $\widehat{C}_{ep}$ such that

$$\begin{aligned}&\|\widehat{\mathbf{f}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{q}_{p,0}\|_{\mathrm{L}^2(\Omega_p)} \\ &\le \widehat{C}_{ep}\big(\|\mathbf{f}_f(0)\|_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{f}_p(0)\|_{\mathbf{L}^2(\Omega_p)} + \|q_f(0)\|_{\mathrm{L}^2(\Omega_f)} + \|q_p(0)\|_{\mathrm{L}^2(\Omega_p)} + \|\mathrm{div}(\mathbf{K}p_{p,0})\|_{\mathrm{L}^2(\Omega_p)}\big),\end{aligned} \tag{4.2.61}$$

completing the proof. □

Theorem 4.2.11. *For each* $(h_{\boldsymbol{\sigma}_p}, h_{p_p}) \in \mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,1}(0,T;\mathrm{W}_p')$*, and* $(\boldsymbol{\sigma}_{p,0}, p_{p,0})$ *satisfying Lemma 4.2.10, there exists a solution to* (4.2.21) *with*

$$(\boldsymbol{\sigma}_p, p_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p) \quad \textit{and} \quad (\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0}).$$

Proof. Applying Theorem 2.2.3 with $\mathcal{N} = \widehat{\mathcal{E}}$, $\mathcal{M} = \mathcal{M}$, $E = \mathbf{E} = \mathbb{X}_p \times \mathrm{W}_p$ and $E_b' = \mathbf{E}_2' = \mathbb{L}^2(\Omega_p) \times \mathrm{W}_p$, and using Theorem 4.2.8 and Lemma 4.2.9, we obtain the existence of a solution to (4.2.21), with $(\boldsymbol{\sigma}_p, p_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ and $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$. □

4.2.2.4 Step 4: The original problem is a special case

Finally, we establish the existence of a solution to (4.1.12) as a direct consequence of Theorem 4.2.11.

Lemma 4.2.12. *If* $(\boldsymbol{\sigma}_p(t), p_p(t)) \in \mathcal{D}$ *solves* (4.2.21) *for*

$$(h_{\boldsymbol{\sigma}_p}, h_{p_p}) = (\mathbf{0}, q_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p),$$

then it also solves (4.1.12)*.*

Proof. Let $(\boldsymbol{\sigma}_p(t), p_p(t)) \in \mathcal{D}$ solves (4.2.21) for $(h_{\boldsymbol{\sigma}_p}, h_{p_p}) = (\mathbf{0}, q_p)$. Note that the resolvent system (4.2.17) from the definition of the domain $\mathcal{D}$ directly implies (4.1.12) when both are tested with $\mathbf{q} = (\mathbf{0}, 0, \mathbf{v}_p, \mathbf{R}_f, \mathbf{v}_f, \boldsymbol{\phi})$ and $\mathbf{s} = (\xi, \mathbf{v}_s, \boldsymbol{\chi}_p)$. Thus it remains to show (4.1.12) with $\mathbf{q} = (\boldsymbol{\tau}_p, w_p, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$.

Since $(\boldsymbol{\sigma}_p(t), p_p(t))$ solves (4.2.21) for $(h_{\boldsymbol{\sigma}_p}, h_{p_p}) = (\mathbf{0}, q_p)$, there exists $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathbb{L}^2(\Omega_p) \times \mathrm{W}_p'$ such that $(\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_p, p_p) \in \mathcal{M}(\boldsymbol{\sigma}_p, p_p)$ satisfies

$$\frac{d}{dt}\widehat{\mathcal{E}}\begin{pmatrix} \boldsymbol{\sigma}_p \\ p_p \end{pmatrix} + \begin{pmatrix} \widehat{\mathbf{f}}_p \\ \widehat{q}_p \end{pmatrix} - \widehat{\mathcal{E}}\begin{pmatrix} \boldsymbol{\sigma}_p \\ p_p \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ q_p \end{pmatrix}. \tag{4.2.62}$$

Then, for all $(\boldsymbol{\tau}_p, w_p) \in \mathbb{X}_p \times \mathrm{W}_p$ there holds

$$\left(\frac{d}{dt}\widehat{\mathcal{E}}\begin{pmatrix} \boldsymbol{\sigma}_p \\ p_p \end{pmatrix}, \begin{pmatrix} \boldsymbol{\tau}_p \\ w_p \end{pmatrix}\right)_{\Omega_p} + \left(\begin{pmatrix} \widehat{\mathbf{f}}_p \\ \widehat{q}_p \end{pmatrix} - \widehat{\mathcal{E}}\begin{pmatrix} \boldsymbol{\sigma}_p \\ p_p \end{pmatrix}, \begin{pmatrix} \boldsymbol{\tau}_p \\ w_p \end{pmatrix}\right)_{\Omega_p} = (q_p, w_p)_{\Omega_p}. \tag{4.2.63}$$

Notice from the first row of (4.2.17) with $\mathbf{q} = (\boldsymbol{\tau}_p, w_p, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \in \mathbf{Q}$, we deduce

$$\begin{aligned}
&\left(\begin{pmatrix} \widehat{\mathbf{f}}_p \\ \widehat{q}_p \end{pmatrix} - \widehat{\mathcal{E}}\begin{pmatrix} \boldsymbol{\sigma}_p \\ p_p \end{pmatrix}, \begin{pmatrix} \boldsymbol{\tau}_p \\ w_p \end{pmatrix}\right)_{\Omega_p} \\
&\quad = (\widehat{\mathbf{f}}_p, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{q}_p, w_p)_{\Omega_p} - a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) - (s_0\, p_p, w_p)_{\Omega_p} \\
&\quad = -\, b_p(\mathbf{u}_p, w_p) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p),
\end{aligned}$$

which together with (4.2.63), yields

$$
\begin{aligned}
&a_e(\partial_t\,\boldsymbol{\sigma}_p, \partial_t\, p_p; \boldsymbol{\tau}_p, w_p) + (s_0\,\partial_t\, p_p, w_p)_{\Omega_p} - b_p(\mathbf{u}_p, w_p)\\
&-\, b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p) \;=\; (q_p, w_p)_{\Omega_p} \quad \forall\,(\boldsymbol{\tau}_p, w_p)\in \mathbb{X}_p\times \mathrm{W}_p,
\end{aligned}
$$

completing the proof. □

We end this section establishing the main result.

Theorem 4.2.13. *For each compatible initial data* $(\mathbf{p}_0, \mathbf{r}_0)\in\mathcal{D}$ *constructed in Lemma 4.2.10 and*

$$
\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'),\quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'),\quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}_f'),\quad q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p')
$$

satisfying (4.2.40)*, there exists a unique solution of* (4.1.12)*,* $(\mathbf{p},\mathbf{r}) : [0,T]\to \mathbf{Q}\times\mathbf{S}$ *with* $\mathbf{u}_f(t) : [0,T]\to \mathbf{W}_r$*,* $(\boldsymbol{\sigma}_p, p_p)\in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p))\times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

Proof. Existence of a solution of (4.1.12) follows from Theorem 4.2.11 and Lemma 4.2.12. In addition, from Lemma 4.2.11 we have that $(\boldsymbol{\sigma}_p, p_p)\in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p))\times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$.

Now, assume that the solution of (4.1.12) is not unique. Let $(\mathbf{p},\mathbf{r})$ and $(\widetilde{\mathbf{p}},\widetilde{\mathbf{r}})$ be two solutions corresponding the same data and denote $\overline{\mathbf{p}} = \mathbf{p}-\widetilde{\mathbf{p}}$ with similar notations for the rest of variables, we deduce that

$$
\begin{aligned}
\partial_t\,\mathcal{E}\,(\overline{\mathbf{p}})\,(\mathbf{q}) + \mathcal{A}\,(\overline{\mathbf{p}})\,(\mathbf{q}) + \mathcal{K}_{\mathbf{u}_f}\,(\overline{\mathbf{p}})\,(\mathbf{q}) + \mathcal{K}_{\overline{\mathbf{u}}_f}\,(\widetilde{\mathbf{p}})\,(\mathbf{q}) + \mathcal{B}'\,(\overline{\mathbf{r}})\,(\mathbf{q}) &= \mathbf{0} \quad \forall\,\mathbf{q}\in\mathbf{Q},\\
-\mathcal{B}\,(\overline{\mathbf{p}})\,(\mathbf{s}) &= \mathbf{0} \quad \forall\,\mathbf{s}\in\mathbf{S}.
\end{aligned}
\tag{4.2.64}
$$

Taking (4.2.64) with $\mathbf{q}=\overline{\mathbf{p}}$ and $\mathbf{s}=\overline{\mathbf{r}}$, making use of continuity of $\mathcal{K}_{\mathbf{w}_f}$ in Lemme 4.2.1 and coercivity of $\mathcal{A}+\mathcal{K}_{\mathbf{w}_f}$ and $\mathcal{E}$ in Lemma 4.2.2, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\,\partial_t\,\left(\|A^{1/2}\,(\overline{\boldsymbol{\sigma}}_p + \alpha_p\,\overline{p}_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|\overline{p}_p\|^2_{\mathrm{W}_p}\right)\\
&\quad+\Big(\alpha_f - C_{\mathcal{K}}\big(\|(\mathbf{T}_f,\mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f} + \|(\widetilde{\mathbf{T}}_f,\widetilde{\mathbf{u}}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\big)\Big)\|(\overline{\mathbf{T}}_f,\overline{\mathbf{u}}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f}\\
&\quad+\mu\,k_{\max}^{-1}\|\overline{\mathbf{u}}_p\|^2_{\mathbf{L}^2(\Omega_p)} + c_{\mathtt{BJS}}|\overline{\mathbf{u}}_f - \overline{\boldsymbol{\phi}}|^2_{\mathtt{BJS}} \leq 0.
\end{aligned}
\tag{4.2.65}
$$

Integrating in time from 0 to $t \in (0,T]$, using $\overline{\boldsymbol{\sigma}}_p(0) = \mathbf{0}$ and $\overline{p}_p(0) = 0$, we obtain

$$\begin{aligned} & \frac{1}{2}\,\left(\|A^{1/2}\,(\overline{\boldsymbol{\sigma}}_p + \alpha_p\,\overline{p}_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|\overline{p}_p\|^2_{\mathrm{W}_p}\right) \\ & \quad + 2\,C_{\mathcal{K}}(r_0 - r)\int_0^t \|(\overline{\mathbf{T}}_f, \overline{\mathbf{u}}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} ds + \int_0^t \mu\,k_{\max}^{-1}\|\overline{\mathbf{u}}_p\|^2_{\mathbf{L}^2(\Omega_p)} ds \leq 0. \end{aligned} \tag{4.2.66}$$

Therefore, it follows from (4.2.66) that $A^{1/2}(\overline{\boldsymbol{\sigma}}_p + \alpha_p\,\overline{p}_p\,\mathbf{I})(t) = \mathbf{0}, \overline{\mathbf{T}}_f(t) = \mathbf{0}, \overline{\mathbf{u}}_f(t) = \mathbf{0}$, $\overline{\mathbf{u}}_p(t) = \mathbf{0}$ for all $t \in (0,T]$.

On the other hand, from the first row of (4.2.64), employing the inf-sup conditions of $\mathcal{B}$ in Lemma 4.2.3 for $\mathbf{v}_s = \overline{\mathbf{u}}_s$, $\boldsymbol{\chi}_p = \overline{\boldsymbol{\gamma}}_p$, $w_p = \overline{p}_p$, $\xi = \overline{\lambda}$, $\boldsymbol{\phi} = \overline{\boldsymbol{\theta}}$, we obtain

$$\begin{aligned} & \beta\|(\overline{\mathbf{u}}_s, \overline{\boldsymbol{\gamma}}_p, \overline{p}_p, \overline{\lambda}, \overline{\boldsymbol{\theta}})\|_{\mathbf{V}_s\times\mathbb{Q}_p\times\mathrm{W}_p\times\Lambda_p\times\mathbf{\Lambda}_s} \\ & \leq \sup_{(\boldsymbol{\tau}_p,\mathbf{v}_p)\in\mathbb{X}_p\times\mathbf{V}_p} \frac{b_s(\boldsymbol{\tau}_p, \overline{\mathbf{u}}_s) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \overline{\boldsymbol{\gamma}}_p) + b_p(\mathbf{v}_p, \overline{p}_p) + b_\Gamma(\mathbf{0}, \mathbf{v}_p, \mathbf{0}; \overline{\lambda}) + b_n^p(\boldsymbol{\tau}_p, \overline{\boldsymbol{\theta}})}{\|(\boldsymbol{\tau}_p, \mathbf{v}_p)\|_{\mathbb{X}_p\times\mathbf{V}_p}} \\ & = \sup_{(\boldsymbol{\tau}_p,\mathbf{v}_p)\in\mathbb{X}_p\times\mathbf{V}_p} \frac{(A\partial_t(\overline{\boldsymbol{\sigma}}_p + \alpha_p\,\overline{p}_p\,\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\mu\,\mathbf{K}^{-1}\,\overline{\mathbf{u}}_p, \mathbf{v}_p)_{\Omega_p}}{\|(\boldsymbol{\tau}_p, \mathbf{v}_p)\|_{\mathbb{X}_p\times\mathbf{V}_p}} = 0. \end{aligned} \tag{4.2.67}$$

Therefore, $\overline{\mathbf{u}}_s(t) = \mathbf{0}$, $\overline{\boldsymbol{\gamma}}_p(t) = \mathbf{0}$, $\overline{p}_p(t) = 0$, $\overline{\lambda}(t) = 0$, $\overline{\boldsymbol{\theta}}(t) = \mathbf{0}$ for all $t \in (0,T]$, which implies $\overline{\boldsymbol{\sigma}}_p(t) = \mathbf{0}$ for all $t \in (0,T]$, so then we can conclude that (4.1.12) has a unique solution. □

Corollary 4.2.14. *Assuming* $\|\mathbf{u}_{f,0}\|_{\mathbf{H}^1(\Omega_f)} \leq r_0$, *the solution of* (4.1.12) *satisfies* $\mathbf{u}_p(0) = \mathbf{u}_{p,0}$, $\mathbf{T}_f(0) = \mathbf{T}_{f,0}$, $\mathbf{u}_f(0) = \mathbf{u}_{f,0}$, $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$ *and* $\lambda(0) = \lambda_0$.

Proof. We let $\overline{\mathbf{u}}_f := \mathbf{u}_f(0) - \mathbf{u}_{f,0}$, and use similar definitions and notations for the rest of the variables. Since Theorem 2.2.3 implies that $\mathcal{M}(u) \in \mathrm{L}^\infty(0,T; E_b')$, we can take $t \to 0^+$ in all equations without time derivatives in (4.1.12). Using that the initial data $(\mathbf{p}_0, \mathbf{r}_0)$ satisfies (4.2.17) at $t = 0$, and that $\overline{\boldsymbol{\sigma}}_p = \mathbf{0}$ and $\overline{p}_p = 0$, we obtain

$$\begin{aligned} & \frac{1}{2\mu}(\overline{\mathbf{T}}_f^{\mathrm{d}}, \mathbf{R}_f^{\mathrm{d}})_{\Omega_f} + \kappa_1\left(\rho_f\,q_f\,\overline{\mathbf{u}}_f + \mathbf{div}(\overline{\mathbf{T}}_f), \mathbf{div}(\mathbf{R}_f)\right)_{\Omega_f} + (\overline{\mathbf{u}}_f, \mathbf{div}(\mathbf{R}_f))_{\Omega_f} - \langle \mathbf{R}_f\mathbf{n}_f, \overline{\mathbf{u}}_f\rangle_{\Gamma_{fp}} \\ & \quad + (\boldsymbol{\gamma}_f(\overline{\mathbf{u}}_f), \mathbf{R}_f)_{\Omega_f} + \frac{\rho_f}{2\mu}((\mathbf{u}_f(0)\otimes\overline{\mathbf{u}}_f)^{\mathrm{d}}, \mathbf{R}_f)_{\Omega_f} + \frac{\rho_f}{2\mu}((\overline{\mathbf{u}}_f\otimes\mathbf{u}_{f,0})^{\mathrm{d}}, \mathbf{R}_f)_{\Omega_f} = 0, \end{aligned} \tag{4.2.68a}$$

$$\begin{aligned} & -\rho_f\,(q_f\,\overline{\mathbf{u}}_f, \mathbf{v}_f)_{\Omega_f} - (\mathbf{v}_f, \mathbf{div}(\overline{\mathbf{T}}_f))_{\Omega_f} + \kappa_2\left(\mathbf{e}(\overline{\mathbf{u}}_f) - \frac{1}{2\,\mu}\,\overline{\mathbf{T}}_f^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f)\right)_{\Omega_f} \\ & \quad - \kappa_2\left(\frac{\rho_f}{2\,\mu}\,(\mathbf{u}_f(0)\otimes\overline{\mathbf{u}}_f)^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f)\right)_{\Omega_f} - \kappa_2\left(\frac{\rho_f}{2\,\mu}\,(\overline{\mathbf{u}}_f\otimes\mathbf{u}_{f,0})^{\mathrm{d}}, \mathbf{e}(\mathbf{v}_f)\right)_{\Omega_f} = 0, \end{aligned} \tag{4.2.68b}$$

$$- (\overline{\mathbf{T}}_f, \boldsymbol{\gamma}_f(\mathbf{v}_f))_{\Omega_f} = 0, \tag{4.2.68c}$$

$$\mu(\mathbf{K}^{-1}\overline{\mathbf{u}}_p, \mathbf{v}_p)_{\Omega_p} + \langle \mathbf{v}_p \cdot \mathbf{n}_p, \overline{\lambda} \rangle_{\Gamma_{fp}} = 0, \tag{4.2.68d}$$

$$- \langle \overline{\mathbf{u}}_f \cdot \mathbf{n}_f + \overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p + \overline{\mathbf{u}}_p \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} = 0, \tag{4.2.68e}$$

$$\langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \overline{\lambda} \rangle_{\Gamma_{fp}} - \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} \left(\overline{\mathbf{u}}_f - \overline{\boldsymbol{\theta}} \right) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} = 0, \tag{4.2.68f}$$

$$\begin{aligned} &\left\langle \overline{\mathbf{T}}_f \mathbf{n}_f, \mathbf{v}_f \right\rangle_{\Gamma_{fp}} + \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} \left(\overline{\mathbf{u}}_f - \overline{\boldsymbol{\theta}} \right) \cdot \mathbf{t}_{f,j}, \mathbf{v}_f \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \\ &\quad + \rho_f \left\langle \mathbf{u}_f(0) \cdot \mathbf{n}_f, \overline{\mathbf{u}}_f \cdot \mathbf{v}_f \right\rangle_{\Gamma_{fp}} + \rho_f \left\langle \overline{\mathbf{u}}_f \cdot \mathbf{n}_f, \mathbf{u}_{f,0} \cdot \mathbf{v}_f \right\rangle_{\Gamma_{fp}} + \left\langle \mathbf{v}_f \cdot \mathbf{n}_f, \overline{\lambda} \right\rangle_{\Gamma_{fp}} = 0. \end{aligned} \tag{4.2.68g}$$

Taking $(\mathbf{v}_p, \mathbf{R}_f, \mathbf{v}_f, \boldsymbol{\phi}, \xi) = (\overline{\mathbf{u}}_p, \overline{\mathbf{T}}_f, \overline{\mathbf{u}}_f, \overline{\boldsymbol{\theta}}, \overline{\lambda})$ and combining the equations results in

$$\|\overline{\mathbf{u}}_p\|^2_{\mathrm{L}^2(\Omega_p)} + \big(\alpha_f - C_{\mathcal{K}}(\|(\mathbf{T}_f(0), \mathbf{u}_f(0))\|_{\mathbb{X}_f \times \mathbf{V}_f} + \|\mathbf{u}_{f,0}\|_{\mathbf{H}^1(\Omega_f)}) \big) \|(\overline{\mathbf{T}}_f, \overline{\mathbf{u}}_f)\|_{\mathbb{X}_f \times \mathbf{V}_f}$$
$$+ |\overline{\mathbf{u}}_f - \overline{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \leq 0,$$

implying $\overline{\mathbf{u}}_p = \mathbf{0}$, $\overline{\mathbf{T}}_f = \mathbf{0}$, $\overline{\mathbf{u}}_f = \mathbf{0}$ and $\overline{\boldsymbol{\theta}} \cdot \mathbf{t}_{f,j} = 0$ since $\|(\mathbf{T}_f(0), \mathbf{u}_f(0))\|_{\mathbb{X}_f \times \mathbf{V}_f}$ are bounded by data, cf. (4.2.41). Then (4.2.68e) implies that $\langle \overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} = 0$ for all $\xi \in \mathrm{H}^{1/2}(\Gamma_{fp})$. Combining with the fact that $\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, we get $\overline{\boldsymbol{\theta}} \cdot \mathbf{n}_p = 0$, thus $\overline{\boldsymbol{\theta}} = \mathbf{0}$. The inf-sup condition (2.2.7), together with (4.2.68d) imply that $\overline{\lambda} = 0$. □

Remark 4.2.2. *As we noted in Remark 4.1.1, the time differentiated equation* (4.1.8d) *can be used to recover the non-differentiated equation* (2.1.12)*. In particular, recalling the initial data construction* (4.2.56)*, let*

$$\forall\, t \in [0,T], \quad \boldsymbol{\eta}_p(t) = \boldsymbol{\eta}_{p,0} + \int_0^t \mathbf{u}_s(s)\, ds,\ \ \boldsymbol{\rho}_p(t) = \boldsymbol{\rho}_{p,0} + \int_0^t \boldsymbol{\gamma}_p(s)\, ds,\ \ \boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \int_0^t \boldsymbol{\theta}(s)\, ds.$$

Then (2.1.12) *follows from integrating* (4.1.8d) *from* 0 *to* $t \in (0,T]$ *and using the first equation in* (4.2.56)*.*

We end this section with a stability bound for the solution of (4.1.12).

Theorem 4.2.15. *For the solution of* (4.1.12)*, assuming sufficient regularity of the data, there exists a positive constant C such that*

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p)} + \sqrt{s_0}\|p_p\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_p\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} + \|\mathbf{u}_p\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} \\
&\quad + \|\mathbf{T}_f\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_f\|_{\mathrm{L}^2(0,T;\mathbf{V}_f)} + |\mathbf{u}_f - \boldsymbol{\theta}|_{\mathrm{L}^2(0,T;\mathtt{BJS})} + \|\boldsymbol{\theta}\|_{\mathrm{L}^2(0,T;\boldsymbol{\Lambda}_s)} + \|\lambda\|_{\mathrm{L}^2(0,T;\Lambda_p)} \\
&\quad + \|\mathbf{u}_s\|_{\mathrm{L}^2(0,T;\mathbf{V}_s)} + \|\boldsymbol{\gamma}_p\|_{\mathrm{L}^2(0,T;\mathbb{Q}_p)} \\
&\leq C\big(\|\mathbf{f}_f\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_f))} + \|q_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega_f))} + \|q_p\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega_p))} + \|\mathbf{f}_p\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} \\
&\quad + (1+\sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \int_0^T \|\partial_t\, q_p\|_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega_p))} dt + \|\mathrm{div}(\mathbf{K}\nabla p_{p,0})\|_{\mathrm{L}^2(\Omega_p)}\big). \qquad (4.2.69)
\end{aligned}$$

Proof. We begin by choosing $(\boldsymbol{\tau}_p, w_p, \mathbf{v}_p, \mathbf{R}_f, \mathbf{v}_f, \boldsymbol{\phi}, \xi, \mathbf{v}_s, \boldsymbol{\chi}_p) = (\boldsymbol{\sigma}_p, p_p, \mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}, \lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p)$ in (4.1.12) to get

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(s_0\|p_p\|^2_{\mathrm{W}_p} + \|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\big) + a_p(\mathbf{u}_p, \mathbf{u}_p) + a_f(\mathbf{T}_f, \mathbf{u}_f; \mathbf{T}_f, \mathbf{u}_f) \\
&\quad + \kappa_{\mathbf{u}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{T}_f, \mathbf{u}_f) + a_{\mathtt{BJS}}(\mathbf{u}_f, \boldsymbol{\theta}; \mathbf{u}_f, \boldsymbol{\theta}) \\
&\quad = -(\mathbf{f}_f, \kappa_1\,\mathbf{div}(\mathbf{T}_f) - \mathbf{u}_f)_{\Omega_f} - \frac{1}{n}\,(q_f\,\mathbf{I}, \mathbf{T}_f)_{\Omega_f} + \frac{\kappa_2}{n}\,(q_f, \mathrm{div}(\mathbf{u}_f))_{\Omega_f} + (q_p, p_p)_{\Omega_p} + (\mathbf{f}_p, \mathbf{u}_s)_{\Omega_p}. \qquad (4.2.70)
\end{aligned}$$

Next, we integrate (4.2.70) from 0 to $t \in (0,T]$, use coercivity bounds (4.2.9) in Lemma 4.2.2, in combination with $\mathbf{u}_f(t) : [0,T] \to \mathbf{W}_r$, cf. (4.2.38), the Cauchy-Schwarz and Young's inequalities, to get

$$\begin{aligned}
&s_0\|p_p(t)\|^2_{\mathrm{W}_p} + \|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \int_0^t \Big(\|\mathbf{u}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{T}_f\|^2_{\mathbb{X}_f} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + |\mathbf{u}_f - \boldsymbol{\theta}|^2_{\mathtt{BJS}}\Big) ds \\
&\leq C\bigg(\int_0^t \Big(\|\mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|q_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)}\Big) ds + s_0\|p_p(0)\|^2_{\mathrm{W}_p} \\
&\quad + \|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)}\bigg) + \delta\int_0^t \Big(\|\mathbf{T}_f\|^2_{\mathbb{X}_f} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s}\Big) ds. \qquad (4.2.71)
\end{aligned}$$

Applying inf-sup conditions (4.2.12)–(4.2.14) in Lemma 4.2.3, and using (4.1.8d) and (4.1.8g), we get

$$\|\mathbf{u}_s\|_{\mathbf{V}_s} + \|\boldsymbol{\gamma}_p\|_{\mathbb{Q}_p} \leq C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_p\in\widehat{\mathbb{X}}_p} \frac{b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p)}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}}$$
$$= C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_p\in\widehat{\mathbb{X}}_p} \frac{-a_e(\partial_t\,\boldsymbol{\sigma}_p, \partial_t\, p_p; \boldsymbol{\tau}_p, 0) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi})}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} \leq C\big(\|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}\big), \tag{4.2.72}$$

$$\|p_p\|_{\mathrm{W}_p} + \|\lambda\|_{\Lambda_p} \leq C \sup_{\mathbf{0}\neq\mathbf{v}_p\in\mathbf{V}_p} \frac{b_p(\mathbf{v}_p, p_p) + b_\Gamma(\mathbf{0}, \mathbf{v}_p, \mathbf{0}; \lambda)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}}$$
$$= C \sup_{\mathbf{0}\neq\mathbf{v}_p\in\mathbf{V}_p} \frac{-a_p(\mathbf{u}_p, \mathbf{v}_p)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}} \leq C\|\mathbf{u}_p\|_{\mathbf{L}^2(\Omega_p)}. \tag{4.2.73}$$

$$\|\boldsymbol{\theta}\|_{\boldsymbol{\Lambda}_s} \leq C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_p\in\widetilde{\mathbb{X}}_p} \frac{b_n^p(\boldsymbol{\tau}_p, \boldsymbol{\phi})}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}} = C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_p\in\widetilde{\mathbb{X}}_p} \frac{-a_e(\partial_t\,\boldsymbol{\sigma}_p, \partial_t\, p_p; \boldsymbol{\tau}_p, 0) - b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) - b_{\mathrm{sk}}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p)}{\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p}}$$
$$\leq C\big(\|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|\boldsymbol{\gamma}_p\|_{\mathbb{Q}_p}\big). \tag{4.2.74}$$

Combining (4.2.71) with (4.2.72)–(4.2.74), and choosing δ small enough lead to

$$s_0\|p_p(t)\|^2_{\mathrm{W}_p} + \|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \int_0^t \Big(\|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_p\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{T}_f\|^2_{\mathbb{X}_f} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f}$$
$$+\ |\mathbf{u}_f - \boldsymbol{\theta}|^2_{\mathtt{BJS}} + \|\boldsymbol{\theta}\|^2_{\boldsymbol{\Lambda}_s} + \|\lambda\|^2_{\Lambda_p} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s} + \|\boldsymbol{\gamma}_p\|^2_{\mathbb{Q}_p}\Big) ds$$
$$\leq C\bigg(\int_0^t \Big(\|\mathbf{f}_f\|^2_{\mathbf{L}^2(\Omega_f)} + \|q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|q_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega_p)}\Big) ds + \|p_p(0)\|^2_{\mathrm{W}_p}$$
$$+\ \|A^{1/2}\boldsymbol{\sigma}_p(0)\|^2_{\mathbb{L}^2(\Omega_p)} + \int_0^t \|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} ds\bigg). \tag{4.2.75}$$

Now, in order to bound the term $\int_0^t \|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} ds$ in (4.2.75), we refer to [74, Theorem IV.4.1(4.3)] applied to problem (4.2.21) with $\mathcal{M}(\boldsymbol{\sigma}_p, p_p) = \{(\widehat{\mathbf{f}}_p, \widehat{q}_p) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_p, p_p)\}$ (c.f. (4.2.20)) and $(h_{\boldsymbol{\sigma}_p}, h_{p_p}) = (\mathbf{0}, q_p)$ (c.f. Lemma 4.2.12), to obtain

$$\|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t p_p\|^2_{\mathrm{W}_p}$$

$$\leq \|\mathcal{M}(\boldsymbol{\sigma}_{p,0}, p_{p,0}) + h_{\boldsymbol{\sigma}_p}(0) + h_{p_p}(0)\|^2 + \left(\int_0^t (\|\partial_s h_{\boldsymbol{\sigma}_p}\| + \|\partial_s h_{p_p}\|) ds\right)^2.$$

Using Lemma 2.2.10, $\mathcal{M}(\boldsymbol{\sigma}_{p,0}, p_{p,0}) = \{(\widehat{\mathbf{f}}_{p,0}, \widehat{q}_{p,0}) - \widehat{\mathcal{E}}(\boldsymbol{\sigma}_{p,0}, p_{p,0})\}$, where $\widehat{\mathbf{f}}_{p,0}$ and $\widehat{q}_{p,0}$ are given in (4.2.59), we obtain

$$\begin{aligned}
&\|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\leq C\Big(\|\widehat{\mathbf{f}}_{p,0}\|^2_{\mathbb{L}^2(\Omega_p)} + \|\widehat{q}_{p,0}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\boldsymbol{\sigma}_{p,0}\|^2_{\mathbb{L}^2(\Omega_p)} + \|p_{p,0}\|^2_{\mathrm{W}_p} \\
&\quad + \|q_p(0)\|^2_{\mathrm{L}^2(\Omega_p)} + \big(\int_0^t \|\partial_s\, q_p\|_{\mathrm{L}^2(\Omega_p)} ds\big)^2\Big).
\end{aligned} \tag{4.2.76}$$

To bound the initial data terms showed up in (4.2.75) and (4.2.76), we recall that $(\boldsymbol{\sigma}_p(0), p_p(0), \mathbf{u}_p(0), \mathbf{T}_f(0), \mathbf{u}_f(0), \boldsymbol{\theta}(0), \lambda(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0}, \mathbf{u}_{p,0}, \mathbf{T}_{f,0}, \mathbf{u}_{f,0}, \boldsymbol{\theta}_0, \lambda_0)$ and the construction of the initial data (4.2.53)–(4.2.56). Combining the three systems and using the steady-state version of the arguments presented in (4.2.70)–(4.2.71), (2.2.7) and (4.2.61), we obtain

$$\begin{aligned}
&\|A^{1/2}\boldsymbol{\sigma}_p(0)\|_{\mathbb{L}^2(\Omega_p)} + \|p_p(0)\|_{\mathrm{W}_p} + \|\mathbf{u}_p(0)\|_{\mathbf{V}_p} + \|\mathbf{u}_f(0)\|_{\mathbf{V}_f} \\
&\leq C\big((1+\sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \|\mathrm{div}(\mathbf{K}\nabla p_{p,0})\|_{\mathrm{L}^2(\Omega_p)} + \|\mathbf{f}_f(0)\|_{\mathbf{L}^2(\Omega_f)} \\
&\quad + \|q_f(0)\|_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{f}_p(0)\|_{\mathbf{L}^2(\Omega_p)}\big)
\end{aligned} \tag{4.2.77}$$

Finally, we derive bounds for $\|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathbf{L}^2(\Omega_p)}$ and $\|\mathrm{div}(\mathbf{u}_p)\|_{\mathrm{L}^2(\Omega_p)}$. In order to do this, we choose $\mathbf{v}_s = \mathbf{div}(\boldsymbol{\sigma}_p)$ in (4.1.8e) and $w_p = \mathrm{div}(\mathbf{u}_p)$ in (4.1.8h) respectively, and apply Cauchy-Schwarz inequality, to get

$$\begin{aligned}
\|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathbb{L}^2(\Omega_p)} &\leq \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega_p)}, \\
\|\mathrm{div}(\mathbf{u}_p)\|_{\mathrm{L}^2(\Omega_p)} &\leq C(\|q_p\|_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}\partial_t(\boldsymbol{\sigma}_p + \alpha_p p_p\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t p_p\|_{\mathrm{W}_p}).
\end{aligned} \tag{4.2.78}$$

Then combining (4.2.75)–(4.2.78), we are able to conclude (4.2.69) and complete the proof. □

4.3 Semi-discrete formulation

In this section we introduce the semidiscrete continuous-in-time approximation of (4.1.13). We state the well-posedness and stability results which can be proved similarly as in Section 4, and we focus on derivation of error estimates with rates of convergence.

