Modular Forms: Constructions & Applications

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Kurzzusammenfassung

Diese Dissertation vereint Resultate aus fünf Forschungsartikeln über Konstruktionen und Anwendungen von Modulformen und ihrer Verallgemeinerungen. Wir beginnen mit der Konstruktion neuer Quantenmodulformen der Tiefe zwei, was Resultate von Bringmann, Kaszian und Milas verallgemeinert. Dazu verbinden wir die Asymptotik gewisser falscher Thetafunktionen binärer quadratischen Formen mit mehrfachen Eichlerintegralen von Thetafunktionen. Quantenmodularität der falschen Thetafunktion folgt aus dem Verhalten dieser Integrale nahe der reellen Achse.

Anschließend betrachten wir das asymptotische Profil eines gewissen Eta-Theta Quotienten, welcher in der Partitionsfunktion der Verschränkungsentropie in der Stringtheorie auftritt. Insbesondere verallgemeinern wir Methoden von Bringmann und Dousse, und von Dousse und Mertens, um die auftretende meromormphe Jacobiform zu behandeln. Durch die Anwendung der Kreismethode nach Wright erhalten wir eine zweidimensionale Asymptotik für die von zwei Variablen abhängigen Koeffizienten des Eta-Theta-Quotienten.

Drittens untersuchen wir das asymptotische Verhalten der Erzeugendenfunktion ganzzahliger Partitionen, wessen Grade zu r kongruent modulo t sind, und wir bezeichnen diese Grade mit $N(r, t; n)$. Indem wir zeigen, dass die Erzeugendenfunktion über einer gewissen Schranke monoton wachsende Koeffizienten aufweist, sind wir in der Lage Inghams Taubersatz anzuwenden. Dies ergibt direkt, dass $N(r, t; n)$ in r bei fixiertem t für $n \to \infty$ gleichverteilt ist, was wiederum eine kürzlich aufgestelle Vermutung von Hou and Jagadeeson über ein Resultat konvexer Art bestätigt.

Das nachfolgende Kapitel ist der Untersuchung von Spuren zyklischer Integrale meromorpher Modulformen und ihrer Beziehung zu Koeffizienten harmonischer Maaßformen gewidmet. Indem wir auf Gittern der Signatur $(1, 2)$ arbeiten, ordnen wir zunächst eine lokal harmonische Maaßform einem Siegel Thetalift zu, unter Einbeziehung des Maaß Steigerungsoperators, durch explizite Berechnung der Steigerung der lokal harmonischen Maaßform und Benutzung des Standardarguments der Entfaltung des Thetaliftes. Danach übernehmen wir Techniken von Bruinier, Ehlen und Yang, um den Thetalift als konstanten Term einer q-Reihe zu berechnen (bis auf Terme, die für gewisse Klassen von Eingabefunktionen verschwinden), was sowohl die Koeffizienten von ξ-Urbildern unärer Thetafunktionen also auch Thetafunktionen beinhaltet. Da solche Urbilder harmonische Maaßformen sind, erhalten wir eine Beschreibung der Spuren in Form der Koeffizienten von Thetafunktionen und harmonischer Maaßformen. Durch die Wahl eines spezifischen Gitters in Beziehung zu quadratischen Formen und durch die Beobachtung, dass die den konstanten Term festlegenden Funktionen mit rationalen Koeffizienten gewählt werden

können, erhalten wir einen neuen Beweis eines kürzlichen Resultates von Alfes-Neumann, Bringmann und Schwagenscheidt.

Abschließend untersuchen wir die Beziehung zwischen Modulformen und selbstkonjugierten t-Kernpartitionen. Wir erhalten die Anzahl selbstkonjugierter 7-Kernpartitionen als einzelne Klassenzahl auf zwei Arten. Die erstere Art zeigen wir anhand von Modularitätsargumenten der Erzeugendenfunktion der Hurwitz-Klassenzahlen. Darüberhinaus bieten wir eine ergänzende kombinatorische Beschreibung zur Erklärung der Gleichheit an. Insbesondere konstruieren wir eine explizite Abbildung zwischen selbstkonjugierten t-Kernpartitionen und quadratischen Formen einer gegebenen Klassengruppe. Zusätzlich zeigen wir, dass das Geschlecht der quadratischen Form eindeutig ist, und wir bestimmen die Anzahl der Urbilder des Geschlechts. Mit Hilfe dieser Resultate können wir die Gleichheit zwischen der Anzahl 4-Kernpartitionen und der Anzahl selbstkonjugierter 7-Kernpartitionen auf bestimmten arithmetischen Progressionen zeigen. Neben dem Fall $t = 4$ betrachten wir auch, ob Gleichheiten zwischen t-Kernpartitionen und selbstkonjugierten $2t - 1$-Kernpartitionen möglich sind. Wir zeigen, dass dies für $t = 2$, 3, 5 nicht so ist, und wir bieten für $t \geq 6$ eine Vermutung sowie Teilresultate an.

Abstract

This thesis combines results of five research papers on the construction and applications of modular forms and their generalisations. We begin by constructing new examples of quantum modular forms of depth two, generalising results of Bringmann, Kaszian, and Milas. To do so, we relate the asymptotics of certain false theta functions of binary quadratic forms to multiple Eichler integrals of theta functions. Quantum modularity of the false theta functions follows from the behaviour of such integrals near the real line.

Next, we turn our attention to the asymptotic profile of a certain eta-theta quotient that arises in the partition function of entanglement entropy in string theory. In particular, we generalise methods of Bringmann and Dousse, and Dousse and Mertens, to deal with the meromorphic Jacobi form at hand. Applying Wright's circle method for Jacobi forms we obtain a bivariate asymptotic for the two-variable coefficients of the eta-theta quotient.

Thirdly, we investigate the asymptotic behaviour of the generating function of integer partitions whose ranks are congruent to r modulo t, denoted by $N(r, t; n)$. By proving that the series has monotonic increasing coefficients above some bound, we are in a position to apply Ingham's Tauberian theorem. This immediately implies that $N(r, t; n)$ is equidistributed in r for fixed t as $n \to \infty$, in turn implying a recent conjecture of Hou and Jagadeeson on a convexity-type result.

The following chapter is dedicated to an investigation of traces of cycle integrals of meromorphic modular forms and their relationship to coefficients of harmonic Maass forms. Working on lattices of signature $(1, 2)$, we first relate a locally harmonic Maass form to a Siegel theta lift involving the Maass raising operator by explicitly computing the raising of the locally harmonic Maass form, and using the usual unfolding argument for the theta lift. We then borrow techniques of Bruinier, Ehlen, and Yang to compute the theta lift as (up to terms that vanish for certain classes of input functions) the constant term in a q-series involving the coefficients of ξ-preimages of unary theta functions as well as theta functions. Since such preimages are harmonic Maass forms, we obtain a description of the traces in terms of coefficients of theta functions and harmonic Maass forms. Choosing a specific lattice related to quadratic forms and noting that the functions determining the constant term can be chosen to have rational coefficients, we obtain a new proof of a recent result of Alfes-Neumann, Bringmann, and Schwagenscheidt.

Finally, we investigate the relationship between modular forms and self-conjugate t-core partitions. We obtain the number of self-conjugate 7-cores as a single class number in two ways. The first we show with modularity arguments on the generating function of Hurwitz class numbers. We also provide a complementary combinatorial description to

explain the equality. In particular, we construct an explicit map between self-conjugate t-cores and quadratic forms in a given class group. Moreover, we show that the genus of the quadratic forms is unique, and determine the number of preimages of the genus. Using these results, we show an equality between the number of 4-cores and the number of self-conjugate 7-cores on specific arithmetic progressions. Aside from the $t = 4$ case, we consider whether equalities between t-cores and self-conjugate $2t-1$-cores are possible. We show for $t = 2, 3, 5$ that they are not, and offer a conjecture and partial results for $t \geq 6$.

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For those I have lost too soon.

Contents

Chapter I

Introduction and Statement of Objectives

I.1 Definitions and previous results

This thesis consists mostly of the research articles [ABMS, BKM, Ma1, Ma2, Ma3] (see page 158 for references for the introduction, statement of results, and outlook) that deal with various aspects of the construction and applications of modular forms and their generalisations. In this section we collect the required definitions, some background, and relevant examples.

I.1.1 Modular forms and Jacobi forms

Modular forms and their generalisations underpin a vast amount of number theory. For example, their Fourier coefficients often encode highly non-trivial information, e.g. the number of certain integer partitions. The values of the coefficients, their asymptotic behaviour, and more general behaviour of modular forms have applications throughout number theory and beyond.

To begin, we introduce some standard notation. We fix the upper half-plane $\mathbb{H} := \{\tau = u + iv \in \mathbb{C} \colon v > 0\}$. We mostly consider the Hecke congruence subgroup of level N[1], defined by

$$\Gamma_0(N) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c \right\}.$$

The group $\Gamma_0(N)$ acts on $\mathbb{H}$ via the Möbius transformation

$$M\tau = \frac{a\tau + b}{c\tau + d}, \qquad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

[1] One may also define modular forms for more general congruence subgroups.

A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight $k \in \mathbb{Z}$, level N, and character χ* if

$$f(M\tau) = \chi(d)(c\tau+d)^k f(\tau) \tag{I.1.1}$$

holds for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and f is bounded at all cusps $\Gamma_0(N)\backslash(\mathbb{Q}\cup\{\infty\})$ of $\Gamma_0(N)$. We write $f \in M_k(N,\chi)$. Functions that satisfy (I.1.1) that are allowed to have poles at cusps are called *weakly holomorphic modular forms*, and those that are allowed isolated poles in points in $\mathbb{H}$ are called *meromorphic modular forms.* We call f a *cusp form* if it vanishes at all cusps, and write $f \in S_k(N,\chi)$.

An extension of this definition is given by *half-integral weight modular forms* where, for $k \in \frac{1}{2}+\mathbb{Z}$, one replaces (I.1.1) by

$$f(M\tau) = \varepsilon_d^{-2k}\left(\frac{c}{d}\right)\chi(d)(c\tau+d)^k f(\tau),$$

where $\varepsilon_d := i$ for $d \equiv 3 \pmod 4$ and $\varepsilon_d := 1$ if $d \equiv 1 \pmod 4$. Here, $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. We also require the $4 \mid N$ for half-integral weight. A prototypical example of a half-integral weight modular form is the modular form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ given by the *Dedekind η-function*, defined as

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{j\geq 1}\left(1-q^j\right),$$

for $q := e^{2\pi i\tau}$.

In this thesis, we are also particularly interested in the two-variable generalisations of modular forms, known as *Jacobi forms.* These are functions $f\colon \mathbb{C}\times\mathbb{H} \to \mathbb{C}$, $(z,\tau)\mapsto f(z;\tau)$ that satisfy a certain modularity property in τ and an elliptic transformation in z (i.e., a simple transformation under shifts by $\mathbb{Z}\tau+\mathbb{Z}$), and certain growth conditions. We refer the reader to the book of Eichler and Zagier [EZ] that first developed the theory of Jacobi forms for their full definitions and properties. As an indication of their importance, note that Eichler and Zagier relied heavily on them on the seminal work proving the Saito–Kurokawa conjecture [EZ, Za4].

Of particular importance for this thesis is the prototypical *Jacobi theta function*, defined for $z \in \mathbb{C}$, $\tau \in \mathbb{H}$ by

$$\vartheta(z;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n\left(z+\frac{1}{2}\right)}.$$

I.1.2 Generalisations of modular forms

In this section we describe some of the generalisations of modular forms found in this thesis. Note that these constructions may be extended to include elliptic variables z in analogy to Jacobi forms. We follow [BFOR, Chapter 4].

I.1.2.1 Harmonic Maass forms and mock modular forms

For $k \in \frac{1}{2}\mathbb{Z}$, the *weight k hyperbolic Laplacian* is given by

$$\Delta_k := -4v^2 \frac{\partial}{\partial \tau}\frac{\partial}{\partial \overline{\tau}} + 2ikv\frac{\partial}{\partial \overline{\tau}}.$$

Weight k harmonic Maass forms are certain real-analytic functions that are annihilated by Δ_k. More precisely, a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0(N)$ (where $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$) is any smooth function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying the following three properties:

(1) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have that

$$f(M\tau) = \begin{cases} (c\tau + d)^k f(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-2k} (c\tau+d)^k f(\tau) & \text{if } k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}, \end{cases}$$

where again $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.

(2) We have that $\Delta_k(f) = 0$.

(3) There exists some polynomial $P_f(\tau) \in \mathbb{C}[q^{-1}]$ such that

$$f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$$

as $v \to \infty$ for some $\varepsilon > 0$. Analagous conditions are required at all cusps.

Further relaxation of the growth condition to be $O(e^{\varepsilon v})$ gives *harmonic Maass forms of manageable growth*, which by a slight abuse we also refer to as a harmonic Maass form in this thesis. Define the operator

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

A seminal paper of Bruinier and Funke [BF] first introduced ξ_k and showed that it maps the space of harmonic Maass forms of weight k to the space of cusp forms of weight $2-k$, and similarly maps the space of harmonic Maass forms of manageable growth of weight k to the space of weakly holomorphic modular forms of weight $2-k$. Moreover, each of these maps is surjective.

A harmonic Maass form f splits naturally into two pieces f^+, f^- where f^+ is holomorphic and f^- is non-holomorphic. In particular, f has a Fourier expansion of the shape [BFOR, Lemma 4.3]

$$f(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n + c_f(0) v^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n) \Gamma(1-k, -4\pi n v) q^n,$$

where $\Gamma(s,z) := \int_z^\infty e^{-t}t^{s-1}dt$ denotes the *incomplete Gamma function.* Then we define

$$f^+(\tau) := \sum_{n \gg -\infty} c_f^+(n)q^n, \qquad f^-(\tau) := c_f(0)v^{1-k} + \sum_{\substack{n \ll \infty \\ n \neq 0}} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n.$$

A *mock modular form* of weight k is the holomorphic part f^+ of a harmonic Maass form of weight k for which the non-holomorphic part f^- is non-trivial. For a mock modular form g, any non-trivial function h for which $g+h$ is modular is known as a *completion* of g. The image of $g+h$ under ξ is known as the *shadow* of $g+h$.

A prototypical example was given by Zagier [Za2] as follows. Let $H(n)$ be the usual Hurwitz class number, which counts the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral binary quadratic forms of discriminant $-n$, weighted by $\frac{1}{2}$ times the order of their automorphism group. Then

$$\mathcal{H}(\tau) := -\frac{1}{12} + \sum_{n \geq 1} H(n)q^n + \frac{1}{8\pi\sqrt{v}} + \sum_{n \geq 1} n\Gamma\left(\frac{1}{2}, 4\pi n^2 v\right) q^{-n^2}$$

is a harmonic Maass form (of manageable growth) of weight $\frac{3}{2}$ on $\Gamma_0(4)$. Thus

$$\mathcal{H}^+(\tau) := -\frac{1}{12} + \sum_{n \geq 1} H(n)q^n$$

is a mock modular form of the same weight. Its shadow is the unary theta function

$$-\frac{1}{16\pi}\sum_{n \in \mathbb{Z}} q^{n^2}.$$

As mentioned, one can generalise these definitions to include an elliptic variable to obtain mock Jacobi forms. A fundamental example of a mock Jacobi form that we make great use of is Zwegers' μ-function [Zw], defined by

$$\mu(z_1, z_2; \tau) := \frac{e^{\pi i z_1}}{\vartheta(z_2;\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau} e^{2\pi i n z_2}}{1 - e^{2\pi i n \tau} e^{2\pi i z_1}}$$

for $z_1, z_2 \in \mathbb{C} \backslash \mathbb{Z}\tau + \mathbb{Z}$. Zwegers showed that the addition of a certain non-holomorphic completion term (written in terms of a Mordell integral) to μ yields a non-holomorphic Jacobi form. This breakthrough unified the theory of the elusive mock theta functions of Ramanujan into a concrete framework, and spurred a vast amount of research in the past two decades.

I.1.2.2 Quantum modular forms

In 2010, Zagier [Za3] introduced a new type of modular object, known as quantum modular forms, following investigations into Kontsevich's "strange" function [Za3, Za5]. A *quantum modular form* is essentially a function $f\colon \mathcal{Q} \to \mathbb{C}$ for some fixed $\mathcal{Q} \subseteq \mathbb{Q}$, whose errors of modularity (for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^k f(M\tau)$$

are in some sense "nicer" than the original function. Often, for example, the original function f is defined only on $\mathbb{Q}$, but the errors of modularity can be defined on some open subset of $\mathbb{R}$. The set $\mathcal{Q}$ is called the quantum set of the function f. One may also consider quantum modular forms for $M \in \Gamma$, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Further, Zagier also considered so-called "strong" quantum modular forms, where one considers asymptotic expansions and not just values. Leaving this definition intentionally vague allowed Zagier to collect many examples in the same heading.

Additional examples of quantum modular forms are given in [BM, CMW], where characters of vertex operator algebras were explored. Further examples arise at the interface of physics and knot theory, after investigations into Kashaev invariants [HK, HL], as well as limits of quantum invariants of 3-manifolds and knots [Za3].

One may assemble strong quantum modular forms into vector-valued versions as follows. For $1 \leq j \leq N \in \mathbb{N}$, a collection of functions $f_j\colon \mathcal{Q} \to \mathbb{C}$ is called a *strong vector-valued quantum modular form* of weight k, multiplier $\chi = (\chi_{j,\ell})_{1 \leq j,\ell \leq N}$, and quantum set $\mathcal{Q}$ for $\mathrm{SL}_2(\mathbb{Z})$ if, for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have that[2]

$$f_j(\tau) - (c\tau + d)^{-k} \sum_{1 \leq \ell \leq N} \chi_{j,\ell}(M) f_\ell(M\tau)$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there.

In Chapter II, extensions to higher depths appear (more specifically, to depth two), following investigations of Bringmann, Kaszian, and Milas [BKMi1, BKMi2] into certain characters of vertex algebras. They considered higher-depth analogues of quantum modular forms, and provided two examples of such forms of depth two. In the simplest case these are functions that satisfy

$$f(\tau) - (c\tau + d)^k f(M\tau) \in \mathsf{Q}_k(\Gamma)\mathcal{O}(R) + \mathcal{O}(R),$$

where $\mathsf{Q}_k(\Gamma)$ is the space of strong quantum modular forms of weight k on Γ, and $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$. As noted in [BKMi1], the easiest (trivial) examples come from multiplying two depth one forms.

[2] Correcting a minor typographical error of [Ma1].

I.1.3 Integer partitions

For a positive integer n a *partition* $\Lambda \coloneqq (\lambda_1, \lambda_2, \dots, \lambda_s)$ of n is a non-increasing sequence of non-negative integers λ_j such that $\sum_{1\leq j\leq s} \lambda_j = n$. The theory of partitions and modular forms is intricately interwoven. For example, let $p(n)$ count the total number of distinct partitions of n. Then the generating function for integer partitions may be written in terms of η by

$$\sum_{n\geq 0} p(n)q^n = \prod_{j\geq 1} \left(1-q^j\right)^{-1} = \frac{q^{\frac{1}{24}}}{\eta(\tau)}.$$

The relationship between coefficients of (generalisations of) modular forms and various partition-theoretic objects has been an area of vast interest in the last century. Hardy and Ramanujan developed their now-ubiquitous circle method to first study the asymptotic behaviour of $p(n)$ [HR]. A further fundamental investigation in this field is focused on the *rank* of a partition, defined to be the largest part minus the number of parts. The rank statistic was first introduced by Dyson [Dy] in an attempt to combinatorially explain the famous Ramanujan congruences [BR], and since its introduction has a storied history, detailed further in Chapter IV. Denoting the number of partitions on n whose rank is congruent to r modulo t by $N(r,t;n)$, a groundbreaking paper of Bringmann and Ono [BO] showed that the generating function of $N(r,t;n)$ is intimately related to the theory of harmonic Maass forms and mock modular forms.

Furthermore, partitions lie at the interface of theory of modular forms, combinatorics, and other areas of mathematics - of particular interest to the current thesis are t-core partitions (defined in Section I.2.5), which encode the representation theory of the symmetric groups S_n and A_n. Techniques from each field have been developed in the last century that build the web of connections between these areas.

I.1.4 Asymptotic expansions of modular forms

The asymptotic behaviour of coefficients of modular forms and their implications in various areas of mathematics is a long-studied problem. For example, in 1918 Hardy and Ramanujan proved their famous asymptotic formula for $p(n)$ [HR], which states that as $n \to \infty$ we have

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}.$$

To show this, they developed the now ubiquitous tool of the Hardy-Ramanujan circle method. We follow the exposition of the circle method given in [BFOR, Proof of Theorem 4.13]. In essence, the idea of the circle method is that for a series $A(q) = \sum_{n\geq 0} a(n)q^n$

with radius of convergence 1, then by Cauchy's theorem, one may extract the coefficients as

$$a(n) = \frac{1}{2\pi i} \int_C \frac{A(q)}{q^{n+1}} dq,$$

where C is an arbitrary path in the unit disc that loops around 0 exactly once in the counterclockwise direction. Many of the applications of the circle method focus on coefficients $a(n)$ where the singularities of the generating function A lie at roots of unity on the unit disc, and which are well-understood. In particular, near these singularities, one can often find nice approximations of A. These provide the main asymptotic terms, and the remaining terms contribute to a smaller error term.

For $p(n)$, for example, Hardy and Ramanujan computed that the pole at $q = 1$ is the dominant one (i.e. that it gives the largest growth), and away from $q = 1$ the contribution is far smaller, and thus contributes to an error term of lower order. Rademacher improved the estimates of Hardy and Ramanujan to provide an exact formula for $p(n)$, demonstrating the power of the circle method. Bringmann used similar techniques, following investigations of Dragonnete [Dr] and Andrews [Andr], to compute the asymptotic behaviour of the generating function for $N(r, t; n)$ for odd t [Bri], a result which was central in the paper of Hou and Jagadeeson [HJ] that inspired the results of Chapter IV.

In the course of this thesis, we see two special cases of the circle method, detailed further below.

I.1.4.1 Ingham's Tauberian Theorem

Ingham's Tauberian theorem is essentially a special case of the Hardy-Ramanujan circle method, where the circle method is computed for a large family of similarly-behaved functions. Consider the case when $a(n)$ are non-negative, real, and (weakly) monotonically increasing[3]. Assuming some technical conditions on the growth of $A(e^{-z})$ for $z \in \mathbb{H}$ such that $z \to 0$ in certain regions, Ingham's Tauberian theorem [In] states that if

$$A(e^{-t}) \sim \lambda \log\left(\frac{1}{t}\right)^{\alpha} t^{\beta} e^{\frac{\gamma}{t}} \text{ as } t \to 0^{+},$$

with $\lambda, \alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > 0$, then

$$a(n) \sim \frac{\lambda \gamma^{\frac{\beta}{2}+\frac{1}{4}} \log(n)^{\alpha}}{2^{\alpha+1}\sqrt{\pi} n^{\frac{\beta}{2}+\frac{3}{4}}} e^{2\sqrt{\gamma n}}, \qquad \text{as } n \to \infty.$$

[3]There is a similar style of statement for the sum of the coefficients if the coefficients $a(n)$ are not monotonic.

As noted in [BJM], the technical conditions alluded to above (as well as the log term) have often been missed in recent applications of the theorem. However, as shown by Bringmann, Jennings-Shaffer, and Mahlburg, for modular objects the technical conditions are essentially always satisfied, and often the log term is not needed.

I.1.4.2 Wright's circle method

Wright [Wr1, Wr2] developed another version of the circle method, which is a middle-ground between the ease of Ingham's Tauberian theorem and the power of the full Hardy-Ramaujan circle method. Essentially, Wright's circle method splits off the main growth term toward the dominant pole(s), and bounds the contribution away from this/these pole(s). Thus one obtains an asymptotic estimate for the main term in a similar fashion to the full circle method, but trades the possibility of obtaining exact formulae for the ease of dealing with the remaining contributions with a uniform bound.

Bringmann and Dousse [BD], and Dousse and Mertens [DM] pioneered the use of a modified version of Wright's circle method to deal with Jacobi forms in their investigations of the partition crank and rank, respectively. A variant of this technique is used heavily in Chapter III where we investigate the bivariate asymptotic behaviour of a certain Jacobi form.

I.2 Statement of objectives

I.2.1 A family of vector-valued quantum modular forms of higher depth

In the first project of this thesis, presented in Chapter II, I obtain an infinite family of examples of a relatively new type of modular object, known as quantum modular forms of higher depth, introduced by Bringmann–Kaszian–Milas in a series of papers [BKMi1, BKMi2, BKMi3]. There, the authors developed the theory after investigations into the character of a vertex operator algebra $W(p)_{A_2}$, where $p \geq 2$, associated to the root lattice of type A_2 of the simple Lie algebra $\mathfrak{sl}_3$, and gave two isolated examples of quantum modular forms of depth two.

For positive definite binary quadratic form $Q(n_1, n_2)$ and a finite set $\mathscr{S}$ of pairs of non-zero rational points α, I study the partial theta function

$$F(q) := \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n_1, n_2 \geq 0} q^{Q(n_1, n_2)},$$

where $\varepsilon\colon \alpha \mapsto \mathbb{R}\backslash\{0\}$. Imposing some symmetry conditions on $\mathscr{S}$ and ε, I show the following theorem, giving an infinite family of vector-valued quantum modular forms of depth two and weight one. As noted in Section II, the techniques presented also immediately give an infinite family of scalar-valued quantum modular form of higher depth.

Theorem I.2.1. *The function F is, up to one-dimensional boundary terms, a sum of components of a vector-valued quantum modular form of depth two and weight one on* $\mathrm{SL}_2(\mathbb{Z})$ *with some explicit quantum set $\mathcal{Q}$ defined in Section II.4. In some special cases, F itself is a single component of a vector-valued form.*

To prove this theorem, I follow the ideas of [BKMi2] in this more general setting. First, I relate $F(q)$ asymptotically to a double Eichler integral of the shape

$$\int_{-\bar{\tau}}^{i\infty}\int_{\omega_1}^{i\infty}\frac{f_1(\omega_1)f_2(\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}}d\omega_2 d\omega_1,$$

where the f_j lie in the space of vector-valued modular forms on $\mathrm{SL}_2(\mathbb{Z})$. Such integrals exhibit higher-depth quantum modularity as τ tends radially to a rational point on the real line, and so $F(q)$ inherits the same behaviour. One difficulty to overcome here is the definition of the quantum set $\mathcal{Q}$. Since the situation is so generic, the definitions are rather technical, using the theory of quadratic Gauss sums to obtain the quantum sets. Nevertheless, these quantum sets are seen to always be infinite (although very occasionally somewhat sparse).

By relating the double Eichler integral to certain non-holomorphic theta functions with coefficients given by double error functions, I then place this in a modular setting by viewing it as the "purely non-holomorphic part" of an indefinite theta function of signature $(2,2)$.

Proposition I.2.2. *The indefinite theta function of signature $(2,2)$ defined in Section II.10 has purely non-holomorphic part $\Theta(\tau)\mathcal{E}(\tau)$, where $\Theta(\tau)$ is a theta function of signature $(2,0)$.*

I.2.2 The asymptotic profile of an eta-theta quotient related to entanglement entropy in string theory

The second paper in this thesis, presented in Chapter III, deals with an application of the theory of modular forms to string theory by determining the asymptotic profile of a certain eta-theta quotient. In particular, the weight minus three and index zero meromorphic Jacobi form

$$f(z;\tau) := \frac{\vartheta(z;\tau)^4}{\eta(\tau)^9\vartheta(2z;\tau)}$$

appears in the partition function for both the Melvin model [RT] and the conical entropy of both the open and closed superstring [HNTW]. To perform certain calculations, for example to find the behaviour of the partition function in the UV limit (that is, as $\tau \to 0^+$), one needs to find the asymptotic behaviour of the partition function, in turn requiring knowledge of the asymptotic behaviour of such Jacobi forms. Letting

$$f(z;\tau) =: \sum_{n\geq 0} b(m,n)\zeta^m q^n,$$

I show how to obtain a uniform bivariate asymptotic behaviour of the coefficients $b(m,n)$ whenever $|m| \leq \frac{\sqrt{n}}{6\pi\sqrt{2}}\log(n)$. In particular, I prove the following theorem.

Theorem I.2.3. *For $\beta := \pi\sqrt{\frac{2}{n}}$ and $|m| \leq \frac{1}{6\beta}\log(n)$ we have that*

$$b(m,n) = (-1)^{m+\delta+\frac{1}{2}}\frac{\beta^5}{2^7\pi^5(2n)^{\frac{1}{4}}}e^{2\pi\sqrt{2n}} + O\left(n^{-\frac{13}{4}}e^{2\pi\sqrt{2n}}\right)$$

as $n \to \infty$. Here, $\delta := 1$ if $m < 0$ and $\delta = 0$ otherwise.

The proof of this theorem relies on Wright's circle method applied to Jacobi forms, a technique that was pioneered by Bringmann-Dousse [BD] and Dousse-Mertens [DM] in their investigations into the partition rank and crank, respectively. However, the fact that $f(z;\tau)$ has a pole at $z = \frac{1}{2}$ necessitates a modification to the arguments.

To combat this, one can deform the path of the integrals determining the Fourier coefficient of ζ^m of f, denoted by f_m, picking up a residue term in the process. Carefully computing the asymptotic behaviour of both the residue term and the remaining expression for f_m toward the dominant pole at $q = 1$ and away from the dominant pole, one obtains strong enough bounds to apply Wright's circle method for Jacobi forms.

I.2.3 Asymptotic equidistribution and convexity for partition ranks

Numerous statistics involving partitions have been introduced in the last century, including the rank of a partition, defined to be the largest part minus the number of parts. We denote the number of partitions of n with rank m by $N(m,n)$, with a refinement given by $N(r,t;n)$. In [BO], Bringmann and Ono showed that the generating function of $N(r,t;n)$ is essentially the holomorphic part of a Maass form on some congruence subgroup, and further used this to give an alternative proof for Ramanujan-type congruences for $N(r,t;n)$ than the proof given by Atkin and Swinnerton-Dyer [AS].

The aim of the project detailed in Chapter IV of this thesis is two-fold; to determine the asymptotic behaviour of $N(r,t;n)$ for all t, and to prove a convexity-type conjecture of Hou and Jagadeeson. The central argument relies on the modular properties of the generating function of $N(r,t;n)$. The first main result is the following.

Theorem I.2.4. *For fixed $0 \leq r < t$ and $t \geq 2$ we have*

$$N(r,t;n) \sim \frac{1}{t}p(n) \sim \frac{1}{4t\sqrt{3}n}e^{2\pi\sqrt{\frac{n}{6}}}$$

as $n \to \infty$. In particular, $N(r,t;n)$ is asymptotically equidistributed (i.e. for large n and fixed t, the asymptotics are independent of the residue class r).

The main idea of the proof is to determine the asymptotic behaviour of $N(r,t;n)$ by using Ingham's Tauberian theorem [In]. To do so, I first show that $N(r,t;n)$ is a monotonic function of n above some bound using standard techniques in analysing coefficients of q-series, generalising results of Chan and Mao [CM]. In order to use the Tauberian theorem, it is necessary to determine the asymptotic behaviour of the generating function $\sum_{n\geq 0} N(r,t;n)q^n$ toward $q=1$. To find this, I rewrite the generating function in terms of the mock-modular higher level Appell functions [Zw], and employ their modular properties as well as their modular completions.

Further, Hou and Jagadeesan [HJ] (in the spirit of Bessenrodt and Ono [BO]) showed that for $0 \leq r \leq 2$ we have

$$N(r,3;a)N(r,3;b) > N(r,3;a+b)$$

for all a, b larger than some specific bound. At the end of the same paper, the authors offered the following conjecture on a more general convexity result.

Conjecture I.2.5. *For $0 \leq r < t$ and $t \geq 2$ then*

$$N(r,t;a)N(r,t;b) > N(r,t;a+b)$$

for sufficiently large a and b.

A simple consequence of the equidistribution of the partition rank is the following theorem.

Theorem I.2.6. *The conjecture of Hou and Jagadeeson is true.*

I.2.4 Cycle integrals of meromorphic modular forms and rationality

The penultimate project in this thesis, presented in Chapter V, concerns the application of modular forms in the theory of theta lifts. In particular, in [ABMS] Alfes-Neumann, Bringmann, Schwagenscheidt, and I show that linear combinations of traces of cycle integrals of meromorphic modular forms may be written in terms of coefficients of harmonic Maass forms.

In [KZ2, KZ1, Za1], Kohnen and Zagier introduced and investigated certain functions $f_{k,d}$ in related a lift between certain spaces of modular forms. The cusp forms $f_{k,d}$, of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, (and their variations) have seen a wide array of applications, for example they appear in the Fourier coefficients of holomorphic kernel functions for the Shimura and Shintani lifts [Ko, KZ2]. Furthermore, $f_{k,d}$ give an important class of functions that have rational periods [KZ1]. In particular, certain linear combinations of the cycle integrals

$$\int_{c_Q} f_{k,d}(\tau)Q(\tau,1)^{k-1}d\tau$$

were shown to be rational. Here, c_Q is the image in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ of the geodesic $\{\tau \in \mathbb{H}\colon a|\tau|^2 + bu + c = 0\}$ where $\tau = u + iv$, and $Q = [a, b, c]$ lies in the space of quadratic forms of discriminant D, $\mathcal{Q}_D$.

More recently, a landmark paper of Duke, Imamoḡlu, and Tóth [DIT] detailed the relationship between $f_{k,d}$ and certain errors of modularity of cycle integrals. A key step was to produce a preimage of a certain generating function of the cycle integrals of weakly holomorphic modular forms.

Bringmann, Kane, and Kohnen [BKK] continued this investigation. However, there was an obstacle to generalising the results of [DIT] to the functions $f_{k,d}$ in finding certain preimages. The authors provided an unprecedented approach in modular forms to allow certain jump singularities of their functions in order to rectify this, introducing the function (for $k \geq 2$ and D a non-square discriminant)

$$\mathcal{F}_{1-k,D}(\tau) = \frac{D^{\frac{1}{2}-k}}{2\binom{2k-2}{k-1}\pi} \sum_{Q=[a,b,c]\in\mathcal{Q}_D} \operatorname{sgn}(a|\tau|^2 + bu + c)Q(\tau,1)^{k-1}\beta\left(\frac{Dv^2}{|Q(\tau,1)|^2}; k-\frac{1}{2}, \frac{1}{2}\right),$$

where $\beta(w; s, r) := \int_0^w t^{s-1}(1-t)^{r-1}dt$ is the incomplete beta function. The function $\mathcal{F}_{1-k,D}$ is an example of a new type of automorphic object - a locally harmonic Maass form. That is, it behaves like a harmonic Maass forms apart from an exceptional set of density 0. While it is known that a harmonic Maass form cannot map to a cusp form under both of the differential operators D^{2k-1} and ξ_{2-2k}, the fact that $\mathcal{F}_{1-k,D}$ has singularities rectifies this. In fact, it was shown that under both operators, $\mathcal{F}_{1-k,D}$ maps to a multiple of $f_{k,D}$ [BKK]. As a simple corollary of this connection, one obtains that the even periods of $f_{k,D}$ are rational.

Alfes-Neumann, Bringmann, and Schwagenscheidt [ABS] extended the rationality result of Kohnen and Zagier to traces of refined versions of $f_{k,D}$. Their results rely on the connection between $f_{k,D}$ and $\mathcal{F}_{1-k,D}$ and the Shintani lift. In Chapter V, we extend the results of [ABS] to a vector-valued setting and showed that the traces can be written as the constant term of a q-expansion involving coefficients of certain harmonic Maass forms and theta function.

Theorem I.2.7. *For all $k \in \mathbb{N}$ certain linear combinations of the traces of the cycle integrals are a constant term of a Fourier expansion involving coefficients of harmonic Maass forms and theta functions.*

The proof involves using that the traces can be written as a special value of the iterated Maass raising operator applied to a certain locally harmonic Maass form. It is then possible to realise this as a regularised theta lift, following Borcherds classical construction [Bor]. Finally, using Stokes' theorem along with techniques of Bruinier, Ehlen, and Yang [BEY] we describe the theta lift as a constant term in the Fourier expansion described.

Since all of the coefficients in the functions determining the Fourier expansion can be chosen to be rational, these results also give an elegant alternative proof to the results of [ABS]. We also give interesting examples that may be immediately concluded from this theorem, e.g. for $L_D(s)$ the usual L-function associated to a non–square discriminant $D > 0$ and Hurwitz class numbers $H(n)$ it may be shown that

$$\operatorname{tr}_{f_{2,[1,0,1]}}(D) = -40L_D(-1) - 4 \sum_{\substack{n,m\in\mathbb{Z} \\ n\equiv D \pmod 2}} H\left(D - n^2 - m^2\right).$$

Further examples of this type, and others involving the smallest parts partition function may also be easily deduced.

I.2.5 On t-core and self-conjugate $(2t-1)$-core partitions in arithmetic progressions

The final project in this thesis concerns the application of modular forms to properties of certain t-core partitions, defined as follows. The *Ferrers–Young diagram* of a partition Λ of n is the s-rowed diagram

$$\begin{array}{llll} \bullet & \bullet & \cdots & \bullet \qquad\qquad \lambda_1 \text{ dots} \\ \bullet & \bullet & \cdots & \bullet \qquad\qquad \lambda_2 \text{ dots} \\ \cdot & & & \\ \cdot & & & \\ \bullet & \cdots & \bullet & \qquad\qquad\quad \lambda_s \text{ dots.} \end{array}$$

We label the cells of the Ferrers–Young diagram as if it were a matrix, and let λ'_k denote the number of dots in column k. The *hook length* of the cell (j,k) in the Ferrers–Young diagram of Λ equals

$$h(j,k) := \lambda_j + \lambda'_k - k - j + 1.$$

If no hook length in any cell of a partition Λ is divisible by t, then Λ is a *t-core partition*. A partition Λ is said to be *self-conjugate* if it remains the same when rows and columns are switched.

Such t-core partitions are intricately linked to various areas of number theory and beyond. For example, Garvan, Kim, and Stanton [GKS] used t-core partitions to investigate special cases of the famous Ramanujan congruences for the partition function. Furthermore, t-core partitions encode the modular representation theory of the symmetric groups S_n and A_n [GO, FS].

For $t, n \in \mathbb{N}$ we let $\mathrm{c}_t(n)$ denote the number of t-core partitions of n, along with $\mathrm{sc}_t(n)$ the number of self-conjugate t-core partitions of n. Ono and Sze [OS] investigated the relation between 4-core partitions and Hurwitz class numbers $H(|D|)$, and showed that if $8n+5$ is square-free, then

$$c_4(n) = \frac{1}{2}H(32n+20).$$

More recently Ono and Raji [OR] showed similar relations between self-conjugate 7-core partitions and certain Hurwitz class numbers. In particular, by combining the results of Ono-Sze and Ono-Raji and using elementary congruence conditions, one may easily show that for $n \not\equiv 4 \pmod 7$ and $56n+21$ square-free, we have

$$2\,\mathrm{sc}_7(8n+1) = \mathrm{c}_4(7n+2). \tag{I.2.1}$$

This fact hints at a deeper relationship between sc_{2t-1} and c_t, which we investigate. Our main results pertain to the case $t=4$. In particular, we give a formula for $\mathrm{sc}_7(n)$ in terms of a single class number. In order to do so, let $\ell \in \mathbb{N}_0$ be chosen maximally such that $n \equiv -2 \pmod{2^{2\ell}}$ and define

$$D_n := \begin{cases} 28n+56 & \text{if } n \equiv 0,1 \pmod 4, \\ 7n+14 & \text{if } n \equiv 3 \pmod 4, \\ D_{\frac{n+2}{2^{2\ell}}-2} & \text{if } n \equiv 2 \pmod 4, \end{cases} \qquad \nu_n := \begin{cases} \frac{1}{4} & \text{if } n \equiv 0,1 \pmod 4, \\ \frac{1}{2} & \text{if } n \equiv 3 \pmod 8, \\ \nu_{\frac{n+2}{2^{2\ell}}-2} & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

A binary quadratic form is called *primitive* if $\gcd(a,b,c)=1$ and, for a prime p, *p-primitive* if $p \nmid \gcd(a,b,c)$. We let $H_p(D)$ count the number of p-primitive classes of integral binary quadratic forms of discriminant $-D$, with the same weighting as $H(D)$. The main result of our work in this direction is the following.

Theorem I.2.8. *For every $n \in \mathbb{N}$ we have*

$$\mathrm{sc}_7(n) = \nu_n H_7(D_n).$$

The first technique we use to show this relies on standard techniques in modular forms, yet yields an elegant and surprisingly simple proof. However, this approach does not yield a combinatorial explanation. To remedy this, and to complement the theorem above, we also provide the following result. The proof relies on constructing a map ϕ between self-conjugate 7-core partitions and binary quadratic forms. In order to construct such a map, we use the combinatorial structures of abaci and extended t-residue diagrams associated to partitions, along with classical results of Gauss connecting solutions of the equation $x^2 + y^2 + z^2 = n$ and the genus of binary quadratic forms in a particular class group.

Theorem I.2.9. *For every $n \in \mathbb{N}$, the image of ϕ is a unique non-principal genus of 7-primitive and 2-totally imprimitive binary quadratic forms with discriminant $-28n - 56$. Moreover, suppose that ℓ is chosen maximally such that $n \equiv -2 \pmod{2^{2\ell}}$ and $\frac{7n+14}{2^{2\ell}}$ has r distinct prime divisors. Then every equivalence class in this genus is the image of $\nu_n 2^r$ many self-conjugate 7-cores of n.*

Aside from the case $t = 4$, we also provide results on whether equalities like (I.2.1) can hold for other t non-trivially. We prove that they cannot in the cases of $t = 2, 3, 5$, and offer the following conjecture and partial results for $t \geq 6$.

Conjecture I.2.10. *The only occurrence of arithmetic progressions for which c_t and sc_{2t-1} agree up to integer multiples non-trivially (even asymptotically) is when $t = 4$.*

Chapter II

A family of vector-valued quantum modular forms of depth two

This chapter is based on a paper published in *The International Journal of Number Theory* [Ma1].

II.1 Introduction and statement of results

In a celebrated paper of Zagier, the concept of quantum modular forms is introduced, following investigations into Kontsevich's "strange" function [22, 23], given by

$$K(q) := 1 + \sum_{n=1}^{\infty} (q;q)_n,$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$ is the q-Pochhammer symbol, and $q := e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$. In particular, $K(q)$ does not converge on any open subset of $\mathbb{C}$, but is seen to be a finite sum at any root of unity. Zagier shows that at roots of unity ζ, the function $K(\zeta)$ agrees to infinite order with the Eichler integral of $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty$ (see page 959 of [22] for the precise definition of the Eichler integral in this context), and hence inherits the Eichler integral's quantum modular properties.

Here we give a brief description of the essence of what a quantum modular form is, and for a full introduction refer the reader to e.g. Chapter 21 of [3]. A quantum modular form is essentially a function $f\colon \mathcal{Q} \to \mathbb{C}$ for some fixed $\mathcal{Q} \subseteq \mathbb{Q}$, whose errors of modularity (for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$)

$$f(\tau) - (c\tau + d)^k f(M\tau) \tag{II.1.1}$$

are in some sense "nicer" than the original function. Often, for example, the original function f is defined only on $\mathbb{Q}$, but the errors of modularity can be defined on some open subset of $\mathbb{R}$. The set $\mathcal{Q}$ is called the quantum set of the function f. One may also consider quantum modular forms for $M \in \Gamma$, a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Further, Zagier also considered so-called "strong" quantum modular forms, where one considers asymptotic expansions and not just values. Leaving this definition of quantum modular forms intentionally vague allowed Zagier to collect many examples in the same heading.

Since their introduction, there has been an explosion of research into quantum modular forms in many guises, and they appear in work in many areas. For example, in [12] the authors consider a certain generalisation of $K(q)$ and investigate its quantum properties. It is shown to have intricate connections to the Habiro ring (introduced in [14]) and implications therein to combinatorics, in particular to the generating function for ranks of strongly unimodal sequences, are explored.

There are also deep connections between quantum modular forms and other areas. For example, the connection between them and mock modular forms (surveyed in e.g. [19]) is investigated in papers such as [4, 9, 10], among others. Furthermore, interesting examples of quantum modular forms exist in the interface of physics and knot theory, see e.g. a study of Kashaev invariants of (p, q)-torus knots in [15, 16] and investigations of Zagier into limits of quantum invariants of 3-manifolds and knots [23] - indeed, this is the reason that Zagier chose the name "quantum" modular forms.

An additional example is given in [8, 11], where characters of vertex operator algebras are explored, and it is shown that natural parts of these characters are quantum modular forms (of depth one). Motivated in part by these discoveries, the authors of [6] consider higher-dimensional analogues, defining so-called higher depth quantum modular forms, and provide two examples of such forms of depth two. In the simplest case, these are functions that satisfy

$$f(\tau) - (c\tau + d)^k f(M\tau) \in \mathsf{Q}_k(\Gamma)\mathcal{O}(R) + \mathcal{O}(R),$$

where $\mathsf{Q}_k(\Gamma)$ is the space of quantum modular forms of weight k on Γ, and $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$. As noted in [6], the easiest (trivial) examples come from multiplying two depth one forms - however, the two examples discussed therein appear to be non-trivial examples.

Again, these examples arise from a physics perspective. In fact, they come from the character of a vertex operator algebra $W(p)_{A_2}$, where $p \geq 2$, associated to the root lattice of type A_2 of the simple Lie algebra $\mathfrak{sl}_3$. The authors show that the character can be decomposed into two distinct functions, each of which are quantum modular forms of depth two on some subgroup of the full modular group. In a follow-up paper [7] the same authors also show that their functions can be viewed as vector-valued quantum modular forms of depth two on all $\mathrm{SL}_2(\mathbb{Z})$ (see Section II.2.4 for definitions).

In this paper we require (II.1.1) to be real-analytic, and the functions we consider will satisfy the properties of strong quantum modular forms. We construct a generalisation of a function called F_1 defined in [6,7]. In doing so, we provide an infinite family of non-trivial vector-valued quantum modular forms of depth two. We define our generalisation F as a sum of three terms, $F(q) \coloneqq F_1(q) + F_2(q) + F_3(q)$ where

$$F_1(q) \coloneqq \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} q^{Q(n)}$$

is a weighted sum of partial theta functions, and where F_2, F_3 are one-dimensional sums arising from the boundary term $n = 0$ in a double Eichler integral. Here, $Q(n)$ is a positive definite integral binary quadratic form, $\mathscr{S}$ is a finite set of pairs in $\mathbb{Q}^2$, and $\varepsilon \colon \mathscr{S} \to \mathbb{R}\backslash\{0\}$. Both $\mathscr{S}$ and ε are required to satisfy some symmetry conditions - see Section II.3 for the full definitions.

Remark 1. The function F_1 of Bringmann, Kaszian, and Milas as defined in [6,7] is a direct specialization of the function F presented here, specialized to a certain set of six pairs of rational points, the specific quadratic form $Q(x) = 3x_1^2 + 3x_1x_2 + x_2^2$, and a fixed ε. In particular we have conflicting notation - note that the functions F_1, F_2 given in [6,7] and the functions F_1, F_2 given in the present paper are different.

Analagously to [7], we show that F satisfies the following (see Theorem II.9.1 for a precise statement).

Theorem II.1.1. *The function F is a sum of components of a vector-valued quantum modular form of depth two and weight one on $SL_2(\mathbb{Z})$ with some explicit quantum set $\mathcal{Q}$ defined in Section II.4. In some special cases, F itself is a single component of a vector-valued form.*

Though here we only show the vector-valued version, we note that it is also possible to show that our function F is a quantum modular form of depth two and weight one itself, on a suitably chosen congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, generalising the situation in [6]. The connection for Theorem II.1.1 is made by relating F asymptotically at certain roots of unity to a double Eichler integral $\mathcal{E}$ of the shape

$$\int_{-\bar{\tau}}^{i\infty} \int_{\omega_1}^{i\infty} \frac{f_1(\omega_1) f_2(\omega_2)}{\sqrt{-i(\omega_1 + \tau)}\sqrt{-i(\omega_2 + \tau)}} d\omega_2 d\omega_1,$$

where the f_j lie in the space of vector-valued modular forms on $\mathrm{SL}_2(\mathbb{Z})$. By a result of [7], such Eichler integrals possess higher depth vector-valued quantum modular properties (see Proposition II.2.1), and so by virtue of the asymptotic agreement at points in $\mathcal{Q}$ of F and $\mathcal{E}$, the function F inherits these properties.

We then place the Eichler integral $\mathcal{E}$ into a modular setting by relating it to an indefinite theta function (see Proposition II.10.1 for a precise statement).

Proposition II.1.2. *The indefinite theta function of signature* $(2, 2)$ *defined in Section II.10 has purely non-holomorphic part* $\Theta(\tau)\mathcal{E}(\tau)$, *where* $\Theta(\tau)$ *is a theta series of signature* $(2, 0)$.

