Continuum of motion equations and control laws for underactuated mechanical systems

by

Constance Ann Lare

B.S., Kansas State University, 2009

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Alan Levin Department of Mechanical and Nuclear Engineering
Carl R. Ice College of Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2021

Abstract

As sizes, lengths, or shapes of a system grow large or shrink to zero, a system will approach limiting forms. As the parameter is allowed to grow or shrink, the system could resemble a simpler system. The sufficient conditions for when the equations of motion will morph from the original system to a target system will be presented. The ball and arc equations of motion morph to those of the ball and beam as the arc's radius is allowed to grow. The equations of motion for the rotary pendulum, pendubot, and two-link robot manipulator will morph to the equations of motion of the inverted pendulum cart.

The effect of a parameter growing large or shrinking to zero has on the controller for the original system will not be fully investigate in this work. A case for when controller morphing might be possible will be examined. A controller for the rotary pendulum will morph to a controller that stabilizes the inverted pendulum cart. Next, a controller for the pendubot will be morphed that does not stabilize the dimensionless inverted pendulum cart. Lastly, a controller for a fully actuated two-link robot manipulator will be morphed to a stabilizing controller for a fully actuated inverted pendulum cart.

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List of Nomenclature

Variable	Definition
$\boldsymbol{a}$	Derivative of $\boldsymbol{v}$
B, B_{rp}	Part of the inverted pendulum cart and rotary pendulum control law
B_N, B_{rpN}	Part of the dimensionless inverted pendulum cart and rotary pendulum control law
$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$	Matrix of Coriolis and centripetal acceleration coefficients, $\in \mathbb{R}^{nxn}$
c, c_{rp}	Control Gains with units of seconds^{-1}
c_G	Center of mass
c_1, c_2, c_3, c_4, c_5	Elements of the equations of motion of the pendubot
c_{1n}, c_{2n}, c_{3n}, c_{4n}, c_{5n}	Elements of the dimensionless equations of motion of the pendubot
D, D_{rp}	$-m_p g l$
D_N, D_{rpN}	$-g/l = -1/\gamma^2$
d	Contact point between the arc and the ball
d_1, d_2, d_3, d_4	Elements of the desired mass matrix for the pendubot
d_{1n}, d_{2n}, d_{3n}, d_{4n}	Elements of the dimensionless desired mass matrix for the pendubot
dL_2	Dimensionless load length
dm_2	Dimensionless load mass
$\boldsymbol{e}_r$	Unit radial vector
$\boldsymbol{e}_t$	Unit tangent vector
F	Force input to drive the cart
F_f	Friction force necessary to sustain roll without slip
F_α, $F_{\alpha N}$	Part of rotary pendulum controller
$\boldsymbol{G}(\boldsymbol{q})$	Vector of gravitational forces and torques, $\in \mathbb{R}^n$
$\boldsymbol{G}_m(\boldsymbol{q})$	Map from the m inputs to the n degrees of freedom, $\in \mathbb{R}^{mxn}$
$\boldsymbol{G}_m^\perp(\boldsymbol{q})$	Left annihilator of $\boldsymbol{G}_m(\boldsymbol{q})$, $\boldsymbol{G}_m^\perp(\boldsymbol{q})\ \boldsymbol{G}_m(\boldsymbol{q}) = 0, \in \mathbb{R}^{nxm}$
g	Acceleration of gravity
H	Distance from the beam's (arc's) rotation point to the beam (arc) surface

Variable	Definition
$H(\boldsymbol{q}, \boldsymbol{p})$	Hamiltonian
$H_d(\boldsymbol{q}, \boldsymbol{p})$	Stabilizing Hamiltonian
I_1	Mass moment of inertia of the first link about fixed axis of rotation
I_2	Mass moment of inertia of the second link about fixed axis of rotation
$\boldsymbol{i}$	Unit vector in x direction
$\boldsymbol{j}$	Unit vector in y direction
$\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$	Skew symmetric matrix, $\in \mathbb{R}^{nxn}$
J_{arc}	Mass moment of inertia of the arc about fixed axis of rotation
J_B	Centroidal mass moment of inertia of the ball
J_{beam}	Mass moment of inertia of the beam about fixed axis of rotation
$\boldsymbol{K}$	Positive definite diagonal gain matrix, $\in \mathbb{R}^{nxn}$
$\boldsymbol{K}_v$	Viscous damping coefficient matrix, $\in \mathbb{R}^{mxm}$
K_{1n}	Dimensionless 1,1 term of $\boldsymbol{K}$
K_{2n}	Dimensionless 2,2 term of $\boldsymbol{K}$
$\boldsymbol{k}$	Unit vector in z direction
k	Gain for the desired mass matrix of the pendubot
k_1	Dimensionless ratio of masses
k_2	Dimensionless ratio of length of pendulum to radius of rotary arm Dimensionless ratio of length of the first link to length of the second link
k_3	Dimensionless ratio of mass moment of inertia of the beam or arc to mR_o^2
k_4	Dimensionless ratio of ball centroidal mass moment of inertia to mR_o^2
k_5	Dimensionless ratio of distance from rotation point to center of mass to the radius of the ball
k_6	Dimensionless ratio of offset distance to the radius of the ball
k_7	Dimensionless ratio of the radius of the arc to that of the radius of the ball
k_8	Dimensionless ratio of inertia of the first link to $m_2L_1^2$
k_9	Dimensionless ratio of length to the center of mass of the first link to the total length of the first link of the pendubot
k_p	Part of the desired potential energy function for the pendubot

Variable	Definition
k_{pn}	Part of the dimensionless desired potential energy function for the pendubot
k_v	Part of the control law for the pendubot
k_{vn}	Part of the dimensionless control law for the pendubot
L	Lagrangian where $L = T + V$
l	Length of the pendulum
L_1	Length of the first link for the pendubot and two-link manipulator
L_2	Length of the second link for the pendubot and two-link manipulator
l_{arc}	Distance from the center of mass of the arc to the rotation point
l_{beam}	Distance from the center of mass of the beam to the rotation point
L_{c1}	Distance to center of mass of the first link from origin of X-Y axis
L_{c2}	Distance to center of mass of the second link from end of the first link
$\boldsymbol{M}(\boldsymbol{q})$	Symmetric, positive definite matrix of inertial and mass terms, $\in \mathbb{R}^{nxn}$
$\boldsymbol{M}_d(\boldsymbol{q})$	Desired mass matrix, $\in \mathbb{R}^{nxn}$
$\boldsymbol{M}_n$	Dimensionless mass matrix for pendubot control law
m	Number of inputs Mass of the ball
m_1	Mass of the first link of the pendubot and two-link manipulator
m_2	Mass of the second link of the pendubot and two-link manipulator
m_a	Mass of the rotary pendulum arm or arc
m_B	Mass of the beam
m_c	Mass of the cart
m_p	Mass of the pendulum
N	Normal force
n	Degrees of freedom
o	Origin for the X-Y coordinate frame for the arc
$\boldsymbol{p}$	Generalized momenta, $\boldsymbol{p} = [p_1, p_2]^{\mathrm{T}}$
$\dot{\boldsymbol{p}}$	Derivative of generalized momenta
$\boldsymbol{p}_n$	Dimensionless generalized momenta, $\boldsymbol{p}_n = [p_{1n}, p_{2n}]^{\mathrm{T}}$

Variable	Definition
p, p_{rp}	Part of the inverted pendulum cart and rotary pendulum control law, units of length
p_N, p_{rpN}	Part of the inverted pendulum cart and rotary pendulum dimensionless control law
$\boldsymbol{Q}$	Generalized forces
$\boldsymbol{q}$	Generalized coordinates
$\boldsymbol{q}^d$	Desired trajectory
$\tilde{\boldsymbol{q}}$	Difference in actual and desired position
$\dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}$	Generalized velocities and accelerations
$q_1, \dot{q}_1, \ddot{q}_1$	Angular displacement, velocity, and acceleration of the first link of the pendubot measured from the positive x axis
$q_2, \dot{q}_2, \ddot{q}_2$	Angular displacement, velocity, and acceleration of the second link of the pendubot measured counterclockwise from a link extending out of the first link
q_2', q_2''	First and second derivative of q_2 with respect to dimensionless time
q_{d1}	Desired position of the first link of the pendubot
R	Radius of the arc or rotary pendulum
$\boldsymbol{r}$	Difference in actual and desired velocities plus lambda times $\tilde{\boldsymbol{q}}$
r	Distance from the center of the ball to the beam longitudinal midpoint
$\dot{\boldsymbol{r}}, \ddot{\boldsymbol{r}}$	Velocity and acceleration of the ball to the beam longitudinal midpoint
$\boldsymbol{r}_{d/o}$	Position vector from the contact point of the ball on arc to the origin
$\boldsymbol{r}_{d/s}$	Position vector from the contact point of the ball on arc to arc's center
R_o	Radius of the ball
$\boldsymbol{r}_{s/o}$	Position vector from the center of the arc to the origin
$\boldsymbol{r}_{u/w}$	Relative position vector of point u with respect to point w
s	Center of the arc
T	Kinetic Energy
t	Time in seconds
$\underline{t}$	Unitless time, t/γ

Variable	Definition
$\boldsymbol{u}$	Control law for the IDA-PBC overview
u	Input torque to drive the first link of the pendubot
u_1, u_{1N}	Control law that converts rotary pendulum to planar pendulum Control law for the first link of the two-link robot manipulator
u_2, u_{2N}	Control law for the second link of the two-link robot manipulator
$\boldsymbol{u}_{2L}$	Control input designed to achieve robustness in Chapter 6
$\boldsymbol{u}_{2Ld}$	Dimensionless control input designed to achieve robustness
u_{2rp}, u_{2rpN}	Control law that stabilizes rotary pendulum
u_d, u_{drp}	Dissipation term, units of mass·length/seconds^2
u_{dN}, u_{drpN}	Dimensionless dissipation term
$\boldsymbol{u}_{di}(\boldsymbol{q}, \boldsymbol{p})$	Damping injection input
$\boldsymbol{u}_{es}(\boldsymbol{q}, \boldsymbol{p})$	Energy shaping input
$V(\boldsymbol{q})$	Potential energy
$V_d(\boldsymbol{q})$	Desired potential energy
V, V_n	Potential energy functions of the pendubot control law
V_d, V_{dn}	Desired potential energy functions of the pendubot control law
$\boldsymbol{v}$	Desired velocity minus lambda times difference in actual and desired position
$\boldsymbol{v}_c$	Velocity of the center of the ball
$\boldsymbol{v}_d$	Velocity of the contact point of ball on arc
$x, \dot{x}, \ddot{x}$	Pendulum cart generalized position, velocity, and acceleration
x_{1c}, y_{1c}, $\dot{x}_{1c}$, $\dot{y}_{1c}$	Position and velocity for the center of mass of the first link of the pendubot
x_{2c}, y_{2c}, $\dot{x}_{2c}$, $\dot{y}_{2c}$	Position and velocity for the center of mass of the second link of the pendubot
X, Y, x, x_c, y, y_c, z	Coordinates for the ball and beam and ball and arc system
$\dot{x}, \dot{x}_c, \dot{y}, \dot{y}_c$	Velocities for the ball and beam system
$\ddot{x}, \ddot{x}_c, \ddot{y}, \ddot{y}_c$	Accelerations for the ball and beam system
$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$	2 x 6 matrix of known functions
$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}, \boldsymbol{a})$	2 x 6 matrix of nominal functions

Variable	Definition
$\boldsymbol{\alpha}$	Part of the control law for the pendubot, $\boldsymbol{\alpha} = [\, \alpha_1, \alpha_2]^T$
$\boldsymbol{\alpha}_n$	Part of the dimensionless control law for the pendubot, $\boldsymbol{\alpha}_n = [\, \alpha_{1n}, \alpha_{2n}]^T$
α, α_{rp}	$m_p l^2$
β	$m_p l$
β_{rp}	$m_p l R$
γ	Conversion factor for dimensionless time, has units of seconds
Δ_d	Determinant of the desired mass matrix
Δ_{dn}	Determinant of the dimensionless desired mass matrix
ΔL_2	Unknown load length on the second link
Δm_2	Unknown load mass on the second link
$\delta q_1, \delta\dot{q}_1, \delta\ddot{q}_1$	Angular position, velocity, and acceleration of the first link relative to the positive y axis
ε, ε_{rp}	Unitless controller gains
ε_{2Li}	Positive constants for the control law of the two-link manipulator
$\theta, \dot{\theta}, \ddot{\theta}$	Beam, arc, and pendulum generalized angular position, velocity, and acceleration
θ', θ''	First and second derivative of the beam, arc, and pendulum generalized angle with respect to unitless time
$\theta_B, \dot{\theta}_B, \ddot{\theta}_B$	Ball's rotation angle, velocity, and acceleration
$\theta_{BA}, \dot{\theta}_{BA}, \ddot{\theta}_{BA}$	Rotation angle, velocity, and acceleration of the ball on the arc
θ_N	Dimensionless system pendulum angle for simulations
κ, κ_{rp}	Unitless control gains
$\boldsymbol{\lambda}$	Positive definite diagonal gain matrix, $\in \mathbb{R}^{nxn}$
λ	$m_p + m_c$
λ_1, λ_2	Part of the pendubot control law
λ_{1n}, λ_{2n}	Part of the pendubot dimensionless control law
λ_N, λ_{rpN}	$(1+k_1)$
λ_{rp}	$(m_p + m_a)R^2$
ξ, ξ_{rp}	Unitless controller gains

Variable	Definition
$\rho, \dot{\rho}$	Dimensionless generalized coordinate and velocity
ρ', ρ''	First and second derivative of the dimensionless generalized coordinate with respect to unitless time
$\boldsymbol{\sigma}$	Constant 6-dimensional column vector of initial inertia parameters
$\boldsymbol{\sigma}_0$	Constant 6-dimensional column vector of nominal inertia parameters
$\boldsymbol{\tau}$	Vector of generalized forces and torques, $\in \mathbb{R}^n$
τ_{BA}	Torque input to drive arc
τ_{BB}	Torque input to drive beam
τ_{IPC}	Force input to drive the cart
τ_N	Dimensionless torque/force
τ_{N2}	Dimensionless torque on link 2 of the two-link manipulator
τ_{RP}	Torque input to drive rotary pendulum arm
ϕ	Angular displacement of the ball's center relative to the arc longitudinal center line Rotary Pendulum base generalized position
$\dot{\phi},\ \ddot{\phi}$	Angular velocity and displacement of the ball's center relative to the arc longitudinal center line Rotary Pendulum base generalized velocity and acceleration
$\boldsymbol{\Psi}$	Difference between nominal and initial inertia parameters

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Chapter 1 - Introduction

1.1 Research Question

A ball on an arc, as the radius of the arc is allowed to grow without bound, will locally resemble a ball on a straight-line segment, once the radius is large enough. As sizes, lengths, or shapes of a system change, the system will approach limiting forms. When these limiting forms are reached, the original mechanical system will morph into a second or target system. What are the conditions for when a system's equations of motion will morph to a simpler system's equations of motion?

The ability to morph from a higher complexity system to a simpler one could aid in controller development. A necessary but not sufficient test of the more complex system dynamics is to check that the more complex system morphs correctly into the simpler one. Should the dynamics and controller of two systems morph, there is no need for a new controller to be designed; it has already been done. When will a controller designed for the original system morph to a controller that works for the simpler system?

This work will present the sufficient conditions for morphing the equations of motion and will investigate morphing a controller that is based on the system's energy. Several examples will be presented evaluating both equations of motion morphing and controller morphing.

1.2 Previous Work

1.2.1 General

Underactuated mechanical systems (UMS) are systems with more degrees of freedom than actuators which usually have nonlinear dynamics. Some examples of such systems include gantry cranes, the inverted pendulum cart, unicycles, and rockets with gimbaled thrusters. There are many ways to design a controller for these systems. One way is through linearizing the system and then the controller can be found using pole placement, LQR, gain scheduling, and other well-known methods. Energy based control techniques, such as Controlled Lagrangians [7] and [8], the Lambda Method [5], and Interconnection Damping Assignment Passivity Based Control (IDA-PBC) [20], design a controller to replace the original system with an asymptotically stable one. In [20], it was shown that Controlled Lagrangians and the Lambda Method were subsets of a general procedure. The Direct Lyapunov Approach (DLA) [29], also uses the control law to replace the original system with an asymptotically stable one where the new system is automatically Lagrangian without having to impose additional constraints. In [30], it was shown that by setting part of the IDA-PBC control law to zero, DLA and IDA-PBC produce the same control law. The survey paper [16] lists these methods and others in greater detail, and contains the equations of motion for some of the more common systems.

1.2.2 Dimensionless Parameters

Many times, a simulation of a real or assumed system might appear in a publication, but there is no mechanism to compare one system to another. One system might be harder to control owing to inherent dynamics and some controllers may or may not be better choices to stabilize a given system. Converting a system to dimensionless form, would provide insight into the effect of parameter changes on nonlinear terms. A control law based on these dimensionless equations could be tuned for the dimensionless ratios and then utilized for different scaled models.

For finding dimensionless parameters, the method that appears most in literature is the Buckingham-Pi Theorem [11], [23], [6], and [21]. In [11], a gain-scheduling controller is designed for a dimensionless gantry system. The dimensionless parameters in [11] have similarity to those used here. Reference [23] uses a dimensionless parameter to design a rate-limiter for a first order system. Dimensionless parameters are utilized to reduce model uncertainty in the bicycle model in [6]. In [21], the dimensionless framework of a bicycle model is investigated to determine the impact of tire size on the model with the aim of doing smaller scale model testing. Reference [9], points out that the Buckingham-Pi theorem does not mention what to do with complex poles.

Some papers appearing in the literature do not explicitly use Buckingham-PI Theorem to arrive at their dimensionless equations. In [25], a spring mass system is rendered dimensionless to classify the stable equilibrium for the system. The effect of emergency lane change maneuvers is examined in [27] using a dimensionless equation for the minimum resultant vehicle force and an optimal state feedback control. In [2], certain conditions are identified for when two systems can be governed by the same control law dependent on their time constants. The system dynamics are rendered dimensionless by manipulating the equations to be independent of the choice of units in [10]. Reference [10] shows that the dimensionless parameters of the passive dynamics of a quadruped robot revealed intrinsic properties that were not observable on the original system. So that a comparison between systems is possible, the process used to obtain dimensionless equations in [15] and [32] will be utilized in this work.

1.2.3 Morphing

The literature contains some examples that utilize morphing. A general dimensionless approach is taken in [27] when looking at switching converters. The authors of [33] consider topological equivalence to examine the stability of different nonlinear time-periodic systems. In

[17], a diffeomorphism is presented that converts a non-straight line reference path to a straight-line path in the transformed domain to simplify motion control for a mobile robot. Similarly, [18] utilizes a feedback equivalence transformation for unmanned aerial vehicles to map curved paths to straight lines, simplifying the controller.

1.3 Organization of the Dissertation

In chapter 2, the process for rendering a system dimensionless will be presented. The dimensionless parameters that will be identified can be utilized for comparing systems, controllers, and the effect of different parameters on the output. Then the sufficient conditions for when the equation of motions of the original system will morph into those of a second target system as a length, size, or shape is changed of the original system is presented.

In chapter 3, the morphing of the equations of motion of the ball and arc to the ball and beam will be examined. The derived equations will be subjected to different sets of assumptions to compare the resulting equations of motion to those in the literature, similar to [15]. For the ball and beam, a math error was discovered in the equations of motion that are used extensively in the literature.