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular [39] and quasi-uniform affine finite element partitions of Ω_f and Ω_p, respectively, where h is the maximum element diameter. The two partitions may be non-matching along the interface Γ_{fp}. For the discretization, we consider the following conforming finite element spaces:

$$\mathbb{X}_{fh} \times \mathbf{V}_{fh} \subset \mathbb{X}_f \times \mathbf{V}_f, \quad \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \subset \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p, \quad \mathbf{V}_{ph} \times \mathrm{W}_{ph} \subset \mathbf{V}_p \times \mathrm{W}_p.$$

We choose $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph})$ to be any stable pair for mixed elasticity with weakly imposed stress symmetry, such as the Amara–Thomas [3], PEERS [12], Stenberg [77], Arnold–Falk–Winther [13,15], or Cockburn–Gopalakrishnan–Guzman [40] families of spaces. And we take $(\mathbf{V}_{ph}, \mathrm{W}_{ph})$ to be any stable mixed finite element Darcy spaces, such as the Raviart–Thomas (RT) or Brezzi-Douglas-Marini (BDM) spaces [23]. We note that these spaces satisfy

$$\mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}, \qquad \mathrm{div}(\mathbf{V}_{ph}) = \mathrm{W}_{ph}. \tag{4.3.1}$$

We also notice that we don't have further requirements for the pair $(\mathbb{X}_{fh}, \mathbf{V}_{fh})$. We could take Raviart-Thomas spaces or Brezzi-Douglas-Marini spaces as an example. For the Lagrange multipliers, we choose non-conforming approximations

$$\Lambda_{ph} := \mathbf{V}_{ph} \cdot \mathbf{n}_p|_{\Gamma_{fp}}, \quad \mathbf{\Lambda}_{sh} := \mathbb{X}_{ph}\mathbf{n}_p|_{\Gamma_{fp}},$$

which consist of discontinuous piecewise polynomials and are equipped with L^2-norms.

Remark 4.3.1. *We note that, since $\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, the continuous variational equations* (4.1.8i) *and* (4.1.8j) *hold for test functions in $\mathrm{L}^2(\Gamma_{fp})$, assuming that the solution is smooth enough. In particular, then hold for $\xi_h \in \Lambda_{ph}$ and $\boldsymbol{\phi}_h \in \mathbf{\Lambda}_{sh}$, respectively.*

Now, we group the spaces, unknowns and test functions similarly to the continuous case:

$$\mathbf{Q}_h := \mathbb{X}_{ph} \times \mathrm{W}_{ph} \times \mathbf{V}_{ph} \times \mathbb{X}_{fh} \times \mathbf{V}_{fh} \times \mathbf{\Lambda}_{sh}, \quad \mathbf{S}_h := \Lambda_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph},$$

$$\mathbf{p}_h := (\boldsymbol{\sigma}_{ph}, p_{ph}, \mathbf{u}_{ph}, \mathbf{T}_{fh}, \mathbf{u}_{fh}, \boldsymbol{\theta}_h) \in \mathbf{Q}_h, \quad \mathbf{r}_h := (\lambda_h, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph}) \in \mathbf{S}_h,$$

$$\mathbf{q}_h := (\boldsymbol{\tau}_{ph}, w_{ph}, \mathbf{v}_{ph}, \mathbf{R}_{fh}, \mathbf{v}_{fh}, \boldsymbol{\phi}_h) \in \mathbf{Q}_h, \quad \mathbf{s}_h := (\xi_h, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}) \in \mathbf{S}_h,$$

where the spaces $\mathbf{Q}$ and $\mathbf{S}$ are respectively endowed with the norms

$$\begin{aligned} \|\mathbf{q}_h\|_{\mathbf{Q}_h} &= \|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p} + \|w_{ph}\|_{\mathrm{W}_p} + \|\mathbf{v}_{ph}\|_{\mathbf{V}_p} + \|\mathbf{R}_{fh}\|_{\mathbb{X}_f} + \|\mathbf{v}_{fh}\|_{\mathbf{V}_f} + \|\boldsymbol{\phi}\|_{\mathbf{\Lambda}_{sh}}, \\ \|\mathbf{s}_h\|_{\mathbf{S}_h} &= \|\xi_h\|_{\Lambda_{ph}} + \|\mathbf{v}_{sh}\|_{\mathbf{V}_p} + \|\boldsymbol{\chi}_{ph}\|_{\mathbb{Q}_p}, \end{aligned}$$

with $\|\boldsymbol{\phi}\|_{\mathbf{\Lambda}_{sh}} = \|\boldsymbol{\phi}\|_{\mathbf{L}^2(\Gamma_{fp})}$ and $\|\xi_h\|_{\Lambda_{ph}} = \|\xi_h\|_{\mathrm{L}^2(\Gamma_{fp})}$. Hence, the semidiscrete continuous-in-time approximation to (4.1.13) is: find $(\mathbf{p}_h, \mathbf{r}_h) : [0,T] \to \mathbf{Q}_h \times \mathbf{S}_h$ such that $(\boldsymbol{\sigma}_{ph}(0), p_{ph}(0)) = (\boldsymbol{\sigma}_{ph,0}, p_{ph,0})$ and for a.e. $t \in (0,T)$,

$$\begin{aligned} \frac{\partial}{\partial t}\,\mathcal{E}\,\mathbf{p}_h(t) + (\mathcal{A} + \mathcal{K}_{\mathbf{u}_{fh}})\,\mathbf{p}_h(t) + \mathcal{B}'\,\mathbf{r}_h(t) &= \mathbf{F}(t) \quad \text{in} \quad \mathbf{Q}_h', \\ -\mathcal{B}\,\mathbf{p}_h(t) &= \mathbf{G}(t) \quad \text{in} \quad \mathbf{S}_h'. \end{aligned} \tag{4.3.2}$$

We next state the discrete inf-sup conditions that are satisfied by the finite element spaces. To do that, we first introduce the space

$$\begin{aligned} \widetilde{\mathbb{X}}_{ph} &:= \Big\{ \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph} : \quad \mathbf{div}(\boldsymbol{\tau}_{ph}) = \mathbf{0} \quad \text{in} \quad \Omega_p \Big\}, \\ \widehat{\mathbb{X}}_{ph} &:= \Big\{ \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph} : \quad \boldsymbol{\tau}_{ph}\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{fp} \Big\}. \end{aligned}$$

Lemma 4.3.1. *There exists constants $\beta_{h,1}$, $\beta_{h,2}$, $\beta_{h,3} > 0$ such that*

$$\beta_{h,1}(\|\mathbf{v}_{sh}\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_{ph}\|_{\mathbb{Q}_p}) \le \sup_{\mathbf{0} \neq \boldsymbol{\tau}_{ph} \in \widehat{\mathbb{X}}_{ph}} \frac{b_s(\boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, \boldsymbol{\chi}_{ph})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}}, \quad \forall \mathbf{v}_{sh} \in \mathbf{V}_{sh}, \boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}, \tag{4.3.3}$$

$$\beta_{h,2}(\|w_{ph}\|_{\mathrm{W}_p} + \|\xi_h\|_{\Lambda_{ph}}) \le \sup_{\mathbf{0} \neq \mathbf{v}_{ph} \in \mathbf{V}_{ph}} \frac{b_p(\mathbf{v}_{ph}, w_{ph}) + b_\Gamma(\mathbf{0}, \mathbf{v}_{ph}, \mathbf{0}; \xi_h)}{\|\mathbf{v}_{ph}\|_{\mathbf{V}_p}}, \quad \forall w_{ph} \in \mathrm{W}_{ph}, \xi_h \in \Lambda_{ph}, \tag{4.3.4}$$

$$\beta_{h,3}\|\boldsymbol{\phi}_h\|_{\mathbf{\Lambda}_{sh}} \le \sup_{\mathbf{0} \neq \boldsymbol{\tau}_{ph} \in \widetilde{\mathbb{X}}_{ph}} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\phi}_p)}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}}, \qquad \forall \phi_h \in \mathbf{\Lambda}_{sh}. \tag{4.3.5}$$

Proof. The first inequality can be shown using the argument in [6, Theorem 4.1]. The second one can be proved similarly as [47]. And the third one can be proved from a slight adaption on [50, Section 5.3]. □

We next discuss the construction of compatible discrete initial data $(\mathbf{p}_{h,0}, \mathbf{r}_{h,0})$.

Lemma 4.3.2. *Assume* $\mathbf{f}_f(0)$, $q_f(0)$ *and* $p_{p,0}$ *satisfy the small data condition* (4.2.52). *Then, there exist discrete initial data* $\mathbf{p}_{h,0} := (\boldsymbol{\sigma}_{ph,0}, p_{ph,0}, \mathbf{u}_{ph,0}, \mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0}) \in \mathbf{Q}_h$ *and* $\mathbf{r}_{h,0} := (\lambda_{h,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbf{S}_h$ *which are compatible in the sense of Lemma 2.2.10:*

$$
\begin{aligned}
& s_0\,(p_{ph,0}, w_{ph})_{\Omega_p} + a_e(\boldsymbol{\sigma}_{ph,0}, p_{ph,0}; \boldsymbol{\tau}_{ph}, w_{ph}) + a_p(\mathbf{u}_{ph,0}, \mathbf{v}_{ph}) + a_f(\mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) \\
& + \kappa_{\mathbf{u}_{fh,0}}(\mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) + a_{\mathtt{BJS}}(\mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0}; \mathbf{v}_{fh}, \boldsymbol{\phi}_h) + b_p(p_{ph,0}, \mathbf{v}_{ph}) - b_p(w_{ph}, \mathbf{u}_{ph,0}) \\
& + b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\phi}_h) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\theta}_{h,0}) + b_s(\mathbf{u}_{sh,0}, \boldsymbol{\tau}_{ph}) + b_{\mathrm{sk}}(\boldsymbol{\gamma}_{ph,0}, \boldsymbol{\tau}_{ph}) + b_\Gamma(\mathbf{v}_{ph}, \mathbf{v}_{fh}, \boldsymbol{\phi}_h; \lambda_{h,0}) \\
& = -(\mathbf{f}_f, \kappa_1\, \mathbf{div}(\mathbf{R}_{fh}) - \mathbf{v}_{fh})_{\Omega_f} - \frac{1}{n}(q_f \mathbf{I}, \mathbf{R}_{fh})_{\Omega_f} + \frac{\kappa_2}{n}(q_f, \mathrm{div}(\mathbf{v}_{fh}))_{\Omega_f} \\
& \quad + (\widehat{\mathbf{f}}_{ph,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + (\widehat{q}_{ph,0}, w_{ph})_{\Omega_p}, \\
& - b_s(\mathbf{v}_{sh}, \boldsymbol{\sigma}_{ph,0}) - b_{\mathrm{sk}}(\boldsymbol{\chi}_{ph}, \boldsymbol{\sigma}_{ph,0}) - b_\Gamma(\mathbf{u}_{ph,0}, \mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0}; \xi_h) \; = \; (\mathbf{f}_p, \mathbf{v}_{sh})_{\Omega_p},
\end{aligned}
\tag{4.3.6}
$$

Equivalently,

$$
\begin{aligned}
(\mathcal{E} + \mathcal{A} + \mathcal{K}_{\mathbf{u}_{fh,0}})\mathbf{p}_{h,0} + \mathcal{B}'\mathbf{r}_{h,0} &= \overline{\mathbf{F}}_0 \quad \text{in} \quad \mathbf{Q}_h', \\
-\mathcal{B}\mathbf{p}_{h,0} &= \mathbf{G}(0) \quad \text{in} \quad \mathbf{S}_h',
\end{aligned}
\tag{4.3.7}
$$

where $\overline{\mathbf{F}}_0$ *is the functional on the right hand side of* (4.3.6)*.*

Proof. The construction is based on a modification of the step-by-step procedure for the continuous initial data.

1. Define

$$
\boldsymbol{\theta}_{h,0} = P_h^{\boldsymbol{\Lambda}_s}\, \boldsymbol{\theta}_0, \tag{4.3.8}
$$

where $P_h^{\boldsymbol{\Lambda}_s} : \boldsymbol{\Lambda}_s \to \boldsymbol{\Lambda}_{sh}$ is the L^2-projection operator, satisfying, for all $\boldsymbol{\phi} \in \mathbf{L}^2(\Gamma_{fp})$,

$$
\langle \boldsymbol{\phi} - P_h^{\boldsymbol{\Lambda}_s} \boldsymbol{\phi}, \boldsymbol{\phi}_h \rangle_{\Gamma_{fp}} = 0 \qquad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}. \tag{4.3.9}
$$

2. Define $(\mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}) \in \mathbb{X}_{fh} \times \mathbf{V}_{fh}$ and $(\mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{ph} \times \mathrm{W}_{ph} \times \Lambda_{ph}$ by solving a coupled Navier Stokes-Darcy problem: for all $\mathbf{R}_{fh} \in \mathbb{X}_{fh}, \mathbf{v}_{fh} \in \mathbf{V}_{fh}$, $\mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $w_p \in \mathrm{W}_{ph}$, $\xi_h \in \Lambda_{ph}$,

$$
\begin{aligned}
& a_f((\mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}), (\mathbf{R}_{fh}, \mathbf{v}_{fh})) + \kappa_{\mathbf{u}_{fh,0}}((\mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}), (\mathbf{R}_{fh}, \mathbf{v}_{fh})) \\
& \quad + \mu \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \langle \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_{fh,0} - \boldsymbol{\theta}_{h,0}) \cdot \mathbf{t}_{f,j}, \mathbf{v}_{fh} \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \mathbf{v}_{fh} \cdot \mathbf{n}_f, \lambda_{h,0} \rangle_{\Gamma_{fp}} \\
& = a_f((\mathbf{T}_{f,0}, \mathbf{u}_{f,0}), (\mathbf{R}_{fh}, \mathbf{v}_{fh})) + \kappa_{\mathbf{u}_{f,0}}((\mathbf{T}_{f,0}, \mathbf{u}_{f,0}), (\mathbf{R}_{fh}, \mathbf{v}_{fh})) \\
& \quad + \mu \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \langle \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_{f,0} - \boldsymbol{\theta}_{0}) \cdot \mathbf{t}_{f,j}, \mathbf{v}_{fh} \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \mathbf{v}_{fh} \cdot \mathbf{n}_f, \lambda_{0} \rangle_{\Gamma_{fp}} \\
& = -(\mathbf{f}_f(0), \kappa_1 \mathbf{div}(\mathbf{R}_{fh}) - \mathbf{v}_{fh})_{\Omega_f} - \frac{1}{n}(q_f(0)\,\mathbf{I}, \mathbf{R}_{fh})_{\Omega_f} + \frac{\kappa_2}{n}(q_f(0), \mathrm{div}(\mathbf{v}_{fh}))_{\Omega_f}, \\
& a_p(\mathbf{u}_{ph,0}, \mathbf{v}_{ph}) + b_p(p_{ph,0}, \mathbf{v}_{ph}) + \langle \mathbf{v}_{ph} \cdot \mathbf{n}_p, \lambda_{h,0} \rangle_{\Gamma_{fp}} \\
& = a_p(\mathbf{u}_{p,0}, \mathbf{v}_{ph}) + b_p(p_{p,0}, \mathbf{v}_{ph}) + \langle \mathbf{v}_{ph} \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0, \\
& - b_p(w_{ph}, \mathbf{u}_{ph,0}) = -b_p(w_{ph}, \mathbf{u}_{p,0}) = -\frac{1}{\mu}(\mathrm{div}(\mathbf{K}\nabla p_{p,0}), w_{ph})_{\Omega_p}, \\
& - \langle \mathbf{u}_{fh,0} \cdot \mathbf{n}_f + (\boldsymbol{\theta}_{h,0} + \mathbf{u}_{ph,0}) \cdot \mathbf{n}_p, \xi_h \rangle_{\Gamma_{fp}} = -\langle \mathbf{u}_{f,0} \cdot \mathbf{n}_f + (\boldsymbol{\theta}_0 + \mathbf{u}_{p,0}) \cdot \mathbf{n}_p, \xi_h \rangle_{\Gamma_{fp}} = 0.
\end{aligned}
\tag{4.3.10}
$$

This is a well-posed problem using fixed point theorem for augmented Navier–Stokes/Darcy coupled problem with small data condition (4.2.52), see [34].

3. Define $(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}, \boldsymbol{\omega}_{h,0}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \times \boldsymbol{\Lambda}_{sh}$ such that, for all $\boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $\mathbf{v}_{sh} \in \mathbf{V}_{sh}$, $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$, $\boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}$,

$$
\begin{aligned}
& (A\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\eta}_{ph,0}, \boldsymbol{\tau}_{ph}) + b_{\mathrm{sk}}(\boldsymbol{\rho}_{ph,0}, \boldsymbol{\tau}_{ph}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\omega}_{h,0}) + (A(\alpha_p\, p_{ph,0}\,\mathbf{I}), \boldsymbol{\tau}_{ph})_{\Omega_p} \\
& = (A\boldsymbol{\sigma}_{p,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\eta}_{p,0}, \boldsymbol{\tau}_{ph}) + b_{\mathrm{sk}}(\boldsymbol{\rho}_{p,0}, \boldsymbol{\tau}_{ph}) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\omega}_{0}) + (A(\alpha_p\, p_{p,0}\,\mathbf{I}), \boldsymbol{\tau}_{ph})_{\Omega_p} = 0, \\
& - b_s(\mathbf{v}_{sh}, \boldsymbol{\sigma}_{ph,0}) = -b_s(\mathbf{v}_{sh}, \boldsymbol{\sigma}_{p,0}) = (\mathbf{f}_p(0), \mathbf{v}_{sh})_{\Omega_p}, \\
& - b_{\mathrm{sk}}(\boldsymbol{\chi}_{ph}, \boldsymbol{\sigma}_{ph,0}) = -b_{\mathrm{sk}}(\boldsymbol{\chi}_{ph}, \boldsymbol{\sigma}_{p,0}) = 0, \\
& b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\phi}_h) - \mu \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \langle \sqrt{\mathbf{K}_j^{-1}} (\mathbf{u}_{fh,0} - \boldsymbol{\theta}_{h,0}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}_h \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \boldsymbol{\phi}_h \cdot \mathbf{n}_p, \lambda_{h,0} \rangle_{\Gamma_{fp}}
\end{aligned}
$$

$$= b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}_h) - \mu\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \langle \sqrt{\mathbf{K}_j^{-1}}(\mathbf{u}_{f,0} - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}_h \cdot \mathbf{t}_{f,j} \rangle_{\Gamma_{fp}} + \langle \boldsymbol{\phi}_h \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0. \tag{4.3.11}$$

It can be shown that the above problem is well-posed using the finite element theory for elasticity with weak stress symmetry [11,13] and the inf-sup condition (4.3.5) for the Lagrange multiplier $\boldsymbol{\psi}_{h,0}$.

4. Define $(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$ such that, for all $\boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $\mathbf{v}_{sh} \in \mathbf{V}_{sh}$, $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$,

$$\begin{aligned}
&(A\widehat{\boldsymbol{\sigma}}_{ph,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\tau}_{ph}, \mathbf{u}_{sh,0}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, \boldsymbol{\gamma}_{ph,0}) = b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\theta}_{h,0}), \\
&- b_s(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{v}_{sh}) = 0, \\
&- b_{\mathrm{sk}}(\widehat{\boldsymbol{\sigma}}_{ph,0}, \boldsymbol{\chi}_{ph}) = 0.
\end{aligned} \tag{4.3.12}$$

This is a well posed discrete mixed elasticity problem [11, 13].

We then define $\mathbf{p}_{h,0} = (\boldsymbol{\sigma}_{ph,0}, p_{ph,0}, \mathbf{u}_{ph,0}, \mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0})$ and $\mathbf{r}_{h,0} = (\lambda_{h,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0})$. According to (4.3.10)–(2.3.13), $\mathbf{p}_{h,0}$ and $\mathbf{r}_{h,0}$ satisfy (4.3.6) with $\widehat{\mathbf{f}}_{ph,0} \in \mathbb{X}'_{p,2}$ and $\widehat{q}_{ph,0} \in \mathrm{W}'_{p,2}$ such that

$$\begin{aligned}
(\widehat{\mathbf{f}}_{ph,0}, \boldsymbol{\tau}_{ph})_{\Omega_p} &= a_e(\boldsymbol{\sigma}_{ph,0}, p_{ph,0}; \boldsymbol{\tau}_{ph}, 0) - (A(\widehat{\boldsymbol{\sigma}}_{ph,0}), \boldsymbol{\tau}_{ph})_{\Omega_p}, \\
(\widehat{q}_{ph,0}, w_{ph})_{\Omega_p} &= (s_0\, p_{ph,0}, w_{ph})_{\Omega_p} + a_e(\boldsymbol{\sigma}_{ph,0}, p_{ph,0}; \mathbf{0}, w_{ph}) - b_p(\mathbf{u}_{ph,0}, w_{ph}),
\end{aligned} \tag{4.3.13}$$

Furthermore, the construction provides compatible initial data for the non-differentiated elasticity variables $(\boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}, \boldsymbol{\psi}_{h,0})$ in the sense of the first equation in (4.2.56). □

4.3.1 Existence and uniqueness of a solution

The well-posedness of problem (4.3.2) follows from similar arguments as in the continuous case.

Theorem 4.3.3. *For each compatible initial data* $(\mathbf{p}_h(0), \mathbf{r}_h(0))$ *satisfying* (4.3.7) *and*

$$\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}_f'), \quad q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p')$$

satisfying (4.2.40)*, there exists a unique solution of* (4.3.2)*,* $(\mathbf{p}_h, \mathbf{r}_h) : [0,T] \to \mathbf{Q}_h \times \mathbf{S}_h$ *with* $\mathbf{u}_{fh}(t) : [0,T] \to \mathbf{W}_r$*,* $(\boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_{ph})$ *and* $(\boldsymbol{\sigma}_{ph}(0), p_{ph}(0), \mathbf{u}_{ph}(0), \mathbf{T}_{fh}(0), \mathbf{u}_{fh}(0), \boldsymbol{\theta}_h(0), \lambda_h(0)) = (\boldsymbol{\sigma}_{ph,0}, p_{ph,0}, \mathbf{u}_{ph,0}, \mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0}, \lambda_{h,0})$*. Moreover, assuming sufficient regularity of the data, there exists a positive constant* C *such that*

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p)} \\
&\quad + \|A^{1/2}\partial_t(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p)} \\
&\quad + \sqrt{s_0}\|p_{ph}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} \\
&\quad + \|\mathbf{T}_{fh}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_{fh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_f)} + |\mathbf{u}_{fh} - \boldsymbol{\theta}_h|_{\mathrm{L}^2(0,T;\mathrm{BJS})} + \|\boldsymbol{\theta}_h\|_{\mathrm{L}^2(0,T;\boldsymbol{\Lambda}_{sh})} \\
&\quad + \|\lambda_h\|_{\mathrm{L}^2(0,T;\Lambda_{ph})} + \|\mathbf{u}_{sh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_s)} + \|\boldsymbol{\gamma}_{ph}\|_{\mathrm{L}^2(0,T;\mathbb{Q}_p)} \\
&\leq C\big(\|\mathbf{f}_f\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_f))} + \|q_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega_f))} + \|q_p\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega_p))} + \|\mathbf{f}_p\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} \\
&\quad + (1+\sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \int_0^T \|\partial_t\, q_p\|_{\mathrm{L}^1(0,t;\mathrm{L}^2(\Omega_p))} dt + \|\mathrm{div}(\mathbf{K}\nabla p_{p,0})\|_{\mathrm{L}^2(\Omega_p)}\big). \qquad (4.3.14)
\end{aligned}$$

Proof. With the discrete inf-sup conditions (4.3.3)–(4.3.5) and the discrete initial data construction described in (4.3.9)–(4.3.11), the proof is similar to the proofs of Theorem 4.2.13, Corollary 4.2.14 and Theorem 4.2.15, with two differences due to non-conforming choices of the Lagrange multiplier spaces equipped with L^2-norms. The first is in the continuity of the bilinear forms $b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\phi}_h)$ and $b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\phi}_h; \xi_h)$, cf. (4.2.5). In particular, using the discrete trace-inverse inequality for piecewise polynomial functions, $\|\varphi\|_{L^2(\Gamma_{fp})} \leq Ch^{-1/2}\|\varphi\|_{L^2(\Omega_p)}$, we have

$$b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\phi}_h) \leq Ch^{-1/2}\|\boldsymbol{\tau}_{ph}\|_{\mathbb{L}^2(\Omega_p)}\|\boldsymbol{\phi}_h\|_{\mathbf{L}^2(\Gamma_{fp})}$$

and

$$b_\Gamma(\mathbf{v}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\phi}_h; \xi_h) \leq C(\|\mathbf{v}_{fh}\|_{\mathbf{H}^1(\Omega_f)} + h^{-1/2}\|\mathbf{v}_{ph}\|_{\mathbf{L}^2(\Omega_p)} + \|\boldsymbol{\phi}_h\|_{\mathbf{L}^2(\Gamma_{fp})})\|\xi_h\|_{\mathrm{L}^2(\Gamma_{fp})}.$$

Therefore these bilinear forms are continuous for any given mesh. Second, the operators L_λ and $R_{\boldsymbol{\theta}}$ from Lemma 4.2.5 are now defined as $L_\lambda : \Lambda_{ph} \to \Lambda'_{ph},\ (L_\lambda\, \lambda_h, \xi_h) := \langle \lambda_h, \xi_h \rangle_{\Gamma_{fp}}$ and $R_{\boldsymbol{\theta}} : \boldsymbol{\Lambda}_{sh} \to \boldsymbol{\Lambda}'_{sh},\ (R_{\boldsymbol{\theta}}\, \boldsymbol{\theta}_h, \boldsymbol{\phi}_h) := \langle \boldsymbol{\theta}_h, \boldsymbol{\phi}_h \rangle_{\Gamma_{fp}}$. The fact that L_λ and $R_{\boldsymbol{\theta}}$ are continuous and coercive follows immediately from their definitions, since $(L_\lambda\, \xi_h, \xi_h) = \|\xi\|^2_{\Lambda_{ph}}$ and $(R_{\boldsymbol{\theta}}\, \boldsymbol{\phi}_h, \boldsymbol{\phi}_h) = \|\boldsymbol{\phi}_h\|^2_{\boldsymbol{\Lambda}_{sh}}$. We note that the proof of Corollary 4.2.14 works in the discrete case due to the choice of the discrete initial data as the elliptic projection of the continuous initial data, cf. (4.3.10) and (4.3.11). □

Remark 4.3.2. *As in the continuous case, we can recover the non-differentiated elasticity variables with*

$$\forall\, t \in [0,T], \quad \boldsymbol{\eta}_{ph}(t) = \boldsymbol{\eta}_{ph,0} + \int_0^t \mathbf{u}_{sh}(s)\, ds,$$

$$\boldsymbol{\rho}_{ph}(t) = \boldsymbol{\rho}_{ph,0} + \int_0^t \boldsymbol{\gamma}_{ph}(s)\, ds, \quad \boldsymbol{\omega}_h(t) = \boldsymbol{\omega}_{h,0} + \int_0^t \boldsymbol{\theta}_h(s)\, ds.$$

Then (2.1.12) *holds discretely, which follows from integrating the equation associated to* $\boldsymbol{\tau}_{ph}$ *in* (4.3.2) *from* 0 *to* $t \in (0,T]$ *and using the discrete version of the first equation in* (4.3.11).

4.3.2 Error analysis

4.3.2.1 Preliminaries

We proceed with establishing rates of convergence. To that end, let us set $\mathrm{V} \in \{\mathrm{W}_p, \mathbf{V}_s, \mathbb{Q}_p\}$, $\Lambda \in \{\boldsymbol{\Lambda}_s, \Lambda_p\}$ and let V_h, Λ_h be the discrete counterparts. Let $P_h^{\mathrm{V}} : \mathrm{V} \to \mathrm{V}_h$ and $P_h^{\Lambda} : \Lambda \to \Lambda_h$ be the L^2-projection operators, satisfying

$$(u - P_h^{\mathrm{V}} u, v_h)_{\Omega_p} \;=\; 0 \quad \forall\, v_h \in \mathrm{V}_h, \qquad \langle \theta - P_h^{\Lambda}\theta, \phi_h \rangle_{\Gamma_{fp}} \;=\; 0 \quad \forall\, \phi_h \in \Lambda_h, \tag{4.3.15}$$

where $u \in \{p_p, \mathbf{u}_s, \boldsymbol{\gamma}_p\}$, $\theta \in \{\boldsymbol{\theta}, \lambda\}$, and v_h, ϕ_h are the corresponding discrete test functions. We have the approximation properties [39]:

$$\|u - P_h^{\mathrm{V}} u\|_{\mathrm{L}^2(\Omega_p)} \;\leq\; C h^{s_u+1}\, \|u\|_{\mathrm{H}^{s_u+1}(\Omega_p)}, \qquad \|\theta - P_h^{\Lambda}\theta\|_{\Lambda_h} \;\leq\; C h^{s_\theta+1}\, \|\theta\|_{\mathrm{H}^{s_\theta+1}(\Gamma_{fp})}, \tag{4.3.16}$$

where $s_u \in \{s_{p_p}, s_{\mathbf{u}_s}, s_{\boldsymbol{\gamma}_p}\}$ and $s_\theta \in \{s_{\boldsymbol{\theta}}, s_\lambda\}$ are the degrees of polynomials in the spaces V_h and Λ_h, respectively.