The paper is organised as follows. We begin in Section II.2 by reviewing basic properties of special functions, and detailing results that will be needed throughout the paper. In Section II.3 we introduce the function F that we concentrate on for the rest of the paper. We define the quantum set $\mathcal{Q}$ in Section II.4 before we find the asymptotic behaviour of F at certain roots of unity in Section II.5. In Section II.6 a double Eichler integral is introduced and shown, via the use of Shimura theta functions, to exhibit modular properties. Next we turn to Section II.7 where we show that the double Eichler integral can be viewed as a piece of a certain indefinite theta series. Given results in this section, we proceed to prove the main results regarding quantum modularity of F in Section II.9. We set the double Eichler integral in a modular setting in Section II.10, using boosted complementary error functions and a result of [1]. Finally, we conclude the paper in Section II.11 with some questions which will be investigated in further work.

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II.2 Preliminaries

We begin by introducing some basic functions along with recalling relevant results pertinent to the rest of the paper.

II.2.1 Error functions

We first define a rescaled version of the usual one-dimensional error function. For $u \in \mathbb{R}$ set

$$E(u) := 2 \int_0^u e^{-\pi\omega^2} d\omega. \tag{II.2.1}$$

This has first derivative given by $E'(u) = 2e^{-\pi u^2}$. The function $E(u)$ may also be written using incomplete gamma functions $\Gamma(a, u) := \int_u^\infty e^{-\omega}\omega^{a-1}d\omega$, with $a > 0$, via the formula

$$E(u) = \operatorname{sgn}(u)\left(1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi u^2\right)\right), \tag{II.2.2}$$

where we set

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We will also make use of an augmented sgn function, defined by $\operatorname{sgn}^*(x) := \operatorname{sgn}(x)$ for $x \neq 0$ and $\operatorname{sgn}^*(0) := 1$.

We also require, for non-zero u, the function

$$M(u) := \frac{i}{\pi}\int_{\mathbb{R}-iu} \frac{e^{-\pi\omega^2 - 2\pi i u\omega}}{\omega} d\omega.$$

A relation between $M(u)$ and $E(u)$, for non-zero u, is given by

$$M(u) = E(u) - \operatorname{sgn}(u). \tag{II.2.3}$$

Therefore, using (II.2.2), we have that

$$M(u) = \frac{-\operatorname{sgn}(u)}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi u^2\right). \tag{II.2.4}$$

We further need the two-dimensional analogues of the above functions. Following [1] and changing notation slightly, we define $E_2 \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$E_2(\kappa; u) := \int_{\mathbb{R}^2} \operatorname{sgn}(\omega_1)\operatorname{sgn}(\omega_2 + \kappa\omega_1) e^{-\pi\left((\omega_1 - u_1)^2 + (\omega_2 - u_2)^2\right)} d\omega_1 d\omega_2,$$

where throughout we denote components of vectors just with subscripts. Note that

$$E_2(\kappa; -u) = E_2(\kappa; u).$$

Again following [1], for $u_2, u_1 - \kappa u_2 \neq 0$, we define

$$M_2(\kappa; u_1, u_2) := -\frac{1}{\pi^2}\int_{\mathbb{R}-iu_2}\int_{\mathbb{R}-iu_1} \frac{e^{-\pi\omega_1^2 - \pi\omega_2^2 - 2\pi i(u_1\omega_1 + u_2\omega_2)}}{\omega_2(\omega_1 - \kappa\omega_2)} d\omega_1 d\omega_2.$$

Then we have that

$$\begin{aligned} M_2(\kappa; u_1, u_2) =& E_2(\kappa; u_1, u_2) - \operatorname{sgn}(u_2) M(u_1) \\ & - \operatorname{sgn}(u_1 - \kappa u_2) M\left(\frac{u_2 + \kappa u_1}{\sqrt{1+\kappa^2}}\right) - \operatorname{sgn}(u_1)\operatorname{sgn}(u_2 + \kappa u_1). \end{aligned} \tag{II.2.5}$$

The relation (II.2.5) extends the definition of $M_2(u)$ to include $u_2 = 0$ or $u_1 = \kappa u_2$ - note however that M_2 is discontinuous across these loci. Putting $x_1 := u_1 - \kappa u_2, x_2 := u_2$ yields

$$\begin{aligned} M_2(\kappa; u_1, u_2) =& E_2(\kappa; x_1 + \kappa x_2, x_2) + \operatorname{sgn}(x_1)\operatorname{sgn}(x_2) \\ & - \operatorname{sgn}(x_2) E(x_1 + \kappa x_2) - \operatorname{sgn}(x_1) E\left(\frac{\kappa x_1}{\sqrt{1+\kappa^2}} + \sqrt{1+\kappa^2} x_2\right). \end{aligned}$$

We also have the first partial derivatives of M_2 as

$$\begin{aligned} M_2^{(1,0)}(\kappa; u_1, u_2) &= 2e^{-\pi u_1^2} M(u_2) + \frac{2\kappa}{\sqrt{1+\kappa^2}} e^{\frac{-\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1 - \kappa u_2}{\sqrt{1+\kappa^2}}\right), \\ M_2^{(0,1)}(\kappa; u_1, u_2) &= \frac{2}{\sqrt{1+\kappa^2}} e^{\frac{-\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} M\left(\frac{u_1 - \kappa u_2}{\sqrt{1+\kappa^2}}\right), \end{aligned}$$

along with the first partial derivatives of E_2

$$\begin{aligned} E_2^{(1,0)}(\kappa; u_1, u_2) &= 2e^{-\pi u_1^2} E(u_2) + \frac{2\kappa}{\sqrt{1+\kappa^2}} e^{\frac{-\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} E\left(\frac{u_1 - \kappa u_2}{\sqrt{1+\kappa^2}}\right), \\ E_2^{(0,1)}(\kappa; u_1, u_2) &= \frac{2}{\sqrt{1+\kappa^2}} e^{\frac{-\pi(u_2+\kappa u_1)^2}{1+\kappa^2}} E\left(\frac{u_1 - \kappa u_2}{\sqrt{1+\kappa^2}}\right), \end{aligned}$$

all of which follow from Proposition 3.3. of [1].

II.2.2 Euler-Maclaurin summation formula

We state two special cases of the Euler-Maclaurin summation formula, in one and two dimensions, as needed for this paper.
Let $B_m(x)$ be the mth Bernoulli polynomial which is defined by $\frac{te^{xt}}{e^t-1} =: \sum_{m\geq 0} B_m(x)\frac{t^m}{m!}$. We recall the property

$$B_m(1-x) = (-1)^m B(x). \tag{II.2.6}$$

The one dimensional case follows a result of Zagier in [21], and it implies that, for $\alpha \in \mathbb{R}$ and $F\colon \mathbb{R} \to \mathbb{R}$ a C^∞ function of rapid decay,

$$\sum_{n\in\mathbb{N}_0} F((n+\alpha)t) \sim \frac{\mathcal{I}_F}{t} - \sum_{n\geq 0} \frac{B_{n+1}(\alpha)}{(n+1)!} F^{(n)}(0) t^n, \tag{II.2.7}$$

where we set $\mathcal{I}_F = \int_0^\infty F(x)dx$. By $\sim$ we mean that the difference between the left- and right-hand side is $O(t^N)$ for any $N \in \mathbb{N}$.

We now turn to the two-dimensional case. Let $\alpha \in \mathbb{R}^2$ and $F\colon \mathbb{R}^2 \to \mathbb{R}$ a C^∞ function of rapid decay. The Euler-Maclaurin summation formula in two dimensions then implies that (generalising another result of Zagier in [21] to include shifts by α)

$$\begin{aligned}\sum_{n\in\mathbb{N}_0^2} F((n+\alpha)t) \sim &\frac{\mathcal{I}_F}{t^2} - \sum_{n_2\geq 0} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} \int_0^\infty F^{(0,n_2)}(x_1,0)dx_1 t^{n_2-1} \\ &- \sum_{n_1\geq 0} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} \int_0^\infty F^{(n_1,0)}(0,x_2)dx_2 t^{n_1-1} \\ &+ \sum_{n_1,n_2\geq 0} \frac{B_{n_2+1}(\alpha_2)}{(n_2+1)!} \frac{B_{n_1+1}(\alpha_1)}{(n_1+1)!} F^{(n_1,n_2)}(0,0) t^{n_1+n_2},\end{aligned} \tag{II.2.8}$$

here with $\mathcal{I}_F = \int_0^\infty \int_0^\infty F(x_1,x_2)dx_1dx_2$.

II.2.3 Shimura's theta functions

In [20] Shimura gave transformation laws of certain theta series, which we require here. For $\nu \in \{0,1\}, h \in \mathbb{Z}$ and $N, A \in \mathbb{N}$ with $A \mid N, N \mid hA$ define

$$\Theta_\nu(A,h,N;\tau) := \sum_{\substack{m\in\mathbb{Z} \\ m\equiv h \pmod N}} m^\nu q^{\frac{Am^2}{2N^2}}, \tag{II.2.9}$$

where $\tau \in \mathbb{H}$ and $q := e^{2\pi i\tau}$, as usual. Then we have the following transformation formula

$$\Theta_\nu(A,h,N;M\tau) = e\left(\frac{abAh^2}{2N^2}\right)\left(\frac{2Ac}{d}\right)\varepsilon_d(c\tau+d)^{\frac{1}{2}+\nu}\Theta_\nu(A,ah,N;\tau), \tag{II.2.10}$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2N)$ with $2 \mid b$. Here $e(x) := e^{2\pi ix}$ and, for odd d, $\varepsilon_d = 1$ or i depending on whether $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$ respectively, and $\left(\frac{c}{d}\right)$ is the extended Jacobi symbol. Further, we have that

$$\Theta_\nu\left(A,h,N;-\frac{1}{\tau}\right) = (-i)^\nu(-i\tau)^{\frac{1}{2}+\nu}A^{-\frac{1}{2}} \sum_{\substack{k \pmod N \\ Ak\equiv 0 \pmod N}} e\left(\frac{Akh}{N^2}\right)\Theta_\nu(A,k,N;\tau). \tag{II.2.11}$$

We also require the transformations

$$\Theta_\nu(A,-h,N;\tau)=(-1)^\nu\Theta_\nu(A,h,N;\tau),$$

and if $h_1\equiv h_2 \pmod N$, then

$$\Theta_\nu(A,h_1,N;\tau)=\Theta_\nu(A,h_2,N;\tau).$$

II.2.4 Vector-valued quantum modular forms

Since the study of vector-valued quantum modular forms has been motivated in the introduction, here we give only the formal definition, following [7]. We begin with the depth one case, before defining those of higher depth.

Definition II.2.1. For $1\le j\le N\in\mathbb{N}$, a collection of functions $f_j\colon \mathcal{Q}\to\mathbb{C}$ is called a vector-valued quantum modular form of weight k and multiplier $\chi=(\chi_{j,\ell})_{1\le j,\ell\le N}$ and quantum set $\mathcal{Q}$ for $\mathrm{SL}_2(\mathbb{Z})$ if, for all $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ we have that[1]

$$f_j(\tau)-(c\tau+d)^{-k}\sum_{1\le\ell\le N}\chi_{j,\ell}(M)f_\ell(M\tau)$$

can be extended to an open subset of $\mathbb{R}$ and is real-analytic there. We denote the vector space of these forms by $\mathsf{Q}_k(\chi)$.

II.2.5 Higher depth vector-valued quantum modular forms

We now consider generalisations of vector-valued quantum modular forms, again following [7].

Definition II.2.2. For $1\le j\le N\in\mathbb{N}$, a collection of functions $f_j\colon\mathcal{Q}\to\mathbb{C}$ is called a vector-valued quantum modular form of depth P, weight k and multiplier $\chi=(\chi_{j,\ell})_{1\le j,\ell\le N}$ and quantum set $\mathcal{Q}$ for Γ if, for all $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ we have that

$$f_j(\tau)-(c\tau+d)^{-k}\sum_{1\le\ell\le N}\chi^{-1}_{\ell,j}(M)f_\ell(M\tau)\in\bigoplus_\ell \mathsf{Q}^{P_\ell}_{\kappa_j}(\chi_\ell)\mathcal{O}(R),$$

where ℓ runs through a finite set, $\kappa_\ell\in\frac12\mathbb{Z}$, $P_\ell\in\mathbb{Z}$ with $\max(P_\ell)=P-1$, χ_l multipliers, $\mathcal{O}_R$ is the space of real analytic functions on $R\subset\mathbb{R}$ which contains an open subset of $\mathbb{R}$. We also define $\mathsf{Q}^1_k(\chi):=\mathsf{Q}_\kappa(\chi)$, $\mathsf{Q}^0_k(\chi):=1$, and let $\mathsf{Q}^P_k(\chi)$ denote the space of forms of weight k, depth P, and multiplier χ for Γ.

[1] Correcting a minor typographical error of [Ma1].

Remark 2. As before, one can consider strong higher depth quantum modular forms, looking at asymptotic expansions and not just values. The functions described in this paper satisfy this stronger condition.

II.2.6 Double Eichler Integrals

In Section II.6 we consider certain double Eichler integrals and investigate their transformation properties. Here, we recall relevant definitions and results.

Let $f_j \in S_{k_j}(\Gamma, \chi_j)$ be a cusp form of weight k with multiplier χ_j on $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. If $k_j = \frac{1}{2}$ then we allow $f_j \in M_{\frac{1}{2}}(\Gamma, \chi_j)$, the space of all holomorphic modular forms of weight $\frac{1}{2}$ with multipler χ_j. We define a double Eichler integral by

$$I_{f_1,f_2}(\tau) := \int_{-\bar{\tau}}^{i\infty} \int_{\omega_1}^{i\infty} \frac{f_1(\omega_1) f_2(\omega_2)}{(-i(\omega_1+\tau))^{2-k_1}(-i(\omega_2+\tau))^{2-k_2}} d\omega_2 d\omega_1,$$

along with the multiple error of modularity ($\frac{d}{c} \in \mathbb{Q}$)

$$r_{f_1,f_2,\frac{d}{c}}(\tau) := \int_{\frac{d}{c}}^{i\infty} \int_{\omega_1}^{\frac{d}{c}} \frac{f_1(\omega_1) f_2(\omega_2)}{(-i(\omega_1+\tau))^{2-k_1}(-i(\omega_2+\tau))^{2-k_2}} d\omega_2 d\omega_1.$$

In [7] the authors also prove the following proposition, which we will make use of.

Proposition II.2.1. *Consider functions f_j, g_ℓ ($1 \le j \le N, 1 \le \ell \le M$) that are vector-valued modular forms which satisfy the transformations*

$$f_j\left(-\frac{1}{\tau}\right) = (-i\tau)^{\kappa_1} \sum_{1\le k\le N} \chi_{j,k}^{-1} f_k(\tau) \quad , \quad g_\ell\left(-\frac{1}{\tau}\right) = (-i\tau)^{\kappa_2} \sum_{1\le m\le N} \psi_{\ell,m}^{-1} g_m(\tau),$$

where $\kappa_1, \kappa_2 \in \frac{1}{2} + \mathbb{N}_0$. Then we have the transformation formula

$$\begin{aligned} &I_{f_j,g_\ell}(\tau) - (-i\tau)^{\kappa_1+\kappa_2-4} \sum_{\substack{1\le k\le N\\ 1\le m\le M}} \chi_{j,k}^{-1} \psi_{\ell,m}^{-1} I_{f_k,g_m}\left(-\frac{1}{\tau}\right) \\ &= \int_0^{i\infty} \int_{\omega_1}^{i\infty} \frac{f_j(\omega_1) g_\ell(\omega_2)}{(-i(\omega_1+\tau))^{2-\kappa_1}(-i(\omega_2+\tau))^{2-\kappa_2}} d\omega_1 d\omega_2 + I_{f_j}(\tau) r_{g_\ell}(\tau) - r_{f_j}(\tau) r_{g_\ell}(\tau). \end{aligned}$$

The one-dimensional version of this proposition can be concluded in a similar way, regarding $g_\ell(\tau)$ as constant. In particular, we define

$$I_{f_j}(\tau) \coloneqq \int_{-\bar{\tau}}^{i\infty} \frac{f_j(\omega)}{(-i(\omega+\tau))^{2-k}} d\omega, \qquad r_{f_j,\frac{d}{d}}(\tau) \coloneqq \int_{\frac{d}{c}}^{\infty} \frac{f_j(\omega)}{(-i(\omega+\tau))^{2-k}} d\omega.$$

If $k = \frac{1}{2}$ then we allow f_j to be in $M_k(\Gamma, \chi)$. The one dimensional Eichler integral I_{f_j} is defined on $\mathbb{H} \cup \mathbb{Q}$, whereas the error of modularity $r_{f_j,\frac{d}{c}}$ exists on all $\mathbb{R}\backslash\{-\frac{d}{c}\}$ and is real-analytic there. If f_j is a cusp form, then $r_{f_j,\frac{d}{c}}$ exists on all $\mathbb{R}$. The transformation property then follows from the above. We note that Proposition II.2.1 implies that the double Eichler integrals above are vector-valued quantum modular forms of depth two.

II.2.7 Gauss Sums

Here we recall, without proof, some relevant results on the vanishing of quadratic Gauss sums, which we will use when investigating the radial asymptotic behaviour of our function in Section II.5.

Let $a, b, c \in \mathbb{N}$ and denote the generalised quadratic Gauss sum by

$$G(a,b,c) \coloneqq \sum_{n \pmod{c}} e^{\frac{2\pi i(an^2+bn)}{c}}.$$

Then we have the following Lemma, which follows from basic properties of Gauss sums - see e.g. Chapter 1 of [2].

Lemma II.2.2. *The following results on the vanishing of $G(a,b,c)$ hold:*

1. *If $\gcd(a,c) > 1$ and $\gcd(a,c) \nmid b$ then $G(a,b,c) = 0$.*
2. *If $4 \mid c$, b is odd, and $\gcd(a,c) = 1$ then $G(a,b,c) = 0$.*
3. *If $c \equiv 2 \pmod 4$ and $\gcd(a,c) = 1$ then $G(a,0,c) = 0$.*

II.2.8 Boosted Error Functions

In Section II.10 we relate a double Eichler integral to a signature $(2,2)$ indefinite theta function. To do so, we use techniques described in [1]. There, the authors consider so-called boosted error functions and use them to find "modular completions" of a certain family of indefinite theta functions in signature $(n-2,2)$. A modular completion of a non-modular holomorphic function $f(\tau)$ is any function $g(\tau)$ such that $\widetilde{f}(\tau) \coloneqq f(\tau) + g(\tau)$ is modular non-trivially, i.e. $g(\tau)$ is non-holomorphic.

We recall the relevant simplified results here for convenience in signature $(2,2)$, noting in particular the change in notation "flips" the conditions of the double null limit situation described in Section 4.3 of [1].

Consider a bilinear form $B(x, y) := x^T Ay$ for a symmetric $m \times m$ matrix A, and its associated quadratic form $Q(x) := \frac{1}{2}B(x, x)$. Assume that $Q(x)$ has signature $(2, 2)$ and also that, for $\mu \in L \subset \mathbb{Z}^4$, we have $Q(\mu) \in \mathbb{Z}$. Take four vectors $C_1, C_2, C_1', C_2' \in \mathbb{R}^4$. Then we define the orthogonal projections

$$C_{1\perp 2} := C_1 - \frac{B(C_1, C_2)}{Q(C_2)} C_2 \quad \text{and} \quad C_{2\perp 1} := C_2 - \frac{B(C_1, C_2)}{Q(C_1)} C_1,$$

along with the discriminant $\Delta(C_1, C_2) := Q(C_1)Q(C_2) - B(C_1, C_2)^2$.

We let $C_{m'} = C_m'$ and let $\Delta_{\mathcal{I}}$ denote the determinant of the Gram matrix $B(C_n, C_m)_{n,m\in\mathcal{I}}$ where $\mathcal{I}$ is a subset of indices $\{1, 1', 2, 2'\}$. Further, let $D_{m,n}$ be off-diagonal cofactors of the Gram matrix $B(C_m, C_n)_{m,n\in\{1,1',2,2'\}}$.

We require

1. $B(C_1, C_2') = B(C_1', C_2') = B(C_1', C_2) = 0$,
2. $Q(C_1) < 0$ and $Q(C_2) < 0$,
3. $Q(C_1') = Q(C_2') = 0$,
4. $B(C_1, C_1') < 0$ and $B(C_2, C_2') < 0$,
5. $\Delta(C_1, C_2) > 0$,
6. M_{00} is positive definite,

where $M_{00} := \left(\begin{smallmatrix} \Delta_{122'} & D_{1'2'} \\ D_{1'2'} & \Delta_{11'2} \end{smallmatrix}\right)$. Then we define boosted complementary error functions in one and two dimensions by

$$M(C; x) := M\left(\frac{B(C, x)}{\sqrt{-Q(C)}}\right),$$
$$M_2(C_1, C_2; x) := M_2\left(\frac{-B(C_1, C_2)}{\sqrt{\Delta(C_1, C_2)}}; \frac{B(C_2, x)}{\sqrt{-Q(C_2)}}, \frac{B(C_{1\perp 2}, x)}{\sqrt{-Q(C_{1\perp 2})}}\right).$$

The authors of [1] then provide the following Theorem, describing the completion of a certain theta function.

Theorem II.2.3. *Under the conditions above, consider the locally constant function given by*

$$\Phi(x) := (\operatorname{sgn}(B(C_1, x)) - (\operatorname{sgn} B(C_1', x)))((\operatorname{sgn} B(C_2, x)) - (\operatorname{sgn} B(C_2', x))).$$

Let $\tau = u + iv \in \mathbb{H}$. Then the theta function

$$\vartheta\left[\Phi(x)\right](\tau) := \sum_{\lambda \in a+\mathbb{Z}^4} \Phi\left(\sqrt{2v}\lambda\right) q^{Q(\lambda)},$$

admits a modular completion to a non-holomorphic theta series

$$\vartheta\left[\widehat{\Phi}(x)\right](\tau) := \sum_{\lambda \in a+\mathbb{Z}^4} \widehat{\Phi}\left(\sqrt{2v}\lambda\right) q^{Q(\lambda)},$$

of weight two, where $a \in \mathbb{R}^4$. The completion (in terms of the function $\Phi(x)$) is given by

$$\begin{aligned}\hat{\Phi}(x) - \Phi(x) =& M_2\left(C_1, C_2; x\right) + \left(\operatorname{sgn}\left(B(C_{2\perp 1}, x)\right) - \left(\operatorname{sgn} B(C_2', x)\right)\right) M\left(C_1; x\right)\\ &+ \left(\operatorname{sgn}\left(B(C_{1\perp 2}, x)\right) - \operatorname{sgn}\left(B(C_1', x)\right)\right) M\left(C_2; x\right).\end{aligned}$$

II.3 The Partial Theta Function F

Throughout, we consider a positive definite integral binary quadratic form $Q(n) := a_1 n_1^2 + a_2 n_1 n_2 + a_3 n_2^2$, where $a_j \in \mathbb{N}$ for $1 \leq j \leq 3$, and $\gcd(a_1, a_2, a_3) = 1$. Let the discriminant of $Q(n)$ be $-D := a_2^2 - 4a_1a_3 < 0$.

Let $s \geq 1$ be some fixed integer. We write elements of $\mathbb{Z}[\frac{1}{s}]$ in the form $r + \frac{x}{s}$ where $r, x \in \mathbb{Z}$ and $-\frac{s}{2} \leq x < \frac{s}{2}$, and let $\alpha^{(k)} := (\alpha_1^{(k)}, \alpha_2^{(k)})$ be a pair of elements in $\mathbb{Z}[\frac{1}{s}] \times \mathbb{Z}[\frac{1}{s}]$, labeled by (k). For $s \neq 1$ we require that α does not lie in $\mathbb{Z}^2$ (we could instead add a condition here similar to (II.3.1) for just elements in $\mathbb{Z}^2$. However, this would be equivalent to breaking the vector-valued form into two separate forms, considering those elements in $\mathbb{Z}^2$ in a separate vector, with $s = 1$).

We let

$$\mathscr{S}^* := \left\{\alpha^{(j)} \mid 1 \leq j \leq N\right\},$$

be a finite set of N such pairs, and define

$$\mathscr{S} := \mathscr{S}^* \cup \left\{(1 - \alpha^{(j)}) \mid \alpha^{(j)} \in \mathscr{S}^*, 1 \leq j \leq N\right\},$$

where $(1 - \alpha^{(k)}) := (1 - \alpha_1^{(k)}, 1 - \alpha_2^{(k)})$ is meant componentwise. For convenience, we will often suppress the superscript on elements $\alpha \in \mathscr{S}$. We are free to assume s is minimal, such that each element α has at least one of x_1, x_2 coprime to s (otherwise, we can reduce each fractional part until we are in this situation, possibly splitting into different sets with two different s_1 and s_2).

We also work with subsets of $\mathscr{S}^*$ given by

$$\mathscr{S}_1^* := \{\alpha \in \mathscr{S}^* \mid \alpha_1 \in \mathbb{Z}\} \quad \text{and} \quad \mathscr{S}_2^* := \{\alpha \in \mathscr{S}^* \mid \alpha_2 \in \mathbb{Z}\}.$$

Consider also a function $\varepsilon\colon \mathscr{S}^* \to \mathbb{R}\backslash\{0\}$ extended to $\mathscr{S}$ by the relation $\varepsilon(1-\alpha) = \varepsilon(\alpha)$, such that

$$\sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) = 0. \tag{II.3.1}$$

For fixed $\mathscr{S}^*$, ε, and $Q(n) = a_1 n_1^2 + a_2 n_1 n_2 + a_3 n_2^2$, the function that we concentrate on in this paper is given by

$$\begin{aligned} F(q) := \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} q^{Q(n)} &- \frac{1}{2} \sum_{\alpha \in \mathscr{S}_1^*} \varepsilon(\alpha) \operatorname{sgn}^*(\alpha_1) \left(\sum_{j \in 1-\alpha_2+\mathbb{N}_0} q^{a_3 j^2} - \sum_{j \in \alpha_2+\mathbb{N}_0} q^{a_3 j^2} \right) \\ &- \frac{1}{2} \sum_{\alpha \in \mathscr{S}_2^*} \varepsilon(\alpha) \operatorname{sgn}^*(\alpha_2) \left(\sum_{j \in 1-\alpha_1+\mathbb{N}_0} q^{a_1 j^2} - \sum_{j \in \alpha_1+\mathbb{N}_0} q^{a_1 j^2} \right). \end{aligned}$$

Throughout, if $\alpha_1, \alpha_2 \in \mathbb{Z}$ then we omit possible $n = (0,0)$ and $j = 0$ terms in summations implicitly. In each case, this is equivalent to subtracting a constant term and so does not affect modularity properties.

We consider three different parts separately, writing $F(q) = F_1(q) + F_2(q) + F_3(q)$, where

$$F_1(q) := \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} q^{Q(n)}.$$

We also define

$$\begin{aligned} F_2(q) &:= -\frac{1}{2} \sum_{\alpha \in \mathscr{S}_1^*} \varepsilon(\alpha) \operatorname{sgn}^*(\alpha_1) \left(\sum_{j \in 1-\alpha_2+\mathbb{N}_0} q^{a_3 j^2} - \sum_{j \in \alpha_2+\mathbb{N}_0} q^{a_3 j^2} \right), \\ F_3(q) &:= -\frac{1}{2} \sum_{\alpha \in \mathscr{S}_2^*} \varepsilon(\alpha) \operatorname{sgn}^*(\alpha_2) \left(\sum_{j \in 1-\alpha_1+\mathbb{N}_0} q^{a_1 j^2} - \sum_{j \in \alpha_1+\mathbb{N}_0} q^{a_1 j^2} \right). \end{aligned}$$

Remark 3. If for each $(a,x) \in \mathscr{S}_1^*$ the element $(b, 1-x)$ is also in $\mathscr{S}_1^*$ and $\operatorname{sgn}^*(a) = \operatorname{sgn}^*(b)$ as well as $\varepsilon(a,x) = \varepsilon(b, 1-x)$, then the function F_2 vanishes identically. A similar statement holds for the function F_3.

Although this definition is only the analogue of the function called F_1 from [6], it is worth noting that results and techniques therein combined with those of the present paper allow us to also consider the obvious generalization of the function called F_2 defined by Bringmann, Kaszian, and Milas in [6], and to give analogous results. Again note that we have conflicting notation, and our function F_2 is different to that in [6].

Remark 4. It is possible to drop the condition (II.3.1) if we are willing to lose the possibility of having quantum set $\mathbb{Q}$. This is essentially the same as using a vector of quantum sets - one for each fixed $\alpha \in \mathscr{S}^*$ - such that the main term in the Euler-Maclaurin summation formula of the element $\sum_{n\in\alpha+\mathbb{N}_0^2} q^{Q(n)} + \sum_{n\in 1-\alpha+\mathbb{N}_0^2} q^{Q(n)}$ vanishes at certain roots of unity dictated by the quantum set. In this case, the largest $\mathrm{SL}_2(\mathbb{Z})$-invariant quantum set would be $\mathcal{Q}_1$ for one fixed α, as defined in the following section. However, this would be empty in some cases, and we would need to work on suitable subgroups of the full modular group to restore an infinite quantum set.

Example II.3.1. *As a running example we consider the positive definite quadratic form $Q(x) = 2x_1^2 + x_1x_2 + x_2^2$ of discriminant $-D = -7$, along with the set*

$$\mathscr{S}^* = \left\{ \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, -\frac{2}{4}\right) \right\}.$$

Further, we set

$$\varepsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha = \left(\frac{1}{4}, \frac{1}{4}\right), \\ -1 & \text{if } \alpha = \left(-\frac{2}{4}, \frac{1}{4}\right). \end{cases}$$

We see that this set satisfies our condition with $s = 4$, and that both $\mathscr{S}_1^*, \mathscr{S}_2^*$ are empty, so we need only consider F_1.

II.4 The Quantum Set $\mathcal{Q}$

Here we describe the quantum set for our function F, the main idea being that the choice of set will force the main term in the Euler-Maclaurin summation formula to vanish so that we do not obtain a growing term in the asymptotic expansions toward certain points in Section II.5.

Throughout we write elements of $\mathbb{Q}$ as h/k with $\gcd(h,k) = 1$, and define $\delta := \gcd(h,s)$ and $\gamma := \gcd(k,s)$. For a fixed $\alpha = (r_1, r_2) + (x_1, x_2)/s \in \mathscr{S}$, set

$$g(x) = g(x_1, x_2) \coloneqq \begin{cases} \gcd(2a_1x_1 + a_2x_2, a_2x_1 + 2a_3x_2) & \text{if } x \neq (0,0), \\ 1 & \text{if } x = (0,0), \end{cases}$$

with the convention that $\gcd(0,t) = t$ for $t \in \mathbb{N}_0$. We define

$$G \coloneqq \{g(x) \mid \alpha \in \mathscr{S}\}.$$

Then the first part of the quantum set is given by

$$\mathcal{Q}_1 \coloneqq \left\{ \frac{h}{k} \;\middle|\; \frac{s}{\delta}, \frac{s}{\gamma} \nmid g \text{ for every } g \in G \right\}.$$

In particular, note that if $g(x) = 1$ for every choice of α then the conditions on $\frac{s}{\delta}$ and $\frac{s}{\gamma}$ are always satisfied away from $h \in s\mathbb{Z}$ or $k \in s\mathbb{Z}$. We also differentiate cases based upon whether or not the following congruence condition holds

$$Q(x) \pmod{s} \quad \text{is constant across } \mathscr{S}. \tag{II.4.1}$$

If in addition we have $s \nmid g$ for every $g \in G$, we set

$$\begin{cases} \mathcal{Q}_2 \coloneqq \left\{ \frac{h}{k} \;\middle|\; h \in s\mathbb{Z} \right\} \text{ and } \mathcal{Q}_3 \coloneqq \left\{ \frac{h}{k} \;\middle|\; k \in s\mathbb{Z} \right\} & \text{if (II.4.1) holds,} \\ \mathcal{Q}_2 \coloneqq \left\{ \frac{h}{k} \;\middle|\; h \in s^2\mathbb{Z} \right\} \text{ and } \mathcal{Q}_3 \coloneqq \left\{ \frac{h}{k} \;\middle|\; k \in s^2\mathbb{Z} \right\} & \text{else.} \end{cases} \tag{II.4.2}$$

If s divides some element in G then the situation is more complicated and we will need to differentiate several cases. In this case, it will be easier to define the "extra" quantum sets $\mathcal{Q}_2$ and $\mathcal{Q}_3$ algorithmically after (II.5.4) is introduced and investigated. In each case, we find a particular $n \in \mathbb{N}$ and define

$$\mathcal{Q}_2 \coloneqq \left\{ \frac{h}{k} \;\middle|\; h \in s^n\mathbb{Z} \right\} \text{ and } \mathcal{Q}_3 \coloneqq \left\{ \frac{h}{k} \;\middle|\; k \in s^n\mathbb{Z} \right\}.$$

The "full" quantum set is then defined as $\mathcal{Q} \coloneqq \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{Q}_3$. Notice that for some choices of $s, Q, \mathscr{S}$ this quantum set is somewhat sparse (e.g. if $s = 2, 4$ and $a_2 \in 2\mathbb{Z}$ and $2^k \in G$ for some $k \in \mathbb{N}$).

The transformation formulae of the double Eichler integrals in Section II.6 currently require $\mathcal{Q} = S\mathcal{Q}$, and so we note the following equalities. First, the action of S on a fraction $\frac{h}{k}$ is given by $S(\frac{h}{k}) = \frac{-k}{h}$. Then it is clear that $S\mathcal{Q}_1 = \mathcal{Q}_1$, since each of the numerator and denominator are assumed to have the same property in the definition of $\mathcal{Q}_1$ above. Then notice that, for a fixed choice of $n \in \mathbb{N}$, we have $S\mathcal{Q}_2 = \mathcal{Q}_3$ as we just switch the numerator and denominator, i.e. for $h \in s^n\mathbb{Z}$ we have that $\gcd(k, s) = 1$ and so $S(\frac{h}{k}) = \frac{-k}{h}$ has a denominator lying in $s^n\mathbb{Z}$ and a numerator co-prime to the denominator by construction. The argument is similar as to why $S\mathcal{Q}_3 = \mathcal{Q}_2$. Hence overall we have that $S\mathcal{Q} = \mathcal{Q}$.

Example II.4.1. *(continued) Continuing our example, we compute* $\gcd(2x_1+x_2, x_1+2x_2)$ *for each of the elements* $x \in \{(1, 1), (1, -2), (-1, -1), (-1, 2)\}$ *and find that* $G = \{1\}$. *Furthermore, we have that* $Q(x) \equiv 0 \pmod 4$ *for every element in* $\mathscr{S}$*, hence we take the quantum set* $\mathbb{Q}$.

II.5 Radial Asymptotic Behaviour of F at Certain Roots of Unity

In this section we aim to find the asymptotic behaviour of the function F at a point $e^{2\pi i \frac{h}{k} - t}$ as $t \to 0^+$, with $h/k \in \mathcal{Q}$. To do so, we rewrite F in a way that we may apply the Euler-Maclaurin summation formula.

II.5.1 Asymptotic Behaviour of F_1

Decomposing F as above, we concentrate firstly on F_1. We have

$$F_1\left(e^{2\pi i \frac{h}{k} - t}\right) = \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{N}_0^2} e^{\left(2\pi i \frac{h}{k} - t\right) Q(n)} = \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{n \in \mathbb{N}_0^2} e^{\left(2\pi i \frac{h}{k} - t\right) Q(n+\alpha)}.$$

Letting $n \mapsto \ell + n\frac{ks}{\delta}$ where $0 \leq \ell_j \leq \frac{ks}{\delta} - 1$ and $\delta := \gcd(h, s)$ gives that the sum on n is

$$\sum_{\substack{0 \leq \ell \leq \frac{ks}{\delta} - 1 \\ n \in \mathbb{N}_0^2}} e^{\left(2\pi i \frac{h}{k} - t\right) Q\left(\ell + n\frac{ks}{\delta} + \alpha\right)}. \tag{II.5.1}$$

Noting that, for $n \in \mathbb{Z}$, we have $e^{2\pi i \frac{h}{k} Q(\ell + n\frac{ks}{\delta} + \alpha)} = e^{2\pi i \frac{h}{k} Q(\ell + \alpha)}$ which is independent of n, we can write (II.5.1) as

$$\sum_{0 \leq \ell \leq \frac{ks}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n \in \frac{\delta}{ks}(\ell + \alpha) + \mathbb{N}_0^2} e^{-tQ\left(n\frac{ks}{\delta}\right)}.$$

Defining $\mathcal{F}_1(x) := e^{-Q(x)}$ we can therefore write

$$F_1\left(e^{2\pi i \frac{h}{k} - t}\right) = \sum_{\alpha \in \mathscr{S}} \varepsilon(\alpha) \sum_{0 \leq \ell \leq \frac{ks}{\delta} - 1} e^{2\pi i \frac{h}{k} Q(\ell + \alpha)} \sum_{n \in \frac{\delta}{ks}(\ell + \alpha) + \mathbb{N}_0^2} \mathcal{F}_1\left(\frac{ks}{\delta} \sqrt{t} n\right). \tag{II.5.2}$$

The main term in the Euler-Maclaurin summation formula (II.2.8) is then given by

$$\frac{\delta^2}{k^2 s^2 t}\mathcal{I}_{\mathcal{F}_1}\sum_{\alpha\in\mathscr{S}}\varepsilon(\alpha)\sum_{0\leq\ell\leq\frac{ks}{\delta}-1}e^{2\pi i\frac{h}{k}Q(\ell+\alpha)},\tag{II.5.3}$$

which we will show below vanishes for $h/k \in \mathcal{Q}$. We may let ℓ run modulo $\frac{ks}{\delta}$, since we have that $e^{2\pi i\frac{h}{k}Q(\ell+n\frac{ks}{\delta}+\alpha)} = e^{2\pi i\frac{h}{k}Q(\ell+\alpha)}$ whenever n is a pair of integers. Hence the sum on ℓ equals (writing $\alpha = r + \frac{x}{s}$ as in Section II.3)

$$\sum_{\ell \pmod{\frac{ks}{\delta}}} e^{2\pi i\frac{h}{k}Q\left(\ell+\frac{x}{s}\right)}.\tag{II.5.4}$$

If $s = 1$ then (II.5.4) is clearly independent of α and hence the main term vanishes. If $s > 1$ we let $\ell = N + \nu k$, again meant componentwise, with $N \pmod{k}$ and $\nu \pmod{\frac{s}{\delta}}$. Similar to previous calculations, we compute that the sum on ℓ in (II.5.3) is equal to

$$\sum_{N \pmod{k}} e^{\frac{2\pi ih}{ks^2}\left((a_1(s^2N_1^2+2sN_1x_1)+a_2(s^2N_1N_2+sx_2N_1+sx_1N_2)+a_3(s^2N_2^2+2sx_2N_2)+Q(x)\right)}$$
$$\times\sum_{\nu \pmod{\frac{s}{\delta}}} e^{2\pi i\frac{h}{s}(\nu_1(2a_1x_1+a_2x_2)+\nu_2(a_2x_1+2a_3x_2))}.$$

Given this, we see that showing (II.5.3) is zero becomes equivalent to showing that the following expression vanishes

$$\sum_{\alpha\in\mathscr{S}}\varepsilon(\alpha)\sum_{N \pmod{k}} e^{\frac{2\pi ih}{ks^2}\left((a_1(s^2N_1^2+2sN_1x_1)+a_2(s^2N_1N_2+sx_2N_1+sx_1N_2)+a_3(s^2N_2^2+2sx_2N_2)+Q(x)\right)}$$
$$\times\sum_{\nu \pmod{\frac{s}{\delta}}} e^{2\pi i\frac{h/\delta}{s/\delta}(\nu_1(2a_1x_1+a_2x_2)+\nu_2(a_2x_1+2a_3x_2))}.\tag{II.5.5}$$

First, consider values $h/k \in \mathcal{Q}_1$. Since $\gcd(\frac{h}{\delta}, \frac{s}{\delta}) = 1$, the sum on ν vanishes unless $\frac{s}{\delta} \mid (2a_1x_1 + a_2x_2)$ and $\frac{s}{\delta} \mid (a_2x_1 + 2a_3x_2)$, implying that $\frac{s}{\delta} \mid g(x)$. By construction, this is a contradiction to our assumption on $\mathcal{Q}_1$. Therefore, for $h/k \in \mathcal{Q}_1$ the main term vanishes. It is also easily seen from here that if $s \nmid g(x)$ for each α then the main term will vanish.

Then it remains to show that the main term is zero for our different choices of sets $\mathcal{Q}_2$ and $\mathcal{Q}_3$. First, we consider values of $h \in s\mathbb{Z}$ or $h \in s^2\mathbb{Z}$ depending on whether (II.4.1) holds or not, respectively. For ease of exposition we show this only in the case where (II.4.1) holds - the second case follows similarly.

Writing $Q(x) = sX + x_0$ for each choice of α, with $0 \leq x_0 < s$ constant across $\mathscr{S}$ by assumption, it follows that it suffices to show

$$\sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha) \sum_{N \pmod k} e^{\frac{2\pi i h/s}{k}\left((a_1(sN_1^2+2N_1x_1)+a_2(sN_1N_2+x_2N_1+x_1N_2)+a_3(sN_2^2+2x_2N_2)+X\right)} = 0.$$

Since $\delta = s$ in $\mathcal{Q}_2$ we see that $\gcd(k,s) = 1$ and so in particular the inverse of s modulo k, which we denote by $\bar{s}$, exists. Making the change of variables $N \mapsto N - \bar{s}x$ and using that $h/s \in \mathbb{Z}$ gives

$$e^{\frac{-2\pi i\bar{s}x_0h/s}{k}} \sum_{N \pmod k} e^{\frac{2\pi ih}{k}Q(N)} \sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha),$$

which vanishes since $\sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha) = 0$.

Next, consider elements in $\mathcal{Q}_3$. First, fix a choice of α. Note that s does not divide both x_1, x_2 for each α by assumption (if this were not the case, then (II.5.4) is seen to be constant and so we would require the same constant across all choices of α, implying that every α lies in $\mathbb{Z}^2$). We have already seen that when $s \nmid g(x)$ the main term vanishes for any value of $k \in \mathbb{Z}$, and so we now assume $s \mid g(x)$ (a similar argument holds for the cases where $s/\delta \mid g(x)$).

We are aiming to find $m \in \mathbb{N}$ such that for $k \in s^m\mathbb{Z}$ the term

$$\sum_{N \pmod k} e^{\frac{2\pi ih}{sk}\left(a_1(sN_1^2+2N_1x_1)+a_2(sN_1N_2+x_2N_1+x_1N_2)+a_3(sN_2^2+2x_2N_2)\right)}$$

vanishes (here we have taken the factor $e^{\frac{2\pi ihQ(x)}{ks^2}}$ out of the sum). Since $s \mid g(x)$ we may define $X_1 \coloneqq (2a_1x_1 + a_2x_2)/s$ and $X_2 \coloneqq (a_2x_1 + 2a_3x_2)/s$, each of which lie in $\mathbb{Z}$, to obtain the two-dimensional Gauss sum (putting $k = s^m$)

$$\sum_{N \pmod{s^m}} e^{\frac{2\pi ih}{s^m}\left(a_1N_1^2+a_2N_1N_2+a_3N_2^2+N_1X_1+N_2X_2\right)}. \tag{II.5.6}$$

The main idea here is to reduce this to a product of one-dimensional Gauss sums and use well-known results. As with most Gauss sums we may reduce (II.5.6) to the product of two-dimensional Gauss sums over prime powers (via the Chinese Remainder theorem), and hence consider

$$\sum_{N \pmod{p^n}} e^{\frac{2\pi ih}{p^n}\left(a_1N_1^2+a_2N_1N_2+a_3N_2^2+N_1X_1+N_2X_2\right)}, \tag{II.5.7}$$

where p is some prime dividing s, $n \in \mathbb{N}$, and $\gcd(h, p) = 1$ by construction. We see that for the main term in the Euler-Maclaurin expansion formula to vanish, it suffices to show that the above sum is zero for any prime dividing s. Since s does not divide both x_1 and x_2 there exists at least one $p^\ell \parallel s$ that does not divide both x_1 and x_2. Fixing such a prime, we see that at least one of a_1 and a_3 admit an inverse modulo p^n. For ease of exposition we assume throughout that $\bar{a}_1$ exists, denoting the inverse of a_1. Next, we differentiate situations depending on the parity of p.

If p is odd then $\bar{2}$ also exists modulo p^n. Completing the square on N_1 in the exponential term of (II.5.7) gives

$$\begin{aligned}&a_1N_1^2 + a_2N_1N_2 + a_3N_2^2 + N_1X_1 + N_2X_2\\ &\equiv a_1(N_1 + \bar{2}\bar{a}_1(a_2N_2 + X_1))^2 - \bar{4}\bar{a}_1(a_2N_2 + X_1)^2 + a_3N_2^2 + X_2N_2 \pmod{p^n}.\end{aligned}$$

Thus the sum on N becomes (up to constants, after making the shift $N_1 \mapsto N_1 + \bar{2}\bar{a}_1(a_2N_2 + X_1)$)

$$\sum_{N_1 \pmod{p^n}} e^{\frac{2\pi ih}{p^n}a_1N_1^2} \sum_{N_2 \pmod{p^n}} e^{\frac{2\pi ih}{p^n}\left(D_1N_2^2+2x_2^*N_2\right)},$$

where $D_1 := a_3 - \bar{4}\bar{a}_1a_2^2$ and $x_2^* := \frac{x_2(a_3-\bar{4}\bar{a}_1a_2^2)}{p^\ell\prod_j q_j^{n_j}} \in \mathbb{Z}$, with $s = p^\ell\prod_j q_j^{n_j}$ written in its prime decomposition. This is now a product of one-dimensional quadratic Gauss sums. Concentrating on the sum on N_2 we consider first the case where D_1 is not coprime with p^n. That is, we assume $\gcd(D_1, p^n) = p^r$ with $r \geq 1$. Then, if $p^r \nmid x_2^*$ we see by part one of Lemma II.2.2 that the Gauss sum vanishes. The second case is where D_1 is coprime with p^n, implying that $p^\ell \mid x_2$ since $x_2^* \in \mathbb{Z}$. Thus the sum vanishes for any $n \in \mathbb{N}$ unless $p^\ell \mid x_2$, which cannot happen as then we would have that p^ℓ divides both x_1 and x_2 since $p \mid g(x)$. Hence the sum vanishes for all $k \in s\mathbb{Z}$.

Next we turn to the case of $p = 2$. In particular, we then have that $2 \mid a_2$. Again we have that at least one of $\bar{a}_1$ or $\bar{a}_3$ exist modulo 2^n for $n \in \mathbb{N}$, and we assume that $\bar{a}_1$ does for ease of exposition. Letting $N_1 \to N_1 - \bar{a}_1\frac{a_2}{2}N_2$ gives us the numerator

$$a_1N_1^2 + X_1N_1 + N_2^2\left(a_3 - \bar{a}_1\frac{a_2^2}{4}\right) + N_2\left(X_2 - X_1\bar{a}_1\frac{a_2}{2}\right) \pmod{2^n},$$

and so again we obtain a product of two one-dimensional Gauss sums. Explicitly, we have

$$\sum_{N_1 \pmod{2^n}} e^{\frac{2\pi ih}{2^n}\left(a_1N_1^2+X_1N_1\right)} \sum_{N_2 \pmod{2^n}} e^{\frac{2\pi ih}{2^n}\left(D_2N_2^2+N_2\left(X_2-X_1\bar{a}_1\frac{a_2}{2}\right)\right)},$$

with $D_2 := a_3 - \bar{a}_1\frac{a_2^2}{4}$. If X_1 is odd then the sum on N_1 will vanish for any $n \geq 2$ using part two of Lemma II.2.2, so take $k \in s\mathbb{Z}$ with $4 \mid k$. If X_1 is even then we put $N_1 \to N_1 + X_1/2$ to give the sum on N_1 as (up to a constant)

$$\sum_{N_1 \pmod{2^n}} e^{\frac{2\pi ih}{2^n}a_1N_1^2}.$$

Using part three of Lemma II.2.2 this vanishes only if $n = 1$, and so we may choose $k \in s\mathbb{Z}$ with $2 \parallel k$. The sum on N_2 is

$$\sum_{N_2 \pmod{2^n}} e^{\frac{2\pi ih}{2^n}\left(D_2N_2^2+2x_2^*N_2\right)},$$

where $x_2^* := \frac{D_2x_2}{2^\ell\prod_j q_j^{n_j}}$ and $2^\ell \parallel s$. In a similar fashion to the case of odd p, this vanishes unless $2^{\ell-1} \mid x_2$. In this case, let r be such that $2^r \parallel D_2$ and put $t := D_2/2^r$ odd, so the sum becomes

$$2^r \sum_{N_2 \pmod{2^{n-r}}} e^{\frac{2\pi ih}{2^{n-r}}\left(tN_2^2+bN_2\right)}.$$

If $b := \frac{2tx_2}{2^\ell\prod_j q_j^{n_j}}$ is odd then this vanishes for all $n \geq r+2$ by part two of Lemma II.2.2, so choose $k \in s\mathbb{Z}$ with $2^{r+2} \mid k$. If it is even then we complete the square by shifting $N_2 \to N_2 - \bar{t}b/2$ to obtain (up to a constant)

$$\sum_{N_2 \pmod{2^{n-r}}} e^{\frac{2\pi ihtN_2^2}{2^{n-r}}},$$

vanishing only when $n = r+1$. Here we may then choose $k \in s\mathbb{Z}$ with $2^{r+1} \parallel k$ by part three of Lemma II.2.2.

