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Asymptotic variational approach to study light propagation in a nonlocal nonlinear medium

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ABSTRACT

We propose and demonstrate analytically, within the framework of a hydrodynamic model, a novel and simpler variational approach to study the asymptotic behavior of a continuous wave (cw) laser beam propagating in a nonlinear nonlocal medium.

The starting point in the analysis is the light propagation in a weakly nonlocal nonlinear defocusing medium described by normalized NLSE

$$i\,\psi_z + \frac{1}{2}\,\psi_{x,x} - \varphi\,\psi = 0\,, \tag{1}$$

where the dimensionless z and x are the spatial evolutionary variable and the transverse coordinates, respectively. Also, ψ is the complex electric field envelop, φ is a real function that denotes the nonlinear nonlocal change of the refractive index depending on the intensity $I = |\psi|^2$. Finally $0 < \epsilon \ll 1$ is a small quantity that deal with the weakly diffracting regime (see [1] for more details). Other examples of light propagating in different media are [2–9].

The above expression is coupled to a diffusion-like equation for the response of the nonlocal medium

$$-\,\sigma^2\,\varphi_{x,x} + \varphi = |\psi|^2, \tag{2}$$

where the parameter σ is a spatial scale (setting the diffusion length) that measures the degree of nonlocality.

We consider small amplitude slowly varying modulations of the steady state given by a continuous wave $\psi = \psi_0\,\exp(-i\,|\psi_0|^2\,z)$, where ψ_0 is an arbitrary complex constant, $|\psi_0|^2 = 1$ and the constant $\varphi = |\psi_0|^2$.

Applying the Mandelung transformation $\psi(z,x) = \rho^{1/2}(z,x)\,\exp[\,i\,h(z,x)\,]$ and retaining leading orders in ϵ, it is possible to obtain the following equations

$$\rho_z + \left(\rho\,h_x\right)_x = 0, \tag{3a}$$

$$h_z + \frac{1}{2}\,h_x^2 + \frac{1}{2}\,\rho^{-1/2}\,\rho_{x,x}^{1/2} + \varphi = 0, \tag{3b}$$

$$-\,\sigma^2\,\varphi_{x,x} + \varphi = \rho, \tag{3c}$$

The above system of equations can be derived from the appropriate Lagrangian density

$$L = \rho\left[\frac{h_x^2}{2} + h_z + \varphi - 1\right] + \frac{1}{2}(\rho_x^{1/2})^2 - \frac{1}{2}\left[\varphi^2 + \left(\sigma\,\varphi_x\right)^2 - 1\right]. \tag{4}$$

Euler–Lagrange variation with respect to h yields (3a) whereas the ρ and φ variations yield (3b) and (3c), respectively.

To discuss the wave envelop dynamics in this long-wavelength limit due to weak nonlinear and weak dispersive effects, we introduce the stretched variables

$$\xi = \epsilon^{1/2}\,(x - z) \quad \text{and} \quad \tau = \epsilon^{3/2}\,z,$$

where ξ allows us to study the system on different, slowly, moving frames and by τ, longer propagation distance z. Also, ϵ is a measure of the deviation from the background ψ_0. Using the perturbation expansions

$$\rho(\xi,\tau) = \rho_0 + \sum_{j=1}^{\infty} \epsilon^j\,\rho^{(j)}(\xi,\tau)\,, \tag{5a}$$

$$\varphi(\xi,\tau) = \varphi_0 + \sum_{j=1}^{\infty} \epsilon^j\,\varphi^{(j)}(\xi,\tau)\,, \tag{5b}$$

$$h(\xi,\tau) = \sum_{j=0}^{\infty} \epsilon^{j+1/2}\,h^{(j+1)}(\xi,\tau)\,. \tag{5c}$$

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where $\rho_0 = 1$, $\varphi_0 = |\psi_0|^2$.

Therefore we can expand the Lagrangian density for small amplitudes following the method in [10,11].

$$L = \epsilon\, L^{(1)} + \epsilon^2\, L^{(2)} + \epsilon^3\, L^{(3)} + \mathcal{O}(\epsilon^4). \tag{6}$$

For ϵ:

$$L^{(1)} = -h^{(1)}_{\xi},$$

from where no relevant information is obtained.