Since the discrete Lagrange multiplier spaces are chosen as $\boldsymbol{\Lambda}_{sh} = \mathbb{X}_{ph}\mathbf{n}_p|_{\Gamma_{fp}}$ and $\Lambda_{ph} = \mathbf{V}_{ph} \cdot \mathbf{n}_p|_{\Gamma_{fp}}$, respectively, we have

$$\langle \boldsymbol{\theta} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\theta}, \boldsymbol{\tau}_{ph}\mathbf{n}_p\rangle_{\Gamma_{fp}} \,=\, 0 \quad \forall\, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}, \qquad \langle \lambda - P_h^{\Lambda_p}\lambda, \mathbf{v}_{ph} \cdot \mathbf{n}_p\rangle_{\Gamma_{fp}} \,=\, 0 \quad \forall\, \mathbf{v}_{ph} \in \mathbf{V}_{ph}. \tag{4.3.17}$$

Next, denote $\mathrm{X} \in \{\mathbb{X}_f, \mathbb{X}_p, \mathbf{V}_p\}$, $\sigma \in \{\mathbf{T}_f, \boldsymbol{\sigma}_p, \mathbf{u}_p\} \in \mathrm{X}$ and let X_h, τ_h be their discrete counterparts. Let $I_h^{\mathrm{X}} : \mathrm{X} \cap \mathrm{H}^1(\Omega_\star) \to \mathrm{X}_h$ be the mixed finite element projection operator [23] satisfying $\forall\, \tau_h \in \mathrm{X}_h$,

$$(\mathrm{div}(I_h^{\mathrm{X}}\sigma), w_h) = (\mathrm{div}(\sigma), w_h) \quad \forall\, w_h \in \mathrm{W}_h, \qquad \left\langle I_h^{\mathrm{X}}(\sigma)\mathbf{n}_\star, \tau_h\mathbf{n}_\star \right\rangle_{\Gamma_{fp}} = \langle \sigma\mathbf{n}_\star, \tau_h\mathbf{n}_\star\rangle_{\Gamma_{fp}}\,, \tag{4.3.18}$$

and

$$\begin{aligned} \|\sigma - I_h^{\mathrm{X}}(\sigma)\|_{\mathrm{L}^2(\Omega_\star)} \,&\le\, C\, h^{s_\sigma+1}\|\sigma\|_{\mathrm{H}^{s_\sigma+1}(\Omega_\star)}, \\ \|\mathrm{div}(\sigma - I_h^{\mathrm{X}}(\sigma))\|_{\mathrm{L}^2(\Omega_\star)} \,&\le\, C\, h^{s_\sigma+1}\|\mathrm{div}(\sigma)\|_{\mathrm{H}^{s_\sigma+1}(\Omega_\star)}, \end{aligned} \tag{4.3.19}$$

where $\star \in \{f, p\}$, $w_h \in \left\{\mathbf{v}_{fh}, \mathbf{v}_{sh}, w_{ph}\right\}$, $\mathrm{W}_h \in \left\{\mathbf{V}_f, \mathbf{V}_s, \mathrm{W}_p\right\}$, and $s_\sigma \in \left\{s_{\mathbf{T}_f}, s_{\mathbf{u}_p}, s_{\boldsymbol{\sigma}_p}\right\}$ – the degrees of polynomials in the spaces X_h.

Finally, let $S_h^{\mathbf{V}_f}$ be the Scott-Zhang interpolation operators onto $\mathbf{V}_{fh}$, satisfying [73]

$$\|\mathbf{v}_f - S_h^{\mathbf{V}_f}(\mathbf{v}_f)\|_{\mathbf{H}^1(\Omega_f)} \,\le\, C\, h^{s_{\mathbf{v}_f}}\|\mathbf{v}_f\|_{\mathrm{H}^{s_{\mathbf{v}_f}+1}(\Omega_f)}, \tag{4.3.20}$$

where $s_{\mathbf{v}_f}$ is the degree of polynomials in the space $\mathbf{V}_f$.

Now, let $(\boldsymbol{\sigma}_p, p_p, \mathbf{u}_p, \mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\theta}, \lambda, \mathbf{u}_s, \boldsymbol{\gamma}_p)$ and $(\boldsymbol{\sigma}_{ph}, p_{ph}, \mathbf{u}_{ph}, \mathbf{T}_{fh}, \mathbf{u}_{fh}, \boldsymbol{\theta}_h, \lambda_h, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{ph})$ be solutions of (4.1.13) and (4.3.2), respectively. We introduce the error terms as the difference of these two solutions and decompose them into approximation and discretization errors using the interpolation operators:

$$
\begin{aligned}
e_{\boldsymbol{\sigma}_p} &:= \boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph} = (\boldsymbol{\sigma}_p - I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p) + (I_h^{\mathbb{X}_p}\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph}) := e_{\boldsymbol{\sigma}_p}^I + e_{\boldsymbol{\sigma}_p}^h, \\
e_{p_p} &:= p_p - p_{ph} = (p_p - P_h^{\mathrm{W}_p} p_p) + (P_h^{\mathrm{W}_p} p_p - p_{ph}) := e_{p_p}^I + e_{p_p}^h, \\
e_{\mathbf{u}_p} &:= \mathbf{u}_p - \mathbf{u}_{ph} = (\mathbf{u}_p - I_h^{\mathbf{V}_p}\mathbf{u}_p) + (I_h^{\mathbf{V}_p}\mathbf{u}_p - \mathbf{u}_{ph}) := e_{\mathbf{u}_p}^I + e_{\mathbf{u}_p}^h, \\
e_{\mathbf{T}_f} &:= \mathbf{T}_f - \mathbf{T}_{fh} = (\mathbf{T}_f - I_h^{\mathbb{X}_f}\mathbf{T}_f) + (I_h^{\mathbb{X}_f}\mathbf{T}_f - \mathbf{T}_{fh}) := e_{\mathbf{T}_f}^I + e_{\mathbf{T}_f}^h, \\
e_{\mathbf{u}_f} &:= \mathbf{u}_f - \mathbf{u}_{fh} = (\mathbf{u}_f - S_h^{\mathbf{V}_f}\mathbf{u}_f) + (S_h^{\mathbf{V}_f}\mathbf{u}_f - \mathbf{u}_{fh}) := e_{\mathbf{u}_f}^I + e_{\mathbf{u}_f}^h, \\
e_{\boldsymbol{\theta}} &:= \boldsymbol{\theta} - \boldsymbol{\theta}_h = (\boldsymbol{\theta} - P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\theta}) + (P_h^{\boldsymbol{\Lambda}_s}\boldsymbol{\theta} - \boldsymbol{\theta}_h) := e_{\boldsymbol{\theta}}^I + e_{\boldsymbol{\theta}}^h, \\
e_{\lambda} &:= \lambda - \lambda_h = (\lambda - P_h^{\boldsymbol{\Lambda}_p}\lambda) + (P_h^{\boldsymbol{\Lambda}_p}\lambda - \lambda_h) := e_{\lambda}^I + e_{\lambda}^h, \\
e_{\mathbf{u}_s} &:= \mathbf{u}_s - \mathbf{u}_{sh} = (\mathbf{u}_s - P_h^{\mathbf{V}_s}\mathbf{u}_s) + (P_h^{\mathbf{V}_s}\mathbf{u}_s - \mathbf{u}_{sh}) := e_{\mathbf{u}_s}^I + e_{\mathbf{u}_s}^h, \\
e_{\boldsymbol{\gamma}_p} &:= \boldsymbol{\gamma}_p - \boldsymbol{\gamma}_{ph} = (\boldsymbol{\gamma}_p - P_h^{\mathbb{Q}_p}\boldsymbol{\gamma}_p) + (P_h^{\mathbb{Q}_p}\boldsymbol{\gamma}_p - \boldsymbol{\gamma}_{ph}) := e_{\boldsymbol{\gamma}_p}^I + e_{\boldsymbol{\gamma}_p}^h.
\end{aligned} \tag{4.3.21}
$$

Then, we set the global errors endowed with above decomposition,

$$
e_{\mathbf{p}} := (e_{\boldsymbol{\sigma}_p}, e_{p_p}, e_{\mathbf{u}_p}, e_{\mathbf{T}_f}, e_{\mathbf{u}_f}, e_{\boldsymbol{\theta}}), \;\; e_{\mathbf{r}} := (e_{\lambda}, e_{\mathbf{u}_s}, e_{\boldsymbol{\gamma}_p}).
$$

We form the error equation by subtracting the discrete equations (4.3.2) from the continuous one (4.1.13):

$$
\begin{aligned}
\partial_t \mathcal{E}\,(e_{\mathbf{p}})(\mathbf{q}_h) + (\mathcal{A} + \mathcal{K}_{\mathbf{u}_{fh}})\,(e_{\mathbf{p}})(\mathbf{q}_h) + \mathcal{B}'\,(e_{\mathbf{r}})(\mathbf{q}_h) &= -\mathcal{K}_{\mathbf{u}_f - \mathbf{u}_{fh}}(\mathbf{p})(\mathbf{q}_h) \quad \forall\, \mathbf{q}_h \in \mathbf{Q}_h, \\
-\mathcal{B}\,(e_{\mathbf{p}})(\mathbf{s}_h) &= \mathbf{0} \quad \forall\, \mathbf{s}_h \in \mathbf{S}_h.
\end{aligned} \tag{4.3.22}
$$

4.3.2.2 A parabolic problem

Before we continue the analysis based on the error equation (4.3.22), we would like to introduce a parabolic problem equivalent to the error equation, which is necessary for the upcoming analysis. The parabolic problem is:

$$\frac{d}{dt}\,\widehat{\mathcal{E}}\begin{pmatrix} e^h_{\boldsymbol{\sigma}_p}(t) \\ e^h_{p_p}(t) \end{pmatrix} + \mathcal{M}_e\begin{pmatrix} e^h_{\boldsymbol{\sigma}_p}(t) \\ e^h_{p_p}(t) \end{pmatrix} \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a.e.\ t\in(0,T]. \tag{4.3.23}$$

where the domain $\mathcal{D}_e$ is defined as
$\mathcal{D}_e := \Big\{(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) \in \mathbb{X}_{ph}\times \mathrm{W}_{ph}$:

there exists $((e^h_{\mathbf{u}_p}, e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}), e^h_{\mathbf{s}}) \in (\mathbf{V}_{ph}\times\mathbb{X}_{fh}\times\mathbf{V}_{fh}\times\boldsymbol{\Lambda}_{sh})\times\mathbf{S}_h$ such that $\forall\,(\mathbf{q}_h,\mathbf{s}_h)\in \mathbf{Q}_h\times\mathbf{S}_h$:

$$\begin{aligned}
& s_0\,(e^h_{p_p}, w_{ph})_{\Omega_p} + a_e(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}; \boldsymbol{\tau}_{ph}, w_{ph}) + a_p(e^h_{\mathbf{u}_p}, \mathbf{v}_{ph}) + a_f(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) \\
&\quad +\ \kappa_{e^h_{\mathbf{u}_f}}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_{fh}, \mathbf{v}_{fh}) + \kappa_{\mathbf{u}_f}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) - \kappa_{e^I_{\mathbf{u}_f}}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) \\
&\quad -\ \kappa_{e^h_{\mathbf{u}_f}}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) - \kappa_{e^h_{\mathbf{u}_f}}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) + a_{\mathtt{BJS}}(e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; \mathbf{v}_{fh}, \boldsymbol{\phi}_h) \\
&\quad +\ b_p(e^h_{p_p}, \mathbf{v}_{ph}) - b_p(w_{ph}, e^h_{\mathbf{u}_p}) + b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, \boldsymbol{\phi}_h) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\theta}}) + b_s(e^h_{\mathbf{u}_s}, \boldsymbol{\tau}_{ph}) \\
&\quad +\ b_{\mathrm{sk}}(e^h_{\boldsymbol{\gamma}_p}, \boldsymbol{\tau}_{ph}) + b_\Gamma(\mathbf{v}_{ph}, \mathbf{v}_{fh}, \boldsymbol{\phi}_h; e^h_\lambda) \\
&\quad =\ -\Big\{ s_0\,(e^I_{p_p}, w_{ph})_{\Omega_p} + a_e(e^I_{\boldsymbol{\sigma}_p}, e^I_{p_p}; \boldsymbol{\tau}_{ph}, w_{ph}) + a_p(e^I_{\mathbf{u}_p}, \mathbf{v}_{ph}) + a_f(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) \\
&\qquad +\ \kappa_{e^I_{\mathbf{u}_f}}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_{fh}, \mathbf{v}_{fh}) + \kappa_{\mathbf{u}_f}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) - \kappa_{e^I_{\mathbf{u}_f}}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; \mathbf{R}_{fh}, \mathbf{v}_{fh}) \\
&\qquad +\ a_{\mathtt{BJS}}(e^I_{\mathbf{u}_f}, e^I_{\boldsymbol{\theta}}; \mathbf{v}_{fh}, \boldsymbol{\phi}_h) + b_p(e^I_{p_p}, \mathbf{v}_{ph}) - b_p(w_{ph}, e^I_{\mathbf{u}_p}) + b_{\mathbf{n}_p}(e^I_{\boldsymbol{\sigma}_p}, \boldsymbol{\phi}_h) - b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^I_{\boldsymbol{\theta}}) \\
&\qquad +\ b_s(e^I_{\mathbf{u}_s}, \boldsymbol{\tau}_{ph}) + b_{\mathrm{sk}}(e^I_{\boldsymbol{\gamma}_p}, \boldsymbol{\tau}_{ph}) + b_\Gamma(\mathbf{v}_{ph}, \mathbf{v}_{fh}, \boldsymbol{\phi}_h; e^I_\lambda)\Big\} + (\widehat{\mathbf{f}}_{p,e}, \boldsymbol{\tau}_{ph})_{\Omega_p} + (\widehat{q}_{p,e}, w_{ph})_{\Omega_p}, \\
& -\ b_s(\mathbf{v}_{sh}, e^h_{\boldsymbol{\sigma}_p}) - b_{\mathrm{sk}}(\boldsymbol{\chi}_{ph}, e^h_{\boldsymbol{\sigma}_p}) - b_\Gamma(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; \xi_h) \\
&\quad =\ b_s(\mathbf{v}_{sh}, e^I_{\boldsymbol{\sigma}_p}) + b_{\mathrm{sk}}(\boldsymbol{\chi}_{ph}, e^I_{\boldsymbol{\sigma}_p}) + b_\Gamma(e^I_{\mathbf{u}_p}, e^I_{\mathbf{u}_f}, e^I_{\boldsymbol{\theta}}; \xi_h)
\end{aligned} \tag{4.3.24}$$

and for some $(\widehat{\mathbf{f}}_{p,e}, \widehat{q}_{p,e}) \in \mathbf{E}_2'$ satisfying

$$\|\widehat{\mathbf{f}}_{p,e}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{q}_{p,e}\|_{\mathrm{L}^2(\Omega_p)} \le \widehat{C}_{ep,e}\big(\|e^I_{\mathbf{p}}\|_{\mathbf{Q}} + \|e^I_{\mathbf{r}}\|_{\mathbf{S}} + \|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\big) \tag{4.3.25}$$

for some constant $\widehat{C}_{ep,e}\Big\} \subset \mathbf{E}_2$,

and the multivalued operator $\mathcal{M}_e(\cdot)$ with domain $\mathcal{D}_e$ is defined by

$$\begin{aligned}\mathcal{M}_e(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) \;:=\; \Big\{(\widehat{\mathbf{f}}_{p,e}, \widehat{q}_{p,e}) - \widehat{\mathcal{E}}(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) :\\ (e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) \text{ satisfies (4.3.24) for } (\widehat{\mathbf{f}}_{p,e}, \widehat{q}_{p,e}) \in \mathbb{L}^2(\Omega_p) \times \mathrm{W}'_p\Big\}.\end{aligned} \tag{4.3.26}$$

Note that the resolvent system (4.3.24) can be written in an operator form as

$$\begin{aligned}(\mathcal{E} + \widetilde{\mathcal{A}} + \widetilde{\mathcal{K}}_{e^h_{\mathbf{u}_f}})\, e^h_{\mathbf{p}} + \mathcal{B}'\, e^h_{\mathbf{r}} &= \mathbf{F}_e \quad \text{in} \quad \mathbf{Q}',\\ -\mathcal{B}\, e^h_{\mathbf{p}} &= \mathbf{G}_e \quad \text{in} \quad \mathbf{S}',\end{aligned} \tag{4.3.27}$$

where $\mathbf{F}_e \in \mathbf{Q}'$ and $\mathbf{G}_e \in \mathbf{S}'$ are the functionals on the right hand side of (4.3.24), and

$$\begin{aligned}\widetilde{\mathcal{A}}(e^h_{\mathbf{p}})(\mathbf{q}) &= \mathcal{A}(e^h_{\mathbf{p}})(\mathbf{q}) + \kappa_{e^h_{\mathbf{u}_f}}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\mathbf{u}_f}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_f, \mathbf{v}_f)\\ &\quad -\kappa_{e^I_{\mathbf{u}_f}}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_f, \mathbf{v}_f) - \kappa_{e^h_{\mathbf{u}_f}}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; \mathbf{R}_f, \mathbf{v}_f),\\ \widetilde{\mathcal{K}}_{e^h_{\mathbf{w}_f}}(e^h_{\mathbf{p}})(\mathbf{q}) &= -\kappa_{e^h_{\mathbf{w}_f}}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; \mathbf{R}_f, \mathbf{v}_f).\end{aligned}$$

In addition, we present a stability result in the following lemma.

Lemma 4.3.4. *Assume the conditions in Lemma 4.2.2, together with* $\|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f} \leq r_{0,e}$, $\|(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f})\|_{\mathbb{X}_f\times\mathbf{V}_f} \leq r_{0,e}$, *and* $\|e^h_{\mathbf{w}_f}\|_{\mathbf{H}^1(\Omega_f)} \leq r_{0,e}$, *where*

$$r_{0,e} := \frac{\alpha_f}{6\, C_{\mathcal{K}}}, \tag{4.3.28}$$

then

$$\begin{aligned}&a_f(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\mathbf{v}_f}(\mathbf{T}_f, \mathbf{u}_f; \mathbf{R}_f, \mathbf{v}_f) + \kappa_{\mathbf{u}_f}(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f)\\ &-\; \kappa_{e^I_{\mathbf{u}_f}}(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f) - \kappa_{\mathbf{v}_f}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; \mathbf{R}_f, \mathbf{v}_f) - \kappa_{e^h_{\mathbf{w}_f}}(\mathbf{R}_f, \mathbf{v}_f; \mathbf{R}_f, \mathbf{v}_f)\\ &\geq \frac{\alpha_f}{6}\|(\mathbf{R}_f, \mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f}.\end{aligned} \tag{4.3.29}$$

Proof. Since a_f is coercive and $\kappa_{\mathbf{w}_f}$ is continuous, we have

$$\begin{aligned}
& a_f(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) + \kappa_{\mathbf{v}_f}(\mathbf{T}_f,\mathbf{u}_f;\mathbf{R}_f,\mathbf{v}_f) + \kappa_{\mathbf{u}_f}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) \\
& \quad - \kappa_{e^I_{\mathbf{u}_f}}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) - \kappa_{\mathbf{v}_f}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f};\mathbf{R}_f,\mathbf{v}_f) - \kappa_{e^h_{\mathbf{w}_f}}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) \\
& \geq a_f(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f) - |\kappa_{\mathbf{v}_f}(\mathbf{T}_f,\mathbf{u}_f;\mathbf{R}_f,\mathbf{v}_f)| - |\kappa_{\mathbf{u}_f}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f)| \\
& \quad - |\kappa_{e^I_{\mathbf{u}_f}}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f)| - |\kappa_{\mathbf{v}_f}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f};\mathbf{R}_f,\mathbf{v}_f)| - |\kappa_{e^h_{\mathbf{w}_f}}(\mathbf{R}_f,\mathbf{v}_f;\mathbf{R}_f,\mathbf{v}_f)| \\
& \geq (\alpha_f - C_{\mathcal{K}}(2\|(\mathbf{T}_f,\mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f} + 2\|(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f})\|_{\mathbb{X}_f\times\mathbf{V}_f} + \|e^h_{\mathbf{w}_f}\|_{\mathbf{H}^1(\Omega_f)}))\|(\mathbf{R}_f,\mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} \\
& \geq \frac{\alpha_f}{6}\|(\mathbf{R}_f,\mathbf{v}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f},
\end{aligned} \tag{4.3.30}$$

where we used the assumptions $\|(\mathbf{T}_f,\mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f} \leq r_{0,e}$, $\|(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f})\|_{\mathbb{X}_f\times\mathbf{V}_f} \leq r_{0,e}$, and $\|e^h_{\mathbf{w}_f}\|_{\mathbf{H}^1(\Omega_f)} \leq r_{0,e}$ in the last inequality. □

We note that combined with Lemma 4.2.2, we obtain the ellipticity of $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{K}}_{e^h_{\mathbf{w}_f}}$, which is used in the upcoming analysis.

We will start by showing that the multivalued operator $\mathcal{M}_e(\cdot)$ is well defined, or equivalently that the domain $\mathcal{D}_e$ is nonempty, using a fixed-point approach similarly as in Section 4.2.2. To do so, we introduce a fixed-point $\mathcal{J}_e : \mathbf{V}_{fh} \to \mathbf{V}_{fh}$ associated to problem (4.3.24) by

$$\mathcal{J}_e(e^h_{\mathbf{w}_f}) := e^h_{\mathbf{u}_f} \quad \forall\, e^h_{\mathbf{w}_f} \in \mathbf{V}_{fh}, \tag{4.3.31}$$

where $e^h_{\mathbf{u}_f}$ is unique solution (to be confirmed below) of the problem: Find $(e^h_{\mathbf{p}}, e^h_{\mathbf{r}}) \in \mathbf{Q}_h \times \mathbf{S}_h$, such that

$$\begin{aligned}
(\mathcal{E} + \widetilde{\mathcal{A}} + \widetilde{\mathcal{K}}_{e^h_{\mathbf{w}_f}})\, e^h_{\mathbf{p}} + \mathcal{B}'\, e^h_{\mathbf{r}} &= \mathbf{F}_e \quad \text{in} \quad \mathbf{Q}', \\
-\mathcal{B}\, e^h_{\mathbf{p}} &= \mathbf{G}_e \quad \text{in} \quad \mathbf{S}'.
\end{aligned} \tag{4.3.32}$$

Thus it is not hard to see that $(e^h_{\mathbf{p}}, e^h_{\mathbf{r}}) \in \mathbf{Q}_h \times \mathbf{S}_h$ is a solution of (4.3.27) if and only if $e^h_{\mathbf{u}_f} \in \mathbf{V}_{fh}$ is a fixed-point of $\mathcal{J}_e$, that is,

$$\mathcal{J}_e(e^h_{\mathbf{u}_f}) = e^h_{\mathbf{u}_f}. \tag{4.3.33}$$

In this way, in what follows we focus on proving that $\mathcal{J}_e$ possesses a unique fixed-point. However, we remark in advance that the definition of $\mathcal{J}_e$ will make sense only in a closed ball of $\mathbf{V}_{fh}$.

We first show the solvability of the resolvent system (4.3.24) using a regularization technique similarly as in Theorem 4.2.6. We present the result without proof.

Theorem 4.3.5. *Let $r_e \in (0, r_{0,e})$ with $r_{0,e}$ given by* (4.3.28)*. Assume conditions in Lemma 4.3.4, then for each $e^h_{\mathbf{w}_f}$ such that $\|e^h_{\mathbf{w}_f}\|_{\mathbf{H}^1(\Omega_f)} \le r_e$ and for each $(\widehat{\mathbf{f}}_{p,e}, \widehat{q}_{p,e})$ satisfying* (4.3.25)*, there exists a unique solution of the resolvent system* (4.3.24)*. Moreover, there exists a constant $C_{\mathcal{J},e} > 0$, independent of $e^h_{\mathbf{w}_f}$ and the data $e^I_{\mathbf{p}}$, $e^I_{\mathbf{r}}$, and $(\mathbf{T}_f, \mathbf{u}_f)$, such that*

$$\|\mathcal{J}_e(e^h_{\mathbf{w}_f})\|_{\mathbf{V}_f} \le \|(e^h_{\mathbf{p}}, e^h_{\mathbf{r}})\|_{\mathbf{Q}\times\mathbf{S}} \le C_{\mathcal{J},e}\big(\|e^I_{\mathbf{p}}\|_{\mathbf{Q}} + \|e^I_{\mathbf{r}}\|_{\mathbf{S}} + \|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\big). \tag{4.3.34}$$

We then claim that $\mathcal{J}_e$ is a contraction mapping according to the lemma as follows.

Lemma 4.3.6. *Let $r_e \in (0, r_{0,e})$ with $r_{0,e}$ given by* (4.3.28) *and let $\mathbf{W}_{r,e}$ be the closed ball defined by*

$$\mathbf{W}_{r,e} := \big\{e^h_{\mathbf{w}_f} \in \mathbf{V}_{fh} : \|e^h_{\mathbf{w}_f}\|_{\mathbf{V}_f} \le r_e\big\}, \tag{4.3.35}$$

and assume conditions in Lemma 4.3.4. Then, for all $e^h_{\mathbf{w}_f}$, $e^h_{\widetilde{\mathbf{w}}_f} \in \mathbf{W}_{r,e}$ there holds

$$\|\mathcal{J}_e(e^h_{\mathbf{w}_f}) - \mathcal{J}_e(e^h_{\widetilde{\mathbf{w}}_f})\|_{\mathbf{V}_f} \le \frac{C_{\mathcal{J},e}}{r_{0,e}}\big(\|e^I_{\mathbf{p}}\|_{\mathbf{Q}} + \|e^I_{\mathbf{r}}\|_{\mathbf{S}} + \|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\big)\|e^h_{\mathbf{w}_f} - e^h_{\widetilde{\mathbf{w}}_f}\|_{\mathbf{V}_f}, \tag{4.3.36}$$

where $C_{\mathcal{J},e}$ is the constant given by (4.3.34)*.*

We are now in position of establishing the fact that the domain $\mathcal{D}_e$, cf. (4.3.24), is nonempty by means of the well known Banach fixed-point theorem.

Theorem 4.3.7. *Given $r_e \in (0, r_{0,e})$, with $r_{0,e}$ given by* (4.3.28)*, we let $\mathbf{W}_{r,e}$ be as in* (4.3.35)*, and assume that the data satisfy*

$$C_{\mathcal{J},e}\big(\|e^I_{\mathbf{p}}\|_{\mathbf{Q}} + \|e^I_{\mathbf{r}}\|_{\mathbf{S}} + \|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\big) \le r_e. \tag{4.3.37}$$

In addition, assume conditions in Lemma 4.3.4, then the problem (4.3.27) *has a unique solution $(e^h_{\mathbf{p}}, e^h_{\mathbf{r}}) \in \mathbf{Q}\times\mathbf{S}$ with $e^h_{\mathbf{u}_f} \in \mathbf{W}_{r,e}$, and there holds*

$$\|(e^h_{\mathbf{p}}, e^h_{\mathbf{r}})\|_{\mathbf{Q}\times\mathbf{S}} \le C_{\mathcal{J},e}\big(\|e^I_{\mathbf{p}}\|_{\mathbf{Q}} + \|e^I_{\mathbf{r}}\|_{\mathbf{S}} + \|(\mathbf{T}_f, \mathbf{u}_f)\|_{\mathbb{X}_f\times\mathbf{V}_f}\big). \tag{4.3.38}$$

Therefore, the multivalued operator $\mathcal{M}_e$ is well defined. We end this section by stating $\mathcal{M}_e$ is monotone, whose proof is similar as the one in Lemma 4.2.9.