The successful morphing of the equations of motion and a controller of the rotary pendulum to the inverted pendulum cart will be presented in chapter 4. The controllers from [19] for the rotary pendulum and inverted pendulum cart will be utilized because [19] presents sufficient information to simulate and replicate results.

The more complex pendubot system will be examined and morphed to the inverted pendulum cart in chapter 5. The controller from [24] will be utilized to provide result replication and this particular IDA-PBC controller will not successfully morph to control the dimensionless inverted pendulum cart.

Chapter 6 will examine a fully actuated two-link robot manipulator to demonstrate that the sufficient conditions for morphing equations of motion work for a fully actuated system. The equations of motion and a controller for the two-link robot manipulator will be morphed to a fully actuated inverted pendulum cart. The controller comes from [28] which has sufficient information to allow for simulations to be performed.

Finally, chapter 7 will present conclusions and discuss future work in this area.

Chapter 2 - Process Overview

As lengths, sizes, or shapes of a system change this can cause the equations of motion of a system to become unwieldy especially if a length or radius becomes large. Converting equations of motion and controllers to dimensionless parameters can alleviate this issue. Once the system's length, size, or shape grows sufficiently large or shrinks to zero, the system would start to resemble a simpler system.

First in this chapter will be a review of the derivation of equation of motion for a general underactuated mechanical system. Then a process for rendering these equations of motion to dimensionless form will be presented. Lastly, a theory for when a more complex system will morph to a simpler system will be presented. This chapter has been previously published see reference [32].

2.1 Equations of Motion

Knowing the potential energy and kinetic energy of a rigid body mechanical system and utilizing Lagrange's equation, the equations of motion are derived. Kinetic energy, T, is made up of the mass matrix, $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{nxn}$ and the generalized velocities, $\dot{\boldsymbol{q}} \in \mathbb{R}^n$, of the mechanical system and is

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}\left(\boldsymbol{q}\right)\dot{\boldsymbol{q}}. \tag{2.1}$$

The mass matrix, $\boldsymbol{M}(\boldsymbol{q})$, is a function of the generalized positions, $\boldsymbol{q} \in \mathbb{R}^n$, due to the dependence of mass moments of inertia on the configuration of the mechanical system. The potential energy, $V(\boldsymbol{q})$, of a mechanical system is a function of positions and relative positions of the n bodies of body mass centers in conservative fields such as gravitational (a function of mass center locations) or strain energy (a function of relative positions of bodies connected by massless springs). The Lagrangian, L, is the difference between kinetic and potential energy, $L = T - V(\boldsymbol{q})$. The motion equations are then determined by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{Q} \tag{2.2}$$

where $\boldsymbol{Q}$ is an n-vector of generalized forces acting on the n rigid bodies and includes applied and frictional forces and torques. The operation of (2.2) yields n equations. This formulation applies to both fully actuated and underactuated systems.

Now that the equations of motion for the system have been derived, the equation can be converted to dimensionless form. The first step is to identify the units and common terms for each equation. Then, divide and simplify each equation by a judiciously-chosen common term with an eye towards the length or shape changing. This process results in dimensionless parameters which are ratios of common units, mass, length, time, etc. These dimensionless parameters demonstrate the impact changing parameters could have on the system dynamics.

The morphing of a mechanical system involves changing dimensions and shapes so that the original system changes into a second or target system. Each system has different equations of motion, and usually, different generalized coordinates and velocities. One result of the morphing is that equations of motion of the original system change into those of the target system. If the equations of motion morph, then so does the Lagrangian. The morphing of the Lagrangian requires the kinetic energy of the original system to morph into the kinetic energy of the target system and the potential energy of the original system to morph into the potential energy of the target system. The kinetic energy morphing requires the mass matrix of the original system to morph into the mass matrix of the target system, and the generalized coordinates and velocities of the original system must change into the generalized coordinates and velocities of the target system.

Therefore, the conditions necessary for the successful morphing of a mechanical system are:

1) The successful morphing of the generalized coordinates and velocities of the original system to the target system.
2) The morphing of the original mass matrix as a function of the original generalized coordinates to the mass matrix of the target system where dependency is now on the target system's generalized coordinates.
3) The original potential energy expressed in terms of the original system's generalized coordinates morphs into the potential energy of the target system expressed in terms of the target system's generalized coordinates.

The satisfaction of these three coordinates is necessary for the dynamics of the original system to morph into the dynamics of the target system. Since the equations of motions for either system depends only on the generalized coordinates and velocities, the mass matrix, and the potential energy, then the successful morphing of these quantities constitutes necessary conditions for the successful morphing of the motion equations.

2.2 Control Law

While the morphing of equations of motion is relatively straight forward, what happens to a controller as the original system morphs into the target system? Are there sufficient conditions for when a controller will morph? These questions will not be entirely answered here but the analysis will demonstrate where controller morphing might be possible. In the control of rigid body mechanical systems, if the control law utilizes the mechanical energies, controller morphing might be possible.

Controllers based on IDA-PBC, [20], for underactuated systems start with the Hamiltonian, $H(\boldsymbol{q}, \boldsymbol{p})$, of the mechanical system where $\boldsymbol{p}$ is the generalized momenta defined as the mass matrix times the generalized velocities. The Hamiltonian is the sum of the kinetic and potential energies. The motion equations then are

$$\dot{\boldsymbol{p}} = -\nabla_{\boldsymbol{q}} H(\boldsymbol{q}, \boldsymbol{p}) + \boldsymbol{G}_m(\boldsymbol{q})\boldsymbol{u} \tag{2.3}$$

where $\nabla_{\boldsymbol{q}}$ denotes the gradient with respect to $\boldsymbol{q}$ and $\boldsymbol{G}_m(\boldsymbol{q}) \in \mathbb{R}^{nxm}$ is a map from the m inputs of $\boldsymbol{u}$ to the various degrees of freedom where $m < n$ because of underactuation. The Hamiltonian is

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{p} + V(\boldsymbol{q}). \tag{2.4}$$

Note (2.2) and (2.4) produce the same equations where $\boldsymbol{G}_m(\boldsymbol{q})\boldsymbol{u}$ is the same as the generalized forces $\boldsymbol{Q}$. The IDA-PBC control law procedure solves for a new Hamiltonian

$$H_d(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p} + V_d(\boldsymbol{q}). \tag{2.5}$$

where $\boldsymbol{M}_d(\boldsymbol{q})$ is the new positive definite symmetric mass matrix and $V_d(\boldsymbol{q})$ is the new potential energy function. In deriving the new Hamiltonian, the generalized coordinates, velocities and momenta of the mechanical system have not changed. The control law takes the form

$$\boldsymbol{u} = \boldsymbol{u}_{es}(\boldsymbol{q}, \boldsymbol{p}) + \boldsymbol{u}_{di}(\boldsymbol{q}, \boldsymbol{p}) \tag{2.6}$$

where $\boldsymbol{u}_{es}(\boldsymbol{q}, \boldsymbol{p})$ is the energy shaping input providing the changes in dynamics and $\boldsymbol{u}_{di}(\boldsymbol{q}, \boldsymbol{p})$ is the damping injection input making the system passive through the generalized inputs. The control law results in the new motion equation

$$\dot{\boldsymbol{p}} = -\boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\nabla_{\boldsymbol{q}} H_d(\boldsymbol{q}, \boldsymbol{p}) + \left(\boldsymbol{J}_2(\boldsymbol{q}, \boldsymbol{p}) - \boldsymbol{G}_m(\boldsymbol{q})\boldsymbol{K}_v\boldsymbol{G}_m^T(\boldsymbol{q})\right)\nabla_{\boldsymbol{p}} H_d(\boldsymbol{q}, \boldsymbol{p}) \tag{2.7}$$

where $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p}) \in \mathbb{R}^{nxn}$ is a skew symmetric matrix and $\boldsymbol{K}_v \in \mathbb{R}^{mxm}$ is a positive definite, symmetric matrix of viscous damping coefficients. The skew symmetric matrix $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ is termed as energy conserving since it vanishes from the product of the system input and output making no contribution to the system's passivity or energy. The main contribution of the matrix $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ is that the designer chooses the elements of the matrix to aid in the process of finding the new mass matrix $\boldsymbol{M}_d(\boldsymbol{q})$. The damping injection input stems from the matrix $\boldsymbol{K}_v$ and is

$$\boldsymbol{u}_{di}(\boldsymbol{q},\boldsymbol{p}) = -\boldsymbol{K}_v \boldsymbol{G}_m^T(\boldsymbol{q}) \nabla_{\boldsymbol{p}} H_d(\boldsymbol{q},\boldsymbol{p}). \tag{2.8}$$

To find the new mass matrix $\boldsymbol{M}_d(\boldsymbol{q})$ and potential energy $\boldsymbol{V}_d(\boldsymbol{q})$, (2.3) is set equal to (2.7) and then (2.8) cancels the term involving $\boldsymbol{K}_v$ resulting in

$$\boldsymbol{G}_m(\boldsymbol{q})\boldsymbol{u}_{es}(\boldsymbol{q},\boldsymbol{p}) = \nabla_{\boldsymbol{q}} H(\boldsymbol{q},\boldsymbol{p}) - \boldsymbol{M}_d \boldsymbol{M}^{-1}(\boldsymbol{q}) \nabla_{\boldsymbol{q}} H_d(\boldsymbol{q},\boldsymbol{p}) + \boldsymbol{J}_2 \boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p} \tag{2.9}$$

To eliminate the input from (2.9), (2.9) is multiplied by the left annihilator $\boldsymbol{G}_m^{\perp}(\boldsymbol{q}) \in \mathbb{R}^{mxn}$ where $\boldsymbol{G}_m^{\perp}(\boldsymbol{q})\, \boldsymbol{G}_m(\boldsymbol{q}) = 0$. Substituting for the Hamiltonians $H(\boldsymbol{q}, \boldsymbol{p})$ and $H_d(\boldsymbol{q}, \boldsymbol{p})$ in (2.9) yields two equations

$$\boldsymbol{G}_m^{\perp}(\boldsymbol{q})\left\{\nabla_{\boldsymbol{q}}\left(\boldsymbol{p}^T \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{p}\right) + 2\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})\boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p} - \boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\nabla_{\boldsymbol{q}}\left(\boldsymbol{p}^T \boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p}\right)\right\} = 0 \tag{2.10}$$

and

$$\boldsymbol{G}_m^{\perp}(\boldsymbol{q})\left[\nabla_{\boldsymbol{q}} V(\boldsymbol{q}) - \boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\nabla_{\boldsymbol{q}} V_d(\boldsymbol{q})\right] = 0 \tag{2.11}$$

each of which is a partial differential equation (PDE). Equation (2.10) determines $\boldsymbol{M}_d(\boldsymbol{q})$ and (2.11) provides $V_d(\boldsymbol{q})$. The energy shaping input then is

$$\begin{aligned}\boldsymbol{u}_{es}(\boldsymbol{q},\boldsymbol{p}) = {}& \left(\boldsymbol{G}_m^T(\boldsymbol{q})\boldsymbol{G}_m(\boldsymbol{q})\right)^{-1}\boldsymbol{G}_m^T(\boldsymbol{q})\left\{\nabla_{\boldsymbol{q}} V(\boldsymbol{q}) - \boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\nabla_{\boldsymbol{q}} V_d(\boldsymbol{q})\right\} + \\ & \left(\boldsymbol{G}_m^T(\boldsymbol{q})\boldsymbol{G}_m(\boldsymbol{q})\right)^{-1}\boldsymbol{G}_m^T(\boldsymbol{q})\left\{\begin{array}{l}\nabla_{\boldsymbol{q}}\left(\frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{p}\right) + \boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})\boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p} \\ -\boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\nabla_{\boldsymbol{q}}\left(\frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p}\right)\end{array}\right\}.\end{aligned} \tag{2.12}$$

The above covers the methods of Controlled Lagrangians, the Lambda Method, and IDA-PBC. For the DLA, [30] shows that requiring the term in braces in (2.10) to vanish produces a new mass matrix, and the energy shaping input for the kinetic energy also vanishes. Then (2.10)-(2.12) also applies to the DLA.

For (2.10)-(2.12), the solution to (2.10) depends on $\boldsymbol{M}(\boldsymbol{q})$ and $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ while the solution of (2.11) depends on $V(\boldsymbol{q})$, $\boldsymbol{M}(\boldsymbol{q})$ and $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$. The matrix $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ is arbitrary for IDA-PBC, as noted by [20], whereas for Controlled Lagrangians, it depends on $\boldsymbol{M}(\boldsymbol{q})$ and $\boldsymbol{M}_d(\boldsymbol{q})$ through

$$\begin{aligned} &\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p}) = \\ &\boldsymbol{M}_d(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left\{\left[\nabla_q\left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p}\right)\right]^T - \nabla_q\left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{M}_d^{-1}(\boldsymbol{q})\boldsymbol{p}\right)\right\}\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{M}_d(\boldsymbol{q}). \end{aligned} \tag{2.13}$$

For Controlled Lagrangians, $\boldsymbol{M}_d(\boldsymbol{q})$ and $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ show a dependence on the kinetic energy of the mechanical system. While morphing the equation of motion of (2.2) is clear, the influence of morphing on $\boldsymbol{M}_d(\boldsymbol{q})$ and $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ is not. Equation (2.2) only requires differentiation whereas solving for $\boldsymbol{M}_d(\boldsymbol{q})$ and $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ requires solving PDEs. The influence these derived values have on the control law as it is morphed from the original system to the target system is uncertain.

2.3 Conclusions

In summary, equations of motion can be derived, knowing a system's kinetic and potential energy function, utilizing (2.2). As a size, length, or shape changes a system will morph to a target system if the necessary conditions for morphing are met:

1) The generalized coordinates and velocities of the original system morph to those of the target system.
2) The mass matrix of the original system morphs to the mass matrix of the target system dependent on the target system's generalized coordinates.

3) The potential energy function of the original system morphs to that of the target system expressed in terms of the target system's generalized coordinates.

Upon solving for the equations of motion, one can render them dimensionless to aid in comparison or ease of modifying the characteristics of a physical system. The effect this morphing has on controllers has not been fully realized yet and a general theory is still in development.

In the case of a system utilizing a control based on the Controlled Lagrangian method controller morphing is possible. When the equation of motion morph from the original to a target system, then $\boldsymbol{M}(\boldsymbol{q})$ and $V(\boldsymbol{q})$ morph. The controller, $\boldsymbol{u}$, utilizes $\boldsymbol{M}(\boldsymbol{q})$, $\boldsymbol{M}_d(\boldsymbol{q})$, $V(\boldsymbol{q})$, $V_d(\boldsymbol{q})$ and $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$, where $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$ is dependent on $\boldsymbol{M}(\boldsymbol{q})$ and $\boldsymbol{M}_d(\boldsymbol{q})$, and $V_d(\boldsymbol{q})$ depends on $\boldsymbol{M}(\boldsymbol{q})$ and $\boldsymbol{M}_d(\boldsymbol{q})$ as well as $V(\boldsymbol{q})$. If it can be shown that if $\boldsymbol{M}_d(\boldsymbol{q})$ successfully morphs for the original system to that of the target system's $\boldsymbol{M}_d(\boldsymbol{q})$, where $\boldsymbol{q}$ is now the target system's generalized coordinates, then the controller $\boldsymbol{u}$ would successfully morph from the original system to the target system.

Chapters 3 through 5 will present underactuated systems whose equations of motion meet the necessary conditions for morphing to a somewhat simpler target system. Chapter 4 will also present a controller which successfully morphs and then present simulations to compare the systems. Chapter 5 will present a controller which upon being morphed does not successfully stabilize the target system. Chapter 6 will present a fully actuated system whose equations of motion and a controller successfully morph. Lastly Chapter 7 will present conclusions.

Chapter 3 - Ball and Arc to Ball and Beam

Now that the sufficient condition for equation of motion to morph has been presented, three sets of systems will be investigated. The first example supporting the sufficient conditions is morphing the ball and arc system to the ball and beam system. Control of the ball and arc has not been investigated to the same extent in the literature as the examples in the subsequent chapters, therefore this chapter will only be examining equations of motion. The ball and beam systems and ball and arc systems that appear in the literature are subjected to many different assumptions, some incorrect, and a full set of equations of motion with few assumptions is not readily available. In this chapter the equations of motion will be derived utilizing both Newton-Euler and Lagrangian-Euler derivations and then compared to often used equations of motion in the literature.

In the first section of this chapter, the ball and beam will be analyzed and equations of motion will be derived. Next, these equations of motion will be compared to models existing in the literature. Then, the equations of motion will be rendered dimensionless.

In the second section, the ball and arc system will be studied and equations of motion will be derived and then compared to the often-cited equations of motion. Lastly, the equations of motion will be rendered dimensionless.

Then the equations of motion for the ball and arc system will be morphed to those of the ball and beam. Lastly, conclusions will be presented about these two systems. The contents of this chapter have been published in *ASME Journal of Dynamic Systems, Measurements, and Control* see reference [15].

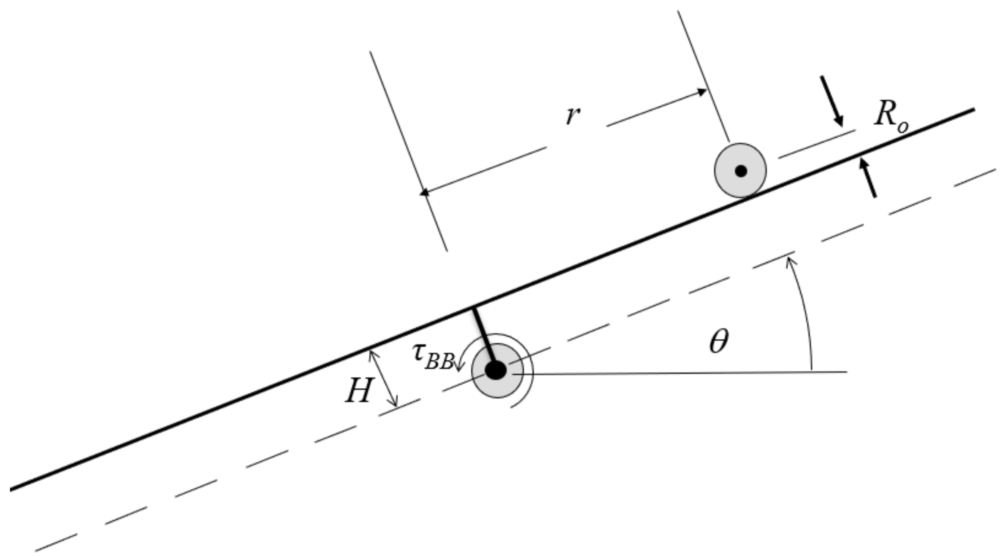


Figure 3.1: Ball and Beam with Offset

3.1 Ball and Beam Analysis

The ball and beam first appeared, to the author's knowledge in the literature in 1989, in [12]. In that work, the ball was modeled as a point mass and the beam rotated about its mass center. An often-cited ball and beam paper that doesn't have these assumptions is [13], but has incorrect equations of motion. In [4], the ball again is not a point mass and the beam rotates about a point offset from the center of mass, but they use an incorrect kinematic analysis to derive their equations of motion. In this section, the equations of motion for the ball and beam will be derived similar to [15], where the ball is not a point mass and the beam rotates about a point offset from the beam by a distance H. The distance l_{beam} measures the length from the rotation point to the center of mass of the beam as shows in the free body diagram of Figure 3.2. Then the equations of motion will be rendered dimensionless. Lastly, the equations of motion will be compared to those in [12], [13], and [4].