Choosing overall the minimum n such that the main term vanishes for each choice of $\alpha \in \mathscr{S}$ (and ensuring that it is at least 1 or 2 depending on (II.4.1)), we may form the extra part of the quantum set. Analogously to (II.4.2) we define

$$\mathcal{Q}_2 := \left\{\frac{h}{k} \;\middle|\; h \in s^n\mathbb{Z}\right\} \text{ and } \mathcal{Q}_3 := \left\{\frac{h}{k} \;\middle|\; k \in s^n\mathbb{Z}\right\},$$

along with any necessary conditions on whether higher powers of s may divide h or k.

Continuing with analysing the asymptotic behaviour of F_1 we next turn to the other terms in the Euler-Maclaurin summation formula (II.2.8). With $h/k \in \mathcal{Q}$, the second term is given by

$$\sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha) \sum_{0\leq \ell\leq \frac{ks}{\delta}-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_2\geq 0} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right)}{(n_2+1)!} \int_0^\infty \mathcal{F}_1^{(0,n_2)}(x_1,0)dx_1 \left(\frac{ks\sqrt{t}}{\delta}\right)^{n_2-1}.$$

In the same way as in [6] we claim that the terms where n_2 is even vanish. To see this we first recall that each $\alpha \in \mathscr{S}$ pairs canonically with $1-\alpha$ by construction. Thus if we show that the expression

$$\sum_{0\leq \ell\leq \frac{ks}{\delta}-1} \left(e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} B_{2n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right) + e^{2\pi i \frac{h}{k} Q(\ell+1-\alpha)} B_{2n_2+1}\left(\frac{\delta(\ell_2+1-\alpha_2)}{ks}\right)\right)$$

vanishes, then the claim will follow immediately. Recalling the behaviour of the Bernoulli polynomials (II.2.6) and shifting the second term via $\ell \mapsto -\ell + (-1+\frac{ks}{\delta})(1,1)$ gives this immediately.

Treating the terms where n_2 is odd (again using the canonical pairing in $\mathscr{S}$) we now see that the Bernoulli polynomial transform no longer cancels, but give the same contribution. Hence the second term in the Euler-Maclaurin summation formula for F_1 is

$$\begin{aligned} -2\sum_{\alpha\in\mathscr{S}^*} \varepsilon(\alpha) \sum_{0\leq \ell\leq \frac{ks}{\delta}-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_2\geq 0} \frac{B_{2n_2+2}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right)}{(2n_2+2)!} \\ \times \int_0^\infty \mathcal{F}_1^{(0,2n_2+1)}(x_1,0)dx_1 \left(\frac{k^2s^2t}{\delta^2}\right)^{n_2}. \end{aligned} \tag{II.5.8}$$

Similarly, the third term in (II.2.8) is given by

$$\begin{aligned} -2\sum_{\alpha\in\mathscr{S}^*} \varepsilon(\alpha) \sum_{0\leq \ell\leq \frac{ks}{\delta}-1} e^{2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_1\geq 0} \frac{B_{2n_1+2}\left(\frac{\delta(\ell_1+\alpha_1)}{ks}\right)}{(2n_1+2)!} \\ \times \int_0^\infty \mathcal{F}_1^{(2n_1+1,0)}(0,x_2)dx_2 \left(\frac{k^2s^2t}{\delta^2}\right)^{n_1}. \end{aligned}$$

The final term of (II.2.8) is equal to

$$\sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha) \sum_{0\le\ell\le\frac{ks}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\ell+\alpha)} \\ \times \sum_{n_1,n_2\ge0} \frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{ks}\right)}{(n_1+1)!} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right)}{(n_2+1)!} \mathcal{F}_1^{(n_1,n_2)}(0,0)\left(\frac{ks\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

Proceeding in the same way, only the terms where $n_1 \equiv n_2 \pmod 2$ are non-zero. Therefore this is equal to

$$2\sum_{\alpha\in\mathscr{S}^*} \varepsilon(\alpha) \sum_{0\le\ell\le\frac{ks}{\delta}-1} e^{2\pi i\frac{h}{k}Q(\ell+\alpha)} \\ \times \sum_{\substack{n_1,n_2\ge0\\ n_1\equiv n_2 \pmod 2}} \frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{ks}\right)}{(n_1+1)!} \frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right)}{(n_2+1)!} \mathcal{F}_1^{(n_1,n_2)}(0,0)\left(\frac{ks\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

II.5.2 Asymptotic Behaviour of F_2 and F_3

We now focus on the function F_2, and use similar techniques to above. Set $\mathcal{F}_2(x) := e^{-a_3x^2}$, rewrite as in (II.5.2), and use the Euler-Macluarin summation formula in one dimension (II.2.7) to obtain the main term as

$$-\frac{\delta}{2ks\sqrt{t}}\mathcal{I}_{\mathcal{F}_2} \sum_{\alpha\in\mathscr{S}_1^*} \varepsilon(\alpha)\,\mathrm{sgn}^*(\alpha_1)\left(\sum_{0\le r\le\frac{ks}{\delta}-1} e^{2\pi i\frac{h}{k}a_3(r+1-\alpha_2)^2} - e^{2\pi i\frac{h}{k}a_3(r+\alpha_2)^2}\right).$$

Letting $r\mapsto\frac{ks}{\delta}-r-1$ in the first term of the inner summand shows that this vanishes identically. The second term in the one-dimensional Euler-Maclaurin formula for F_2 is given by (pairing even terms and noting odd terms vanish as above)

$$\frac{1}{2}\sum_{\alpha\in\mathscr{S}_1^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_1)\sum_{0\le r\le\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_3(r+(1-\alpha_2))^2}\sum_{m\ge0}\frac{B_{2m+1}\left(\frac{\delta(r+(1-\alpha_2))}{ks}\right)}{(2m+1)!}$$
$$\times\mathcal{F}_2^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m$$
$$-\frac{1}{2}\sum_{\alpha\in\mathscr{S}_1^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_1)\sum_{0\le r\le\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_3(r+\alpha_2)^2}\sum_{m\ge0}\frac{B_{2m+1}\left(\frac{\delta(r+\alpha_2)}{ks}\right)}{(2m+1)!}$$
$$\times\mathcal{F}_2^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m.$$

The same argument runs for the function F_3 with setting $\mathcal{F}_3(x):=e^{-a_1x^2}$, yielding

$$\frac{1}{2}\sum_{\alpha\in\mathscr{S}_2^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_2)\sum_{0\le r\le\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_1(r+(1-\alpha_1))^2}\sum_{m\ge0}\frac{B_{2m+1}\left(\frac{\delta(r+(1-\alpha_1))}{ks}\right)}{(2m+1)!}$$
$$\times\mathcal{F}_3^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m$$
$$-\frac{1}{2}\sum_{\alpha\in\mathscr{S}_2^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_2)\sum_{0\le r\le\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_1(r+\alpha_1)^2}\sum_{m\ge0}\frac{B_{2m+1}\left(\frac{\delta(r+\alpha_1)}{ks}\right)}{(2m+1)!}$$
$$\times\mathcal{F}_3^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m.$$

II.6 Double Eichler Integrals of Weight One

Here we introduce and study a family of double Eichler integrals of weight 1, and show that they are a part of a vector-valued quantum modular form of depth two and weight one.

Recalling that $Q(n)$ has non-zero coefficients a_j and has discriminant $-D<0$, for $\alpha\in\mathscr{S}^*$, $\omega_j\in\mathbb{H}$ we set

$$\mathcal{E}_\alpha(\tau):=-\frac{\sqrt{D}}{4}\int_{-\bar{\tau}}^{i\infty}\int_{\omega_1}^{i\infty}\frac{\theta_1(\alpha;\omega_1,\omega_2)+\theta_2(\alpha;\omega_1,\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}}d\omega_2d\omega_1, \qquad \text{(II.6.1)}$$

along with theta functions

$$\theta_1(\alpha;\omega_1,\omega_2) := \frac{1}{a_1}\sum_{n\in\alpha+\mathbb{Z}^2}(2a_1n_1+a_2n_2)n_2e^{\frac{\pi i(2a_1n_1+a_2n_2)^2\omega_1}{2a_1}+\frac{\pi iDn_2^2\omega_2}{2a_1}}$$

and

$$\theta_2(\alpha;\omega_1,\omega_2) := \frac{1}{a_3}\sum_{n\in\alpha+\mathbb{Z}^2}(a_2n_1+2a_3n_2)n_1e^{\frac{\pi i(a_2n_1+2a_3n_2)^2\omega_1}{2a_3}+\frac{\pi iDn_1^2\omega_2}{2a_3}}.$$

In particular, we note that if $\alpha\in\mathbb{Z}^2$, the term $n=(0,0)$ vanishes in each of the theta functions, and therefore so does $\mathcal{E}_\alpha(\tau)$ at $n=(0,0)$. We aim to show the following proposition.

Proposition II.6.1. *The function*

$$\mathcal{E}(\tau) := \sum_{\alpha\in\mathscr{S}^*}\varepsilon(\alpha)\mathcal{E}_\alpha(s\tau)$$

is a linear combination of components of a vector-valued quantum modular form of depth two and weight one for $SL_2(\mathbb{Z})$.

Remark 5. Though we do not explore the situation here, for a fixed α the term $\mathcal{E}_\alpha(\tau)$ can itself be viewed as a modular form on a suitable subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. As mentioned in Section II.4 a larger quantum set can be used here (if it is not already $\mathbb{Q}$), modifying the level of Γ where appropriate.

Proof of Proposition II.6.1. We start by rewriting $\mathcal{E}(\tau)$ in terms of Shimura theta functions $\Theta_1(A,h,N;\tau)$ - see Section II.2.3 for the relevant definitions. For θ_1 set $\nu_1=2a_1n_1+a_2n_2\in 2a_1\alpha_1+a_2\alpha_2+\mathbb{Z}$ and $\nu_2=n_2\in\alpha_2+\mathbb{Z}$. We further have that $\nu_1-a_2\nu_2=2a_1n_1\in 2a_1\alpha_1+2a_1\mathbb{Z}$.

Putting these into the definition we obtain that

$$\theta_1(\alpha;\omega_1,\omega_2) = \frac{1}{a_1}\sum_{\substack{\nu\in(2a_1\alpha_1+a_2\alpha_2,\alpha_2)+\mathbb{Z}^2\\ \nu_1-a_2\nu_2\in 2a_1\alpha_1+2a_1\mathbb{Z}}}\nu_1\nu_2e^{\frac{\pi i\nu_1^2\omega_1}{2a_1}+\frac{\pi iD\nu_2^2\omega_2}{2a_1}}.$$

We then rewrite $\theta_1(\alpha;\omega_1,\omega_2)$ as

$$\frac{1}{a_1}\sum_{\varrho\in\{0,1,\dots,2a_1-1\}}\left(\sum_{\nu_1\in 2a_1\alpha_1+a_2\alpha_2+a_2\varrho+2a_1\mathbb{Z}}\nu_1e^{\frac{\pi i\nu_1^2\omega_1}{2a_1}}\sum_{\nu_2\in\alpha_2+\varrho+2a_1\mathbb{Z}}\nu_2e^{\frac{\pi iD\nu_2^2\omega_2}{2a_1}}\right).$$

Summing over α in the set $\mathscr{S}^*$ then gives

$$\begin{aligned}
&\sum_{\alpha\in\mathscr{S}^*} \varepsilon(\alpha)\theta_1(\alpha;\omega_1,\omega_2)\\
&\qquad=\frac{1}{a_1s^2}\sum_{A\in\mathcal{A}}\varepsilon_1(A)\sum_{\nu_1\equiv A_1 \pmod{2a_1s}}\nu_1 e^{\frac{\pi i\nu_1^2\omega_1}{2a_1s^2}}\sum_{\nu_2\equiv A_2 \pmod{2a_1s}}\nu_2 e^{\frac{\pi iD\nu_2^2\omega_2}{2a_1s^2}}\\
&\qquad=\frac{1}{a_1s^2}\sum_{A\in\mathcal{A}}\varepsilon_1(A)\Theta_1\left(2a_1s,A_1,2a_1s;\frac{\omega_1}{s}\right)\Theta_1\left(2a_1s,A_2,2a_1s;\frac{D\omega_2}{s}\right),
\end{aligned}$$

where

$$\mathcal{A}:=\{(2a_1s\alpha_1+a_2s\alpha_2+a_2\varrho s, s\alpha_2+\varrho s)\mid \alpha\in\mathscr{S}^*, 0\le\varrho\le 2a_1-1\}\pmod{2a_1s}$$

and $\varepsilon_1(A):=\varepsilon(\frac{A_1-a_2A_2}{2a_1s},\frac{A_2}{s})$. Note that $\mathcal{A}$ has size $2a_1N$, where we count elements with multiplicity.

There is a similar situation for θ_2, where we let

$$\mathcal{B}:=\{(2a_3s\alpha_2+a_2s\alpha_1+a_2\varrho s, s\alpha_1+\varrho s)\mid \alpha\in\mathscr{S}^*, 0\le\varrho\le 2a_3-1\}\pmod{2a_3s}$$

of size $2a_3N$ along with $\varepsilon_2(B):=\varepsilon(\frac{B_2}{s},\frac{B_1-a_2B_2}{2a_3s})$. We obtain that $\mathcal{E}(\tau)$ is given by the expression

$$\begin{aligned}
&-\frac{\sqrt{D}}{4a_1s^2}\sum_{A\in\mathcal{A}}\varepsilon_1(A)\int_{-\bar\tau}^{i\infty}\int_{\omega_1}^{i\infty}\frac{\Theta_1(2a_1s,A_1,2a_1s;\omega_1)\Theta_1(2a_1s,A_2,2a_1s;D\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}}d\omega_2 d\omega_1\\
&-\frac{\sqrt{D}}{4a_3s^2}\sum_{B\in\mathcal{B}}\varepsilon_2(B)\int_{-\bar\tau}^{i\infty}\int_{\omega_1}^{i\infty}\frac{\Theta_1\left(2a_3s,B_1,2a_3s;\omega_1\right)\Theta_1\left(2a_3s,B_2,2a_3s;D\omega_2\right)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}}d\omega_2 d\omega_1.
\end{aligned}$$

For $n\in\mathbb{N}$, we note the equality

$$\begin{aligned}
\Theta_1(a,b,a;n\tau)=\sum_{j\in\mathbb{Z}}(aj+b)q^{\frac{n}{2a}(aj+b)^2} &=\frac{1}{n}\sum_{j\in\mathbb{Z}}(anj+bn)q^{\frac{1}{2an}(anj+bn)^2}\\
&=\frac{1}{n}\Theta_1(na,nb,na;\tau).
\end{aligned}\tag{II.6.2}$$

We split $\mathcal{E}(\tau)=\mathcal{E}_A(\tau)+\mathcal{E}_B(\tau)$ where

$$\mathcal{E}_A(\tau):=-\frac{\sqrt{D}}{4a_1s^2}\sum_{A\in\mathcal{A}}\varepsilon_1(A)$$

$$\times \int_{-\bar{\tau}}^{i\infty} \int_{\omega_1}^{i\infty} \frac{\Theta_1(2a_1s, A_1, 2a_1s; \omega_1)\Theta_1(2a_1s, A_2, 2a_1s; D\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}} d\omega_2 d\omega_1,$$

$$\mathcal{E}_B(\tau) \coloneqq -\frac{\sqrt{D}}{4a_3s^2} \sum_{B\in\mathcal{B}} \varepsilon_2(B) \times \int_{-\bar{\tau}}^{i\infty} \int_{\omega_1}^{i\infty} \frac{\Theta_1\left(2a_3s, B_1, 2a_3s; \omega_1\right)\Theta_1\left(2a_3s, B_2, 2a_3s; D\omega_2\right)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}} d\omega_2 d\omega_1.$$

We concentrate firstly on $\mathcal{E}_A(\tau)$ and, for $k_1 \pmod{2a_1s}$ and $k_2 \pmod{2Da_1s}$, set

$$I_{k_1,k_2}(\tau) \coloneqq I_{\Theta_1(2a_1s,k_1,2a_1s;\cdot),\Theta_1(2Da_1s,Dk_2,2Da_1s;\cdot)}(\tau).$$

Via (II.2.11) we compute the transformations of the two Shimura theta functions as

$$\Theta_1\left(2a_1s, k_1, 2a_1s; -\frac{1}{\tau}\right) = \frac{(-i)(-i\tau)^{\frac{3}{2}}}{\sqrt{2a_1s}} \sum_{j \pmod{2a_1s}} e\left(\frac{jk_1}{2a_1s}\right)\Theta_1(2a_1s, j, 2a_1s; \tau)$$

and

$$\Theta_1\left(2Da_1s, Dk_1, 2Da_1s; -\frac{1}{\tau}\right) = \frac{(-i)(-i\tau)^{\frac{3}{2}}}{\sqrt{2Da_1s}} \sum_{j \pmod{2Da_1s}} e\left(\frac{jk_2}{2a_1s}\right) \times \Theta_1(2Da_1s, j, 2Da_1s; \tau).$$

Using (II.6.2) we find

$$\mathcal{E}_A(\tau) = -\frac{1}{4a_1s^2\sqrt{D}} \sum_{\alpha\in\mathscr{S}^*} \sum_{A\in\mathcal{A}_\alpha} \varepsilon_1(A) I_{A_1,A_2}(\tau),$$

where for a fixed $\alpha \in \mathscr{S}^*$ we define

$$\mathcal{A}_\alpha \coloneqq \{(2a_1s\alpha_1 + a_2s\alpha_2 + a_2\varrho s, s\alpha_2 + \varrho s) \mid 0 \le \varrho \le 2a_1 - 1\} \pmod{2a_1s}.$$

Then using Proposition II.2.1 we obtain the transformation formula

$$\begin{aligned}
&\sum_{\alpha\in\mathscr{S}^*}\sum_{A\in\mathcal{A}_\alpha}\varepsilon_1(A)I_{A_1,A_2}(\tau)\\
&-\frac{(-i\tau)^{-1}}{2a_1s\sqrt{D}}\sum_{\alpha\in\mathscr{S}^*}\sum_{A\in\mathcal{A}_\alpha}\varepsilon_1(A)\sum_{\substack{k_1 \pmod{2a_1s}\\ k_2 \pmod{2Da_1s}}} e\left(\frac{k_1A_1+k_2A_2}{2a_1s}\right)I_{k_1,\frac{k_2}{D}}\left(-\frac{1}{\tau}\right)\\
=&\sum_{\alpha\in\mathscr{S}^*}\sum_{A\in\mathcal{A}_\alpha}\varepsilon_1(A)\Bigg(\int_0^{i\infty}\int_{\omega_1}^{i\infty}\frac{\Theta_1(2a_1s,A_1,2a_1s;\omega_1)\Theta_1(2Da_1s,DA_2,2Da_1s;\omega_2)}{\sqrt{-i(\omega_1+\tau)}\sqrt{-i(\omega_2+\tau)}}d\omega_1d\omega_2\\
&\qquad+I_{\Theta_1(2a_1s,A_1,2a_1s;\cdot)}(\tau)r_{\Theta_1(2Da_1s,DA_2,2Da_1s;\cdot)}(\tau)\\
&\qquad-r_{\Theta_1(2a_1s,A_1,2a_1s;\cdot)}(\tau)r_{\Theta_1(2Da_1s,DA_2,2Da_1s;\cdot)}(\tau)\Bigg).
\end{aligned}$$

Choosing $(k_1,k_2)=(A_1,DA_2)$ in the second term then returns our original Eichler integral. Each choice of $A\in\mathcal{A}_\alpha$ is then seen to be a component of a vector valued quantum modular form. In cases where $e(A):=e\left(\frac{A_1^2+DA_2^2}{2a_1s}\right)$ is the same across choices of $A\in\mathcal{A}_\alpha$, one can take this outside of the sum on A as a constant factor, and so $\sum_{A\in\mathcal{A}_\alpha}\varepsilon_1(A)I_{A_1,A_2}(\tau)$ can be seen as a single component of a vector-valued quantum modular form. Furthermore, if $e(A)$ is also constant across choices of $\alpha\in\mathscr{S}^*$ then we view all of $\mathcal{E}_A(\tau)$ as a single component.

A similar statement holds for $\mathcal{E}_B$, and then one can easily put all components into a single vector-valued form in the obvious way. □

Example II.6.2. *(continued) Returning to our example we see that we set*

$$\theta_1(\alpha;\omega_1,\omega_2):=\frac{1}{2}\sum_{n\in\alpha+\mathbb{Z}^2}(4n_1+n_2)n_2e^{\frac{\pi i(4n_1+n_2)^2\omega_1}{4}+\frac{7\pi i n_2^2\omega_2}{4}}$$

along with the similar expression for θ_2. Working through, we set $\nu_1=4n_1+n_2$ and $\nu_2=n_2$ so that $\nu_1-\nu_2=4n_1$, giving the expression in terms of Shimura theta functions as

$$\sum_{\alpha\in\mathscr{S}^*}\varepsilon(\alpha)\theta_1(\alpha;\omega_1,\omega_2)=\frac{1}{32}\sum_{A\in\mathcal{A}}\varepsilon_1(A)\Theta_1\left(16,A_1,16;\frac{\omega_1}{4}\right)\Theta_1\left(16,A_2,16;\frac{7\omega_2}{4}\right),$$

where, after a little calculation, we have the set

$$\mathcal{A}=\{(5,1),(6,2),(9,5),(10,6),(13,9),(14,10),(1,13),(2,14)\}\pmod{16}.$$

Further, we set $\varepsilon_1(A) = \varepsilon(\frac{A_1-A_2}{16}, \frac{A_2}{4})$*. It is then simple to check that* $e(A)$ *is constant across the set* $\mathcal{A}$*, and hence* $\mathcal{E}_A$ *is a single component. We also find that the similarly defined function* $e(B)$ *is constant across the set*

$$\mathcal{B} = \{(5,1),(3,1),(1,5),(7,5)\} \pmod 8 .$$

Hence we view our Eichler integral as a single component of the vector-valued form.

II.7 Indefinite Theta Functions

Here we realise the double Eichler integrals as pieces of indefinite theta functions, with coefficients given by double error functions. We first write $\mathbb{E}(\tau) := \mathcal{E}(\frac{\tau}{s})$ in such a way that we can apply the Euler-Maclaurin summation formula.

Lemma II.7.1. *Let* $u(n_1,n_2) := (u_1,u_2) = (\sqrt{v}(2\sqrt{a_1}n_1 + \frac{a_2}{\sqrt{a_1}}n_2), \sqrt{v}mn_2)$*, with* $m := \sqrt{4a_3 - \frac{a_2^2}{a_1}}$*, and* $\kappa := \frac{a_2}{m\sqrt{a_1}} = \frac{a_2}{\sqrt{D}}$ *. We have that*

$$\mathbb{E}(\tau) = \frac{1}{2} \sum_{\alpha \in \mathscr{S}^*} \varepsilon(\alpha) \sum_{n \in \alpha + \mathbb{Z}^2} M_2(\kappa; u) q^{-Q(n)}.$$

Proof. The claim follows once we have shown that

$$M_2(\kappa;u) =$$
$$- \frac{\sqrt{D}n_2(2a_1n_1 + a_2n_2)}{2a_1} q^{Q(n)} \int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{\pi i (2a_1n_1+a_2n_2)^2\omega_1}{2a_1}}}{\sqrt{-i(\omega_1+\tau)}} \int_{\omega_1}^{i\infty} \frac{e^{\frac{\pi i D n_2^2 \omega_2}{2a_1}}}{\sqrt{-i(\omega_2+\tau)}} d\omega_2 d\omega_1$$
$$- \frac{\sqrt{D}n_1(a_2n_1 + 2a_3n_2)}{2a_3} q^{Q(n)} \int_{-\bar{\tau}}^{i\infty} \frac{e^{\frac{\pi i (a_2n_1+2a_3n_2)^2\omega_1}{2a_3}}}{\sqrt{-i(\omega_1+\tau)}} \int_{\omega_1}^{i\infty} \frac{e^{\frac{\pi i D n_1^2 \omega_2}{2a_3}}}{\sqrt{-i(\omega_2+\tau)}} d\omega_2 d\omega_1.$$

There are three different cases to consider, since we do not have the term $n = (0,0)$:

1. Both $n_1 \neq 0$ and $n_2 \neq 0$.
2. We have $n_1 = 0$ and $n_2 \neq 0 \iff u_1 - \kappa u_2 = 0$ and $u_2 \neq 0$.
3. We have $n_1 \neq 0$ and $n_2 = 0 \iff u_1 - \kappa u_2 \neq 0$ and $u_2 = 0$.

We argue as in [6], and for the first case obtain that

$$\begin{aligned}M_2(\kappa;u) =& -\frac{u_1}{2\sqrt{v}}\frac{u_2}{\sqrt{v}}q^{\frac{u_1^2}{4v}+\frac{u_2^2}{4v}}\int_{-\bar{\tau}}^{i\infty}\frac{e^{\frac{\pi i u_1^2\omega_1}{2v}}}{\sqrt{-i(\omega_1+\tau)}}\int_{\omega_1}^{i\infty}\frac{e^{\frac{\pi i u_2^2\omega_2}{2v}}}{\sqrt{-i(\omega_2+\tau)}}d\omega_2 d\omega_1\\ &- \frac{u_1-\kappa u_2}{2\sqrt{(1+\kappa^2)v}}\frac{u_2+\kappa u_1}{\sqrt{(1+\kappa^2)v}}q^{\frac{(u_2+\kappa u_1)^2}{4(1+\kappa^2)v}+\frac{(u_1-\kappa u_2)^2}{4(1+\kappa^2)v}}\int_{-\bar{\tau}}^{i\infty}\frac{e^{\frac{\pi i(u_2+\kappa u_1)^2\omega_1}{2(1+\kappa^2)v}}}{\sqrt{-i(\omega_1+\tau)}}\\ &\times\int_{\omega_1}^{i\infty}\frac{e^{\frac{\pi i(u_1-\kappa u_2)^2\omega_2}{2(1+\kappa^2)v}}}{\sqrt{-i(\omega_2+\tau)}}d\omega_2 d\omega_1.\end{aligned}$$

Plugging in the definitions of u and κ here yields the result directly.

For case 2 we set $f_1(v) := M_2(\kappa; \frac{a_2}{\sqrt{a_1}}\sqrt{v}n_2, m\sqrt{v}n_2)$ and we want to prove the equality

$$f_1(v) = -\frac{\sqrt{D}a_2n_2^2}{2a_1}e^{2\pi i a_3 n_2^2\tau}\int_{-\bar{\tau}}^{i\infty}\frac{e^{\frac{\pi i(a_2n_2)^2\omega_1}{2a_1}}}{\sqrt{-i(\omega_1+\tau)}}\int_{\omega_1}^{i\infty}\frac{e^{\frac{\pi i D n_2^2\omega_2}{2a_1}}}{\sqrt{-i(\omega_2+\tau)}}d\omega_2 d\omega_1.$$

Letting $\omega_1 \mapsto \omega_1-\tau$ and $\omega_2 \mapsto \omega_2-\tau$ where $\tau = u+iv$ the right-hand side becomes

$$\begin{aligned}&-\frac{\sqrt{D}a_2}{2a_1}n_2^2\int_{2iv}^{i\infty}\frac{e^{\frac{\pi i(a_2n_2)^2\omega_1}{2a_1}}}{\sqrt{-i\omega_1}}\int_{\omega_1}^{i\infty}\frac{e^{\frac{\pi i D n_2^2\omega_2}{2a_1}}}{\sqrt{-i\omega_2}}d\omega_2 d\omega_1\\ =&\frac{\sqrt{D}a_2}{a_1}n_2^2\int_{v}^{\infty}\frac{e^{\frac{-\pi(a_2n_2)^2\omega_1}{a_1}}}{\sqrt{\omega_1}}\int_{\omega_1}^{\infty}\frac{e^{\frac{-\pi D n_2^2\omega_2}{a_1}}}{\sqrt{\omega_2}}d\omega_2 d\omega_1 =: f_2(v).\end{aligned}$$

By (II.2.5) we have that

$$f_1(v) = E_2\left(\kappa; \frac{a_2}{\sqrt{a_1}}\sqrt{v}n_2, m\sqrt{v}n_2\right) - \operatorname{sgn}(n_2)E_1\left(\frac{a_2}{\sqrt{a_1}}\sqrt{v}n_2\right).$$

Considering differentials in v we obtain

$$
\begin{aligned}
f_1'(v) =& \frac{n_2}{2\sqrt{v}}\left(\frac{a_2}{\sqrt{a_1}}E_2^{(1,0)}\left(\kappa;\frac{a_2}{\sqrt{a_1}}\sqrt{v}n_2, m\sqrt{v}n_2\right) + mE_2^{(0,1)}\left(\kappa;\frac{a_2}{\sqrt{a_1}}\sqrt{v}n_2, m\sqrt{v}n_2\right)\right)\\
&- \frac{n_2}{2\sqrt{v}}\operatorname{sgn}(n_2)\frac{a_2}{\sqrt{a_1}}E_1'\left(\frac{a_2}{\sqrt{a_1}}\sqrt{v}n_2\right)\\
=& \frac{n_2}{2\sqrt{v}}\Bigg(\frac{2a_2}{\sqrt{a_1}}e^{\frac{-\pi a_2^2 v n_2^2}{a_1}}E\left(m\sqrt{v}n_2\right) + \frac{2(\kappa+1)}{\sqrt{1+\kappa^2}}e^{\frac{-\pi\left(m\sqrt{v}n_2+\frac{\kappa a_2}{\sqrt{a_1}}\sqrt{v}n_2\right)^2}{1+\kappa^2}}E(0)\\
&- \operatorname{sgn}(n_2)\frac{a_2}{\sqrt{a_1}}2e^{\frac{-\pi a_2^2 v n_2^2}{a_1}}\Bigg)\\
=& \frac{a_2 n_2}{\sqrt{v a_1}}e^{\frac{-\pi a_2^2 v n_2^2}{a_1}}\left(E\left(m\sqrt{v}n_2\right) - \operatorname{sgn}(n_2)\right).
\end{aligned}
$$

Since $m > 0$ we have

$$
E\left(m\sqrt{v}n_2\right) - \operatorname{sgn}(n_2) = M\left(m\sqrt{v}n_2\right) = \frac{-\operatorname{sgn}(n_2)}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \pi m^2 v n_2^2\right),
$$

using (II.2.4). Thus we obtain

$$
f_1'(v) = -\frac{a_2|n_2|}{\sqrt{v a_1 \pi}}e^{\frac{-\pi a_2^2 v n_2^2}{a_1}}\Gamma\left(\frac{1}{2}, \pi m^2 v n_2^2\right).
$$

We then consider the differential of $f_2(v)$. Computing directly we obtain

$$
\begin{aligned}
f_2'(v) &= \frac{\sqrt{D}a_2}{a_1}n_2^2(-1)\frac{e^{\frac{-\pi(a_2n_2)^2 v}{a_1}}}{\sqrt{v}}\int_v^\infty e^{\frac{-\pi(Dn_2^2)\omega_2}{a_1}}\sqrt{\omega_2}\frac{d\omega_2}{\omega_2}\\
&= \frac{\sqrt{D}a_2}{a_1}n_2^2(-1)\frac{e^{\frac{-\pi(a_2n_2)^2 v}{a_1}}}{\sqrt{v}}\frac{\sqrt{a_1}}{\sqrt{\pi D n_2^2}}\Gamma\left(\frac{1}{2}, m^2\pi v n_2^2\right)\\
&= \frac{-a_2|n_2|}{\sqrt{a_1 v \pi}}e^{\frac{-\pi(a_2n_2)^2 v}{a_1}}\Gamma\left(\frac{1}{2}, m^2\pi v n_2^2\right).
\end{aligned}
$$

Setting $f(v) = f_1(v) - f_2(v)$ we see that $f'(v) = 0$, and since $\lim_{v\to\infty} f(v) = 0$ we obtain that $f_1(v) = f_2(v)$ as required.

For case 3 a similar argument holds, setting $f_3(v) \coloneqq M_2(\kappa; 2\sqrt{a_1}\sqrt{v}n_1, 0)$. The claim now follows. □

II.8 Asymptotic behaviour of the double Eichler integral

In this section we relate the functions $\mathbb{E}$ and F. Letting $F(e^{2\pi i \frac{h}{k} - t}) =: \sum_{m\geq 0} a_{h,k}(m)t^m$ as $t \to 0^+$, we prove the following Theorem.

Theorem II.8.1. *For $h, k \in \mathcal{Q}$ as determined by Section II.4 we have that*

$$\mathbb{E}\left(\frac{h}{k} + \frac{it}{2\pi}\right) \sim \sum_{m\geq 0} a_{-h,k}(m)(-t)^m.$$

Proof. Using Lemma II.7.1 and that M_2 is an even function we have

$$\begin{aligned}\mathbb{E}(\tau) =& \frac{1}{2}\sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha) \sum_{n\in\alpha+\mathbb{N}_0^2} M_2(\kappa; u(n_1,n_2))q^{-Q(n_1,n_2)} \\ &+ \frac{1}{2}\sum_{\alpha\in\widetilde{\mathscr{S}}} \widetilde{\varepsilon}(\alpha) \sum_{n\in\alpha+\mathbb{N}_0^2} M_2(\kappa; u(-n_1,n_2))q^{-Q(-n_1,n_2)},\end{aligned}$$

with $\widetilde{\mathscr{J}} := \{(1-\alpha_1,\alpha_2) \mid \alpha \in \mathscr{S}\}$ and $\widetilde{\varepsilon}(\alpha_1,\alpha_2) := \varepsilon(1-\alpha_1,\alpha_2)$.

In order to be able to apply the Euler-Maclaurin summation formula, we define $M_2^*(\kappa; x_1, x_2)$ by replacing each sgn with sgn*. Explicitly, we set

$$\begin{aligned}M_2^*(\kappa; u_1, u_2) :=& \operatorname{sgn}^*(x_1)\operatorname{sgn}^*(x_2) + E_2(\kappa; x_1 + kx_2, x_2) - \operatorname{sgn}^*(x_2)E(x_1+\kappa x_2) \\ &- \operatorname{sgn}^*(x_1)E\left(\frac{\kappa x_1}{\sqrt{1+\kappa^2}} + \sqrt{1+\kappa^2}x_2\right).\end{aligned} \tag{II.8.1}$$

It is easy to see that, using (II.2.5) and (II.8.1), we have

$$\begin{aligned}&M_2(\kappa; u_1(0,x_2), u_2(x_2)) - \lim_{x_1\to 0^+} M_2^*(\kappa; u_1(\pm x_1, x_2), u_2(x_2)) = \pm M\left(\sqrt{1+\kappa^2}x_2\right), \\ &M_2(\kappa; u_1(x_1,0), u_2(0)) - \lim_{x_2\to 0^+} M_2^*(\kappa; u_1(\pm x_1, x_2), u_2(x_2)) = \pm M(x_1).\end{aligned}$$

We then rewrite $\mathbb{E}(\tau) = \mathcal{E}^*(\tau) + H_1(\tau) + H_2(\tau)$, defining

$$\begin{aligned}\mathcal{E}^*(\tau) :=& \frac{1}{2}\sum_{\alpha\in\mathscr{S}} \varepsilon(\alpha) \sum_{n\in\alpha+\mathbb{N}_0^2} M_2^*(\kappa; u(n_1,n_2))q^{-Q(n_1,n_2)} \\ &+ \frac{1}{2}\sum_{\alpha\in\widetilde{\mathscr{S}}} \widetilde{\varepsilon}(\alpha) \sum_{n\in\alpha+\mathbb{N}_0^2} M_2^*(\kappa; u(-n_1,n_2))q^{-Q(-n_1,n_2)},\end{aligned}$$

along with the boundary terms

$$H_1(\tau) := -\frac{1}{2}\sum_{\alpha\in\mathscr{S}_1^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_1) \times \left(\sum_{j\in\alpha_2+\mathbb{N}_0} M\left(j\sqrt{(1+\kappa^2)v}\right)q^{-a_3j^2} - \sum_{j\in 1-\alpha_2+\mathbb{N}_0} M\left(j\sqrt{(1+\kappa^2)v}\right)q^{-a_3j^2}\right)$$

and

$$H_2(\tau) := -\frac{1}{2}\sum_{\alpha\in\mathscr{S}_2^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_2) \times \left(\sum_{j\in\alpha_1+\mathbb{N}_0} M\left(j\sqrt{v}\right)q^{-a_1j^2} - \sum_{j\in 1-\alpha_1+\mathbb{N}_0} M\left(j\sqrt{v}\right)q^{-a_1j^2}\right).$$

If $\alpha_1\in\mathbb{Z}$ (resp. $\alpha_2\in\mathbb{Z}$) then for the $n_1=0$ (resp. $n_2=0$) we take the limit $n_1\to 0$ (resp. $n_2\to 0$) in the M_2^* functions.

Remark 6. In the case that for every (a,x) in $\mathscr{S}_1^*$, the element $(b,1-x)$ also exists in $\mathscr{S}_1^*$, along with the conditions $\operatorname{sgn}^*(a)=\operatorname{sgn}^*(b)$ and $\varepsilon(a,x)=\varepsilon(b,1-x)$, then $H_1=0$ identically. A similar statement holds for the function H_2.

Using techniques similar to those in Section II.5 we next determine the asymptotic behaviour of $\mathcal{E}^*, H_1$ and H_2. First we rewrite $\mathcal{E}^*$ as

$$\mathcal{E}^*\left(\frac{h}{k}+\frac{it}{2\pi}\right) = \sum_{\alpha\in\mathscr{S}}\varepsilon(\alpha)\sum_{0\le\ell\le\frac{ks}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(\ell_1+\alpha_1,\ell_2+\alpha_2)}\sum_{n\in\frac{\delta(\ell+\alpha)}{ks}+\mathbb{N}_0^2}\mathcal{F}_4\left(\frac{ks}{\delta}\sqrt{t}n\right) + \sum_{\alpha\in\widetilde{\mathscr{S}}}\widetilde{\varepsilon}(\alpha)\sum_{0\le\ell\le\frac{ks}{\delta}-1} e^{-2\pi i\frac{h}{k}Q(-(\ell_1+\alpha_1),\ell_2+\alpha_2)}\sum_{n\in\frac{\delta(\ell+\alpha)}{ks}+\mathbb{N}_0^2}\widetilde{\mathcal{F}_4}\left(\frac{ks}{\delta}\sqrt{t}n\right),$$

with $\mathcal{F}_4(x) := \frac{1}{2}M_2^*(\kappa;\frac{1}{\sqrt{2\pi}}(u(x_1,x_2)))e^{Q(x)}$ and $\widetilde{\mathcal{F}}_4(x) := \mathcal{F}_4(u(-x_1,x_2))$.

Then the contribution from the $\mathcal{F}_4$ term to the main term in the Euler-Maclaurin summation formula is given by

$$\frac{\delta^2}{k^2s^2t}\mathcal{I}_{\mathcal{F}_4}\sum_{\alpha\in\mathscr{S}}\varepsilon(\alpha)\sum_{0\le\ell\le\frac{ks}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\ell+\alpha)},$$

which vanishes, conjugating a result from Section II.5. Similarly, the contribution from the $\widetilde{\mathcal{F}}_4$ to the main term of the Euler-Maclaurin summation formula also vanishes.

The second term of (II.2.8) is (again noting as in Section II.5 that terms where n_2 is even vanish)

$$-2 \sum_{\alpha \in \mathscr{S}^*} \varepsilon(\alpha) \sum_{0 \le \ell \le \frac{ks}{\delta}-1} e^{-2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_2 \ge 0} \frac{B_{2n_2+2}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right)}{(2n_2+2)!}$$
$$\times \int_0^\infty \left(\mathcal{F}_4^{(0,2n_2+1)}(x_1,0) + \widetilde{\mathcal{F}}_4^{(0,2n_2+1)}(x_1,0)\right) dx_1 \left(\frac{k^2 s^2 t}{\delta^2}\right)^{n_2}.$$

We now claim that

$$\int_0^\infty \left(\mathcal{F}_4^{(0,2n_2+1)}(x_1,0) + \widetilde{\mathcal{F}_4}^{(0,2n_2+1)}(x_1,0)\right) dx_1 = (-1)^{n_2} \int_0^\infty \mathcal{F}_1^{(0,2n_2+1)}(x_1,0) dx_1, \tag{II.8.2}$$

corresponding to the terms arising in equation (II.5.8). First, we simplify the right-hand side of (II.8.2)

$$\begin{aligned}(-1)^{n_2} \int_0^\infty \mathcal{F}_1^{(0,2n_2+1)}(x_1,0) dx_1 &= \left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \int_0^\infty \mathcal{F}_1(x_1,x_2) dx_1\right]_{x_2=0} \\ &= \left[\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} e^{\frac{-m^2 x_2^2}{4}} \int_0^\infty e^{-\left(\sqrt{a_1}x_1 + \frac{a_2 x_2}{2\sqrt{a_1}}\right)^2} dx_1\right]_{x_2=0}.\end{aligned}$$

Taking the integral without differentiating and substituting $\omega = \frac{1}{\sqrt{\pi}}\left(\sqrt{a_1}x_1 + \frac{a_2 x_2}{2\sqrt{a_1}}\right)$ we get

$$\int_0^\infty e^{-\left(\sqrt{a_1}x_1 + \frac{a_2 x_2}{2\sqrt{a_1}}\right)^2} dx_1 = \sqrt{\frac{\pi}{a_1}} \int_{\frac{a_2 x_2}{2\sqrt{a_1 \pi}}}^\infty e^{-\pi\omega^2} d\omega = \frac{\sqrt{\pi}}{2\sqrt{a_1}}\left(1 - E\left(\frac{a_2 x_2}{2\sqrt{a_1 \pi}}\right)\right).$$

Therefore the right-hand side of (II.8.2) is given by

$$\left[\frac{\sqrt{\pi}}{2\sqrt{a_1}} \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} e^{\frac{-m^2 x_2^2}{4}} \left(1 - E\left(\frac{a_2 x_2}{2\sqrt{a_1 \pi}}\right)\right)\right]_{x_2=0}$$

$$= -\left[\frac{\sqrt{\pi}}{2\sqrt{a_1}}\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} e^{\frac{-m^2x_2^2}{4}} E\left(\frac{a_2x_2}{2\sqrt{a_1\pi}}\right)\right]_{x_2=0},$$

since the other terms vanish under differentiation and setting $x_2 = 0$.

Next we concentrate on the left-hand side of (II.8.2), and to ease notation we set

$$\begin{aligned}
h_1(x_1, x_2) &:= E_2(\kappa; u(x_1, x_2)),\\
h_2(x_1, x_2) &:= E(u_1(x_1, x_2)),\\
h_3(x_1, x_2) &:= E\left(\frac{\kappa x_1}{\sqrt{1+\kappa^2}} + \sqrt{1+\kappa^2}x_2\right).
\end{aligned}$$

We also define

$$\begin{aligned}
c_0(x_1, x_2) &:= e^{Q(x_1,x_2)},\\
c_j(x_1, x_2) &:= h_j\left(\frac{1}{\sqrt{2\pi}}(x_1, x_2)\right) e^{Q(x_1,x_2)},
\end{aligned}$$

for $j = 1, 2, 3$.

By definition of $M_2^*(\kappa; u)$ we compute that

$$\begin{aligned}
&\mathcal{F}_4^{(0,2n_2+1)}(x_1, 0) + \widetilde{\mathcal{F}}_4^{(0,2n_2+1)}(x_1, 0)\\
=&\frac{1}{2}(c_0^{(0,2n_2+1)}(x_1, 0) + c_1^{(0,2n_2+1)}(x_1, 0) - c_2^{(0,2n_2+1)}(x_1, 0) - c_3^{(0,2n_2+1)}(x_1, 0))\\
&+ \frac{1}{2}(-c_0^{(0,2n_2+1)}(-x_1, 0) + c_1^{(0,2n_2+1)}(-x_1, 0) - c_2^{(0,2n_2+1)}(-x_1, 0) + c_3^{(0,2n_2+1)}(-x_1, 0))\\
=&c_0^{(0,2n_2+1)}(x_1, 0) - c_2^{(0,2n_2+1)}(x_1, 0),
\end{aligned}$$

using that c_0 and c_1 are even, whereas c_2 and c_3 are odd.

Then we are considering the expression

$$\begin{aligned}
&-\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left[\int_0^\infty \left(e^{Q(x_1,x_2)} - e^{Q(x_1,x_2)}E\left(\frac{1}{\sqrt{2\pi}}u_1(x_1, x_2)\right)\right) dx_1\right]_{x_2=0}\\
=&-\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left[\int_0^\infty e^{Q(x_1,x_2)}M\left(\frac{1}{\sqrt{2\pi}}u_1(x_1, x_2)\right) dx_1\right]_{x_2=0}\\
=&-\frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}}\left[e^{a_3x_2^2}\int_0^\infty e^{a_1x_1^2+a_2x_1x_2}M\left(\frac{1}{\sqrt{2\pi}}u_1(x_1, x_2)\right) dx_1\right]_{x_2=0}.
\end{aligned}$$

Taking the integral without differentiating, and letting $\omega = \frac{1}{\sqrt{2\pi}} u_1(x_1, x_2)$ we obtain

$$\begin{aligned}
&- e^{\frac{m^2 x_2^2}{4}} \int_{\frac{a_2 x_2}{\sqrt{2\pi a_1}}}^{\infty} M(\omega) e^{\frac{\pi\omega^2}{2}} \sqrt{\frac{\pi}{2a_1}} d\omega \\
&= - \sqrt{\frac{\pi}{2a_1}} e^{\frac{m^2 x_2^2}{4}} \left(\int_0^\infty M(\omega) e^{\frac{\pi\omega^2}{2}} d\omega - \int_0^{\frac{a_2 x_2}{\sqrt{2\pi a_1}}} M(\omega) e^{\frac{\pi\omega^2}{2}} d\omega \right).
\end{aligned}$$

After differentiating an odd number of times and evaluating at $x_2 = 0$ the terms arising from the first intgeral here vanish, and we decompose the second integral using $M(\omega) = E(\omega) - 1$. Since $E(\omega)$ is odd, the contributions from this term also vanish, and overall we are left with

$$\begin{aligned}
&- \sqrt{\frac{\pi}{2a_1}} \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left[e^{\frac{m^2 x_2^2}{4}} \int_0^{\frac{a_2 x_2}{\sqrt{2\pi a_1}}} e^{\frac{\pi\omega^2}{2}} d\omega \right]_{x_2=0} \\
&= - \sqrt{\frac{\pi}{2a_1}} i^{-2n_2-1} \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left[e^{-\frac{m^2 x_2^2}{4}} \int_0^{\frac{a_2 x_2 i}{\sqrt{2\pi a_1}}} e^{\frac{\pi\omega^2}{2}} d\omega \right]_{x_2=0}.
\end{aligned}$$

The integral in question is therefore given by

$$i\sqrt{2} \int_0^{\frac{a_2 x_2}{2\sqrt{\pi a_1}}} e^{-\pi\omega^2} d\omega = \frac{i}{\sqrt{2}} E\left(\frac{a_2 x_2}{2\sqrt{a_1 \pi}} \right).$$

Given this, it is easy to conclude that the left-hand side of (II.8.2) is given by

$$-\sqrt{\frac{\pi}{4a_1}} (-1)^{n_2} \frac{\partial^{2n_2+1}}{\partial x_2^{2n_2+1}} \left[e^{-\frac{m^2 x_2^2}{4}} E\left(\frac{a_2 x_2}{2\sqrt{a_1 \pi}} \right) \right]_{x_2=0}$$

as claimed.