Lemma 4.3.8. *Let $r_e \in (0, r_{0,e})$ with $r_{0,e}$ defined by* (4.3.28), *and assume that the data satisfy* (4.3.37). *In addition, assume conditions in Lemma 4.3.4, then the operator $\mathcal{M}_e$ defined by* (4.3.26) *is monotone.*

4.3.2.3 A priori error estimates

We start the analysis by adding up the equations in (4.3.22), then taking $(\boldsymbol{\tau}_{ph}, w_{ph}, \mathbf{v}_{ph}, \mathbf{R}_{fh}, \mathbf{v}_{fh}, \boldsymbol{\phi}_h, \xi_h, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}) = (e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}, e^h_{\mathbf{u}_p}, e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}, e^h_{\lambda}, e^h_{\mathbf{u}_s}, e^h_{\boldsymbol{\gamma}_p})$, to obtain

$$\begin{aligned}
&\frac{1}{2} s_0\, \partial_t\, (e^h_{p_p}, e^h_{p_p})_{\Omega_p} + \frac{1}{2}\partial_t\, a_e(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) + a_p(e^h_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) + a_f(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}) \\
&\quad + \kappa_{\mathbf{u}_{fh}}(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}; e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}) + \kappa_{e^h_{\mathbf{u}_f}}(\mathbf{T}_f, \mathbf{u}_f; e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}) + a_{\mathtt{BJS}}(e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) \\
&= - a_e(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, \partial_t\, e^I_{p_p}; e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) - a_p(e^I_{\mathbf{u}_p}, e^h_{\mathbf{u}_p}) - a_f(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}, e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}) \\
&\quad - \kappa_{\mathbf{u}_{fh}}(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f}; e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}) - \kappa_{e^I_{\mathbf{u}_f}}(\mathbf{T}_f, \mathbf{u}_f; e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f}) - a_{\mathtt{BJS}}(e^I_{\mathbf{u}_f}, e^I_{\boldsymbol{\theta}}; e^h_{\mathbf{u}_f}, e^h_{\boldsymbol{\theta}}) \\
&\quad - b_{\mathrm{sk}}(e^I_{\boldsymbol{\gamma}_p}, e^h_{\boldsymbol{\sigma}_p}) + b_{\mathrm{sk}}(e^h_{\boldsymbol{\gamma}_p}, e^I_{\boldsymbol{\sigma}_p}) - \langle e^h_{\mathbf{u}_f} \cdot \mathbf{n}_f, e^I_{\lambda}\rangle_{\Gamma_{fp}} - \langle e^h_{\boldsymbol{\theta}} \cdot \mathbf{n}_p, e^I_{\lambda}\rangle_{\Gamma_{fp}} \\
&\quad + \langle e^I_{\mathbf{u}_f} \cdot \mathbf{n}_f, e^h_{\lambda}\rangle_{\Gamma_{fp}} + \langle e^I_{\boldsymbol{\theta}} \cdot \mathbf{n}_p, e^h_{\lambda}\rangle_{\Gamma_{fp}},
\end{aligned} \tag{4.3.39}$$

where, the following terms vanish due to the projection properties (4.3.15), (4.3.17), (4.3.18), and the fact that $\mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}$, $\operatorname{div}(\mathbf{V}_{ph}) = \mathrm{W}_{ph}$, cf. (4.3.1),

$$\begin{gathered}
s_0(\partial_t\, e^I_{p_p}, e^h_{p_p})_{\Omega_p},\ b_p(e^I_{p_p}, e^h_{\mathbf{u}_p}),\ b_p(e^h_{p_p}, e^I_{\mathbf{u}_p}),\ b_{\mathbf{n}_p}(e^I_{\boldsymbol{\sigma}_p}, e^h_{\boldsymbol{\theta}}),\ b_{\mathbf{n}_p}(e^h_{\boldsymbol{\sigma}_p}, e^I_{\boldsymbol{\theta}}), \\
b_s(e^I_{\mathbf{u}_s}, e^h_{\boldsymbol{\sigma}_p}),\ b_s(e^h_{\mathbf{u}_s}, e^I_{\boldsymbol{\sigma}_p}),\ \langle e^h_{\mathbf{u}_p} \cdot \mathbf{n}_p, e^I_{\lambda}\rangle_{\Gamma_{fp}},\ \langle e^I_{\mathbf{u}_p} \cdot \mathbf{n}_p, e^h_{\lambda}\rangle_{\Gamma_{fp}}.
\end{gathered}$$

Then, applying ellipticity properties of $a_f + \kappa_{\mathbf{w}_f}$ and a_p, the semi-positive definiteness of $a_{\mathtt{BJS}}$, c.f. (4.2.9) and (4.2.10), continuity bounds of the bilinear forms in Lemma 4.2.1, in combination with $\mathbf{u}_f(t), \mathbf{u}_{fh}(t) : [0, T] \to \mathbf{W}_r$, cf. (4.2.38), the Cauchy-Schwarz and Young's inequalities, we get

$$\frac{1}{2} s_0 \partial_t \|e^h_{p_p}\|^2_{\mathrm{W}_p} + \frac{1}{2}\partial_t \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \mu k_{\max}^{-1} \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)}$$

$$
\begin{aligned}
&+ \alpha_f(1-\frac{r}{r_0})\|(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f})\|^2_{\mathbb{X}_f\times\mathbf{V}_f} + c_{\mathtt{BJS}}|e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \\
&\leq C\Big(\|e^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)} + \|\partial_t A^{1/2}(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f})\|^2_{\mathbb{X}_f\times\mathbf{V}_f} \\
&\quad + \|\mathbf{u}_{fh}\|^2_{\mathbf{V}_f}\|(e^I_{\mathbf{T}_f}, e^I_{\mathbf{u}_f})\|^2_{\mathbb{X}_f\times\mathbf{V}_f} + \|e^I_{\mathbf{u}_f}\|^2_{\mathbf{V}_f}\|(\mathbf{T}_f, \mathbf{u}_f)\|^2_{\mathbb{X}_f\times\mathbf{V}_f} + \|e^I_{\mathbf{u}_f}\|^2_{\mathbf{V}_f} + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \\
&\quad + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} + \|e^I_{\lambda}\|^2_{\Lambda_{ph}} + \|e^I_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}}\Big) + \delta_1\Big(\|e^h_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|(e^h_{\mathbf{T}_f}, e^h_{\mathbf{u}_f})\|^2_{\mathbb{X}_f\times\mathbf{V}_f} + \|e^h_{\mathbf{u}_f}\|^2_{\mathbf{V}_f} \\
&\quad + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}}\Big) + \delta_2\Big(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}e^h_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \\
&\quad + \|e^h_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^h_{\lambda}\|^2_{\Lambda_{ph}}\Big), \qquad (4.3.40)
\end{aligned}
$$

where we also used

$$
b_{\mathrm{sk}}(e^I_{\boldsymbol{\gamma}_p}, e^h_{\boldsymbol{\sigma}_p}) = \frac{1}{c}(Ae^I_{\boldsymbol{\gamma}_p}, e^h_{\boldsymbol{\sigma}_p})_{\Omega_p} = \frac{1}{c}(A^{1/2}e^I_{\boldsymbol{\gamma}_p}, A^{1/2}e^h_{\boldsymbol{\sigma}_p})_{\Omega_p} \leq C\|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}\|A^{1/2}e^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega_p)} \quad (4.3.41)
$$

by the definition of A due to the extension from $\mathbb{S}$ to $\mathbb{M}$ as in [63]. Next, we choose δ_1 small enough, take integration from 0 to $t \in (0, T]$, and use the stability results of $(\mathbf{T}_f, \mathbf{u}_f)$ in Theorem 4.2.15 and $\mathbf{u}_{fh}$ in Theorem 4.3.3, we find

$$
\begin{aligned}
&\|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \int_0^t \Big(\|e^h_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^h_{\mathbf{T}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_f}\|^2_{\mathbf{V}_f} + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}}\Big)ds \\
&\leq C\bigg(\int_0^t \Big(\|e^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)} + \|\partial_t A^{1/2}(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^I_{\mathbf{T}_f}\|^2_{\mathbb{X}_f} + \|e^I_{\mathbf{u}_f}\|^2_{\mathbf{V}_f} \\
&\quad + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^I_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^I_{\lambda}\|^2_{\Lambda_{ph}} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p}\Big)ds\bigg) + \delta_2\int_0^t \Big(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \\
&\quad + \|e^h_{p_p}\|^2_{\mathrm{W}_p} + \|e^h_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^h_{\lambda}\|^2_{\Lambda_{ph})} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p}\Big)ds + \|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} \\
&\quad + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)}, \qquad (4.3.42)
\end{aligned}
$$

where we also used

$$
\|A^{1/2}e^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega_p)} \leq C(\|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^h_{p_p}\|_{\mathrm{W}_p}). \quad (4.3.43)
$$

On the other hand, from discrete inf-sup conditions (4.3.3)–(4.3.5) in Lemma 4.3.1, we have

$$\begin{aligned}
&\|e^h_{\mathbf{u}_s}\|_{\mathbf{V}_s} + \|e^h_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p} \le C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_{ph}\in\widehat{\mathbb{X}}_{ph}} \frac{b_s(\boldsymbol{\tau}_{ph}, e^h_{\mathbf{u}_s}) + b_{\mathrm{sk}}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\gamma}_p})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&= C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_{ph}\in\widehat{\mathbb{X}}_{ph}} \frac{-a_e(\partial_t e^I_{\boldsymbol{\sigma}_p}, \partial_t e^I_{p_p}; \boldsymbol{\tau}_{ph}, 0) - a_e(\partial_t e^h_{\boldsymbol{\sigma}_p}, \partial_t e^h_{p_p}; \boldsymbol{\tau}_{ph}, 0) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\theta}}) - b_{\mathrm{sk}}(e^I_{\boldsymbol{\gamma}_p}, \boldsymbol{\tau}_{ph})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&\le C\big(\|\partial_t A^{1/2}(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}\big),
\end{aligned} \tag{4.3.44}$$

$$\begin{aligned}
\|e^h_{p_p}\|_{\mathrm{W}_p} + \|e^h_\lambda\|_{\Lambda_{ph}} &\le C \sup_{\mathbf{0}\neq\mathbf{v}_{ph}\in\mathbf{V}_{ph}} \frac{b_p(\mathbf{v}_{ph}, e^h_{p_p}) + b_\Gamma(\mathbf{0}, \mathbf{v}_{ph}, \mathbf{0}; e^h_\lambda)}{\|\mathbf{v}_{ph}\|_{\mathbf{V}_p}} \\
&= C \sup_{\mathbf{0}\neq\mathbf{v}_{ph}\in\mathbf{V}_{ph}} \frac{-a_p(e^I_{\mathbf{u}_p}, \mathbf{v}_{ph}) - a_p(e^h_{\mathbf{u}_p}, \mathbf{v}_{ph})}{\|\mathbf{v}_{ph}\|_{\mathbf{V}_p}} \le C(\|e^I_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)}),
\end{aligned} \tag{4.3.45}$$

$$\begin{aligned}
&\|e^h_{\boldsymbol{\theta}}\|_{\boldsymbol{\Lambda}_{sh}} \le C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_{ph}\in\widetilde{\mathbb{X}}_{ph}} \frac{b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, e^h_{\boldsymbol{\theta}})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&= C \sup_{\mathbf{0}\neq\boldsymbol{\tau}_{ph}\in\widetilde{\mathbb{X}}_{ph}} \Big(\frac{-a_e(\partial_t\, e^I_{\boldsymbol{\sigma}_p}, \partial_t\, e^I_{p_p}; \boldsymbol{\tau}_{ph}, 0) - a_e(\partial_t\, e^h_{\boldsymbol{\sigma}_p}, \partial_t\, e^h_{p_p}; \boldsymbol{\tau}_{ph}, 0) - b_{\mathrm{sk}}(e^I_{\boldsymbol{\gamma}_p}, \boldsymbol{\tau}_{ph})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \\
&\quad + \frac{-b_{\mathrm{sk}}(e^h_{\boldsymbol{\gamma}_p}, \boldsymbol{\tau}_{ph}) - b_s(\boldsymbol{\tau}_{ph}, e^h_{\mathbf{u}_s})}{\|\boldsymbol{\tau}_{ph}\|_{\mathbb{X}_p}} \Big) \\
&\le C\big(\|\partial_t A^{1/2}(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p} + \|e^h_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}\big).
\end{aligned} \tag{4.3.46}$$

We next derive bounds for $\|\mathrm{div}(e^h_{\mathbf{u}_p})\|_{\mathrm{L}^2(\Omega_p)}$ and $\|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega_p)}$. Due to (4.3.1), we can choose $w_{ph} = \mathrm{div}(e^h_{\mathbf{u}_p})$ in (4.3.22), obtaining

$$\begin{aligned}
\|\mathrm{div}(e^h_{\mathbf{u}_p})\|^2_{\mathrm{L}^2(\Omega_p)} &= -(s_0\partial_t e^h_{p_p}, \mathrm{div}(e^h_{\mathbf{u}_p}))_{\Omega_p} - (A\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha e^h_{p_p}\mathbf{I}), \mathrm{div}(e^h_{\mathbf{u}_p}))_{\Omega_p} \\
&\quad - (A\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha e^I_{p_p}\mathbf{I}), \mathrm{div}(e^h_{\mathbf{u}_p}))_{\Omega_p} \\
&\le (s_0\|\partial_t e^h_{p_p}\|_{\mathrm{W}_p} + a_{\max}^{1/2}\|A^{1/2}\partial_t(e^h_{\boldsymbol{\sigma}_p} + \alpha e^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \\
&\quad + a_{\max}^{1/2}\|A^{1/2}\partial_t(e^I_{\boldsymbol{\sigma}_p} + \alpha e^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)})\|\nabla\cdot e^h_{\mathbf{u}_p}\|_{\mathrm{L}^2(\Omega_p)}.
\end{aligned} \tag{4.3.47}$$

Similarly, the choice of $\mathbf{v}_{sh} = \mathbf{div}(e^h_{\boldsymbol{\sigma}_p})$ in (4.3.22) gives

$$\|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})(t)\|_{\mathbf{L}^2(\Omega_p)} = 0 \quad \text{and} \quad \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,t;\mathbf{L}^2(\Omega_p))} = 0. \tag{4.3.48}$$

Combining (4.3.42) with (4.3.44)–(4.3.48), choosing δ_2 small enough, and employing the Gronwall's inequality for $\displaystyle\int_0^t \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} ds$, we obtain

$$\begin{aligned}
&\|e^h_{p_p}(t)\|^2_{\mathrm{W}_p} + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p}(t))\|_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \int_0^t \Big(\|e^h_{p_p}\|^2_{\mathrm{W}_p} + \|\mathbf{div}(e^h_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega_p)} + \|e^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + \|e^h_{\mathbf{T}_f}\|^2_{\mathbb{X}_f} + \|e^h_{\mathbf{u}_f}\|^2_{\mathbf{V}_f} \\
&\quad + |e^h_{\mathbf{u}_f} - e^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^h_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^h_\lambda\|^2_{\Lambda_{ph}} + \|e^h_{\mathbf{u}_s}\|^2_{\mathbf{V}_s} + \|e^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p}\Big) ds \\
&\leq C\exp(T)\bigg(\int_0^t \Big(\|e^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)} + \|\partial_t A^{1/2}(e^I_{\boldsymbol{\sigma}_p} + \alpha_p e^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^I_{\mathbf{u}_p}\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \|e^I_{\mathbf{T}_f}\|^2_{\mathbb{X}_f} + \|e^I_{\mathbf{u}_f}\|^2_{\mathbf{V}_f} + |e^I_{\mathbf{u}_f} - e^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|e^I_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|e^I_\lambda\|^2_{\Lambda_{ph}} + \|e^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \\
&\quad + \|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\Big) ds + \|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} \\
&\quad + \|A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)}\bigg).
\end{aligned} \tag{4.3.49}$$

In order to bound the term $\|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$, we note that the error equation (4.3.22) is equivalent to the parabolic problem (4.3.23). Therefore by referring to [74, Theorem IV.4.1(4.3)] applied to problem (4.3.23) with $\mathcal{M}_e(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p}) = \{(\widehat{\mathbf{f}}_{p,e}, \widehat{q}_{p,e}) - \widehat{\mathcal{E}}(e^h_{\boldsymbol{\sigma}_p}, e^h_{p_p})\}$ (c.f. (4.3.26)), we obtain

$$\|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t e^h_{p_p}\|^2_{\mathrm{W}_p} \leq \|\mathcal{M}_e(e^h_{\boldsymbol{\sigma}_p,0}, e^h_{p_p,0})\|^2.$$

Using $\mathcal{M}_e(e^h_{\boldsymbol{\sigma}_p,0}, e^h_{p_p,0}) = \{(\widehat{\mathbf{f}}^0_{p,e}, \widehat{q}^0_{p,e}) - \widehat{\mathcal{E}}(e^h_{\boldsymbol{\sigma}_p,0}, e^h_{p_p,0})\}$ with

$$(\widehat{\mathbf{f}}^0_{p,e}, \boldsymbol{\tau}_{ph})_{\Omega_p} = a_e(e_{\boldsymbol{\sigma}_p,0}, e_{p_p,0}; \boldsymbol{\tau}_{ph}, 0) - (A(\widehat{\boldsymbol{\sigma}}_{p,0} - \widehat{\boldsymbol{\sigma}}_{ph,0}), \boldsymbol{\tau}_{ph})_{\Omega_p},$$

$$(\widehat{q}^0_{p,e}, w_{ph})_{\Omega_p} = (s_0\, e_{p_p,0}, w_{ph})_{\Omega_p} + a_e(e_{\boldsymbol{\sigma}_p,0}, e_{p_p,0}; \mathbf{0}, w_{ph}) - b_p(e_{\mathbf{u}_p,0}, w_{ph}),$$

according to (4.2.59) and (4.3.13), we get

$$\|\partial_t A^{1/2}(e^h_{\boldsymbol{\sigma}_p} + \alpha_p e^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$$

$$\leq C\big(\|e^h_{\boldsymbol{\sigma}_p,0}\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{p_p,0}\|^2_{\mathrm{W}_p} + \|e^h_{\mathbf{u}_p,0}\|^2_{\mathbf{L}^2(\Omega_p)} + \|e^I_{\boldsymbol{\sigma}_p,0}\|^2_{\mathbb{L}^2(\Omega_p)}$$
$$+ \|e^I_{p_p,0}\|^2_{\mathrm{W}_p} + \|e^I_{\mathbf{u}_p,0}\|^2_{\mathbf{L}^2(\Omega_p)} + \|\widehat{\boldsymbol{\sigma}}_{p,0} - \widehat{\boldsymbol{\sigma}}_{ph,0}\|^2_{\mathbb{L}^2(\Omega_p)}\big). \tag{4.3.50}$$

To bound the initial data terms above, we recall that $(\boldsymbol{\sigma}_p(0), p_p(0), \mathbf{u}_p(0), \mathbf{T}_f(0), \mathbf{u}_f(0), \boldsymbol{\theta}(0), \lambda(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0}, \mathbf{u}_{p,0}, \mathbf{T}_{f,0}, \mathbf{u}_{f,0}, \boldsymbol{\theta}_0, \lambda_0)$, c.f. Corollary 4.2.14, and $(\boldsymbol{\sigma}_{ph}(0), p_{ph}(0), \mathbf{u}_{ph}(0), \mathbf{T}_{fh}(0), \mathbf{u}_{fh}(0), \boldsymbol{\theta}_h(0), \lambda_h(0)) = (\boldsymbol{\sigma}_{ph,0}, p_{ph,0}, \mathbf{u}_{ph,0}, \mathbf{T}_{fh,0}, \mathbf{u}_{fh,0}, \boldsymbol{\theta}_{h,0}, \lambda_{h,0})$, c.f. Theorem 4.3.3. Similarly to (4.3.49), we obtain

$$\|A^{1/2} e^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{p_p}(0)\|^2_{\mathrm{W}_p} + \|e^h_{\boldsymbol{\sigma}_p,0}\|^2_{\mathbb{L}^2(\Omega_p)} + \|e^h_{p_p,0}\|^2_{\mathrm{W}_p} + \|e^h_{\mathbf{u}_p,0}\|^2_{\mathbf{L}^2(\Omega_p)}$$
$$+ \|\widehat{\boldsymbol{\sigma}}_{p,0} - \widehat{\boldsymbol{\sigma}}_{ph,0}\|^2_{\mathbb{L}^2(\Omega_p)}$$
$$\leq C\big(\|e^I_{\boldsymbol{\sigma}_p,0}\|_{\mathbb{X}_p} + \|e^I_{p_p,0}\|_{\mathrm{W}_p} + \|e^I_{\mathbf{u}_p,0}\|_{\mathbf{V}_p} + \|e^I_{\mathbf{T}_f,0}\|_{\mathbb{X}_f} + \|e^I_{\mathbf{u}_f,0}\|_{\mathbf{V}_f} + |e^I_{\mathbf{u}_f,0} - e^I_{\boldsymbol{\theta},0}|^2_{\mathtt{BJS}}$$
$$+ \|e^I_{\boldsymbol{\theta},0}\|_{\boldsymbol{\Lambda}_{sh}} + \|e^I_{\lambda,0}\|_{\Lambda_p} + \|e^I_{\boldsymbol{\psi},0}\|_{\mathbf{V}_s} + \|e^I_{\boldsymbol{\rho}_p,0}\|_{\mathbb{Q}_p}\big). \tag{4.3.51}$$

Combining (4.3.49)–(4.3.51), and making use of triangle inequality and the approximation properties (4.3.16), (4.3.19), and (4.3.20) results in the following theorem.

Theorem 4.3.9. *For the solutions of the continuous and discrete problems* (4.1.13) *and* (4.3.2)*, respectively, assuming sufficient regularity of the data which satisfy* (4.2.40) *and compatible initial data* $(\mathbf{p}_h(0), \mathbf{r}_h(0))$*, then there exists a positive constant* C *independent of* h*, such that*

$$\|A^{1/2}((\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}) - (\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I}))\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))}$$
$$+ \|\mathbf{div}(\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \|\partial_t A^{1/2}((\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}) - (\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I}))\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))}$$
$$+ \|p_p - p_{ph}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_p - p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} + \|\mathbf{u}_p - \mathbf{u}_{ph}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)}$$
$$+ \|\mathbf{T}_f - \mathbf{T}_{fh}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_f - \mathbf{u}_{fh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_f)} + |(\mathbf{u}_f - \boldsymbol{\theta}) - (\mathbf{u}_{fh} - \boldsymbol{\theta}_h)|_{\mathrm{L}^2(0,T;\mathtt{BJS})}$$
$$+ \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{\mathrm{L}^2(0,T;\boldsymbol{\Lambda}_{sh})} + \|\lambda - \lambda_h\|_{\mathrm{L}^2(0,T;\Lambda_{ph})} + \|\mathbf{u}_s - \mathbf{u}_{sh}\|_{\mathrm{L}^2(0,T;\mathbf{V}_s)} + \|\boldsymbol{\gamma}_p - \boldsymbol{\gamma}_{ph}\|_{\mathrm{L}^2(0,T;\mathbb{Q}_p)}$$
$$\leq C\big(h^{s_{\underline{u}}+1} + h^{s_{\underline{\theta}}+1} + h^{s_{\underline{\sigma}}+1} + h^{s_{\mathbf{v}_f}+1}\big) \tag{4.3.52}$$

where $s_{\underline{u}} = \min\{s_{p_p}, s_{\mathbf{u}_s}, s_{\boldsymbol{\gamma}_p}\}$*,* $s_{\underline{\theta}} = \min\{s_{\boldsymbol{\theta}}, s_\lambda,\}$*, and* $s_{\underline{\sigma}} = \min\{s_{\mathbf{T}_f}, s_{\mathbf{u}_p}, s_{\boldsymbol{\sigma}_p}\}$.

4.4 Numerical results

For the fully discrete method, we employ the backward Euler method for the time discretization. Let Δt be the time step, $T = N\Delta t$, $t_n = n\Delta t$, $n = 0, \cdots, N$. Let $d_t\, u^n := (\Delta t)^{-1}(u^n - u^{n-1})$ be the first order (backward) discrete time derivative, where $u^n := u(t_n)$. Then the fully discrete model reads: Given $(\mathbf{p}_h^0, \mathbf{r}_h^0) = (\mathbf{p}_h(0), \mathbf{r}_h(0))$ satisfying (4.3.7), find $(\mathbf{p}_h^n, \mathbf{r}_h^n) \in \mathbf{Q}_h \times \mathbf{S}_h$, $n = 1, \cdots, N$, such that for all $(\mathbf{q}_h, \mathbf{s}_h) \in \mathbf{Q}_h \times \mathbf{S}_h$,

$$d_t\, \mathcal{E}\,(\mathbf{p}_h^n)(\mathbf{q}_h) + (\mathcal{A} + \mathcal{K}_{\mathbf{u}_{fh}^n})\,(\mathbf{p}_h^n)(\mathbf{q}_h) + \mathcal{B}'\,(\mathbf{r}_h^n)(\mathbf{q}_h) = \mathbf{F}^n(\mathbf{q}_h)$$
$$-\mathcal{B}\,(\mathbf{p}_h^n)(\mathbf{s}_h) = \mathbf{G}^n(\mathbf{s}_h). \qquad (4.4.1)$$

In this section we present numerical results that illustrate the behavior of the fully discrete method (4.4.1). To solve this non-linear problem, we use a Newton-Rhapson method. Our implementation is based on a `FreeFem++` code [55], in conjunction with the direct linear solver `UMFPACK` [41]. For spatial discretization we use the $(\mathbb{BDM}_1 - \mathbf{P}_1) - (\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1) - (\mathbf{BDM}_1 - \mathrm{P}_0) - (\mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}})$ approximation for the Navier–Stokes – Biot model.

The examples considered in this section are described next. Example 1 is used to corroborate the rate of convergence in a two dimensional domain. In Example 2 we present a simulation of blood flow in an artery bifurcation.

4.4.1 Convergence test

In this test we study the convergence for the space discretization using an analytical solution. The domain is $\Omega = \Omega_f \cup \Gamma_{fp} \cup \Omega_p$, where $\Omega_f = (0,1) \times (0,1), \Gamma_{fp} = (0,1) \times \{0\}$, and $\Omega_p = (0,1) \times (-1,0)$. We associate the upper half with the Stokes flow, while the lower half represents the flow in the poroelastic structure governed by the Biot system. The appropriate interface conditions are enforced along the interface Γ_{fp}. The solution in the Navier–Stokes region is

$$\mathbf{u}_f = e^t \begin{pmatrix} \sin(\pi\, x)\cos(\pi\, y) \\ -\sin(\pi\, y)\cos(\pi\, x) \end{pmatrix}, \quad p_f = e^t \sin(\pi x)\cos(\frac{\pi y}{2}) + 2\pi\cos(\pi t).$$

The Biot solution is chosen accordingly to satisfy the interface conditions:

$$\mathbf{u}_p = \pi e^t \begin{pmatrix} -\cos(\pi x)\cos(\frac{\pi y}{2}) \\ \frac{1}{2}\sin(\pi x)\sin(\frac{\pi y}{2}) \end{pmatrix}, \; p_p = e^t \sin(\pi x)\cos(\frac{\pi y}{2}), \; \boldsymbol{\eta}_p = \sin(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y+1 \end{pmatrix}.$$

The right hand side functions $\mathbf{f}_f, q_f, \mathbf{f}_p$ and q_p are computed from (4.1.1) and (4.1.3) using the above solution. The model problem is then complemented with the appropriate mixed boundary conditions and initial data. Notice that the boundary conditions for $\boldsymbol{\sigma}_f, \mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p$, and $\boldsymbol{\eta}_p$, cf. (4.1.1) and (4.1.3) are not homogeneous and therefore the right-hand side of the resulting system must be modified accordingly. Tables 4.4.1 show the convergence history for a sequence of quasi-uniform mesh refinements in no-matching grids. In the tables, h_f and h_p denote the mesh sizes in Ω_f and Ω_p, respectively, while the mesh sizes for their traces on Γ_{fp} are h_{tf} and h_{tp}, satisfying $h_{tf} = \frac{5}{8}h_{tp}$. We note that the Navier–Stokes pressure and displacement at t_n are recovered by the post-processed formulae $p_f^n = -\dfrac{1}{n}\left(\mathrm{tr}(\mathbf{T}_f^n) + \rho_f\,\mathrm{tr}(\mathbf{u}_f^n \otimes \mathbf{u}_f^n) - 2\,\mu\, q_f^n\right)$ and $\boldsymbol{\eta}_p^n = \Delta t\, \mathbf{u}_s^n + \boldsymbol{\eta}_p^{n-1}$, respectively. The results illustrate that at least the optimal spatial rates of convergence $\mathcal{O}(h)$ provided by Theorem 4.3.9 are attained for all subdomain variables in their natural norms. The Lagrange multiplier variables, which are approximated in $\mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$, exhibit a rates of convergence $\mathcal{O}(h^2)$ in the L^2-norm on Γ_{fp}, which is consistent with the order of approximation.

h_f	$\|\mathbf{e}_{\mathbf{T}_f}\|_{\ell^2(0,T;\mathbb{X}_f)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_f}\|_{\ell^2(0,T;\mathbf{V}_f)}$ error	rate	$\|\mathbf{e}_{p_f}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega_f))}$ error	rate
0.1964	1.79E-01	–	4.57E-02	–	3.16E-03	–
0.0997	9.11E-02	0.9958	2.33E-02	0.9961	1.22E-03	1.4058
0.0487	4.43E-02	1.0056	1.18E-02	0.9451	5.28E-04	1.1652
0.0250	2.23E-02	1.0295	5.89E-03	1.0418	2.38E-04	1.1915
0.0136	1.11E-02	1.1422	2.93E-03	1.1452	1.15E-04	1.1893
0.0072	5.50E-03	1.1061	1.46E-03	1.0981	4.87E-05	1.3567

h_p	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{X}_p)}$ error	rate	$\|\mathbf{e}_{p_p}\|_{\ell^\infty(0,T;\mathrm{W}_p)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_p}\|_{\ell^2(0,T;\mathbf{V}_p)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^2(0,T;\mathbf{V}_s)}$ error	rate
0.2828	2.73E-01	–	7.54E-02	–	1.04E-01	–	4.31E-02	–
0.1646	1.37E-01	1.2731	3.84E-02	1.2480	5.01E-02	1.3516	2.22E-02	1.2250
0.0779	6.67E-02	0.9650	1.91E-02	0.9328	2.39E-02	0.9887	1.08E-02	0.9616
0.0434	3.37E-02	1.1690	9.39E-03	1.2150	1.16E-02	1.2359	5.41E-03	1.1865
0.0227	1.69E-02	1.0634	4.70E-03	1.0658	5.79E-03	1.0738	2.71E-03	1.0667
0.0124	8.43E-03	1.1462	2.35E-03	1.1429	2.89E-03	1.1452	1.35E-03	1.1456

$\|\mathbf{e}_{\boldsymbol{\gamma}_p}\|_{\ell^2(0,T;\mathbb{Q}_p)}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\eta}_p}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega_p))}$ error	rate	h_{tp}	$\|\mathbf{e}_{\boldsymbol{\theta}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	$\|\mathbf{e}_{\lambda}\|_{\ell^2(0,T;\mathrm{L}^2(\Gamma_{fp}))}$ error	rate	iter
5.03E-02	–	2.67E-04	–	0.2000	6.81E-03	–	1.07E-03	–	
1.41E-02	2.3537	1.38E-04	1.2235	0.1000	2.41E-03	1.5016	2.68E-04	2.0005	2.2
3.00E-03	2.0649	6.72E-05	0.9613	0.0500	5.77E-04	2.0587	6.71E-05	2.0004	2.2
7.27E-04	2.4264	3.36E-05	1.1864	0.0250	1.45E-04	1.9912	1.68E-05	1.9939	2.2
1.80E-04	2.1524	1.68E-05	1.0667	0.0125	3.62E-05	2.0051	4.26E-06	1.9829	2.2
4.80E-05	2.1814	8.40E-06	1.1456	0.0063	9.21E-06	1.9743	1.09E-06	1.9629	2.2

Table 4.4.1: Example 1, Mesh sizes, errors, rates of convergences and average Newton iterations for the fully discrete system in no-matching grids.

4.4.2 A blood flow example in an artery bifurcation

In this example, we study numerically a simulation of blood flow in an artery bifurcation. We use the fully dynamic Navier-Stokes – Biot model for a better numerical performance. In particular, the Navier-Stokes momentum equation in the fluid region is

$$\rho_f \, \partial_t \mathbf{u}_f - \rho_f (\nabla \mathbf{u}_f)\mathbf{u}_f - \mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f,$$

and the linear elasticity equation in the Biot system is

$$\rho_p \, \partial_t^2 \boldsymbol{\eta}_p - \beta \, \boldsymbol{\eta}_p - \mathbf{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p.$$

The additional term $\beta \, \boldsymbol{\eta}_p$ comes from the axially symmetric formulation, accounting for the recoil due to the circumferential strain [26]. The physical parameters are chosen based on [26] and fall within the range of physiological values for blood flow:

$$\mu = 0.035 \text{ g/cm-s}, \quad \rho_f = 1 \text{ g/cm}^3, \quad s_0 = 5 \times 10^{-6} \text{ cm}^2/\text{dyn}, \quad \mathbf{K} = 10^{-9} \times \mathbf{I} \text{ cm}^2,$$

$$\rho_p = 1.1 \text{ g/cm}^3, \lambda_p = 4.28 \times 10^6 \text{ dyn/cm}^2, \quad \mu_p = 1.07 \times 10^6 \text{ dyn/cm}^2,$$

$$\beta = 5 \times 10^7 \text{ dyn/cm}^4, \quad \alpha = 1, \quad \alpha_{\mathsf{BJS}} = 1.$$

The body force terms and external source are set to be zero, as well as the initial conditions. The flow is driven by the time-dependent pressure data

$$p_{in}(t) = \begin{cases} \dfrac{P_{\max}}{2}\Big(1 - \cos\big(\dfrac{2\pi t}{T_{\max}}\big)\Big), & \text{if } t \leq T_{\max}; \\ 0, & \text{if } t > T_{\max}, \end{cases} \tag{4.4.2}$$

where $P_{\max} = 13{,}334$ dyn/cm^2 and $T_{\max} = 0.003$ s. We specify the boundary conditions as follows,

$$\boldsymbol{\sigma}_f \, \mathbf{n}_f = -p_{in} \, \mathbf{n}_f \quad \text{on} \quad \Gamma_f^{in} \quad \text{and} \quad \boldsymbol{\sigma}_f \, \mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{out},$$

$$\mathbf{u}_s = \mathbf{0} \quad \text{on} \quad \Gamma_p^{in} \cup \Gamma_p^{out} \quad \text{and} \quad \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_p^{ext},$$

$$\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{in} \cup \Gamma_p^{out} \quad \text{and} \quad p_p = 0 \quad \text{on} \quad \Gamma_p^{ext},$$

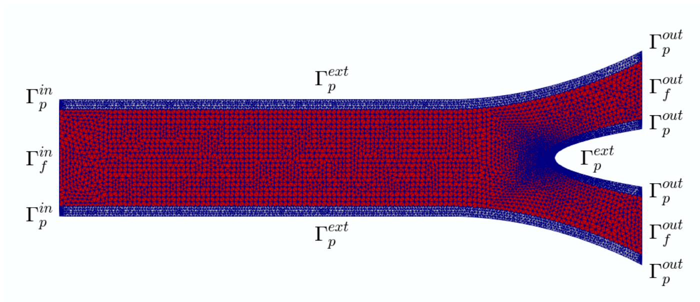


Figure 4.4.1: Simulation domain.