For ϵ^2 :

$$L^{(2)} = \frac{1}{2} h^{(1)^2}_{\xi} - 2\rho^{(1)} h^{(1)}_{\xi} + \rho^{(1)}\varphi^{(1)} + h^{(1)}_{\tau} - h^{(2)}_{\xi} - \frac{1}{2}\varphi^{(1)^2} \tag{7}$$

from where we have obtained the following expression as Euler–Lagrange equations

$$\delta\rho^{(1)} \;:\quad h^{(1)}_{\xi} = \varphi^{(1)}, \tag{8a}$$

$$\delta\varphi^{(1)} \;:\quad \rho^{(1)} = \varphi^{(1)}, \tag{8b}$$

$$\delta h^{(1)} \;:\quad \rho^{(1)} = h^{(1)}_{\xi}, \tag{8c}$$

and the relation

$$h^{(2)}_{\xi} = -\rho^{(1)}\, h^{(1)}_{\xi}. \tag{9}$$

The ϵ^3 final Lagrangian is obtained with help of (8) and (9) as

$$L^{(3)} = \frac{1}{2} h^{(1)^3}_{\xi} - \rho^{(2)} h^{(1)}_{\xi} + 2h^{(1)}_{\tau} h^{(1)}_{\xi} + \frac{\gamma}{8} h^{(1)^2}_{\xi\xi} + h^{(2)}_{\tau} \tag{10}$$

providing the condition

$$h^{(2)}_{\tau} = -\rho^{(2)}\, h^{(1)}_{\xi}. \tag{11}$$

where $\gamma = \left(1 - 4\,\sigma^2\right)$ is the optical analogue to surface tension [1]. Second-approximation terms $h^{(2)}$ and $\rho^{(2)}$ could be obtained and studied [12] using the expressions (9)–(11).

Assuming $u = h_x$, the preceding equation yields, as its Euler–Lagrange equation, a KdV type [1,13]

$$u_\tau + \frac{3}{2}\, u\, u_\xi - \frac{\gamma}{8}\, u_{\xi\xi\xi} = 0. \tag{12}$$

The solution of (12) is given by

$$\rho(\xi,\tau) \equiv u = N\, sech^2 \left[\sqrt{\frac{N}{2\,\gamma}} \left(\xi - \frac{N}{4}\,\tau\right)\right]. \tag{13}$$

where N is the soliton amplitude. In original coordinates

$$u(z,x) = N\, sech^2 \left\{ \frac{1}{4}\sqrt{\frac{\epsilon\, N}{\gamma}} \left[x - \left(1 + \frac{\epsilon N}{8}\right) z\right]\right\}, \tag{14}$$

and $h(z,x)$ can be obtained readily from (8c),

$$h = -\frac{4\gamma}{\epsilon}\sqrt{\frac{\epsilon N}{\gamma}} \tanh \left\{ \frac{1}{4}\sqrt{\frac{\epsilon N}{\gamma}} \left[x - \left(1 + \frac{\epsilon N}{8}\right) z\right]\right\}. \tag{15}$$

In the original (dimensionless) x and z, one may write down an approximate [up to order $\mathcal{O}(\epsilon)$] solution for the macroscopic wavefunction ψ

$$\psi = \psi_0\, \sqrt{\rho_0 + \epsilon\, \rho_1}\quad \exp\left[-i\, |\psi_0|^2 z + i\, h(z,x)\right], \tag{16}$$

$$\varphi = |\psi_0|^2 + \epsilon\, \varphi_1\,, \tag{17}$$

where φ_1 is written as (8b) and (14).

Conclusions

We have explored theoretically light propagation in a nonlocal nonlinear defocusing media through a proposed alternative simpler method, the asymptotic variational multiscale approach. The obtained KdV equation is similar to the one derived using reductive multiscale technique. Our results advance the understanding of nonlinear phenomena.

CRediT authorship contribution statement

Artorix de la Cruz: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Michael Cada:** Project administration, Supervision. **Jaromir Pistora:** Supervision. **Tamara Diaz-Chang:** Review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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