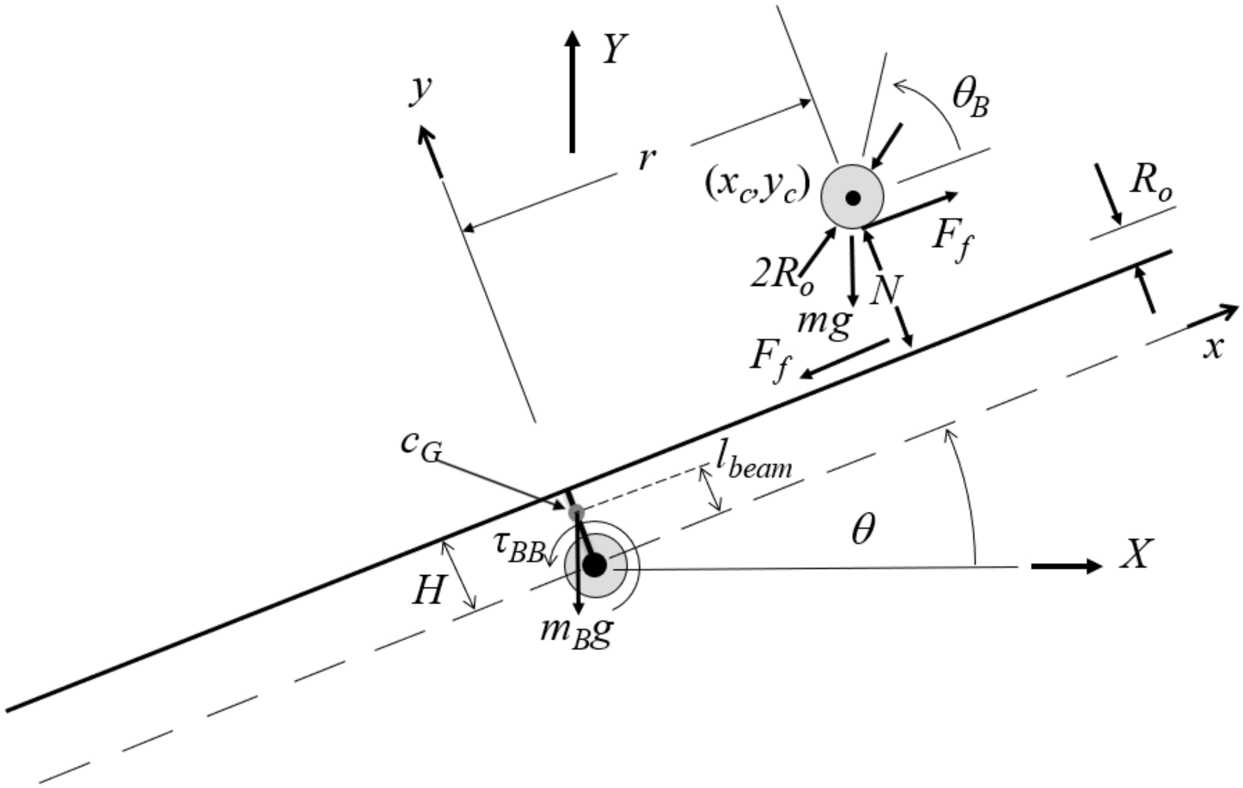


Figure 3.2: Ball and Beam Free Body Diagram

3.1.1 Equations of Motion

3.1.1.1 Newton-Euler Derivation

Figure 3.2 shows a free body diagram of the ball and beam. Summing the forces on the ball in the *x* direction, Newton's second law shows

$$F_f - mg\sin(\theta) = m\ddot{x} \tag{3.1.1}$$

and doing the same in the *y* direction produces

$$-mg\cos(\theta) + N = m\ddot{y}. \tag{3.1.2}$$

Summing the moments about a line passing through the ball center parallel to the z axis, which has a positive, right hand direction out of the plane of Figure 3.2, Euler's equation yields

$$R_o F_f = J_B \ddot{\theta}_B. \tag{3.1.3}$$

Summing the moments acting on the beam about the point of rotation using the same positive direction used for (3.1.3) and adding (3.1.3) shows that

$$\tau_{BB} - Nr + F_f H + g m_B l_{beam} \sin(\theta) - J_B \ddot{\theta}_B + R_o F_f = J_{beam} \ddot{\theta}. \tag{3.1.4}$$

Referring to the ball and beam in Figure 3.2, the ball's center location coordinates relative to the *X*-*Y* frame origin are

$$x_c = r\cos(\theta) - (R_o + H)\sin(\theta) \tag{3.1.5}$$

and

$$y_c = r\sin(\theta) + (R_o + H)\cos(\theta). \tag{3.1.6}$$

The ball's center velocity components in the *X*-*Y* frame are

$$\dot{x}_c = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta} - (R_o + H)\cos(\theta)\dot{\theta} \tag{3.1.7}$$

and

$$\dot{y}_c = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta} - (R_o + H)\sin(\theta)\dot{\theta}. \tag{3.1.8}$$

The ball's center acceleration in the *X*-*Y* frame is found by differentiating (3.1.7) and (3.1.8) with respect to time to obtain

$$\ddot{x}_c = \ddot{r}\cos(\theta) - 2\dot{r}\sin(\theta)\dot{\theta} - r\cos(\theta)\dot{\theta}^2 - r\sin(\theta)\ddot{\theta} + (R_o + H)\left(\sin(\theta)\dot{\theta}^2 - \cos(\theta)\ddot{\theta}\right) \tag{3.1.9}$$

and

$$\ddot{y}_c = \ddot{r}\sin(\theta) + 2\dot{r}\cos(\theta)\dot{\theta} - r\sin(\theta)\dot{\theta}^2 + r\cos(\theta)\ddot{\theta} - (R_o + H)\left(\cos(\theta)\dot{\theta}^2 + \sin(\theta)\ddot{\theta}\right). \tag{3.1.10}$$

Let *x* and *y* represent the ball center coordinates in the *x, y* frame. Referring the ball kinematics to the *x*-*y* coordinate system, the velocity and acceleration components of the ball center become

$$\dot{x} = \dot{x}_c\cos(\theta) + \dot{y}_c\sin(\theta) = \dot{r} - (R_o + H)\dot{\theta}, \tag{3.1.11}$$

$$\ddot{x} = \ddot{x}_c\cos(\theta) + \ddot{y}_c\sin(\theta) = \ddot{r} - r\dot{\theta}^2 - (R_o + H)\ddot{\theta}, \tag{3.1.12}$$

$$\dot{y} = -\dot{x}_c \sin(\theta) + \dot{y}_c \cos(\theta) = r\dot{\theta}, \tag{3.1.13}$$

and

$$\ddot{y} = -\ddot{x}_c \sin(\theta) + \ddot{y}_c \cos(\theta) = 2\dot{r}\dot{\theta} + r\ddot{\theta} - (R_o + H)\dot{\theta}^2. \tag{3.1.14}$$

The angular velocity and acceleration of the ball stem from the time derivatives of the ball orientation angle given by

$$\theta_B = \theta - \frac{r}{R_o} \tag{3.1.15}$$

to get

$$\dot{\theta}_B = \dot{\theta} - \frac{\dot{r}}{R_o} \tag{3.1.16}$$

and

$$\ddot{\theta}_B = \ddot{\theta} - \frac{\ddot{r}}{R_o}. \tag{3.1.17}$$

In [4], the authors list the orientation angle of the ball, here given by (3.1.15), as just r/R_o neglecting to add θ, the rotation of the beam. When the beam rotates to $\pi/4$ and the ball does not rotate, the kinematic equation in [4] would have the angular position of the ball as zero when it should be, from (3.1.15), $\pi/4$. The time derivative of the kinematic analysis in [4], $\theta = -\dot{r}/R_o$, also shows that the angular velocity of the beam is not included in the angular velocity of the ball.

Substituting into (3.1.3) for the angular acceleration of the ball in terms of $\ddot{\theta}$ and $\ddot{r}$ by using (3.1.17) including the accelerations from (3.1.12) and (3.1.14), substituting F_f from (3.1.1), and N from (3.1.2), the equation of motion for the ball becomes

$$-\left(R_o m(R_o + H) + J_B\right)\ddot{\theta} + \left(mR_o + \frac{J_B}{R_o}\right)\ddot{r} - rmR_o\dot{\theta}^2 + mgR_o \sin(\theta) = 0. \tag{3.1.18}$$

Dividing the last result by R_o shows the ball equation is

$$-\left(m\left(R_o+H\right)+\frac{J_B}{R_o}\right)\ddot{\theta}+\left(m+\frac{J_B}{R_o^2}\right)\ddot{r}-rm\dot{\theta}^2+mg\sin\left(\theta\right)=0, \tag{3.1.19}$$

a step done to eventually provide a symmetric mass matrix. Substituting into (3.1.4) using N from (3.1.2), F_f from (3.1.1), together with (3.1.17), the beam's dynamic equation becomes

$$\begin{aligned}&\left(\left(R_o+H\right)^2 m+J_B+J_{beam}+mr^2\right)\ddot{\theta}-\left(\left(H+R_o\right)m+\frac{J_B}{R_o}\right)\ddot{r}+2mr\dot{r}\dot{\theta}\\&+rmg\cos\left(\theta\right)-\left(H+R_o\right)mg\sin\left(\theta\right)-gm_B l_{beam}\sin\left(\theta\right)-\tau_{BB}=0.\end{aligned} \tag{3.1.20}$$

The dynamic equations of motion take the form of

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}+\boldsymbol{G}(\boldsymbol{q})=\boldsymbol{\tau} \tag{3.1.21}$$

where $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{nxn}$ is a symmetric, positive definite matrix of inertial and mass terms, where n is the number of degrees of freedom, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^{nxn}$ is a matrix of Coriolis and centripetal acceleration coefficients, $\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^n$ is a vector of gravitational forces and torques, $\tau \in \mathbb{R}^n$ is a vector of actuations, and $\boldsymbol{q} \in \mathbb{R}^n$ are the generalized coordinates. Time derivatives of $\boldsymbol{q}$ provide the generalized velocities and accelerations, denoted as $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$, respectively. Using (3.1.19) and (3.1.20) the matrices for (3.1.21) are

$$\boldsymbol{M}\left(\boldsymbol{q}\right)=\begin{bmatrix}\left(R_o+H\right)^2 m+J_B+J_{beam}+mr^2 & -\left(H+R_o\right)m-\dfrac{J_B}{R_o}\\ -\left(H+R_o\right)m-\dfrac{J_B}{R_o} & m+\dfrac{J_B}{R_o^2}\end{bmatrix}, \tag{3.1.22}$$

$$\boldsymbol{C}\left(\boldsymbol{q},\dot{\boldsymbol{q}}\right)=\begin{bmatrix}rm\dot{r} & rm\dot{\theta}\\ -rm\dot{\theta} & 0\end{bmatrix}, \tag{3.1.23}$$

$$\boldsymbol{G}\left(\boldsymbol{q}\right)=\begin{bmatrix}-\left(R_o+H\right)mg\sin\left(\theta\right)+rmg\cos\left(\theta\right)-gm_B l_{beam}\sin\left(\theta\right)\\ mg\sin\left(\theta\right)\end{bmatrix}, \tag{3.1.24}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{BB} \\ 0 \end{bmatrix}. \tag{3.1.25}$$

3.1.1.2 Lagrangian Formulation

The kinetic energy for the ball and beam is

$$T = \frac{1}{2} J_{beam} \dot{\theta}^2 + \frac{1}{2} J_B \dot{\theta}_B^2 + \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) \tag{3.1.26}$$

where $\dot{x}$ and $\dot{y}$ are specified in (3.1.11) and (3.1.13), respectively, and using (3.1.16), the kinetic energy becomes

$$\begin{gathered} T = \frac{1}{2} J_{beam} \dot{\theta}^2 + \frac{1}{2} J_B \dot{\theta}^2 - J_B \dot{\theta} \frac{\dot{r}}{R_o} + \frac{1}{2} J_B \frac{\dot{r}^2}{R_o^2} + \frac{1}{2} m\dot{r}^2 - m\left(R_o + H\right)\dot{r}\dot{\theta} \\ + \frac{1}{2} mr^2 \dot{\theta}^2 + \frac{1}{2} m\dot{\theta}^2 R_o^2 + m\dot{\theta}^2 R_o H + \frac{1}{2} m\dot{\theta}^2 H^2. \end{gathered} \tag{3.1.27}$$

The gravitational potential energy is

$$V = mgr\sin\left(\theta\right) + mg\left(R_o + H\right)\cos\left(\theta\right) + m_B g l_{beam} \cos\left(\theta\right). \tag{3.1.28}$$

Utilizing $L = T - V$ and (2.2), the ball position equation is

$$-\left(\frac{J_B}{R_o} + m\left(R_o + H\right)\right)\ddot{\theta} + \left(\frac{J_B}{R_o^2} + m\right)\ddot{r} - mr\dot{\theta}^2 + mg\sin\left(\theta\right) = 0. \tag{3.1.29}$$

The beam position equation is

$$\begin{gathered} \left(J_B + m\left(R_o + H\right)^2 + mr^2 + J_{beam}\right)\ddot{\theta} - \left(\frac{J_B}{R_o} + m\left(R_o + H\right)\right)\ddot{r} \\ + 2mr\dot{r}\dot{\theta} + mgr\cos\left(\theta\right) - mg\sin\left(\theta\right)\left(R_o + H\right) - gm_B l_{beam}\sin\left(\theta\right) = \tau_{BB}. \end{gathered} \tag{3.1.30}$$

Equations (3.1.29) and (3.1.30) are the same as (3.1.19) and (3.1.20), respectively.

3.1.2 Comparison to Models in Literature

To compare the equations of motion of equation (3.1.29) and (3.1.30) to those of [12], the ball's inertia J_B, radius R_o, and offset H, are set to zero, the center of mass of the beam is moved to the rotation point, and the ball equation is divided through by the mass of the ball, m. Then (3.1.29) and (3.1.30) become

$$\ddot{r} - r\dot{\theta}^2 + g\sin\left(\theta\right) = 0 \tag{3.1.31}$$

and

$$\left(mr^2 + J_{beam}\right)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\left(\theta\right) = \tau_{BB}, \tag{3.1.32}$$

which match the equations of motion from [12]. The mass matrix presented in [13], has a ball of mass m and radius R_o rolling on a beam rotating about a point in line with the center of mass of the ball is

$$\boldsymbol{M}\left(\boldsymbol{q}\right) = \begin{bmatrix} J_B + J_{beam} + mr^2 & 0 \\ 0 & m + \dfrac{J_B}{R_o^2} \end{bmatrix}. \tag{3.1.33}$$

This matrix is using the inertia and radius of the ball together with the roll without slip condition but the off-diagonal terms are zero. In an attempt of simplifying the mass matrix of (3.1.22), three different assumptions are examined. First, assume a slider, moving on a frictionless beam surface, replaces the ball, where radius R_o, equal to half the slider thickness, is not zero, and the offset H is -R_o. This assumption changes the kinematics of (3.1.5)-(3.1.8) to

$$x_c = r\cos\left(\theta\right), \tag{3.1.34}$$

$$y_c = r\sin\left(\theta\right), \tag{3.1.35}$$

$$\dot{x}_c = \dot{r}\cos\left(\theta\right) - r\sin\left(\theta\right)\dot{\theta}, \tag{3.1.36}$$

and

$$\dot{y}_c = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta}\,. \tag{3.1.37}$$

Since the slider cannot rotate, (3.1.16) becomes

$$\dot{\theta}_B = \dot{\theta}. \tag{3.1.38}$$

Then the kinetic energy of (3.1.26) becomes

$$T = \frac{1}{2}J_{beam}\dot{\theta}^2 + \frac{1}{2}J_B\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \tag{3.1.39}$$

leading to a mass matrix of

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} J_B + J_{beam} + mr^2 & 0 \\ 0 & m \end{bmatrix} \tag{3.1.40}$$

which does not match the mass matrix of [13]. Next, assume the ball rotates with radius R_o, rolls without slip, and the offset H is -R_o, then using (3.1.34)-(3.1.37) and (3.1.16), (3.1.26) becomes

$$T = \frac{1}{2}J_{beam}\dot{\theta}^2 + \frac{1}{2}J_B\dot{\theta}^2 - J_B\dot{\theta}\frac{\dot{r}}{R_o} + \frac{1}{2}J_B\frac{\dot{r}^2}{R_o^2} + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right). \tag{3.1.41}$$

The corresponding mass matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} J_{beam} + J_B + mr^2 & -\dfrac{J_B}{R_o} \\ -\dfrac{J_B}{R_o} & \dfrac{J_B}{R_o^2} + m \end{bmatrix}, \tag{3.1.42}$$

which is not the same as in [13]. Lastly, solve (3.1.22) for the value of H that causes the off-diagonal terms to be zero. Setting H to

$$H = -\frac{J_b + mR_o^2}{mR_o} \tag{3.1.43}$$

results in a mass matrix of

$$M(q) = \begin{bmatrix} mr^2 + J_{beam} + J_B + \dfrac{J_B^2}{mR_o^2} & 0 \\ 0 & \dfrac{J_B}{R_o^2} + m \end{bmatrix} \tag{3.1.44}$$

which also does not match that of [13]. There is not a set of assumptions that yields the same equations of motion as those presented in [13].

3.1.3 Dimensionless Equations of Motion

Now to transform the ball and beam equations from (3.1.22)-(3.1.25) into dimensionless equations, divide the beam (first) equation by mR_o^2 and ball (second) equation by mR_o. By canceling like coefficients, the terms for (3.1.21) become

$$M(q)\ddot{q} = \begin{bmatrix} \dfrac{J_B}{mR_o^2} + 1 + \dfrac{r^2}{R_o^2} + \dfrac{2H}{R_o} + \dfrac{H^2}{R_o^2} + \dfrac{J_{beam}}{mR_o^2} & -\left(\dfrac{J_B}{mR_o^2} + 1 + \dfrac{H}{R_o}\right) \\ -\left(\dfrac{J_B}{mR_o^2} + 1 + \dfrac{H}{R_o}\right) & \dfrac{J_B}{mR_o^2} + 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \dfrac{\ddot{r}}{R_o} \end{bmatrix}, \tag{3.1.45}$$

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} \dfrac{r}{R_o}\dfrac{\dot{r}}{R_o} & \dfrac{r}{R_o}\dot{\theta} \\ -\dfrac{r}{R_o}\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dfrac{\dot{r}}{R_o} \end{bmatrix}, \tag{3.1.46}$$

$$G(q) = \begin{bmatrix} \dfrac{g}{R_o}\left(-\dfrac{(R_o + H)}{R_o}\sin(\theta) + \dfrac{r}{R_o}\cos(\theta) - \dfrac{m_B l_{beam}}{mR_o}\sin(\theta)\right) \\ \dfrac{g}{R_o}\sin(\theta) \end{bmatrix}, \tag{3.1.47}$$

and

$$\tau = \begin{bmatrix} \dfrac{\tau_{BB}}{mR_o^2} \\ 0 \end{bmatrix}. \tag{3.1.48}$$

Table 3.1 shows the dimensionless parameters that will be utilized for this chapter. Utilizing Table 3.1 and multiplying both equations by γ^2 to change the time scale to unitless time, $\underline{t}$, (3.1.45)-(3.1.48) become

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'' = \begin{bmatrix} k_4 + (1+k_6)^2 + \rho^2 + k_3 & -k_4 - 1 - k_6 \\ -k_4 - 1 - k_6 & k_4 + 1 \end{bmatrix} \begin{bmatrix} \theta'' \\ \rho'' \end{bmatrix}, \tag{3.1.49}$$

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}')\boldsymbol{q}' = \begin{bmatrix} \rho\rho' & \rho\theta' \\ -\rho\theta' & 0 \end{bmatrix} \begin{bmatrix} \theta' \\ \rho' \end{bmatrix}, \tag{3.1.50}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \rho\cos(\theta) - (1 + k_6 + k_1 k_5)\sin(\theta) \\ \sin(\theta) \end{bmatrix}, \tag{3.1.51}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ 0 \end{bmatrix}, \tag{3.1.52}$$

where $\theta' = \dfrac{d\theta}{d\underline{t}}$, $\theta'' = \dfrac{d^2\theta}{d\underline{t}^2}$, $\rho' = \dfrac{d\rho}{d\underline{t}}$, and $\rho'' = \dfrac{d^2\rho}{d\underline{t}^2}$.