The red area is fluid region Ω_f and the grey areas are structure regions Ω_p.

where the boundaries are shown in the figure below.

The total simulation time is $T = 0.006$ s with a time step of size $\Delta t = 0.0001$ s. The final time T is chosen so that the pressure wave could barely reach the outflow section.

We present the computed velocity and pressure waves along the channel at time $t = 1.8$, 3.6, 5.4 ms in Figure 4.4.2. On the top, the arrows represent the velocity vectors $\mathbf{u}_{fh}$ and $\mathbf{u}_{ph}$ in the fluid and structure regions, while the color shows the magnitudes of these vectors. The bottom plots presents the fluid pressure p_{fh} and Darcy pressure p_{ph} in the corresponding regions. From the plots, we could clearly see a wave propagates from left to right. As the flow in the fluid region moves to the outflow region, some are penetrating into the structure region, causing relatively larger pressure along the wave. We also observe singularity of $|\mathbf{u}_{fh}|$ near the splitting point of the fluid region at $t = 5.4$ ms, which is typical for bifurcation geometry. In addition, the magnitudes of pressure match the order of that for inflow pressure, indicating the accuracy of our finite element method.

4.4.3 An industrial filter example

In this example, we study the flow of air through an industrial filter numerically, which is similar to the one that has been presented in [72]. We consider a two-dimensional rectangular channel with length 0.75 m and width 0.25 m, which in the bottom center is partially blocked by a rectangular porous medium of length 0.25 m and width 0.2 m. The parameters are set

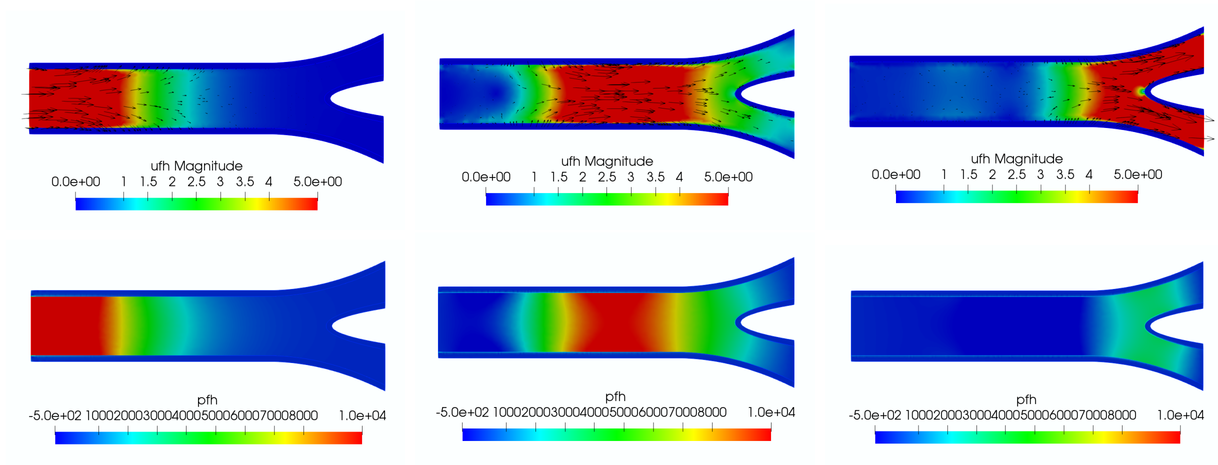


Figure 4.4.2: Computed solution at time t=1.8 ms, t=3.6 ms and t=5.4 ms.

Top: velocities $\mathbf{u}_{fh}$ and $\mathbf{u}_{ph}$ (arrows), $|\mathbf{u}_{fh}|$ and $|\mathbf{u}_{ph}|$ (color); bottom: pressures p_{fh} and p_{ph} (color).

as

$$\mu = 1.81 \times 10^{-8} \text{ kPa s}, \quad \rho_f = 1.225 \times 10^{-3} \text{ Mg/m}^3, \quad s_0 = 7 \times 10^{-2} \text{ kPa}^{-1},$$
$$\mathbf{K} = [0.505, \pm 0.495; \pm 0.495, 0.505] \times 10^{-6} \text{ m}^2, \quad \alpha_{\mathtt{BJS}} = 1.0, \quad \alpha = 1.0.$$

Notice that μ and ρ_f are chosen to feature the compressible fluid air, and the permeability tensor $\mathbf{K}$ in the porous medium is considered in two cases to study the influence of the anisotropy on the total mass fluxes based on rotation angle to be $-45°$ and $45°$ respectively.

The top and bottom of the domain are considered as rigid, impermeable walls with velocity $\mathbf{v} = \mathbf{0}$ (including the wall part below the porous box). Flow is driven by a pressure difference between the left and right boundary which is set to $\Delta p = 10^{-9}$ kPa. The body force terms and external source are set to be zero. The following boundary conditions are imposed,

$$\mathbf{T}_f \, \mathbf{n}_f = -p_{in} \, \mathbf{n}_f \quad \text{on} \quad \Gamma_f^{in}, \qquad \mathbf{T}_f \, \mathbf{n}_f = -p_{out} \, \mathbf{n}_f \quad \text{on} \quad \Gamma_f^{out},$$

$$\mathbf{u}_f = 0 \quad \text{on} \quad \Gamma_f^{top} \cup \Gamma_f^{bottom},$$

$$\mathbf{u}_s = \mathbf{0} \quad \text{and} \quad \mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{bottom}.$$

where

$$p_{in} = p_{ref} + 10^{-9} \text{ kPa}, \quad p_{out} = p_{ref} = 100 \text{ kPa}.$$

For the initial condition, we consider

$$p_{p,0} = 100 \text{ kPa}, \quad \boldsymbol{\sigma}_{p,0} = -\alpha_p \, p_{p,0} \, \mathbf{I}, \quad \mathbf{u}_{f,0} = \mathbf{0} \text{ m/s}.$$

The total simulation time is $T = 80$ s with $\Delta t = 1$ s.

We first consider the hard material in the poroelastic region with parameters

$$\lambda_p = 1 \times 10^5 \text{ kPa}, \quad \mu_p = 1 \times 10^4 \text{ kPa}.$$

We then consider the soft material with parameters

$$\lambda_p = 1 \times 10^3 \text{ kPa}, \quad \mu_p = 1 \times 10^2 \text{ kPa}.$$

We present the computed solutions all at the final time $T = 80$ s. The plots on the left are corresponding to rotation angle 45° and the plots are the right are for rotation angle −45°. Since the pressure variation is small relative to its value, for visualization purpose we plot its difference from the reference pressure, $p_f - 100$ and $p_p - 100$ in the corresponding region. We do the same thing for stress tensors, that is, we present $\boldsymbol{\sigma}_f + \alpha \, p_{ref} \mathbf{I}$ and $\boldsymbol{\sigma}_p + \alpha \, p_{ref} \mathbf{I}$ respectively.

From the velocity plots, we could see that most of the air passes the porous block through the constricted section above the block due to the flow resistance imposed by the porous medium, thus leading to relatively higher flow velocities there. The effect of anisotropy is clearly visible as the flow follows the inclined principal direction of the permeability tensor. In particular, the rotation angle affects the structure velocities while exhibiting no such difference on the displacement. Furthermore, changing the material parameters has a significant effect on most of the computed solutions, including velocities, stress tensors, displacement and structure velocities. We note that the material parameters make a difference not only on the magnitude of the displacement, but also on the flow outside of the structure. When the material of the obstacle is softer, we observe recirculation zone formed on the right side of the block. In addition, in the hard material, the structure velocity has larger magnitude on the left plot, while for the soft material, it is larger on the right plot. This is related to the larger vortex being formed behind the obstacle for the soft material with the rotation angle −45°. Thus we conclude that using a poroelastic model would contribute on capturing important flow characteristics compared with Navier-Stokes – Darcy model as in [72].

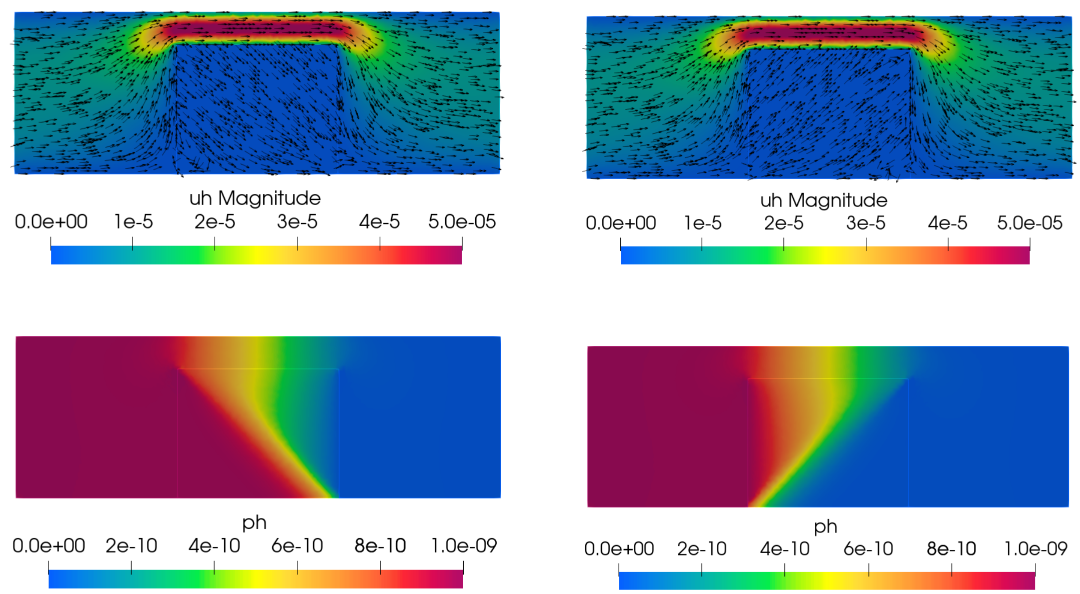


Figure 4.4.3: Computed velocities and pressures (left with angle 45 and right with angle -45) for the hard material at time T=80 s.

Top: velocities (arrows) and their magnitudes (color); bottom: pressures (color).

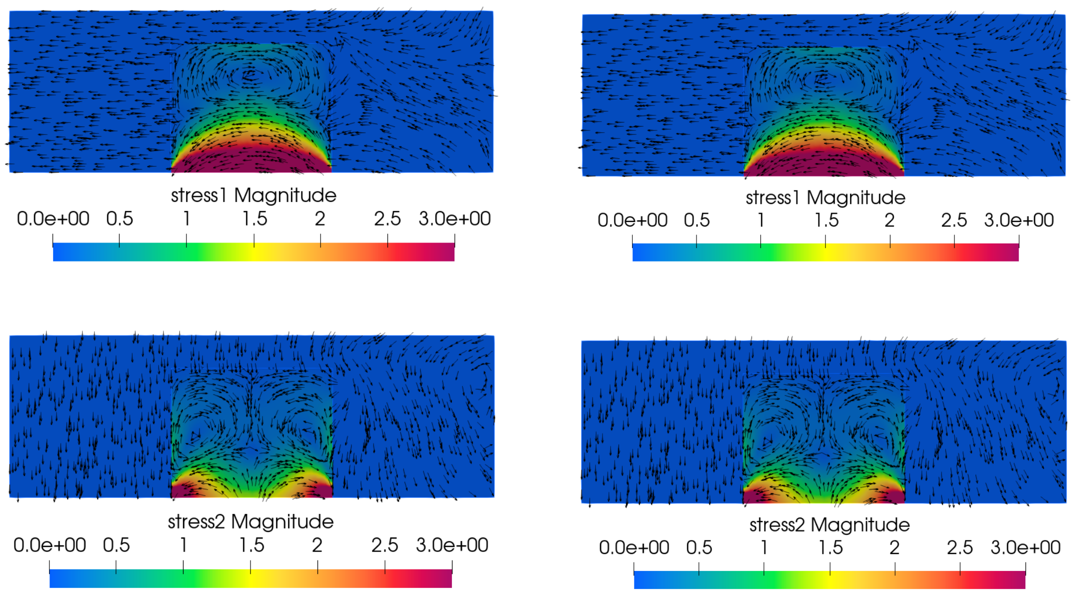


Figure 4.4.4: Computed stress tensors (left with angle 45 and right with angle -45) for the hard material at time T=80 s.

Top: first row of the stress tensors (arrows) and their magnitudes (color); bottom: second row of the stress tensors (arrows) and their magnitudes (color).

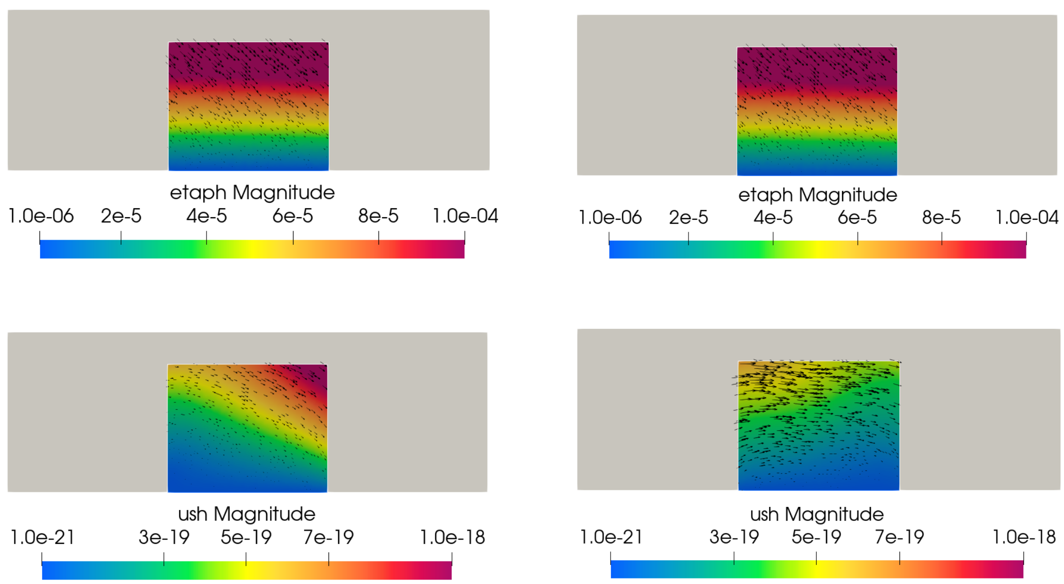


Figure 4.4.5: Computed displacement and structure velocities (left with angle 45 and right with angle -45) for the hard material at time T=80 s.

Top: displacement (arrows) and their magnitudes (color); bottom: structure velocities (arrows) and their magnitudes (color).

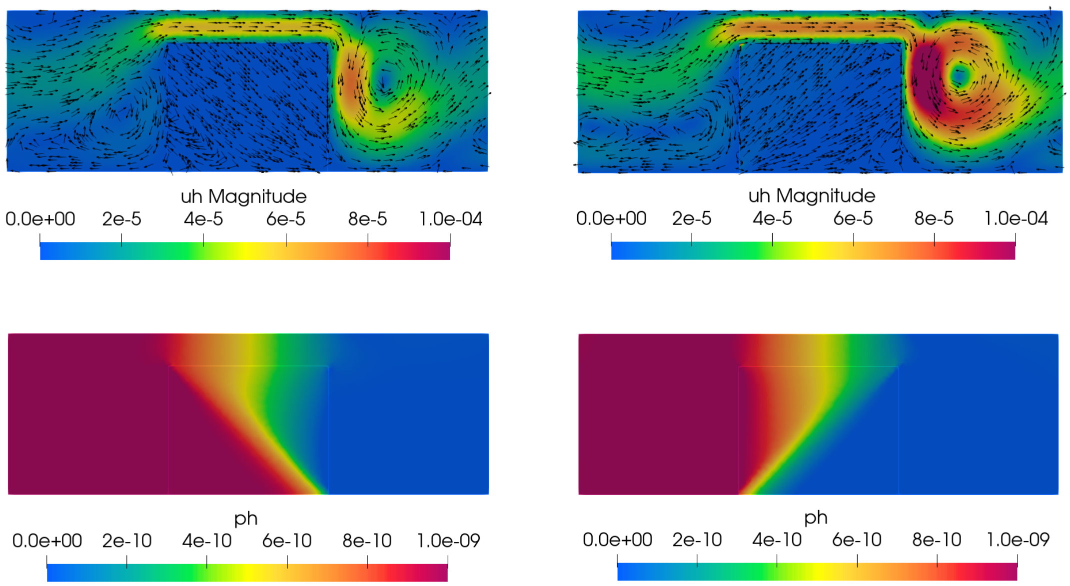


Figure 4.4.6: Computed velocities and pressures (left with angle 45 and right with angle -45) for the soft material at time T=80 s.

Top: velocities (arrows) and their magnitudes (color); bottom: pressures (color).

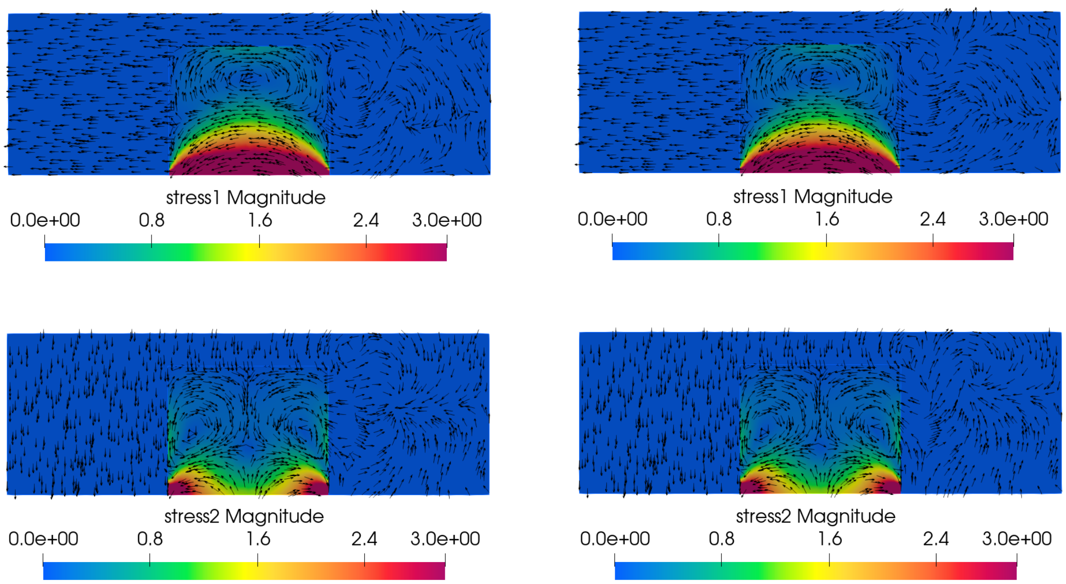


Figure 4.4.7: Computed stress tensors (left with angle 45 and right with angle -45) for the soft material at time T=80 s.

Top: first row of the stress tensors (arrows) and their magnitudes (color); bottom: second row of the stress tensors (arrows) and their magnitudes (color).

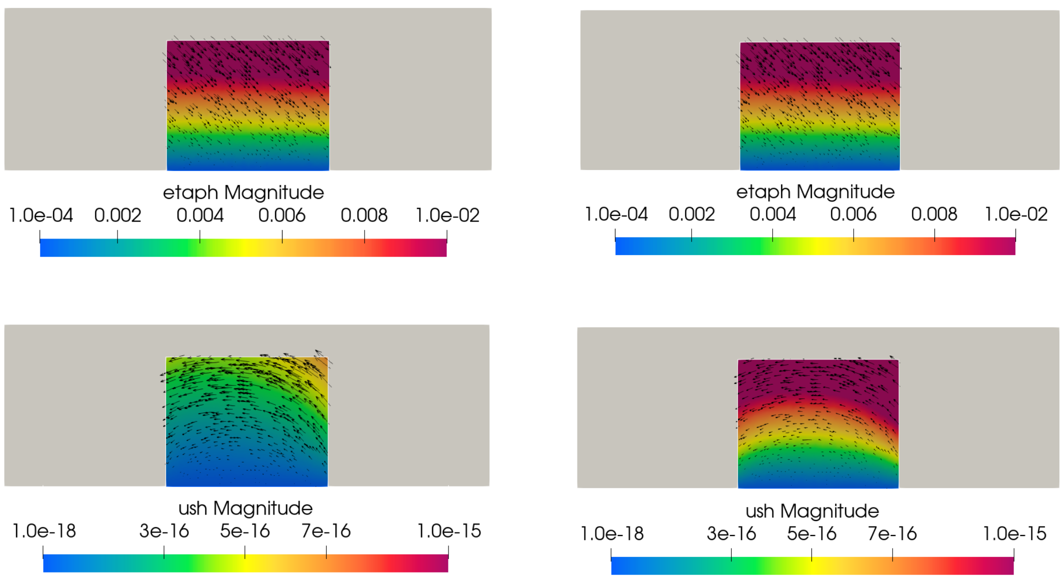


Figure 4.4.8: Computed displacement and structure velocities (left with angle 45 and right with angle -45) for the soft material at time T=80 s.

Top: displacement (arrows) and their magnitudes (color); bottom: structure velocities (arrows) and their magnitudes (color).

5.0 A cell-centered finite volume method for the Navier-Stokes – Biot model

5.1 The model problem and weak formulation

We consider the same domain and set up for terms as in Section 4.1. We assume that the flow in Ω_f is governed by the Navier–Stokes equations with constant density and viscosity, which are written in the following nonstandard pseudostress-velocity-pressure formulation:

$$\begin{aligned} \mathbf{T}_f \;&=\; -p_f\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_f) - \rho_f\,(\mathbf{u}_f \otimes \mathbf{u}_f), \quad \operatorname{div}(\mathbf{u}_f) \;=\; q_f \quad \text{in} \quad \Omega_f \times (0,T], \\ \rho_f \left(\frac{\partial\,\mathbf{u}_f}{\partial\,t} + (\nabla \mathbf{u}_f)\,\mathbf{u}_f \right) &- \mathbf{div}\big(-p_f\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_f)\big) \;=\; \mathbf{f}_f \quad \text{in} \quad \Omega_f \times (0,T], \end{aligned} \tag{5.1.1}$$

with boundary conditions $\mathbf{T}_f\mathbf{n}_f = \mathbf{0}$ on $\Gamma_f^{\mathrm{N}} \times (0,T]$, $\mathbf{u}_f = \mathbf{0}$ on $\Gamma_f^{\mathrm{D}} \times (0,T]$, where $\mathbf{T}_f$ is the nonlinear pseudostress tensor, $\mathbf{e}(\mathbf{u}_f) := \left(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^{\mathrm{t}}\right)/2$ stands for the deformation rate tensor, $\Gamma_f = \Gamma_f^{\mathrm{D}} \cup \Gamma_f^{\mathrm{N}}$, and $T > 0$ is the final time.

As in [31], we first observe that, due to tr $\mathbf{e}(\mathbf{u}_f) = \operatorname{div}(\mathbf{u}_f) = q_f$, there hold

$$\mathbf{div}(\mathbf{u}_f \otimes \mathbf{u}_f) \;=\; (\nabla \mathbf{u}_f)\,\mathbf{u}_f + q_f\,\mathbf{u}_f, \quad \operatorname{tr}(\mathbf{T}_f) \;=\; -\,n\,p_f + 2\,\mu\,q_f - \rho_f \operatorname{tr}(\mathbf{u}_f \otimes \mathbf{u}_f). \tag{5.1.2}$$

In particular, the pressure p_f can be written in terms of $\mathbf{u}_f$, $\mathbf{T}_f$ and q_f as

$$p_f \;=\; -\frac{1}{n}\left(\operatorname{tr}(\mathbf{T}_f) + \rho_f \operatorname{tr}(\mathbf{u}_f \otimes \mathbf{u}_f) - 2\,\mu\,q_f\right), \tag{5.1.3}$$

and hence, eliminating the pressure p_f, which can be recovered by (5.1.3), and employing the identities (5.1.2), problem (5.1.1) can be rewritten as

$$\begin{aligned} \mathbf{T}_f^{\mathrm{d}} \;&=\; 2\,\mu\,\mathbf{e}(\mathbf{u}_f) - \rho_f\,(\mathbf{u}_f \otimes \mathbf{u}_f)^{\mathrm{d}} - \frac{2\,\mu}{n}\,q_f\,\mathbf{I} \quad \text{in} \quad \Omega_f \times (0,T], \\ \rho_f\,\frac{\partial\,\mathbf{u}_f}{\partial\,t} - \rho_f\,q_f\,\mathbf{u}_f &- \mathbf{div}(\mathbf{T}_f) \;=\; \mathbf{f}_f \quad \text{in} \quad \Omega_f \times (0,T]. \end{aligned} \tag{5.1.4}$$

Next, in order to impose weakly the symmetry of $\mathbf{T}_f$, we introduce

$$\boldsymbol{\gamma}_f \;:=\; \frac{1}{2}\left(\nabla \mathbf{u}_f - (\nabla \mathbf{u}_f)^{\mathrm{t}}\right),$$

which represents the vorticity (or skew-symmetric part of the velocity gradient). Instead of (5.1.4), in the sequel we consider the problem with unknowns $\mathbf{T}_f, \boldsymbol{\gamma}_f$ and $\mathbf{u}_f$,

$$\begin{aligned} \frac{1}{2\,\mu}\,\mathbf{T}_f^{\mathrm{d}} \,&=\, \nabla\mathbf{u}_f - \boldsymbol{\gamma}_f - \frac{\rho_f}{2\,\mu}\,(\mathbf{u}_f\otimes\mathbf{u}_f)^{\mathrm{d}} - \frac{1}{n}\,q_f\,\mathbf{I} \quad \text{in} \quad \Omega_f\times(0,T], \\ \mathbf{T}_f \,&=\, \boldsymbol{\sigma}_f^{\mathrm{t}}, \quad \rho_f\,\frac{\partial\,\mathbf{u}_f}{\partial\,t} - \rho_f\,q_f\,\mathbf{u}_f - \mathbf{div}(\mathbf{T}_f) \,=\, \mathbf{f}_f \quad \text{in} \quad \Omega_f\times(0,T]. \end{aligned} \tag{5.1.5}$$

The Biot system is the same as the one in Section 4.1. We present them here for completeness.

$$-\mathrm{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p \quad \text{in} \quad \Omega_p\times(0,T], \quad \mu\,\mathbf{K}^{-1}\,\mathbf{u}_p + \nabla\,p_p = \mathbf{0} \quad \text{in} \quad \Omega_p\times(0,T], \tag{5.1.6a}$$

$$\frac{\partial}{\partial t}\big(s_0\,p_p + \alpha_p\,\mathrm{div}(\boldsymbol{\eta}_p)\big) + \mathrm{div}(\mathbf{u}_p) = q_p \quad \text{in} \quad \Omega_p\times(0,T], \tag{5.1.6b}$$

$$\mathbf{u}_p\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}\times(0,T], \quad p_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}}\times(0,T], \quad \boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \Gamma_p\times(0,T]. \tag{5.1.6c}$$

Next, we introduce the transmission conditions on the interface $\Gamma_{fp}\times(0,T]$ [4,10]:

$$\begin{aligned} &\mathbf{u}_f\cdot\mathbf{n}_f + \left(\frac{\partial\,\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot\mathbf{n}_p \,=\, 0, \quad \mathbf{T}_f\mathbf{n}_f + \boldsymbol{\sigma}_p\mathbf{n}_p \,=\, \mathbf{0}, \\ &(\mathbf{T}_f\mathbf{n}_f)\cdot\mathbf{n}_f \,=\, -\,p_p, \quad (\mathbf{T}_f\mathbf{n}_f)\cdot\mathbf{t}_{f,j} \,=\, -\,\mu\,\alpha_{\mathtt{BJS}}\,\sqrt{\mathbf{K}_j^{-1}}\left(\mathbf{u}_f - \frac{\partial\,\boldsymbol{\eta}_p}{\partial t}\right)\cdot\mathbf{t}_{f,j}, \end{aligned} \tag{5.1.7}$$

where $\mathbf{t}_{f,j}$, $1\le j\le n-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $\mathbf{K}_j = (\mathbf{K}\,\mathbf{t}_{f,j})\cdot\mathbf{t}_{f,j}$, and $\alpha_{\mathtt{BJS}}\ge 0$ is an experimentally determined friction coefficient. Finally, the above system of equations is complemented by the initial conditions $\mathbf{u}_f(\mathbf{x},0) = \mathbf{u}_{f,0}(\mathbf{x})$ in Ω_f and $p_p(\mathbf{x},0) = p_{p,0}(\mathbf{x})$ in Ω_p.