The third term in the Euler-Maclaurin summation formula is given by

$$\begin{aligned}
-2 \sum_{\alpha \in \mathscr{S}^*} \varepsilon(\alpha) \sum_{0 \le \ell \le \frac{ks}{\delta} - 1} e^{-2\pi i \frac{h}{k} Q(\ell+\alpha)} \sum_{n_1 \ge 0} \frac{B_{2n_1+2}\left(\frac{\delta(l_1+\alpha_1)}{ks}\right)}{(2n_1+2)!} \\
\times \int_0^\infty \left(\mathcal{F}_4^{(2n_1+1,0)}(0, x_2) + \widetilde{\mathcal{F}}_4^{(2n_2+1,0)}(0, x_2) \right) dx_2 \left(\frac{k^2 s^2 t}{\delta^2} \right)^{n_1},
\end{aligned}$$

again observing that the terms with even n_1 vanish. A similar argument to before gives us that this is equal to

$$-2\sum_{\alpha\in\mathscr{S}^*}\varepsilon(\alpha)\sum_{0\le\ell\le\frac{ks}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\ell+\alpha)}\sum_{n_1\ge0}\frac{B_{2n_1+2}\left(\frac{\delta(\ell_1+\alpha_1)}{ks}\right)}{(2n_1+2)!}\\ \times\int_0^\infty\mathcal{F}_1^{(2n_1+1,0)}(0,x_2)(-1)^{n_1}dx_2\left(\frac{k^2s^2t}{\delta^2}\right)^{n_1}.$$

The final term in (II.2.8) is

$$2\sum_{\alpha\in\mathscr{S}^*}\varepsilon(\alpha)\sum_{0\le\ell\le\frac{ks}{\delta}-1}e^{-2\pi i\frac{h}{k}Q(\ell+\alpha)}\sum_{\substack{n_1,n_2\ge0\\ n_1\equiv n_2 \mod 0}}\frac{B_{n_1+1}\left(\frac{\delta(\ell_1+\alpha_1)}{ks}\right)}{(n_1+1)!}\frac{B_{n_2+1}\left(\frac{\delta(\ell_2+\alpha_2)}{ks}\right)}{(n_2+1)!}\\ \times\left(\mathcal{F}_4^{(n_1,n_2)}(0,0)-(-1)^{n_1}\widetilde{\mathcal{F}}_4^{(n_1,n_2)}(0,0)\right)\left(\frac{ks\sqrt{t}}{\delta}\right)^{n_1+n_2}.$$

Via a similar argument to the one in [6], we have that

$$\mathcal{F}_4^{(n_1,n_2)}(0,0)-(-1)^{n_1}\widetilde{\mathcal{F}_4}^{(n_1,n_2)}(0,0)=i^{n_1+n_2}\mathcal{F}_1^{(n_1,n_2)}(0,0).$$

We can see this by decomposing the left-hand side as

$$\mathcal{F}_4^{(n_1,n_2)}(0,0)-(-1)^{n_1}\widetilde{\mathcal{F}_4}^{(n_1,n_2)}(0,0)=c_0^{(n_1,n_2)}(0,0)-c_3^{(n_1,n_2)}(0,0).$$

Using $c_3(-x_1,-x_2)=-c_3(x_1,x_2)$ we have that

$$c_3^{(n_1,n_2)}(0,0)=(-1)^{n_1+n_2+1}c_3^{(n_1,n_2)}(0,0).$$

Since we have only cases where $n_1\equiv n_2 \pmod 2$ the contribution from the c_3 terms vanishes. Thus we are left with

$$c_0^{(n_1,n_2)}(0,0)=i^{n_1+n_2}\left[\frac{\partial^{n_1}}{\partial x_1^{n_1}}\frac{\partial^{n_2}}{\partial x_2^{n_2}}e^{-Q(x_1,x_2)}\right]_{x_1=x_2=0}=i^{n_1+n_2}\mathcal{F}_1(0,0).$$

Now consider the asympototic behaviour of the functions H_1 and H_2. Set $\mathcal{F}_5(x)\coloneqq M(\frac{\sqrt{2}}{\sqrt{\pi}}x)e^{a_3x^2}$ and note that

$$\mathcal{F}_5^{2m}(0)=(-1)^{m+1}\left[\frac{\partial^{2m}}{\partial x^{2m}}e^{-a_3x^2}\right]=(-1)^{m+1}\mathcal{F}_2^{2m}(0).$$

The contribution to the Euler-Maclaurin summation formula in one dimension (II.2.7) arising from the H_1 term is then given by

$$\frac{1}{2}\sum_{\alpha\in\mathscr{S}_1^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_1)\sum_{0\leq r\leq\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_3(r+(1-\alpha_2))^2}\sum_{m\geq 0}\frac{B_{2m+1}\left(\frac{\delta(r+(1-\alpha_2))}{ks}\right)}{(2m+1)!}\times\mathcal{F}_5^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m$$
$$-\frac{1}{2}\sum_{\alpha\in\mathscr{S}_1^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_1)\sum_{0\leq r\leq\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_3(r+\alpha_2)^2}\sum_{m\geq 0}\frac{B_{2m+1}\left(\frac{\delta(r+\alpha_2)}{ks}\right)}{(2m+1)!}\times\mathcal{F}_5^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m.$$

Similarly, for the H_2 term we obtain the contribution

$$\frac{1}{2}\sum_{\alpha\in\mathscr{S}_2^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_2)\sum_{0\leq r\leq\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_1(r+(1-\alpha_1))^2}\sum_{m\geq 0}\frac{B_{2m+1}\left(\frac{\delta(r+(1-\alpha_1))}{ks}\right)}{(2m+1)!}\times\mathcal{F}_6^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m$$
$$-\frac{1}{2}\sum_{\alpha\in\mathscr{S}_2^*}\varepsilon(\alpha)\operatorname{sgn}^*(\alpha_2)\sum_{0\leq r\leq\frac{ks}{\delta}-1}e^{\frac{2\pi ih}{k}a_1(r+\alpha_1)^2}\sum_{m\geq 0}\frac{B_{2m+1}\left(\frac{\delta(r+\alpha_1)}{ks}\right)}{(2m+1)!}\times\mathcal{F}_6^{(2m)}(0)\left(\frac{k^2s^2t}{\delta^2}\right)^m,$$

where $\mathcal{F}_6(x) := M(\frac{\sqrt{2}}{\sqrt{\pi}}x)e^{a_1x^2}$. Noting that

$$\mathcal{F}_6^{2m}(0) = (-1)^{m+1}\left[\frac{\partial^{2m}}{\partial x^{2m}}e^{-a_1x^2}\right] = (-1)^{m+1}\mathcal{F}_3^{2m}(0).$$

gives the claim. □

II.9 Proof of Theorem 1.2

We are now ready to prove a refined version of Theorem II.1.1.

Theorem II.9.1. *Let $\mathcal{Q}$ be as in Section II.4. The function $\widehat{F}\colon \mathcal{Q} \to \mathbb{C}$ defined by $\widehat{F}(\frac{h}{k}) := F(e^{2\pi i s \frac{h}{k}})$ is a sum of components of a vector-valued quantum modular form of depth two and weight one.*
Moreover, with $\mathcal{A}$ and $\mathcal{B}$ as in Section II.6, if the functions

$$e\left(\frac{A_1^2 + DA_2^2}{2a_1 s}\right) \quad , \quad e\left(\frac{B_1^2 + DB_2^2}{2a_3 s}\right)$$

are constant across all choices of $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $\widehat{F}(\frac{h}{k})$ is a single component of a vector-valued quantum modular form of depth two and weight one.

Proof. By Theorem II.8.1 we have that

$$\widehat{F}\left(\frac{h}{k}\right) = \lim_{t \to 0^+} F\left(e^{2\pi i \frac{sh}{k} - t}\right) = a_{hs_1, \frac{k}{s_2}} = \lim_{t \to 0^+} \mathbb{E}\left(-\frac{h}{k} + \frac{it}{2\pi}\right),$$

where $s_1 := s/\gcd(s,k)$ and $s_2 := \gcd(s,k)$. The claim then follows from Proposition II.6.1. □

Example II.9.2. *(continued) Returning again to our example, and using that $e(A), e(B)$ are constant across $\mathcal{A}, \mathcal{B}$ (as seen in Section II.2.6), the above Theorem shows us that this example is a single component of a vector-valued quantum modular form of depth two and weight one with quantum set $\mathcal{Q} = \mathbb{Q}$ on $SL_2(\mathbb{Z})$.*

II.10 Completed Indefinite Theta Functions

We now view the function $\mathbb{E}(\tau)$ as the "purely non-holomorphic" part of an indefinite theta series. For $A \in M_m(\mathbb{Z})$ a non-singular $m \times m$ matrix, $P\colon \mathbb{R}^m \to \mathbb{C}$, and $a \in \mathbb{Q}^m$ we define the theta function

$$\Theta_{A,P,a}(\tau) := \sum_{n \in a + \mathbb{Z}^m} P(n) q^{\frac{1}{2} n^T A n}.$$

Set $A_1 := \begin{pmatrix} 2a_1 & a_2 & 2a_1 & a_2 \\ a_2 & 2a_3 & a_2 & 2a_3 \\ 2a_1 & a_2 & 0 & 0 \\ a_2 & 2a_3 & 0 & 0 \end{pmatrix}$ with associated bilinear form $B_1(x,y) = x^T A_1 y$ and quadratic form $Q_1(x) := \frac{1}{2} B_1(x,x)$. We also set $A_0 := \begin{pmatrix} 2a_1 & a_2 \\ a_2 & 2a_3 \end{pmatrix}$ and the function $P_0(n) := M_2(\kappa; 2\sqrt{a_1} n_1 + \frac{a_2}{\sqrt{a_1}} n_2, m n_2)$, and for $n \in \mathbb{R}^4$ put

$$\begin{aligned}P(n) \coloneqq{}& M_2\left(\kappa; 2\sqrt{a_1}n_3 + \frac{a_2}{\sqrt{a_1}}n_4, mn_4\right) + (\operatorname{sgn}(n_1) + \operatorname{sgn}(n_3))M\left(\frac{a_2n_3 + 2a_3n_4}{\sqrt{a_3}}\right)\\ &+ (\operatorname{sgn}(n_2) + \operatorname{sgn}(n_4))M\left(\frac{2a_1n_3 + a_2n_4}{\sqrt{a_1}}\right)\\ &+ (\operatorname{sgn}(2a_1n_3 + a_2n_4) + \operatorname{sgn}(n_1))(\operatorname{sgn}(a_2n_3 + 2a_3n_4) + \operatorname{sgn}(n_2)).\end{aligned}$$

Note that for $\alpha \in \mathscr{S}^*$ we have

$$2\mathcal{E}_\alpha(\tau) = \Theta_{-A_0,P_0,\alpha}(\tau).$$

In particular, we have the following Proposition.

Proposition II.10.1. *The function $\mathcal{E}_\alpha$ can be viewed as the "purely non-holomorphic part" of*

$$\Theta_{A_1,P,a}(\tau) = \sum_{n\in a+\mathbb{Z}^4} P(\sqrt{v}n)q^{Q_1(n)},$$

where $a \in \frac{1}{s}A_1^{-1}\mathbb{Z}^4$ with $(a_3, a_4) = (\alpha_1, \alpha_2)$. Moreover, $\Theta_{A_1,P,a}(\tau)$ is an indefinite theta function of signature $(2, 2)$.

Proof. To see the first claim, we put $P^- \coloneqq M_2(\kappa; 2\sqrt{a_1}n_3 + \frac{a_2}{\sqrt{a_1}}n_4, mn_4)$ and we then have

$$\Theta_{A_1,P^-,a}(\tau) = 2\mathcal{E}_\alpha(\tau)\Theta_{A_0,1,(a_1-a_3,a_2-a_4)}(\tau).$$

The authors in [6] give a framework to proceed directly to prove the convergence and modularity properties of $\Theta_{A_1,P,a}(\tau)$ via a Theorem of Vignéras, and so instead here we employ Section II.2.8 of the present paper, which follows from [1].

Choosing $C_1 = (0, 1, 0, -1)^T$, $C_2 = (1, 0, -1, 0)^T$, $C_1' = (0, 0, a_2, -2a_1)^T$, and $C_2' = (0, 0, -2a_3, a_2)^T$ one can verify that the conditions in Section II.2.8 hold. Computing the completion given in Theorem II.2.3 along with those in the locally constant product of sign functions gives exactly the terms in $P(n)$ up to a factor of $\frac{1}{\sqrt{2}}$. Therefore, by Theorem II.2.3 we see that $\Theta_{A_1,P,a}(\tau)$ is a non-holomorphic theta series of weight 2, clearly of signature $(2, 2)$. □

Example II.10.2. *(continued) For our example we set $A_1 \coloneqq \left(\begin{smallmatrix}4&1&4&1\\1&2&1&2\\4&1&0&0\\1&2&0&0\end{smallmatrix}\right)$ along with $A_0 \coloneqq \left(\begin{smallmatrix}4&1\\1&2\end{smallmatrix}\right)$. We also set*

$$\begin{aligned}P(n) :=&M_2\left(\frac{1}{7}; 2\sqrt{2}n_3 + \frac{1}{\sqrt{2}}n_4, \frac{\sqrt{7}}{\sqrt{2}}n_4\right) + (\operatorname{sgn}(n_1) + \operatorname{sgn}(n_3))M\left(n_3 + 2n_4\right)\\ &+ (\operatorname{sgn}(n_2) + \operatorname{sgn}(n_4))M\left(\frac{4n_3 + n_4}{\sqrt{2}}\right)\\ &+ (\operatorname{sgn}(4n_3 + n_4) + \operatorname{sgn}(n_1))(\operatorname{sgn}(n_3 + 2n_4) + \operatorname{sgn}(n_2)).\end{aligned}$$

Then, for example, the function $\mathcal{E}_{\left(\frac{1}{4},\frac{1}{4}\right)}$ is seen to be the non-holomorphic part of

$$\Theta_{A_1,P,a}(\tau) = \sum_{n\in a+\mathbb{Z}^4} P(\sqrt{v}n)q^{Q_1(n)},$$

with $a \in \frac{1}{4}A_1^{-1}\mathbb{Z}^4$ such that $(a_3, a_4) = \left(\frac{1}{4}, \frac{1}{4}\right)$.

II.11 Further questions

We end by commenting on some related questions.

1. Here we discussed only the case where we had a binary quadratic form. It is expected that a somewhat similar situation occurs for more general forms exists, though it is expected to be much more technical. Systematic study of higher depth analogues of F is planned, however this will require more careful treatment due to a variety of factors. Work of Nazaroglu on higher dimension error functions [18] gives a possible pathway to this. Discussions with Kaszian have also revealed some difficulty in defining the quantum set for higher dimensional versions of F in general. In particular, in the present paper we exploited the fact that

$$B_m(1 - x) = (-1)^m B(x)$$

when determining the asymptotic behaviour of F. We were then able to deduce vanishing results for possible growing terms via the use of the Euler-Maclaurin summation formula in two dimensions. In higher dimensions this relationship becomes more complex, and using the n-dimensional Euler-Maclaurin formula (see e.g. Equation 3.1. in [5]) we find more growing terms that we require to vanish. In general, it is anticipated that this will place many restrictions on the quantum set.

2. Further generalisation of the situation will be explored, in particular to include the holomorphic part of certain indefinite theta functions. That is, replace $Q(n)$ by an indefinite quadratic form of signature $(1, 1)$, and take the holomorphic part of a

certain family of indefinite theta functions following Zwegers' construction in [24]. Generically these look like

$$\sum_{n\in a+\mathbb{Z}^r} \left(\operatorname{sgn}(B(c_1,n)) - \operatorname{sgn}(B(c_2,n))\right) e^{2\pi i B(n,b)} q^{Q(n)},$$

where a, b, c_1, c_2 lie in $\mathbb{Z}^r$ and satisfy certain conditions to ensure convergence (see [24] for a full definition). Our question is then: are there certain families of these indefinite theta functions with higher depth quantum modularity?

3. It is anticipated that in a following paper certain examples of the quantum modular forms here are viewed as q-series, using Larson's "translation" [17] of work by Griffin, Ono, and Warnaar on generalised Andrews-Gordon identities [13].

4. In this paper we only realised the double Eichler integral (and therefore F) as components of a vector-valued quantum modular form, but we did not compute the other components of the vector-valued form explicitly. Further work in this direction is expected, and its implications in representation theory explored.

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Chapter III

The asymptotic profile of an eta-theta quotient related to entanglement entropy in string theory

This chapter is based on a paper published in *Research in Number Theory* [Ma2].

III.1 Introduction and Statement of Results

Modern mathematical physics in the direction of string theory and black holes is intricately linked to number theory. For example, work of Dabholkar, Murthy, and Zagier relates certain mock modular forms to physical phenomena such as quantum black holes and wall crossing [6]. Similarly, the connections between automorphic forms and a second quantised string theory are described in [7], and modular forms for certain elliptic curves and their realisation in string theory is discussed in [14]. Further, the recent paper [11] discusses in-depth the links between work of the enigmatic Ramanujan in relation to modular forms and their generalisations and string theoretic objects (and indeed, why such links should be expected).

Knowledge of the behaviour of the modular objects aids the descriptions of physical phenomena. For instance, in [10], the authors use the classical number-theoretic Jacobi triple product identity to demonstrate the supersymmetry of the open-string spectrum using RNS fermions in light-cone gauge (see also [21]). In particular, parts of physical partition functions are often modular or mock modular objects. For example, the partition functions of the Melvin model [18] and the conical entropy of both the open and closed superstring [12] both involve the weight -3 and index 0 meromorphic Jacobi form

$$f(z;\tau) := \frac{\vartheta(z;\tau)^4}{\eta(\tau)^9\vartheta(2z;\tau)},$$

where η is the Dedekind eta function given by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n)\,,$$

and

$$\vartheta(z;\tau) := i\zeta^{\frac{1}{2}} q^{\frac{1}{8}} \prod_{n\geq 1} (1-q^n)(1-\zeta q^n)(1-\zeta^{-1}q^{n-1})$$

is the Jacobi theta function, with $\zeta := e^{2\pi i z}$ for $z \in \mathbb{C}$, and $q := e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$, the upper half-plane.

We are particularly interested in the coefficients of the q-expansion of f where $0 \leq z \leq 1$, away from the pole at $z = 1/2$, where the residue of f is calculated in [21] - the other residues may be calculated using the elliptic transformation formulae for f. For instance, the asymptotic behaviour of the coefficients is required in order to investigate the UV limit. For a fixed value of z the problem of finding the asymptotics of the coefficients is elementary, as [12] notes. In particular, fixing $z = \frac{h}{k}$ a rational number with $\gcd(h,k) = 1$ and $0 \leq h < \frac{k}{2}$, then classical results in the theory of modular forms (see Theorem 15.10 of [3] for example) give that the coefficients of $f(\frac{h}{k};\tau) = \sum_{n\geq 0} a_{h,k}(n)q^n$ behave asymptotically as

$$a_{h,k}(n) \sim \frac{\left(\frac{h}{k}\right)^{\frac{7}{4}}}{2\sqrt{2}\pi} n^{-\frac{9}{4}} e^{4\pi\sqrt{\frac{hn}{k}}}.$$

In the present paper, we let

$$f(z;\tau) =: \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} b(m,n)\zeta^m q^n.$$

and investigate the coefficients $b(m,n)$; in particular we want to compute the bivariate asymptotic profile of $b(m,n)$ for a certain range of m.

In [2], the authors introduce techniques in order to compute the bivariate asymptotic behaviour of coefficients for a Jacobi form in order to answer Dyson's conjecture on the bivariate asymptotic behaviour of the partition crank. This method is used in numerous other papers - for example, in relation to the rank of a partition [8], ranks and cranks of cubic partitions [13], and certain genera of Hilbert schemes [15] (a result that has recently been extended to a complete classification with exact formulae using the Hardy-Ramanujan circle method [9]), along with many other partition-related statistics.

Using Wright's circle method [22, 23] and following the same approach as [2] we show the following theorem.

Theorem III.1.1. *For $\beta := \pi\sqrt{\frac{2}{n}}$ and $|m| \leq \frac{1}{6\beta}\log(n)$ we have that*

$$b(m,n) = (-1)^{m+\delta+\frac{1}{2}} \frac{\beta^5}{2^7\pi^5(2n)^{\frac{1}{4}}} e^{2\pi\sqrt{2n}} + O\left(n^{-\frac{13}{4}} e^{2\pi\sqrt{2n}}\right)$$

as $n \to \infty$. Here, $\delta := 1$ if $m < 0$ and $\delta = 0$ otherwise.

Remark 7. Although our approach is similar to [2, 8], in some places we require a little more care since finding the Fourier coefficients requires taking an integral over a path where f has a pole. In this case, we turn to the framework of [6] - this is explained explicitly in Section III.3.

We begin in Section V.2 by recalling relevant results that are pertinent to the rest of the paper. In Section III.3.1 we investigate the behaviour of f toward the dominant pole $q = 1$. We follow this in Section III.3.2 by bounding the contribution away from the pole at $q = 1$. We finish in Section III.4 by applying Wright's circle method to find the asymptotic behaviour of $b(m,n)$ and hence prove Theorem III.1.1.

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III.2 Preliminaries

Here we recall relevant definitions and results which will be used throughout the rest of the paper.

III.2.1 Properties of ϑ and η

When determining the asymptotic behaviour of f we will require the modularity behaviour of both ϑ and η. It is well-known that ϑ satisfies the following lemma (see e.g. [16]).

Lemma III.2.1. *The function ϑ satisfies the following transformation properties.*

1. $\vartheta(-z;\tau) = -\vartheta(z;\tau)$

2. $\vartheta(z+1;\tau) = -\vartheta(z;\tau)$

3. $\vartheta(z;\tau) = \frac{i}{\sqrt{-i\tau}} e^{\frac{-\pi i z^2}{\tau}} \vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)$

Further, we have the following well-known modular transformation formula of η (see e.g. [19]).

Lemma III.2.2. *We have that*

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

III.2.2 Euler Polynomials

We will also make use properties of the Euler polynomials E_r, defined by the generating function

$$\frac{2e^{xt}}{1+e^t} =: \sum_{r\geq 0} E_r(x)\frac{t^r}{r!}.$$

Lemma 2.2 of [2] shows that the following lemma holds.

Lemma III.2.3. *We have*

$$-\frac{1}{2}\operatorname{sech}^2\left(\frac{t}{2}\right) = \sum_{r\geq 0} E_{2r+1}(0)\frac{t^{2r}}{(2r)!}.$$

Further, Lemma 2.3 of [2] gives the following integral representation for the Euler polynomials.

Lemma III.2.4. *We have that*

$$\mathcal{E}_j := \int_0^\infty \frac{w^{2j+1}}{\sinh(\pi w)}dw = \frac{(-1)^{j+1}E_{2j+1}(0)}{2}.$$

III.2.3 A particular bound

In Section III.3.2 we require a bound on the size of

$$P(q) := \frac{q^{\frac{1}{24}}}{\eta(\tau)},$$

away from the pole at $q = 1$. For this we use the following lemma which is shown to hold in Lemma 3.5 of [2].

Lemma III.2.5. *Let $\tau = u + iv \in \mathbb{H}$ with $Mv \leq u \leq \frac{1}{2}$ for $u > 0$ and $v \to 0$. Then*

$$|P(q)| \ll \sqrt{v} \exp\left[\frac{1}{v}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

In particular, with $v = \frac{\beta}{2\pi}$, $u = \frac{\beta m^{-\frac{1}{3}} x}{2\pi}$ and $M = m^{-\frac{1}{3}}$ this gives for $1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ the bound

$$|P(q)| \ll n^{-\frac{1}{4}} \exp\left[\frac{2\pi}{\beta}\left(\frac{\pi}{12} - \frac{1}{2\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right)\right]. \qquad \text{(III.2.1)}$$

III.2.4 I-Bessel functions

Here we recall relevant results on the I-Bessel function defined by

$$I_\ell(x) := \frac{1}{2\pi i}\int_\Gamma t^{-\ell-1} e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} dt,$$

where Γ is a contour which starts in the lower half plane at $-\infty$, surrounds the origin counterclockwise and returns to $-\infty$ in the upper half-plane. We are particularly interested in the asymptotic behaviour of I_ℓ, given in the following lemma (see e.g. (4.12.7) of [1]).

Lemma III.2.6. *For fixed ℓ we have*

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right)$$

as $x \to \infty$.

We also require the behaviour of an integral related to the I-Bessel function. Define

$$P_s := \frac{1}{2\pi i}\int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^s e^{\pi\sqrt{2n}\left(v+\frac{1}{v}\right)} dv.$$

Then Lemma 4.2 of [2] reads as follows.

Lemma III.2.7. *For $|m| \leq \frac{1}{6\beta}\log(n)$ we have*

$$P_s = I_{-s-1}\left(2\pi\sqrt{2n}\right) + O\left(\exp\left(\pi\sqrt{2n}\left(1 + \frac{1}{1+|m|^{-\frac{2}{3}}}\right)\right)\right)$$

as $n \to \infty$.

III.3 Asymptotic behaviour of f

The aim of this Section is to determine the asymptotic behaviour of f. To do so we consider two separate cases: when q tends toward the pole $q = 1$, and when q is away from this pole. It is shown that the behaviour toward the pole at $q = 1$ gives the dominant contribution when applying the circle method in Section III.4.

First note that Lemma III.2.1 implies that $f(-z;\tau) = -f(z;\tau)$, and so $b(-m,n) = -b(m,n)$. We now restrict our attention to the case of $m \geq 0$.

We next find the Fourier coefficient of ζ^m of f, following the framework of [6]. Since there is a pole of f at $z = \frac{1}{2}$, we define

$$\begin{aligned} f_m^{\pm}(\tau) &:= \int_0^{\frac{1}{2}-a} f(z;\tau)e^{-2\pi imz}dz + \int_{\frac{1}{2}+a}^{1} f(z;\tau)e^{-2\pi imz}dz + G^{\pm} \\ &= -2i\int_0^{\frac{1}{2}-a} f(z;\tau)\sin(2\pi mz)dz + G^{\pm}, \end{aligned}$$

where $a > 0$ is small, and

$$G^{\pm} := \int_{\frac{1}{2}-a}^{\frac{1}{2}+a} f(z;\tau)e^{-2\pi imz}dz.$$

For G^+ the integral is taken over a semi-circular path passing above the pole. Similarly, G^- is taken over a semi-circular path passing below the pole. Then the Fourier coefficient of ζ^m of f is

$$f_m(\tau) := \frac{f_m^+ + f_m^-}{2} = -2i\int_0^{\frac{1}{2}-a} f(z;\tau)\sin(2\pi mz)dz + \frac{G^+ + G^-}{2}. \tag{III.3.1}$$

Shifting $z \mapsto z - \frac{1}{2}$ and parameterising the semi-circle we see that

$$G^+ = \lim_{a\to 0^+}\int_{-\pi}^{0} aie^{i\theta} f\left(ae^{i\theta} + \frac{1}{2};\tau\right) e^{-2\pi im\left(ae^{i\theta}+\frac{1}{2}\right)}d\theta.$$

Next we insert the Taylor expansion of the exponential function $e^{-2\pi imae^{i\theta}}$ and note that terms of order $O\left(a^2 f\left(ae^{i\theta} + \frac{1}{2};\tau\right)\right)$ vanish as $a \to 0^+$ since f has only a simple pole at $z = \frac{1}{2}$. Thus we obtain

$$G^+ = i(-1)^m \int_{-\pi}^{0} \lim_{a\to 0^+}\left(af\left(ae^{i\theta} + \frac{1}{2};\tau\right)\right) e^{i\theta}d\theta.$$

We next note that using L'Hôpital's rule and Lemma III.2.1 gives

$$\lim_{a\to 0^+}\left(af\left(ae^{i\theta}+\frac{1}{2};\tau\right)\right)=-e^{-i\theta}\frac{\vartheta\left(\frac{1}{2};\tau\right)^4}{2\eta(\tau)^9\vartheta'(0;\tau)}.$$

Therefore we see that

$$G^+=i(-1)^{m+1}\pi\frac{\vartheta\left(\frac{1}{2};\tau\right)^4}{2\eta(\tau)^9\vartheta'(0;\tau)}=4(-1)^{m+\frac{1}{2}}\frac{\eta(2\tau)^8}{\eta(\tau)^{16}},$$

where we have used the well-known facts that

$$\vartheta\left(\frac{1}{2};\tau\right)=-2\frac{\eta(2\tau)^2}{\eta(\tau)},$$

and

$$\vartheta'(0;\tau)=-2\pi\eta(\tau)^3.$$

A similar calculation shows that $G^-=G^+$. Hence the contribution of the final term in (III.3.1) is given by

$$\frac{G^++G^-}{2}=4(-1)^{m+\frac{1}{2}}\frac{\eta(2\tau)^8}{\eta(\tau)^{16}}.$$

Remark 8. The residue term $\frac{\eta(2\tau)^8}{\eta(\tau)^{16}}$ is the generating function for 8-tuple partitions [20]. Various number-theoretic properties of similar overpartition tuple functions are studied in [4, 5] for example. The physical interpretation of the residue term is discussed in Section 3.2 of [21], and it is an interesting question as to whether further number-theoretic properties (aside from asymptotics) of $\frac{\eta(2\tau)^8}{\eta(\tau)^{16}}$ also have a physical interpretation.

In the following two subsections we determine the asymptotic behaviour of f toward and away from the dominant pole at $q=1$ respectively. Throughout, we will let $\tau=\frac{i\varepsilon}{2\pi}$, $\varepsilon:=\beta(1+ixm^{-\frac{1}{3}})$ and $\beta:=\pi\sqrt{\frac{2}{n}}$. We determine asymptotics as $n\to\infty$.

III.3.1 Bounds towards the dominant pole

Here we find the asymptotic behaviour of f toward the dominant pole at $q=1$, shown in the following lemma.

Lemma III.3.1. *Let $\tau=\frac{i\varepsilon}{2\pi}$, with $0<\mathrm{Re}(\varepsilon)\ll 1$, and $0<z<\frac{1}{2}$. Then we have that*

$$f\left(z;\frac{i\varepsilon}{2\pi}\right)=-\frac{\varepsilon^3}{\pi^3}\frac{\sinh\left(\frac{2\pi^2z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2z}{\varepsilon}\right)}\left(1+e^{-4\pi^2\mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-2z)}+O\left(e^{-4\pi^2\mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-z)}\right)\right).$$

Proof. Using the modularity of f (which follows from Lemmas III.2.1 and III.2.2) and setting $q_0 := e^{-\frac{2\pi i}{\tau}}$, we have that

$$\begin{aligned} f(z;\tau) &= \frac{\tau^3 \zeta^{\frac{2}{\tau}} \prod_{n\geq 1} \left(1-\zeta^{\frac{1}{\tau}} q_0^n\right)^4 \left(1-\zeta^{-\frac{1}{\tau}} q_0^{n-1}\right)^4}{i\zeta^{\frac{1}{\tau}} \prod_{n\geq 1} (1-q_0^n)^6 \left(1-\zeta^{\frac{2}{\tau}} q_0^n\right)\left(1-\zeta^{\frac{-2}{\tau}} q_0^{n-1}\right)} \\ &= \frac{\tau^3 \left(\zeta^{\frac{1}{2\tau}} - \zeta^{-\frac{1}{2\tau}}\right)^4}{i\left(\zeta^{\frac{1}{\tau}} - \zeta^{\frac{-1}{\tau}}\right)} \prod_{n\geq 1} \frac{\left(1-\zeta^{\frac{1}{\tau}} q_0^n\right)^4 \left(1-\zeta^{-\frac{1}{\tau}} q_0^n\right)^4}{(1-q_0^n)^6 \left(1-\zeta^{\frac{2}{\tau}} q_0^n\right)\left(1-\zeta^{\frac{-2}{\tau}} q_0^n\right)}. \end{aligned}$$

This gives

$$-\frac{\varepsilon^3}{\pi^3} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \prod_{n\geq 1} \frac{\left(1-e^{\frac{4\pi^2}{\varepsilon}(z-n)}\right)^4 \left(1-e^{\frac{4\pi^2}{\varepsilon}(-z-n)}\right)^4}{\left(1-e^{\frac{-4\pi^2 n}{\varepsilon}}\right)^6 \left(1-e^{\frac{4\pi^2}{\varepsilon}(2z-n)}\right)\left(1-e^{\frac{4\pi^2}{\varepsilon}(-2z-n)}\right)}.$$

In order to find a bound we expand the denominator using geometric series. For $0 < z < \frac{1}{2}$ we see that $|e^{\frac{4\pi^2}{\varepsilon}(\pm 2z-n)}| < 1$ for all $n \geq 1$, and so we expand the denominator to obtain the product as

$$\prod_{n\geq 1} \left(1-e^{\frac{4\pi^2}{\varepsilon}(z-n)}\right)^4 \left(1-e^{\frac{-4\pi^2}{\varepsilon}(z+n)}\right)^4 \sum_{j\geq 0} e^{\frac{4j\pi^2}{\varepsilon}(2z-n)} \sum_{k\geq 0} e^{\frac{-4k\pi^2}{\varepsilon}(2z+n)} \left(\sum_{\ell\geq 0} e^{\frac{-4\pi^2 \ell n}{\varepsilon}}\right)^6,$$

which, for $0 < \mathrm{Re}(\varepsilon) \ll 1$, is of order

$$1 + e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-2z)} + O\left(e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-z)}\right).$$

Hence overall we find that

$$f\left(z;\frac{i\varepsilon}{2\pi}\right) = -\frac{\varepsilon^3}{\pi^3} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \left(1 + e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-2z)} + O\left(e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-z)}\right)\right),$$

yielding the claim. □

Remark 9. It is easy to see that this gives the same main term as noted in Section 4.5 of [12] (up to sign, which the authors there do not make use of).

Since $f(z;\tau) = -f(1-z;\tau)$ we see this immediately also implies the following lemma.

Lemma III.3.2. *Let $\tau = \frac{i\varepsilon}{2\pi}$, with $0 < \mathrm{Re}(\varepsilon) \ll 1$, and $\frac{1}{2} < z < 1$. Then we have that*

$$f\left(z; \frac{i\varepsilon}{2\pi}\right) = \frac{\varepsilon^3}{\pi^3} \frac{\sinh\left(\frac{2\pi^2(1-z)}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2(1-z)}{\varepsilon}\right)} \left(1 + e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(2z-1)} + O\left(e^{-4\pi^2\mathrm{Re}\left(\frac{1}{\varepsilon}\right)z}\right)\right).$$

We now look to find the behaviour of f_m toward the pole at $q = 1$. We begin with the contribution from the residue term

$$4(-1)^{m+\frac{1}{2}} \frac{\eta(2\tau)^8}{\eta(\tau)^{16}}.$$

Lemma III.3.3. *As $n \to \infty$ we have*

$$4(-1)^{m+\frac{1}{2}} \frac{\eta(\frac{i\varepsilon}{\pi})^8}{\eta(\frac{i\varepsilon}{2\pi})^{16}} = (-1)^{m+\frac{1}{2}} \frac{\varepsilon^4}{2^6\pi^4}\left(e^{\frac{2\pi^2}{\varepsilon}} + O(1)\right).$$

Proof. Using the modularity of η given in Lemma III.2.2 we see that

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}} = \left(\frac{-i\tau}{2}\right)^4 \frac{\eta\left(-\frac{1}{2\tau}\right)^8}{\eta\left(-\frac{1}{\tau}\right)^{16}} = \left(\frac{\tau}{2}\right)^4 \left(e^{\frac{\pi i}{\tau}} + O(1)\right).$$

As $\tau = \frac{i\varepsilon}{2\pi}$ this yields

$$\frac{\varepsilon^4}{2^8\pi^4}\left(e^{\frac{2\pi^2}{\varepsilon}} + O(1)\right).$$

$\square$

To estimate the contribution from the first integral in (III.3.1), we follow the approach of [2, 8], and define three further integrals

$$g_{m,1} := -\frac{\varepsilon^3}{\pi^3}\int_0^{\frac{1}{2}-a} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \sin(2\pi m z) dz, \tag{III.3.2}$$

along with

$$g_{m,2} := -\frac{\varepsilon^3}{\pi^3}\int_0^{\frac{1}{2}-a} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} e^{-4\pi^2\mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-2z)} \sin(2\pi m z) dz, \tag{III.3.3}$$

and

$$g_{m,3} := \int_0^{\frac{1}{2}-a} \left(f\left(z; \frac{i\varepsilon}{2\pi}\right) + \frac{\varepsilon^3}{\pi^3} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \left(1 + e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-2z)}\right)\right) \sin(2\pi m z) dz.$$

We now investigate the contribution from $g_{m,1}$ and show the following proposition.

Proposition III.3.4. *Assume that $|x| \leq 1$. Then for $a \to 0^+$, and with $\tau = \frac{i\varepsilon}{2\pi}$ we have that*

$$g_{m,1} = O\left(\beta^4\right)$$

as $n \to \infty$.

Proof. We first use the Taylor series representation of $\sinh(x)^4$ and $\sin(x)$ which are given by

$$\sinh(x)^4 = \sum_{n\geq 0} \frac{\left(4^{n+2}\left(4^{n+2}-4\right)\right)x^{2n+4}}{8\cdot(2n+4)!},$$
$$\sin(x) = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Thus we see that

$$\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4 \sin(2\pi m z) = \sum_{k\geq 0} \frac{\left(4^{k+2}\left(4^{k+2}-4\right)\right)\left(\frac{2\pi^2 z}{\varepsilon}\right)^{2k+4}}{8\cdot(2k+4)!} \sum_{r\geq 0} \frac{(-1)^r (2\pi m z)^{2r+1}}{(2r+1)!}.$$

Substituting this into (III.3.2) we find that

$$g_{m,1}(\tau) = -\frac{\varepsilon^3}{\pi^3} \sum_{k,r\geq 0} \frac{(-1)^r\left(4^{k+2}\left(4^{k+2}-4\right)\right)(2\pi m)^{2r+1}\left(\frac{2\pi^2}{\varepsilon}\right)^{2k+4}}{8\cdot(2k+4)!(2r+1)!} I_{k+r+2},$$

where

$$I_\ell := \lim_{a\to 0^+} \int_0^{\frac{1}{2}-a} \frac{z^{2\ell+1}}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} dz = \int_0^{\frac{1}{2}} \frac{z^{2\ell+1}}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} dz.$$

Following the ideas of [2] we further define I_ℓ' by

$$I_\ell' := \int_0^{\infty} \frac{z^{2\ell+1}}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} dz - I_\ell.$$

Then

$$I'_\ell = \int_{\frac{1}{2}}^{\infty} \frac{z^{2\ell+1}}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} dz \ll \int_{\frac{1}{2}}^{\infty} z^{2\ell+1} e^{-4\pi^2 z \mathrm{Re}\left(\frac{1}{\varepsilon}\right)} dz$$
$$\ll \left(4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)\right)^{-2\ell-2} \Gamma\left(2\ell+2; 2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)\right),$$

where we use the incomplete gamma function $\Gamma(\alpha;x) \coloneqq \int_x^\infty e^{-w} w^{\alpha-1} dw$. Since we have the asymptotic behaviour of

$$\Gamma(\ell;x) \sim x^{\ell-1} e^{-x}$$

as $x \to \infty$ we find that

$$\Gamma\left(2\ell+2; 2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)\right) \sim \left(2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)\right)^{2\ell+1} e^{-2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)}.$$

Hence we may conclude that

$$I'_\ell \ll \mathrm{Re}\left(\frac{1}{\varepsilon}\right)^{-1} e^{-2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)} \ll e^{-2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)}.$$

Now under the substitution $z \mapsto \frac{z\varepsilon}{4\pi}$ we find that

$$\int_0^\infty \frac{z^{2\ell+1}}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} dz = \left(\frac{\varepsilon}{4\pi}\right)^{2\ell+2} \int_0^\infty \frac{z^{2\ell+1}}{\sinh(\pi z)} dz = \left(\frac{\varepsilon}{4\pi}\right)^{2\ell+2} \mathcal{E}_\ell$$
$$= \left(\frac{\varepsilon}{4\pi}\right)^{2\ell+2} \frac{(-1)^{\ell+1} E_{2\ell+1}(0)}{2}.$$

Then we obtain that

$$g_{m,1} = -\frac{\varepsilon^3}{2\pi^3} \sum_{k,r\geq 0} \frac{(-1)^{2r+k+3}\left(4^{k+2}\left(4^{k+2}-4\right)\right)(2\pi m)^{2r+1}\left(\frac{2\pi^2}{\varepsilon}\right)^{2k+4}}{8\cdot(2k+4)!(2r+1)!} \left(\frac{\varepsilon}{4\pi}\right)^{2r+2k+6}$$
$$\times \left[E_{2r+2k+5}(0) + O\left((4\pi)^{2r+2k+6} |\varepsilon|^{-2r-2k-6} e^{-2\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)}\right)\right].$$

Letting $m' \coloneqq m/2$ and bounding terms where $k \geq 1$, we see that

$$g_{m,1} = \frac{\varepsilon^4}{2^7} \sum_{r\geq 0} \frac{(m'\varepsilon)^{2r+1}}{(2r+1)!} \left[E_{2r+5}(0) + O\left(|\varepsilon|^2\right)\right].$$

Next, using Lemma III.2.3 we recognise that

$$\sum_{r\geq 0}\frac{(m'\varepsilon)^{2r+1}}{(2r+1)!}E_{2r+5}(0)=\frac{1}{m'^3}\frac{\partial^3}{\partial\varepsilon^3}\sum_{r\geq 0}\frac{(m'\varepsilon)^{2r+4}}{(2r+4)!}E_{2r+5}(0)=-\frac{1}{2m'^3}\frac{\partial^3}{\partial\varepsilon^3}\operatorname{sech}^2\left(\frac{m'\varepsilon}{2}\right)$$
$$=-\frac{4}{m^3}\frac{\partial^3}{\partial\varepsilon^3}\operatorname{sech}^2\left(\frac{m\varepsilon}{4}\right).$$

We therefore obtain

$$g_{m,1}=-\frac{\varepsilon^4}{2^5m^3}\left(\frac{\partial^3}{\partial\varepsilon^3}\operatorname{sech}^2\left(\frac{m\varepsilon}{4}\right)+O\left(|\varepsilon|^2\cosh\left(m\varepsilon\right)\right)\right).$$

Further, we have that

$$\frac{\partial^3}{\partial\varepsilon^3}\operatorname{sech}^2\left(\frac{m\varepsilon}{4}\right)=-\frac{m^3\left(\cosh^2\left(\frac{m\varepsilon}{4}\right)-3\right)\sinh\left(\frac{m\varepsilon}{4}\right)}{8\cosh^5\left(\frac{m\varepsilon}{4}\right)}.$$

It is clear that

$$-\frac{\left(\cosh^2\left(\frac{m\varepsilon}{4}\right)-3\right)\sinh\left(\frac{m\varepsilon}{4}\right)}{8\cosh^5\left(\frac{m\varepsilon}{4}\right)}=O(1).$$

Therefore, we see that

$$g_{m,1}=\frac{\varepsilon^4}{2^5}\left(O(1)+O\left(\varepsilon^2m^{-3}\cosh(m\varepsilon)\right)\right).$$

Further, recall that $\varepsilon=\beta(1+ixm^{-\frac{1}{3}})$. Then we have that

$$\cosh(m\varepsilon)=\cosh\left(\beta m+i\beta m^{\frac{2}{3}}x\right)=\cosh(\beta m)\left(1+O\left(\beta m^{\frac{2}{3}}\right)\right).$$

Hence we obtain

$$g_{m,1}=\frac{\varepsilon^4}{2^5}\left(O(1)+O(\varepsilon^2m^{-3}\cosh(m\beta)\right)=O(\varepsilon^4)=O(\beta^4),$$

where for the last equality we use that $\varepsilon\ll\beta$. $\square$

To bound the contribution of $g_{m,2}$ we note the following trivial lemma.

Lemma III.3.5. *For $|x|\leq 1$ we have that*

$$|g_{m,2}|\ll g_{m,1}.$$

Next we bound the contribution from $g_{m,3}$.

Proposition III.3.6. *For $|x| \leq 1$, we have that*

$$|g_{m,3}| \ll \frac{\varepsilon^3}{\pi^3}.$$

Proof. We see that

$$|g_{m,3}| = \left| \int_0^{\frac{1}{2}-a} \left(f\left(z; \frac{i\varepsilon}{2\pi}\right) + \frac{\varepsilon^3}{\pi^3} \frac{\sinh\left(\frac{2\pi^2 z}{\varepsilon}\right)^4}{\sinh\left(\frac{4\pi^2 z}{\varepsilon}\right)} \left(1 + e^{-4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(1-2z)}\right)\right) \sin(2\pi m z) dz \right|.$$

We estimate the right-hand side using Lemma III.3.1, to find that

$$|g_{m,3}| \ll \int_0^{\frac{1}{2}-a} \frac{\varepsilon^3}{\pi^3} \left| \frac{\sin(2\pi m z)}{1 - e^{\frac{-8\pi^2 z}{\varepsilon}}} \right| e^{4\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)(2z-1)} dz.$$

We see that

$$\frac{\sin(2\pi m z)}{1 - e^{\frac{-8\pi^2 z}{\varepsilon}}} \ll 1,$$

and so it follows that $|g_{m,3}| \ll \frac{\varepsilon^3}{\pi^3} e^{-4a\pi^2 \mathrm{Re}\left(\frac{1}{\varepsilon}\right)} \ll \frac{\varepsilon^3}{\pi^3}$. □

Combining Lemmas III.3.3 and III.3.5, and Propositions III.3.4 and III.3.6, we obtain the following theorem regarding f_m as defined in (III.3.1).

Theorem III.3.7. *For $|x| \leq 1$ we have that*

$$f_m\left(\frac{i\varepsilon}{2\pi}\right) = (-1)^{m+\frac{1}{2}} \frac{\varepsilon^4}{2^6 \pi^4} e^{\frac{2\pi^2}{\varepsilon}} + O\left(\beta^3\right)$$

as $n \to \infty$.

III.3.2 Bounds away from the dominant pole

We next investigate the behaviour of f_m away from the pole $q = 1$, by assuming that $1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$. In the following lemma we bound the residue term

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}},$$

away from the pole $q = 1$.

Lemma III.3.8. *For* $1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ *we have that*

$$\left|\frac{\eta\left(\frac{i\varepsilon}{\pi}\right)^8}{\eta\left(\frac{i\varepsilon}{2\pi}\right)^{16}}\right| \ll n^{-2}\exp\left[\pi\sqrt{2n} - \frac{8\sqrt{2n}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right]$$

as $n \to \infty$.

Proof. We first write

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}} = \frac{\eta(2\tau)^8}{q^{\frac{2}{3}}}\frac{q^{\frac{2}{3}}}{\eta(\tau)^{16}}.$$

Using equation (III.2.1) directly we find that (with $\tau = \frac{i\varepsilon}{2\pi}$)

$$\frac{q^{\frac{2}{3}}}{\eta(\tau)^{16}} = P(q)^{16} \ll n^{-4}\exp\left[\frac{4\pi\sqrt{2n}}{3} - \frac{8\sqrt{2n}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right].$$

It remains to consider the behaviour of $e^{\varepsilon/12}\eta(\frac{i\varepsilon}{\pi})$. Using the transformation formula of η given in Lemma III.2.2 along with the well-known summation representation of η, we see that as $n \to \infty$ we obtain

$$e^{\frac{\varepsilon}{12}}\eta\left(\frac{i\varepsilon}{\pi}\right) = e^{\frac{\varepsilon}{12}}\sqrt{\frac{\pi}{\varepsilon}}e^{-\frac{\pi^2}{12\varepsilon}}\sum_{j\in\mathbb{Z}}(-1)^j e^{-\frac{\pi^2(3j^2-j)}{\varepsilon}} \ll \sqrt{\frac{\pi}{\varepsilon}}e^{-\frac{\pi^2}{12\varepsilon}}.$$

We hence have

$$\left|\frac{\eta\left(\frac{i\varepsilon}{\pi}\right)}{e^{\frac{2\varepsilon}{3}}}\right|^8 \ll \left|\sqrt{\frac{\pi}{\varepsilon}}e^{-\frac{\pi^2}{12\varepsilon}}\right|^8 \ll \left(\frac{\pi}{\beta}\right)^4 e^{-\frac{2\pi^2}{3\beta}} \ll n^2 e^{-\frac{\pi\sqrt{2n}}{3}}.$$

Combining the two bounds yields the result. ∎

Next, we investigate the contribution of

$$\left|\int_0^{\frac{1}{2}-a} f(z;\tau)\sin(2\pi mz)dz\right| \ll \int_0^{\frac{1}{2}-a}|f(z;\tau)\sin(2\pi mz)|dz.$$

Then we want to bound

$$|f(z;\tau)\sin(2\pi mz)| = \left|\frac{\sin(2\pi mz)\vartheta(z;\tau)^4}{\eta(\tau)^9\vartheta(2z;\tau)}\right|$$

away from the dominant pole. For $0 < b < \frac{1}{2}$ far from $\frac{1}{2}$ we see that we may bound the integrand in modulus by

$$|f(b;\tau)\sin(2\pi mb)| \ll |P(q)|^9 \left| q^{-\frac{3}{8}} \frac{\vartheta(b;\tau)^4}{\vartheta(2b;\tau)} \right| \ll |P(q)|^9 \sum_{n\in\mathbb{Z}} |q|^{\frac{n^2+n}{2}} \ll |P(q)|^9 \sum_{n\in\mathbb{Z}} e^{-\beta n^2}.$$

As $z \to \frac{1}{2}$ we apply L'Hôpital's rule to the integrand $|f(z;\tau)\sin(2\pi mz)|$ which yields the bound

$$\left| \int_0^{\frac{1}{2}-a} f(z;\tau)\sin(2\pi mz)dz \right| \ll \frac{\eta(2\tau)^8}{\eta(\tau)^{16}}.$$

Hence, away from the dominant pole in q, we have shown the following proposition.