Table 3.1: Dimensionless Ball and Beam and Ball and Arc Parameters

Dimensionless Parameter	k_1	k_3	k_4	k_5	k_6	k_7	ρ	γ^2	τ_N
Ball and Beam	$\frac{m_B}{m}$	$\frac{J_{beam}}{mR_o^2}$	$\frac{J_B}{mR_o^2}$	$\frac{l_{beam}}{R_o}$	$\frac{H}{R_o}$	--	$\frac{r}{R_o}$	$\frac{R_o}{g}$	$\frac{\gamma^2\tau_{BB}}{mR_o^2}$
Ball and Arc	$\frac{m_a}{m}$	$\frac{J_{arc}}{mR_o^2}$	$\frac{J_B}{mR_o^2}$	$\frac{l_{arc}}{R_o}$	$\frac{H}{R_o}$	$\frac{R}{R_o}$	$\frac{(R+R_o)\phi}{R_o}$	$\frac{R_o}{g}$	$\frac{\gamma^2\tau_{BA}}{mR_o^2}$

3.2 Ball and Arc

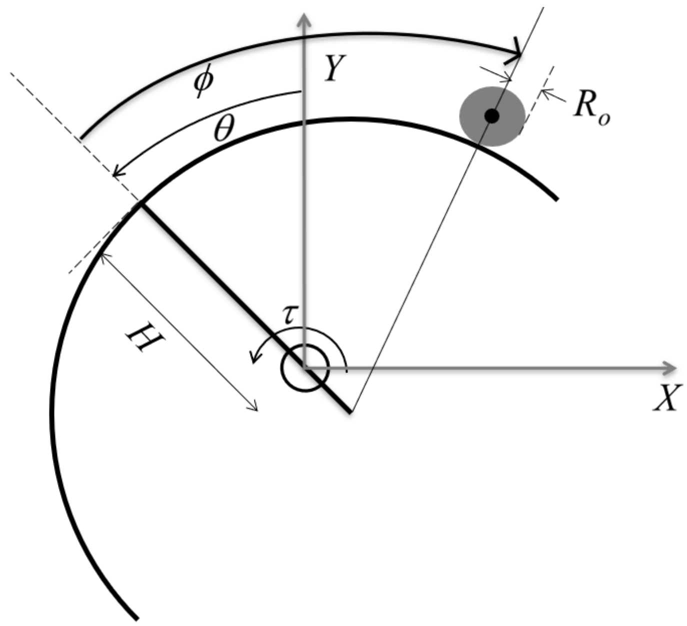


Figure 3.3: Ball and Arc

The ball and arc first appeared, to the author's knowledge, in 2007 in [3]. In that paper the ball is modeled having a non-zero radius rolling along a circular beam which was rotating about a point that was not the center of mass. This paper has the same kinematic error for the ball's rotational angle as [4]. The ball and arc of [26] modeled the ball as a point mass and constrained the center of mass of the arc to the rotation point. The ball and arc model in Figure 3.2, consists of a ball of mass m and radius R_o rolling without slip along an arc of radius R with mass m_a. The arc rotates about a point o that is a distance of H from the arc's edge and a distance of l_{arc} from the center of mass of the arc. In this section, the equations of motion for the ball and arc of Figure 3.2 will be derived using both Newton-Euler and Euler-Lagrangian analyses. Then the equations of motion will be made dimensionless.

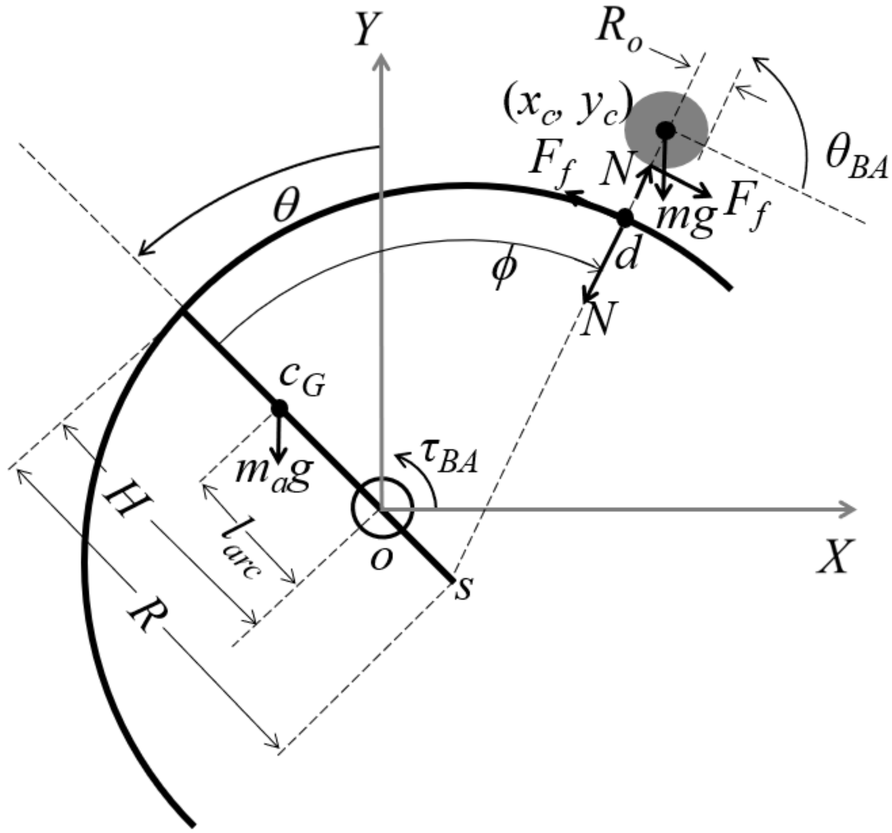


Figure 3.4: Ball and Arc Free Body Diagram

3.2.1 Equations of Motion

3.2.1.1 Newton-Euler Derivation

For the ball and arc in Figure 3.4, the ball's center relative to the *X*-*Y* frame origin is

$$x_c = (R + R_o)\sin(\phi - \theta) + (R - H)\sin(\theta) \tag{3.2.1}$$

and

$$y_c = (R + R_o)\cos(\phi - \theta) - (R - H)\cos(\theta). \tag{3.2.2}$$

In the *X*-*Y* frame, the ball's center velocity components are

$$\dot{x}_c = (R + R_o)\cos(\phi - \theta)(\dot{\phi} - \dot{\theta}) + (R - H)\cos(\theta)\dot{\theta} \tag{3.2.3}$$

and

$$\dot{y}_c = -(R+R_o)\sin(\phi-\theta)(\dot{\phi}-\dot{\theta}) + (R-H)\sin(\theta)\dot{\theta}. \tag{3.2.4}$$

In the *X*-*Y* frame, the ball's center acceleration is

$$\begin{aligned}\ddot{x}_c &= (R+R_o)\left(\cos(\phi-\theta)(\ddot{\phi}-\ddot{\theta}) - \sin(\phi-\theta)(\dot{\phi}-\dot{\theta})^2\right)\\ &+(R-H)\left(\cos(\theta)\ddot{\theta} - \sin(\theta)\dot{\theta}^2\right)\end{aligned} \tag{3.2.5}$$

and

$$\begin{aligned}\ddot{y}_c &= -(R+R_o)\left(\cos(\phi-\theta)(\dot{\phi}-\dot{\theta})^2 + \sin(\phi-\theta)(\ddot{\phi}-\ddot{\theta})\right)\\ &+(R-H)\left(\sin(\theta)\ddot{\theta} + \cos(\theta)\dot{\theta}^2\right).\end{aligned} \tag{3.2.6}$$

Summing the forces in the *X* direction, Newton's second law shows

$$F_f\cos(\phi-\theta) + N\sin(\phi-\theta) = m\ddot{x}_c \tag{3.2.7}$$

and, similarly in the *Y* direction, produces

$$-mg - F_f\sin(\phi-\theta) + N\cos(\phi-\theta) = m\ddot{y}_c. \tag{3.2.8}$$

For a line passing through the ball, parallel to the *z* axis which has a positive, right hand direction out of the plane, summing the moments shown in Figure 3.4 and using Euler's equation yields

$$-R_oF_f = J_B\ddot{\theta}_{BA}. \tag{3.2.9}$$

The angular velocity of the ball, $\dot{\theta}_{BA}$, stems from calculating the velocity of the ball and the velocity of the contact point. To do this, first the analysis finds the position vector from the origin to the contact point using

$$\boldsymbol{r}_{d/o} = \boldsymbol{r}_{s/o} + \boldsymbol{r}_{d/s}. \tag{3.2.10}$$

where

$$\boldsymbol{r}_{s/o} = (R-H)\begin{bmatrix}\sin(\theta)\\ -\cos(\theta)\\ 0\end{bmatrix} \tag{3.2.11}$$

and

$$\boldsymbol{r}_{d/s} = \begin{bmatrix} R\sin(\phi-\theta) \\ R\cos(\phi-\theta) \\ 0 \end{bmatrix} \tag{3.2.12}$$

where $\boldsymbol{r}_{u/w}$ denotes the relative position vector to point u with respect to point w. Next, the unit radial vector $\boldsymbol{e}_r$ is $\boldsymbol{e}_r = \dfrac{\boldsymbol{r}_{d/s}}{\|\boldsymbol{r}_{d/s}\|}$, where $\|\boldsymbol{r}_{d/s}\| = R$ and the unit tangent vector $\boldsymbol{e}_t$ is $\boldsymbol{e}_t = -\boldsymbol{k}\times\boldsymbol{e}_r$. The velocity of the contact point is $\boldsymbol{v}_d = \dot{\theta}(\boldsymbol{k})\times\boldsymbol{r}_{d/o}$ and the velocity of the ball is $\boldsymbol{v}_c = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$.

Finally, $\dot{\theta}_{BA}$ is

$$\dot{\theta}_{BA} = -\frac{(\boldsymbol{v}_c - \boldsymbol{v}_d)}{R_o}\cdot\boldsymbol{e}_t. \tag{3.2.13}$$

The operation of (3.2.13) produces

$$\dot{\theta}_{BA} = -\frac{\dot{\phi}(R+R_o)}{R_o} + \dot{\theta}. \tag{3.2.14}$$

The time derivative of (3.2.14) shows

$$\ddot{\theta}_{BA} = -\frac{\ddot{\phi}(R+R_o)}{R_o} + \ddot{\theta}. \tag{3.2.15}$$

Integrating with respect to time (3.2.14), the orientation angle of the ball becomes

$$\theta_{BA} = -\frac{\phi(R+R_o)}{R_o} + \theta. \tag{3.2.16}$$

Substituting the accelerations from (3.2.5) and (3.2.6) into (3.2.7) and (3.2.8), along with F_f from (3.2.7), N from (3.2.8), and $\dot{\theta}_{BA}$ from (3.2.15) into (3.2.9), the equation of motion for the ball becomes

$$\begin{aligned}&\left(-mR_o^2-RR_om-J_B+\left(R-H\right)R_om\cos\left(\phi\right)\right)\ddot{\theta}+\left(RR_om+R_o^2m+J_B\left(\frac{R}{R_o}+1\right)\right)\ddot{\phi}\\&-\left(R_om\left(R-H\right)\sin\left(\phi\right)\right)\dot{\theta}^2-g\sin\left(\phi-\theta\right)R_om=0.\end{aligned}\tag{3.2.17}$$

Multiplying the last result by $(R + R_o)/R_o$ produces a symmetric mass matrix, where the ball equation is

$$\begin{aligned}&\left(R+R_o\right)\left(-mR_o-Rm-\frac{J_B}{R_o}+\left(R-H\right)m\cos\left(\phi\right)\right)\ddot{\theta}+\left(R+R_o\right)^2\left(m+\frac{J_B}{R_o^2}\right)\ddot{\phi}\\&-\left(\left(R+R_o\right)m\left(R-H\right)\sin\left(\phi\right)\right)\dot{\theta}^2-\left(R+R_o\right)g\sin\left(\phi-\theta\right)m=0.\end{aligned}\tag{3.2.18}$$

Summing the moments acting on the arc about the fixed point of rotation using the same positive direction used for (3.2.9) and using Euler's equation shows that

$$-\left(R-H\right)N\sin\left(\phi\right)+F_f\left(R-\left(R-H\right)\cos\left(\phi\right)\right)+gm_al_{arc}\sin\left(\theta\right)+\tau_{BA}=J_{arc}\ddot{\theta}.\tag{3.2.19}$$

Using F_f from (3.2.7), N from (3.2.8), and adding (3.2.9) to (3.2.19) shows the arc equation of motion is

$$\begin{aligned}&\left(-2\left(R+R_o\right)\left(R-H\right)m\cos\left(\phi\right)+J_{arc}+\left(2R^2-2HR+2RR_o+R_o^2+H^2\right)m+J_B\right)\ddot{\theta}\\&-\left(R+R_o\right)\left(mR_o+Rm+\frac{J_B}{R_o}-\left(R-H\right)m\cos\left(\phi\right)\right)\ddot{\phi}\\&+\left(2\left(R+R_o\right)m\left(R-H\right)\sin\left(\phi\right)\right)\dot{\phi}\dot{\theta}-\left(R+R_o\right)m\left(R-H\right)\sin\left(\phi\right)\dot{\phi}^2\\&+g\left(R+R_o\right)m\sin\left(\phi-\theta\right)+g\left(\left(R-H\right)m-l_{arc}m_a\right)\sin\left(\theta\right)=\tau_{BA}.\end{aligned}\tag{3.2.20}$$

From (3.2.18) and (3.2.20), the matrices for (3.1.21) are

$$\boldsymbol{M}\left(\boldsymbol{q}\right)=\begin{bmatrix}M1 & M2\\ M2 & \left(R+R_o\right)^2\left(m+\dfrac{J_B}{R_o^2}\right)\end{bmatrix}\tag{3.2.21}$$

where

$$M1 = -2m(R+R_o)(R-H)\cos(\phi) + J_{arc} + (2R^2 - 2HR + 2RR_o + R_o^2 + H^2)m + J_B,$$
$$M2 = -(R+R_o)\left(-m(R-H)\cos(\phi) + mR + mR_o + \frac{J_B}{R_o}\right), \tag{3.2.22}$$

plus

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} \dot{\phi}(R-H)\sin(\phi)m(R+R_o) & -(R-H)\sin(\phi)m(\dot{\phi}-\dot{\theta})(R+R_o) \\ -\dot{\theta}(R-H)\sin(\phi)m(R+R_o) & 0 \end{bmatrix}, \tag{3.2.23}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} g(R_o+R)m\sin(\phi-\theta) + g((R-H)m - m_a l_{arc})\sin(\theta) \\ -mg(R_o+R)\sin(\phi-\theta) \end{bmatrix}, \tag{3.2.24}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{BA} \\ 0 \end{bmatrix}. \tag{3.2.25}$$

3.2.1.2 Lagrangian Formulation

The kinetic energy of the ball and arc is

$$T = \frac{1}{2}J_{arc}\dot{\theta}^2 + \frac{1}{2}J_B\dot{\theta}_{BA}^2 + \frac{1}{2}m(\dot{x}_c^2 + \dot{y}_c^2). \tag{3.2.26}$$

With $\dot{x}_c$ and $\dot{y}_c$ from (3.2.3) and (3.2.4), respectively, and $\dot{\theta}_{BA}$ from (3.2.14), the kinetic energy becomes

$$T = \frac{1}{2R_o^2}\begin{pmatrix} 2m\dot{\theta}R_o^2(\dot{\phi}-\dot{\theta})(R+R_o)(R-H)\cos(\phi) + J_{arc}R_o^2\dot{\theta}^2 + J_B\dot{\theta}^2R_o^2 + \\ mR_o^4(\dot{\phi}-\dot{\theta})^2 + 2mRR_o^3(\dot{\phi}-\dot{\theta})^2 + \dot{\phi}^2(J_B + mR^2)R_o^2 + J_BR^2\dot{\phi}^2 + \\ (-2RH + 2R^2 + H^2)m\dot{\theta}^2R_o^2 - 2\dot{\phi}(J_B + mR^2)\dot{\theta}R_o^2 + 2R\dot{\phi}J_B(\dot{\phi}-\dot{\theta})R_o \end{pmatrix}. \tag{3.2.27}$$

The potential energy is

$$V = mg((R+R_o)\cos(\phi-\theta) - (R-H)\cos(\theta)) + m_a g l_{arc}\cos(\theta). \tag{3.2.28}$$

Then using (2.2), the ball position equation is

$$-\left(R+R_o\right)\left(mR_o+Rm+\frac{J_B}{R_o}-\left(R-H\right)m\cos\left(\phi\right)\right)\ddot{\theta}+\left(R+R_o\right)^2\left(m+\frac{J_B}{R_o^2}\right)\ddot{\phi}$$
$$-\left(R+R_o\right)m\left(R-H\right)\sin\left(\phi\right)\dot{\theta}^2-\left(R+R_o\right)g\sin\left(\phi-\theta\right)m=0 \quad (3.2.29)$$

and the arc position equation is

$$\left(-2\left(R+R_o\right)\left(R-H\right)m\cos\left(\phi\right)+J_{arc}+\left(2R^2-2HR+2RR_o+R_o^2+H^2\right)m+J_B\right)\ddot{\theta}$$
$$-\left(R+R_o\right)\left(mR_o+Rm+\frac{J_B}{R_o}-\left(R-H\right)m\cos\left(\phi\right)\right)\ddot{\phi}$$
$$+\left(2\left(R+R_o\right)m\left(R-H\right)\sin\left(\phi\right)\right)\dot{\phi}\dot{\theta}-\left(R+R_o\right)m\left(R-H\right)\sin\left(\phi\right)\dot{\phi}^2$$
$$+g\left(R+R_o\right)m\sin\left(\phi-\theta\right)+g\left(\left(R-H\right)m-l_{arc}m_a\right)\sin\left(\theta\right)=\tau_{BA}. \quad (3.2.30)$$

Equation (3.2.29) and (3.2.30) are the same as (3.2.18) and (3.2.20), respectively.