We then proceed analogously to [4, Section 3] (see also [50]) and derive a weak formulation of the coupled problem given by (5.1.5), (5.1.6), and (5.1.7). Similarly to [31], we employ suitable Banach spaces to deal with the nonlinear stress tensor and velocity of the Navier-Stokes equation, together with the subspace of skew-symmetric tensors of $\mathbb{L}^2(\Omega_f)$ for the vorticity:

$$\mathbb{X}_f \,:=\, \Big\{\mathbf{R}_f\in\mathbb{L}^2(\Omega_f) : \quad \mathbf{div}(\mathbf{R}_f)\in\mathbf{L}^{4/3}(\Omega_f) \quad \text{and} \quad \mathbf{R}_f\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}}\Big\},$$

$$\mathbf{V}_f \,:=\, \mathbf{L}^4(\Omega_f), \quad \mathbb{Q}_f \,:=\, \Big\{\boldsymbol{\chi}_f\in\mathbb{L}^2(\Omega_f) : \quad \boldsymbol{\chi}_f^{\mathrm{t}} = -\boldsymbol{\chi}_f\Big\}.$$

In turn, we introduce the structure velocity $\mathbf{u}_s := \partial_t \boldsymbol{\eta}_p \in \mathbf{V}_s$ in the Biot system, and take the Hilbert spaces:

$$\mathbb{X}_p := \mathbb{H}(\mathbf{div};\Omega_p), \quad \mathbf{V}_s := \mathbf{L}^2(\Omega_p), \quad \mathbb{Q}_p := \Big\{ \boldsymbol{\chi}_p \in \mathbb{L}^2(\Omega_p) : \quad \boldsymbol{\chi}_p^{\mathrm{t}} = -\boldsymbol{\chi}_p \Big\},$$

$$\mathbf{V}_p := \Big\{ \mathbf{v}_p \in \mathbf{H}(\mathrm{div};\Omega_p) : \quad \mathbf{v}_p \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}} \Big\}, \quad \mathrm{W}_p := \mathrm{L}^2(\Omega_p).$$

Finally, as in [4, 10, 50], we introduce three Lagrange multipliers

$$\boldsymbol{\varphi} := \mathbf{u}_f|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_f, \quad \boldsymbol{\theta} := \mathbf{u}_s|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_s, \quad \text{and} \quad \lambda := p_p|_{\Gamma_{fp}} \in \Lambda_p.$$

with the spaces of traces $\Lambda_p := \mathrm{H}^{1/2}(\Gamma_{fp})$, $\boldsymbol{\Lambda}_f := \mathbf{H}^{1/2}(\Gamma_{fp})$, and $\boldsymbol{\Lambda}_s := \mathbf{H}_{00}^{1/2}(\Gamma_{fp}) := \Big\{ v|_{\Gamma_{fp}} : v \in (\mathrm{H}^1(\Omega_p))^n, v = 0 \text{ on } \Gamma_p \Big\}$.

Then, similarly to [4, 10, 50], we obtain the following variational problem. Find $(\mathbf{T}_f, \mathbf{u}_f, \boldsymbol{\gamma}_f, \boldsymbol{\varphi}, \boldsymbol{\sigma}_p, \mathbf{u}_s, \boldsymbol{\gamma}_p, \boldsymbol{\theta}, \mathbf{u}_p, p_p, \lambda) : [0,T] \mapsto \mathbb{X}_f \times \mathbf{V}_f \times \mathbb{Q}_f \times \boldsymbol{\Lambda}_f \times \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \boldsymbol{\Lambda}_s \times \mathbf{V}_p \times \mathrm{W}_p \times \Lambda_p$ such that for all $(\mathbf{R}_f, \mathbf{v}_f, \boldsymbol{\chi}_f, \boldsymbol{\psi}, \boldsymbol{\tau}_p, \mathbf{v}_s, \boldsymbol{\chi}_p, \boldsymbol{\phi}, \mathbf{v}_p, w_p, \xi)$,

$$\begin{aligned}
&\frac{1}{2\,\mu} (\mathbf{T}_f^{\mathrm{d}}, \mathbf{R}_f^{\mathrm{d}})_{\Omega_f} - \langle \boldsymbol{\varphi}, \mathbf{R}_f \mathbf{n}_f \rangle_{\Gamma_{fp}} + (\mathbf{u}_f, \mathbf{div}\,\mathbf{R}_f)_{\Omega_f} \\
&\qquad + \frac{\rho_f}{2\,\mu} ((\mathbf{u}_f \otimes \mathbf{u}_f)^{\mathrm{d}}, \mathbf{R}_f)_{\Omega_f} + (\boldsymbol{\gamma}_f, \mathbf{R}_f)_{\Omega_f} = -\frac{1}{n} (q_f, \mathrm{tr}(\mathbf{R}_f))_{\Omega_f}, \\
&\rho_f (\partial_t \mathbf{u}_f, \mathbf{v}_f)_{\Omega_f} - \rho_f (q_f \mathbf{u}_f, \mathbf{v}_f)_{\Omega_f} - (\mathbf{div}\,\mathbf{T}_f, \mathbf{v}_f)_{\Omega_f} = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}, \\
&(\mathbf{T}_f, \boldsymbol{\chi}_f)_{\Omega_f} = 0, \\
&(\partial_t A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} - \langle \boldsymbol{\theta}, \boldsymbol{\tau}_p \mathbf{n}_p \rangle_{\Gamma_{fp}} + (\mathbf{u}_s, \mathbf{div}\,\boldsymbol{\tau}_p)_{\Omega_p} + (\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p)_{\Omega_p} = 0, \\
&(\mathbf{div}\,\boldsymbol{\sigma}_p, \mathbf{v}_s)_{\Omega_p} = (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}, \\
&(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p)_{\Omega_p} = 0, \\
&\mu\, (\mathbf{K}^{-1} \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p} - (p_p, \mathrm{div}\,\mathbf{v}_p)_{\Omega_p} + \langle \lambda, \mathbf{v}_p \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}} = 0, \\
&(s_0\, \partial_t p_p, w_p)_{\Omega_p} + \alpha_p (\partial_t A(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I}), w_p \mathbf{I})_{\Omega_p} + (w_p, \mathrm{div}\,\mathbf{u}_p)_{\Omega_p} = (q_p, w_p)_{\Omega_p}, \\
&\langle \boldsymbol{\varphi} \cdot \mathbf{n}_f + (\boldsymbol{\theta} + \mathbf{u}_p) \cdot \mathbf{n}_p, \xi \rangle_{\Gamma_{fp}} = 0, \\
&\langle \boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\phi} \rangle_{\Gamma_{fp}} - \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} (\boldsymbol{\varphi} - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \langle \lambda, \boldsymbol{\phi} \cdot \mathbf{n}_p, \rangle_{\Gamma_{fp}} = 0,
\end{aligned}$$

$$\langle \mathbf{T}_f \mathbf{n}_f, \boldsymbol{\psi} \rangle_{\Gamma_{fp}} + \mu \, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} \, (\boldsymbol{\varphi} - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\psi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \langle \lambda . \boldsymbol{\psi} \cdot \mathbf{n}_f \rangle_{\Gamma_{fp}} = 0. \quad (5.1.8)$$

For the well posedness of the problem, compatible initial data is needed for all variables. It can be obtained from $\mathbf{u}_{f,0}$ and $p_{p,0}$ using that the equations without time derivatives hold at $t = 0$, see [4, 35].

5.2 Numerical methods

We employ a mixed finite element approximation of the weak formulation (5.1.8). Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be affine finite element partitions of Ω_f and Ω_p, respectively, which may be non-matching along the interface Γ_{fp}. For the spatial discretization, we consider the conforming finite element spaces $\mathbb{X}_{fh} \times \mathbf{V}_{fh} \times \mathbb{Q}_{fh} = \mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1$, $\mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} = \mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1$, and $\mathbf{V}_{ph} \times \mathrm{W}_{ph} = \mathbf{BDM}_1 - \mathrm{P}_0$, where $\mathbf{BDM}_1$ denotes the first order Brezzi-Douglas-Marini space [22]. For the Lagrange multiplier spaces on Γ_{fp} we take $\boldsymbol{\Lambda}_{fh} = \mathbb{X}_{fh} \, \mathbf{n}_f$, $\boldsymbol{\Lambda}_{sh} = \mathbb{X}_{ph} \, \mathbf{n}_p$, and $\Lambda_{ph} = \mathbf{V}_{ph} \cdot \mathbf{n}_p$, resulting in $\boldsymbol{\Lambda}_{fh} \times \boldsymbol{\Lambda}_{sh} \times \Lambda_{ph} = \mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$. For the time discretization we employ the backward Euler method. The straightforward application of the MFE method results, on each time step, in a large 11-field saddle point problem. In order to reduce the computational cost, we employ the vertex quadrature rule for some of the terms in (5.1.8), which allows for local elimination of certain variables. For a pair of tensor or vector valued functions (φ, ψ) and a linear operator L, define the quadrature rule

$$(L(\varphi), \psi)_{Q,\Omega_\star} := \sum_{E \in \mathcal{T}_h^\star} (L(\varphi), \psi)_{Q,E} = \sum_{E \in \mathcal{T}_h^\star} \frac{|E|}{s} \sum_{i=1}^{s} L(\varphi(\mathbf{r}_i)) : \psi(\mathbf{r}_i),$$

where $\star \in \{f, p\}$, $s = 3$ on triangles, $s = 4$ on tetrahedra or rectangles, and $\mathbf{r}_i$ are the vertices of E. The quadrature rule is applied to the terms

$$(\mathbf{T}_f^{\mathrm{d}}, \mathbf{R}_f^{\mathrm{d}})_{\Omega_f}, \quad (\boldsymbol{\gamma}_f, \mathbf{R}_f)_{\Omega_f}, \quad (\mathbf{T}_f, \boldsymbol{\chi}_f)_{\Omega_f}, \quad (\partial_t \, A(\boldsymbol{\sigma}_p + \alpha_p \, p_p \, \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p w_p \, \mathbf{I})_{\Omega_p},$$
$$(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p)_{\Omega_p}, \quad (\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p)_{\Omega_p}, \quad (\mathbf{K}^{-1} \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}.$$

Since the $\mathbf{BDM}_1$ degrees of freedom on each edge of face can be associated with the vertices, the quadrature rule results in block-diagonal stress and Darcy velocity matrices with one block per vertex. Therefore $\mathbf{T}_f$, $\boldsymbol{\sigma}_p$, and $\mathbf{u}_p$ can be easily eliminated. The resulting matrices for the vorticity γ_f and the rotation γ_p are also block-diagonal, due the quadrature rule and the vertex degrees of freedom of these variables. They can also be eliminated, resulting in a cell-centered positive definite system for $\mathbf{u}_f$, $\mathbf{u}_s$, and p_p, coupled through the Lagrange multipliers $\boldsymbol{\varphi}$, $\boldsymbol{\theta}$, and λ. After solving this system, the rest of the variables are recovered from their elimination expressions. We refer to [35] for further details. The numerical method for the Stokes-Biot model is analyzed in [35], where first order convergence for all variables in their natural norms is shown. The analysis of the method presented in this thesis for the nonlinear Navier-Stokes/Biot model will be developed in future work.

5.3 Numerical results

In this section we study numerically the convergence in space, using unstructured triangular grids. The total simulation time is $T = 0.01\,\mathrm{s}$ and the time step is $\Delta\, t = 10^{-3}\,\mathrm{s}$, which is sufficiently small, so that the time discretization error does not affect the convergence rates. The domain is $\Omega = \Omega_f \cup \Gamma_{fp} \cup \Omega_p$, where $\Omega_f = (0,1)\times(0,1), \Gamma_{fp} = (0,1)\times\{0\}$, and $\Omega_p = (0,1)\times(-1,0)$. We take $\Gamma_f^D = (0,1)\times\{1\}$ and $\Gamma_p^D = (0,1)\times\{-1\}$. The solution in the Navier-Stokes region is

$$\mathbf{u}_f = \pi\,\cos(\pi\,t)\begin{pmatrix} -3x+\cos(y) \\ y+1 \end{pmatrix}, \quad p_f = \exp(t)\,\sin(\pi\,x)\,\cos\left(\frac{\pi\,y}{2}\right) + 2\,\pi\,\cos(\pi\,t).$$

The Biot solution is chosen accordingly to satisfy the interface conditions (5.1.7):

$$p_p = \exp(t)\,\sin(\pi\,x)\,\cos\left(\frac{\pi\,y}{2}\right), \quad \mathbf{u}_p = -\frac{1}{\mu}\,\mathbf{K}\,\nabla p_p, \quad \boldsymbol{\eta}_p = \sin(\pi\,t)\begin{pmatrix} -3x+\cos(y) \\ y+1 \end{pmatrix}.$$

We run a sequence of mesh refinements with non-matching grids along Γ_{fp}. The results are reported on Table 5.3.1. We note that the displacement at t_n is recovered by the formula $\boldsymbol{\eta}_p^n = \Delta t\,\mathbf{u}_s^n + \boldsymbol{\eta}_p^{n-1}$. As expected, we observe at least first order convergence for all subdomain variables in their natural norms. The Lagrange multiplier variables, which are approximated in $\mathbf{P}_1^{\mathrm{dc}}-\mathbf{P}_1^{\mathrm{dc}}-\mathrm{P}_1^{\mathrm{dc}}$, exhibit second order convergence in the L^2-norm on Γ_{fp}, which is consistent with the order of approximation.

h_f	$\|\mathbf{e}_{\mathbf{T}_f}\|_{\ell^2(0,T;\mathbb{X}_f)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_f}\|_{\ell^2(0,T;\mathbf{V}_f)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_f}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega_f))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\gamma}_f}\|_{\ell^2(0,T;\mathbb{Q}_f)}$ error	rate	$\|\mathbf{e}_{p_f}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega_f))}$ error	rate
0.1964	5.1E-01	–	3.4E-02	–	2.7E-01	–	3.2E-02	–	1.7E-01	–
0.0997	2.4E-01	1.1136	1.7E-02	0.9965	1.4E-01	1.0044	1.0E-02	1.6752	8.2E-02	1.0411
0.0487	1.2E-01	1.0327	8.5E-03	0.9978	6.8E-02	0.9943	4.2E-03	1.2504	3.9E-02	1.0249
0.0250	5.6E-02	1.0665	4.2E-03	1.0420	3.4E-02	1.0436	1.5E-03	1.4745	2.0E-02	1.0111
0.0136	2.8E-02	1.1521	2.1E-03	1.1458	1.7E-02	1.1449	6.5E-04	1.4287	1.0E-02	1.1489
0.0072	1.4E-02	1.0895	1.0E-03	1.1040	8.4E-03	1.0971	2.8E-04	1.3025	4.8E-03	1.1392

h_p	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{X}_p)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^2(0,T;\mathbf{V}_s)}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\gamma}_p}\|_{\ell^2(0,T;\mathbb{Q}_p)}$ error	rate	$\|\mathbf{e}_{\mathbf{u}_p}\|_{\ell^2(0,T;\mathbf{V}_p)}$ error	rate	$\|\mathbf{e}_{p_p}\|_{\ell^\infty(0,T;\mathrm{W}_p)}$ error	rate
0.2828	2.7E-01	–	4.3E-02	–	3.6E-02	–	1.0E-01	–	7.5E-02	–
0.1646	1.4E-01	1.2737	2.2E-02	1.2289	9.9E-03	2.3678	5.2E-02	1.2576	3.8E-02	1.2486
0.0779	6.7E-02	0.9651	1.1E-02	0.9623	2.3E-03	1.9774	2.5E-02	1.0003	1.9E-02	0.9335
0.0434	3.4E-02	1.1690	5.4E-03	1.1865	6.2E-04	2.1958	1.2E-02	1.2373	9.4E-03	1.2151
0.0227	1.7E-02	1.0635	2.7E-03	1.0668	2.0E-04	1.7255	5.9E-03	1.0816	4.7E-03	1.0659
0.0124	8.4E-03	1.1462	1.4E-03	1.1456	8.2E-05	1.5042	2.9E-03	1.1486	2.4E-03	1.1429

$\|\mathbf{e}_{\boldsymbol{\eta}_p}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega_p))}$ error	rate	h_{tf}	$\|\mathbf{e}_{\boldsymbol{\varphi}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	h_{tp}	$\|\mathbf{e}_{\boldsymbol{\theta}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	$\|\mathbf{e}_{\lambda}\|_{\ell^2(0,T;\mathrm{L}^2(\Gamma_{fp}))}$ error	rate	iter
2.7E-04	–	1/8	8.4E-03	–	1/5	1.0E-02	–	1.2E-03	–	4
1.4E-04	1.2275	1/16	2.1E-03	2.0195	1/10	3.3E-03	1.6431	3.2E-04	1.8656	4
6.7E-05	0.9623	1/32	4.7E-04	2.1340	1/20	6.1E-04	2.4481	7.7E-05	2.0334	4
3.4E-05	1.1865	1/64	1.2E-04	1.9659	1/40	1.7E-04	1.8741	1.9E-05	2.0006	4
1.7E-05	1.0668	1/128	2.8E-05	2.1140	1/80	3.9E-05	2.0897	4.9E-06	1.9817	4
8.4E-06	1.1456	1/256	7.7E-06	1.8636	1/160	9.0E-06	2.1194	1.2E-06	2.0796	4

Table 5.3.1: EXAMPLE 1, Mesh sizes, errors, rates of convergences, and average number of Newton iterations.

6.0 Conclusions

In this thesis we have studied mixed finite element methods for the coupled Stokes or Navier-Stokes – Biot problems arising in the interaction between free fluid flow and flow in deformable poroelastic medium, motivated by a wide range of applications. We have developed various formulations and conducted theoretical analysis such as well-posedness, stability and error analysis for the formulations. We also proposed finite element methods for their numerical solutions focusing on accuracy, physical fidelity, and computational efficiency. We finally implemented the methods using finite element packages and conducted a series of numerical experiments to validate our convergence results and benchmark the performance of the methods in applications to geosciences and bioengineering.

First, we developed and analyzed a new mixed elasticity formulation for the Stokes–Biot problem, as well as its mixed finite element approximation. We consider a five-field Biot formulation based on a weakly symmetric stress–displacement–rotation elasticity formulation and a mixed velocity–pressure Darcy formulation. The classical velocity–pressure formulation is used for the Stokes system. Suitable Lagrange multipliers are introduced to enforce weakly the balance of force, slip with friction, and continuity of normal flux on the interface. The advantages of the resulting mixed finite element method, compared to previous works, include local momentum conservation, accurate stress with continuous normal component, and robustness with respect to the physical parameters. In particular, the numerical results indicate locking-free and oscillation-free behavior in the regimes of small storativity and permeability, as well as for almost incompressible media.

Second, we presented and analyzed the first, to the best of our knowledge, fully dual mixed formulation of the quasi-static Stokes-Biot model, and its mixed finite element approximation, using a weakly symmetric stress-velocity-vorticity Stokes formulation, a velocity-pressure Darcy formulation, and a weakly symmetric stress-displacement-rotation elasticity formulation. Essential-type interface conditions are imposed via suitable Lagrange multipliers. The numerical method features accurate stresses and Darcy velocity with local mass and momentum conservation. Furthermore, a new multipoint stress-flux mixed finite element method is developed that allows for local elimination of the Darcy velocity, the fluid and poroelastic stresses, the vorticity, and the rotation, resulting in a reduced positive definite cell-centered pressure-velocities-traces system. The theoretical results are complemented by a series of numerical experiments that illustrate the convergence rates for all variables in their natural norms, as well as the ability of the method to simulate physically realistic problems motivated by applications to coupled surface-subsurface flows and flows in fractured poroelastic media with parameter values in locking regimes.

We then introduce and analyze an augmented fully-mixed finite element method for the quasi-static Navier-Stokes – Biot model, together with its mixed finite element approximation. We adopt a pesudostress-velocity formulation for the Navier-Stokes equations and a five-field Biot formulation, with interface conditions being imposed through suitable Lagrange multipliers. We further augment the resulting formulation by redundant Garlerkin-type types to relax the hypotheses of the corresponding discrete subspaces. The numerical experiments indicates the ability of our method to handle computationally challenging problems involving fast flows of scientific and engineering interests such as blood flow and industrial filters.

Finally, we derived a fully mixed formulation for the Navier-Stokes – Biot model. Focusing on the efficiency of the solution of this problem, we proposed a cell-centered finite volume method based on the multipoint stress-flux mixed finite element method for the Stokes-Biot model we derived earlier. We implemented the method and verified numerically its convergence in space. The theoretical analysis of the method will be developed in future work.

Another direction for the future work is on coupling FPSI with transport, as these are fundamental processes arising in many applications such as tracking and cleaning up groundwater contaminants, modeling drug delivery, and transport of low-density lipoprotein. In particular, a time-dependent Navier-Stokes – Biot system coupled with transport model, to the best of our knowledge, has not been studied in the literature. It is worth studying the model as it is more suitable, for example, to describe blood flow in an aorta.