Proposition III.3.9. *For $1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ we have that*

$$\left| f\left(z; \frac{i\varepsilon}{2\pi}\right) \right| \ll n^{-2} \exp\left[\pi\sqrt{2n} - \frac{8\sqrt{2n}}{\pi} \left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}} \right) \right]$$

as $n \to \infty$.

III.4 The Circle Method

In this section we use Wright's variant of the Circle Method to complete the proof of Theorem III.1.1. We start by noting that Cauchy's theorem implies that

$$b(m,n) = \frac{1}{2\pi i} \int_C \frac{f_m(\tau)}{q^{n+1}} dq,$$

where $C := \{q \in \mathbb{C} \mid |q| = e^{-\beta}\}$ is a circle centred at the origin of radius less than 1, with the path taken in the counter-clockwise direction. Making a change of variables, changing the direction of the path of the integral, and recalling that $\varepsilon = \beta(1 + ixm^{-\frac{1}{3}})$ we have

$$b(m,n) = \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}} f_m\left(\frac{i\varepsilon}{2\pi}\right) e^{\varepsilon n} dx.$$

Splitting this integral into two pieces, we have $b(m,n) = M + E$ where

$$M := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x|\leq 1} f_m\left(\frac{i\varepsilon}{2\pi}\right) e^{\varepsilon n} dx,$$

and

$$E := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{1\leq|x|\leq \frac{\pi m^{\frac{1}{3}}}{\beta}} f_m\left(\frac{i\varepsilon}{2\pi}\right) e^{\varepsilon n} dx.$$

Next we determine the contributions of each of the integrals M and E, and see that M contributes to the main asymptotic term, while E is part of the error term.

III.4.1 The major arc

First we concentrate on the contribution M. Then we obtain the following proposition.

Proposition III.4.1. *We have that*

$$M = (-1)^{m+\frac{1}{2}} \frac{\beta^5}{2^7\pi^5(2n)^{\frac{1}{4}}} e^{2\pi\sqrt{2n}} + O\left(m^{-\frac{2}{3}} n^{-\frac{13}{4}} e^{2\pi\sqrt{2n}}\right)$$

as $n \to \infty$.

Proof. By Theorem III.3.7 and making the change of variables $v = 1 + ixm^{-\frac{1}{3}}$ we obtain

$$M = (-1)^{m+\frac{1}{2}} \frac{\beta^5}{2^6\pi^4} P_4 + O\left(\beta^4 e^{\pi\sqrt{2n}}\right).$$

Now we rewrite P_4 in terms of the I-Bessel function using Lemma III.2.7, yielding

$$M = (-1)^{m+\frac{1}{2}} \frac{\beta^5}{2^6\pi^4} I_{-5}(2\pi\sqrt{2n}) + O\left(\beta^5 e^{\pi\sqrt{2n}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)}\right) + O\left(\beta^4 e^{\pi\sqrt{2n}}\right).$$

The asymptotic behaviour of the I-Bessel function given in Lemma III.2.6 gives that

$$\begin{aligned} M =& (-1)^{m+\frac{1}{2}} \frac{\beta^5}{2^7\pi^5(2n)^{\frac{1}{4}}} e^{2\pi\sqrt{2n}} + O\left(n^{-\frac{13}{4}} e^{2\pi\sqrt{2n}}\right) + O\left(\beta^5 e^{\pi\sqrt{2n}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)}\right) \\ &+ O\left(\beta^4 e^{\pi\sqrt{2n}}\right). \end{aligned}$$

It is clear that the first error term is the dominant one, and the result follows. □

III.4.2 The error arc

Now we bound E as follows.

Proposition III.4.2. *As* $n \to \infty$

$$E \ll n^{-2} \exp\left[2\pi\sqrt{2n} - \frac{8\sqrt{2n}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right].$$

Proof. By Proposition III.3.9 we see that the main term in the error arc is given by the residue. Hence we may bound

$$\begin{aligned} E &\ll \int_{1 \leq x \leq \frac{\pi m^{\frac{1}{3}}}{\beta}} n^{-2} \exp\left[\pi\sqrt{2n} - \frac{8\sqrt{2n}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right] e^{\varepsilon n} dx \\ &\ll n^{-2} \exp\left[2\pi\sqrt{2n} - \frac{8\sqrt{2n}}{\pi}\left(1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}}\right)\right]. \end{aligned}$$

Noting that this is exponentially smaller than M finishes the proof of Theorem III.1.1. □

III.5 Open questions

We end by commenting on some questions related to the results presented above.

1. Here we discuss the asymptotic profile of the coefficients $b(m, n)$ for $|m| \leq \frac{1}{6\beta}\log(n)$. We are also interested in the profile when m is larger than this bound, and so in future it would be instructive to investigate the asymptotic profile of $b(m, n)$ for large $|m|$. For example, similar results in this direction for the crank of a partition are given in [17].

2. In the present paper, we provide a framework for investigating the profile of eta-theta quotients. In particular, we deal with the case of a function with a single simple pole on the path of integration. Future research is planned in order to expand this framework for a family of meromorphic eta-theta quotients with a finite number of (not necessarily single) poles on the path of integration. This should include similar eta-theta quotients that appear in other physical partition functions.

3. In showing Theorem III.1.1 we see that the main asymptotic term arises from the pole at $z = 1/2$, and in turn from the residue term $\frac{\eta(2\tau)^8}{\eta(\tau)^{16}}$; is there a physical interpretation for the fact that these terms give the largest contribution to the asymptotic behaviour of $b(m, n)$?

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Chapter IV

Asymptotic Equidistribution and Convexity for Partition Ranks

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IV.1 Introduction and statement of results

A familiar statistic in combinatorics is the number of partitions of an integer n, denoted by $p(n)$. The function $p(n)$ has been studied extensively, giving rise to results such as the famous Ramanujan congruences [13]. Of particular interest to the current paper is the asymptotic behaviour of the number of partitions, proven by Hardy and Ramanujan in [8]. They showed that as $n \to \infty$

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{2\pi\sqrt{\frac{n}{6}}}.$$

Other statistics involving partitions have been introduced since, the most pertinent of which for us is the rank of a partition, defined to be the largest part minus the number of parts. We denote the number of partitions of n with rank m by $N(m,n)$. By standard combinatorial arguments it can be shown that the generating function of $N(m,n)$ is given by (see equation 7.2 of [7] for example)

$$R(\zeta;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}} N(m,n)\zeta^m q^n = \sum_{n\geq 0} \frac{q^{n^2}}{(\zeta q, \zeta^{-1}q; q)_n},$$

where $\zeta := e^{2\pi i z}$, $q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$ the upper half plane, and $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. Further, to ease notation we set $(a_1,a_2;q)_n := (a_1;q)_n(a_2;q)_n$. First introduced by Dyson in [6] as an attempt to describe the Ramanujan congruences combinatorially, the rank statistic has a storied history. For example, we have that

$$R(-1;q) = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

which is the famous mock theta function $f(q)$, defined by Ramanujan and Watson in the early twentieth century.

As a further refinement of $N(m,n)$ we let $N(r,t;n)$ be the number of partitions of n with rank congruent to r modulo t. It is well-known that for nonnegative integers r,t we have the following equation that relates the generating function for $N(r,t;n)$ to the generating functions of $p(n)$ and $N(m,n)$ (see e.g. Section 14.3.3 of [3])

$$\sum_{n\geq 0} N(r,t;n)q^n = \frac{1}{t}\left[\sum_{n\geq 0} p(n)q^n + \sum_{j=1}^{t-1} \zeta_t^{-rj} R(\zeta_t^j;q)\right], \tag{IV.1.1}$$

where $\zeta_t := e^{2\pi i/t}$.

In [4] it was remarked that the results therein may be employed to obtain asymptotics of $N(r,t;n)$. This question was explored by Bringmann in [2] for odd t, via use of the circle method. However, while the formulae obtained therein are stronger than our asymptotics, the present paper requires less strict results and hence we have somewhat shorter proofs. While the theorem we present can be concluded from the results of Bringmann in [2] for odd t, we give results for all $t \geq 2$ and prove the following result.

Theorem IV.1.1. *For fixed $0 \leq r < t$ and $t \geq 2$ we have that*

$$N(r,t;n) \sim \frac{1}{t}p(n) \sim \frac{1}{4tn\sqrt{3}} e^{2\pi\sqrt{\frac{n}{6}}}$$

as $n \to \infty$. Hence for fixed t the number of partitions of rank congruent to r modulo t is equidistributed in the limit.

Recently, in [1] Ono and Bessenrodt showed that the partition function satisfies the following convexity result. If $a,b \geq 1$ and $a+b \geq 9$ then

$$p(a)p(b) > p(a+b).$$

A natural question to ask is then: does $N(r,t;n)$ satisfy a similar property? In [9] Hou and Jagadeesan provide an answer if $t=3$. They showed that for $0 \leq r \leq 2$ we have

$$N(r,3;a)N(r,3;b) > N(r,3;a+b)$$

for all a, b larger than some specific bound. Further, at the end of the same paper, the authors offer the following conjecture on a more general convexity result.

Conjecture IV.1.2. *For $0 \leq r < t$ and $t \geq 2$ then*

$$N(r,t;a)N(r,t;b) > N(r,t;a+b)$$

for sufficiently large a and b.

As a simple consequence of Theorem IV.1.1 we prove the following theorem.

Theorem IV.1.3. *Conjecture IV.1.2 is true.*

Remark 10. We note that unlike in [9] our proof of Theorem IV.1.3 does not give an explicit lower bound on a and b. To yield such a bound one could employ similar techniques to those in [9], relying on the asymptotics found in [2]. However, since [2] gives results only for odd t one could only find such bounds directly for odd t. Further, to find an explicit bound for general t is a difficult problem.

The paper is organised as follows. In Section V.2 we give some preliminary results needed for the rest of the paper. We begin by showing the strict monotonicity in n of $N(m,n)$ in Section IV.3 which then allows us to prove a monotonicity result of $N(r,t;n)$ in Section IV.4. Section IV.5 serves to find the asymptotic behaviour of the level three Appell function. In Section IV.6 we prove Theorem IV.1.1. We are then able to conclude Theorem IV.1.3 in Section IV.7.

IV.2 Preliminaries

IV.2.1 Appell functions

We make extensive use of properties of Appell functions in Section IV.5, and so here we recall relevant results without proof. In his celebrated thesis [14] Zwegers studied the Appell function

$$\mu(u,z;\tau) := \frac{e^{\pi i u}}{\vartheta(z;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{\pi i (n^2+n)\tau} e^{2\pi i n z}}{1 - e^{2\pi i n \tau} e^{2\pi i u}},$$

where

$$\vartheta(z;\tau) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n (z+\frac{1}{2})},$$

with $z \in \mathbb{C}$, is a Jacobi theta function. It is well-known that ϑ satisfies the following two transformation formulae (see e.g. [12]);

$$\vartheta(z+1;\tau) = -\vartheta(z;\tau),$$

and

$$\vartheta(z;\tau) = \frac{i}{\sqrt{-i\tau}} e^{\frac{-\pi i z^2}{\tau}} \vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right).$$

Zwegers used this to then show that μ satisfies

$$\mu(u+1, v;\tau) = -\mu(u, v;\tau),$$

and

$$\mu(u, v;\tau) = \frac{-1}{\sqrt{-i\tau}} e^{\frac{\pi i (u-v)^2}{\tau}} \mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) + \frac{1}{2i} h(u-v;\tau),$$

where h is the Mordell integral

$$h(z;\tau) := \int_{\mathbb{R}} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{\cosh(\pi x)} dx.$$

Further, Zwegers showed the following two transformation properties of h;

$$h(z;\tau) = \frac{1}{\sqrt{-i\tau}} e^{\frac{\pi i z^2}{\tau}} h\left(\frac{z}{\tau}; -\frac{1}{\tau}\right), \tag{IV.2.1}$$

and

$$h(z;\tau) + e^{-2\pi i z - \pi i \tau} h(z+\tau;\tau) = 2 e^{-\pi i z - \frac{\pi i \tau}{4}}. \tag{IV.2.2}$$

In more recent work [15] Zwegers introduced Appell functions of higher level and showed that they also exhibit similar transformation formulae. We define the level ℓ Appell function by

$$A_\ell(u, v;\tau) := e^{\pi i \ell u} \sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n} q^{\frac{\ell n(n+1)}{2}} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}.$$

Then it is shown that

$$\begin{aligned} A_\ell(u, v;\tau) &= \sum_{k=0}^{\ell-1} e^{2\pi i u k} A_1\left(\ell u, v + k\tau + \frac{\ell-1}{2}; \ell\tau\right) \\ &= \sum_{k=0}^{\ell-1} e^{2\pi i u k} \vartheta\left(v + k\tau + \frac{\ell-1}{2}; \ell\tau\right) \mu\left(\ell u, v + k\tau + \frac{\ell-1}{2}; \ell\tau\right), \end{aligned}$$

and so A_ℓ inherits transformation properties from ϑ and μ.

IV.2.2 A bound on h

In Section IV.5 we investigate asymptotic properties of h, and make use of a bound given in [11]. Proposition 5.2 therein reads as follows.

Proposition IV.2.1. *Let κ be a positive integer, $\alpha, \beta \in \mathbb{R}$ with $|\alpha| < \frac{1}{2}$ and $-\frac{1}{2} \leq \beta < \frac{1}{2}$, and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Then*

$$\left| h\left(\frac{i\beta}{\kappa z} + \alpha; \frac{i}{\kappa z}\right)\right| \leq \begin{cases} |\sec(\pi\beta)|\kappa^{\frac{1}{2}} \operatorname{Re}\left(\frac{1}{z}\right)^{-\frac{1}{2}} e^{-\frac{\pi\beta^2}{\kappa}\operatorname{Re}\left(\frac{1}{z}\right)+\pi\kappa\alpha^2\operatorname{Re}\left(\frac{1}{z}\right)^{-1}} & \text{if } \beta \neq -\frac{1}{2}, \\ \left(1 + \kappa^{\frac{1}{2}} \operatorname{Re}\left(\frac{1}{z}\right)^{-\frac{1}{2}}\right) e^{-\frac{\pi}{4\kappa}\operatorname{Re}\left(\frac{1}{z}\right)} & \text{if } \beta = -\frac{1}{2}. \end{cases}$$

In particular, we will use this to show that all but finitely many terms arising from a particular Appell function are exponentially decaying in the asymptotic limit.

IV.2.3 Ingham's Tauberian Theorem

To conclude our main result, we use the following theorem of Ingham [10] that gives an asymptotic formula for the coefficients of certain power series.

Theorem IV.2.2. *Let $f(q) := \sum_{n\geq 0} a(n)q^n$ be a power series with weakly increasing non-negative coefficients and radius of convergence equal to one. If there exist constants $A > 0$, $\lambda, \alpha \in \mathbb{R}$ such that*

$$f(e^{-\varepsilon}) \sim \lambda \varepsilon^{\alpha} e^{\frac{A}{\varepsilon}}$$

as $\varepsilon \to 0^+$, then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

IV.3 Strict monotonicity of $N(m, n)$

In this section we show strict monotonicity of $N(m, n)$ for $n \geq 2m + 25$. This follows work of Chan and Mao in [5] in which the following theorem regarding weak monotonicity of $N(m, n)$ is shown.

Theorem IV.3.1. *For all non-negative integers m and positive integers n we have that*

$$N(m, n) \geq N(m, n - 1),$$

except when $(m, n) = (\pm 1, 7), (0, 8), (\pm 3, 11)$ and when $n = m + 2$.

First, we state without proof some relevant results, beginning with the following trivial lemma which is an example of the famous Postage Stamp Problem.

Lemma IV.3.2. *The coefficient of q^n with $n \geq 18$ in the expression*

$$\sum_{j\geq 0} q^{3j} \sum_{k\geq 0} q^{4k}$$

is greater than or equal to two.

We also have the following result, see Lemma 10 of [5].

Lemma IV.3.3. *The expression*

$$\frac{1-q^{m+1}}{(1-q^2)(1-q^3)}$$

has non-negative power series coefficients for any positive integer m.

Lemma 9 of [5] reads as follows.

Lemma IV.3.4. *With $(a)_n := (a;q)_n$, we have that*

$$\frac{1-q}{(aq)_1(q/a)_1} = \sum_{n\geq 0} \sum_{m=-n}^{n} (-1)^{m+n} a^m q^n,$$

and

$$\begin{aligned}\frac{1-q}{(aq)_2(q/a)_2} =& -q + \frac{1}{1-q^3} + \frac{q^2}{1-q^4} + \frac{q^8}{(1-q^3)(1-q^4)} \\ &+ \sum_{m\geq 1} (a^m + a^{-m}) q^m \left(\frac{1-q^{m+1}}{(1-q^2)(1-q^3)} + \frac{q^{m+3}}{(1-q^3)(1-q^4)} \right).\end{aligned}$$

We use results of [5] to show that, for sufficiently large n, the coefficients of $a^m q^n$ in the series

$$\sum_{n\geq 0} \frac{(1-q)q^{n^2}}{(aq)_n(q/a)_n}$$

are strictly positive for $n \neq m+2$. This then implies the following proposition.

Proposition IV.3.5. *For positive m and $n \geq 2m+25$, or $m=0$ and $n \geq 30$, we have that*

$$N(m,n) > N(m,n-1).$$

Proof. As in [5] we define

$$\sum_{m\in\mathbb{Z}} a^m f_{m,k}(q) := \frac{1-q}{(aq)_k(q/a)_k}.$$

Then

$$\begin{aligned}\sum_{n\geq 0}\frac{(1-q)q^{n^2}}{(aq)_n(q/a)_n} =& 1-q+\sum_{n\geq 1} q^{n^2} f_{0,n}(q) \\ &+\sum_{m\geq 1}\left(a^m+a^{-m}\right)\left(qf_{m,1}(q)+q^4 f_{m,2}(q)+\sum_{n\geq 3} q^{n^2} f_{m,n}(q)\right).\end{aligned} \tag{IV.3.1}$$

The main idea of [5] is to show that these combinations of $f_{m,n}(q)$ have non-negative coefficients of q^n and $a^m q^n$ for n large enough, and away from $n = m+2$ (since $N(n-2,n) = 0$ trivially). Here, we simply observe that for some larger bound on n the coefficients are in fact strictly positive, implying our result.

Concentrating firstly on the first sum in the right-hand side of (IV.3.1), the proof of Lemma 13 in [5] gives (correcting a minor error of [5])

$$\begin{aligned}\sum_{n\geq 1} q^{n^2} f_{0,n}(q) = \sum_{n\geq 0}(-1)^n q^{n+1} + q^4\left(-q+\frac{1}{1-q^3}+\frac{q^2}{1-q^4}+\frac{q^8}{(1-q^3)(1-q^4)}\right) \\ -\sum_{n\geq 3} q^{n^2+1}+\sum_{n\geq 0} b_n q^n,\end{aligned}$$

for some nonnegative sequence $\{b_n\}_{n\geq 0}$. Thus, if we show that

$$\sum_{n\geq 0}(-1)^n q^{n+1} + q^4\left(-q+\frac{1}{1-q^3}+\frac{q^2}{1-q^4}+\frac{q^8}{(1-q^3)(1-q^4)}\right)-\sum_{n\geq 3} q^{n^2+1}$$

has strictly positive coefficients of q^n for large enough n then we are done for this term. Expanding the above expression gives

$$\sum_{n\geq 0} q^{2n+1}-\sum_{n\geq 0} q^{2n+2}-q^5+\frac{q^4}{1-q^3}+\frac{q^6}{1-q^4}+\frac{q^{12}}{(1-q^3)(1-q^4)}-\sum_{n\geq 2} q^{4n^2+1}-\sum_{n\geq 1} q^{4n^2+4n+2}.$$

As in [5] we note that both of the expressions

$$\sum_{n\geq 0} q^{2n+1}-\sum_{n\geq 2} q^{4n^2+1}, \qquad \frac{q^6}{1-q^4}-\sum_{n\geq 1} q^{4n^2+4n+2}$$

have non-negative coefficients. So, it remains to show that

$$\frac{q^{12}}{(1-q^3)(1-q^4)} - \sum_{n\geq 0} q^{2n+2} \tag{IV.3.2}$$

has strictly positive coefficients for every n large enough. Using Lemma IV.3.2 it is easy to see that for $n \geq 30$ the coefficients of q^n in (IV.3.2) are strictly positive.

We next consider the second sum in the right-hand side of (IV.3.1) i.e. the expression

$$\sum_{m\geq 1} \left(a^m + a^{-m}\right) \left(q f_{m,1}(q) + q^4 f_{m,2}(q) + \sum_{n\geq 3} q^{n^2} f_{m,n}(q) \right),$$

and we wish to show that, for n sufficiently large and $n \neq m+2$, the coefficients of $a^m q^n$ are strictly positive.

Consider first the terms

$$\frac{q(1-q)}{(aq)_1(q/a)_1} + \frac{q^4(1-q)}{(aq)_2(q/a)_2}.$$

We now show that these have positive coefficients of q^n for large enough n. This will imply that

$$q f_{m,1}(q) + q^4 f_{m,2}(q)$$

also has positive coefficients for large enough n and $m \geq 1$. Unlike in [5] we do not need to split this into three cases. Then, by Lemma IV.3.4, we want to show that

$$q \sum_{n\geq 0} \sum_{\substack{m=-n \\ m\neq 0}}^{n} (-1)^{m+n} a^m q^n + q^4 \sum_{m\geq 1} (a^m + a^{-m}) q^m \left(\frac{1-q^{m+1}}{(1-q^2)(1-q^3)} + \frac{q^{m+3}}{(1-q^3)(1-q^4)} \right)$$

has positive coefficients for n large enough. By Lemma IV.3.3 it clearly suffices to choose n such that the coefficient of q^n in

$$\frac{q^{2m+7}}{(1-q^3)(1-q^4)}$$

is at least two. By Lemma IV.3.2 we see that choosing $n \geq 2m+25$ will suffice. Therefore the coefficients of q^n with $n \geq 2m+25$ and $m \geq 1$ in the expression

$$\sum_{m\geq 1} \left(a^m + a^{-m}\right) \left(q f_{m,1}(q) + q^4 f_{m,2}(q) \right)$$

are strictly positive.

From [5] we have that $\sum_{k\geq 3} q^{k^2} f_{m,k}(q)$ has non-negative coefficients for all n, and so we conclude overall that

$$N(0,n) > N(0,n-1) \text{ for } n \geq 30,$$
$$N(m,n) > N(m,n-1) \text{ for } m \geq 1, n \geq 2m+25.$$

□

IV.4 Monotonicity of $N(r,t;n)$

Using results of the previous section we now prove the following theorem.

Theorem IV.4.1. *Let $0 \leq r < t$ and $n \geq M$ where $M \coloneqq \max(2r+25, 2(t-r)+25)$. Then we have that*

$$N(r,t;n) \geq N(r,t;n-1).$$

Proof. We first rewrite $N(r,t;n)$ as

$$N(r,t;n) = \sum_{k\in\mathbb{Z}} N(r+kt,n), \tag{IV.4.1}$$

in particular noting that this is a finite sum, since for $|r+kt| > n$ we have $N(r+kt,n) = 0$. We differentiate two separate cases, depending on whether $r=0$ or $r \neq 0$. If $r+kt+2 \neq n$ for any $k \in \mathbb{Z}$ then we use Theorem IV.3.1 directly to conclude that $N(r,t;n) \geq N(r,t;n-1)$.

Now assume that there exists a term where $r+kt+2 = n$. First, let $r \neq 0$. We want to show that

$$\sum_{k\in\mathbb{Z}} N(r+kt,n) \geq \sum_{k\in\mathbb{Z}} N(r+kt,n-1).$$

Since $N(-m,n) = N(m,n)$ we see that there are at most two terms that vanish on the left-hand side, given by $N(n-2,n)$ and $N(2-n,n)$. Then their counterparts on the right-hand side satisfy $N(n-2,n-1) = N(2-n,n-1) = 1$. Since $r \neq 0$ and $n \geq M$ there must be at least two non-zero intermediate terms e.g. $N(r,n)$ and $N(r-t,n)$. For each of these intermediate terms we apply Proposition IV.3.5 and conclude our result for $n \geq M$.

We now turn to the case of $r=0$. Then (IV.4.1) becomes

$$N(0,n) + 2N(t,n) + \cdots + 2N(n-2,n),$$

where again the last term vanishes. We want to show that this expression is greater than or equal to

$$N(0,n-1) + 2N(t,n-1) + \cdots + 2N(n-1,n-1),$$

where the last term is equal to two. Then it is enough to use that $N(0, n) \geq N(0, n-1)+2$ for large enough n. Further, it is easy to see that we may adapt the proof of Proposition IV.3.5 to show that $\sum_{n\geq 1} f_{0,n}(q)q^{n^2}$ has coefficients strictly greater than one for all $n \geq 42$, implying that

$$N(0, n) \geq N(0, n-1) + 2,$$

for $n \geq 42$. For values of n between 1 and 42 we test on MAPLE the expression $N(0, n) - N(0, n-1)$ and can show for all $n \geq 15$ we have that $N(0, n) \geq N(0, n-1) + 2$.

Therefore for $n \geq 15$ we have that

$$N(0, t; n) \geq N(0, t; n-1).$$

Combining the above arguments finishes the proof.

□

IV.5 Asymptotic behaviour of the Appell function $A_3(u, -\tau; \tau)$

In this section we investigate the asymptotic behaviour of the Appell function $A_3(u, -\tau; \tau)$ when we let $\tau = \frac{i\varepsilon}{2\pi}$ and $\varepsilon \to 0^+$. We further impose that $0 < u \leq \frac{1}{2}$ throughout. We prove the following theorem.

Theorem IV.5.1. *Let $0 < u \leq \frac{1}{2}$ and $\tau = \frac{i\varepsilon}{2\pi}$. Then*

$$A_3(u, -\tau; \tau) \to 0$$

as $\varepsilon \to 0^+$.

Proof. Using the transformation formulae given in Section IV.2.1 we rewrite the level three Appell function

$$\begin{aligned} A_3(u, v; \tau) =& \frac{1}{3\tau}\sum_{k=0}^{2} e^{\frac{\pi i u(3u-2v)}{\tau}} \vartheta\left(\frac{v}{3\tau} + \frac{k}{3}; -\frac{1}{3\tau}\right)\mu\left(\frac{u}{\tau}, \frac{v}{3\tau} + \frac{k}{3}; -\frac{1}{3\tau}\right) \\ &+ \frac{1}{\sqrt{-12i\tau}}\sum_{k=0}^{2} e^{\pi i\left(\frac{-k^2\tau}{3} + \frac{6uk-2vk}{3} - \frac{v^2}{3\tau}\right)}\vartheta\left(\frac{v}{3\tau} + \frac{k}{3}; -\frac{1}{3\tau}\right) h(3u - v - k\tau; 3\tau). \end{aligned}$$

Specialising to $v = -\tau$ we obtain

$$\begin{aligned} A_3(u, -\tau; \tau) =& \frac{1}{3\tau} e^{\frac{\pi i u(3u+2\tau)}{\tau}} \sum_{k=0}^{2} \vartheta\left(-\frac{1}{3} + \frac{k}{3}; -\frac{1}{3\tau}\right)\mu\left(\frac{u}{\tau}, -\frac{1}{3} + \frac{k}{3}; -\frac{1}{3\tau}\right) \\ &+ \frac{e^{\frac{-\pi i\tau}{3}}}{\sqrt{-12i\tau}}\sum_{k=0}^{2} e^{\pi i\left(\frac{-k^2\tau}{3} + \frac{6uk+2\tau k}{3}\right)}\vartheta\left(\frac{k-1}{3}; -\frac{1}{3\tau}\right) h(3u + \tau - k\tau; 3\tau). \end{aligned}$$

We write $A_3(u, -\tau; \tau) = S_1 + S_2$ with

$$S_1 := \frac{1}{3\tau} e^{\frac{\pi i u(3u+2\tau)}{\tau}} \sum_{k=0}^{2} \vartheta\left(-\frac{1}{3} + \frac{k}{3}; -\frac{1}{3\tau}\right) \mu\left(\frac{u}{\tau}, -\frac{1}{3} + \frac{k}{3}; -\frac{1}{3\tau}\right) \tag{IV.5.1}$$

and

$$S_2 := \frac{e^{\frac{-\pi i \tau}{3}}}{\sqrt{-12i\tau}} \sum_{k=0}^{2} e^{\pi i\left(\frac{-k^2\tau}{3} + \frac{6uk+2\tau k}{3}\right)} \vartheta\left(\frac{k-1}{3}; -\frac{1}{3\tau}\right) h(3u + \tau - k\tau; 3\tau).$$

We first investigate the terms from S_1. By definition we know that

$$\vartheta(z_2; \tau)\mu(z_1, z_2; \tau) = e^{\pi i z_1} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau} e^{2\pi i n z_2}}{1 - e^{2\pi i n \tau} e^{2\pi i z_1}},$$

and so

$$\vartheta\left(z; -\frac{1}{3\tau}\right) \mu\left(\frac{u}{\tau}, z; -\frac{1}{3\tau}\right) = e^{\frac{\pi i u}{\tau}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{-\frac{\pi i(n^2+n)}{3\tau}} e^{2\pi i n z}}{1 - e^{-\frac{2\pi i n}{3\tau}} e^{\frac{2\pi i u}{\tau}}} = q_0^{-\frac{u}{2}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2+n}{6}} \zeta^n}{1 - q_0^{\frac{n}{3} - u}},$$

where $q_0 := e^{\frac{-2\pi i}{\tau}}$. Thus, with $z = \frac{k-1}{3}$, we have

$$S_1 = \frac{1}{3\tau} q_0^{-\frac{1}{2}(3u^2+u)} e^{2\pi i u} \sum_{k=0}^{2} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2+n}{6}} \zeta^n}{1 - q_0^{\frac{n}{3} - u}}. \tag{IV.5.2}$$

First we check the behaviour of S_1 at possible poles. Assume $u = \frac{1}{3}$, so that the $n = 1$ term has a pole of order one. Then (IV.5.1) is

$$\frac{1}{3\tau} \rho q_0^{-\frac{1}{3}} \sum_{k=0}^{2} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q_0^{\frac{n^2+n}{6}} e^{\frac{2\pi i n(k-1)}{3}}}{1 - q_0^{\frac{n-1}{3}}},$$

where $\rho := e^{\frac{2\pi i}{3}}$. The only issues are the $n = 1$ terms in this sum, and so we investigate the numerator

$$-\frac{1}{3\tau} \rho \sum_{k=0}^{2} e^{\frac{2\pi i(k-1)}{3}}.$$

From here it is clear that we have a zero of order one in the numerator and hence have a removable singularity at $u = \frac{1}{3}$. It is clear that for the $n = 1$ terms, the limit as u

approaches $\frac{1}{3}$ from both above and below is zero, since the numerator is always zero and the denominator is non-zero away from $u = \frac{1}{3}$.

Furthermore, it is clear that

$$\sum_{k=0}^{2} \zeta^n = \begin{cases} 3 \text{ if } n \equiv 0 \pmod 3, \\ 0 \text{ else.} \end{cases}$$

Thus (IV.5.2) is equal to

$$\frac{1}{\tau} q_0^{-\frac{1}{2}(3u^2+u)} e^{2\pi i u} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q_0^{\frac{3n^2+n}{2}}}{1-q_0^{n-u}}.$$

We want to find the lowest power of q_0 in this sum, since negative powers of q_0 give growing terms in the asymptotic limit. Considering only the inner sum without the prefactor, the $n = 0$ term is

$$\frac{1}{1-q_0^{-u}} = \frac{-q_0^u}{1-q_0^u} = -q_0^u - q_0^{2u} + \dots,$$

where we have used that $u \in (0, \frac{1}{2}]$ and $\tau \in \mathbb{H}$. It is clear that any term $n \geq 1$ will have terms of order $q_0^{\frac{5}{2}}$ or higher.

When $n < 0$ we have that $n - u < 0$ and hence the term

$$\begin{aligned} \frac{(-1)^n q_0^{\frac{3n^2+n}{2}}}{1-q_0^{n-u}} = \frac{(-1)^{n+1} q_0^{\frac{3n^2+n}{2}} q_0^{u-n}}{1-q_0^{u-n}} &= (-1)^{n+1} q_0^{\frac{3n^2+n}{2}} q_0^{u-n} \sum_{j\geq 0} q_0^{j(u-n)} \\ &= (-1)^{n+1} q_0^{\frac{3n^2-n+2u}{2}} \sum_{j\geq 0} q_0^{j(u-n)}, \end{aligned}$$

with the lowest order term $(-1)^{n+1} q_0^{\frac{3n^2-n+2u}{2}}$. We note that $\frac{3n^2-n+2u}{2} \geq 2 + u$.

We then see that, for $u \in (0, \frac{1}{2}]$, the most negative power of q_0 is given by the $n = 0$ term and is

$$-\frac{1}{\tau} q_0^{-\frac{1}{2}(3u^2+u)} e^{2\pi i u} q_0^u = -\frac{1}{\tau} q_0^{-\frac{1}{2}(3u^2-u)} e^{2\pi i u}. \tag{IV.5.3}$$

Note in particular that for $0 < u \leq \frac{1}{6}$ we have that $3u^2 - u < 0$ and so here we have a positive power of q_0, hence in this case (IV.5.3) tends to 0 in our asymptotic limit.

We now investigate the second-smallest power of q_0 giving a non-zero contribution to the asymptotic behaviour. This is given by the second term in the $n = 0$ expansion, and is

$$-\frac{1}{\tau} q_0^{-\frac{1}{2}(3u^2+u)} e^{2\pi i u} q_0^{2u} = -\frac{1}{\tau} q_0^{-\frac{3}{2}(u^2-u)} e^{2\pi i u}.$$

Since $u^2 - u = u(u-1) < 0$ the power of q_0 is positive and hence this term gives vanishing contribution to the asymptotic behaviour. In a similar way, all further terms give no contribution, since the power of q_0 increases as we take $|n|$ larger in (IV.5.2).

Now we look to find the contribution of the error of modularity terms S_2 to the asymptotic behaviour of A_3. First, we note that the smallest power of q_0 appearing in $\vartheta\left(\frac{k-1}{3}; -\frac{1}{3\tau}\right)$ is given by

$$-i q_0^{\frac{1}{24}} \left(e^{-\frac{\pi i (k-1)}{3}} - e^{\frac{\pi i (k-1)}{3}}\right). \tag{IV.5.4}$$

Using (IV.2.1) we find that

$$\begin{aligned} h(3u + (1-k)\tau; 3\tau) &= \frac{1}{\sqrt{-3i\tau}} e^{\frac{\pi i (3u+(1-k)\tau)^2}{3\tau}} h\left(\frac{u}{\tau} + \frac{1-k}{3}; -\frac{1}{3\tau}\right) \\ &= \frac{1}{\sqrt{-3i\tau}} e^{\frac{\pi i \tau (k-1)^2}{3} + \frac{3\pi i u^2}{\tau} + 2\pi i u(1-k)} h\left(\frac{u}{\tau} + \frac{1-k}{3}; -\frac{1}{3\tau}\right). \end{aligned}$$

Hence we have

$$S_2 = \frac{i}{6\tau} \sum_{k=0}^{2} e^{\pi i \left(2u + \frac{3u^2}{\tau}\right)} \vartheta\left(\frac{k-1}{3}; -\frac{1}{3\tau}\right) h\left(\frac{u}{\tau} + \frac{1-k}{3}; -\frac{1}{3\tau}\right). \tag{IV.5.5}$$

If $u \leq \frac{1}{6}$ we rewrite

$$h\left(\frac{u}{\tau} + \frac{1-k}{3}; -\frac{1}{3\tau}\right) = h\left(\frac{-3u}{-3\tau} + \frac{1-k}{3}; -\frac{1}{3\tau}\right).$$

Then writing $\tau = \frac{i\varepsilon}{2\pi}$ we see that Proposition IV.2.1 with $\kappa = 1$, $z = \frac{3\varepsilon}{2\pi}$, $\beta = -3u$, and $\alpha = \frac{1-k}{3}$ gives the bound as $\varepsilon \to 0^+$ of

$$\left| h\left(\frac{u}{\tau} + \frac{1-k}{3}; -\frac{1}{3\tau}\right) \right| \leq \begin{cases} |\sec(-3\pi u)| \left(\frac{2\pi}{3\varepsilon}\right)^{-\frac{1}{2}} e^{-\frac{6\pi^2 u^2}{\varepsilon} + \frac{(1-k)^2 \varepsilon}{6}} & \text{if } -3u \neq -\frac{1}{2}, \\ \left(1 + \left(\frac{2\pi}{3\varepsilon}\right)^{-\frac{1}{2}}\right) e^{-\frac{\pi^2}{6\varepsilon}} & \text{if } -3u = -\frac{1}{2}. \end{cases}$$

Combining the above we see that for $u < \frac{1}{6}$ the contribution of S_2 to the overall asymptotic behaviour is bounded in modulus by

$$\frac{2\pi |\sec(-3\pi u)|}{3\varepsilon} \sum_{k=0}^{2} e^{-\frac{\pi^2}{6\varepsilon}} \left(\frac{2\pi}{3\varepsilon}\right)^{-\frac{1}{2}} e^{\frac{(1-k)^2 \varepsilon}{6}}.$$

It is easy to see that as $\varepsilon \to 0^+$ this contribution vanishes. In a similar way, the contribution from S_2 to the overall asymptotics vanishes when $u = \frac{1}{6}$.

We now consider $u > \frac{1}{6}$. In order to apply Proposition IV.2.1 we need to shift the function h. Using (IV.2.2) gives

$$\begin{aligned}
&h(\frac{-3u}{-3\tau}+\frac{1-k}{3};-\frac{1}{3\tau})\\
&\qquad = h\left(\frac{-3u+1}{-3\tau}+\frac{1}{3\tau}+\frac{1-k}{3};-\frac{1}{3\tau}\right)\\
&\qquad = -e^{-2\pi i\left(\frac{u}{\tau}+\frac{1-k}{3}\right)+\frac{\pi i}{3\tau}} h\left(\frac{1-3u}{-3\tau}+\frac{1-k}{3};-\frac{1}{3\tau}\right)+2e^{-\pi i\left(\frac{u}{\tau}+\frac{1-k}{3}\right)+\frac{\pi i}{12\tau}}\\
&\qquad = -e^{-2\pi i\left(\frac{u}{\tau}+\frac{1-k}{3}\right)+\frac{\pi i}{3\tau}} h\left(\frac{1-3u}{-3\tau}+\frac{1-k}{3};-\frac{1}{3\tau}\right)+2e^{\frac{\pi i(k-1)}{3}} q_0^{\frac{u}{2}-\frac{1}{24}}.
\end{aligned}$$

Then we write $S_2 = S_{2,1} + S_{2,2}$, where

$$S_{2,1} := \frac{-i}{6\tau}\sum_{k=0}^{2} e^{\pi i\left(2u+\frac{3u^2}{\tau}\right)} e^{-2\pi i\left(\frac{u}{\tau}+\frac{1-k}{3}\right)+\frac{\pi i}{3\tau}} \vartheta\left(\frac{k-1}{3};-\frac{1}{3\tau}\right) h\left(\frac{1-3u}{-3\tau}+\frac{1-k}{3};-\frac{1}{3\tau}\right)$$

and

$$\begin{aligned}
S_{2,2} :=& \frac{i}{3\tau}\sum_{k=0}^{2} e^{\pi i\left(2u+\frac{3u^2}{\tau}\right)} \vartheta\left(\frac{k-1}{3};-\frac{1}{3\tau}\right) e^{\frac{\pi i(k-1)}{3}} q_0^{\frac{u}{2}-\frac{1}{24}}\\
=& \frac{i}{3\tau}\sum_{k=0}^{2} e^{2\pi i u} q_0^{-\frac{3u^2}{2}+\frac{u}{2}-\frac{1}{24}} \vartheta\left(\frac{k-1}{3};-\frac{1}{3\tau}\right) e^{\frac{\pi i(k-1)}{3}}.
\end{aligned}$$

We concentrate firstly on $S_{2,1}$. Recalling that $u \leq \frac{1}{2}$ and using Proposition IV.2.1 with $\kappa = 1$, $z = \frac{3\varepsilon}{2\pi}$, $\beta = 1-3u$, and $\alpha = \frac{1-k}{3}$ gives the bound

$$\left| h\left(\frac{1-3u}{-3\tau}+\frac{1-k}{3};-\frac{1}{3\tau}\right)\right| \leq |\sec(\pi(1-3u))| \left(\frac{2\pi}{3\varepsilon}\right)^{-\frac{1}{2}} e^{-\frac{2\pi^2(3u-1)^2}{3\varepsilon}+\frac{(1-k)^2\varepsilon}{6}}.$$

Then we see that the contribution of $S_{2,1}$ to the overall asymptotic behaviour is bounded in modulus by

$$\frac{2\pi|\sec(\pi(1-3u))|}{3\varepsilon}\sum_{k=0}^{2} e^{\frac{-\pi^2}{6\varepsilon}} \left(\frac{2\pi}{3\varepsilon}\right)^{-\frac{1}{2}} e^{\frac{(1-k)^2\varepsilon}{6}}.$$

It is easy to see that as $\varepsilon \to 0^+$ this contribution vanishes. We are left to consider the contribution of $S_{2,2}$. Using the behaviour of ϑ given in (IV.5.4) the lowest power of q_0

arising from this sum is

$$\frac{1}{3\tau}\sum_{k=0}^{2} e^{2\pi i u} q_0^{-\frac{3u^2}{2}+\frac{u}{2}-\frac{1}{24}} q_0^{\frac{1}{24}} \left(1-e^{\frac{2\pi i(k-1)}{3}}\right) = \frac{1}{\tau} e^{2\pi i u} q_0^{-\frac{1}{2}(3u^2-u)},$$

exactly canceling the contribution from the first term of the Appell function given in (IV.5.3). So, when $u > \frac{1}{6}$ we must investigate the second-largest non-zero term of both the Appell function and $S_{2,2}$, since all terms in $S_{2,1}$ are exponentially suppressed in the limit.

It is easily seen from the definition of ϑ that the power of q_0 in $\vartheta\left(\frac{k-1}{3}; -\frac{1}{3\tau}\right)$ is greater than or equal to $\frac{1}{24}+\frac{1}{3}$ for other terms. Then the power of q_0 in $S_{2,2}$ is seen to be positive, since $-\frac{1}{2}(3u^2-u) \geq -\frac{1}{8}$ for $\frac{1}{6} < u \leq \frac{1}{2}$. Hence these terms give no contribution in the limiting situation. Further, we have already seen that there are no other non-vanishing contributions from (IV.5.2). The claimed result now follows. $\square$

IV.6 Proof of Theorem IV.1.1

In this section we prove the following.

Theorem IV.1.1. *For fixed $0 \leq r < t$ and $t \geq 2$ we have that*

$$N(r,t;n) \sim \frac{1}{t}p(n) \sim \frac{1}{4tn\sqrt{3}} e^{2\pi\sqrt{\frac{n}{6}}}$$

as $n \to \infty$. Hence for fixed t the number of partitions of rank congruent to r modulo t is equidistributed in the limit.

Proof. From Theorem IV.4.1 we know that the power series

$$\sum_{n\geq M} N(r,t;n)q^n$$

has weakly increasing coefficients. We are therefore in the situation where we may apply Theorem IV.2.2, and so we investigate the asymptotic behaviour

$$\lim_{\varepsilon\to 0^+} \sum_{n\geq 1} N(r,t;n)e^{-\varepsilon n}.$$

Using (IV.1.1) and the fact that $R(\zeta;q) = R(\zeta^{-1};q)$ we have that

$$\sum_{n\geq 0} N(r,t;n)q^n = \frac{1}{t}\left[\sum_{n=0}^{\infty} p(n)q^n + \sum_{j=1}^{\lfloor\frac{t-1}{2}\rfloor}\left(\zeta_t^{rj}+\zeta_t^{-rj}\right)R(\zeta_t^j;q) + \delta_t(-1)^r R(-1;q)\right],$$

where $\delta_t := 1$ if t is even, and 0 otherwise. We next note that it is possible to rewrite

$$R(\zeta;q) = \frac{(1-\zeta)}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-\zeta q^n} = (1-\zeta)\phi(\tau)^{-1} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}} q^{-n}}{1-\zeta q^n}$$
$$= \left(\zeta^{-\frac{3}{2}} - \zeta^{-\frac{1}{2}}\right) \frac{1}{\phi(\tau)} A_3(z,-\tau;\tau),$$

where $\phi(\tau) := \prod_{n\geq 1}(1-q^n)$.

Considering generating functions we therefore want to investigate the behaviour of

$$\frac{1}{t\phi(\tau)}\left[1 + \sum_{j=1}^{\lfloor\frac{t-1}{2}\rfloor} \left(\zeta_t^{rj} + \zeta_t^{-rj}\right)\left(\zeta_{2t}^{-3j} - \zeta_{2t}^{-j}\right) A_3\left(\frac{j}{t}, -\tau;\tau\right) + 2i\delta_t(-1)^r A_3\left(\frac{1}{2}, -\tau;\tau\right)\right]. \tag{IV.6.1}$$

Let $\tau = \frac{i\varepsilon}{2\pi}$ and consider $\varepsilon \to 0^+$. We use Theorem IV.5.1 with $u = \frac{j}{t}$ and see that the term in square brackets is asymptotically equal to 1 in this limit. Hence we have that (IV.6.1) behaves as

$$\frac{1}{t\phi\left(\frac{i\varepsilon}{2\pi}\right)} \sim \frac{1}{t\sqrt{2\pi}}\varepsilon^{\frac{1}{2}} e^{\frac{\pi^2}{6\varepsilon}}.$$

Then using Theorem IV.2.2 we see that as $n \to \infty$

$$N(r,t;n) \sim \frac{1}{t}p(n) \sim \frac{1}{4tn\sqrt{3}} e^{2\pi\sqrt{\frac{n}{6}}}.$$

The claim now follows. $\square$

IV.7 Proof of Theorem IV.1.3

As a simple application of Theorem IV.1.1 we prove the following theorem.

Theorem IV.1.3. *Conjecture IV.1.2 is true.*

Proof. Consider the ratio

$$\frac{N(r,t;a)N(r,t;b)}{N(r,t;a+b)}$$

as $a, b \to \infty$. By Theorem IV.1.1 we have

$$\frac{N(r,t;a)N(r,t;b)}{N(r,t;a+b)} \sim \frac{\frac{1}{4ta\sqrt{3}} e^{2\pi\sqrt{\frac{a}{6}}} \frac{1}{4tb\sqrt{3}} e^{2\pi\sqrt{\frac{b}{6}}}}{\frac{1}{4t(a+b)\sqrt{3}} e^{2\pi\sqrt{\frac{(a+b)}{6}}}} = \frac{(4ta\sqrt{3} + 4tb\sqrt{3})}{48t^2ab} \frac{e^{2\pi\sqrt{\frac{a}{6}}\sqrt{\frac{b}{6}}}}{e^{2\pi\sqrt{\frac{(a+b)}{6}}}} > 1$$

as $a, b \to \infty$. $\square$

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Chapter V

Cycle integrals of meromorphic modular forms and coefficients of harmonic Maass forms

This chapter is based on a paper [ABMS], accepted for publication in *Journal of Mathematical Analysis and Applications*. This is joint work with Prof. Dr. Claudia Alfes-Neumann, Prof. Dr. Kathrin Bringmann, and Dr. Markus Schwagenscheidt.