3.2.2 Comparison to Models in Literature

For the ball and arc in this paper to match those of [26], change the ball to a point mass, m, with the ball's inertia J_B and radius R_o set to zero, and move the point of rotation to the arc center of mass by setting l_{arc} to zero. Then the kinetic energy of (3.2.26) becomes

$$T=\frac{1}{2}J_{arc}\dot{\theta}^2+\frac{1}{2}m\left(\dot{x}_c^2+\dot{y}_c^2\right). \quad (3.2.31)$$

and the equations for the velocities of (3.2.3) and (3.2.4) become

$$\dot{x}_c=R\cos\left(\phi-\theta\right)\left(\dot{\phi}-\dot{\theta}\right)+\left(R-H\right)\cos\left(\theta\right)\dot{\theta} \quad (3.2.32)$$

and

$$\dot{y}_c=-R\sin\left(\phi-\theta\right)\left(\dot{\phi}-\dot{\theta}\right)+\left(R-H\right)\sin\left(\theta\right)\dot{\theta}\,. \quad (3.2.33)$$

For the potential energy, let $l_{arc}=0$ and then (3.2.28) becomes

$$V=mgy \quad (3.2.34)$$

where $y=R\cos\left(\phi-\theta\right)-\left(R-H\right)\cos\left(\theta\right)$. Then the matrices of (3.2.21)-(3.2.25) become

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} M1 & -Rm\left(R-(R-H)\cos(\phi)\right) \\ -Rm\left(R-(R-H)\cos(\phi)\right) & R^2 m \end{bmatrix}, \tag{3.2.35}$$

where

$$M1 = -2Rm(R-H)\cos(\phi) + J_{arc} + \left(R^2 + (R-H)^2\right)m, \tag{3.2.36}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} Rm(R-H)\sin(\phi)\dot{\phi} & -Rm(R-H)\sin(\phi)\left(\dot{\phi}-\dot{\theta}\right) \\ -Rm(R-H)\sin(\phi)\dot{\theta} & 0 \end{bmatrix}, \tag{3.2.37}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} gRm\sin(\phi-\theta) + g(R-H)m\sin(\theta) \\ -Rg\sin(\phi-\theta)m \end{bmatrix}, \tag{3.2.38}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{BA} \\ 0 \end{bmatrix} \tag{3.2.39}$$

which, when put into (3.1.21) and solved for $\ddot{\boldsymbol{q}}$, produce the same generalized accelerations as in [26].

3.2.3 Dimensionless Equations of Motion

For the ball and arc system of (3.2.21)-(3.2.25), divide the arc (first) equation by mR_o^2 and the ball (second) equation by $mR_o(R+R_o)$. Then the terms for (3.1.21) become

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{bmatrix} M1 & M2 \\ M2 & \left(1+\dfrac{J_B}{mR_o^2}\right) \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \dfrac{(R+R_o)}{R_o}\ddot{\phi} \end{bmatrix} \tag{3.2.40}$$

where

$$
\begin{aligned}
M1 &= -2\frac{(R+R_o)}{R_o}\frac{(R-H)}{R_o}\cos(\phi)+\frac{J_{arc}}{mR_o^2}+\left(\frac{2R^2}{R_o^2}-\frac{2HR}{R_o^2}+\frac{2R}{R_o}+1+\frac{H^2}{R_o^2}\right)+\frac{J_B}{mR_o^2},\\
M2 &= -\left(-\frac{(R-H)}{R_o}\cos(\phi)+\frac{R}{R_o}+1+\frac{J_B}{mR_o}\right),
\end{aligned}
\tag{3.2.41}
$$

plus

$$
\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\phi}\frac{(R-H)}{R_o}\frac{(R+R_o)}{R_o}\sin(\phi) & -\frac{(R-H)}{R_o}\sin(\phi)(\dot{\phi}-\dot{\theta}) \\ -\dot{\theta}\frac{(R-H)}{R_o}\sin(\phi) & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \frac{(R+R_o)}{R_o}\dot{\phi} \end{bmatrix}, \tag{3.2.42}
$$

$$
\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \frac{g}{R_o}\left(\frac{(R_o+R)}{R_o}\sin(\phi-\theta)+\left(\frac{(R-H)}{R_o}-\frac{m_a}{m}\frac{l_{arc}}{R_o}\right)\sin(\theta)\right) \\ -\frac{g}{R_o}\sin(\phi-\theta) \end{bmatrix}, \tag{3.2.43}
$$

and

$$
\boldsymbol{\tau} = \begin{bmatrix} \frac{\tau_{BA}}{mR_o^2} \\ 0 \end{bmatrix}. \tag{3.2.44}
$$

With the parameter definitions in Table 3.1 and multiplying both equations by γ^2, (3.2.40)-(3.2.44) become

$$
\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'' = \begin{bmatrix} M1 & M2 \\ M2 & (1+k_4) \end{bmatrix}\begin{bmatrix} \theta'' \\ (k_7+1)\phi'' \end{bmatrix} \tag{3.2.45}
$$

where

$$
\begin{aligned}
M1 &= -2(k_7-k_6)(k_7+1)\cos(\phi)+k_3+2k_7^2-2k_7k_6+2k_7+1+k_6^2+k_4,\\
M2 &= -((-k_7+k_6)\cos(\phi)+k_7+1+k_4),
\end{aligned}
\tag{3.2.46}
$$

plus

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}')\boldsymbol{q}' = \begin{bmatrix} \phi'(k_7-k_6)(k_7+1)\sin(\phi) & -(k_7-k_6)\sin(\phi)(\phi'-\theta') \\ -\theta'(k_7-k_6)\sin(\phi) & 0 \end{bmatrix} \begin{bmatrix} \theta' \\ (k_7+1)\phi' \end{bmatrix}, \tag{3.2.47}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} (k_7+1)\sin(\phi-\theta)+(k_7-k_6-k_1k_5)\sin(\theta) \\ -\sin(\phi-\theta) \end{bmatrix}, \tag{3.2.48}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ 0 \end{bmatrix} \tag{3.2.49}$$

where $\theta' = \dfrac{d\theta}{d\underline{t}}$, $\theta'' = \dfrac{d^2\theta}{d\underline{t}^2}$, $\phi' = \dfrac{d\phi}{d\underline{t}}$, and $\phi'' = \dfrac{d^2\phi}{d\underline{t}^2}$.

3.3 Morphing

To show that as R gets large the ball and arc morphs into the ball and beam, first note that as R grows, ϕ becomes small. To check the sufficient conditions necessary for morphing the equations of motion for the ball and arc to ball and beam first approximate $\sin(\phi)\approx\phi$, $\cos(\phi)\approx 1-\frac{1}{2}\phi^2$, then adding and subtracting $(1+k_7)\phi^2$ in $M1$, and simplifying (3.2.45)-(3.2.49) produces

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'' = \begin{bmatrix} M1 & M2 \\ M2 & (1+k_4) \end{bmatrix} \begin{bmatrix} \theta'' \\ (k_7+1)\phi'' \end{bmatrix} \tag{3.3.1}$$

where

$$\begin{aligned} M1 &= k_3+(1+k_6)^2+k_4-(k_7+k_6+1+k_7k_6)\phi^2+(k_7+1)^2\phi^2, \\ M2 &= -(k_6+1+k_4)-\frac{1}{2}(k_7-k_6)\phi^2, \end{aligned} \tag{3.3.2}$$

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}')\boldsymbol{q}' = \begin{bmatrix} \phi'(k_7-k_6)(k_7+1)\phi & -\phi(k_7-k_6)(\phi'-\theta') \\ -\theta'(k_7-k_6)\phi & 0 \end{bmatrix} \begin{bmatrix} \theta' \\ (k_7+1)\phi' \end{bmatrix}, \tag{3.3.3}$$

$$G(q) = \begin{bmatrix} (k_7+1)\phi\cos(\theta) - (1+k_6+k_1k_5)\sin(\theta) \\ -\left(\phi\cos(\theta) - \sin(\theta)\left(1-\frac{1}{2}\phi^2\right)\right) \end{bmatrix}, \tag{3.3.4}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ 0 \end{bmatrix}. \tag{3.3.5}$$

To convert the second generalized coordinate, ϕ, to arc length, first it must be noted that $(R+R_o)\phi = r$, then from Table 3.1 $\rho = r/R_o = (k_7 + 1)\,\phi$. Similarly, $\rho' = (k_7 + 1)\phi'$ and $\rho'' = (k_7 + 1)\phi''$. As $R \to \infty$, the quantities ϕ and $\dot{\phi}$ approach zero and k_7 grows but the product $k_7\phi$ becomes the constant value ρ, then $(k_7 + 1)\phi \to k_7\phi \to \rho$, and $(k_7 + 1)\phi' \to k_7\phi' \to \rho'$ resulting in (3.3.1)-(3.3.5) becoming

$$M(q)q'' = \begin{bmatrix} k_3 + (1+k_6)^2 + k_4 + \rho^2 & -(k_6+1+k_4) \\ -(k_6+1+k_4) & 1+k_4 \end{bmatrix} \begin{bmatrix} \theta'' \\ \rho'' \end{bmatrix}, \tag{3.3.6}$$

$$C(q, q')q' = \begin{bmatrix} \rho'\rho & \rho'\theta' \\ -\theta'\rho' & 0 \end{bmatrix} \begin{bmatrix} \theta' \\ \rho' \end{bmatrix}, \tag{3.3.7}$$

$$G(q) = \begin{bmatrix} \rho\cos(\theta) - (1+k_6+k_1k_5)\sin(\theta) \\ \sin(\theta) \end{bmatrix}, \tag{3.3.8}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ 0 \end{bmatrix}. \tag{3.3.9}$$

As $R \to \infty$ for the ball and arc, $J_{arc} \to J_{Beam}$, $m_a \to m_B$, and $l_{arc} \to l_{beam}$, then (3.3.6)-(3.3.9) exactly match those of (3.1.49)-(3.1.52). Since (3.3.6), (3.3.8), and the generalized coordinates and velocities of the ball and beam matches those of (3.1.49), (3.1.51), and the generalized coordinates and velocities of the ball and beam, the sufficient conditions for morphing are met.

3.4 Conclusion

In this chapter, the equations of motion were derived for both the ball and beam and ball and arc systems with few assumptions. Then the equations of motion were compared to existing models in the literature. The comparison revealed several errors occurring in other previous dynamic analyses. Finally, with the conditions for morphing theory being met, the equations of motion for the ball and arc system successfully morph to those of the ball and beam.

Chapter 4 - Rotary Pendulum to Inverted Pendulum Cart

The next example supporting the sufficient conditions for morphing is morphing the rotary pendulum to the inverted pendulum cart. In this chapter, it will be shown that the equations of motion, as well as a controller, for the rotary pendulum cart successfully morph to the equations of motion and a controller for the inverted pendulum cart.

The equations for the inverted pendulum cart and rotary pendulum cart will be derived and rendered dimensionless similar to [32]. Next, a controller from [19] will be presented for both systems and converted to dimensionless form. Lastly, simulations will be performed demonstrating that the process of converting to dimensionless quantities was successful.

Then the radius of the arm of the rotary pendulum will be allowed to grow without bound to show that the equations of motion and controller for the rotary pendulum cart morph to the respective quantities for the inverted pendulum cart. Then simulations will be performed to showcase the successful morphing. Lastly, conclusion about the rotary pendulum morphing will be presented. This chapter has been previously published see reference [32].

4.1 Inverted Pendulum Cart Analysis

The inverted pendulum cart is an often-used example in the control literature. The inverted pendulum cart of Figure 4.1 from [32], is modeled with a cart of mass m_c to which a pendulum of mass m_p and length l is attached. The pendulum is modeled as a point mass at the end of a long slender, massless rod. In this section, the equations of motion for the inverted pendulum cart will be derived using Euler-Lagrange and then the equations will be made dimensionless. Next, a controller from [19] will be presented and rendered dimensionless. Lastly, simulations will be performed of the original system and the dimensionless system. This controller was chosen because [19] also has a controller for the rotary pendulum. In [19], simulation results are presented and that will be used as a check that the equations of motion, dimensionless system, and ultimately the morphed system of this work match those existing, accepted results.

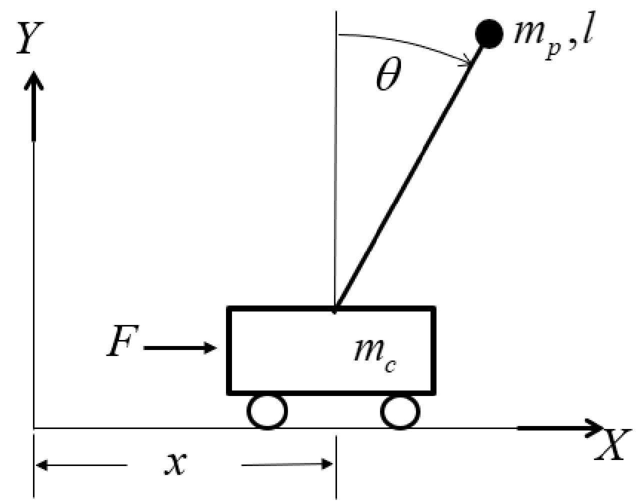


Figure 4.1: Inverted Pendulum Cart

4.1.1 Equations of Motion

The kinetic energy for the inverted pendulum cart of Figure 3.1 is

$$T = \frac{1}{2} m_p l^2 \dot{\theta}^2 + \frac{1}{2}\left(m_c + m_p\right)\dot{x}^2 + m_p l \cos\left(\theta\right)\dot{\theta}\dot{x} \tag{4.1.1}$$

while the gravitational potential energy is

$$V = m_p g l \cos(\theta). \tag{4.1.2}$$

Utilizing (2.2), the cart position equation is

$$\left(m_c + m_p\right)\ddot{x} + m_p l \cos\left(\theta\right)\ddot{\theta} - m_p l \sin\left(\theta\right)\dot{\theta}^2 = \tau_{IPC} \tag{4.1.3}$$

and pendulum position equation is

$$m_p l\cos(\theta)\ddot{x}+m_p l^2\ddot{\theta}-m_p lg\sin(\theta)=0. \tag{4.1.4}$$

For the inverted pendulum cart, where the pendulum is modeled as a point mass, the matrices of (3.1.21) are

$$\boldsymbol{M}(\boldsymbol{q})=\begin{bmatrix} m_c+m_p & m_p l\cos(\theta) \\ m_p l\cos(\theta) & m_p l^2 \end{bmatrix}, \tag{4.1.5}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{bmatrix} 0 & -m_p l\sin(\theta)\dot{\theta} \\ 0 & 0 \end{bmatrix}, \tag{4.1.6}$$

$$\boldsymbol{G}(\boldsymbol{q})=\begin{bmatrix} 0 \\ -m_p lg\sin(\theta) \end{bmatrix}, \tag{4.1.7}$$

and

$$\boldsymbol{\tau}=\begin{bmatrix} \tau_{IPC} \\ 0 \end{bmatrix}. \tag{4.1.8}$$

4.1.2 Dimensionless Equations of Motion

To transform (4.1.3) and (4.1.4) into dimensionless equations, divide (4.1.3) by $m_p l$ and (4.1.4) by $m_p l^2$ to cancel units of mass and length from the equations, resulting in

$$\frac{(m_c+m_p)}{m_p}\frac{\ddot{x}}{l}+\cos(\theta)\ddot{\theta}-\sin(\theta)\dot{\theta}^2=\frac{\tau_{IPC}}{m_p l} \tag{4.1.9}$$

and

$$\cos(\theta)\frac{\ddot{x}}{l}+\ddot{\theta}-\frac{g}{l}\sin(\theta)=0. \tag{4.1.10}$$

Then, multiplying (4.1.9) and (4.1.10) by l/g, having units of second^2, transforms time, t, to unitless time, $\underline{t}$. Using the parameters from Table 4.1, the dimensionless equations of motion for the cart and pendulum are

$$(k_1+1)\rho''+\cos(\theta)\theta''-\sin(\theta)\theta'^2=\tau_N \tag{4.1.11}$$

and

$$\cos(\theta)\rho''+\theta''-\sin(\theta)=0. \tag{4.1.12}$$

where $\theta'=\frac{d\theta}{d\underline{t}}$, $\theta''=\frac{d^2\theta}{d\underline{t}^2}$, and $\rho''=\frac{d^2\rho}{d\underline{t}^2}$. The resulting mass matrix is

$$\boldsymbol{M}(\boldsymbol{q})=\begin{bmatrix} k_1+1 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix}. \tag{4.1.13}$$

Table 4.1: Dimensionless Inverted Pendulum Cart and Rotary Pendulum Parameters

Dimensionless Parameter	k_1	k_2	ρ	γ^2	τ_N
Inverted Pendulum Cart	$\frac{m_c}{m_p}$	--	$\frac{x}{l}$	$\frac{l}{g}$	$\frac{\gamma^2\tau_{IPC}}{m_p l}$
Rotary Pendulum	$\frac{m_a}{m_p}$	$\frac{l}{R}$	$\frac{\phi R}{l}$	$\frac{l}{g}$	$\frac{\gamma^2\tau_{RP}}{m_p lR}$

4.1.3 Controller

The controller, designed in [19] for the inverted pendulum cart, is

$$\tau_{IPC}=\frac{\kappa\beta\sin(\theta)\left(\alpha\dot{\theta}^2+D\cos(\theta)\right)-\frac{B\varepsilon D\lambda^2 x}{\beta^2}+Bu_d}{\alpha-\frac{\beta^2(\kappa+1)\cos^2(\theta)}{\lambda}} \tag{4.1.14}$$

where $\alpha = m_p l^2$, $\beta = m_p l$, $\lambda = m_p + m_c$, $D = -m_p g l$, κ and ε are unitless constants, and

$$u_d = c\lambda(\dot{x}+p\cos(\theta)\dot{\theta}). \tag{4.1.15}$$

With $c > 0$ is a constant with units of seconds^{-1},

$$B = \frac{1}{\xi}\left(\alpha - \frac{\beta^2 \cos^2(\theta)}{\lambda}\right), \tag{4.1.16}$$

and

$$p = \frac{\beta}{\lambda}\left(\kappa + \frac{\xi - 1}{\xi}\right) > 0 \tag{4.1.17}$$

where ξ is a unitless constant.

Now that an energy-based controller for the rotary pendulum cart has been identified, the controller will be rendered dimensionless based on τ_N for the rotary pendulum from Table 4.1.

4.1.4 Dimensionless Controllers

Utilizing the definition of τ_N, defined in Table 4.1, and dividing by $m_p l$ converts the controller of (4.1.14) to a dimensionless controller. Making this substitution and simplifying yields

$$\frac{\tau_N}{\gamma^2} = \frac{\left(\kappa \sin(\theta)\left(\dot{\theta}^2 + D_N \cos(\theta)\right) - B_N \varepsilon D_N \lambda_N^2 \frac{x}{l} + B_N u_{dN}\right)\lambda_N}{\lambda_N - (\kappa + 1)\cos^2(\theta)} \tag{4.1.18}$$

where $\lambda_N = 1 + k_I$, $D_N = -1/\gamma^2$,

$$u_{dN} = c\lambda_N\left(\frac{\dot{x}}{l} + p_N \cos(\theta)\dot{\theta}\right). \tag{4.1.19}$$

$$B_N = \frac{1}{\xi}\left(1 - \frac{\cos^2(\theta)}{\lambda_N}\right), \tag{4.1.20}$$

and

$$p_N = \frac{1}{\lambda_N}\left(\kappa + \frac{\xi - 1}{\xi}\right) > 0. \tag{4.1.21}$$

Lastly, multiply (4.1.18) by γ^2 and utilizing the definition for ρ from Table 4.1, obtains the dimensionless control law of

$$\tau_N = \frac{\left(\kappa \sin(\theta)\left(\theta'^2 - \cos(\theta)\right) + B_N \varepsilon \lambda_N^2 \rho + B_N u_{dN}\right)\lambda_N}{\lambda_N - (\kappa+1)\cos^2(\theta)} \tag{4.1.22}$$

where $\rho' = \dfrac{d\rho}{d\underline{t}}$ and

$$u_{dN} = \gamma c \lambda_N \left(\rho' + p_N \cos(\theta)\theta'\right). \tag{4.1.23}$$

4.1.5 Simulation

Simulations of the inverted pendulum equations of motion of (4.1.3) and (4.1.4) with the controller of (4.1.14) were performed using as control parameters c = 0.015, κ = 20, ε = 0.00001, and ξ = -0.02, where c has units of seconds^{-1} , κ, ξ, and ε are unitless, and initial conditions of x = 3, $\dot{x}$ = 0, where x has units of meters and θ = π/6, $\dot{\theta}$ = 0, where the angles are measured in radians, radians are dimension. Also simulations with the same control parameters and initial conditions for the dimensionless equations of motion described by (4.1.11) and (4.1.12) using the dimensionless controller of (4.1.22) were executed. Figure 4.2a shows cart position x and ρ responses of these two simulations while Figure 4.2b compares the pendulum angle θ and θ_N responses. The axis scales in Figure 4.2, for the dimensionless quantities ρ and $\underline{t}$ are modified according to Table 4.1 so the responses are the same size as x and t.