Appendix FREEFEM++ CODE

We first present FreeFem++ code for convergence test with matching grids with the mixed elasticity formulation.

```
load "Element_Mixte"
load "iovtk"
load "medit"
load "MUMPS"
load "Element_P3"

// MACRO:
macro div(ax,ay) (dx(ax)+dy(ay))  //
macro cdot(ax,ay,bx,by) (ax*bx+ay*by)  // dot product of two given vectors
macro tgx(ax,ay) (ax-cdot(ax,ay,N.x,N.y)*N.x) //
macro tgy(ax,ay) (ay-cdot(ax,ay,N.x,N.y)*N.y) //x and y coordinate of tangent
    component
// tangential component is computed by the formula tang(v)=v-(v dot n)n;
// where (v dot n)n is the normal component of v

// TIME:
real T=0.01;          //total time T=0.01;
real delt=0.001;      //delta t=0.001;
real t=0;             //initialize t
func NN=T/delt;       //number of time interval
int pr=1;    // for vtk. files


// Flags:
bool converg=1; // true for convergence test
bool plotflag=false;   // true for making .vtk files

int cm,cn,cl;
if(converg){
    cm=128;
    cl=8;
} else{
    cm=24;
    cl=cm;
    }

int number = log(real(cm/cl))/log(2.0) + 1;
cout << "Number of steps: " << number << endl;

int nMeshes = number;
int count=0;

real[int] error1(nMeshes);  error1 = 0; // L inf H1 for u_f fluid velovity
real[int] error2(nMeshes);  error2 = 0; // L2 H1 for u_f fluid velocity
real[int] error3(nMeshes);  error3 = 0; // L2 L2 for u_p darcy velocity
```

```
real[int] error4(nMeshes);  error4 = 0; // L2 L2 for u_s structure velocity
real[int] error5(nMeshes);  error5 = 0; // L2 Hdiv sigma_p for elasticity
real[int] error6(nMeshes);  error6 = 0; // L2 L2 for p_f fluid pressure
real[int] error7(nMeshes);  error7 = 0; // L2 L2 for p_p darcy pressure
real[int] error8(nMeshes);  error8 = 0; // L2 L2 for sigma_pdiv elasticity
real[int] error9(nMeshes);  error9 = 0; // L2 L2 gamma_p
real[int] error10(nMeshes);  error10 = 0; // L inf L2 for sigma_p
real[int] error11(nMeshes);  error11 = 0; // L2 L2 for lambda
real[int] error12(nMeshes);  error12 = 0; // L2 L2 for theta
real[int] error13(nMeshes);  error13 = 0; // L2 L2 for div up
real[int] errorq1(nMeshes);  errorq1 = 0;
real[int] errorq2(nMeshes);  errorq2 = 0;

real[int] error1tmp(NN); error1tmp=0;
real[int] abs2(nMeshes);  abs2 = 0;
real[int] abs3(nMeshes);  abs3 = 0;
real[int] abs4(nMeshes);  abs4 = 0;
real[int] abs5(nMeshes);  abs5 = 0;
real[int] abs6(nMeshes);  abs6 = 0;
real[int] abs7(nMeshes);  abs7 = 0;
real[int] abs8(nMeshes);  abs8 = 0;
real[int] abs9(nMeshes);  abs9 = 0;
real[int] error10tmp(NN); error10tmp=0;
real[int] abs11(nMeshes);  abs11 = 0;
real[int] abs12(nMeshes);  abs12 = 0;
real[int] abs13(nMeshes);  abs13 = 0;
real[int] absq1(nMeshes);  absq1 = 0;
real[int] absq2(nMeshes);  absq2 = 0;


// convergence test loop:
for(int cn=cl; cn<=cm; cn*=2){

t=0;

cout<<"n is "<<cn<<endl;

mesh ThF = square(cn,cn, flags=3);
mesh ThS1 = square(cn,cn, flags=3); // the structure region
ThS1 = movemesh(ThS1, [x,y-1]);
mesh ThL = emptymesh(ThS1);

// FINITE ELEMENT SPACES:
// fluid:
fespace VFh(ThF,[P1b, P1b, P1]); // fluid velocity (x, y) and pressure
// structure:
fespace VM1h(ThS1,[RT0, P0]);  // poroelastic velocity (x, y) and pressure
// displacement
fespace VS1h(ThS1,[P0,P0]);    // eta (x, y) --> structure velocity (x, y)
// elasticity
fespace VE1h(ThS1,[BDM1,BDM1]);  // elasticity tensor
// lagrange (rotation operator)
fespace LL1h(ThS1, P1);   // lagrange: rotation operator
fespace LL2h(ThL, [P1, P1, P0]);  // lagrange: trace
```

```
// VARIABLES:
VFh [uFx,uFy,pF], [vFx,vFy,wF], [uFoldx,uFoldy,pFold];
VM1h [uP1x,uP1y,pP1], [vP1x,vP1y,wP1], [uP1oldx,uP1oldy,pP1old];
VS1h [uS1x,uS1y],[vS1x,vS1y],[uS1oldx,uS1oldy];
VE1h [sigmap1xx, sigmap1xy, sigmap1yx, sigmap1yy], [taup1xx, taup1xy, taup1yx,
    taup1yy],
        [sigmap1oldxx, sigmap1oldxy, sigmap1oldyx, sigmap1oldyy];
LL1h gamma, theta, gammaold;
LL2h [phix,phiy,lambda],[psix,psiy,mu],[phioldx,phioldy,lambdaold];

// DATA
func lambdaS = 1.0; //lame coefficient lambda_p
func muS = 1.0; // lame coefficient miu_p
// alpha = inv (K) = 1 in the solution
real alpha = 1.0; // Biot-Willis constant alpha

real s0=1.0;  // mass storativity
real muF = 1.0; // fluid viscosity mu
real Kxx=1.0;
real Kyy=1.0; // symmetric and uniformly positive definite rock permeability
   tensor
real kappaxx=muF/Kxx;
real kappayy=muF/Kyy; // muK^(-1)

real alfabjs=1.0;  //BJS coefficient, experimentally determined friction
   coefficient
real bjs=muF*alfabjs*sqrt(2)/sqrt(Kxx+Kyy);


// TRUE SOLUTION
func ufx0 = pi*cos(pi*t)*(-3*x+cos(y));
func ufy0 = pi*cos(pi*t)*(y+1);  // fluid velocity

func dxufx0 = pi*cos(pi*t)*(-3);
func dyufx0 = pi*cos(pi*t)*(-sin(y));
func dxufy0 = 0;
func dyufy0 = pi*cos(pi*t);

func pf0 = exp(t)*sin(pi*x)*cos(pi*y/2) + 2*pi*cos(pi*t);    // fluid pressure

func upx0 = -exp(t)*pi*cos(pi*x)*cos(pi*y/2);
func upy0 = exp(t)*pi/2*sin(pi*x)*sin(pi*y/2);    // poroelastic velocity

func dxupx0 = exp(t)*pi^2*sin(pi*x)*cos(pi*y/2);
func dyupy0 = (1./4)*exp(t)*pi^2*sin(pi*x)*cos(pi*y/2);

func updiv0 = (5./4)*pi^2*sin(pi*x)*cos((pi/2.)*y);

func pp0 = exp(t)*sin(pi*x)*cos(pi*y/2);    // poroelastic pressure

func eta0x = sin(pi*t)*(-3*x+cos(y));
func eta0y = sin(pi*t)*(y+1);     // displacement
```

```
func uS0x = pi*cos(pi*t)*(-3*x+cos(y));
func uS0y = pi*cos(pi*t)*(y+1);

func sigmap0xx = -8*sin(pi*t)-exp(t)*sin(pi*x)*cos((pi*y)/2);
func sigmap0xy = -sin(pi*t)*sin(y);
func sigmap0yx = -sin(pi*t)*sin(y);
func sigmap0yy = -exp(t)*sin(pi*x)*cos((pi*y)/2);     // poroelastic stress
    tensor

func gamma0=-0.5*sin(pi*t)*sin(y); //rotation operator or lagrange multiplier

func phi0x = pi*cos(pi*t)*(-3*x+cos(y));
func phi0y = pi*cos(pi*t)*(y+1);  // lagrange multiplier for u_s
func lambda0 = exp(t)*sin(pi*x)*cos(pi*y/2);  // lagrange multiplier for p_p

// solve right hand side
func ffx = pi*exp(t)*cos(pi*x)*cos((pi*y)/2) + pi*cos(pi*t)*cos(y);
func ffy = -(pi/2)*exp(t)*sin(pi*x)*sin((pi*y)/2);

func qf = -2*pi*cos(pi*t);

func fpx = sin(pi*t)*cos(y) + pi*exp(t)*cos(pi*x)*cos((pi*y)/2);
func fpy = -(pi*exp(t)*sin(pi*x)*sin((pi*y)/2))/2;

func qp = exp(t)*cos((pi*y)/2)*sin(pi*x) - 2*pi*cos(pi*t) + (5*pi^2*exp(t)*cos
    ((pi*y)/2)*sin(pi*x))/4;

///////////////////////////////////
//Matrix formulation
///////////////////////////////////
/***************************************/
varf AFsum([uFx,uFy,pF],[vFx,vFy,wF],init=1)=
int2d(ThF)( 2.0*muF*( dx(uFx)*dx(vFx) + dy(uFy)*dy(vFy) ) )
+int2d(ThF)( muF*( dy(uFx)+dx(uFy) )*( dy(vFx)+dx(vFy) ) ) + int2d(ThF)(1.e
    -8*pF*wF)+ on(2,3,4,uFx=ufx0, uFy=ufy0);
matrix AF=AFsum(VFh,VFh);

varf BPFTsum([uFx,uFy,pF],[vFx,vFy,wF],init=1)=
-int2d(ThF)(pF*div(vFx,vFy));
matrix BPFT=BPFTsum(VFh,VFh);

varf BGAM1sum([phix,phiy,lambda],[vFx,vFy,wF],init=1)=
-int1d(ThL,3)(lambda*cdot(vFx,vFy,N.x,N.y));
matrix BGAM1=BGAM1sum(LL2h,VFh);

varf ABJS1sum([uFx,uFy,pF],[vFx,vFy,wF],init=1)=
int1d(ThF,1)(bjs*cdot(tgx(uFx,uFy),tgy(uFx,uFy),tgx(vFx,vFy),tgy(vFx,vFy)));
matrix ABJS1=ABJS1sum(VFh,VFh);

varf ABJS2sum([phix,phiy,lambda],[vFx,vFy,wF],init=1)=
-int1d(ThF,1)(bjs*cdot(tgx(phix,phiy),tgy(phix,phiy),tgx(vFx,vFy),tgy(vFx,vFy)
    ));
matrix ABJS2=ABJS2sum(LL2h,VFh);
```

```
/*****************************************/
varf BPFsum([uFx,uFy,pF],[vFx,vFy,wF],init=1)=
int2d(ThF)(wF*div(uFx,uFy));
matrix BPF=BPFsum(VFh,VFh);

/*****************************************/
varf BESTsum([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[vS1x,vS1y],init=1)=
-int2d(ThS1)(cdot(div(sigmap1xx,sigmap1xy),div(sigmap1yx,sigmap1yy),vS1x,vS1y)
    );
matrix BEST=BESTsum(VE1h,VS1h);

/*****************************************/
varf AQ1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)(cdot(kappaxx*uP1x,kappayy*uP1y,vP1x,vP1y)) + int2d(ThS1)(1.e-8*pP1
    *wP1);
matrix AQ1=AQ1sum(VM1h,VM1h);

varf BPQT1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1)=
-int2d(ThS1)(1*pP1*div(vP1x,vP1y));
matrix BPQT1=BPQT1sum(VM1h,VM1h);

varf BGAM2sum([phix,phiy,lambda],[vP1x,vP1y,wP1],init=1)=
int1d(ThL,3)(lambda*cdot(vP1x,vP1y,N.x,N.y));
matrix BGAM2=BGAM2sum(LL2h,VM1h);


/*****************************************/
varf MASSP1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)((s0/delt)*(wP1*pP1));
matrix MASSP1=MASSP1sum(VM1h,VM1h);

varf AAEPsum([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[vP1x,vP1y,wP1],init=1)
    =
int2d(ThS1)((alpha/(2*muS+2*lambdaS))*(1/delt)*(sigmap1xx+sigmap1yy)*wP1);
matrix AAEP=AAEPsum(VE1h,VM1h);

varf APP1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)((alpha^2/(muS+lambdaS))*(1/delt)*pP1*wP1);
matrix APP1=APP1sum(VM1h,VM1h);

varf BPQ1sum([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)(wP1*div(uP1x,uP1y));
matrix BPQ1=BPQ1sum(VM1h,VM1h);

/*****************************************/
varf AEsum([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[taup1xx,taup1xy,taup1yx,
    taup1yy],init=1)=
int2d(ThS1)((1.0/(2*muS))*(1/delt)*((sigmap1xx-(lambdaS/(2*muS+2*lambdaS))*(
    sigmap1xx+sigmap1yy))*taup1xx
                                +sigmap1xy*taup1xy
                                +sigmap1yx*taup1yx
                                +(sigmap1yy-(lambdaS/(2*muS+2*lambdaS))*(sigmap1xx
                                    +sigmap1yy))*taup1yy));
matrix AE=AEsum(VE1h,VE1h);
```

```
varf AAEPTsum([uP1x,uP1y,pP1],[taup1xx,taup1xy,taup1yx,taup1yy],init=1)=
int2d(ThS1)((alpha/(2*muS+2*lambdaS))*(1.0/delt)*pP1*(taup1xx+taup1yy));
matrix AAEPT=AAEPTsum(VM1h,VE1h);

varf ALEsum([gamma],[taup1xx,taup1xy,taup1yx,taup1yy],init=1)=
int2d(ThS1)((taup1xy-taup1yx)*gamma*(1.0/delt));
matrix ALE=ALEsum(LL1h,VE1h);

varf BESsum([uS1x,uS1y],[taup1xx,taup1xy,taup1yx,taup1yy],init=1)=
int2d(ThS1)(cdot(div(taup1xx,taup1xy),div(taup1yx,taup1yy),uS1x,uS1y));
matrix BES=BESsum(VS1h,VE1h);

varf BLAGsum([phix,phiy,lambda],[taup1xx,taup1xy,taup1yx,taup1yy],init=1)=
-int1d(ThL,3)(cdot(phix,phiy,(taup1xx*N.x+taup1xy*N.y),(taup1yx*N.x+taup1yy*N.
    y)));
matrix BLAG=BLAGsum(LL2h,VE1h);


/****************************************/
varf ALETsum([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[theta],init=1)=
-int2d(ThS1)((sigmap1xy-sigmap1yx)*theta);
matrix ALET=ALETsum(VE1h,LL1h);

/****************************************/
varf BGAM1Tsum([uFx,uFy,pF],[psix,psiy,mu],init=1)=
int1d(ThL,3)(mu*cdot(uFx,uFy,N.x,N.y));
matrix BGAM1T=BGAM1Tsum(VFh,LL2h);

varf BGAM3Tsum([phix,phiy,lambda],[psix,psiy,mu],init=1)=
-int1d(ThL,3)(mu*cdot(phix,phiy,N.x,N.y));
matrix BGAM3T=BGAM3Tsum(LL2h,LL2h);

varf BGAM2Tsum([uP1x,uP1y,wP1],[psix,psiy,mu],init=1)=
-int1d(ThL,3)(mu*cdot(uP1x,uP1y,N.x,N.y));
matrix BGAM2T=BGAM2Tsum(VM1h,LL2h);

/****************************************/
varf BGAM3sum([phix,phiy,lambda],[psix,psiy,mu],init=1)=
int1d(ThL,3)(lambda*cdot(psix,psiy,N.x,N.y));
matrix BGAM3=BGAM3sum(LL2h,LL2h);

varf ABJS3sum([uFx,uFy,pF],[psix,psiy,mu],init=1)=
-int1d(ThF,1)(bjs*cdot(tgx(uFx,uFy),tgy(uFx,uFy),tgx(psix,psiy),tgy(psix,psiy)
    ));
matrix ABJS3=ABJS3sum(VFh,LL2h);

varf ABJS4sum([phix,phiy,lambda],[psix,psiy,mu],init=1)=
int1d(ThF,1)(bjs*cdot(tgx(phix,phiy),tgy(phix,phiy),tgx(psix,psiy),tgy(psix,
    psiy)));
matrix ABJS4=ABJS4sum(LL2h,LL2h);

varf BLAGTsum([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[psix,psiy,mu],init=1)
    =
```

```
int1d(ThL,3)(cdot((sigmap1xx*N.x+sigmap1xy*N.y),(sigmap1yx*N.x+sigmap1yy*N.y),
    psix,psiy));
matrix BLAGT=BLAGTsum(VE1h,LL2h);

/****************************************/
varf stabetasum([uS1x,uS1y],[vS1x,vS1y],init=1)=
int2d(ThS1)( 0*1.e-8*(uS1x*vS1x+uS1y*vS1y));
matrix stabeta=stabetasum(VS1h,VS1h);

varf stabgamsum([gamma],[theta],init=1)=
int2d(ThS1)( 0*1.e-8*gamma*theta );
matrix stabgam=stabgamsum(LL1h,LL1h);

varf stabsigsum([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[taup1xx,taup1xy,
    taup1yx,taup1yy],init=1)=
int2d(ThS1)( 1.e-10*(sigmap1xx*taup1xx+sigmap1xy*taup1xy+sigmap1yx*taup1yx+
    sigmap1yy*taup1yy));
matrix stabsig=stabsigsum(VE1h,VE1h);

varf stablagsum([phix,phiy,lambda],[psix,psiy,mu],init=1)=
int2d(ThS1)( 1.e-16*(phix*psix+phiy*psiy+lambda*mu));
//varf stablagsum([phix,phiy,lambda],[psix,psiy,mu],init=1)=
//intalledges(ThL)(1.e-13*lambda*mu)+int1d(ThL,2,1,4)(1.e-13*lambda*mu)+int2d(
    ThS1)(1.e-13*(phix*psix+phiy*psiy));
matrix stablag=stablagsum(LL2h,LL2h);

/****************************************/
matrix FF1mono=AF+ABJS1+BPFT+BPF;
matrix LF1mono=ABJS2+BGAM1;

matrix MM1mono=AQ1+BPQT1+BPQ1+APP1+MASSP1;
matrix EM1mono=AAEP;
matrix LM1mono=BGAM2;

matrix ME1mono=AAEPT;
matrix EE1mono=AE+stabsig;
matrix LE1mono=BLAG;

matrix FL1mono=ABJS3+BGAM1T;
matrix ML1mono=BGAM2T;
matrix LL2mono=ABJS4+BGAM3+BGAM3T+stablag;

matrix mono=
[
    [FF1mono,  0,        0,        0,        0,        LF1mono ],
    [0,        MM1mono,  0,        AAEP,     0,        BGAM2   ],
    [0,        0,        stabeta,  BEST,     0,        0       ],
    [0,        AAEPT,    BES,      EE1mono,  ALE,      BLAG    ],
    [0,        0,        0,        ALET,     stabgam,  0       ],
    [FL1mono,  BGAM2T,   0,        BLAGT,    0,        LL2mono ]
];

//ofstream matout("matmono.txt");
//matout << mono<<endl;
```

```
///////////////////////////////////////
//OLD matrix formulation
///////////////////////////////////////
varf MASSP1sumold([uP1oldx,uP1oldy,pP1old],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)((s0/delt)*(wP1*pP1old));
matrix MASSP1old=MASSP1sumold(VM1h,VM1h);

varf AAEPsumold([sigmap1oldxx,sigmap1oldxy,sigmap1oldyx,sigmap1oldyy],[vP1x,
    vP1y,wP1],init=1)=
int2d(ThS1)((alpha/(2*muS+2*lambdaS))*(1/delt)*(sigmap1oldxx+sigmap1oldyy)*wP1
    );
matrix AAEPold=AAEPsumold(VE1h,VM1h);

varf APP1sumold([uP1oldx,uP1oldy,pP1old],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)((alpha^2/(muS+lambdaS))*(1/delt)*pP1old*wP1);
matrix APP1old=APP1sumold(VM1h,VM1h);

varf AEsumold([sigmap1oldxx,sigmap1oldxy,sigmap1oldyx,sigmap1oldyy],[taup1xx,
    taup1xy,taup1yx,taup1yy],init=1)=
int2d(ThS1)((1.0/(2*muS))*(1.0/delt)*((sigmap1oldxx-(lambdaS/(2*muS+2*lambdaS)
    )*(sigmap1oldxx+sigmap1oldyy))*taup1xx
                                +sigmap1oldxy*taup1xy
                                +sigmap1oldyx*taup1yx
                                +(sigmap1oldyy-(lambdaS/(2*muS+2*lambdaS))*(
                                    sigmap1oldxx+sigmap1oldyy))*taup1yy));
matrix AEold=AEsumold(VE1h,VE1h);

varf AAEPTsumold([uP1oldx,uP1oldy,pP1old],[taup1xx,taup1xy,taup1yx,taup1yy],
    init=1)=
int2d(ThS1)((alpha/(2*muS+2*lambdaS))*(1.0/delt)*pP1old*(taup1xx+taup1yy));
matrix AAEPTold=AAEPTsumold(VM1h,VE1h);

varf ALEsumold([gammaold],[taup1xx,taup1xy,taup1yx,taup1yy],init=1)=
int2d(ThS1)((taup1xy-taup1yx)*gammaold*(1.0/delt));
matrix ALEold=ALEsumold(LL1h,VE1h);

matrix MM1monoold=APP1old+MASSP1old;

matrix tmp1 = 0*FF1mono;
matrix tmp2 = 0*stabeta;
matrix tmp3 = 0*stabgam;
matrix tmp4 = 0*LL2mono;


matrix monoold=
[
      [tmp1,      0,            0,         0,         0,       0       ],
      [0,         MM1monoold,   0,         AAEPold,   0,       0       ],
      [0,         0,            tmp2,      0,         0,       0       ],
      [0,         AAEPTold,     0,         AEold,     ALEold,  0       ],
      [0,         0,            0,         0,         tmp3,    0       ],
      [0,         0,            0,         0,         0,       tmp4    ]
];
```

```
//ofstream matoutold("matmonoold.txt");
//matoutold << monoold<<endl;

varf BCinSuf([uFx,uFy,pF],[vFx,vFy,wF],init=1)=
int2d(ThF)(ffx*vFx + ffy*vFy) + int2d(ThF)(qf*wF) + on(2,3,4,uFx=ufx0, uFy=
    ufy0);

varf BCinSup([uP1x,uP1y,pP1],[vP1x,vP1y,wP1],init=1)=
int2d(ThS1)(qp*wP1) - int1d(ThS1,1,2,4) ( pp0*(vP1x*N.x+vP1y*N.y));

varf BCinSus([uS1x,uS1y],[vS1x,vS1y],init=1)=
int2d(ThS1)(cdot(fpx,fpy,vS1x,vS1y));

varf BCinSsigma([sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy],[taup1xx,taup1xy,
    taup1yx,taup1yy],init=1)=
int1d(ThS1,1,2,4)( uS0x*(taup1xx*N.x + taup1xy*N.y) + uS0y*(taup1yx*N.x +
    taup1yy*N.y) );
 //    + on(1,2,4,sigmap1xx=sigmap0xx,sigmap1xy=sigmap0xy,sigmap1yx=sigmap0yx,
     sigmap1yy=sigmap0yy);


// vector of RHS
real[int] xxf(FF1mono.n), xxfold(FF1mono.n), xxfmono(FF1mono.n);
real[int] xxm(MM1mono.n), xxmold(MM1mono.n), xxmmono(MM1mono.n);
real[int] xxu(stabeta.n), xxuold(stabeta.n), xxumono(stabeta.n);
real[int] xxs(EE1mono.n), xxsold(EE1mono.n), xxsmono(EE1mono.n);
real[int] xxl1(stabgam.n), xxl1old(stabgam.n), xxl1mono(stabgam.n);
real[int] xxl2(LL2mono.n), xxl2old(LL2mono.n), xxl2mono(LL2mono.n);

real[int] pfake1(stabgam.n);
real[int] pfake2(LL2mono.n);
pfake1=0;
pfake2=0;

varf l1(unused, VFh) = BCinSuf;
varf l2(unused, VM1h) = BCinSup;
varf l3(unused, VS1h) = BCinSus;
varf l4(unused, VE1h) = BCinSsigma;

// set the initialized value:
//[uFx,uFy,pF]=[ufx0,ufy0,pf0];
[uP1x,uP1y,pP1]=[upx0,upy0,pp0];
//[uS1x,uS1y] = [uS0x,uS0y];
[sigmap1xx,sigmap1xy,sigmap1yx,sigmap1yy]=[sigmap0xx,sigmap0xy,sigmap0yx,
    sigmap0yy];
//gamma = gamma0;
//[phix,phiy,lambda] = [phi0x,phi0y,lambda0];

xxf=0; xxm=0; xxu=0; xxs=0; xxl1=0; xxl2=0;
xxfmono=0; xxmmono=0; xxumono=0; xxsmono=0; xxl1mono=0; xxl2mono=0;
xxfold=uFx[];
xxmold=uP1x[];
xxuold=uS1x[];
```

```
xxsold=sigmap1xx[];
xxl1old=gamma[];
xxl2old=phix[];


real[int] xx=[xxf, xxm, xxu, xxs, xxl1, xxl2];
real[int] xxold=[xxfold, xxmold, xxuold, xxsold, xxl1old, xxl2old];

int br=1; // for vtk.files

for (int k=1;k<=NN;++k){
    t=t+delt;
    //cout<<" *** t *** "<<t<<endl;

    // RHS data (change in time)
    real[int] BCin1=l1(0, VFh);
    real[int] BCin2=l2(0, VM1h);
    real[int] BCin3=l3(0, VS1h);
    real[int] BCin4=l4(0, VE1h);
    real[int] b=[BCin1, BCin2, BCin3, BCin4, pfake1, pfake2];
    b+=(monoold)*xxold;

    set(mono, solver=sparsesolver);
    xx=mono^(-1)*b;

    xxold=xx;
    [xxfmono, xxmmono,xxumono,xxsmono,xxl1mono,xxl2mono]=xx;

    // split solution
    uFx[]=xxfmono;
    uP1x[]=xxmmono;
    uS1x[]=xxumono;
    sigmap1xx[]=xxsmono;
    gamma[]=xxl1mono;
    phix[]=xxl2mono;

    // compute errors
    // error: fluid velocity L inf in time L2 in space
    error1tmp[k-1] = (int2d(ThF))( ((uFx - ufx0)^2 + (uFy - ufy0)^2 + (dx(uFx)
        - dxufx0)^2 + (dy(uFx) - dyufx0)^2 + (dx(uFy) - dxufy0)^2 + (dy(uFy)
       - dyufy0)^2 )/( ufx0^2+ufy0^2+dxufx0^2 + dyufx0^2 + dxufy0^2 + dyufy0
       ^2 ));

    // error: fluid velocity L2 in time H1 in space
    error2[count] += int2d(ThF)( (uFx - ufx0)^2 + (uFy - ufy0)^2  + (dx(uFx) -
        dxufx0)^2 + (dy(uFx) - dyufx0)^2 + (dx(uFy) - dxufy0)^2 + (dy(uFy) -
       dyufy0)^2 );
    abs2[count]  += int2d(ThF)( ufx0^2 + ufy0^2 + dxufx0^2 + dxufy0^2 + dyufx0
       ^2 + dyufy0^2 );

    // error: darcy velocity L2 in time H div in space
    //error3[count] += int2d(ThS1)( (uP1x - upx0)^2 + (uP1y - upy0)^2 + (dx(
       uP1x)+dy(uP1y) - updiv0)^2);
    //abs3[count]  += int2d(ThS1)( upx0^2 + upy0^2 + updiv0^2);
```

```
// error: darcy velocity L2 in time L2 in space
error3[count] += int2d(ThS1)( (uP1x - upx0)^2 + (uP1y - upy0)^2 );
abs3[count]  += int2d(ThS1)( upx0^2 + upy0^2);


// error: structure velocity L2 in time L2 in space
error4[count] += int2d(ThS1) ((uS1x-uS0x)^2+(uS1y-uS0y)^2);
abs4[count]  += int2d(ThS1) (uS0x^2+uS0y^2);

// error: elasticity L2 in time H div in space
//error5 += int2d(ThS1)((sigmap1xx-sigmap0xx)^2+(sigmap1xy-sigmap0xy)^2+(
    sigmap1yx-sigmap0yx)^2+(sigmap1yy-sigmap0yy)^2
//                    + (dx(sigmap1xx) + dy(sigmap1yy) + fpx)^2 + (dx(
    sigmap1yx)+ dy(sigmap1yy) +fpy)^2 );
//abs5   += int2d(ThS1)(sigmap0xx^2+sigmap0xy^2+sigmap0yx^2+sigmap0yy^2 +
    fpx^2 + fpy^2);
// error: elasticity L2 in time L2 in space
error5[count] += int2d(ThS1)((sigmap1xx-sigmap0xx)^2+(sigmap1xy-sigmap0xy)
    ^2+(sigmap1yx-sigmap0yx)^2+(sigmap1yy-sigmap0yy)^2);
abs5[count]   += int2d(ThS1)(sigmap0xx^2+sigmap0xy^2+sigmap0yx^2+sigmap0yy
    ^2);


// error: fluid pressure L2 in time L2 in space
error6[count] += int2d(ThF) ( (pF - pf0)^2);
abs6[count] += int2d(ThF) ( pf0^2);

// error: darcy pressure L2 in time L2 in space
error7[count] += int2d(ThS1) ( (pP1 - pp0)^2);
abs7[count] += int2d(ThS1) ( pp0^2);

// error: div sigma_p L2 in time L2 in space
error8[count] += int2d(ThS1)( (dx(sigmap1xx) + dy(sigmap1xy) + fpx)^2 + (
    dx(sigmap1yx)+ dy(sigmap1yy) +fpy)^2 );
abs8[count]   += int2d(ThS1)( fpx^2 + fpy^2);

// error: gamma L2 in time L2 in space
error9[count] += int2d(ThS1)((gamma-gamma0)^2);
abs9[count] += int2d(ThS1) (gamma0^2);

// error: elasticy L inf in time L2 in space
error10tmp[k-1] = (int2d(ThS1))( ((sigmap1xx-sigmap0xx)^2+(sigmap1xy-
    sigmap0xy)^2+(sigmap1yx-sigmap0yx)^2+(sigmap1yy-sigmap0yy)^2)/(
    sigmap0xx^2+sigmap0xy^2+sigmap0yx^2+sigmap0yy^2));

// error: lambda L2 L2
error11[count] += int1d(ThL,3)((lambda-lambda0)^2);
abs11[count] += int1d(ThL,3) (lambda0^2);

// error: theta L2 L2
error12[count] += int1d(ThL,3)((phix - phi0x)^2+ (phiy - phi0y)^2);
abs12[count] += int1d(ThL,3) ( phi0x^2 + phi0y^2);

// error: div darcy velocity L2 in time L2 in space
```

```
    error13[count] += int2d(ThS1)( (dx(uP1x)+dy(uP1y) - updiv0)^2 );
    abs13[count]  += int2d(ThS1)( updiv0^2);

    // error: structure velocity
    errorq1[count] += int2d(ThS1,qft=qf1pT) ((uS1x-uS0x)^2+(uS1y-uS0y)^2);
    absq1[count]  += int2d(ThS1,qft=qf1pT) (uS0x^2+uS0y^2);

    // error: darcy pressure
    errorq2[count] += int2d(ThS1,qft=qf1pT) ( (pP1 - pp0)^2);
    absq2[count]  += int2d(ThS1,qft=qf1pT) ( pp0^2);

int[int] forder = [1,0];
int[int] sorder = [1,0,1];
if(k%pr==0 && plotflag)
        {
            savevtk("paraview_convergence/Fluid_"+string(br)+".vtk", ThF, [uFx
                ,uFy,0],pF,
                    order=forder,dataname="Velocity_Pressure");
            savevtk("paraview_convergence/Structure_"+string(br)+".vtk", ThS1,
                 [uP1x,uP1y,0], pP1, [uS1x, uS1y, 0],
                    order=sorder,dataname="Velocity_Pressure_Displacement");

            br=br+1;
        }


}

error1[count]= error1tmp.max;
error10[count]= error10tmp.max;
count +=1;
}

real[int] err1(nMeshes); err1=0;
real[int] err2(nMeshes); err2=0;
real[int] err3(nMeshes); err3=0;
real[int] err4(nMeshes); err4=0;
real[int] err5(nMeshes); err5=0;
real[int] err6(nMeshes); err6=0;
real[int] err7(nMeshes); err7=0;
real[int] err8(nMeshes); err8=0;
real[int] err9(nMeshes); err9=0;
real[int] err10(nMeshes); err10=0;
real[int] err11(nMeshes); err11=0;
real[int] err12(nMeshes); err12=0;
real[int] err13(nMeshes); err13=0;
real[int] errq1(nMeshes); errq1=0;
real[int] errq2(nMeshes); errq2=0;

real[int] rate1(nMeshes); rate1=0;
real[int] rate2(nMeshes); rate2=0;
real[int] rate3(nMeshes); rate3=0;
real[int] rate4(nMeshes); rate4=0;
real[int] rate5(nMeshes); rate5=0;
```

```
real[int] rate6(nMeshes); rate6=0;
real[int] rate7(nMeshes); rate7=0;
real[int] rate8(nMeshes); rate8=0;
real[int] rate9(nMeshes); rate9=0;
real[int] rate10(nMeshes); rate10=0;
real[int] rate11(nMeshes); rate11=0;
real[int] rate12(nMeshes); rate12=0;
real[int] rate13(nMeshes); rate13=0;
real[int] rateq1(nMeshes); rateq1=0;
real[int] rateq2(nMeshes); rateq2=0;

for (int k=0; k<error1.n; ++k){
    err1(k) = sqrt(error1(k));
    err2(k) = sqrt(error2(k)/abs2(k));
    err3(k) = sqrt(error3(k)/abs3(k));
    err4(k) = sqrt(error4(k)/abs4(k));
    err5(k) = sqrt(error5(k)/abs5(k));
    err6(k) = sqrt(error6(k)/abs6(k));
    err7(k) = sqrt(error7(k)/abs7(k));
    err8(k) = sqrt(error8(k)/abs8(k));
    err9(k) = sqrt(error9(k)/abs9(k));
    err10(k) = sqrt(error10(k));
    err11(k) = sqrt(error11(k)/abs11(k));
    err12(k) = sqrt(error12(k)/abs12(k));
    err13(k) = sqrt(error13(k)/abs13(k));
    errq1(k) = sqrt(errorq1(k)/absq1(k));
    errq2(k) = sqrt(errorq2(k)/absq2(k));

    if (k == 0)
    {
        rate1(k) = 0.0;
        rate2(k) = 0.0;
        rate3(k) = 0.0;
        rate4(k) = 0.0;
        rate5(k) = 0.0;
        rate6(k) = 0.0;
        rate7(k) = 0.0;
        rate8(k) = 0.0;
        rate9(k) = 0.0;
        rate10(k) = 0.0;
        rate11(k) = 0.0;
        rate12(k) = 0.0;
        rate13(k) = 0.0;
        rateq1(k) = 0.0;
        rateq2(k) = 0.0;
    }
    else
    {
        rate1(k) = log(err1(k-1)/err1(k))/log(2.0);
        rate2(k) = log(err2(k-1)/err2(k))/log(2.0);
        rate3(k) = log(err3(k-1)/err3(k))/log(2.0);
        rate4(k) = log(err4(k-1)/err4(k))/log(2.0);
        rate5(k) = log(err5(k-1)/err5(k))/log(2.0);
        rate6(k) = log(err6(k-1)/err6(k))/log(2.0);
```

```
            rate7(k) = log(err7(k−1)/err7(k))/log(2.0);
            rate8(k) = log(err8(k−1)/err8(k))/log(2.0);
            rate9(k) = log(err9(k−1)/err9(k))/log(2.0);
            rate10(k) = log(err10(k−1)/err10(k))/log(2.0);
            rate11(k) = log(err11(k−1)/err11(k))/log(2.0);
            rate12(k) = log(err12(k−1)/err12(k))/log(2.0);
            rate13(k) = log(err13(k−1)/err13(k))/log(2.0);
            rateq1(k) = log(errq1(k−1)/errq1(k))/log(2.0);
            rateq2(k) = log(errq2(k−1)/errq2(k))/log(2.0);
        }
}

// OUTPUT ERRORS:
/*if(converg){
    matrix errors=[[(err1), (rate1), (err2), (rate2), (err3), (rate3), (err4),
         (rate4), (err5), (rate5), (err6), (rate6),(err7), (rate7),(err8), (
        rate8),(err9), (rate9),(err10), (rate10),(err11), (rate11),(err12), (
        rate12),(errq1), (rateq1),(errq2), (rateq2) ]];
    {
        ofstream errOut("errorsrates.txt");
        errOut<<errors;
    }
    matrix errors1=[[(error1), (error2), (error3), (error4), (error5), (error6
        ), (error7), (error8), (error9),(error10),(error11),(error12), (
        errorq1), (errorq2)]];
    {
        ofstream errout("errors.txt");
        errout << errors1;
    }
}*/

// Print results
cout << "==================================================================" <<
    endl;
cout << "Errors and rates" << endl;
cout << "|u_f(H1)|" << "   rate   "
     << "|u_f(l2H1)|" << "   rate   "
     << "|u_p(L2)|" << "   rate   "
     << "|u_s(L2)|" << "   rate   "
     << "|e_p(L2)|" << "   rate   "
     <<endl;
for (int i=0; i<err1.n; i++){
    // Stokes velocity
    cout.precision(3);
    cout.scientific << err1[i] << "   ";
    cout.precision(1);
    cout.fixed << rate1[i] << "    ";
    // Stokes pressure
    cout.precision(3);
    cout.scientific << err2[i] << "   ";
    cout.precision(1);
    cout.fixed << rate2[i] << "    ";
    // Darcy velocity
    cout.precision(3);
```

```
        cout.scientific << err3[i] << "   ";
        cout.precision(1);
        cout.fixed << rate3[i] << "    ";
        // Darcy pressure
        cout.precision(3);
        cout.scientific << err4[i] << "   ";
        cout.precision(1);
        cout.fixed << rate4[i] << "    ";
        // Displacement
        cout.precision(3);
        cout.scientific << err5[i] << "   ";
        cout.precision(1);
        cout.fixed << rate5[i] << "    ";
        cout << endl;
    }
    cout << "|p_f(L2)|" << "   rate   "
         << "|p_p(L2)|" << "   rate   "
         << "|div e_p(L2)|" << "  rate  "
         << "|gam_p(L2)|" << "  rate  "
         << "|u_s(qft)|" << "   rate   "
         << "|p_p(qft)|" << "   rate   "
         << endl;
    for (int i=0; i<err1.n; i++){
        // Darcy pressure
        cout.precision(3);
        cout.scientific << err6[i] << "   ";
        cout.precision(1);
        cout.fixed << rate6[i] << "    ";
        // Displacement
        cout.precision(3);
        cout.scientific << err7[i] << "   ";
        cout.precision(1);
        cout.fixed << rate7[i] << "    ";
        //
        cout.precision(3);
        cout.scientific << err8[i] << "   ";
        cout.precision(1);
        cout.fixed << rate8[i] << "    ";
        //
        cout.precision(3);
        cout.scientific << err9[i] << "   ";
        cout.precision(1);
        cout.fixed << rate9[i] << "    ";
        //
        cout.precision(3);
        cout.scientific << errq1[i] << "   ";
        cout.precision(1);
        cout.fixed << rateq1[i] << "    ";
        //
        cout.precision(3);
        cout.scientific << errq2[i] << "   ";
        cout.precision(1);
        cout.fixed << rateq2[i] << "    ";
        cout << endl;
```

```
}
cout << "|sigma_p(linfL2)|" << "   rate   "
     << "|lambda_p(L2)|" << "   rate   "
     << "|theta(L2)|" << "  rate  "
     << "|div u_p(L2)|" << "  rate  "
     << endl;
for (int i=0; i<err1.n; i++){
    //
    cout.precision(3);
    cout.scientific << err10[i] << "   ";
    cout.precision(1);
    cout.fixed << rate10[i] << "    ";
    //
    cout.precision(3);
    cout.scientific << err11[i] << "   ";
    cout.precision(1);
    cout.fixed << rate11[i] << "    ";
    //
    cout.precision(3);
    cout.scientific << err12[i] << "   ";
    cout.precision(1);
    cout.fixed << rate12[i] << "    ";
    //
    cout.precision(3);
    cout.scientific << err13[i] << "   ";
    cout.precision(1);
    cout.fixed << rate13[i] << "    ";
    cout << endl;
}

cout << "====================================================================" <<
    endl;
```