V.1 Introduction and statement of results

A classical result of Kohnen and Zagier [14] asserts that certain simple linear combinations of geodesic cycle integrals of the weight $2k$ cusp forms[1]

$$f_{k,\mathcal{B}}(z) := \frac{D^{\frac{k+1}{2}}}{\pi} \sum_{Q \in \mathcal{B}} Q(z,1)^{-k}$$

are rational. Here $k \in \mathbb{N}_{\geq 2}$ is even and $\mathcal{B}$ denotes an equivalence class of indefinite integral binary quadratic forms of discriminant $D > 0$. On the other hand, if $\mathcal{A}$ is an equivalence class of positive definite quadratic forms of discriminant $d < 0$, then the functions

$$f_{k,\mathcal{A}}(z) := \frac{|d|^{\frac{k+1}{2}}}{\pi} \sum_{Q \in \mathcal{A}} Q(z,1)^{-k}, \tag{V.1.1}$$

are meromorphic modular forms of weight $2k$ for $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ which decay like cusp forms towards $i\infty$. Inspired by the results of Kohnen and Zagier, three of the authors showed in [2] that certain linear combinations of traces of cycle integrals of the meromorphic

[1]Kohnen and Zagier used a slightly different normalisation to the present paper.

modular forms $f_{k,\mathcal{A}}$

$$\operatorname{tr}_{f_{k,\mathcal{A}}}(D) \coloneqq \sum_{Q\in\mathcal{Q}_D/\Gamma} \int_{c_Q} f_{k,\mathcal{A}}(z)Q(z,1)^{k-1}dz$$

are rational. Here $\mathcal{Q}_D$ denotes the set of integral binary quadratic forms of non-square discriminant $D>0$, and $c_Q \coloneqq \Gamma_Q\backslash C_Q$ (Γ_Q the stabilizer of Q in Γ) is the image in $\Gamma\backslash\mathbb{H}$ of the geodesic $C_Q \coloneqq \left\{z = x+iy \in \mathbb{H} : a|z|^2+bx+c=0\right\}$ associated to $Q=[a,b,c]\in\mathcal{Q}_D$. Note that the cycle integrals have to be defined using the Cauchy principal value as explained in [2] if a pole of $f_{k,\mathcal{A}}$ lies on a geodesic c_Q for $Q\in\mathcal{Q}_D$.

In the present paper, we relate the traces of cycle integrals of the meromorphic modular forms $f_{k,\mathcal{A}}$ to Fourier coefficients of so-called harmonic Maass forms. Below we state our main results in terms of vector-valued harmonic Maass forms for the Weil representation associated with an even lattice. In the introduction, we however restrict to the lattice of signature $(1,2)$

$$L \coloneqq \left\{X = \begin{pmatrix} -b & -c \\ a & b\end{pmatrix} : a,b,c\in\mathbb{Z}\right\},$$

equipped with the quadratic form $q(X)\coloneqq \det(X)$. The significance of the lattice L lies in the fact that its dual lattice L' can be identified with the set of all integral binary quadratic forms, with $-4q(X)$ corresponding to the discriminant. We let $\mathbb{C}[L'/L]$ be the group ring of the discriminant form L'/L, and we denote by $\mathcal{D}$ the Grassmannian of positive definite lines in $L\otimes\mathbb{R}$, which can be identified with the complex upper half-plane $\mathbb{H}$ by sending $z\in\mathbb{H}$ to the positive line generated by $\left(\begin{smallmatrix} -x & x^2+y^2 \\ -1 & x\end{smallmatrix}\right)$.

Let $\mathcal{A}$ be a fixed class of positive definite integral binary quadratic forms of discriminant $d<0$. We let $z_{\mathcal{A}}\in\mathbb{H}$ denote the CM point associated to the unique reduced form $Q_0\in\mathcal{A}$, which means that $z_{\mathcal{A}}$ is the unique solution of $Q_0(z_{\mathcal{A}},1)=0$ in $\mathbb{H}$. For simplicity, we denote the corresponding positive line in $\mathcal{D}$ by the same symbol $z_{\mathcal{A}}$, and we let $z_{\mathcal{A}}^{\perp}$ denote its orthogonal complement in $L\otimes\mathbb{R}$. Since $z_{\mathcal{A}}$ is a CM point, the corresponding positive line in $\mathcal{D}$ and its orthogonal complement are defined over $\mathbb{Q}$, and we may define two sublattices of L by

$$P\coloneqq L\cap z_{\mathcal{A}}, \qquad N\coloneqq L\cap z_{\mathcal{A}}^{\perp},$$

which are one-dimensional positive definite and two-dimensional negative definite sublattices, respectively. Note that $P\oplus N$ has finite index in L.

The usual vector-valued theta function Θ_P associated to P is a holomorphic modular form of weight $\frac{1}{2}$ for the Weil representation of P. We denote by $\mathcal{G}_P^+$ the holomorphic part of a harmonic Maass form $\mathcal{G}_P$ of weight $\frac{3}{2}$ for the dual Weil representation of P that maps to Θ_P under $\xi_{\frac{3}{2}}$, where $\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}}$ with $\tau = u + iv \in \mathbb{H}$. Furthermore, for $k \in 2\mathbb{N}$ we let

$$f(\tau) = \sum_{n \gg -\infty} c_f(n) e(n\tau)$$

(with $e(w) := e^{2\pi i w}$ for $w \in \mathbb{C}$) be a fixed weakly holomorphic modular form of weight $\frac{3}{2} - k$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition $c_f(n) = 0$ for $n \equiv 1, 2 \pmod 4$. We also assume that $c_f(-D) = 0$ if $D > 0$ is a square. Let $\mathfrak{e}_0, \mathfrak{e}_1$ denote the standard basis of $\mathbb{C}[L'/L] \cong \mathbb{Z}/2\mathbb{Z}$. By the results of [12, Section 5], vector-valued modular forms for the Weil representation of L can be identified with scalar-valued modular forms (of the same weight) satisfying the Kohnen plus space condition by the map

$$g_0(\tau)\,\mathfrak{e}_0 + g_1(\tau)\,\mathfrak{e}_1 \mapsto g_0(4\tau) + g_1(4\tau). \tag{V.1.2}$$

For simplicity, we use the same symbol for a scalar-valued weakly holomorphic modular form and its vector-valued version. Finally, since $P \oplus N$ has finite index in L, one can naturally view (the vector-valued version of) f as a weakly holomorphic modular form for the Weil representation of $P \oplus N$, which we denote by $f_{P \oplus N}$. We refer the reader to Section V.2.4 for the precise definition of $f_{P \oplus N}$.

The following formula is the main result of this paper; the general result for arbitrary congruence subgroups and both even and odd k can be found in Theorem V.5.1.

Theorem V.1.1. *Let $k \in 2\mathbb{N}$ and assume that $z_{\mathcal{A}}$ does not lie on any of the geodesics c_Q for $Q \in \mathcal{Q}_D$ if $c_f(-D) \neq 0$. Then we have that*

$$\sum_{D > 0} c_f(-D) \operatorname{tr}_{f_{k,\mathcal{A}}}(D) = \frac{2^{k-3} |d|^{\frac{1}{2}}}{\pi \left| \overline{\Gamma}_{z_{\mathcal{A}}} \right|} \operatorname{CT}\left(\left\langle f_{P \oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau) \right]_{\frac{k}{2} - 1} \right\rangle \right),$$

where $\overline{\Gamma}_{z_{\mathcal{A}}}$ is the stabilizer of $z_{\mathcal{A}}$ in $\overline{\Gamma} = \Gamma/\{\pm 1\}$, CT denotes the constant term in a Fourier expansion, $\langle \cdot, \cdot \rangle$ is the natural bilinear form on the group ring of $P \oplus N$, $[\cdot, \cdot]_n$ denotes the n-th Rankin–Cohen bracket, and Θ_{N^-} is the holomorphic theta function associated to the positive definite lattice $N^- := (N, -q)$.

We remark that by [2, Theorem 1.1] the left-hand side of Theorem V.1.1 is rational if the coefficients of f are rational. By [10, Theorem 4.3], one can choose $\mathcal{G}_P$ such that the coefficients of its holomorphic part $\mathcal{G}_P^+$ lie in $\pi |d|^{-\frac{1}{2}} \mathbb{Q}$. In particular, combining Theorem V.1.1 and [10, Theorem 4.3] we obtain a new proof for the rationality of the linear combinations of cycle integrals of the meromorphic modular forms $f_{k,\mathcal{A}}$ in Theorem V.1.1. Moreover, by comparing [2, Theorem 1.2] with Theorem V.1.1 above one can obtain interesting identities between two finite sums involving coefficients of harmonic Maass forms.

Example V.1.2. *As an illustrating example of Theorem V.1.1, we consider the class $\mathcal{A}$ of the quadratic form $[1,0,1]$ of discriminant -4, with the associated CM point $z_{\mathcal{A}} = i$. The corresponding positive line in $\mathcal{D}$ is spanned by the vector $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The lattice P is also spanned by this vector, and is therefore isomorphic to $(\mathbb{Z}, n^2)$. The lattice N consists of those $X \in L$ with $a = -c$, and hence is isomorphic to $(\mathbb{Z}^2, -n^2 - m^2)$. Note that $P'/P \cong \mathbb{Z}/2\mathbb{Z}$ and $N'/N \cong (\mathbb{Z}/2\mathbb{Z})^2$. The vector-valued theta functions associated to P and N^- are given by*

$$\Theta_P(\tau) = \sum_{n\in\mathbb{Z}} e\left(\frac{n^2\tau}{4}\right)\mathfrak{e}_n, \qquad \Theta_{N^-}(\tau) = \sum_{m,n\in\mathbb{Z}} e\left(\frac{(m^2+n^2)\tau}{4}\right)\mathfrak{e}_{(m,n)}.$$

Note that Θ_P can be identified with the Jacobi theta function $\theta(\tau) \coloneqq \sum_{n\in\mathbb{Z}} e(n^2\tau)$ under the map defined in (V.1.2)*. It follows from work of Zagier* [16] *that the $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$-valued generating function*

$$\mathcal{G}_P^+(\tau) := -8\pi \sum_{n\geq 0} H(n) e\left(\frac{n\tau}{4}\right)\mathfrak{e}_n \tag{V.1.3}$$

of Hurwitz class numbers $H(n)$ (with $H(0) \coloneqq -\frac{1}{12}$) is the holomorphic part of a harmonic Maass form of weight $\frac{3}{2}$ for the dual Weil representation of P which maps to Θ_P under $\xi_{\frac{3}{2}}$.

Now we choose $k = 2$. In this case, for every non-square discriminant $D > 0$ there exists a unique weakly holomorphic modular form g_D of weight $-\frac{1}{2}$ satisfying the Kohnen plus space condition and having a Fourier expansion of the form $g_D(\tau) = e(-D\tau) + O(1)$. Using the Cohen-Eisenstein series of weight $\frac{5}{2}$, one can show that the constant term of g_D is given by $-120L_D(-1)$, where $L_D(s)$ denotes the usual L-function associated to a non-square discriminant $D > 0$. We denote the vector-valued modular form corresponding to g_D via (V.1.2) *by the same symbol g_D. Now choosing $f = g_D$ and $\mathcal{G}_P^+$ as in* (V.1.3)*, Theorem V.1.1 yields the formula*

$$\mathrm{tr}_{f_{2,[1,0,1]}}(D) = -40L_D(-1) - 4 \sum_{\substack{n,m\in\mathbb{Z} \\ n\equiv D \pmod 2}} H\left(D - n^2 - m^2\right)$$

for any non-square discriminant $D > 0$ if i does not lie on any of the geodesics c_Q for $Q \in \mathcal{Q}_D$. Similarly, by computing the Rankin–Cohen bracket, for $k = 4$ we obtain

$$\mathrm{tr}_{f_{4,[1,0,1]}}(D) = \sum_{\substack{n,m\in\mathbb{Z} \\ n\equiv D \pmod 2}} \left(4D - 10n^2 - 10m^2\right) H\left(D - n^2 - m^2\right).$$

The proof of Theorem V.1.1 consists of three main steps. For the first one, we use the fact that $\mathrm{tr}_{f_{k,\mathcal{A}}}(D)$ can be written as a special value of the iterated raising operator applied to a locally harmonic Maass form $\mathcal{F}_{1-k,D}$, which was first introduced by Kane, Kohnen, and one of the authors [4] and whose precise definition in the vector-valued setup is recalled in Section V.3. Namely, [15, Corollary 4.3] implies that

$$\mathrm{tr}_{f_{k,\mathcal{A}}}(D) \doteq D^{k-\frac{1}{2}} R_{2-2k}^{k-1}(\mathcal{F}_{1-k,D})(z_{\mathcal{A}}),$$

where $R_\kappa^n := R_{\kappa+2n-2} \circ \cdots \circ R_\kappa$ with $R_\kappa^0 := \mathrm{id}$ is an iterated version of the *Maass raising operator* $R_\kappa := 2i\frac{\partial}{\partial \tau} + \frac{\kappa}{v}$, and the symbol $\doteq$ means equality up to a non-zero multiplicative constant.

In the second step, we write the function $R_{2-2k}^{k-1}(\mathcal{F}_{1-k,D})$ as a regularized theta lift, following Borcherds [3]. Namely, in Theorem V.3.2 we show that

$$\sum_{D>0} c_f(-D) D^{k-\frac{1}{2}} R_{2-2k}^{k-1}(\mathcal{F}_{1-k,D})(z) \doteq \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{\frac{3}{2}-k}^{\frac{k}{2}-1}(f)(\tau), \overline{\Theta_L(\tau,z)} \right\rangle v^{-\frac{1}{2}} \frac{du dv}{v^2},$$

where the integral is taken over the standard fundamental domain $\mathcal{F}$ of Γ and has to be regularized as explained in Section V.3, and $\Theta_L(\tau, z)$ denotes the Siegel theta function associated to L.

Finally, in the third step, we use the fact that the evaluation of the Siegel theta function $\Theta_L(\tau, z_{\mathcal{A}})$ at the CM point $z_{\mathcal{A}}$ essentially splits as a tensor product of the holomorphic theta functions Θ_P and Θ_{N^-} associated to the lattices P and N^-. Then using Stokes' Theorem, the regularized theta integral can be evaluated as

$$\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{\frac{3}{2}-k}^{\frac{k}{2}-1}(f)(\tau), \overline{\Theta_L(\tau,z_{\mathcal{A}})} \right\rangle v^{-\frac{1}{2}} \frac{du dv}{v^2} \doteq \mathrm{CT}\left(\left\langle f(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)\right]_{\frac{k}{2}-1} \right\rangle\right),$$

see Theorem V.4.1 below. Our strategy to prove the last formula closely follows methods from recent work of Bruinier, Ehlen, and Yang [8]. Combining these three steps gives Theorem V.1.1.

The paper is organized as follows. We begin in Section V.2 by recalling preliminaries which are pertinent to the rest of the paper. Section V.3 is dedicated to the study of the regularized theta lift alluded to above. The evaluation of the theta lift at CM points is discussed in Section V.4. Finally, in Section V.5 we give the proof of Theorem V.1.1 and its generalisation to higher level and arbitrary weight.

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V.2 Preliminaries

V.2.1 The Weil representation

The *metaplectic extension of* $\mathrm{SL}_2(\mathbb{Z})$ is defined as

$$\begin{aligned}\widetilde{\Gamma} &\coloneqq \mathrm{Mp}_2(\mathbb{Z})\\ &\coloneqq \left\{(\gamma,\phi)\colon \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \phi\colon \mathbb{H}\to\mathbb{C} \text{ holomorphic}, \phi^2(\tau) = c\tau+d\right\}.\end{aligned}$$

It is generated by $\widetilde{T} := \left(\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right), 1\right)$ and $\widetilde{S} := \left(\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right), \sqrt{\tau}\right)$. We let $\widetilde{\Gamma}_\infty$ denote the subgroup generated by $\widetilde{T}$.

Let L be an even lattice of signature (r,s) with quadratic form q and associated bilinear form $(\cdot,\cdot)$. Let L' denote its dual lattice, $\mathbb{C}[L'/L]$ be the group ring of L'/L with standard basis elements $\mathfrak{e}_\mu$ for $\mu\in L'/L$, and $\langle\cdot,\cdot\rangle$ be the natural bilinear form on $\mathbb{C}[L'/L]$ given by $\langle \mathfrak{e}_\mu, \mathfrak{e}_\nu\rangle = \delta_{\mu,\nu}$. The *Weil representation* ρ_L associated with L is the representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L\left(\widetilde{T}\right)(\mathfrak{e}_\mu) \coloneqq e(q(\mu))\,\mathfrak{e}_\mu, \qquad \rho_L\left(\widetilde{S}\right)(\mathfrak{e}_\mu) \coloneqq \frac{e\left(\frac{1}{8}(s-r)\right)}{\sqrt{|L'/L|}}\sum_{\nu\in L'/L} e(-(\nu,\mu))\,\mathfrak{e}_\nu\,.$$

The Weil representation ρ_{L^-} associated to the lattice $L^- = (L,-q)$ is called the *dual Weil representation* associated to L.

V.2.2 Harmonic Maass forms

Let $\kappa\in\frac{1}{2}\mathbb{Z}$ and define the *slash-operator* by

$$f\mid_{\kappa,\rho_L}(\gamma,\phi)(\tau) \coloneqq \phi(\tau)^{-2\kappa}\rho_L^{-1}(\gamma,\phi)f(\gamma\tau),$$

for a function $f\colon\mathbb{H}\to\mathbb{C}[L'/L]$ and $(\gamma,\phi)\in\widetilde{\Gamma}$. Following [9], we call a smooth function $f\colon\mathbb{H}\to\mathbb{C}[L'/L]$ a *harmonic Maass form* of weight κ with respect to ρ_L if it is annihilated by the *weight* κ *Laplace operator*

$$\Delta_\kappa \coloneqq -v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) + i\kappa v\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right),$$

if it is invariant under the slash-operator $|_{\kappa,\rho_L}$, and if there exists a $\mathbb{C}[L'/L]$-valued Fourier polynomial (the *principal part* of f)

$$P_f(\tau) \coloneqq \sum_{\mu\in L'/L}\sum_{n\leq 0} c_f^+(\mu,n)e(n\tau)\,\mathfrak{e}_\mu$$

such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. We denote the vector space of harmonic Maass forms of weight κ with respect to ρ_L by $H_{\kappa,L}$, and we let $M^!_{\kappa,L}$ be the subspace of weakly holomorphic modular forms. Every $f \in H_{\kappa,L}$ can be written as a sum $f = f^+ + f^-$ of a *holomorphic* and a *non-holomorphic part*, having Fourier expansions of the form

$$f^+(\tau) = \sum_{\mu \in L'/L} \sum_{n \gg -\infty} c_f^+(\mu, n) e(n\tau)\, \mathfrak{e}_\mu,$$
$$f^-(\tau) = \sum_{\mu \in L'/L} \sum_{n<0} c_f^-(\mu, n) \Gamma(1-\kappa, 4\pi|n|v) e(n\tau)\, \mathfrak{e}_\mu,$$

where $\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt$ denotes the *incomplete Gamma function.*

The antilinear differential operator $\xi_\kappa = 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}}$ from the introduction maps a harmonic Maass form $f \in H_{\kappa,L}$ to a cusp form of weight $2-\kappa$ for ρ_{L^-}. We further require the *lowering* and *raising operators* $L_\kappa := -2iv^2 \frac{\partial}{\partial \overline{\tau}}$ and $R_\kappa = 2i\frac{\partial}{\partial \tau} + \frac{\kappa}{v}$, which lower and raise the weight of a smooth function transforming like a modular form of weight κ for ρ_L by two.

V.2.3 Maass–Poincaré series

Let $\kappa \in \frac{1}{2}\mathbb{Z}$ with $\kappa < 0$, and denote by $M_{\mu,\nu}$ the usual M-Whittaker function (see [1, equation 13.1.32]). We define, for $s \in \mathbb{C}$ and $y \in \mathbb{R}\backslash\{0\}$,

$$\mathcal{M}_{\kappa,s}(y) := |y|^{-\frac{\kappa}{2}} M_{\operatorname{sgn}(y)\frac{\kappa}{2}, s-\frac{1}{2}}(|y|). \tag{V.2.1}$$

Following [7], for $\mu \in L'/L$ and $m \in \mathbb{Z} - q(\mu)$ with $m > 0$ we define the vector-valued Maass–Poincaré series

$$F_{\mu,-m,\kappa,s}(\tau) := \frac{1}{2\Gamma(2s)} \sum_{(\gamma,\phi) \in \widetilde{\Gamma}_\infty \backslash \widetilde{\Gamma}} (\mathcal{M}_{\kappa,s}(-4\pi m v) e(-mu)\, \mathfrak{e}_\mu) \mid_{\kappa,\rho_L} (\gamma,\phi)(\tau).$$

The series converges absolutely for $\mathrm{Re}(s) > 1$, and at the special point $s = 1 - \frac{\kappa}{2}$, the function

$$F_{\mu,-m,\kappa}(\tau) := F_{\mu,-m,\kappa,1-\frac{\kappa}{2}}(\tau)$$

defines a harmonic Maass form in $H_{\kappa,L}$ with principal part $e(m\tau)(\mathfrak{e}_\mu + \mathfrak{e}_{-\mu}) + \mathfrak{c}$ for some constant $\mathfrak{c} \in \mathbb{C}[L'/L]$. In particular, every harmonic Maass form $f \in H_{\kappa,L}$ can be written as a linear combination

$$f(\tau) = \frac{1}{2} \sum_{\mu \in L'/L} \sum_{m>0} c_f^+(\mu, -m) F_{\mu,-m,\kappa}(\tau). \tag{V.2.2}$$

The following lemma follows inductively from [8, Proposition 3.4].

Lemma V.2.1. *For $n \in \mathbb{N}_0$ we have that*

$$R_\kappa^n \left(F_{\mu,-m,\kappa,s}\right)(\tau) = (4\pi m)^n \frac{\Gamma\left(s+n+\frac{\kappa}{2}\right)}{\Gamma\left(s+\frac{\kappa}{2}\right)} F_{\mu,-m,\kappa+2n,s}(\tau).$$

V.2.4 Operators on vector-valued modular forms

For an even lattice L we let $A_{\kappa,L}$ be the space of $\mathbb{C}[L'/L]$-valued smooth modular forms (i.e., modular forms which possess derivatives of all orders) of weight κ with respect to the representation ρ_L.

Let $K \subset L$ be a sublattice of finite index. Since we have the inclusions $K \subset L \subset L' \subset K'$ we therefore have $L/K \subset L'/K \subset K'/K$, hence the natural map $L'/K \to L'/L, \mu \mapsto \bar{\mu}$. For $\mu \in K'/K$ and $f \in A_{\kappa,L}$, and $g \in A_{\kappa,K}$, define

$$(f_K)_\mu := \begin{cases} f_{\bar{\mu}} & \text{if } \mu \in L'/K, \\ 0 & \text{if } \mu \notin L'/K, \end{cases} \qquad \left(g^L\right)_{\bar{\mu}} = \sum_{\alpha \in L/K} g_{\alpha+\mu},$$

where μ is a fixed preimage of $\bar{\mu}$ in L'/K. For the proof of the following lemma we refer the reader to [11, Section 3].

Lemma V.2.2. *There are two natural maps*

$$\mathrm{res}_{L/K}\colon A_{\kappa,L} \to A_{\kappa,K}, \quad f \mapsto f_K, \qquad\qquad \mathrm{tr}_{L/K}\colon A_{\kappa,K} \to A_{\kappa,L}, \quad g \mapsto g^L,$$

such that for any $f \in A_{\kappa,L}$ and $g \in A_{\kappa,K}$, we have $\langle f, \overline{g}^L \rangle = \langle f_K, \overline{g} \rangle$.

V.2.5 Rankin–Cohen brackets

Let K and L be even lattices. For $n \in \mathbb{N}_0$ and functions $f \in A_{\kappa,K}$ and $g \in A_{\ell,L}$ with $\kappa, \ell \in \frac{1}{2}\mathbb{Z}$ we define the n-th *Rankin–Cohen bracket*

$$[f,g]_n := \frac{1}{(2\pi i)^n} \sum_{\substack{r,s \geq 0 \\ r+s=n}} (-1)^r \frac{\Gamma(\kappa+n)\Gamma(\ell+n)}{\Gamma(s+1)\Gamma(\kappa+n-s)\Gamma(r+1)\Gamma(\ell+n-r)} f^{(r)} \otimes g^{(s)},$$

where the tensor product of two vector-valued functions $f = \sum_\mu f_\mu \mathfrak{e}_\mu \in A_{\kappa,K}$ and $g = \sum_\nu g_\nu \mathfrak{e}_\nu \in A_{\ell,L}$ is defined by

$$f \otimes g := \sum_{\mu,\nu} f_\mu g_\nu \, \mathfrak{e}_{\mu+\nu} \in A_{\kappa+\ell, K \oplus L}.$$

The proof of the following formula can be found in [8, Proposition 3.6].

Proposition V.2.3. *Let $f \in H_{\kappa,K}$ and $g \in H_{\ell,L}$ be harmonic Maass forms. For $n \in \mathbb{N}_0$ we have*

$$(-4\pi)^n L_{\kappa+\ell+2n}\left([f,g]_n\right) = \frac{\Gamma(\kappa+n)}{\Gamma(n+1)\Gamma(\kappa)} L_\kappa(f) \otimes R_\ell^n(g) + (-1)^n \frac{\Gamma(\ell+n)}{\Gamma(n+1)\Gamma(\ell)} R_\kappa^n(f) \otimes L_\ell(g).$$

V.2.6 A quadratic space of signature $(1,2)$

For $M \in \mathbb{N}$ we consider the rational quadratic space

$$V := \left\{ X = \begin{pmatrix} -\frac{b}{2M} & -\frac{c}{M} \\ a & \frac{b}{2M} \end{pmatrix} : a,b,c \in \mathbb{Q} \right\}$$

along with the quadratic form $q(X) := M\det(X)$ and the corresponding bilinear form $(X,Y) := -M\operatorname{tr}(XY)$ for $X, Y \in V$; it has signature $(1,2)$. Furthermore, its elements can be identified with rational quadratic forms $Q_X = [aM, b, c]$, where the discriminant of Q_X corresponds to $-4Mq(X)$. The group $\mathrm{SL}_2(\mathbb{Q})$ acts as isometries on V via $gX := gXg^{-1}$. Let $\mathcal{D}$ be the Grassmannian of lines in $V \otimes \mathbb{R}$ on which q is positive definite. We may identify $\mathcal{D}$ with the upper half-plane $\mathbb{H}$ by associating to $z \in \mathbb{H}$ the positive line generated by

$$X_1(z) := \frac{1}{\sqrt{2My}} \begin{pmatrix} -x & x^2+y^2 \\ -1 & x \end{pmatrix}.$$

Then $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations, and the identification is $\mathrm{SL}_2(\mathbb{R})$-invariant, i.e., $gX_1(z) = X_1(gz)$. Furthermore, define

$$X_2(z) := \frac{1}{\sqrt{2My}} \begin{pmatrix} x & -x^2+y^2 \\ 1 & -x \end{pmatrix}, \qquad X_3(z) := \frac{1}{\sqrt{2My}} \begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix}.$$

Along with $X_1(z)$, these form an orthogonal basis of $V \otimes \mathbb{R}$. For $X \in V$ and $z \in \mathbb{H}$ we define the quantities

$$p_X(z) := -\sqrt{2M}(X, X_1(z)) = \frac{1}{y}\left(aM|z|^2 + bx + c\right),$$
$$Q_X(z) := \sqrt{2M}y(X, X_2(z) + iX_3(z)) = aMz^2 + bz + c.$$

We let X_z and $X_{z^\perp}$ denote the orthogonal projections of X to the line $\mathbb{R}X_1(z)$ and its orthogonal complement, respectively. We have the useful formulas

$$q(X_z) = \frac{1}{4M} p_X(z)^2, \qquad q(X_{z^\perp}) = -\frac{1}{4My^2}|Q_X(z)|^2. \tag{V.2.3}$$

V.2.7 Theta functions

For a positive definite lattice (K, q) of rank n we define the vector-valued theta function

$$\Theta_K(\tau) \coloneqq \sum_{\mu \in K'/K} \sum_{X \in K+\mu} e(q(X)\tau)\, \mathfrak{e}_\mu .$$

The function Θ_K is a holomorphic modular form of weight $\frac{n}{2}$ for the Weil representation ρ_K.

For the rest of this section we let L be an even lattice in the rational quadratic space V of signature $(1, 2)$ defined in Section V.2.6. For $\tau, z \in \mathbb{H}$ we define the *Siegel theta function*

$$\Theta_L(\tau, z) \coloneqq v \sum_{\mu \in L'/L} \sum_{X \in L+\mu} e(q(X_z)\tau + q(X_{z^\perp})\overline{\tau})\, \mathfrak{e}_\mu . \tag{V.2.4}$$

By [3, Theorem 4.1], the Siegel theta function Θ_L transforms like a modular form of weight $-\frac{1}{2}$ for the Weil representation ρ_L in τ. Similarly, we define the *Millson theta function*

$$\Theta^*_L(\tau, z) \coloneqq v \sum_{\mu \in L'/L} \sum_{X \in L+\mu} p_X(z) e(q(X_z)\tau + q(X_{z^\perp})\overline{\tau})\, \mathfrak{e}_\mu .$$

Again using [3, Theorem 4.1], we see that the Millson theta function Θ^*_L transforms like a modular form of weight $\frac{1}{2}$ for ρ_L in τ. Note that both theta functions can be rewritten using (V.2.3). Both theta functions are invariant in z under the subgroup Γ_L of the orthogonal group $\mathrm{O}(L)$ which fixes the classes of L'/L.

If $K \subset L$ is a sublattice of finite index, then Lemma V.2.2 implies that

$$\Theta_L = (\Theta_K)^L, \qquad \Theta^*_L = (\Theta^*_K)^L. \tag{V.2.5}$$

Now fix some $X_0 \in L'$ with $q(X_0) > 0$, let $\mathcal{A} = \Gamma_L X_0$ be its Γ_L-class and let $z_\mathcal{A} = \mathbb{R}X_0 \in \mathcal{D}$ be the positive line spanned by X_0. Recall that we can also view $z_\mathcal{A}$ as a point in $\mathbb{H}$, which we call a *CM point* by a slight abuse of notation. Furthermore, let $P = L \cap z_\mathcal{A}$ and $N = L \cap z_\mathcal{A}^\perp$ be the corresponding positive definite one-dimensional and negative definite two-dimensional sublattices of L. A direct computation shows that the evaluation of the Siegel and the Millson theta functions at $z_\mathcal{A}$ split as

$$\Theta_{P\oplus N}(\tau, z_\mathcal{A}) = \Theta_P(\tau) \otimes v\overline{\Theta_{N^-}(\tau)}, \qquad \Theta^*_{P\oplus N}(\tau, z_\mathcal{A}) = \Theta^*_P(\tau) \otimes v\overline{\Theta_{N^-}(\tau)}, \tag{V.2.6}$$

where

$$\Theta^*_P(\tau) \coloneqq \sum_{\mu \in P'/P} \sum_{X \in P+\mu} p_X(z_\mathcal{A}) e(q(X)\tau)\, \mathfrak{e}_\mu$$

is a holomorphic unary theta series of weight $\frac{3}{2}$ for ρ_P.

V.3 Locally harmonic Maass forms and theta lifts

In this section we compute the action of the iterated raising operator on a certain locally harmonic Maass form and show that the resulting function can be written as the image of a suitable regularized theta lift. From now on, L denotes an even lattice of full rank in the quadratic space V of signature $(1,2)$ defined in Section V.2.6, and Γ_L is the subgroup of $\mathrm{O}(L)$ which fixes the classes of L'/L. Furthermore, throughout we let $k \in \mathbb{N}_{\geq 2}$.

Let $\mu \in L'/L$ and $m \in \mathbb{Z} - q(\mu)$ with $m > 0$ such that Mm is not a square. Following [5] (where a scalar-valued version was used) we define the function

$$\mathcal{F}_{1-k,\mu,m}(z) := \frac{(-1)^k (4Mm)^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi(2k-1)} \sum_{\substack{X \in L+\mu \\ q(X)=-m}} \operatorname{sgn}(p_X(z)) Q_X(z)^{k-1} \left(\frac{4Mmy^2}{|Q_X(z)|^2}\right)^{k-\frac{1}{2}} \times {}_2F_1\left(\frac{1}{2}, k-\frac{1}{2}; k+\frac{1}{2}; \frac{4Mmy^2}{|Q_X(z)|^2}\right).$$

The Euler integral representation of the hypergeometric function (see [1, equation 15.3.1]) yields

$$\mathcal{F}_{1-k,\mu,m}(z) = \frac{(-1)^k (4Mm)^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \sum_{\substack{X \in L+\mu \\ q(X)=-m}} \operatorname{sgn}(p_X(z)) Q_X(z)^{k-1} \psi\left(\frac{4Mmy^2}{|Q_X(z)|^2}\right),$$

where $\psi(v) := \frac{1}{2}\beta(v; k-\frac{1}{2}, \frac{1}{2})$ is a special value of the incomplete β-function $\beta(w;s,r) := \int_0^w t^{s-1}(1-t)^{r-1} dt$. In particular, by the same arguments as in [4] the function $\mathcal{F}_{1-k,m,\mu}$ converges absolutely and defines a locally harmonic Maass form of weight $2-2k$ for Γ_L. We recover the function $\mathcal{F}_{1-k,D}$ from [4] if we choose $M=1, D=4m$, and the lattice L from the introduction. We have the following series representation of $R_{2-2k}^{k-1}(\mathcal{F}_{1-k,\mu,m})$.

Proposition V.3.1. *Assume that $p_X(z) \neq 0$ for every $X \in L+\mu$ with $q(X) = -m$. Then*

$$R_{2-2k}^{k-1}\left(\mathcal{F}_{1-k,\mu,m}\right)(z) = \frac{(-1)^k (k-1)! y^k}{\binom{2k-2}{k-1}\pi(2k-1)} \sum_{\substack{X \in L+\mu \\ q(X)=-m}} \operatorname{sgn}(p_X(z))^k |Q_X(z)|^{-k} {}_2F_1\left(\frac{k}{2}, \frac{k}{2}; k+\frac{1}{2}; \frac{4Mmy^2}{|Q_X(z)|^2}\right).$$

Proof. It suffices to show that

$$\begin{aligned}
&R_{2-2k}^{k-1}\left(Q_X(z)^{k-1}\left(\frac{4Mmy^2}{|Q_X(z)|^2}\right)^{k-\frac{1}{2}}{}_2F_1\left(\frac{1}{2};k-\frac{1}{2};k+\frac{1}{2};\frac{4Mmy^2}{|Q_X(z)|^2}\right)\right)\\
&\qquad=(k-1)!(4Mm)^{k-\frac{1}{2}}\operatorname{sgn}(p_X(z))^{k-1}y^k|Q_X(z)|^{-k}{}_2F_1\left(\frac{k}{2},\frac{k}{2};k+\frac{1}{2};\frac{4Mmy^2}{|Q_X(z)|^2}\right).
\end{aligned}$$

Let $w:=\frac{4Mmy^2}{|Q_X(z)|^2}$. Using the Euler transformation

$$ {}_2F_1(a,b;c;Z)=(1-Z)^{c-a-b}{}_2F_1(c-a,c-b;c;Z) \tag{V.3.1}$$

and the identity (see [1, equation 15.2.3])

$$\partial_Z(Z^a{}_2F_1(a,b;c;Z))=aZ^{a-1}{}_2F_1(a+1,b;c;Z)$$

it may be shown by induction that for $j\in\mathbb{N}_0$

$$\begin{aligned}
&R_0^j\left(w^{\frac{k}{2}}{}_2F_1\left(\frac{k}{2},\frac{k}{2};k+\frac{1}{2};w\right)\right)\\
&=\frac{(k+j-1)!}{(k-1)!}(4Mm)^{-\frac{j}{2}}\operatorname{sgn}(p_X(z))^j\left(\frac{Q_X(\bar{z})}{y^2}\right)^j w^{\frac{k+j}{2}}{}_2F_1\left(\frac{k-j}{2},\frac{k+j}{2};k+\frac{1}{2};w\right).
\end{aligned}\tag{V.3.2}$$

In proving this induction, it is useful to note that

$$R_0(w)=-w\frac{R_{-2}Q_X(z)}{Q_X(z)}=-w\frac{2p_X(z)}{Q_X(z)},$$

and that one uses (V.2.3) to obtain the $\operatorname{sgn}(p_X(z))$-factor.

In particular, for $j=k-1$ equation (V.3.2) becomes

$$\begin{aligned}
&R_0^{k-1}\left(w^{\frac{k}{2}}{}_2F_1\left(\frac{k}{2},\frac{k}{2};k+\frac{1}{2};w\right)\right)\\
&=\frac{(2k-2)!}{(k-1)!}(4Mm)^{-\frac{k-1}{2}}\operatorname{sgn}(p_X(z))^{k-1}\left(\frac{Q_X(\bar{z})}{y^2}\right)^{k-1}w^{k-\frac{1}{2}}{}_2F_1\left(\frac{1}{2},k-\frac{1}{2};k+\frac{1}{2};w\right).
\end{aligned}\tag{V.3.3}$$

Furthermore, it is possible to show that $w^{\frac{k}{2}}{}_2F_1(\frac{k}{2},\frac{k}{2};k+\frac{1}{2},w)$ is an eigenfunction under the Laplace operator Δ_0 with eigenvalue $k(1-k)$. Now [6, Lemma 2.1] states that for $j\in\mathbb{N}_0$ and $g:\mathbb{H}\to\mathbb{C}$ satisfying $\Delta_0(g)=\lambda g$ we have

$$R_{2-2k}^{k-1}\left(y^{2k-2}\overline{R_0^{k-1}(g)}\right)(z)=\prod_{\ell=1}^{k-1}\left(-\overline{\lambda}-\ell(\ell-1)\right)\overline{g(z)}.$$

Substituting into (V.3.3) we find that

$$\frac{(k-1)!}{(2k-2)!}(4Mm)^{\frac{k-1}{2}}R_{2-2k}^{k-1}\left(y^{2k-2}\overline{R_0^{k-1}\left(w^{\frac{k}{2}}{}_2F_1\left(\frac{k}{2},\frac{k}{2};k+\frac{1}{2};w\right)\right)}\right)$$
$$=(k-1)!(4Mm)^{k-\frac{1}{2}}\operatorname{sgn}(p_X(z))^{k-1}y^k|Q_X(z)|^{-k}{}_2F_1\left(\frac{k}{2},\frac{k}{2};k+\frac{1}{2};w\right),$$

where we are using $\prod_{\ell=1}^{k-1}(k(1-k)-\ell(\ell-1))=(2k-2)!$ and are inserting the definition of w. □

For a harmonic Maass form $f\in H_{\frac{3}{2}-k,L}$ we consider the regularized theta lift

$$\Lambda^{\mathrm{reg}}(f,z):=\begin{cases}\displaystyle\int_{\mathcal{F}}^{\mathrm{reg}}\left\langle R_{\frac{3}{2}-k}^{\frac{k}{2}-1}(f)(\tau),\overline{\Theta_L(\tau,z)}\right\rangle v^{-\frac{1}{2}}d\mu(\tau), & \text{if } k \text{ is even},\\ \displaystyle\int_{\mathcal{F}}^{\mathrm{reg}}\left\langle R_{\frac{3}{2}-k}^{\frac{k-1}{2}}(f)(\tau),\overline{\Theta_L^*(\tau,z)}\right\rangle v^{\frac{1}{2}}d\mu(\tau), & \text{if } k \text{ is odd},\end{cases}$$

where $d\mu(\tau):=\frac{dudv}{v^2}$ denotes the invariant measure on $\mathbb{H}$, and the regularised integral is defined by $\int_{\mathcal{F}}^{\mathrm{reg}}:=\lim_{T\to\infty}\int_{\mathcal{F}_T}$, where $\mathcal{F}_T$ denotes the standard fundamental domain for Γ truncated at height T. By the results of [7, Section 2.3] for the Siegel theta function (corresponding to k even) and by [8, Section 7.3] for the Millson theta function (corresponding to k odd), the integral converges for every $z\in\mathbb{H}$.

We now compute the lift of the Maass–Poincaré series by unfolding against it. Thereby we obtain the following representation of $R_{2-2k}^{k-1}(\mathcal{F}_{1-k,\mu,m})$ as a regularized theta lift.

Theorem V.3.2. *Assume that $p_X(z)\neq 0$ for every $X\in L+\mu$ with $q(X)=-m$. If $k\in\mathbb{N}$ is even, then*

$$R_{2-2k}^{k-1}\left(\mathcal{F}_{1-k,\mu,m}\right)(z)=\frac{(k-1)!^2(4Mm)^{\frac{1}{2}-k}M^{\frac{k-1}{2}}}{2^{2k}\pi^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)^2}\Lambda^{\mathrm{reg}}\left(F_{\mu,-m,\frac{3}{2}-k},z\right).$$

If $k\in\mathbb{N}$ is odd, then

$$R_{2-2k}^{k-1}(\mathcal{F}_{1-k,\mu,m})(z)=-\frac{(k-1)!^2(4Mm)^{\frac{1}{2}-k}M^{\frac{k}{2}-1}}{2^{2k}\pi^{\frac{k-1}{2}}\Gamma\left(\frac{k+1}{2}\right)^2}\Lambda^{\mathrm{reg}}\left(F_{\mu,-m,\frac{3}{2}-k},z\right).$$

Proof. A similar result was proved in [7, Theorem 2.14] and [8, Theorem 7.9]. Here we give a sketch of the proof for k even for the convenience of the reader; the case k odd follows similarly (using a single application of (V.3.1) in the final step). We consider the regularized theta lift of the Maass Poincaré series $F_{\mu,-m,\frac{3}{2}-k,s}$. Applying Lemma V.2.1 we obtain

$$\begin{aligned}&\Lambda^{\mathrm{reg}}\left(F_{\mu,-m,\frac{3}{2}-k,s},z\right)\\ =&(4\pi m)^{\frac{k}{2}-1}\frac{\Gamma\left(s-\frac{1}{4}\right)}{\Gamma\left(s+\frac{3}{4}-\frac{k}{2}\right)}\int_{\mathcal{F}}^{\mathrm{reg}}\left\langle F_{\mu,-m,-\frac{1}{2},s}(\tau),\overline{\Theta_L(\tau,z)}\right\rangle v^{-\frac{1}{2}}d\mu(\tau).\end{aligned}$$

By the usual unfolding argument the above expression can be written as

$$2(4\pi m)^{\frac{k}{2}-1}\frac{\Gamma\left(s-\frac{1}{4}\right)}{\Gamma(2s)\Gamma\left(s+\frac{3}{4}-\frac{k}{2}\right)}\int_0^\infty\int_0^1\mathcal{M}_{-\frac{1}{2},s}(-4\pi mv)e(-mu)\overline{\Theta_{L,\mu}(\tau,z)}v^{-\frac{5}{2}}dudv,$$

where $\Theta_{L,\mu}$ denotes the μ-th component of Θ_L. Inserting the Fourier expansion of Θ_L given in (V.2.4) and the definition of $\mathcal{M}_{-\frac{1}{2},s}$ given in (V.2.1), and evaluating the integral over u, this becomes

$$2(4\pi m)^{\frac{k}{2}-\frac{3}{4}}\frac{\Gamma\left(s-\frac{1}{4}\right)}{\Gamma(2s)\Gamma\left(s+\frac{3}{4}-\frac{k}{2}\right)}\sum_{\substack{X\in L+\mu\\ q(X)=-m}}\int_0^\infty M_{\frac{1}{4},s-\frac{1}{2}}(4\pi mv)v^{-\frac{5}{4}}e^{-2\pi v\left(q(X_z)-q\left(X_{z^\perp}\right)\right)}dv.$$

The integral is an inverse Laplace transform and can be computed using equation (11) on page 215 of [13]. We obtain

$$2(4\pi m)^{\frac{k-1}{2}}\frac{\Gamma\left(s-\frac{1}{4}\right)^2}{\Gamma(2s)\Gamma\left(s+\frac{3}{4}-\frac{k}{2}\right)}\sum_{\substack{X\in L+\mu\\ q(X)=-m}}\left(\frac{m}{|q(X_{z^\perp})|}\right)^{s-\frac{1}{4}}{}_2F_1\left(s-\frac{1}{4},s-\frac{1}{4};2s;\frac{m}{|q(X_{z^\perp})|}\right).$$

Plugging in the formula $q(X_{z^\perp})=-\frac{1}{4Mmy^2}|Q_X(z)|^2$ (see (V.2.3)) and the special value $s=\frac{k}{2}+\frac{1}{4}$, we arrive at

$$2(4\pi m)^{\frac{k-1}{2}}\frac{\Gamma\left(\frac{k}{2}\right)^2}{\Gamma\left(k+\frac{1}{2}\right)}\sum_{\substack{X\in L+\mu\\ q(X)=-m}}\left(\frac{4Mmy^2}{|Q_X(z)|^2}\right)^{\frac{k}{2}}{}_2F_1\left(\frac{k}{2},\frac{k}{2};k+\frac{1}{2};\frac{4Mmy^2}{|Q_X(z)|^2}\right).$$

Using the Legendre duplication formula $\pi^{\frac{1}{2}}\Gamma(2k)=2^{2k-1}\Gamma(k)\Gamma(k+\frac{1}{2})$ and comparing the above expression with Proposition V.3.1, we obtain the stated result. $\square$

V.4 Evaluation of the theta lift at CM points

We now evaluate the theta integral at CM points. As in the previous section we let L denote an even lattice of full rank in the signature $(1,2)$ quadratic space V

from Section V.2.6, and we let Γ_L be the subgroup of $\mathrm{O}(L)$ which fixes the classes of L'/L. Moreover, we fix some $X_0 \in L'$ with $q(X_0) > 0$, and we set $\mathcal{A} = \Gamma_L X_0$ and $z_{\mathcal{A}} = \mathbb{R}X_0 \in \mathcal{D} \cong \mathbb{H}$. Then we have the sublattices $P = L \cap z_{\mathcal{A}}$ and $N = L \cap z_{\mathcal{A}}^{\perp}$.

Recall that $\mathcal{G}_P$ denotes a harmonic Maass form of weight $\frac{3}{2}$ for ρ_P that maps to Θ_P under $\xi_{\frac{3}{2}}$. Similarly, we let $\mathcal{G}_P^*$ be a harmonic Maass form of weight $\frac{1}{2}$ for ρ_P that maps to Θ_P^* under $\xi_{\frac{1}{2}}$. For simplicity, we now assume that the input f for the regularized theta lift is weakly holomorphic. We have the following theorem, which is inspired by a similar recent result of Bruinier, Ehlen, and Yang (compare [8, Theorem 5.4]).

Theorem V.4.1. *Let $f \in M^!_{\frac{3}{2}-k,L}$. For k even we have*

$$\Lambda^{\mathrm{reg}}(f, z_{\mathcal{A}}) = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)}{2(4\pi)^{1-\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right)} \mathrm{CT}\left(\left\langle f_{P\oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)\right]_{\frac{k}{2}-1}\right\rangle\right).$$

For k odd we have

$$\Lambda^{\mathrm{reg}}(f, z_{\mathcal{A}}) = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{(4\pi)^{\frac{1-k}{2}}\Gamma\left(\frac{k}{2}\right)} \mathrm{CT}\left(\left\langle f_{P\oplus N}(\tau), \left[\mathcal{G}_P^{*,+}(\tau), \Theta_{N^-}(\tau)\right]_{\frac{k-1}{2}}\right\rangle\right).$$

Proof. We give the details of the proof for k even, since the proof for k odd is very similar. Note that Lemma V.2.2 and (V.2.5) imply that

$$\langle f, \Theta_L\rangle = \left\langle f, (\Theta_{P\oplus N})^L\right\rangle = \langle f_{P\oplus N}, \Theta_{P\oplus N}\rangle.$$

Thus we may assume that $L = P \oplus N$ if we replace f by $f_{P\oplus N}$. For simplicity, we write just f instead of $f_{P\oplus N}$ throughout the proof.

First, using the self-adjointness of the raising operator (see [7, Lemma 4.2]) we obtain

$$\begin{aligned}&\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R^{\frac{k}{2}-1}_{\frac{3}{2}-k}(f)(\tau), \overline{\Theta(\tau, z_{\mathcal{A}})}\right\rangle v^{-\frac{1}{2}} d\mu(\tau)\\ =&(-1)^{\frac{k}{2}-1}\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), R^{\frac{k}{2}-1}_{\frac{1}{2}}\left(v^{-\frac{1}{2}}\overline{\Theta_L(\tau, z_{\mathcal{A}})}\right)\right\rangle d\mu(\tau).\end{aligned}$$

Note that the apparent boundary term appearing disappears in the same way as in the proof of [7, Lemma 4.4]. Using the splitting (V.2.6) of the Siegel theta function and the formula

$$R_{\ell-\kappa}\left(v^{\kappa}\overline{g(\tau)} \otimes h(\tau)\right) = v^{\kappa}\overline{g(\tau)} \otimes R_{\ell}(h)(\tau)$$

which holds for every holomorphic function g, every smooth function h, and $\kappa, \ell \in \mathbb{R}$, we obtain

$$R^{\frac{k}{2}-1}_{\frac{1}{2}}\left(v^{-\frac{1}{2}}\overline{\Theta_{P\oplus N}(\tau, z_{\mathcal{A}})}\right) = L_{\frac{3}{2}}(\mathcal{G}_P)(\tau) \otimes R^{\frac{k}{2}-1}_{1}(\Theta_{N^-})(\tau).$$

Since $L_1(\Theta_{N^-}) = 0$, Proposition V.2.3 implies that

$$L_{\frac{3}{2}}(\mathcal{G}_P)(\tau) \otimes R_1^{\frac{k}{2}-1}(\Theta_{N^-})(\tau) = \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)}{2\Gamma\left(\frac{k+1}{2}\right)}(-4\pi)^{\frac{k}{2}-1} L_{k+\frac{1}{2}}\left([\mathcal{G}_P(\tau), \Theta_{N^-}(\tau)]_{\frac{k}{2}-1}\right).$$

Hence we have that

$$\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle R_{\frac{3}{2}-k}^{\frac{k}{2}-1}(f)(\tau), \overline{\Theta_L(\tau, z_{\mathcal{A}})}\right\rangle v^{-\frac{1}{2}} d\mu(\tau)$$
$$= \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)}{2\Gamma\left(\frac{k+1}{2}\right)}(4\pi)^{\frac{k}{2}-1} \int_{\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), L_{k+\frac{1}{2}}\left([\mathcal{G}_P(\tau), \Theta_{N^-}(\tau)]_{\frac{k}{2}-1}\right)\right\rangle d\mu(\tau).$$

Now a standard application of Stokes' Theorem as in the proof of [9, Proposition 3.5] gives the stated formula. □

V.5 Statement of the main results and the proof of Theorem V.1.1

We are now ready to state and prove our main result, which is a more general version of Theorem V.1.1 for arbitrary congruence subgroups and both even and odd $k \in \mathbb{N}_{\geq 2}$.