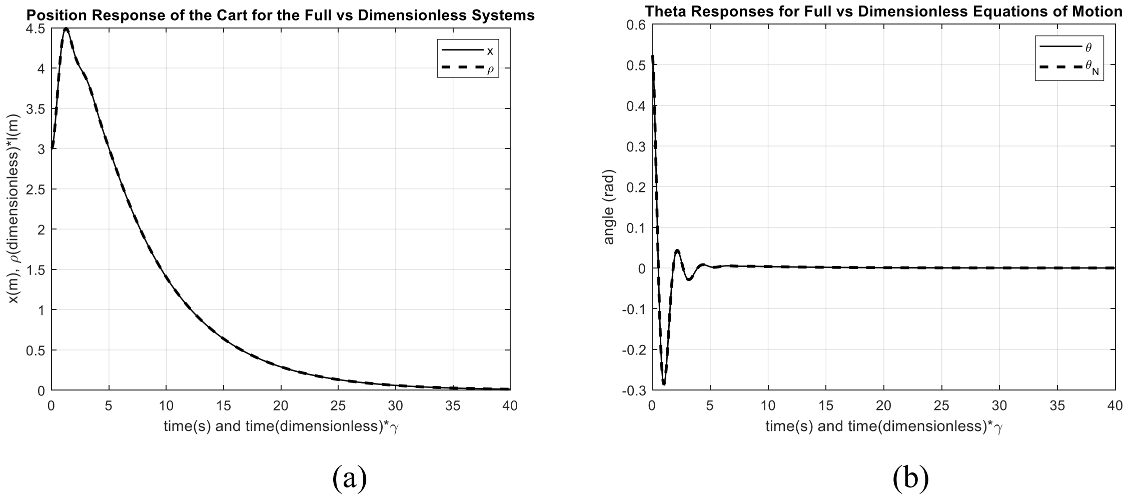


Figure 4.2: Simulation Results for the Inverted Pendulum Cart

4.2 Rotary Pendulum Analysis

For the rotary pendulum of Figure 4.3, the pendulum is modeled as a massless rod of length l with a point mass, m_p, and the arm is a point mass, m_a, located a distance R from the rotation point. In this section, the equations of motion will be derived and then rendered dimensionless. The controller for the rotary pendulum from [19] will be presented and then made dimensionless. Next simulations will be presented of the original and dimensionless systems.

4.2.1 Equations of Motion

The kinetic energy for the rotary pendulum of Figure 4.3 from [32] is

$$T = \frac{1}{2}m_p l^2\dot{\theta}^2 + \frac{1}{2}\left(m_p + m_a\right)R^2\dot{\phi}^2 + m_p lR\cos\left(\theta\right)\dot{\phi}\dot{\theta} + \frac{1}{2}m_p l^2\sin^2\left(\theta\right)\dot{\phi}^2 \tag{4.2.1}$$

while the gravitational potential energy is

$$V = m_p gl\cos\left(\theta\right). \tag{4.2.2}$$

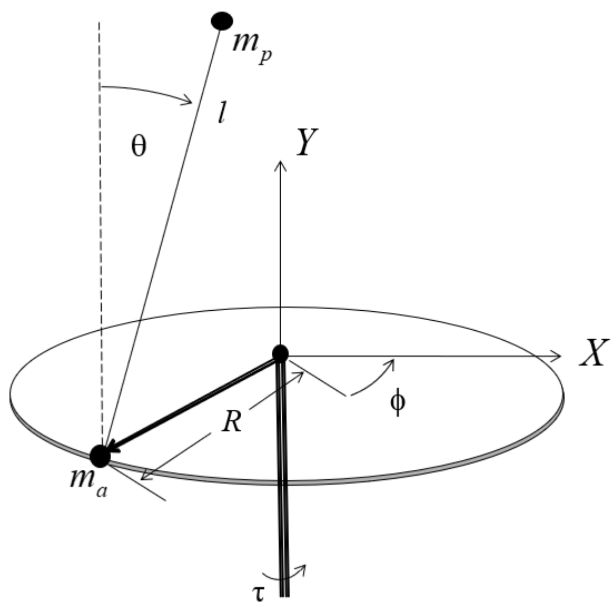


Figure 4.3: Rotary Pendulum Cart

The position equation for the arm, found using Lagrange's equation and (2.2), is

$$\begin{aligned}&\left(\left(m_p+m_a\right)R^2+m_pl^2\sin^2\left(\theta\right)\right)\ddot{\phi}+m_plR\cos\left(\theta\right)\ddot{\theta}+\\&2m_pl^2\sin\left(\theta\right)\cos\left(\theta\right)\dot{\phi}\dot{\theta}-m_plR\sin\left(\theta\right)\dot{\theta}^2=\tau_{RP}\end{aligned} \tag{4.2.3}$$

and the ball position equation is

$$m_pl^2\ddot{\theta}+m_plR\cos\left(\theta\right)\ddot{\phi}-m_pl^2\sin\left(\theta\right)\cos\left(\theta\right)\dot{\phi}^2-m_pgl\sin\left(\theta\right)=0. \tag{4.2.4}$$

The matrices of (3.1.21) for this system are

$$\boldsymbol{M}(\boldsymbol{q})=\begin{bmatrix}\left(m_p+m_a\right)R^2+m_pl^2\sin^2\left(\theta\right) & m_plR\cos\left(\theta\right)\\ m_plR\cos\left(\theta\right) & m_pl^2\end{bmatrix}, \tag{4.2.5}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{bmatrix}m_pl^2\sin\left(\theta\right)\cos\left(\theta\right)\dot{\theta} & -m_plR\sin\left(\theta\right)\dot{\theta}+m_pl^2\sin\left(\theta\right)\cos\left(\theta\right)\dot{\phi}\\ -m_pl^2\sin\left(\theta\right)\cos\left(\theta\right)\dot{\phi} & 0\end{bmatrix}, \tag{4.2.6}$$

$$\boldsymbol{G}(\boldsymbol{q})=\begin{bmatrix}0\\ -m_plg\sin\left(\theta\right)\end{bmatrix}, \tag{4.2.7}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{RP} \\ 0 \end{bmatrix}. \tag{4.2.8}$$

4.2.2 Dimensionless Equations of Motion

To transform (4.2.3) and (4.2.4) into dimensionless equations, divide (4.2.3) by $m_p lR$ and (4.2.4) by $m_p l^2$. Then, the dimensionless arm equation is

$$\begin{aligned} &\frac{(m_p + m_a)}{m_p}\left(\frac{R}{l}\ddot{\phi}\right) + \left(\frac{l}{R}\right)^2 \sin^2(\theta)\left(\frac{R}{l}\ddot{\phi}\right) + \cos(\theta)\ddot{\theta} \\ &+2\left(\frac{l}{R}\right)^2 \sin(\theta)\cos(\theta)\left(\frac{R}{l}\dot{\phi}\right)\dot{\theta} - \sin(\theta)\dot{\theta}^2 = \frac{\tau_{RP}}{m_p lR} \end{aligned} \tag{4.2.9}$$

and the dimensionless pendulum equation is

$$\ddot{\theta} + \cos(\theta)\left(\frac{R}{l}\ddot{\phi}\right) - \left(\frac{l}{R}\right)^2 \sin(\theta)\cos(\theta)\left(\frac{R}{l}\dot{\phi}\right)^2 - \frac{g}{l}\sin(\theta) = 0. \tag{4.2.10}$$

Letting $R/l\phi = \rho$, which also holds for the first and second derivatives, and multiplying (4.2.9) and (4.2.10) by γ^2, transforms time, t, to unitless time, $\underline{t}$. With Table 4.1, the dimensionless equations of motion for the rotary pendulum are

$$\left(1 + k_1 + k_2^2 \sin^2(\theta)\right)\rho'' + \cos(\theta)\theta'' + 2k_2^2 \sin(\theta)\cos(\theta)\rho'\theta' - \sin(\theta)(\theta')^2 = \tau_N \tag{4.2.11}$$

and

$$\theta'' + \cos(\theta)\rho'' - k_2^2 \sin(\theta)\cos(\theta)(\rho')^2 - \sin(\theta) = 0. \tag{4.2.12}$$

4.2.3 Controller

The full control law from [19] is

$$\tau_{RP} = u_1 + u_{2rp} \tag{4.2.13}$$

where u_1 converts the system through partial feedback linearization to cancel out the nonlinear terms and u_{2rp} stabilizes the resulting system. To convert the equations of motion so that the parameters morph into recognizable quantities, first define $\alpha_{rp} = m_p l^2$, $\beta_{rp} = m_p lR$, $\lambda_{rp} = (m_p + m_a)R^2$, and $D_{rp} = -m_p gl$. Note, κ_{rp} and ε_{rp} are unitless constants and

$$u_1 = -\alpha_{rp}\sin(\theta)\cos(\theta)\dot{\phi}\dot{\theta} + \alpha_{rp}\sin^2(\theta)\ddot{\phi}. \tag{4.2.14}$$

Next, the stabilizing controller is

$$u_{2rp} = \frac{\kappa_{rp}\beta_{rp}\sin(\theta)\left(\alpha_{rp}\dot{\theta}^2 + D_{rp}\cos(\theta)\right) - \dfrac{B_{rp}\varepsilon_{rp}D_{rp}\lambda_{rp}^2\phi}{\beta_{rp}^2} + B_{rp}u_{drp} - \kappa_{rp}\beta_{rp}F_\alpha\cos(\theta)}{\alpha_{rp} - \dfrac{\beta_{rp}^2(\kappa_{rp}+1)\cos^2(\theta)}{\lambda_{rp}}} \tag{4.2.15}$$

where $c_{rp} > 0$ is a constant with units of seconds^{-1},

$$F_\alpha = \alpha_{rp}\sin(\theta)\cos(\theta)\dot{\phi}^2, \tag{4.2.16}$$

$$p_{rp} = \frac{\beta_{rp}}{\lambda_{rp}}\left(\kappa_{rp} + \frac{\xi_{rp}-1}{\xi_{rp}}\right), \tag{4.2.17}$$

$$B_{rp} = \frac{1}{\xi_{rp}}\left(\alpha_{rp} - \frac{\beta_{rp}^2}{\lambda_{rp}}\cos^2(\theta)\right), \tag{4.2.18}$$

and

$$u_{drp} = c_{rp}\lambda_{rp}\left(\dot{\phi} + p_{rp}\cos(\theta)\dot{\theta}\right) \tag{4.2.19}$$

where ξ_{rp} is a unitless constant.

4.2.4 Dimensionless Controllers

To transform the controller of (4.2.13) to a dimensionless controller, divide (4.2.13) by $m_p lR$ and simplify using the definitions of k_1 and k_2. Utilizing the dimensionless parameters $\lambda_{rpN} = (1+k_1)$ and $D_{rpN} = -1/\gamma^2$, the dimensionless controller is

$$\frac{\tau_N}{\gamma^2} = u_{1N} + u_{2rpN} \tag{4.2.20}$$

where

$$u_{1N} = k_2^2\left(-\sin(\theta)\cos(\theta)\left(\frac{R}{l}\dot{\phi}\right)\dot{\theta} + \sin^2(\theta)\left(\frac{R}{l}\ddot{\phi}\right)\right), \tag{4.2.21}$$

$$\begin{aligned} u_{2rpN} &= \frac{\lambda_{rpN}\kappa_{rp}\sin(\theta)\left(\dot{\theta}^2 + D_{rpN}\cos(\theta)\right)}{\lambda_{rpN} - \left(\kappa_{rp}+1\right)\cos^2(\theta)} \\ &+ \frac{\lambda_{rpN}\left(-B_{rpN}\varepsilon_{rp}D_{rpN}\lambda_{rpN}^2\left(\frac{R}{l}\phi\right) + B_{rpN}u_{drpN} - \kappa_{rp}k_2^2 F_{\alpha N}\cos(\theta)\right)}{\lambda_{rpN} - \left(\kappa_{rp}+1\right)\cos^2(\theta)}, \end{aligned} \tag{4.2.22}$$

$$F_{\alpha N} = \sin(\theta)\cos(\theta)\left(\frac{R}{l}\dot{\phi}\right)^2, \tag{4.2.23}$$

$$u_{drpN} = c_{rp}\lambda_{rpN}\left(\left(\frac{R}{l}\dot{\phi}\right) + p_{rpN}\cos(\theta)\dot{\theta}\right), \tag{4.2.24}$$

$$p_{rpN} = \frac{1}{\lambda_{rpN}}\left(\kappa_{rp} + \frac{\xi_{rp}-1}{\xi_{rp}}\right), \tag{4.2.25}$$

and

$$B_{rpN} = \frac{1}{\xi_{rp}}\left(1 - \frac{1}{\lambda_{rpN}}\cos^2(\theta)\right). \tag{4.2.26}$$

Finally multiply (4.2.20) through by γ^2 and using the definition of ρ from Table 4.1, the dimensionless controller is

$$\tau_N = u_{1N} + u_{2rpN} \tag{4.2.27}$$

where

$$F_{\alpha N} = \sin(\theta)\cos(\theta)\rho'^2, \tag{4.2.28}$$

$$u_{1N} = k_2^2\left(-\sin(\theta)\cos(\theta)\rho'\theta' + \sin^2(\theta)\rho''\right), \tag{4.2.29}$$

$$\begin{aligned} u_{2rpN} = {} & \frac{\lambda_{rpN}\kappa_{rp}\sin(\theta)\left(\theta'^2 - \cos(\theta)\right)}{\lambda_{rpN} - \left(\kappa_{rp}+1\right)\cos^2(\theta)} \\ & + \frac{\lambda_{rpN}\left(B_{rpN}\varepsilon_{rp}\lambda_{rpN}^2\rho + B_{rpN}u_{drpN}\gamma - \kappa_{rp}k_2^2F_{\alpha N}\cos(\theta)\right)}{\lambda_{rpN} - \left(\kappa_{rp}+1\right)\cos^2(\theta)}, \end{aligned} \tag{4.2.30}$$

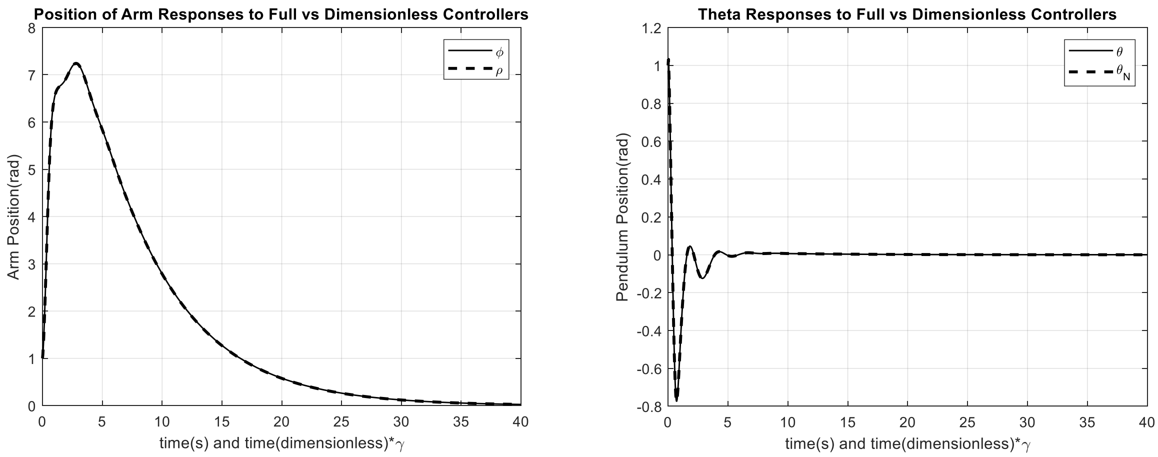
and

$$u_{drpN} = \lambda_{rpN}c_{rp}\left(\rho' + p_{rpN}\cos(\theta)\theta'\right). \tag{4.2.31}$$

4.2.5 Simulation

The analysis produced simulations of the rotary pendulum equations of motion of (4.2.3) and (4.2.4) with the controller of (4.2.13) together with the dimensionless equations of motion of (4.2.11) and (4.2.12) with the dimensionless controller of (4.2.27) to validate the dimensionless process was performed correctly. For both simulations, the control gains were c_{rp} = 0.015, κ_{rp} = 25, ε_{rp} = 0.00001, and ξ_{rp} = -0.02, the same as used in [19]. For the simulation using controller (4.2.13), the initial conditions were $\phi = 1$, $\dot{\phi} = 2$, $\theta = 1$, and $\dot{\theta}=2$, where the angles are measures in radians, and for the controller of (4.2.27) $\rho = R/l$, $\rho' = 2R\sqrt{l/g}/l$, $\theta_N = 1$, and $\theta_N' = 2\sqrt{l/g}$. Figure 4.4 compares the responses of these two simulations where Figure 4.4a compares the rotary pendulum arm angle ϕ and ρ while Figure 4.4b compares the pendulum angular position θ

and θ_N. As done for Figure 4.2, the axis scale for the dimensionless quantity $\underline{t}$ was modified according to Table 4.1 so the responses are the same size.



(a) (b)

Figure 4.4: Simulation Results for the Rotary Pendulum

4.3 Morphing

As the radius of the rotary pendulum arm is allowed to grow without bound, do the equations of motion and the controller of the rotary pendulum become those of the inverted pendulum cart? To check the sufficient conditions necessary for morphing the equations of motion for the rotary pendulum to the inverted pendulum cart, first examine the generalized coordinates. The second coordinate for both systems measures the angular displacement of the pendulum. The first generalized coordinate for the rotary pendulum in the dimensionless system is $\phi R/l$. As the radius of the pendulum arm base is allowed to grow without bound ϕ grows small to cover the same distance, then ϕR becomes a straight-line displacement and $\phi R/l$ morphs to the ρ of the inverted pendulum cart. Similarly, the generalized velocities of the rotary pendulum morph to those of the inverted pendulum cart. The mass matrix of the dimensionless rotary pendulum cart is

$$M = \begin{bmatrix} 1+k_1+k_2^2\sin^2(\theta) & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix} \tag{4.3.1}$$

and as R grows large, k_2 goes to zero showing (4.3.1) matches the mass matrix of the inverted pendulum cart, (4.1.13), provided k_1 is the same ratio of the cart mass to the rotary pendulum mass. Lastly, the potential energy of the dimensionless rotary pendulum from (4.2.11) and (4.2.12) is

$$V = \cos(\theta) \tag{4.3.2}$$

which is the same as the potential energy function of the dimensionless inverted pendulum cart. Therefore, the sufficient conditions for morphing the rotary pendulum dynamics to that of the inverted pendulum cart are met.