We then present FreeFem++ code for convergence test with the multipoint stress-flux mixed finite element method, writing in a different structure.

```
//
// This code solves a multipoint stress-flux mixed finite element method
// for the Stokes-Biot model
//
// authors: Sergio Caucao, Tongtong Li, Ivan Yotov
//
// Global information
load "iovtk";          // for saving data in paraview format
load "UMFPACK64";      // UMFPACK solver
load "Element_Mixte"; // for using BDM1

//--------------------------------------
//               Initial parameters
//--------------------------------------

//------ Global parameters
int nref = 5;
```

```
real mvphi1;
real mvphi2;
real mtheta1;
real mtheta2;
real mlam1;
real mlam2;
real t;
real T = 0.01;    //total time T=0.01;
real dt = 0.001;  //delta t=0.001;
real NN = T/dt;  //number of time interval

//——— Stokes
real[int] Hdivsigf(nref);
real[int] L2uf(nref);
real[int] L2gamf(nref);
real[int] L2pf(nref);
real[int] hF(nref);
real[int] DOFf(nref);

//——— Biot
real[int] Hdivsigp(nref);
real[int] eauxsigp(NN);
real[int] Hdivup(nref);
real[int] L2pp(nref);
real[int] eauxpp(NN);
real[int] L2us(nref);
real[int] L2gamp(nref);
real[int] hP(nref);
real[int] DOFp(nref);

//——— Interface
real[int] vphierror1(nref);
real[int] vphierror2(nref);
real[int] thetaerror1(nref);
real[int] thetaerror2(nref);
real[int] lamerror1(nref);
real[int] lamerror2(nref);
real[int] htf(nref);
real[int] htp(nref);

//——— rate of convergence
real[int] sigfrate(nref-1);
real[int] ufrate(nref-1);
real[int] gamfrate(nref-1);
real[int] pfrate(nref-1);

real[int] sigprate(nref-1);
real[int] uprate(nref-1);
real[int] pprate(nref-1);
real[int] usrate(nref-1);
real[int] gamprate(nref-1);

real[int] vphirate1(nref-1);
real[int] vphirate2(nref-1);
```

```
real[int] thetarate1(nref-1);
real[int] thetarate2(nref-1);
real[int] lamrate1(nref-1);
real[int] lamrate2(nref-1);

//------------------------------------------------------------
//                          Global data
//------------------------------------------------------------

//------ Stokes
real mu = 1.;

func pf  =  (2.*pi)*cos(pi*t) + exp(t)*sin(pi*x)*cos((pi/2.)*y);
func pfx =  pi*exp(t)*cos(pi*x)*cos((pi/2.)*y);
func pfy = -(pi/2.)*exp(t)*sin(pi*x)*sin((pi/2.)*y);

func uf1   =  pi*cos(pi*t)*(-3.*x + cos(y));
func uf2   =  pi*cos(pi*t)*(y + 1.);
func uf1x  = -(3.*pi)*cos(pi*t);
func uf1y  = -pi*cos(pi*t)*sin(y);
func uf2x  =  0.;
func uf2y  =  pi*cos(pi*t);
func uf1xx =  0.;
func uf1xy =  0.;
func uf1yy = -pi*cos(pi*t)*cos(y);
func uf2xx =  0.;
func uf2xy =  0.;
func uf2yy =  0.;

func gamf  = (uf1y - uf2x)/2.;

func sigf1 = 2.*mu*uf1x - pf;
func sigf2 = mu*(uf1y + uf2x);
func sigf3 = sigf2;
func sigf4 = 2.*mu*uf2y - pf;

func gf  = uf1x + uf2y;
func ff1 = -mu*(2.*uf1xx + uf1yy + uf2xy) + pfx;
func ff2 = -mu*(uf1xy + uf2xx + 2.*uf2yy) + pfy;

//------ Biot
real k1 = 1.; // matrix K=[[k1,k2],[k2,k3]]
real k2 = 0.;
real k3 = 1.;
real s0 = 1.;
real omi = 1.;
real mup = 1.;
real lamp = 1.;
real trAI = (1./(mup+lamp));
real lamup = lamp/(2.*(mup+lamp));
real alphap = 1.;

func pp  =  exp(t)*sin(pi*x)*cos((pi/2.)*y);
func ppx =  pi*exp(t)*cos(pi*x)*cos((pi/2.)*y);
```

```
func ppy = -(pi/2.)*exp(t)*sin(pi*x)*sin((pi/2.)*y);
func ppt =  exp(t)*sin(pi*x)*cos((pi/2.)*y);

func up1  = -(k1*ppx)/mu;
func up2  = -(k3*ppy)/mu;
func up1x =  ((k1*pi^2)/mu)*exp(t)*sin(pi*x)*cos((pi/2.)*y);
func up2y =  ((k3*pi^2)/(4.*mu))*exp(t)*sin(pi*x)*cos((pi/2.)*y);

func etap1   =  sin(pi*t)*(-3.*x + cos(y));
func etap2   =  sin(pi*t)*(y + 1.);
func etap1x  = -3.*sin(pi*t);
func etap1y  = -sin(pi*t)*sin(y);
func etap2x  =  0.;
func etap2y  =  sin(pi*t);
func etap1xx =  0.;
func etap1xy =  0.;
func etap1yy = -sin(pi*t)*cos(y);
func etap2xx =  0.;
func etap2xy =  0.;
func etap2yy =  0.;

func us1  = pi*cos(pi*t)*(-3.*x + cos(y));
func us2  = pi*cos(pi*t)*(y + 1.);
func us1x = -(3.*pi)*cos(pi*t);
func us1y = -pi*cos(pi*t)*sin(y);
func us2x = 0.;
func us2y = pi*cos(pi*t);

func gamp  = (us1y - us2x)/2.;

func sigp1 = (lamp+2.*mup)*etap1x + lamp*etap2y - alphap*pp;
func sigp2 = mup*(etap1y + etap2x);
func sigp3 = sigp2;
func sigp4 = lamp*etap1x + (lamp+2.*mup)*etap2y - alphap*pp;

func divetapt = -(2.*pi)*cos(pi*t);
func divup = up1x + up2y;
func gp  = s0*ppt + alphap*divetapt + divup;
func fp1 = -((lamp+2.*mup)*etap1xx + (lamp+mup)*etap2xy + mup*etap1yy) +
    alphap*ppx;
func fp2 = -((lamp+2.*mup)*etap2yy + (lamp+mup)*etap1xy + mup*etap2xx) +
    alphap*ppy;

//------ Global macros
macro uf [uf1,uf2] //
macro up [up1,up2] //
macro us [us1,us2] //
macro gpp [ppx,ppy] //
macro Guf1 [uf1x,uf1y] //
macro Guf2 [uf2x,uf2y] //
macro Gus1 [us1x,us1y] //
macro Gus2 [us2x,us2y] //
macro sigf [sigf1,sigf2,sigf3,sigf4] //
macro sigp [sigp1,sigp2,sigp3,sigp4] //
```

```
macro Ff [ff1,ff2] //
macro Fp [fp1,fp2] //
macro Ki [[k3/(k1*k3-k2^2),-k2/(k1*k3-k2^2)],[-k2/(k1*k3-k2^2),k1/(k1*k3-k2^2)
    ]] //

macro sigfh [sigfh1,sigfh2,sigfh3,sigfh4] //
macro taufh [taufh1,taufh2,taufh3,taufh4] //

macro sigph [sigph1,sigph2,sigph3,sigph4] //
macro tauph [tauph1,tauph2,tauph3,tauph4] //
macro sigphold [sigphold1,sigphold2,sigphold3,sigphold4] //

macro ufh [ufh1,ufh2] //
macro vfh [vfh1,vfh2] //

macro uph [uph1,uph2] //
macro vph [vph1,vph2] //

macro ush [ush1,ush2] //
macro vsh [vsh1,vsh2] //

macro vphih [vphih1,vphih2] //
macro psih [psih1,psih2] //

macro auxfh [auxfh1,auxfh2] //
macro xaufh [xaufh1,xaufh2] //

macro thetah [thetah1,thetah2] //
macro phih [phih1,phih2] //

macro norm [N.x,N.y] //
macro tgt  [-N.y,N.x] //

macro div(vph) (dx(vph[0]) + dy(vph[1])) //
macro grad(xih) [dx(xih),dy(xih)] //
macro Grad(vfh) [dx(vfh[0]),dy(vfh[0]),dx(vfh[1]),dy(vfh[1])] //
macro tr(taufh) (taufh[0] + taufh[3]) //
macro trA(tauph) (tr(tauph)/(2.*(mup+lamp))) //
macro dev(taufh) [0.5*(taufh[0] - taufh[3]),taufh[1],taufh[2],0.5*(taufh[3] -
    taufh[0])] //
macro Div(taufh) [dx(taufh[0]) + dy(taufh[1]),dx(taufh[2]) + dy(taufh[3])] //
macro A(tauph) [(tauph[0]-lamup*tr(tauph))/(2.*mup),tauph[1]/(2.*mup),tauph
    [2]/(2.*mup),(tauph[3]-lamup*tr(tauph))/(2.*mup)] //
macro pfh(taufh,gf) (-0.5*tr(taufh) + mu*gf ) //

//------------------------------------------------------------
//                               Defining the domain
//------------------------------------------------------------
for(int n = 0; n < nref; n++){

int sizef = 2^(n + 3);
int sizep = (5./8.)*sizef;

int gammafp = 1;
```

```
int gammafD = 21;
int gammafN = 22;
int gammapD = 31;
int gammapN = 32;

//------ Omegaf
border Gammaf1(t=0,1){x=1; y=t; label = gammafN;};
border Gammaf2(t=1,0){x=t; y=1; label = gammafD;};
border Gammaf3(t=1,0){x=0; y=t; label = gammafN;};

//------ Interface
border Gammafp(t=0,1){x=t; y=0; label = gammafp;};

//------ Omegap
border Gammap1(t=0,-1){x=0; y=t; label = gammapN;};
border Gammap2(t=0,1){x=t; y=-1; label = gammapD;};
border Gammap3(t=-1,0){x=1; y=t; label = gammapN;};

//------ Meshes
mesh Thf = buildmesh(Gammaf1(sizef) + Gammaf2(sizef) + Gammaf3(sizef) +
    Gammafp(sizef));
mesh Thp = buildmesh(Gammap1(sizep) + Gammap2(sizep) + Gammap3(sizep) +
    Gammafp(-sizep));

mesh Shf = emptymesh(Thf);
mesh Shp = emptymesh(Thp);

//                  Gammaf^D
//              1-----------------
//              |                |
//              |                |
//   Gammaf^N   |     Omegaf     |Gammaf^N
//              |                |
//              |                |
//              0----------------1  Gammafp
//              |                |
//              |                |
//   Gammap^N   |     Omegap     |Gammap^N
//              |                |
//              |                |
//             -1-----------------
//                  Gammap^D

//plot(Thf,Thp,wait=true);
//------------------------------------------------
//                    Finite element spaces
//------------------------------------------------
fespace Qhsigf(Thf,[BDM1,BDM1]);
fespace Qhup(Thp,BDM1);
fespace Qhsigp(Thp,[BDM1,BDM1]);
fespace Qhpp(Thp,P0);

fespace Shuf(Thf,[P0,P0]);
fespace Shus(Thp,[P0,P0]);
```

```
fespace Shgamf(Thf,P1);
fespace Shgamp(Thp,P1);

fespace Lhf(Shf,[P1,P1]);
fespace Lhs(Shp,[P1,P1]);
fespace Lhp(Shp,P1);
fespace Auxf(Shf,[P1,P1]);
fespace Auxp(Shp,P1);

fespace Phf(Thf,P1);
fespace Php(Thp,P1);

//---------------------------------------------------------
//                Defining the bilinear forms
//---------------------------------------------------------
Qhsigf sigfh;
Qhup uph;
Qhpp pph, pphold;
Qhsigp sigph, sigphold;
Shuf ufh;
Shus ush;
Shgamf gamfh;
Shgamp gamph;
Lhf vphih;
Lhs thetah;
Lhp lamh;

real eps  = 1.e-12;
real epsI = 1.e-12;
//------- bilinear forms
varf a1(sigfh,taufh) = int2d(Thf,qft=qf1pTlump)( (dev(sigfh)'*dev(taufh))/(2.*
    mu) - eps*(tr(sigfh)*tr(taufh)) );
varf a2(uph,vph)      = int2d(Thp,qft=qf1pTlump)( mu*((Ki*uph)'*vph) );
varf a3([pph],vph)    = int2d(Thp)( -(pph*div(vph)) );
varf a4(uph,[qph])    = int2d(Thp)( qph*div(uph) );
varf a5(sigph,tauph) = int2d(Thp,qft=qf1pTlump)( (A(sigph)'*tauph)/dt );
varf a6([pph],tauph) = int2d(Thp)( (alphap/dt)*(pph*trA(tauph)) );
varf a7(sigph,[qph]) = int2d(Thp)( (alphap/dt)*(trA(sigph)*qph) );
varf a8(pph,qph)      = int2d(Thp)( ((s0 + (alphap^2)*trAI)/dt)*(pph*qph) - eps
    *(pph*qph) );

varf b1(vphih,taufh)  = int1d(Thf,gammafp)( -(vphih'*([[taufh[0],taufh[1]],[
    taufh[2],taufh[3]]]*norm)) );
varf b2(thetah,tauph) = int1d(Thp,gammafp)( -(thetah'*([[tauph[0],tauph[1]],[
    tauph[2],tauph[3]]]*norm)) );
varf b3([lamh],vph)   = int1d(Thp,gammafp)( lamh*(vph'*norm) );

varf c1(vphih,psih)  = int1d(Shf,gammafp)( -omi*(vphih'*tgt)*(psih'*tgt) ) +
    int1d(Shf)( epsI*(vphih'*psih) );
varf c2(thetah,psih) = int1d(Shf,gammafp)( (thetah'*tgt)*(psih'*tgt) );
varf c3([lamh],psih) = int1d(Shf,gammafp)( -lamh*(psih'*norm) );
varf c4(vphih,phih)  = int1d(Shp,gammafp)( (vphih'*tgt)*(phih'*tgt) );
varf c5(thetah,phih) = int1d(Shp,gammafp)( -omi*(thetah'*tgt)*(phih'*tgt) ) +
    int1d(Shp)( epsI*(thetah'*phih) );
```

```
varf c6([lamh],phih) = int1d(Shp,gammafp)( -lamh*(phih'*norm) );
varf c7(vphih,[xih]) = int1d(Shp,gammafp)( -xih*(vphih'*norm) );
varf c8(thetah,[xih])= int1d(Shp,gammafp)( xih*(thetah'*norm) );
varf penI(lamh,xih)  = int1d(Shp)( epsI*(lamh*xih) );

varf B1(ufh,taufh)      = int2d(Thf)( ufh'*Div(taufh) );
varf B2(ush,tauph)      = int2d(Thp)( ush'*Div(tauph) );
varf B3([gamfh],taufh) = int2d(Thf,qft=qf1pTlump)( gamfh*(taufh[1] - taufh[2]) );
varf B4([gamph],tauph) = int2d(Thp,qft=qf1pTlump)( gamph*(tauph[1] - tauph[2]) );
varf B5(auxfh,taufh)    = int1d(Thf,gammafN)( -(auxfh'*([[taufh[0],taufh[1]],[taufh[2],taufh[3]]]*norm)) );
varf B6([auxph],vph)    = int1d(Thp,gammapN)( auxph*(vph'*norm) );
varf faux(auxfh,xaufh) = int1d(Shf)( epsI*(auxfh'*xaufh) );
varf paux(auxph,xauph) = int1d(Shp)( epsI*(auxph*xauph) );

//------ RHS
varf rhs1(sigfh,taufh) = int2d(Thf,qft=qf1pTlump)( -0.5*(gf*tr(taufh)) ) + int1d(Thf,gammafD)( uf'*([[taufh[0],taufh[1]],[taufh[2],taufh[3]]]*norm) );
varf rhs2(uph,vph)      = int1d(Thp,gammapD)( -(pp*(vph'*norm)) );
varf rhs3(sigph,tauph) = int2d(Thp,qft=qf1pTlump)( (alphap/dt)*(pphold*trA(tauph)) + (A(sigphold)'*tauph)/dt ) + int1d(Thp,gammapD,gammapN)( us'*([[tauph[0],tauph[1]],[tauph[2],tauph[3]]]*norm) );
varf rhs4(pph,qph)      = int2d(Thp)( (gp + ((s0 + (alphap^2)*trAI)/dt)*pphold + (alphap/dt)*trA(sigphold))*qph );
varf rhs5(ufh,vfh)      = int2d(Thf)( -(Ff'*vfh) );
varf rhs6(ush,vsh)      = int2d(Thp)( -(Fp'*vsh) );
varf bjs1(vphih,psih)  = int1d(Shf)( epsI*(uf'*psih) );
varf bjs2(thetah,phih) = int1d(Shp)( epsI*(us'*phih) );
varf lpen(lamh,xih)     = int1d(Shp)( epsI*(pp*xih) );
varf lauxf(auxfh,xaufh) = int1d(Thf,gammafN)( -(xaufh'*([[sigf[0],sigf[1]],[sigf[2],sigf[3]]]*norm)) ) + int1d(Shf)( epsI*(uf'*xaufh) );
varf lauxp(auxph,xauph) = int1d(Thp,gammapN)( xauph*(up'*norm) ) + int1d(Shp)( epsI*(pp*xauph) );

//------------------------------------------------------
//                      Stiff matrix
//------------------------------------------------------
matrix aa1 = a1(Qhsigf,Qhsigf);
matrix aa2 = a2(Qhup,Qhup);
matrix aa3 = a3(Qhpp,Qhup);
matrix aa4 = a4(Qhup,Qhpp);
matrix aa5 = a5(Qhsigp,Qhsigp);
matrix aa6 = a6(Qhpp,Qhsigp);
matrix aa7 = a7(Qhsigp,Qhpp);
matrix aa8 = a8(Qhpp,Qhpp);

matrix bb1 = b1(Lhf,Qhsigf);
matrix bb2 = b2(Lhs,Qhsigp);
matrix bb3 = b3(Lhp,Qhup);

matrix cc1 = c1(Lhf,Lhf);
```

```
matrix cc2 = c2(Lhs,Lhf);
matrix cc3 = c3(Lhp,Lhf);
matrix cc4 = c4(Lhf,Lhs);
matrix cc5 = c5(Lhs,Lhs);
matrix cc6 = c6(Lhp,Lhs);
matrix cc7 = c7(Lhf,Lhp);
matrix cc8 = c8(Lhs,Lhp);
matrix PENI = penI(Lhp,Lhp);

matrix BB1 = B1(Shuf,Qhsigf);
matrix BB2 = B2(Shus,Qhsigp);
matrix BB3 = B3(Shgamf,Qhsigf);
matrix BB4 = B4(Shgamp,Qhsigp);
matrix BB5 = B5(Auxf,Qhsigf);
matrix BB6 = B6(Auxp,Qhup);
matrix PAF = faux(Auxf,Auxf);
matrix PAP = paux(Auxp,Auxp);

matrix M;{
M = [[   aa1,     0,     0,    0,   bb1,    0,     0, BB1,    0, BB3,    0, BB5,    0],
        [     0,   aa2,     0,  aa3,     0,    0,   bb3,    0,    0,    0,    0,    0,
          BB6],
        [     0,     0,   aa5,  aa6,     0,  bb2,     0,    0,  BB2,    0,  BB4,    0,
          0],
     [     0,   aa4,   aa7,  aa8,     0,    0,     0,    0,    0,    0,    0,    0,    0],
     [  bb1',     0,     0,    0,   cc1,  cc2,   cc3,    0,    0,    0,    0,    0,    0],
     [     0,     0,  bb2',    0,   cc4,  cc5,   cc6,    0,    0,    0,    0,    0,    0],
     [     0,  bb3',     0,    0,   cc7,  cc8,  PENI,    0,    0,    0,    0,    0,    0],
     [  BB1',     0,     0,    0,     0,    0,     0,    0,    0,    0,    0,    0,    0],
     [     0,     0,  BB2',    0,     0,    0,     0,    0,    0,    0,    0,    0,    0],
     [  BB3',     0,     0,    0,     0,    0,     0,    0,    0,    0,    0,    0,    0],
     [     0,     0,  BB4',    0,     0,    0,     0,    0,    0,    0,    0,    0,    0],
     [  BB5',     0,     0,    0,     0,    0,     0,    0,    0,    0,    0,  PAF,    0],
     [     0,  BB6',     0,    0,     0,    0,     0,    0,    0,    0,    0,    0,  PAP
        ]];}

//------ Initial condition
t = 0.;
pphold = pp;
sigphold = [sigp1,sigp2,sigp3,sigp4];

real[int] sol1(Qhsigf.ndof), sol2(Qhup.ndof), sol3(Qhsigp.ndof), sol4(Qhpp.
   ndof);
real[int] sol5(Lhf.ndof), sol6(Lhs.ndof), sol7(Lhp.ndof);
real[int] sol8(Shuf.ndof), sol9(Shus.ndof), sol10(Shgamf.ndof), sol11(Shgamp.
   ndof), sol12(Auxf.ndof), sol13(Auxp.ndof);

for(int k = 0; k < NN; k++){  // loop in the number of time interval
    t = t + dt;

//------ RHS data change in time
    real[int] RHS1 = rhs1(0,Qhsigf);
    real[int] RHS2 = rhs2(0,Qhup);
    real[int] RHS3 = rhs3(0,Qhsigp);
```

```
    real[int] RHS4 = rhs4(0,Qhpp);
    real[int] BJS1 = bjs1(0,Lhf);
    real[int] BJS2 = bjs2(0,Lhs);
    real[int] LPEN = lpen(0,Lhp);
    real[int] RHS5 = rhs5(0,Shuf);
    real[int] RHS6 = rhs6(0,Shus);
    real[int] ZZ1(Shgamf.ndof); ZZ1 = 0.;
    real[int] ZZ2(Shgamp.ndof); ZZ2 = 0.;
    real[int] LAUXF = lauxf(0,Auxf);
    real[int] LAUXP = lauxp(0,Auxp);

    real[int] L = [RHS1,RHS2,RHS3,RHS4,BJS1,BJS2,LPEN,RHS5,RHS6,ZZ1,ZZ2,LAUXF,
      LAUXP];

    set(M, solver = sparsesolver);
    real[int] solt = M^-1*L;
    [sol1,sol2,sol3,sol4,sol5,sol6,sol7,sol8,sol9,sol10,sol11,sol12,sol13] =
      solt;

//------- Approximation of the solution
    sigfh1[] = sol1;
    uph1[]   = sol2;
    sigph1[] = sol3;
    pph[]    = sol4;
    vphih1[] = sol5;
    thetah1[]= sol6;
    lamh[]   = sol7;
    ufh1[]   = sol8;
    ush1[]   = sol9;
    gamfh[]  = sol10;
    gamph[]  = sol11;

//------- calculating the errors
    Hdivsigf[n] += int2d(Thf)( (sigf - sigfh)'*(sigf - sigfh) + (Ff + Div(
      sigfh))'*(Ff + Div(sigfh)) );
    L2uf[n]     += int2d(Thf)( (uf - ufh)'*(uf - ufh) );
     L2gamf[n]    += int2d(Thf)( 2.*square(gamf - gamfh) );
     L2pf[n]      += int2d(Thf)( square(pf - pfh(sigfh,gf)) );

     eauxsigp[k] = sqrt(int2d(Thp)( (sigp - sigph)'*(sigp - sigph) + (Fp + Div(
         sigph))'*(Fp + Div(sigph)) ));
     Hdivup[n]   += int2d(Thp)( (up - uph)'*(up - uph) + square(divup - div(uph)
         ) );
    eauxpp[k]   = sqrt(int2d(Thp)( square(pp - pph) ));
    L2us[n]     += int2d(Thp)( (us - ush)'*(us - ush) );
     L2gamp[n]   += int2d(Thp)( 2.*square(gamp - gamph) );

     mvphi1 = sqrt( int1d(Shf,gammafp)( (uf - vphih)'*(uf - vphih) ) );
    mvphi2 = sqrt( mvphi1^2 + int1d(Shf,gammafp)( square((Guf1 - grad(vphih
      [0]))'*tgt)*(tgt'*tgt) + square((Guf2 - grad(vphih[1]))'*tgt)*(tgt'*tgt) )
      );
    vphierror1[n] += mvphi1*mvphi2;
    vphierror2[n] += int1d(Shf,gammafp)( (uf - vphih)'*(uf - vphih) );
```

```
    mtheta1 = sqrt( int1d(Shp,gammafp)( (us - thetah)'*(us␣-␣thetah)␣)␣);
␣␣␣␣mtheta2␣=␣sqrt(␣mtheta1^2␣+␣int1d(Shp,gammafp)(␣square((Gus1␣-␣grad(thetah
    [0]))'*tgt)*(tgt'*tgt)␣+␣square((Gus2␣-␣grad(thetah[1]))'*tgt)*(tgt'*tgt)␣
    )␣);
␣␣␣␣thetaerror1[n]␣+=␣mtheta1*mtheta2;
␣␣␣␣thetaerror2[n]␣+=␣int1d(Shp,gammafp)(␣(us␣-␣thetah)'*(us - thetah) );

    mlam1 = sqrt( int1d(Shp,gammafp)( square(pp - lamh) ) );
    mlam2 = sqrt( mlam1^2 + int1d(Shp,gammafp)( square((gpp - grad(lamh))'*tgt
        )*(tgt'*tgt) ) );
    lamerror1[n] += mlam1*mlam2;
    lamerror2[n] += int1d(Shp,gammafp)( square(pp - lamh) );

//------ updating RHS
    pphold = pph;
    sigphold = [sigph1,sigph2,sigph3,sigph4];
}

Hdivsigf[n] = sqrt(dt*Hdivsigf[n]);
L2uf[n] = sqrt(dt*L2uf[n]);
L2gamf[n] = sqrt(dt*L2gamf[n]);
L2pf[n] = sqrt(dt*L2pf[n]);

Hdivsigp[n] = eauxsigp.max;
Hdivup[n] = sqrt(dt*Hdivup[n]);
L2pp[n] = eauxpp.max;
L2us[n] = sqrt(dt*L2us[n]);
L2gamp[n] = sqrt(dt*L2gamp[n]);

vphierror1[n] = sqrt(dt*vphierror1[n]);
vphierror2[n] = sqrt(dt*vphierror2[n]);
thetaerror1[n] = sqrt(dt*thetaerror1[n]);
thetaerror2[n] = sqrt(dt*thetaerror2[n]);
lamerror1[n] = sqrt(dt*lamerror1[n]);
lamerror2[n] = sqrt(dt*lamerror2[n]);

//------ for the meshsize in Omega
Phf hf = hTriangle;
hF[n]  = hf[].max;

Php hp = hTriangle;
hP[n]  = hp[].max;

htf[n] = 1.0 / sizef;
htp[n] = 1.0 / sizep;

DOFf[n] = Qhsigf.ndof + Shuf.ndof + Shgamf.ndof + Lhf.ndof;
DOFp[n] = Qhsigp.ndof + Qhup.ndof + Qhpp.ndof + Shus.ndof + Shgamp.ndof + Lhs.
    ndof + Lhp.ndof;

//------ exporting to Praraview
// savevtk("Data_Paraview_2D/Stokes_aprox"+n+".vtk",Thf,[sigfh1,sigfh2,0],[
    sigfh3,sigfh4,0],[ufh1,ufh2,0],gamfh,pfh(sigfh,gf),dataname="sigfh1␣sigfh2
    ␣ufh␣gamfh␣pfh");
```

```
// savevtk("Data_Paraview_2D/Biot_approx"+n+".vtk",Thp,[sigph1,sigph2,0],[
    sigph3,sigph4,0],[uph1,uph2,0],[ush1,ush2,0],gamph,pph,dataname="sigph1 
    sigph2 uph ush gamph pph");
// savevtk("Data_Paraview_2D/Stokes_exact"+n+".vtk",Thf,[sigf1,sigf2,0],[sigf3
    ,sigf4,0],[uf1,uf2,0],gamf,pf,dataname="sigf1 sigf2 uf gamf pf");
// savevtk("Data_Paraview_2D/Biot_exact"+n+".vtk",Thp,[sigp1,sigp2,0],[sigp3,
    sigp4,0],[up1,up2,0],[us1,us2,0],gamp,pp,dataname="sigp1 sigp2 up us gamp 
    pp");
}

//-------------------------------------------------------
//                    showing the tables
//-------------------------------------------------------
cout << " sigferror = " << Hdivsigf <<endl;
for(int n = 1; n < nref; n++)
sigfrate[n-1] = log(Hdivsigf[n-1]/Hdivsigf[n]) / log(hF[n-1]/hF[n]);
cout <<" convergence rate sigf = "<< sigfrate <<endl;

cout << " uferror = " << L2uf <<endl;
for(int n = 1; n < nref; n++)
ufrate[n-1] = log(L2uf[n-1]/L2uf[n]) / log(hF[n-1]/hF[n]);
cout << " convergence rate uf = " << ufrate <<endl;

cout << " gamferror = " << L2gamf <<endl;
for(int n = 1; n < nref; n++)
gamfrate[n-1] = log(L2gamf[n-1]/L2gamf[n]) / log(hF[n-1]/hF[n]);
cout << " convergence rate gamf = " << gamfrate <<endl;

cout << " pferror = " << L2pf <<endl;
for(int n = 1; n < nref; n++)
pfrate[n-1] = log(L2pf[n-1]/L2pf[n]) / log(hF[n-1]/hF[n]);
cout << " convergence rate pf = " << pfrate <<endl;
//
cout << " sigperror = " << Hdivsigp <<endl;
for(int n = 1; n < nref; n++)
sigprate[n-1] = log(Hdivsigp[n-1]/Hdivsigp[n]) / log(hP[n-1]/hP[n]);
cout << " convergence rate sigp = " << sigprate <<endl;

cout << " userror = " << L2us <<endl;
for(int n = 1; n < nref; n++)
usrate[n-1] = log(L2us[n-1]/L2us[n]) / log(hP[n-1]/hP[n]);
cout << " convergence rate us = " << usrate <<endl;

cout << " gamperror = " << L2gamp <<endl;
for(int n = 1; n < nref; n++)
gamprate[n-1] = log(L2gamp[n-1]/L2gamp[n]) / log(hP[n-1]/hP[n]);
cout << " convergence rate gamp = " << gamprate <<endl;

cout << " uperror = " << Hdivup <<endl;
for(int n = 1; n < nref; n++)
uprate[n-1] = log(Hdivup[n-1]/Hdivup[n]) / log(hP[n-1]/hP[n]);
cout << " convergence rate up = " << uprate <<endl;

cout << " pperror = " << L2pp <<endl;
```

```
for(int n = 1; n < nref; n++)
pprate[n-1] = log(L2pp[n-1]/L2pp[n]) / log(hP[n-1]/hP[n]);
cout << " convergence rate pp = " << pprate <<endl;

cout << " vphierror in H^1/2 = " << vphierror1 <<endl;
for(int n = 1; n < nref; n++)
vphirate1[n-1] = log(vphierror1[n-1]/vphierror1[n]) / log(htf[n-1]/htf[n]);
cout << " convergence rate vphi in H^1/2 = " << vphirate1 <<endl;

cout << " vphierror in L2 = " << vphierror2 <<endl;
for(int n = 1; n < nref; n++)
vphirate2[n-1] = log(vphierror2[n-1]/vphierror2[n]) / log(htf[n-1]/htf[n]);
cout << " convergence rate vphi in L2 = " << vphirate2 <<endl;

cout << " thetaerror in H^1/2 = " << thetaerror1 <<endl;
for(int n = 1; n < nref; n++)
thetarate1[n-1] = log(thetaerror1[n-1]/thetaerror1[n]) / log(htp[n-1]/htp[n]);
cout << " convergence rate theta in H^1/2 = " << thetarate1 <<endl;

cout << " thetaerror in L2 = " << thetaerror2 <<endl;
for(int n = 1; n < nref; n++)
thetarate2[n-1] = log(thetaerror2[n-1]/thetaerror2[n]) / log(htp[n-1]/htp[n]);
cout << " convergence rate theta in L2 = " << thetarate2 <<endl;

cout << " lamerror in H^1/2 = " << lamerror1 <<endl;
for(int n = 1; n < nref; n++)
lamrate1[n-1] = log(lamerror1[n-1]/lamerror1[n]) / log(htp[n-1]/htp[n]);
cout << " convergence rate lam in H^1/2 = " << lamrate1 <<endl;

cout << " lamerror in L2 = " << lamerror2 <<endl;
for(int n = 1; n < nref; n++)
lamrate2[n-1] = log(lamerror2[n-1]/lamerror2[n]) / log(htp[n-1]/htp[n]);
cout << " convergence rate lam in L2 = " << lamrate2 <<endl;

cout << " mesh size Of = " << hF <<endl;
cout << " mesh size Op = " << hP <<endl;
cout << " mesh size Gammafp in Of = " << htf <<endl;
cout << " mesh size Gammafp in Op = " << htp <<endl;
cout << " degrees of freedom Of = " << DOFf <<endl;
cout << " degrees of freedom Op = " << DOFp <<endl;
```

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