As before we let L denote an even lattice of signature $(1, 2)$ in the quadratic space V from Section V.2.6, and we let Γ_L be the subgroup of $\mathrm{O}(L)$ which fixes the classes of L'/L. We can view Γ_L as a subgroup of $\mathrm{SL}_2(\mathbb{R})$, the action on $\mathcal{D}$ corresponding to fractional linear transformations on $\mathbb{H}$. Moreover, we fix some $X_0 \in L'$ with $q(X_0) > 0$, and we set $\mathcal{A} = \Gamma_L X_0$ and $z_{\mathcal{A}} = \mathbb{R}X_0 \in \mathcal{D} \cong \mathbb{H}$. We have the corresponding sublattices $P = L \cap z_{\mathcal{A}}$ and $N = L \cap z_{\mathcal{A}}^{\perp}$.

Generalising (V.1.1) we define the meromorphic modular form

$$f_{k,\mathcal{A}}(z) := \frac{(4Mq(\mathcal{A}))^{\frac{k+1}{2}}}{\pi} \sum_{X \in \mathcal{A}} Q_X(z, 1)^{-k}$$

of weight $2k$ for Γ_L. Furthermore, for $\mu \in L'/L$ and $m \in \mathbb{Z} - q(\mu)$ with $m > 0$ and Mm not being a square, we define the trace of cycle integrals

$$\mathrm{tr}_{f_{k,\mathcal{A}}}(\mu, m) := \sum_{X \in \Gamma_L \backslash L_{\mu,-m}} \int_{c_X} f_{k,\mathcal{A}}(z) Q_X(z, 1)^{k-1} dz,$$

where $L_{\mu,-m}$ denotes the set of all $X \in L + \mu$ with $q(X) = -m$, and $c_X := (\Gamma_L)_X \backslash C_X$ with the geodesic

$$C_X := \{z \in \mathbb{H} : p_X(z) = 0\} = \{z \in \mathbb{H} : aM|z|^2 + bx + c = 0\}.$$

We let $\mathcal{G}_P$ be a harmonic Maass form of weight $\frac{3}{2}$ for ρ_P that maps to Θ_P under $\xi_{\frac{3}{2}}$. Similarly, we let $\mathcal{G}_P^*$ be a harmonic Maass form of weight $\frac{1}{2}$ for ρ_P that maps to Θ_P^* under $\xi_{\frac{1}{2}}$. Finally, we let $f \in M^!_{\frac{3}{2}-k,L}$ be a weakly holomorphic modular form with Fourier coefficients $c_f(\mu, m)$ and we assume that $c_f(\mu, -m) = 0$ if $m > 0$ and Mm is a square.

Theorem V.5.1. *Assume that $z_{\mathcal{A}}$ does not lie on any of the geodesics c_X for $X \in L_{\mu,-m}$ if $c_f(\mu,-m) \neq 0$. For k even we have*

$$\sum_{\mu \in L'/L} \sum_{m>0} c_f(\mu,-m) \operatorname{tr}_{f_{k,\mathcal{A}}}(\mu,-m)$$
$$= \frac{2^{k-3}\left(4Mq(\mathcal{A})\right)^{\frac{1}{2}}}{\pi M^{\frac{1-k}{2}} \left|\left(\overline{\Gamma}_L\right)_{z_{\mathcal{A}}}\right|} \operatorname{CT}\left(\left\langle f_{P\oplus N}(\tau), \left[\mathcal{G}_P^+(\tau), \Theta_{N^-}(\tau)\right]_{\frac{k}{2}-1}\right\rangle\right).$$

For k odd we have

$$\sum_{\mu \in L'/L} \sum_{m>0} c_f(\mu,m) \operatorname{tr}_{f_{k,\mathcal{A}}}(\mu,m)$$
$$= -\frac{2^{k-1}\left(4Mq(\mathcal{A})\right)^{\frac{1}{2}}}{M^{1-\frac{k}{2}} \left|\left(\overline{\Gamma}_L\right)_{z_{\mathcal{A}}}\right|} \operatorname{CT}\left(\left\langle f_{P\oplus N}(\tau), \left[\mathcal{G}_P^{*,+}(\tau), \Theta_{N^-}(\tau)\right]_{\frac{k-1}{2}}\right\rangle\right).$$

Remark 11. For $M = 1$ and the lattice L from the introduction, with $\Gamma_L \cong \mathrm{SL}_2(\mathbb{Z})$, $L_{\mu,-m} = \mathcal{Q}_{4m}$, and $d = -4Mq(\mathcal{A})$, we recover Theorem V.1.1. We remark that by combining the results of this paper and the methods from [8, Section 7] one can also derive similar formulas for twisted traces of cycle integrals of $f_{k,\mathcal{A}}$.

Proof of Theorem V.5.1. We prove the case of even k, and the case of odd k follows similarly. First, by [15, Corollary 4.3] we have that

$$\operatorname{tr}_{f_{k,\mathcal{A}}}(\mu,m) = \frac{2^k\left(4Mq(\mathcal{A})\right)^{\frac{1}{2}}(4Mm)^{k-\frac{1}{2}}}{(k-1)! \left|\left(\overline{\Gamma}_L\right)_{z_{\mathcal{A}}}\right|} R^{k-1}_{2-2k}(\mathcal{F}_{1-k,\mu,m})(z_{\mathcal{A}}).$$

Note that we are using a different normalisation of $f_{k,\mathcal{A}}$, and that the results of [15] are formulated in a more classical language. However, the exact same arguments as in the proof of [15, Corollary 4.3] work in the general case that we need.

Next, recall that we can write f as a linear combination of Maass Poincaré series as in (V.2.2). Hence, we obtain from Theorem V.3.2 the formula

$$\sum_{\mu \in L'/L} \sum_{m>0} c_f(\mu,m)(4Mm)^{k-\frac{1}{2}} R^{k-1}_{2-2k}(\mathcal{F}_{1-k,\mu,m})(z_{\mathcal{A}}) = \frac{(k-1)!^2 M^{\frac{k-1}{2}}}{2^{2k-1}\pi^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)^2} \Lambda^{\mathrm{reg}}\left(f, z_{\mathcal{A}}\right).$$

Finally, by Theorem V.4.1 we have the evaluation

$$\Lambda^{\mathrm{reg}}\left(f, z_{\mathcal{A}}\right)=\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)}{2(4\pi)^{1-\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right)}\operatorname{CT}\left(\left\langle f_{P\oplus N}(\tau),\left[\mathcal{G}_P^{+}(\tau),\Theta_{N^{-}}(\tau)\right]_{\frac{k}{2}-1}\right\rangle\right).$$

If we put all the constants together and use the Legendre duplication formula $\pi^{\frac{1}{2}}\Gamma(k)=2^{k-1}\Gamma(\frac{k}{2})\Gamma(\frac{k+1}{2})$, we obtain the stated formula. □

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Chapter VI

On t-core and self-conjugate $2t-1$-core partitions in arithmetic progressions

This chapter is based on a manuscript submitted for publication [BKM]. This is joint work with Prof. Dr. Kathrin Bringmann and Prof. Dr. Ben Kane.

VI.1 Introduction and Statement of Results

A *partition* Λ of $n \in \mathbb{N}$ is a non-increasing sequence $\Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of non-negative integers λ_j such that $\sum_{1 \le j \le s} \lambda_j = n$. The *Ferrers–Young diagram* of Λ is the s-rowed diagram

$$\begin{array}{llll} \bullet & \bullet & \cdots & \bullet \qquad\qquad \lambda_1 \text{ dots} \\ \bullet & \bullet & \cdots & \bullet \qquad\qquad \lambda_2 \text{ dots} \\ \cdot & & & \\ \cdot & & & \\ \bullet & \cdots & \bullet & \qquad\qquad\quad \lambda_s \text{ dots.} \end{array}$$

We label the cells of the Ferrers–Young diagram as if it were a matrix, and let λ'_k denote the number of dots in column k. The *hook length* of the cell (j,k) in the Ferrers–Young diagram of Λ equals

$$h(j,k) := \lambda_j + \lambda'_k - k - j + 1.$$

If no hook length in any cell of a partition Λ is divisible by t, then Λ is a *t-core partition.* A partition Λ is said to be *self-conjugate* if it remains the same when rows and columns are switched.

Example VI.1.1. *The partition $\Lambda = (3,2,1)$ of 6 has the Ferrers–Young diagram*

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$$

and has hook lengths $h(1,1)=5$, $h(1,2)=3$, $h(1,3)=1$, $h(2,1)=3$, $h(2,2)=1$, and $h(3,1)=1$. Therefore, Λ is a t-core partition for all $t \notin \{1,3,5\}$. Furthermore, switching rows and columns leaves Λ unaltered, and so Λ is self-conjugate.

The theory of t-core partitions is intricately linked to various areas of number theory and beyond. For example, Garvan, Kim, and Stanton [6] used t-core partitions to investigate special cases of the famous Ramanujan congruences for the partition function $p(n)$. Furthermore, t-core partitions encode the modular representation theory of symmetric groups S_n and A_n (see e.g. [5, 8])

For $t, n \in \mathbb{N}$ we let $\mathrm{c}_t(n)$ denote the number of t-core partitions of n, along with $\mathrm{sc}_t(n)$ the number of self-conjugate t-core partitions of n. In 1997, Ono and Sze [15] investigated the relation between 4-core partitions and class numbers. Denote by $H(|D|)$ ($D>0$ a discriminant) the D-th Hurwitz class number, which counts the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of integral binary quadratic forms of discriminant D, weighted by $\frac{1}{2}$ times the order of their automorphism group.[1] Then Ono and Sze proved the following theorem.

Theorem VI.1.2 (Theorem 2 of [15])**.** *If $8n+5$ is square-free, then*

$$c_4(n) = \frac{1}{2} H(32n+20).$$

More recently Ono and Raji [14] showed similar relations between self-conjugate 7-core partitions and class numbers. To state their result, let

$$D_n := \begin{cases} 28n+56 & \text{if } n \equiv 1 \pmod 4, \\ 7n+14 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Theorem VI.1.3 (Theorem 1 of [14])**.** *Let $n \not\equiv -2 \pmod 7$ be a positive odd integer. Then*

$$\mathrm{sc}_7(n) = \begin{cases} \frac{1}{4} H(D_n) & \text{if } n \equiv 1 \pmod 4, \\ \frac{1}{2} H(D_n) & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

In particular, by combining Theorems VI.1.2 and VI.1.3 and using elementary congruence conditions, one may easily show that for $n \not\equiv 5 \pmod 7$ and $8n+5$ square-free,

$$2\,\mathrm{sc}_7(8n+1) = \mathrm{c}_4(7n+2). \tag{VI.1.1}$$

This fact hints at a deeper relationship between sc_{2t-1} and c_t, which we investigate. Our main results pertain to the case of $t=4$. We begin by extending recent results of

[1] Some authors write $H(D)$ instead of $H(|D|)$; in particular this notation was used in [14].

Ono and Raji [14]. Letting $\mathrm{sc}_7(n)$ denote the number of self-conjugate 7-core partitions of n and $\left(\frac{\cdot}{\cdot}\right)$ denote the extended Jacobi Symbol, we may state our first theorem. For this, for $n \in \mathbb{Q}$ we set $H(n) := 0$ if $n \notin \mathbb{Z}$ or $-n$ is not a discriminant.

Theorem VI.1.4. *For every $n \in \mathbb{N}$, we have*

$$\mathrm{sc}_7(n) = \frac{1}{4}\left(H(28n+56) - H\left(\frac{4n+8}{7}\right) - 2H(7n+14) + 2H\left(\frac{n+2}{7}\right)\right).$$

While Theorem VI.1.4 gives a uniform formula for $\mathrm{sc}_7(n)$ as a linear combination of Hurwitz class numbers, it is also desirable to obtain a formula in terms of a single class number. For this, let $\ell \in \mathbb{N}_0$ be chosen maximally such that $n \equiv -2 \pmod{2^{2\ell}}$ and extend the definition of D_n to

$$D_n := \begin{cases} 28n+56 & \text{if } n \equiv 0,1 \pmod 4, \\ 7n+14 & \text{if } n \equiv 3 \pmod 4, \\ D_{\frac{n+2}{2^{2\ell}}-2} & \text{if } n \equiv 2 \pmod 4, \end{cases} \tag{VI.1.2}$$

and

$$\nu_n := \begin{cases} \frac{1}{4} & \text{if } n \equiv 0,1 \pmod 4, \\ \frac{1}{2} & \text{if } n \equiv 3 \pmod 8, \\ \nu_{\frac{n+2}{2^{2\ell}}-2} & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{VI.1.3}$$

A binary quadratic form $[a,b,c]$ is called *primitive* if $\gcd(a,b,c)=1$ and, for a prime p, *p-primitive* if $p \nmid \gcd(a,b,c)$. We let $H_p(D)$ count the number of p-primitive classes of integral binary quadratic forms of discriminant $-D$, with the same weighting as $H(D)$.

Corollary VI.1.5. *For every $n \in \mathbb{N}$ we have*

$$\mathrm{sc}_7(n) = \nu_n H_7(D_n).$$

Remark. For $n \not\equiv -2 \pmod 7$, one has $H(D_n) = H_7(D_n)$ and hence the cases $n \equiv 1,3 \pmod 4$ of Corollary VI.1.5 with $n \not\equiv -2 \pmod 7$ are covered by Theorem VI.1.3.

For $n+2$ squarefree, we may use Dirichlet's class number formula to obtain another representation for $\mathrm{sc}_7(n)$; Ono and Raji [14, Corollary 2] covered the case that $n \not\equiv -2 \pmod 7$ is odd.

Corollary VI.1.6. *If $n \in \mathbb{N}$ is an integer for which $n+2$ is squarefree, then*

$$\mathrm{sc}_7(n) = -\frac{\nu_n}{D_n}\begin{cases} \sum_{m=1}^{D_n-1}\left(\frac{-D_n}{m}\right)m & \text{if } n \not\equiv -2 \pmod 7, \\ 7^2\left(7+\left(\frac{\frac{D_n}{7^2}}{7}\right)\right)\sum_{m=1}^{\frac{D_n}{7^2}-1}\left(\frac{-\frac{D_n}{7^2}}{m}\right)m & \text{if } n \equiv -2 \pmod 7. \end{cases}$$

The following corollary relates $\mathrm{sc}_7(m)$ with $m+2$ not necessarily squarefree to $\mathrm{sc}_7(n)$ with $n+2$ squarefree, for which Corollary VI.1.6 applies. The cases $\ell = r = 0$ with $n \not\equiv -2 \pmod{7}$ odd were proven in [14, Corollary 3]. For this μ denotes the Möbius function and $\sigma(n) := \sum_{d|n} d$.

Corollary VI.1.7. *If $n \in \mathbb{N}$ satisfies $n+2$ squarefree, $\ell, r \in \mathbb{N}_0$, and $f \in \mathbb{N}$ with $\gcd(f, 14) = 1$, then*

$$\mathrm{sc}_7\left((n+2)2^{2\ell}f^2 7^{2r} - 2\right) = 7^r\, \mathrm{sc}_7(n) \sum_{d|f} \mu(d) \left(\frac{-D_n}{d}\right) \sigma\left(\frac{f}{d}\right).$$

We also provide a combinatorial explanation for Corollary VI.1.5. To do so, we first extend techniques of Ono and Sze [15] and explicitly describe the possible abaci (defined in Section VI.4) of self-conjugate 7-core partitions . Then, in (VI.4.1) below we construct an explicit map ϕ sending self-conjugate 7-core partitions to binary quadratic forms, via abaci and extended t-residue diagrams (defined in Section VI.4).

In order to describe the image of this map, for a prime p and a discriminant $D = \Delta f^2$ with Δ fundamental, we call a binary quadratic form of discriminant D *p-totally imprimitive* if the power of p dividing $\gcd(a, b, c)$ equals the power of p dividing f (i.e., if the power of p dividing $\gcd(a, b, c)$ is maximal). Furthermore, recall that two binary quadratic forms of discriminant D are said to be in the same *genus* if they represent the same values in $(\mathbb{Z}/D\mathbb{Z})^*$. We call the genus containing the principal binary quadratic form of discriminant D the *principal genus*. The image of ϕ is then described in the following theorem.

Theorem VI.1.8. *For every $n \in \mathbb{N}$, the image of ϕ is a unique non-principal genus of 7-primitive and 2-totally imprimitive binary quadratic forms with discriminant $-28n-56$. Moreover, suppose that ℓ is chosen maximally such that $n \equiv -2 \pmod{2^{2\ell}}$ and $\frac{7n+14}{2^{2\ell}}$ has r distinct prime divisors. Then every equivalence class in this genus is the image of $\nu_n 2^r$ many self-conjugate 7-cores of n.*

Note that Theorem VI.1.8 along with [15, Theorem 6] provides a combinatorial explanation for (VI.1.1). The cases $t \in \{2, 3\}$ are simple to describe, and immediately imply that relationships similar to (VI.1.1) along arithmetic progressions do not exist for $t \in \{2, 3\}$, which we see in Section VI.5.1. We prove a similar result for $t = 5$ in Proposition VI.5.3. Based on these results we offer the following conjecture, along with partial results on possible values of $t \pmod{6}$ along with the possible shapes of arithmetic progressions in Section VI.5.3.

Conjecture VI.1.9. *The only occurrence of arithmetic progressions for which c_t and sc_{2t-1} agree up to integer multiples non-trivially (even asymptotically) is when $t = 4$.*

The paper is organised as follows. In Section VI.2, we provide proofs for Theorem VI.1.4 and Corollary VI.1.6, Corollaries VI.1.5 and VI.1.7 are shown in Section VI.3. Section VI.4 is dedicated to providing a combinatorial explanation of Theorem VI.1.3 and its generalization in Corollary VI.1.5. In Section VI.5 we prove Conjecture VI.1.9 in the cases $t \in \{2, 3, 5\}$ and provides partial results for larger t.

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VI.2 Proofs of Theorem VI.1.4 and Corollary VI.1.6

Our investigation for the case $t = 4$ begins by packaging the number of self-conjugate 7-cores into a generating function and using the fact that it is a modular form to relate $\mathrm{sc}_7(n)$ to class numbers. We thus define

$$S(\tau) := \sum_{n\geq 0} \mathrm{sc}_7(n) q^{n+2}.$$

As stated on [14, page 4], S is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(28)$ with character $\left(\frac{28}{\cdot}\right)$.

VI.2.1 Proof of Theorem VI.1.4

To prove Theorem VI.1.4, we let

$$\mathcal{H}_{\ell_1,\ell_2}(\tau) := \mathcal{H}\big|(U_{\ell_1,\ell_2} - \ell_2 U_{\ell_1} V_{\ell_2})(\tau).$$

Here for $f(\tau) := \sum_{n\in\mathbb{Z}} c_f(n) q^n$

$$f\big|U_d(\tau) := \sum_{n\in\mathbb{Z}} c_f(dn) q^n, \qquad f\big|V_d(\tau) := \sum_{n\in\mathbb{Z}} c_f(n) q^{dn},$$

and

$$\mathcal{H}(\tau) := \sum_{\substack{D\geq 0 \\ D\equiv 0,3 \pmod 4}} H(D) q^D.$$

Proof of Theorem VI.1.4. Shifting $n \mapsto n-2$ in Theorem VI.1.4 and taking the generating function of both sides, the claim of the theorem is equivalent to

$$S = \frac{1}{4}\mathcal{H}_{1,2}\big| \left(U_{14} - U_2\big|V_7\right). \tag{VI.2.1}$$

By [3, Lemma 2.3 and Lemma 2.6], both sides of (VI.2.1) are modular forms of weight $\frac{3}{2}$ on $\Gamma_0(56)$ with character $\left(\frac{28}{\cdot}\right)$. By the valence formula, it thus suffices to check (VI.2.1) for the first 12 coefficients; this has been done by computer, yielding (VI.2.1) and hence Theorem VI.1.4. $\square$

VI.2.2 Rewriting $\mathrm{sc}_7(n)$ in terms of representation numbers

The next lemma rewrites $\mathrm{sc}_7(n)$ in terms of the representation numbers ($m \in \mathbb{N}_0$)

$$r_3(m) := \#\left\{\boldsymbol{x} \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = m\right\}.$$

For $m \in \mathbb{Q} \setminus \mathbb{N}_0$, we furthermore set $r_3(m) := 0$ for ease of notation.

Lemma VI.2.1.

(1) *For $n \in \mathbb{N}$, we have*

$$\mathrm{sc}_7(n) = \frac{1}{48}\left(r_3(7n+14) - r_3\left(\frac{n+2}{7}\right)\right).$$

(2) *If $n \equiv -2 \pmod 7$, then we have*

$$\mathrm{sc}_7(n) = \frac{1}{48}\left(\left(7 + \left(\frac{\frac{D_n}{7^2}}{7}\right)\right) r_3\left(\frac{n+2}{7}\right) - 7r_3\left(\frac{n+2}{7^3}\right)\right).$$

Proof. (1) By the proof of [3, Lemma 4.1] we have

$$\Theta^3(\tau) = \sum_{n \geqslant 0} r_3(n)q^n = 12\mathcal{H}_{1,2}\big|U_2(\tau),$$

where $\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}$ is the usual theta function. Plugging this into (VI.2.1), the claim follows after picking off the Fourier coefficients and shifting $n \mapsto n+2$.
(2) Recall that for $f(\tau) = \sum_{n\in\mathbb{Z}} c_f(n)q^n$ a modular form of weight $\lambda + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$, the p^2-*th Hecke operator* is defined as

$$f|T_{p^2}(\tau) = \sum_{n\geqslant 0}\left(c_f\left(pn^2\right) + \left(\frac{(-1)^\lambda n}{p}\right)p^{\lambda-1}c_f(n) + p^{2\lambda-1}c_f\left(\frac{n}{p^2}\right)\right)q^n.$$

It is well-known that

$$\Theta^3|T_{p^2} = (p+1)\Theta^3. \tag{VI.2.2}$$

Rearranging (VI.2.2) and comparing coefficients we obtain, by (VI.2.2), for $m := n+2 \equiv 0 \pmod 7$,

$$r_3(7m) = 8r_3\left(\frac{m}{7}\right) - \left(\frac{-\frac{m}{7}}{7}\right) r_3\left(\frac{m}{7}\right) - 7r_3\left(\frac{m}{7^3}\right).$$

The claim follows by (1). □

VI.2.3 Formulas in terms of single class numbers

We next turn to formulas for $\mathrm{sc}_7(n)$ in terms of a single class number.

Corollary VI.2.2.

(1) *For $n \not\equiv -2 \pmod 7$ and $n \not\equiv 2 \pmod 4$, we have*

$$\mathrm{sc}_7(n) = \nu_n H(D_n).$$

(2) *For $n \equiv -2 \pmod 7$, $n \not\equiv -2 \pmod{7^3}$, and $n \not\equiv 2 \pmod 4$, we have*

$$\mathrm{sc}_7(n) = \left(7 + \left(\frac{\frac{D_n}{7^2}}{7}\right)\right) \nu_n H\left(\frac{D_n}{7^2}\right).$$

(3) *If $n \equiv 2 \pmod 4$, then*

$$\mathrm{sc}_7(n) = \mathrm{sc}_7\left(\frac{n+2}{4} - 2\right).$$

(4) *If $n \equiv -2 \pmod{7^2}$, then*

$$\mathrm{sc}_7(n) = 7\,\mathrm{sc}_7\left(\frac{n+2}{7^2} - 2\right).$$

Remark. For $n \not\equiv 2 \pmod 4$, we have $7(n+2) \mid D_n$, so $n \equiv -2 \pmod 7$ implies that $7^2 \mid D_n$, and hence Corollary VI.2.2 (2) is meaningful.

Proof of Corollary VI.2.2. (1) Since $n \not\equiv -2 \pmod 7$, the final term in Lemma VI.2.1 (1) vanishes, giving

$$\mathrm{sc}_7(n) = \frac{1}{48} r_3(7n+14).$$

The claim then follows immediately by plugging in the well-known formula of Gauss (see e.g. [13, Theorem 8.5])

$$r_3(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1,2 \pmod 4, \\ 24H(n) & \text{if } n \equiv 3 \pmod 8, \\ r_3\left(\frac{n}{4}\right) & \text{if } 4 \mid n, \\ 0 & \text{otherwise.} \end{cases} \tag{VI.2.3}$$

(2) Since $7^3 \nmid (n+2)$, the final term in Lemma VI.2.1 (2) vanishes, giving

$$\mathrm{sc}_7(n) = \frac{1}{48}\left(7 + \left(\frac{\frac{D_n}{7^2}}{7}\right)\right) r_3\left(\frac{n+2}{7}\right).$$

The claim then immediately follows by plugging in (VI.2.3).
(3) Since $n \equiv 2 \pmod 4$, we have $4 \mid (n+2)$, and hence (VI.2.3) and Lemma VI.2.1 (1) imply the claim.
(4) Since $n \equiv -2 \pmod{7^2}$, $7^3 \mid D_n$, so $7 \mid \frac{D_n}{7^2}$. Hence Lemma VI.2.1 (1), (2) imply the claim. □

VI.2.4 Proof of Corollary VI.1.6

We next consider the special case that $n+2$ is squarefree and use Dirichlet's class number formula to obtain another formula for $\mathrm{sc}_7(n)$.

Proof of Corollary VI.1.6. Note that since $n+2$ is squarefree, either $-D_n$ is fundamental (for $n \not\equiv -2 \pmod 7$) or $-\frac{D_n}{7^2}$ is fundamental (for $n \equiv -2 \pmod 7$). Dirichlet's class number formula (see e.g. [16, Satz 3]) states that

$$H(|D|) = -\frac{1}{|D|}\sum_{m=1}^{|D|-1}\left(\frac{D}{m}\right)m. \tag{VI.2.4}$$

By Corollary VI.2.2 (1), (2) (the conditions given there are satisfied because $n+2$ is squarefree and thus neither $n \equiv 2 \pmod 4$ nor $n \equiv -2 \pmod{7^3}$), we have

$$\mathrm{sc}_7(n) = \nu_n \begin{cases} H(D_n) & \text{if } n \not\equiv -2 \pmod 7, \\ \left(7 + \left(\frac{\frac{D_n}{7^2}}{7}\right)\right) H\left(\frac{D_n}{7^2}\right) & \text{if } n \equiv -2 \pmod 7. \end{cases} \tag{VI.2.5}$$

Since $-D_n$ is fundamental in the first case and $-\frac{D_n}{7^2}$ is fundamental in the second case, we may plug in (VI.2.4) with $D = -D_n$ in the first case and $D = -\frac{D_n}{7^2}$ in the second case.

Thus for $n \not\equiv -2 \pmod 7$ we plug

$$H\left(D_n\right) = -\frac{1}{D_n} \sum_{m=1}^{D_n-1} \left(\frac{-D_n}{m}\right) m$$

into (VI.2.5), while for $n \equiv -2 \pmod 7$ we plug in

$$H\left(\frac{D_n}{7^2}\right) = -\frac{7^2}{D_n} \sum_{m=1}^{\frac{D_n}{7^2}-1} \left(\frac{-\frac{D_n}{7^2}}{m}\right) m.$$

This yields the claim. $\square$

VI.3 Proofs of Corollaries VI.1.5 and VI.1.7

This section relates $\mathrm{sc}_7(m)$ and $\mathrm{sc}_7(n)$ if $\frac{m+2}{n+2}$ is a square.

VI.3.1 A recursion for $\mathrm{sc}_7(n)$

In this subsection, we consider the case $\frac{m+2}{n+2} = 2^{2j}7^{2\ell}$.

Lemma VI.3.1. *Let $\ell \in \mathbb{N}_0$ and $n \in \mathbb{N}$.*

(1) *We have*

$$\mathrm{sc}_7\left((n+2)2^{2\ell} - 2\right) = \mathrm{sc}_7(n).$$

(2) *We have*

$$\mathrm{sc}_7\left((n+2)7^{2\ell} - 2\right) = 7^\ell \,\mathrm{sc}_7(n).$$

Proof. (1) Corollary VI.2.2 (3) gives inductively that for $0 \le j \le \ell$ we have

$$\mathrm{sc}_7\left((n+2)2^{2\ell} - 2\right) = \mathrm{sc}_7\left((n+2)2^{2(\ell-j)} - 2\right).$$

In particular, $j = \ell$ yields the claim.
(2) The claim is trivial if $\ell = 0$. For $\ell \ge 1$, Corollary VI.2.2 (4) inductively yields that for $0 \le j \le \ell$

$$\mathrm{sc}_7\left((n+2)7^{2\ell} - 2\right) = 7^j \,\mathrm{sc}_7\left((n+2)7^{2(\ell-j)} - 2\right).$$

The case $j = \ell$ is precisely the claim. $\square$

VI.3.2 Proof of Corollary VI.1.7

We are now ready to prove Corollary VI.1.7.

Proof of Corollary VI.1.7. We first use Lemma VI.3.1 (1), (2) to obtain that

$$\mathrm{sc}_7\left((n+2)2^{2\ell}f^2 7^{2r}-2\right)=7^r\,\mathrm{sc}_7\left((n+2)f^2-2\right). \tag{VI.3.1}$$

We split into the case $n\not\equiv -2\pmod 7$ (in which case $-D_n$ is fundamental) and $n\equiv -2\pmod 7$ (in which case $-\frac{D_n}{7^2}$ is fundamental).

First suppose that $n\not\equiv -2\pmod 7$. We use Corollary VI.2.2 (1) to obtain

$$\mathrm{sc}_7\left((n+2)f^2-2\right)=\nu_n H\left(D_n f^2\right)$$

We then plug in [4, p. 273] ($-D$ a fundamental discriminant)

$$H\left(Df^2\right)=H(D)\sum_{1\leq d|f}\mu(d)\left(\frac{-D}{d}\right)\sigma\left(\frac{f}{d}\right). \tag{VI.3.2}$$

Hence by Corollary VI.2.2 (1)

$$\mathrm{sc}_7\left((n+2)f^2-2\right)=\mathrm{sc}_7(n)\sum_{1\leq d|f}\mu(d)\left(\frac{-D_n}{d}\right)\sigma\left(\frac{f}{d}\right),$$

and plugging back into (VI.3.1) yields the corollary in that case.

We next suppose that $n\equiv -2\pmod 7$. First note that since $7\nmid f$ and $n+2$ is squarefree, $(n+2)f^2-2\not\equiv -2\pmod{7^3}$ and $n\not\equiv 2\pmod 4$. We plug in Corollary VI.2.2 (2), use (VI.3.2) (recall that $-\frac{D_n}{7^2}$ is fundamental), and note that $\left(\frac{\frac{D_nf^2}{7^2}}{7}\right)=\left(\frac{\frac{D_n}{7^2}}{7}\right)$ to obtain that

$$\mathrm{sc}_7\left((n+2)f^2-2\right)=\left(7+\left(\frac{\frac{D_n}{7^2}}{7}\right)\right)\nu_n H\left(\frac{D_n}{7^2}\right)\sum_{1\leq d|f}\mu(d)\left(\frac{-\frac{D_n}{7^2}}{d}\right)\sigma\left(\frac{f}{d}\right).$$

We then use Corollary VI.2.2 (2) again and plug back into (VI.3.1) to conclude that

$$\mathrm{sc}_7\left((n+2)2^{2\ell}f^2 7^{2r}-2\right)=7^r\,\mathrm{sc}_7(n)\sum_{1\leq d|f}\mu(d)\left(\frac{-\frac{D_n}{7^2}}{d}\right)\sigma\left(\frac{f}{d}\right).$$

Since $7\nmid f$, we have $\left(\frac{-\frac{D_n}{7^2}}{d}\right)=\left(\frac{-D_n}{d}\right)$ for $d\mid f$. Therefore the corollary follows. $\square$

VI.3.3 Proof of Corollary VI.1.5

We next rewrite Corollary VI.2.2 (2) in order to uniformly package Corollary VI.2.2 (1), (2), and (3). We first require a lemma relating the 7-primitive class numbers H_7 and the Hurwitz class numbers.

Lemma VI.3.2. *For a discriminant $-D$, we have*

$$H_7(D) = H(D) - H\left(\frac{D}{7^2}\right).$$

Proof. To rewrite the right-hand side, we write $D = \Delta 7^{2\ell} f^2$ with $7 \nmid f$ and $-\Delta$ fundamental discriminant and then plug in the well-known identity

$$H(D) = \sum_{d^2 \mid D} \frac{h\left(-\frac{D}{d^2}\right)}{\omega_{-\frac{D}{d^2}}},$$

where as usual $h(-\frac{D}{d^2})$ counts the number of classes of primitive quadratic forms $[a,b,c]$ with discriminant $-\frac{D}{d^2}$ and $\gcd(a,b,c)=1$. This yields

$$\begin{aligned} H(D) - H\left(\frac{D}{7^2}\right) &= \sum_{d \mid 7^\ell f} \frac{h\left(-\frac{D}{d^2}\right)}{\omega_{-\frac{D}{d^2}}} - \sum_{d \mid 7^{\ell-1} f} \frac{h\left(-\frac{D}{7^2 d^2}\right)}{\omega_{-\frac{D}{7^2 d^2}}} = \sum_{d \mid 7^\ell f} \frac{h\left(-\frac{D}{d^2}\right)}{\omega_{-\frac{D}{d^2}}} - \sum_{\substack{d \mid 7^\ell f \\ 7 \mid d}} \frac{h\left(-\frac{D}{d^2}\right)}{\omega_{-\frac{D}{d^2}}} \\ &= \sum_{\substack{d \mid 7^\ell f \\ 7 \nmid d}} \frac{h\left(-\frac{D}{d^2}\right)}{\omega_{-\frac{D}{d^2}}}. \end{aligned} \tag{VI.3.3}$$

The claim of the lemma is thus equivalent to showing that the right-hand side of (VI.3.3) equals $H_7(D)$. Multiplying each form counted by $h(-\frac{D}{d^2})$ by d, we see that (VI.3.3) precisely counts those quadratic forms $[a,b,c]$ of discriminant $-D$ with $7 \nmid \gcd(a,b,c)$, weighted in the usual way. $\square$

To finish the proof of Corollary VI.1.5, for a fundamental discriminant $-\Delta$, we also require the evaluation of

$$C_{r,\Delta} := \sum_{d \mid 7^r} \mu(d) \left(\frac{-\Delta}{d}\right) \sigma\left(\frac{7^r}{d}\right) - \sum_{d \mid 7^{r-1}} \mu(d) \left(\frac{-\Delta}{d}\right) \sigma\left(\frac{7^{r-1}}{d}\right).$$

A straightforward calculation gives the following lemma.

Lemma VI.3.3. *For $r \in \mathbb{N}$ we have*

$$C_{r,\Delta} = 7^{r-1}\left(7 + \left(\frac{\Delta}{7}\right)\right).$$

We are now ready to prove Corollary VI.1.5.

Proof of Corollary VI.1.5. We first consider the case that $n \not\equiv 2 \pmod 4$. If $n \not\equiv -2 \pmod 7$, then Corollary VI.1.5 follows directly from Corollary VI.2.2 (1) and Lemma VI.3.2.

For $n \equiv -2 \pmod 7$, we choose $r_n \in \mathbb{N}_0$ maximally such that $n \equiv -2 \pmod{7^{2r_n+1}}$ and proceed by induction on r_n. For $r_n = 0$ we have $D_n = \Delta_n f^2 7^2$ with $-\Delta_n$ a fundamental discriminant and $7 \nmid f$. Since $7 \nmid f$, we have

$$\left(\frac{-\Delta_n f^2}{7}\right) = \left(\frac{-\Delta_n}{7}\right),$$

and hence combining Corollary VI.2.2 (2), (VI.3.2), and Lemma VI.3.3 gives

$$\mathrm{sc}_7(n) = \nu_n H(\Delta_n)\left(\sum_{d|7}\mu(d)\left(\frac{-\Delta_n}{d}\right)\sigma\left(\frac{7}{d}\right) - 1\right)\sum_{d|f}\mu(d)\left(\frac{-\Delta_n}{d}\right)\sigma\left(\frac{f}{d}\right).$$

Noting that $7 \nmid f$ and

$$\sum_{d|f}\mu(d)\left(\frac{-\Delta_n}{d}\right)\sigma\left(\frac{f}{d}\right) \tag{VI.3.4}$$

is multiplicative, we obtain

$$\mathrm{sc}_7(n) = \nu_n H(\Delta_n)\left(\sum_{d|7f}\mu(d)\left(\frac{-\Delta_n}{d}\right)\sigma\left(\frac{7f}{d}\right) - \sum_{d|f}\mu(d)\left(\frac{-\Delta_n}{d}\right)\sigma\left(\frac{f}{d}\right)\right).$$

We then apply (VI.3.2) again and use Lemma VI.3.2 to obtain Corollary VI.1.5 in this case. This completes the base case $r_n = 0$ of the induction.

Let $r \geq 1$ be given and assume the inductive hypothesis that that Corollary VI.1.5 holds for all n with $r_n < r$. We then let n be arbitrary with $r_n = r$ and show that Corollary VI.1.5 holds for n. By Corollary VI.2.2 (4), we have

$$\mathrm{sc}_7(n) = 7\,\mathrm{sc}_7\left(\frac{n+2}{7^2} - 2\right). \tag{VI.3.5}$$

By the maximality of r_n, $7^{2r-1} \mid \frac{n+2}{7^2}$ but $7^{2r+1} \nmid \frac{n+2}{7^2}$, so $r_{\frac{n+2}{7^2}-2} = r-1 < r$ and hence by induction we may plug Corollary VI.1.5 into the right-hand side of (VI.3.5) to obtain

$$\mathrm{sc}_7(n) = 7\nu_{\frac{n+2}{7^2}-2} H_7\left(D_{\frac{n+2}{7^2}-2}\right). \tag{VI.3.6}$$

A straightforward calculation shows that

$$\nu_{\frac{n+2}{7^2}-2} = \nu_n \quad \text{and} \quad D_{\frac{n+2}{7^2}-2} = \frac{D_n}{7^2}$$

and hence (VI.3.6) implies that

$$\mathrm{sc}_7(n) = 7\nu_n H_7\left(\frac{D_n}{7^2}\right).$$

Hence Corollary VI.1.5 in this case is equivalent to showing that

$$H_7\left(D_n\right) = 7H_7\left(\frac{D_n}{7^2}\right). \tag{VI.3.7}$$

Plugging Lemma VI.3.2 and then (VI.3.2) into both sides of (VI.3.7), cancelling $H(\Delta_n)$, and again using the multiplicativity of (VI.3.4), one obtains that (VI.3.7) is equivalent to $C_{r+1,\Delta_n} = 7C_{r,\Delta_n}$. Since $r \geq 1$, we have $r+1 \geq 2$, and Lemma VI.3.3 implies that $C_{r+1,\Delta_n} = 7C_{r,\Delta_n}$, yielding Corollary VI.1.5 for all $n \not\equiv 2 \pmod 4$.

We finally consider the case $n \equiv 2 \pmod 4$. We choose ℓ maximally such that $n \equiv -2 \pmod{2^{2\ell}}$. Lemma VI.3.1 (1) implies that

$$\mathrm{sc}_7(n) = \mathrm{sc}_7\left(\left(\frac{n+2}{2^{2\ell}} - 2 + 2\right)2^{2\ell} - 2\right) = \mathrm{sc}_7\left(\frac{n+2}{2^{2\ell}} - 2\right).$$

The choice of ℓ implies that $\frac{n+2}{2^{2\ell}} - 2 \not\equiv 2 \pmod 4$. We may therefore plug in Corollary VI.1.5 and the definitions (VI.1.2) and (VI.1.3) to conclude that

$$\mathrm{sc}_7\left(\frac{n+2}{2^{2\ell}} - 2\right) = \nu_{\frac{n+2}{2^{2\ell}}-2} H_7\left(D_{\frac{n+2}{2^{2\ell}}-2}\right) = \nu_n H_7\left(D_n\right). \qquad \square$$

VI.4 A combinatorial explanation of Corollary VI.1.5

Here we provide a combinatorial explanation for Corollary VI.1.5. We use the theory of abaci, following the construction in [15].

VI.4.1 Abaci, extended t-residue diagrams, and self-conjugate t-cores

Given a partition $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ of a positive integer n and a positive integer t, we next describe the *t-abacus* associated to Λ. This consists of s beads on t rods constructed in the following way [15]. For every $1 \leq j \leq s$ define *structure numbers* by

$$B_j := \lambda_j - j + s.$$

For each B_j there are unique integers (r_j, c_j) such that

$$B_j = t(r_j - 1) + c_j,$$

and $0 \leq c_j < t - 1$. The *abacus* for the partition Λ is then formed by placing one bead for each B_j in row r_j and column c_j. The *extended t-residue diagram* associated to a t-core partition Λ is constructed as follows (see [6, page 3]). Label a cell in the j-th row and k-th column of the Ferrers–Young diagram of Λ by $k - j \pmod t$. We also label the cells in column 0 in the same way. A cell is called *exposed* if it is at the end of a row. The *region r* of the extended t-residue diagram of Λ is the set of cells (j, k) satisfying $t(r-1) \leq k - j < tr$. Then we define n_j to be the maximum region of Λ which contains an exposed cell labeled j. As noted in [6], this is well-defined since column 0 contains infinitely many exposed cells.

Example VI.4.1. *Let $t = 4$ and construct the abacus and 4-residue diagram for the partition $\Lambda = (3, 2, 1)$. We begin with the abacus, computing the structure numbers $B_1 = 5$, $B_2 = 3$, and $B_3 = 1$. Then diagrammatically the abacus is*

$$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 1 & & B_3 & & B_2 \\ 2 & & B_1 & & \end{array}$$

The extended 4-residue diagram of the partition is

$$\begin{array}{ccccc} & 0 & 1 & 2 & 3 \\ 1 & {}_3 & \bullet_0 & \bullet_1 & \bullet_2 \\ 2 & {}_2 & \bullet_3 & \bullet_0 & \\ 3 & {}_1 & \bullet_2 & & \end{array}$$

Then the exposed cells in this diagram are $(1, 3)$, $(2, 2)$, and $(3, 1)$. One may then determine the region of these cells in the prescribed fashion. For example, the exposed cell $(1, 3)$ labeled by 2 belongs to the region 1, and hence $n_2 = 1$.

Using this construction, [15, Theorem 4] reads as follows.

Theorem VI.4.2. *Let A be an abacus for a partition Λ, and let m_j denote the number of beads in column j. Then Λ is a t-core partition if and only if the m_j beads in column j are the beads in positions $(1, j), (2, j), \ldots, (m_j, j)$.*

Furthermore, using extended t-residue diagrams, the authors of [6] showed the following result.

Lemma VI.4.3 (Bijection 2 of [6])**.** *Let $P_t(n)$ be the set of t-core partitions of n. There is a bijection $P_t(n) \to \{N := [n_0, \ldots, n_{t-1}] \colon n_j \in \mathbb{Z}, n_0 + \cdots + n_{t-1} = 0\}$ such that*

$$|\Lambda| = \frac{t|N|^2}{2} + B \cdot N, \qquad B := [0, 1, \ldots, t-1].$$

When computing the norm and dot-product, we consider N, B as elements in $\mathbb{Z}^t$.

We call N the *list associated* to the t-residue diagram. We now show a relationship between abaci and lists of a partition.

Proposition VI.4.4. *Let $N = [n_0, \ldots, n_{t-1}]$ be the list associated to the extended t-residue diagram of a t-core partition Λ. Let $\ell + s = \alpha_\ell t + \beta_\ell$ with $0 \leq \beta_\ell \leq t-1$. Then N also uniquely represents the abacus $(\ldots, n_{t-1} + \alpha_{t-1}, n_0 + \alpha_0, n_1 + \alpha_1, \ldots)$, where $n_\ell + \alpha_\ell$ occurs in position β_ℓ of the abacus.*

Proof. The largest part λ_1 corresponds to the maximum region of the t-residue diagram, and also the lowest right-hand bead on the abacus. Let $m_1 := \max\{n_0, \ldots, n_{t-1}\}$ be achieved at n_{ℓ_1}. Then $\lambda_1 = t(m_1 - 1) + \ell_1 + 1$. For the abacus, we correspondingly find $B_1 = \lambda_1 - 1 + s = t(m_1 - 1) + \ell_1 + s = t(m_1 + \alpha_1 - 1) + \beta_1$, where $\ell_1 + s = \alpha_1 t + \beta_1$ with $0 \leq \beta_1 \leq t - 1$. Hence we place a bead in the abacus at the slot $(m_1 + \alpha_1, \beta_1)$. Since this is a t-core partition, we also know that there are beads in all places above this slot. These beads correspond to other parts in the partition whose labels of exposed cells in the t-residue diagram are ℓ_1 but where the exposed cells themselves lie in a lower region. Thus the β_1-th entry in the abacus takes value $m_1 + \alpha_1$.

Then removing the element n_{ℓ_1} from the list we are left with $[n_0, \ldots, n_{\ell_1 - 1}, n_{\ell_1 + 1}, \ldots, n_{t-1}]$. We use the same technique as before, identifying $m_2 := \max\{n_0, \ldots, n_{\ell_1 - 1}, n_{\ell_1 + 1}, \ldots, n_{t-1}\}$, achieved at n_{ℓ_2}. We have $k - j \equiv \ell_2 \pmod{t}$ such that $t(m_2 - 1) \leq k - j < tm_2$, meaning that $\lambda_j = k = t(m_2 - 1) + \ell_2 + j$. Plugging this in to the formula for the structure numbers we find that $B_j = t(m_2 - 1) + \ell_2 + s = t(m_2 + \alpha_2 - s) + \beta_2$, where $\ell_2 = \alpha_2 t + \beta_2$ with $0 \leq \beta < t$. Hence we place a bead in the abacus in the slot $(m_2 + \alpha_2, \beta_2)$ and all other slots vertically above this, and so the β_2-nd entry in the abacus list is given by $m_2 + \alpha_2$. This process continues for each entry of the list.

If this process gives a non-positive value for the slots of the abacus in which beads are to be placed, we define the value in that column of the abacus list to be 0 (it is seen that these values arise from the exposed cells in column 0 of the extended t-residue diagram and hence are not a part of the partition). It is clear that the β_ℓ run through exactly a complete set of residues modulo t, and hence each column in the abacus is represented exactly once. It is easily seen that this process defines a unique abacus for each list N (up to equivalency by Lemma VI.4.6). The converse is also seen to hold. □

Remark 12. If the resulting abacus A that appears under an application of Proposition VI.4.4 has a non-zero first column, we may use Lemma VI.4.6 to rewrite A as an equivalent abacus with a 0 in the first place.

We use Proposition VI.4.4 to restrict the possible shapes of abaci associated to self-conjugate t-core partitions.

Lemma VI.4.5. *With the notation defined as in Proposition VI.4.4, an abacus is self-conjugate if and only if it is of the form* $(\dots,-n_1+\alpha_1,-n_0+\alpha_0,n_0+\alpha_0,n_1+\alpha_1,\dots)$.

Proof. The proof of [6, Bijection 2] implies that the elements in the list $[n_0,\dots,n_{t-1}]$ associated to a self-conjugate partition satisfy the relations $n_\ell=-n_{t-\ell-1}$ for every $0\le\ell\le t-1$. Combining this with Proposition VI.4.4 immediately yields the claim. □

VI.4.2 Self-conjugate 7-cores

We now restrict our attention to abaci of self-conjugate 7-cores. We require [15, Lemma 1], which allows us to form a system of canonical representatives for abaci associated to 7-core partitions.

Lemma VI.4.6. *The two abaci* $A_1=(m_0,m_1,\dots,m_6)$ *and* $A_2=(m_6+1,m_0,\dots,m_5)$ *represent the same* 7*-core partition.*

Thus every 7-core partition may be represented by an abacus of the form $(0,a,b,c,d,e,f)$. Then in a similar fashion to Ono and Sze, we find that there is a one-to-one correspondence

$$(0,a,b,c,d,e,f)\leftrightarrow\{\text{all 7-core partitions}\},$$

where a,b,c,d,e, and f are non-negative integers. We thus assume that the first column in each abacus has no beads. We next use Lemma VI.4.5 to considerably reduce the number of abaci we need to consider.