4.3.1 Equations of Motion and Controller

As the radius of the arm of the rotary pendulum grows R becomes large and then $k_2 \to 0$. This transforms equations (4.2.11) and (4.2.12) to

$$(1+k_1)\rho'' + \cos(\theta)\theta'' - \sin(\theta)\theta'^2 = \tau_N \tag{4.3.3}$$

and

$$\theta'' + \cos(\theta)\rho'' - \sin(\theta) = 0. \tag{4.3.4}$$

These equations match those of the dimensionless inverted pendulum cart (4.1.11) and (4.1.12). Next the controller of (4.2.27) becomes

$$\tau_N = u_{1N} + u_{2rpN} \tag{4.3.5}$$

where

$$u_{1N} = 0, \tag{4.3.6}$$

$$u_{2rpN} = \frac{\lambda_{rpN}\left(\kappa_{rp}\sin(\theta)\left(\theta'^2 - \cos(\theta) + B_{rpN}\varepsilon_{rp}\lambda_{rpN}^2\rho + B_{rpN}u_{drpN}\gamma\right)\right)}{\lambda_{rpN} - \left(\kappa_{rp}+1\right)\cos^2(\theta)}, \tag{4.3.7}$$

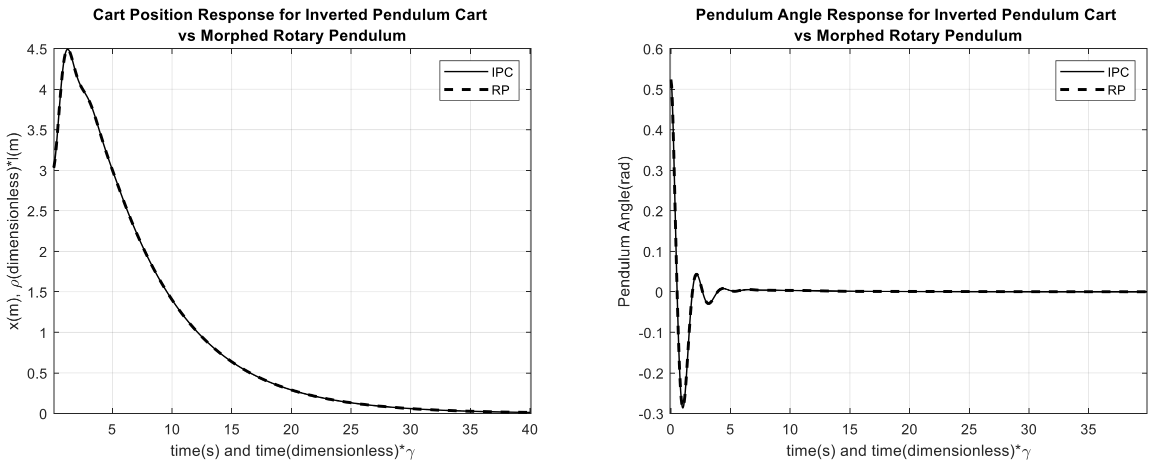
and

$$u_{drpN} = \lambda_{rpN}c_{rp}\left(\rho' + p_{rpN}\cos(\theta)\theta'\right). \tag{4.3.8}$$

The dimensionless, morphed controller of (4.3.5) matches that of the dimensionless cart controller of (4.1.22).

4.3.2. Simulations

A simulation of the controller of (4.3.5) applied to the morphed equations of motion of (4.3.3) and (4.3.4) was performed. The simulations of the rotary pendulum cart and inverted pendulum cart used the same constants except for κ which was equal to 25 for the rotary pendulum cart and 20 for the inverted pendulum cart. The simulation used the same constants, c, κ, ε, and ξ, as those used for the cart with initial conditions of $\rho = 3/l$, $\rho' = 0$, $\theta_N = \pi/6$, and $\theta_N' = 0$ and produced the results of Figure 4.5. Figure 4.5a shows the cart position for the inverted pendulum cart of Figure 4.2a compared to the morphed rotary pendulum's position. Figure 4.5b shows the pendulum angular position of Figure 4.2b compared to the morphed rotary pendulum. For the morphed systems of Figure 4.5, dimensionless time was multiplied by γ to have units of seconds for ease of comparison. The morphed cart position, ρ, was multiplied by the length of the pendulum in Figure 4.5a to scale the response to compare to the cart position of the inverted pendulum cart.



(a) (b)

Figure 4.5: Morphed Rotary Pendulum Simulation Results

4.4 Conclusion

In this chapter, the equations of motion for the rotary pendulum cart were successfully morphed to match the equations of motion for the inverted pendulum cart. Also, an energy-based controller for the rotary pendulum cart was successfully morphed to a controller for the inverted pendulum cart. The process of morphing the controller did not cause the constants to be the same as the inverted pendulum cart, just the symbolic form. This chapter has shown one set of systems which supports the sufficient conditions for equations of motion to morph.

Chapter 5 - Pendubot

As a final underactuated example supporting the conditions of morphing, the Pendubot will be investigated. First, the equations of motion will be derived and then the coordinates will be modified to match those of the inverted pendulum cart. Next, the equations of motion will be rendered dimensionless and morphed to those of the inverted pendulum cart. Then a controller will be presented that has simulated results in the literature. The controller will be rendered dimensionless and then the length of the first link will be allowed to grow large. Lastly, the equations of motion and controller for the full and dimensionless systems will be simulated.

5.1 Equations of Motion

The pendubot contains two links where the first link is subject to actuation. For the system of Figure 5.1, the first link is of length L_1, with mass m_1, subject to actuation u, and its rotational displacement, q_1, is measured counterclockwise from the negative Y axis. The second link is pinned to the end of the first and has length L_2, mass m_2, and its rotational displacement, q_2, is measured counterclockwise from a line extending out of the first link. In this section, the equations of motion will be derived, rendered dimensionless, and then morphed to those of the inverted pendulum cart.

5.1.1 Full Equations of Motion

For the pendubot of Figure 5.1, the kinetic energy is

$$T = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} m_1 \left(\dot{x}_{1c}^2 + \dot{y}_{1c}^2 \right) + \frac{1}{2} m_2 \left(\dot{x}_{2c}^2 + \dot{y}_{2c}^2 \right) + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2 \right)^2 \tag{5.1.1}$$

while the gravitational potential energy is

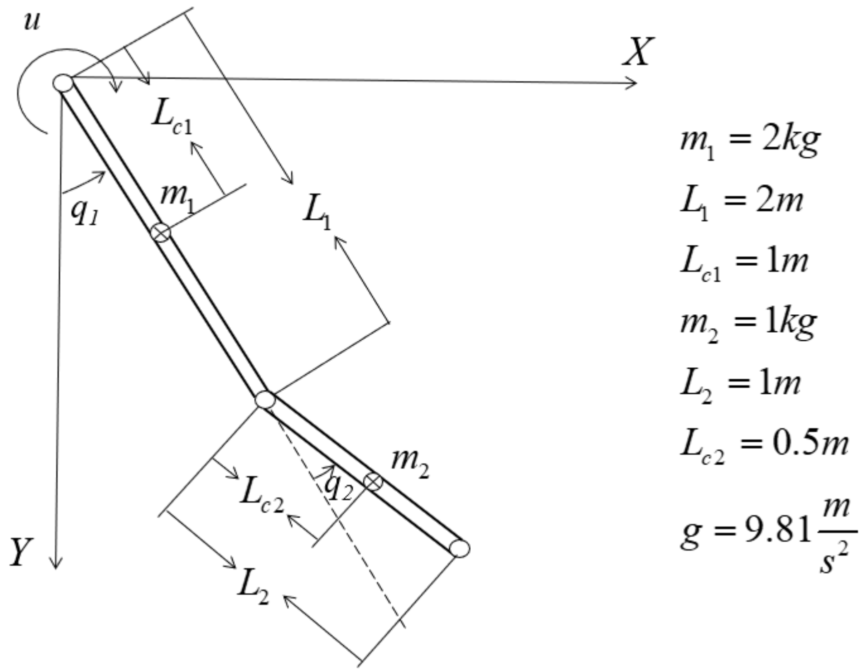


Figure 5.1: Pendubot Diagram

$$V = m_1 g y_{1c} + m_2 g y_{2c}. \tag{5.1.2}$$

The position and velocity for the center of mass of the first link are

$$x_{1c} = L_{c1}\sin(q_1), \tag{5.1.3}$$

$$y_{1c} = -L_{c1}\cos(q_1), \tag{5.1.4}$$

$$\dot{x}_{1c} = L_{c1}\cos(q_1)\dot{q}_1, \tag{5.1.5}$$

and

$$\dot{y}_{1c} = L_{c1}\sin(q_1)\dot{q}_1. \tag{5.1.6}$$

The position and velocity for the center of mass of the second link are

$$x_{2c} = L_1\sin(q_1) + L_{c2}\sin(q_1+q_2), \tag{5.1.7}$$

$$y_{2c} = -L_1\cos(q_1) - L_{c2}\cos(q_1+q_2), \tag{5.1.8}$$

$$\dot{x}_{2c} = L_1 \cos(q_1)\dot{q}_1 + L_{c2}\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2), \tag{5.1.9}$$

and

$$\dot{y}_{2c} = L_1 \sin(q_1)\dot{q}_1 + L_{c2}\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2). \tag{5.1.10}$$

The position equation for the first link, found using Lagrange's equation, (2.2), is

$$\begin{aligned}&\left(2L_{c2}m_2L_1\cos(q_2)+\left(L_1^2+L_{c2}^2\right)m_2+L_{c1}^2m_1+I_1+I_2\right)\ddot{q}_1+\\&\left(L_{c2}m_2L_1\cos(q_2)+m_2L_{c2}^2+I_2\right)\ddot{q}_2-2m_2L_1L_{c2}\sin(q_2)\dot{q}_1\dot{q}_2-\\&m_2L_1L_{c2}\sin(q_2)\dot{q}_2^2+g\left(m_2L_{c2}\sin(q_1+q_2)+\left(L_1m_2+L_{c1}m_1\right)\sin(q_1)\right)=u\end{aligned} \tag{5.1.11}$$

and the second link's position equation is

$$\begin{aligned}&\left(m_2L_{c2}L_1cos(q_2)+m_2L_{c2}^2+I_2\right)\ddot{q}_1+\left(L_{c2}^2m_2+I_2\right)\ddot{q}_2\\&+m_2L_1L_{c2}sin(q_2)\dot{q}_1^2+gm_2L_{c2}sin(q_1+q_2)=0.\end{aligned} \tag{5.1.12}$$

The matrices of (3.1.21) for this system are

$$\begin{aligned}&\boldsymbol{M}(\boldsymbol{q})=\\&\begin{bmatrix}\left(2L_{c2}L_1\cos(q_2)+L_1^2+L_{c2}^2\right)m_2+L_{c1}^2m_1+I_1+I_2 & \left(L_1\cos(q_2)+L_{c2}\right)m_2L_{c2}+I_2\\ \left(L_1\cos(q_2)+L_{c2}\right)m_2L_{c2}+I_2 & L_{c2}^2m_2+I_2\end{bmatrix},\end{aligned} \tag{5.1.13}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{bmatrix}-m_2L_1L_{c2}\sin(q_2)\dot{q}_2 & -m_2L_1L_{c2}\sin(q_2)(\dot{q}_1+\dot{q}_2)\\ m_2L_1L_{c2}\sin(q_2)\dot{q}_1 & 0\end{bmatrix}, \tag{5.1.14}$$

$$\boldsymbol{G}(\boldsymbol{q})=\begin{bmatrix}g\left(m_2L_{c2}\sin(q_1+q_2)+\left(L_1m_2+L_{c1}m_1\right)\sin(q_1)\right)\\ gm_2L_{c2}\sin(q_1+q_2)\end{bmatrix}, \tag{5.1.15}$$

and

$$\boldsymbol{\tau}=\begin{bmatrix}u\\0\end{bmatrix}. \tag{5.1.16}$$

5.1.2 Change of Coordinates

With an eye towards this system morphing to the inverted pendulum cart, move the mass center of the second link to the end of a very light rod. In doing this, set the quantity L_2 equal to L_{c2} and call it L_2 for simplicity, and set I_2 to zero. Modifying the first coordinate to measure the angular displacement relative to the positive vertical axis changes q_1 to $\pi + \delta q_1$. Then to convert the coordinate to circumferential displacement in the clockwise direction, factor out the quantity $-L_1$ from terms multiplying the time derivatives of q_1. Then after simplifying, (5.1.13)-(5.1.16) become

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{bmatrix} -\left(2L_2\cos(q_2) + L_1 + \frac{L_2^2}{L_1}\right)m_2 - \frac{L_{c1}^2}{L_1}m_1 - \frac{I_1}{L_1} & (L_1\cos(q_2) + L_2)m_2L_2 \\ -\left(\cos(q_2) + \frac{L_2}{L_1}\right)m_2L_2 & L_2^2 m_2 \end{bmatrix} \begin{bmatrix} -L_1\delta\ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}, \quad (5.1.17)$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{bmatrix} m_2L_2\sin(q_2)\dot{q}_2 & -m_2L_2\sin(q_2)\left(L_1\dot{q}_2 - (-L_1\delta\dot{q}_1)\right) \\ m_2\frac{L_2}{L_1}\sin(q_2)(-L_1\delta\dot{q}_1) & 0 \end{bmatrix} \begin{bmatrix} -L_1\delta\dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad (5.1.18)$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} -g\left(m_2L_2\sin(\delta q_1 + q_2) + (L_1m_2 + L_{c1}m_1)\sin(\delta q_1)\right) \\ -L_2\sin(\delta q_1 + q_2)gm_2 \end{bmatrix}, \quad (5.1.19)$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} u \\ 0 \end{bmatrix}. \quad (5.1.20)$$

In order for the mass matrix to be symmetric, divide the first link equation by $-L_1$, then the matrices are

$$M(q)\ddot{q} = \begin{bmatrix} \left(2\frac{L_2}{L_1}\cos(q_2)+1+\frac{L_2^2}{L_1^2}\right)m_2+\frac{L_{c1}^2}{L_1^2}m_1+\frac{I_1}{L_1^2} & -\left(\cos(q_2)+\frac{L_2}{L_1}\right)m_2L_2 \\ -\left(\cos(q_2)+\frac{L_2}{L_1}\right)m_2L_2 & L_2^2m_2 \end{bmatrix}\begin{bmatrix} -L_1\delta\ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}, \quad (5.1.21)$$

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} -m_2\frac{L_2}{L_1}\sin(q_2)\dot{q}_2 & m_2\sin(q_2)\left(L_2\dot{q}_2-\frac{L_2}{L_1}(-L_1\delta\dot{q}_1)\right) \\ m_2\frac{L_2}{L_1}\sin(q_2)(-L_1\delta\dot{q}_1) & 0 \end{bmatrix}\begin{bmatrix} -L_1\delta\dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad (5.1.22)$$

$$G(q) = \begin{bmatrix} g\left(m_2\frac{L_2}{L_1}\sin(\delta q_1+q_2)+\left(m_2+\frac{L_{c1}}{L_1}m_1\right)\sin(\delta q_1)\right) \\ -gm_2L_2\sin(\delta q_1+q_2) \end{bmatrix}, \quad (5.1.23)$$

and

$$\tau = \begin{bmatrix} -\frac{u}{L_1} \\ 0 \end{bmatrix}. \quad (5.1.24)$$

5.1.3 Dimensionless Equations of Motion

To convert to dimensionless equations of motion, divide the first link equation by m_2L_2 and the second link equation by $m_2L_2^2$ to obtain

$$M(q)\ddot{q} = \begin{bmatrix} 2\frac{L_2}{L_1}\cos(q_2)+1+\frac{L_2^2}{L_1^2}+\frac{L_{c1}^2}{L_1^2}\frac{m_1}{m_2}+\frac{I_1}{m_2L_1^2} & -\left(\cos(q_2)+\frac{L_2}{L_1}\right) \\ -\left(\cos(q_2)+\frac{L_2}{L_1}\right) & 1 \end{bmatrix}\begin{bmatrix} -\frac{L_1}{L_2}\delta\ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}, \quad (5.1.25)$$

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} -\frac{L_2}{L_1}\sin(q_2)\dot{q}_2 & \sin(q_2)\left(\dot{q}_2-\frac{L_2}{L_1}\left(-\frac{L_1}{L_2}\delta\dot{q}_1\right)\right) \\ \frac{L_2}{L_1}\sin(q_2)\left(-\frac{L_1}{L_2}\delta\dot{q}_1\right) & 0 \end{bmatrix}\begin{bmatrix} -\frac{L_1}{L_2}\delta\dot{q}_1 \\ \dot{q}_2 \end{bmatrix}, \quad (5.1.26)$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \dfrac{g}{L_2}\left(\dfrac{L_2}{L_1}\sin\left(-\dfrac{L_2}{L_1}\left(-\dfrac{L_1}{L_2}\delta q_1\right)+q_2\right)+\left(1+\dfrac{L_{c1}}{L_1}\dfrac{m_1}{m_2}\right)\sin\left(-\dfrac{L_2}{L_1}\left(-\dfrac{L_1}{L_2}\delta q_1\right)\right)\right) \\ -\dfrac{g}{L_2}\sin\left(-\dfrac{L_2}{L_1}\left(-\dfrac{L_1}{L_2}\delta q_1\right)+q_2\right) \end{bmatrix}, \quad (5.1.27)$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} -\dfrac{u}{m_2 L_1 L_2} \\ 0 \end{bmatrix}. \quad (5.1.28)$$

Multiplying (5.1.25)-(5.1.28) by L_2/g, transforms time to unitless time, $\underline{t}$. Utilizing the dimensionless parameters from Table 5.1, the matrices for the equations of motion of (3.1.21) become

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'' = \begin{bmatrix} 2k_2\cos(q_2)+1+k_2^2+k_9^2k_1+k_8 & -(\cos(q_2)+k_2) \\ -(\cos(q_2)+k_2) & 1 \end{bmatrix}\begin{bmatrix} \rho'' \\ q_2'' \end{bmatrix}, \quad (5.1.29)$$

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}')\boldsymbol{q}' = \begin{bmatrix} -k_2\sin(q_2)q_2' & \sin(q_2)(q_2'-k_2\rho') \\ k_2\sin(q_2)\rho' & 0 \end{bmatrix}\begin{bmatrix} \rho' \\ q_2' \end{bmatrix}, \quad (5.1.30)$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} k_2\sin(-k_2\rho+q_2)+(1+k_9k_1)\sin(-k_2\rho) \\ -\sin(-k_2\rho+q_2) \end{bmatrix}, \quad (5.1.31)$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ 0 \end{bmatrix}. \quad (5.1.32)$$

where $q_2' = \dfrac{dq_2}{d\underline{t}}$ and $q_2'' = \dfrac{dq_2^2}{d^2\underline{t}}$.