Lemma VI.4.7. *Assume that* $A=(0,a,b,c,d,e,f)$ *is an abacus for a self-conjugate* 7*-core partition and recall that* $s=a+b+c+d+e+f$*. Let* $s\not\equiv 4 \pmod 7$ *and* $r\in\mathbb{N}_0$*.*

(1) *Assume that* $s = 7r$. *Then* $f = 2r$, $a + e = 2r$, $b + d = 2r$, $c = r$.

(2) *Assume that* $s = 7r + 1$. *Then* $a = 2r + 1$, $b + f = 2r$, $c + e = 2r$, $d = r$.

(3) *Assume that* $s = 7r + 2$. *Then* $b + a = 2r + 1$, $c = 2r + 1$, $d + f = 2r$, $e = r$.

(4) *Assume that* $s = 7r + 3$. *Then* $b + c = 2r + 1$, $a + d = 2r + 1$, $e = 2r + 1$, $f = r$.

(5) *Assume that* $s = 7r + 5$. *Then* $d + e = 2r + 1$, $c + f = 2r + 1$, $b = 2r + 2$, $a = r + 1$.

(6) *Assume that* $s = 7r + 6$. *Then* $e + f = 2r + 1$, $d = 2r + 2$, $a + c = 2r + 2$, $b = r + 1$.

Proof. We prove (1). By Proposition VI.4.4 we see that A corresponds to the list $[-r, a - r, b - r, c - r, d - r, e - r, f - r]$. Using Lemma VI.4.5 and the fact that $s = 7r$, the conditions are easy to determine. The other cases follow in the same way. $\square$

Remarks.

1. It is clear how a similar result to Lemma VI.4.7 may be obtained for all self-conjugate t-cores.

2. The lack of the case $s \equiv 4 \pmod 7$ in Lemma VI.4.7 follows from the fact that there are no self-conjugate $2t-1$-core partitions with $s \equiv t \pmod{(2t-1)}$, which may be seen by inspecting the upper-left cell in the Ferrers–Young diagram of such a partition.

Lemma VI.4.7 shows that the abaci of self-conjugate 7-core partitions naturally fall into one of the distinct families given in Table VI.1, enumerated with parameters $a, b, r \in \mathbb{N}_0$.

Type of Partition	Shape of Abaci
I	$(0, a, b, r, 2r - b, 2r - a, 2r)$
II	$(0, 2r + 1, a, b, r, 2r - b, 2r - a)$
III	$(0, a, 2r + 1 - a, 2r + 1, b, r, 2r - b)$
IV	$(0, a, b, 2r + 1 - b, 2r + 1 - a, 2r + 1, r)$
V	$(0, r + 1, 2r + 2, a, b, 2r + 1 - b, 2r + 1 - a)$
VI	$(0, a, r + 1, 2r + 2 - a, 2r + 2, b, 2r + 1 - b)$

Table VI.1: The different types of abaci for self-conjugate 7-core partitions.

We relate the families of partitions to quadratic forms, with the relationship shown in the following proposition. For brevity, we write only triples without $\pm$ signs - it is clear that changing the sign on any entry preserves the result.

Proposition VI.4.8. *Let $n \in \mathbb{N}$ and $a, b, r \in \mathbb{N}_0$ be given.*

(1) *The Type I partition with parameters a, b, and r is a partition of n if and only if*

$$7n + 14 = (7r+3)^2 + (7r+2-7a)^2 + (7r+1-7b)^2.$$

(2) *The Type II partition with parameters a, b, and r is a partition of n if and only if*

$$7n + 14 = (7r+4)^2 + (7r+2-7a)^2 + (7r+1-7b)^2.$$

(3) *The Type III partition with parameters a, b, and r is a partition of n if and only if*

$$7n + 14 = (7r+5)^2 + (7r+4-7a)^2 + (7r+1-7b)^2.$$

(4) *The Type IV partition with parameters a, b, and r is a partition of n if and only if*

$$7n + 14 = (7r+6)^2 + (7r+5-7a)^2 + (7r+4-7b)^2.$$

(5) *The Type V partition with parameters a, b, and r is a partition of n if and only if*

$$7n + 14 = (7r+8)^2 + (7r+5-7a)^2 + (7r+4-7b)^2.$$

(6) *The Type VI partition with parameters a, b, and r is a partition of n if and only if*

$$7n + 14 = (7r+9)^2 + (7r+8-7a)^2 + (7r+4-7b)^2.$$

Proof. We only prove (1). Combining the definition with Proposition VI.4.4, the Type I partition Λ with parameters a, b, and r has the associated list $[-r, a-r, b-r, 0, r-b, r-a, r]$. By Lemma VI.4.3, we thus have

$$n = |\Lambda| = 7\left(r^2 + (a-r)^2 + (b-r)^2\right) + (a-r) + 2(b-r) + 4(r-b) + 5(r-a).$$

Hence we see that

$$\begin{aligned} &7n + 14 \\ =&49\left(r^2 + (a-r)^2 + (b-r)^2\right) + 7\left(a - r + 2(b-r) + 4(r-b) + 5(r-a) + 6r\right) + 14 \\ =&147r^2 + 49a^2 + 49b^2 + 84r - 98ar - 98br - 28a - 14b + 14. \end{aligned}$$

This is exactly the expansion of

$$(7r+3)^2 + (7r+2-7a)^2 + (7r+1-7b)^2.$$

The other cases follow in the same way, using the associated lists in Table VI.2. $\square$

Type of Partition	Shape of Associated list
I	$[-r, a-r, b-r, 0, r-b, r-a, r]$
II	$[r+1, a-r, b-r, 0, r-b, r-a, -r-1]$
III	$[a-r, r+1, b-r, 0, r-b, -r-1, r-a]$
IV	$[a-r, b-r, r+1, 0, -r-1, r-b, r-a]$
V	$[a-r, b-r, -r-1, 0, r+1, r-b, r-a]$
VI	$[b-r, -r-1, a-r-1, 0, r+1-a, r+1, r-b]$

Table VI.2: The different types of associated lists for self-conjugate 7-core partitions.

Proposition VI.4.8 shows that for each self-conjugate 7-core of n there is a representation of $7n+14 = x^2+y^2+z^2$ as the sum of three squares with none of x, y, z divisible by 7. Define

$$J(7n+14) \coloneqq \{(x,y,z) \in \mathbb{Z}^3 \colon x^2+y^2+z^2 = 7n+14, \text{ and } x,y,z \not\equiv 0 \pmod 7\}.$$

Let $K(7n+14) \coloneqq J(7n+14)/\sim$ where $(x,y,z) \sim (x',y',z')$ if (x',y',z') is any permutation of the triple (x,y,z), including minus signs i.e., $(-x,y,z) \sim (x,y,z)$. Then it is easy to see that we obtain the following corollary.

Corollary VI.4.9. *There is an isomorphism between self-conjugate* 7*-core partitions and* $K(7n+14)$.

Remark 13. Corollary VI.4.9 gives a combinatorial explanation for Lemma VI.2.1 (1). We then obtain an explanation of Corollary VI.1.5 via the following exposition, using Gauss' bijective map from solutions of the equation $x^2+y^2+z^2 = 7n+14$ to primitive binary quadratic forms in certain class groups.

We elucidate the case $n \equiv 0, 1 \pmod 4$ (a similar story holds for $n \equiv 3 \pmod 8$). By Gauss's [7, article 278], for each representation of $7n+14$ as the sum of three squares there corresponds a primitive binary quadratic form of discriminant $-28n-56$. This correspondence is invariant under a pair of simultaneous sign changes on the triple (x,y,z). Explicitly, the correspondence is given by the following. For $(x,y,z) \in J(7n+14)$ let $(m_0, m_1, m_2, n_0, n_1, n_2)$ be an integral solution to

$$x = m_1 n_2 - m_2 n_1, \qquad y = m_2 n_0 - m_0 n_2 \qquad z = m_0 n_1 - m_1 n_0,$$

where a solution is guaranteed by Gauss's [7, article 279]. Then

$$(m_0 u + n_0 v)^2 + (m_1 u + n_1 v)^2 + (m_2 u + n_2 v)^2$$

is a form in $\mathrm{CL}(-28n-56)$. Further, this map is independent of $(m_0, m_1, m_2, n_0, n_1, n_2)$. Hence, similarly to [15], we find a map ϕ taking self-conjugate 7-cores to binary quadratic forms of discriminant $-28n-56$ given by

$$\phi\colon \Lambda \to A \to N \to (x,y,z) \to (m_0, m_1, m_2, n_0, n_1, n_2) \to \text{binary quadratic form.} \tag{VI.4.1}$$

We are now in a position to prove Theorem VI.1.8.

Proof of Theorem VI.1.8. We first assume that $n \equiv 0,1 \pmod 4$. It is well-known (see e.g. [7, article 291]) that we have $|\mathrm{CL}(-28n-56)| = 2^{r-1}k$, where k is the number of classes per genus, and 2^{r-1} is the number of genera in $\mathrm{CL}(-28n-56)$. Fix $f_1, \ldots, f_k$ to be representatives of the k classes of the unique genus of $\mathrm{CL}(-28n-56)$ that ϕ maps onto. As in [15] we say that (x,y,z) and f_j are *represented* by $(m_0, m_1, m_2, n_0, n_1, n_2)$ if

$$x = m_1n_2 - m_2n_1, \qquad y = m_2n_0 - m_0n_2, \qquad z = m_0n_1 - m_1n_0,$$

and

$$(m_0u + n_0v)^2 + (m_1u + n_1v)^2 + (m_2u + n_2v)^2 = f_j,$$

respectively. Let $\mathfrak{M}$ denote the set of all tuples $(m_0, m_1, m_2, n_0, n_1, n_2)$ that represent some pair (x,y,z) and f_j. By Gauss's [7, article 291], we have $|\mathfrak{M}| = 3\cdot 2^{r+3}k$, and each f_j is representable by $3 \cdot 2^{r+3}$ elements in $\mathfrak{M}$. It is clear that all representatives f_j have $(x,y,z) \in J(7n+14)$. Note that the elements $(m_0, m_1, m_2, n_0, n_1, n_2)$ and $(-m_0, -m_1, -m_2, -n_0, -n_1, -n_2)$ both map to the same form in $K(7n+14)$, and there are no other such relations. Since each class in $K(7n+14)$ corresponds to $8\cdot 6$ different triples, we see that each element in $K(7n+14)$ has $\frac{3\cdot 2^{r+3}}{8\cdot 6\cdot 2} = 2^{r-2}$ different preimages. Hence the set of self-conjugate 7-cores is a 2^{r-2}-fold cover of this genus. To see that the genus is non-principal, we note as in [15, Remark 3 ii)] that to be in the principal genus, one of x, y, z would need to vanish. However, this is guaranteed to not happen by the congruence conditions on elements in $K(7n+14)$. The case where $n \equiv 3 \pmod 8$ is similar.

Finally, for $n \equiv 2 \pmod 4$, one uses the simple fact that if the sum of three squares is congruent to 0 modulo 4, then all squares must be even. Iterating this eventually reduces it to one of the cases covered above or the $n \equiv 7 \pmod 8$ case. □

VI.5 Other t and Conjecture VI.1.9

In this section we consider other values of t, proving Conjecture VI.1.9 in the cases $t \in \{2,3,5\}$ and offering partial results if $t > 5$.

VI.5.1 The cases $t \in \{2, 3\}$

With $\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n)$ the usual *Dedekind eta-function*, [15, (3)] and [1, Theorem 13] give the generating functions of $\mathrm{c}_2(n)$ and $\mathrm{sc}_3(n)$ as

$$\sum_{n\geq 1} \mathrm{c}_2(n)q^n = q^{-\frac{1}{8}}\frac{\eta(2\tau)^2}{\eta(\tau)}, \qquad \sum_{n\geq 1} \mathrm{sc}_3(n)q^n = q^{-\frac{1}{3}}\frac{\eta(2\tau)^2\eta(3\tau)\eta(12\tau)}{\eta(\tau)\eta(4\tau)\eta(6\tau)}.$$

These are modular forms of weight $\frac{1}{2}$ and levels 2 and 12, respectively. It is a classical fact that each is a lacunary series, i.e., that the asymptotic density of its non-zero coefficients is zero (for example, see the discussion after [8, (2)]). We immediately see that

$$\mathrm{c}_2(n) = \begin{cases} 1 & \text{if } n = \frac{j(j+1)}{2} \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, [6, (7.4)] stated that

$$\mathrm{sc}_3(n) = \begin{cases} 1 & \text{if } n = j(3j \pm 2) \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

From these, we immediately obtain the following corollary.

Corollary VI.5.1. *For any $n \in \mathbb{N}$ that is both a triangular number and satisfies $n = j(3j \pm 2)$ for some $j \in \mathbb{Z}$ we have that $\mathrm{sc}_3(n) = \mathrm{c}_2(n) = 1$.*

Clearly, there are progressions on which both $\mathrm{sc}_3(n)$ and $\mathrm{c}_2(n)$ trivially vanish. For example, we have $\mathrm{sc}_3(4n+3) = \mathrm{c}_2(3n+2) = 0$. For $t = 3$ we simply observe the following corollary.

Corollary VI.5.2. *There are no arithmetic progressions on which c_3 and sc_5 are integer multiples of one-another, even asymptotically.*

Proof. Comparing the explicit descriptions for $\mathrm{c}_3(n)$ and $\mathrm{sc}_5(n)$ given in [10, Theorem 6] and [6, Theorem 7] respectively immediately yields the claim. □

VI.5.2 The case $t = 5$

In [6, Theorem 4], Garvan, Kim, and Stanton proved that

$$\mathrm{c}_5(n) = \sigma_5(n+1),$$

where $\sigma_5(n) := \sum_{d|n} (\frac{d}{5}) \frac{n}{d}$ denotes the usual twisted divisor sum. Furthermore, Alpoge provided an exact formula for $\mathrm{sc}_9(n)$ in [1, Theorem 10]:

$27\,\mathrm{sc}_9(n)$

$$= \begin{cases} \sigma(3n+10) + a_{3n+10}(36a) - a_{3n+10}(54a) - a_{3n+10}(108a) & \text{if } n \equiv 1,3 \pmod 4, \\ \sigma(3n+10) + a_{3n+10}(36a) - 3a_{3n+10}(54a) - a_{3n+10}(108a) & \text{if } n \equiv 0 \pmod 4, \\ \sigma(k) + a_{3n+10}(36a) - 3a_{3n+10}(54a) - a_{3n+10}(108a) & \text{if } n \equiv 2 \pmod 4, \end{cases}$$

where k is odd and is defined by $3n+10 = 2^e k$ where $e \in \mathbb{N}_0$ is maximal such that $2^e \mid (3n+10)$. Here, the $a_n(E)$ are the coefficients appearing in the Dirichlet series for the L-function of the elliptic curve E. The curve $36a$ is $y^2 = x^3+1$, the curve $54a$ is $y^2+xy = x^3-x^2+12x+8$, and the curve $108a$ is $y^2 = x^3+4$.

Proposition VI.5.3. *There are no arithmetic progressions on which* $27\,\mathrm{sc}_9(n)$ *and* $\mathrm{c}_5(n)$ *are asymptotically equal up to an integral multiplicative factor.*

Proof. Applying the Hasse–Weil bound for counting points on elliptic curves as in [1, (13)] and letting $n \to \infty$ we have, for $n \not\equiv 2 \pmod 4$, that

$$\frac{27\,\mathrm{sc}_9(n)}{\mathrm{c}_5(3n+9)} \sim \frac{\sigma(3n+10)}{\sigma_5(3n+10)},$$

and for $n \equiv 2 \pmod 4$

$$\frac{27\,\mathrm{sc}_9(n)}{\mathrm{c}_5(n)} \sim \frac{\sigma(k)}{\sigma_5(n+1)}.$$

It is then enough to show that σ_5 is never constant along arithmetic progressions, i.e., the limit is not constant. To see this, consider an arithmetic progression $n \equiv m \pmod M$. Let ℓ be any prime which does not divide $(3m+10)M$ and for which $\left(\frac{\ell}{5}\right) = -1$. For each prime $p \neq \ell$ that lies in the congruence class of the inverse of $\ell \pmod{3M}$ and is relatively prime to $5(3m+10)$ we may construct $n(p) = n_\ell(p)$ such that

$$3n(p) + 10 = (3m+10)p\ell.$$

Note that $3n(p)+10$ lies in the arithmetic progression. A straightforward calculation shows that if the limit exists, then

$$\lim_{p\to\infty} \frac{\sigma(3n(p)+10)}{\sigma_5(3n(p)+10)} = \pm \frac{1+\ell}{1-\ell} \frac{\sum_{d\mid(3m+10)} d}{\sum_{d\mid(3m+10)} \left(\frac{d}{5}\right) d}.$$

Since ℓ is arbitrary and there are infinitely many choices of ℓ by Dirchlet's primes in arithmetic progressions theorem, this is a contradiction. □

VI.5.3 The case $t \geq 6$

Anderson showed in [2, Theorem 2] that, for $t \geq 6$ and $n \to \infty$,

$$\mathrm{c}_t(n) = \frac{(2\pi)^{\frac{t-1}{2}} A_t(n)}{t^{\frac{t}{2}} \Gamma\left(\frac{t-1}{2}\right)} \left(n + \frac{t^2-1}{24}\right)^{\frac{t-3}{2}} + O\left(n^{\frac{t-1}{2}}\right), \tag{VI.5.1}$$

where

$$A_t(n) := \sum_{\substack{k \geq 1 \\ \gcd(t,k)=1}} k^{\frac{1-t}{2}} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} e\left(-\frac{hn}{k}\right) \psi_{h,k}$$

for a certain $24k$-th root of unity $\psi_{h,k}$ independent of n. As Anderson remarked, it is possible to show that $0.05 < A_t(n) < 2.62$, although A_t varies depending on both t and n.

In a similar vein, Alpoge showed in [1, Theorem 3] that, for $r \geq 10$ odd and $n \to \infty$, we have

$$\mathrm{sc}_r(n) = \frac{(2\pi)^{\frac{r-1}{4}} \mathcal{A}_r(n)}{(2r)^{\frac{r-1}{4}} \Gamma\left(\frac{r-1}{4}\right)} \left(n + \frac{r^2-1}{24}\right)^{\frac{r-1}{4}-1} + O_r\left(n^{\frac{r-1}{8}}\right), \tag{VI.5.2}$$

where

$$\mathcal{A}_r(n) := \sum_{\substack{\gcd(k,r)=1 \\ k \not\equiv 2 \pmod 4}} (2,k)^{\frac{r-1}{2}} k^{\frac{1-r}{4}} \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} e\left(-\frac{hn}{k}\right) \chi_{h,k}$$

with $\chi_{h,k}$ a particular 24-th root of unity independent of n. Moreover, [1, (86) and (87)] imply that $0.14 < \mathcal{A}_r(n) < 1.86$.

Remark 14. Inspecting the asymptotic behaviours given in (VI.5.1) and (VI.5.2), it is clear that the only possibility of arithmetic progressions where the two asymptotics of $\mathrm{c}_t(n)$ and $\mathrm{sc}_r(n)$ are integer multiples of one another is $r = 2t-1$.

The following lemma provides partial results on Conjecture VI.1.9.

Lemma VI.5.4. *For $t \geq 6$ and $t \not\equiv 1 \pmod 6$ there are no arithmetic progressions on which $\mathrm{c}_t(n)$ and $\mathrm{sc}_{2t-1}(n)$ are integer multiples of one another.*

Proof. Using equations (VI.5.1) and (VI.5.2) we find that, for $M_1, M_2, m_1, m_2 \in \mathbb{N}$,

$$\frac{\mathrm{c}_t(M_1 n + m_1)}{\mathrm{sc}_{2t-1}(M_2 n + m_2)} \sim \frac{(4t-2)^{\frac{t-1}{2}} A_t(M_1 n + m_1)}{4^{\frac{t-3}{2}} t^{\frac{t}{2}} \mathcal{A}_{2t-1}(M_2 n + m_2)} \frac{\left(24(M_1 n + m_1) + t^2 - 1\right)^{\frac{t-3}{2}}}{(6(M_2 n + m_2) + t^2 - t)^{\frac{t-3}{2}}},$$

as $n \to \infty$. Furthermore, for the two growing powers of n to be equal and cancel on arithmetic progressions, it is not difficult to see that we must also have that $t \equiv 1 \pmod 6$. $\square$

To prove Conjecture VI.1.9 it remains to consider the cases where $t \equiv 1 \pmod 6$. We easily find that for the powers of n to be equal we must have

$$M_2 = 4M_1, \qquad m_2 = 4m_1 + \frac{t^2-1}{6}.$$

It would therefore suffice to show that

$$\frac{(4t-2)^{\frac{t-1}{2}} A_t(M_1 n + m_1)}{4^{\frac{t-3}{2}} t^{\frac{t}{2}} \mathcal{A}_{2t-1}\left(4M_1 n + 4m_1 + \frac{t^2-1}{6}\right)}$$

is never constant as n runs. However, this appears to be a difficult problem, and we leave Conjecture VI.1.9 open.

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Chapter VII

Summary and Discussion

In this chapter, the results of this thesis are summarised, and an outlook on related problems is discussed.

VII.1 A family of vector-valued quantum modular forms of depth two

In Chapter II, we saw how to construct an infinite family of vector-valued quantum modular forms of depth two on $\mathrm{SL}_2(\mathbb{Z})$ with some quantum set. These arise as (finite sums of) two-dimensional partial theta functions attached to a positive-definite quadratic form. To show quantum modularity, we followed techniques of Bringmann-Kaszian-Milas [BKMi1, BKMi2], asymptotically relating the two-dimensional theta functions to double Eichler integrals, which are in turn the purely non-holomorphic part of an indefinite theta function of signature $(2,2)$.

Further examples of this shape were explored in relation to plumbed 3-manifolds [BMM]. As remarked in Chapter II, to describe infinite families of similar quantum modular forms of depth $d > 2$ using similar techniques would require some more technical results to obtain non-sparse quantum sets. However, a theory of partial theta functions was recently developed by Bringmann and Nazaroglu [BN] that describes the general situation. Explicitly, let L be a lattice of rank N, $B\colon L \times L \to \mathbb{Z}$ a positive definite bilinear form, with $Q(\boldsymbol{n}) = \frac{1}{2}B(\boldsymbol{n},\boldsymbol{n})$ for $\boldsymbol{n} \in L$. Furthermore, let $\boldsymbol{\ell}$ be a characteristic vector in L, that is $Q(\boldsymbol{n}) + \frac{1}{2}B(\boldsymbol{\ell},\boldsymbol{n}) \in \mathbb{Z}$ for all $\boldsymbol{n} \in L$, along with $\boldsymbol{c}$ a vector with $Q(\boldsymbol{c}) = 1$.

Then for $\boldsymbol{\mu}$ in the dual lattice of L, Bringmann and Nazaroglu define

$$\Psi_{Q,\boldsymbol{\mu}}(\boldsymbol{z},\tau) := \sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}+L} \operatorname{sgn}\left(B(\boldsymbol{c},\boldsymbol{n})\right) q^{\boldsymbol{Q}(\boldsymbol{n})} e^{2\pi i B\left(\boldsymbol{n},\boldsymbol{z}+\frac{\boldsymbol{\ell}}{2}\right)}$$

They show that $\Psi_{Q,\boldsymbol{\mu}}$ is a mock Jacobi form, and give its completion explicitly. Note that this covers the situation of the function F from Chapter II as a certain specialisation. Since mock modularity is a stricter condition that quantum modularity, the results of Bringmann and Nazaroglu thus supersede those on the modularity of the functions investigated in [BKMi1, BKMi2, Ma1].

VII.2 The asymptotic profile of an eta-theta quotient related to entanglement entropy in string theory

In Chapter III, I investigated the bivariate asymptotic profile of a certain eta-theta quotient that arose from the partition function of entanglement entropy in string theory. Of course, this is not the only eta-theta quotient that appears in settings where it is important to investigate certain asymptotic limits. For example, they appear naturally in the investigations of superconformal characters related to black holes [DMZ, Section 7.4]. In more generality, it would be instructive to provide a resolution to the following problem.

Problem VII.2.1. *Extend the framework of* [Ma3] *to infinite families of eta-theta quotients.*

For $a_j, c, d \in \mathbb{N}$ and $b_j \in \mathbb{Z}$ consider a general family of the shape

$$\prod_{j=1}^{N} \eta(a_j\tau)^{b_j} \frac{\vartheta(z;\tau)}{\vartheta(cz;d\tau)}.$$

At the time of writing, an ongoing project joint with Cesana investigates the bivariate asymptotic profile of this family. In particular, this explicitly covers the cases of eta-theta quotients with simple poles, where similar techniques to [Ma3] can be used, along with Wright's circle method for Jacobi forms. The theta function in the numerator is purely for convenience, and we also explain how to perform similar calculations for powers of theta functions in both the numerator and denominator.

Further families of similarly-shaped meromorphic eta-theta quotients in a single elliptic variable should yield bivariate asymptotics in a similar manner. One would need to make minor adjustments to the definitions of Fourier coefficients for those with poles of order higher than one, following [DMZ]. More refined techniques may be required to deal with two elliptic variables. Such functions appear in numerous places, e.g. planar polygons arising from investigations into homological mirror symmetry for elliptic curves [BKZ]. To define the Fourier coefficients of such meromorphic Jacobi forms would require new insights, as there are now planes of discontinuity where we cannot deform the integral defining the Fourier coefficients around.

Problem VII.2.2. *Investigate bivariate asymptotics for families of Jacobi forms with multiple elliptic variables.*

While the mathematical descriptions of the bivariate behaviour of the coefficients can be tackled with the techniques described above, the physical meaning of these descriptions is yet to be explored.

Problem VII.2.3. *What do the descriptions of the bivariate asymptotics described here mean from a physics perspective?*

For example, it seems curious that in Chapter III the main growth came from the residue term

$$\frac{\eta(2\tau)^8}{\eta(\tau)^{16}},$$

which is both the generating function of 8-tuple partitions and has a physical interpretation in terms of the supersymmetry of the open-string spectrum using RNS fermions in light-cone gauge [GSO, Wi]. Is there a deeper picture here?

VII.3 Asymptotic equidistribution and convexity for partition ranks

In Chapter IV, the nature of the asymptotics of the ranks of partitions was investigated. We first saw that $N(r,t;n)$ is monotonic in n above some bound for fixed r,t. Using this we determined the asymptotic equidistribution of $N(r,t;n)$ by using Ingham's Tauberian theorem. This in turn lead to a resolution of a recent conjecture of Hou and Jagadeeson [HJ].

However, we showed only the asymptotic behaviour of the partition rank. For $t=3$, the authors of [HJ] proved that

$$N(r,3;a)N(r,3;b) > N(r,3;a+b)$$

for all a,b greater than 11. The techniques used by the authors rely on Bringmann's precise asymptotic for $N(r,t;n)$ with t odd, given in [Bri].

Problem VII.3.1. *For all t, determine precise lower bounds $n(t)$ such that $N(r,t;a)N(r,t;b) > N(r,t;a+b)$ for all a,b greater than $n(t)$.*

To use similar techniques to Hou and Jagadeeson, one would first need to perform a close inspection similar to that of [Bri] for even t. Although this would be a lengthy computation, the essence of the calculations is well-understood. However, to provide a bound in a uniform way across all t appears to be a very delicate problem.

Recently, Gomez and Zhu [GZ] gave a different way to construct the lower bound in the case $t=2$. Their techniques rely on an algebraic description of the generating function of $N(0,2;n) - N(1,2;n)$, which is Ramanujan's order three mock theta function $f(q)$, as the trace

$$S(n) := \sum_{[Q]} F(\tau_Q).$$

Here, the sum is over $\Gamma_0(6)$ equivalence classes of quadratic forms $Q = [a,b,c]$ with $6 \mid a$ and $b \equiv 1 \pmod{12}$, the function F is a certain weight zero modular form on $\Gamma_0(6)$, and τ_Q is the Heegner point associated to Q. The central connection is then given by the fact that $N(0,2;n) - N(1,2;n)$ can be recognised as

$$-\frac{1}{\sqrt{24n-1}}\mathrm{Im}(S(n)),$$

a result of Bruinier and Schwagenscheidt [BS1]. Their proof relies on recognising $f(q)$ as the holomorphic part of a vector-valued harmonic Maass form of weight $\frac{1}{2}$, constructing this harmonic Maass form as the Millson lift of F, and then comparing the Fourier coefficients of either side.

Problem VII.3.2. *Can one provide an algebraic toolkit to obtain explicit bounds for convexity of $N(r,t;n)$ for other values of t?*

To investigate this problem, one would first need to define families of generating functions that capture the equidistribution of the partition rank, and then determine whether they may be described by certain lifts in a similar process to that of [BS1]. One would then need to carefully determine the Fourier coefficients of the theta lifts in the usual way. While the idea may persist for small values of t, it seems fanciful to expect that this would yield a uniform explanation, and a more general technique may be required.

VII.4 Cycle integrals of meromorphic modular forms and rationality

In Chapter V we proved that the traces of certain meromorphic modular forms may be written as the constant term of a Fourier expansion involving the coefficients of theta functions and harmonic Maass forms. A central object in our proofs was the locally harmonic Maass form $\mathcal{F}_{1-k,D}$ of Bringmann, Kane, and Kohnen.

For $k=1$, Hövel also independently discovered the function $\mathcal{F}_{0,D}$ [Hö] (in the vector-valued setting). Furthermore, in [EGKR], Ehlen, Guerzhoy, Kane, and Rolen discussed the properties of the weight 0 case in the scalar-valued setting; in particular proving that $\mathcal{F}_{0,D}$ is a locally harmonic Maass form via Hecke's trick. They then used this to describe results on vanishing of twisted central L-values. This is intricately related to the famous congruent number problem, and in turn the Birch and Swinnerton-Dyer conjecture.

The Kudla-Millson lift, its constructions and applications have been studied in detail in recent years, e.g. [BES, BF, BS1, FH]. It takes modular objects of weight 0 to objects of weight $\frac{3}{2}$. In contrast to the lift considered in the project described in Chapter V, the Kudla-Millson lift is a lift on the Grassmanian and hence has very different properties. Given the properties and importance of the functions $\mathcal{F}_{1-k,D}$, a natural problem to consider is the following.

Problem VII.4.1. *Define and investigate the Kudla-Millson lift of the (vector-valued generalisation of the) locally harmonic Maass form $\mathcal{F}_{0,D}$.*

Another avenue to consider is defining locally harmonic Maass forms in signatures other than $(1, 2)$. In signatures other than $(n, 2)$ we lose the complex structure of the underlying manifold, and so in particular we lose the underlying arithmetic geometry since the manifold is no longer a modular variety. However, one may still define a Laplacian on n-dimensional hyperbolic space $\mathbb{H}^n$, and there are still interesting properties of such functions. For example, the same theta lift as in [BEY] but on even lattices of signature $(1, n)$ was studied by Bruinier in his celebrated habilitation [Bru][1]. Under the action of the hyperbolic Laplacian, the lifts studied by Bruinier are not harmonic, but are eigenfunctions with a certain eigenvalue depending on the signature of the lattice [Bru, Section 3.1.1].

In signature $(1, 2)$ Hövel studied a modified theta lift [Hö]. In particular, he included a particular polynomial factor in the Siegel theta function which had the effect of making the lift harmonic. A major novelty in Hövel's work was the inclusion of the twist as defined by Alfes-Neumann and Ehlen [AE], in turn relying on a genus character defined by Gross, Kohnen, and Zagier [GKZ]. Similar untwisted theta lifts were also studied by Bruinier and Funke in [BF] for general signature (p, q) lattices and general choices of polynomial. In particular, they also showed that such lifts also satisfy a current equation (see their Theorem 1.5).

Given the results of [Bru], it is a natural question to ask whether the phenomenon observed by Hövel generalises to even lattices of signature $(1, n)$. In an ongoing project with Neururer, Scharf, and Schwagenscheidt we have embarked on a part of this investigation. In particular, we are able to define locally harmonic Maass forms in hyperbolic n-space $\mathbb{H}^n$, by way of constructing the theta lift Φ_L^* of an explicit modified Siegel theta function θ_L^*. In doing so, we obtain a series representation (as a sum over a signature $(1, n)$ lattice with summands involving Gauss hypergeometric functions) in a similar fashion to [Bru, Hö]. This gives hints to a natural way to define more refined locally

[1]Actually, Bruinier did not include Maass raising operators, but given the results of [BEY] this is an easy consequence.

harmonic Maass forms in $\mathbb{H}^n$ without the use of a theta lift. In particular when $n = 3$ the construction should yield Bianchi modular forms, which are historically difficult to construct.

In [ABMS, BEY, BS2], at certain points in the Grassmanian the theta lifts studied had close ties to coefficients of mock theta functions, in turn offering rationality results. For example, explicit results in the case of $n = 2$ have been obtained in [ABMS, BS2] for Zagier's weight $\frac{3}{2}$ non-holomorphic Eisenstein series whose coefficients are Hurwitz class numbers, and similar results can be concluded for the classical smallest parts partition function from the same papers. We would like to apply the techniques of these papers to Φ_L^*. However, the proofs in [BEY, ABMS] rely critically on the splitting of the Siegel theta function over natural positive- and negative-definite sublattices of L at such points.

Problem VII.4.2. *Determine the splitting behaviour of the modified Siegel theta function θ_L^* at certain points in the Grassmanian of L. Use this to provide connections to mock theta functions and rationality results.*

Furthermore, the ability to twist theta lifts would allow one to greatly generalise the results, both within the scope of this particular overarching project and beyond (currently, many results only offer twisting in signature $(1, 2)$, see e.g. [BEY, Hö]). To do so in signature $(1, n)$ with $n \geq 3$ would require defining a similar character to that of Gross, Kohnen, and Zagier but acting on more general hermitian forms.

Problem VII.4.3. *Define a twist for signatures with $n \geq 3$.*

VII.5 On t-core and self-conjugate $2t-1$-core partitions in arithmetic progressions

In Chapter VI we discussed the relationships between t-cores and self-conjugate $(2t-1)$-cores on arithmetic progressions. Our main results pertained to the case of $t = 4$, where we showed that one may write the number of self-conjugate 7-cores as a single class number. Furthermore, we offered a combinatorial explanation for this fact, relying on the combinatorial structures of abaci and extended t-residue diagrams, as well as Gauss' map between solutions to ternary quadratic equations and binary quadratic forms in certain class groups. We also discussed whether equations of a similar shape to (I.2.1) can hold for other values of t, proving that they cannot hold if $t = 2, 3, 5$ and offering a conjecture and partial results for $t > 6$.

As noted in Chapter VI, t-core partitions encode the modular representation theory of the symmetric groups S_n and A_n. Thus, information regarding the existence and approximate number of such representations can be rephrased in terms of the positivity and asymptotic behaviour of $c_t(n)$. Such questions have been discussed in detail, for example in [GO] Granville and Ono showed that if $t \geq 4$ then every positive integer n has at least one t-core partition.

The asymptotic behaviour of the number of t-core partitions is also well-studied. Exact formulae are known for $c_t(n)$ when $t = 2, 3, 4, 5$, which immediately yield (non)-vanishing results and estimates for the number of such t-cores. For $t \geq 6$ Anderson [Ande] provided an asymptotic formula relying on the Hardy-Ramanujan circle method. More refined details may be found by considering congruences and divisibility properties satisfied by t-cores. An explicit example can be found by combining Ramanujan's congruences modulo $5, 7, 11$ for the partition function with knowledge of the generating function for t-cores, which yields that

$$c_5(5n-1) \equiv 0 \pmod{5}, \qquad c_7(7n-2) \equiv 0 \pmod{7}, \qquad c_{11}(11n-5) \equiv 0 \pmod{11}.$$

Garvan, Kim, and Stanton extended these kind of relations, and showed in [Ga, GKS] that more general congruences hold for $t = 5, 7, 11$. A natural question highlighted recently by Stanton, among others, is whether congruences of this form exist for other values of t. For $t = 2, 3$ the answer follows directly from their explicit formulae, but for higher t the theory is more complex. While some authors have sporadically given congruence and divisibility properties for 4-cores and self-conjugate 7-cores - e.g. [GCGL, HS, OS], a unifying theory is still missing.

Problem VII.5.1. *Completely classify arithmetic progressions on which the number of* 4*-cores (resp. self-conjugate* 7*-cores) is always divisible by some prime* ℓ*. Provide a combinatorial explanation for this.*

The disproof of many arithmetic progressions may be quickly resolved by brute force by computer, but finding arithmetic progressions where such a statement holds will be more delicate. Recall from Chapter VI that 4-cores are intricately related with class numbers. Recently Beckwith, Raum, and Richter [BRR] gave a description of conditions for when $H(an+b)$ is divisible by a prime ℓ for all n and fixed $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ such that $-b$ is a square modulo a. Their arguments depend on combining a holomorphic projection argument pioneered in [IRR] and a powerful method developed by Serre [Se1, Se2] based on the Chebotarev Density Theorem that was later used by Ono in his celebrated paper [On] that described the distribution of the partition function modulo primes. Combining Beckwith, Raum, and Richter's arguments with those of [OS, OR, BKM] suggests that Problem VII.5.1 is tractable.

A natural extension of Problem VII.5.1 is to try to apply similar techniques to other values of t. While we lose the connection to Hurwitz class numbers and therefore the explicit results of [BRR], the underlying philosophy carries over to this setting. However, the increase of weights in the generating functions for t-cores with $t \geq 6$ may require new insights, and so a further expansion of the framework discussed above may be required.

Problem VII.5.2. *By expanding the results of* [BRR], *allow the value of* t *to vary in Problem VII.5.1.*

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[Ma2] ——, *Asymptotic equidistribution and convexity for partition ranks*, Ramanujan J. (2020).

[Ma3] ——, *The asymptotic profile of an eta-theta quotient related to entanglement entropy in string theory*, Res. Number Theory **6** (2020), no. 1, Paper No. 15, 14.

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Declaration

I hereby declare that the article *Cycle integrals of meromorphic modular forms and coefficients of harmonic Maass forms* [ABMS] was jointly written with Prof. Dr. Claudia Alfes-Neumann, Prof. Dr. Kathrin Bringmann, and Dr. Markus Schwagenscheidt and my share of the work amounted to 25%. The article *On t-core and self-conjugate* $(2t-1)$*-core partitions in arithmetic progressions* [BKM] was jointly written with Prof. Dr. Kathrin Bringmann and Prof. Dr. Ben Kane, and my share of the work amounted to 33%. The articles *A family of vector-valued quantum modular forms of depth two* [Ma1], *Asymptotic equidistribution and convexity for partition ranks* [Ma2], and *The asymptotic profile of an eta-theta quotient related to entanglement entropy in string theory* [Ma3] were written by myself as sole author and are 100% my own work.

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Eklräung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne die Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Werken dem Wortlaut oder dem Sinn nach entnommen wurden, sind als solche kenntlich gemacht. Ich versichere an Eides statt, dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen und eingebundenen Artikeln und Manuskripten - noch nicht veröffentlicht worden ist sowie, dass ich eine Veröffentlichung der Dissertation vor Abschluss der Promotion nicht ohne Genehmigung des Promotionsausschusses vornehmen werde. Die Bestimmungen dieser Ordnung sind mir bekannt. Darüber hinaus erkläre ich hiermit, dass ich die Ordnung zur Sicherung guter wissenschaftlicher Praxis und zum Umgang mit wissenschaftlichem Fehlverhalten der Universität zu Köln gelesen und sie bei der Durchführung der Dissertation zugrundeliegenden Arbeiten und der schriftlich verfassten Dissertation beachtet habe und verpflichte mich hiermit, die dort genannten Vorgaben bei allen wissenschaftlichen Tätigkeiten zu beachten und umzusetzen. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Teilpublikationen:

1. C. Alfes-Neumann, K. Bringmann, J. Males, M. Schwagenscheidt, *Cycle integrals of meromorphic modular forms and coefficients of harmonic Maass forms*, J. Math. Anal. Appl., **497** (2021), no. 2, 124898.

2. K. Bringmann, B. Kane, J. Males, *On t-core and self-conjugate $2t-1$-core partitions in arithmetic progressions*, 2020, preprint, available at `https://arxiv.org/abs/2005.07020`.

3. J. Males, *A family of vector-valued quantum modular forms of depth two*, Int. J. Number Theory **16** (2020), no. 1, 29–64.

4. J. Males, *The asymptotic profile of an eta-theta quotient related to entanglement entropy in string theory*, Res. Number Theory **6** (2020), no. 1, Paper no. 15, 14 pp.

5. J. Males, *Asymptotic equidistribution and convexity for partition ranks*, Ramanujan J. (2020).

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Curriculum Vitae

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Education

2017 - 2021 **University of Cologne**, Ph.D. candidate.
Advisor: Kathrin Bringmann. Thesis titled *Modular forms: Constructions and Applications.*

2012 - 2016 **Durham University** , MMath (integrated bachelors and masters), 1st class honours.
Advisor: Jens Funke. Thesis titled *Topics in Modular Forms - Automorphic Forms and L-functions over the Rationals.*

Employment

2017 - 2021 **University of Cologne**, Wissenschaftlicher Mitarbeiter (Research assistant).

Research Interests

Number theory and combinatorics - in particular; modular forms and their generalisations e.g. harmonic Maass forms, partitions, and their applications to number theory and beyond.

Publications

submitted G. Cesana and J. Males, *Bivariate asymptotics for eta-theta quotients with a simple pole.*

submitted J. Males, *A short note computing higher Mordell integrals associated to higher depth quantum modular forms.*

submitted J. Males, A. Mono, and L. Rolen, *Higher depth mock theta functions and q-hypergeometric series.*

submitted J. Males and Z. Tripp, *Combinatorial results on t-cores and sums of squares.*

submitted J. Males, A. Mono, and L. Rolen, *Polar harmonic Maass forms and holomorphic projection of mixed harmonic Maass forms.*

submitted K. Bringmann, B. Kane, and J. Males, *On t-core and self-conjugate $(2t-1)$-core partitions in arithmetic progressions.*

2021 C. Alfes-Neumann, K. Bringmann, J. Males, and M. Schwagenscheidt, *Cycle integrals of meromorphic modular forms and coefficients of harmonic Maass forms*, J. Math. Anal. Appl., **497**, no. 2, 124898.

2020 J. Males, *The asymptotic profile of an eta-theta quotient appearing in entanglement entropy in string theory*, Res. Number Theory **6**, 1:15.

2020 J. Males, *Asymptotic equidistribution and convexity for partition ranks*, Ramanujan J.

2020 J. Males, *A family of vector-valued quantum modular forms of depth two*, Int. J. Number Theory **16**, No. 1, 29–64.

Co-Advising

Masters

2019-2020 Cagdas Demir, *The Riemann hypothesis for period polynomials.*

2018 Robin Sauer, *Modular forms associated to real quadratic fields.*

Bachelors

2017 - 2018 Carsten Mitzkus, *Fourier coefficients of a certain η-quotient.*

Teaching

Winter 2020 **Asymptotic expansions of modular forms**, Teaching assistant.
Undergraduate and masters seminar, University of Cologne.

Winter 2020 **Linear Algebra**, Teaching assistant.
Undergraduate course, University of Cologne.

Summer 2020 **Elementary Number Theory**, Teaching assistant.
Undergraduate course, University of Cologne.

Winter 2019 **Algebra**, Teaching assistant.
Undergraduate course, University of Cologne.

Summer 2019 **Algebraic number theory**, Teaching assistant.
Undergraduate and masters course, University of Cologne.

Summer 2019 **L-functions**, Teaching assistant.
Undergraduate and masters seminar, University of Cologne.

2019–present **Ph.D. reading seminar**, Session leader.
University of Cologne.

Winter 2018 **Theta functions**, Teaching assistant.
Undergraduate seminar, University of Cologne.

2015–2016 **Mathematical physics**, Marker.
Undergraduate course, Durham University.

2015–2016 **Statistical concepts II**, Marker.
Undergraduate course, Durham University.

2015–present Private tutor.
Various pre-university and undergraduate topics.

Professional Service

Refereeing: Adv. Math., Ann. Comb., Mathematical Reviews, Ramanujan J., Res. Math. Sci., Res. Number Theory, The L-Functions and Modular Forms Database.

Sep. 2020 - present **Co-organiser**: POINT - New developments in number theory.

2019 - 2021 **Organiser**: Ph.D. reading seminar, University of Cologne.

Grants

Nov. 2020 KinderUni Koeln, *Outreach grant.*
A small grant awarded to run an outreach programme on integer partitions in the children's university in early 2021. Joint with Andreas Mono.

May - Oct 2020 HYPATIA.SCIENCE, *Harmonic Maass forms in hyperbolic n-space.*
A six month grant designed to enable supervision of a female masters student in a live research project. Joint with Pauline Scharf and Markus Schwagenscheidt.

Talks

Dec. 2020 Traces of cycle integrals of meromorphic modular forms and harmonic Maass forms.
Palmetto Number Theory Series.

Sep. 2020 Self-conjugate t-core partitions and class numbers.
Prerecorded talk, Junior Mathematician Research Archive.

Sep. 2020 Self-conjugate t-core partitions and class numbers.
Prerecorded talk, Québec-Maine Number Theory Conference 2020.

Sep. 2020 Self-conjugate t-core partitions and class numbers.
Palmetto Number Theory Series.

Sep. 2020 Self-conjugate t-core partitions and class numbers*.
Number Theory Lunch Seminar, Max Planck Institute for Mathematics, Bonn, Germany.

Aug. 2020 Self-conjugate t-core partitions, class numbers, and arithmetic progressions*.
Elementare und Analytische Zahlentheorie (postponed).

Jul. 2020 Traces of cycle integrals and harmonic Maass forms.
Building Bridges 5th EU/US Summer School and Workshop (postponed).

May. 2020 Traces of cycle integrals and harmonic Maass forms.
The Sixth Mini Symposium of the Roman Number Theory Association (cancelled).

Dec. 2019 Asymptotic equidistribution for partition ranks*.
SASTRA Ramanujan Conference, SASTRA University, India.

Nov. 2019 Constructing quantum modular forms of higher depth*.
Arithmetic study group seminar, Durham University, UK.

Feb. 2019 Constructing quantum modular forms of higher depth*.
Geometry seminar, Vanderbilt University, USA.

Jan. 2019 - present Various topics (6).
Mock theta functions reading seminar, University of Cologne, Germany.

Conferences and Research Visits

May. 2021 Hausdorff School: The Circle Method♯, Bonn, Germany.
Summer school with places awarded on a competitive basis.

Dec. 2020 Luxembourg Number Theory Day♯.

Dec. 2020 Palmetto Number Theory Series♯.

Sep. 2020 Québec-Maine Number Theory Conference♯.

Sep. 2020 Palmetto Number Theory Series♯.

Sep. 2020 Heilbronn Annual Conference♯.

Aug. 2020 Elementare und Analytische Zahlentheorie (ELAZ), Poznan, Poland (postponed).

Jul. 2020 Building Bridges 5th EU/US Summer School and Workshop on Automorphic Forms and Related Topics, University of Sarajevo, Bosnia and Herzegovina (postponed).

Jun. 2020 Chicago Number Theory Day♯.

Jun. 2020 Online Conference in Automorphic Forms♯.

Mar. 2020 AMS sectional meeting, University of Virginia, USA (cancelled).

Feb. - Mar. 2020 Research visit to University of Virginia, Virginia, USA.

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*Denotes invited talk.
♯ Denotes an online conference.
† Denotes competitive or invitational funding awarded for travel and/or accommodation.

Feb. 2020 ABKLS, Aachen, Germany.

Dec. 2019 SASTRA Ramanujan conference†, Kumbakonam, India.

Sep. 2019 ABKLS, Cologne, Germany.

Jul. 2019 The first JNT biennial conference†, Cetraro, Italy.

Jul. 2019 Number theoretic methods in quantum physics, Bonn, Germany.

Feb. 2019 Research visit to Vanderbilt University, Tennessee, USA.

Oct. 2018 Srinivasa Ramanujan: in celebration of the centenary of his election as FRS, Royal Society, London, UK.

Sep. 2018 Elementare und analytische Zahlentheorie (ELAZ), Bonn, Germany.

Nov. 2017 3rd Japanese-German number theory workshop, Bonn, Germany.

Sep. 2017 ABKLS, Cologne, Germany.

May. 2017 Modular forms are everywhere†, Bonn, Germany.
Summer school and conference.

Apr. 2017 British mathematical colloquium, Durham, UK.

Aug. 2015 New moonshine, mock modular forms and string theory†, Durham, UK.

Other Skills

Programming: Python, Sage, Maple, Blender, basic C++.

Languages: Native English, intermediate German, beginner French.

Trained examiner (Cambridge International Examinations).