Table 5.1: Dimensionless Pendubot Parameters

Dimensionless Parameter	k_1	k_2	k_8	k_9	ρ	γ^2	τ_N
Pendubot	$\frac{m_1}{m_2}$	$\frac{L_2}{L_1}$	$\frac{I_1}{m_2 L_1^2}$	$\frac{L_{c1}}{L_1}$	$-\frac{\delta q_1 L_1}{L_2}$	$\frac{L_2}{g}$	$-\frac{\gamma^2 u}{m_2 L_2 L_1}$

5.1.4 Morphed Equations of Motion

To check the sufficient conditions necessary for morphing the equations of motion for the pendubot to the inverted pendulum cart, first examine the generalized coordinates. The first generalized coordinate for the pendubot, in the dimensionless system, is $\delta q_1 L_1/L_2$. As the length of the first link is allowed to grow without bound δq_1 grows small to cover the same circumferential distance, then $\delta q_1 L_1$ becomes a straight-line displacement and $-\delta q_1 L_1/L_2$ morphs to ρ which matches that of the inverted pendulum cart. The second coordinate for the pendubot measures the angular displacement of the second link counterclockwise whereas the pendulum of the inverted pendulum cart of chapter 4 is measured clockwise. For the second coordinate to morph, then q_2 will need to be multiplied by -1 to have the same directionality of θ. In a similar manner, the generalized velocities of the pendubot morph to those of the inverted pendulum cart. The mass matrix of the dimensionless pendubot is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} 2k_2\cos(q_2) + 1 + k_2 + k_9^2 k_1 + k_8 & -(\cos(q_2) + k_2) \\ -(\cos(q_2) + k_2) & 1 \end{bmatrix}. \tag{5.1.33}$$

As L_1 grows large, k_2 and k_8 go to zero, and k_9 goes to one, then the mass matrix can be simplified to

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} 1 + k_1 & -\cos(q_2) \\ -\cos(q_2) & 1 \end{bmatrix}. \tag{5.1.34}$$

Now substitute in $-\theta$ for q_2 then, q_2'' becomes $-\theta''$, and (5.1.34) is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} 1+k_1 & \cos(\theta) \\ -\cos(\theta) & -1 \end{bmatrix}. \tag{5.1.35}$$

For a symmetric mass matrix, multiply the second row of (5.1.35) by -1, and then the resulting mass matrix matches the mass matrix of the dimensionless inverted pendulum cart, (4.1.13), provided k_1 is the same ratio of masses. Lastly, the gradient of the potential energy of the dimensionless pendubot from (5.1.31) is

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} k_2 \sin(-k_2\rho + q_2) + (1 + k_9 k_1)\sin(-k_2\rho) \\ -\sin(-k_2\rho + q_2) \end{bmatrix}, \tag{5.1.36}$$

As L_1 grows large, k_2 and ρ go to zero, and then integrating (5.1.36) results in a potential energy function that will morph to the potential energy function of the dimensionless inverted pendulum cart. Therefore, the sufficient conditions for morphing the pendubot dynamics to that of the inverted pendulum cart are met.

As the length of the first link, L_1, is allowed to grow, the dimensionless parameters k_9 will go to one, and k_8 will go to zero. As the first length grows large in order to maintain the same circumferential displacement δq_1 will become small, then using q_2 equals $-\theta$ and multiplying the bottom row by -1, (5.1.29)-(5.1.32) become

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'' = \begin{bmatrix} 1+k_1 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix} \begin{bmatrix} \rho'' \\ \theta'' \end{bmatrix}, \tag{5.1.37}$$

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}')\boldsymbol{q}' = \begin{bmatrix} 0 & \sin(\theta)\theta' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \rho' \\ \theta' \end{bmatrix}, \tag{5.1.38}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ -\sin(\theta) \end{bmatrix}, \tag{5.1.39}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ 0 \end{bmatrix}. \tag{5.1.40}$$

Putting these matrices into (3.1.21) yields equations of motion that match those of (4.1.11) and (4.1.12).

5.2 IDA-PBC Controller

5.2.1 Full Controller

The IDA-PBC controller from [24] will be applied to the pendubot equations of motion since there are enough details to perform simulations. The full controller is given by

$$\begin{aligned} u = \nabla_1 V - \left[\lambda_1 \nabla_1 V_d + \lambda_2 \left[\frac{1}{2} \nabla_2 \left(\boldsymbol{p}^T \boldsymbol{M}_d^{-1} \boldsymbol{p} \right) + \nabla_2 V_d \right] \right] \\ + \boldsymbol{p}^T \boldsymbol{M}_d^{-1} \boldsymbol{\alpha} \left[\frac{d_1 p_2 - d_3 p_1}{\Delta_d} \right] + k_v \left[\frac{d_2 p_2 - d_4 p_1}{\Delta_d} \right], \end{aligned} \tag{5.2.1}$$

where k_v is a positive constant. The desired mass matrix is

$$\boldsymbol{M}_d = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}, \tag{5.2.2}$$

where $\Delta_d = d_1 d_4 - d_2 d_3$, $d_1 = k\varphi$, $d_2 = d_2 = k(c_1\text{-}c_2)$, $d_4 = k(c_3\cos(q_2)\text{-}c_2)$, $c_1 = m_1 L_{c1}^2 + m_2 L_1^2 + I_1$, $c_2 = m_2 L_{c2}^2 + I_2$, $c_3 = m_2 L_1 L_{c2}$, $c_4 = m_1 L_{c1} + m_2 L_1$, $c_5 = m_2 L_{c2}$, and k is a positive constant. Utilizing the potential energy function of the pendubot of (5.1.2), the mass matrix of (5.1.13), and the desired mass matrix, the desired potential energy function is found to be

$$V_d(q_1, q_2) = \frac{c_5}{k} g \left(\cos(q_1 + q_2) + 1 \right) + \frac{k_p}{2} \left(q_2 + 2q_1 - (\pi + q_{d1}) \right)^2, \tag{5.2.3}$$

where k_p is a positive constant. The derivatives of V and V_d then are

$$\nabla_1 V = c_4 g \sin(q_1) + c_5 g \sin(q_1 + q_2), \tag{5.2.4}$$

$$\nabla_1 V_d = -\frac{c_5 g}{k}\sin\left(q_1+q_2\right)+2k_p\left(q_2+2q_1-\left(\pi+q_{d1}\right)\right), \tag{5.2.5}$$

and

$$\nabla_2 V_d = -\frac{c_5 g}{k}\sin\left(q_1+q_2\right)+k_p\left(q_2+2q_1-\left(\pi+q_{d1}\right)\right). \tag{5.2.6}$$

Next, the generalized momenta is $\boldsymbol{p}=\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$. The matrix J_2 is skew symmetric and $J_2(1,2)$ is

defined to be $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{M}_d^{-1}\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}=\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, with

$$\alpha_1 = c_3\sin\left(q_2\right)\left(\lambda_1^2+\lambda_1\lambda_2\right), \tag{5.2.7}$$

$$\alpha_2 = c_3\sin\left(q_2\right)k^2\frac{c_3\left(2c_1-2c_2-\phi\right)\cos\left(q_2\right)+c_1^2-c_2^2-c_2\phi}{-\cos^2\left(q_2\right)c_3^2+c_1c_2}, \tag{5.2.8}$$

$$\lambda_1 = \frac{k\left(\cos\left(q_2\right)c_3\left(c_1-c_2\right)+c_1c_2-c_2^2-c_2\phi\right)}{\cos^2\left(q_2\right)c_3^2-c_1c_2}, \tag{5.2.9}$$

and

$$\lambda_2 = \frac{k\left(c_3\left(2c_1-2c_2-\phi\right)\cos\left(q2\right)+c_1^2-c_2^2-c_2\phi\right)}{\left(-\cos\left(q2\right)^2c_3^2+c_1c_2\right)}. \tag{5.2.10}$$

Lastly, the derivative of $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{M}_d^{-1}\boldsymbol{p}$ is

$$\nabla_2\left(\boldsymbol{p}^T\boldsymbol{M}_d^{-1}\boldsymbol{p}\right)=\frac{c_3\left(\left(c_1-c_2\right)p_1-p_2\phi\right)^2\sin\left(q2\right)}{\left(-\cos\left(q_2\right)c_3\phi+c_2^2+\left(-2c_1+\phi\right)c_2+c_1^2\right)^2k}. \tag{5.2.11}$$

5.2.2 Dimensionless Controller

Next, to convert the pendubot controller to have the second link modeled as point mass, let L_{c2} and L_2 be equal and I_2 to be zero. For the first link to measure the circumferential displacement from the vertical y-axis, let q_1 be equal to $\pi - \rho L_2/L_1$. Now to render the controller of (5.2.1) dimensionless, after invoking the substitutions above, multiply u by $-\gamma^2/(m_2L_2L_1)$ which results in the dimensionless controller

$$\begin{aligned}\tau_N &= -\frac{1}{m_2L_1g}\nabla_1 V + \lambda_1\frac{1}{m_2L_1g}\nabla_1 V_d + \lambda_2\frac{1}{2}\frac{1}{m_2L_1g}\nabla_2\left(\boldsymbol{p}^T\boldsymbol{M}_d^{-1}\boldsymbol{p}\right)\\ &+\lambda_2\frac{1}{m_2L_1g}\nabla_2 V_d - \frac{1}{m_2L_1g}\boldsymbol{p}^T\boldsymbol{M}_d^{-1}\boldsymbol{\alpha}\left[\frac{-d_3p_1+d_1p_2}{\Delta_d}\right] + \frac{1}{m_2L_1g}k_v\left[\frac{d_4p_1-d_2p_2}{\Delta_d}\right].\end{aligned} \tag{5.2.12}$$

Simplifying the terms in (5.2.12) and factoring out $m_2L_1^2$, the c_is become $c_{1n} = k_1k_9^2+1+k_8$, $c_{2n} = k_2^2$, and $c_{3n} = k_2$, and the d_is become $d_{1n} = k\varphi_n$, $d_{2n} = d_{3n} = k(c_{1n} - c_{2n})$, and $d_{4n} = k(c_{3n}\cos(q_2) - c_{2n})$, where $\varphi_n = \varphi/(m_2L_1^2)$. Then the desired mass matrix is

$$\boldsymbol{M}_{dn} = \begin{bmatrix} d_{1n} & d_{2n} \\ d_{3n} & d_{4n}\end{bmatrix}, \tag{5.2.13}$$

and the determinate of the mass matrix is

$$\Delta_{dn} = d_{1n}d_{4n} - d_{2n}d_{3n}. \tag{5.2.14}$$

Factoring out m_2L_1 from c_4 and c_5 leaves $c_{4n} = k_1k_9+1$ and $c_{5n} = k2$. Next the derivative of the potential energy function of the original system with respect to the first variable of $\boldsymbol{q}$, can be rewritten as

$$\nabla_1 V_n = c_{4n}\sin\left(k_2\rho\right) - c_{5n}\sin\left(q_2 - k_2\rho\right), \tag{5.2.15}$$

and the derivatives of the desired potential energy function are

$$\nabla_1 V_{dn} = \frac{c_{5n}}{k}\sin\left(q_2 - k_2\rho\right) + 2k_{pn}\left(q_2 + 2\left(-k_2\rho\right) + \pi - q_{d1}\right), \tag{5.2.16}$$

and

$$\nabla_2 V_{dn} = \frac{c_{5n}}{k}\sin\left(q_2 - k_2\rho\right) + k_{pn}\left(q_2 + 2\left(-k_2\rho\right) + \pi - q_{d1}\right), \tag{5.2.17}$$

with $k_{pn} = k_p/(m_2L_1g)$. The elements of $\boldsymbol{\alpha}_n$ are

$$\alpha_{1n} = c_{3n}\sin\left(q_2\right)\left(\lambda_{1n}^2 + \lambda_{1n}\lambda_{2n}\right) \tag{5.2.18}$$

and

$$\alpha_{2n} = c_{3n}\sin\left(q_2\right)k^2 \frac{c_{3n}\left(2c_{1n} - 2c_{2n} - \phi_n\right)\cos\left(q_2\right) + c_{1n}^2 - c_{2n}^2 - c_{2n}\phi_n}{-\cos^2\left(q_2\right)c_{3n}^2 + c_{1n}c_{2n}}, \tag{5.2.19}$$

where

$$\lambda_{1n} = \frac{k\left(\cos\left(q_2\right)c_{3n}\left(c_{1n} - c_{2n}\right) + c_{1n}c_{2n} - c_{2n}^2 - c_{2n}\phi_n\right)}{\cos^2\left(q_2\right)c_{3n}^2 - c_{1n}c_{2n}} \tag{5.2.20}$$

and

$$\lambda_{2n} = \frac{k\left(c_{3n}\left(2c_{1n} - 2c_{2n} - \phi_n\right)\cos\left(q2\right) + c_{1n}^2 - c_{2n}^2 - c_{2n}\phi_n\right)}{\left(-\cos\left(q2\right)^2 c_{3n}^2 + c_{1n}c_{2n}\right)}. \tag{5.2.21}$$

Finally, the generalized momenta, which are defined as $\boldsymbol{p}_n = \boldsymbol{M}_n\dot{\boldsymbol{q}}$, become

$$p_{1n} = -\left(2c_{3n}\cos\left(q_2\right) + c_{2n} + c_{1n}\right)\rho' + \left(\cos\left(q_2\right) + c_{3n}\right)q_2' \tag{5.2.22}$$

and

$$p_{2n} = -\left(\cos\left(q_2\right) + c_{3n}\right)\rho' + k_2 q_2' \tag{5.2.23}$$

Lastly, the derivative of $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{M}_d^{-1}\boldsymbol{p}$ simplifies as

$$\nabla_2\left(\boldsymbol{p}^T\boldsymbol{M}_d^{-1}\boldsymbol{p}\right)_n = \frac{c_{3n}\left(\left(c_{1n} - c_{2n}\right)p_{1n} - p_{2n}\phi_n\right)^2\sin\left(q2\right)}{\left(-\cos\left(q_2\right)c_{3n}\phi_n + c_{2n}^2 + \left(-2c_{1n} + \phi_n\right)c_{2n} + c_{1n}^2\right)^2 k} \tag{5.2.24}$$

Now (5.2.12) simplifies to

$$\begin{aligned} \tau_N = & -\nabla_1 V_n + \lambda_{1n}\nabla_1 V_{dn} + k_2\lambda_{2n}\nabla_2 V_{dn} + k_2\lambda_{2n}\frac{1}{2}\nabla_2\left(\boldsymbol{p}^T\boldsymbol{M}_d^{-1}\boldsymbol{p}\right)_n \\ & -k_2\boldsymbol{p}_n^T\boldsymbol{M}_d^{-1}\boldsymbol{\alpha}_n\left[\frac{-d_{3n}p_{1n} + d_{1n}p_{2n}}{\Delta_{dn}}\right] + k_{vn}\left[\frac{d_{4n}p_{1n} - d_{2n}p_{2n}}{\Delta_{dn}}\right]. \end{aligned} \tag{5.2.25}$$

where $k_{vn} = \gamma k_v/(m_2 L_1^2)$.

5.2.3 Controller Morphing

As L_1 is allowed to grow, k_9 goes to one, k_2 and k_8 go to zero, and ρ becomes small to maintain the same circumferential displacement. Then c_{2n}, c_{3n}, c_{5n} are equal to zero, c_{1n} and c_{4n} are $k_1 + 1$. This makes $d_{1n} = k\varphi_n$, $d_{2n} = d_{3n} = k(k_1 + 1)$, and $d_{4n} = 0$. The dimensionless control law of (5.2.25) then becomes

$$\tau_N = -\nabla_1 V_n + \lambda_{1n} \nabla_1 V_{dn} + k_{vn} \left[\frac{-p_{2n}}{k\left(k_1 + 1\right)} \right] \tag{5.2.26}$$

where

$$\nabla_1 V_n = \left(k_1 + 1\right) \sin\left(0\right), \tag{5.2.27}$$

$$\nabla_1 V_{dn} = 2k_{pn} \left(q_2 + \pi - q_{d1}\right), \tag{5.2.28}$$

$$p_{2n} = -\cos\left(q_2\right) \rho' \tag{5.2.29}$$

and

$$\lambda_{1n} = -k. \tag{5.2.30}$$

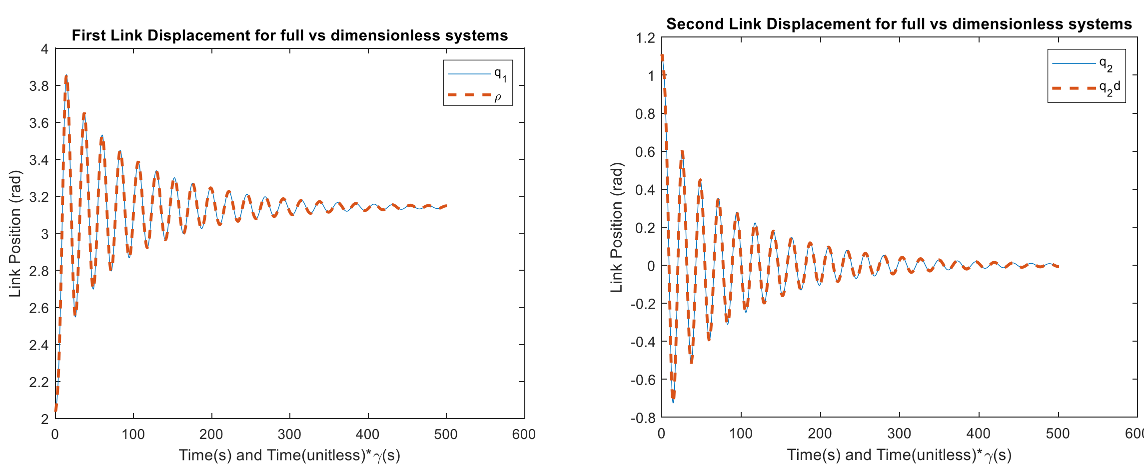
Let q_{di} be equal to π and q_2 equal to $-\theta$, (5.2.26) simplifies to

$$\tau_N = 2kk_{pn}\theta + k_{vn} \frac{\cos\left(\theta\right)\rho'}{k\left(k_1 + 1\right)}. \tag{5.2.31}$$

5.3 Simulations

After the morphed controller was derived simulations were performed. First the simulation of the original equations of motion of (5.1.11) and (5.1.12) with the controller of (5.2.1) was performed using initial conditions of $q_1 = \pi - 1.1$, $q_2 = 1.1$, and $\dot{q}_1 = \dot{q}_2 = 0$, with control gains of

$\varphi = 500$, $k = 0.0033$, $k_p = 30$, $k_v = 20$, and $q_{di} = \pi$, similar to [24]. Next, a simulation was executed on the dimensionless equations of motion represented by (5.1.29)-(5.1.32) with the dimensionless controller (5.2.25). The dimensionless simulations had controller gains of $\varphi_d = 125$, $k = 0.0033$, $k_{pd} = 1.5291$, $k_{vd} = 1.1288$, and initial conditions of $\rho = 4.4$, $q_2 = 1.1$, and $\dot{\rho} = \dot{q}_2 = 0$. Figure 5.2 compares the results of these two simulations to validate the dimensionless process was performed correctly. For both plots in this figure, the time axis for the dimensionless simulation was scaled by γ to compare to time in seconds. For Figure 5.2a, the y-axis for the rho values was scaled by L_2/L_1 and then π was added to the link one position to compare it with the original link displacement. Lastly, the morphed equations of motion represented by the matrices of (5.1.37)-(5.1.40) were simulated using the morphed controller of (5.2.31), utilizing initial conditions of $\rho = 3/L_2$, $q_2 = \pi/6$, and $\dot{\rho} = \dot{q}_2 = 0$, to match the initial conditions of the dimensionless inverted pendulum cart system of section 4.1.5, and the same controller gains as the dimensionless simulation utilized. This simulation produced unstable results.



(a) (b)

Figure 5.2: Comparison of the Original and Dimensionless Pendubot Systems

For this IDA-PBC controller, why did the morphed control law with the morphed equations of motion not produce stable results when the dimensionless system did produce stable results? Looking at the dimensionless potential energy function of (5.2.28), once the first link is allowed to grow large and δq_1 goes to zero the new potential energy function then morphs to

$$V_d = k_{pn} q_2^2 \tag{5.3.1}$$

which is positive for all values of q_2. Using the definition of the d_is, letting the first link grow, the dimensionless desired mass matrix becomes

$$\boldsymbol{M}_d = \begin{bmatrix} k\phi_n & k\left(k_1+1\right) \\ k\left(k_1+1\right) & 0 \end{bmatrix} \tag{5.3.2}$$

which is not a positive definite matrix. This is one reason why going from the dimensionless control law of (5.2.25) to the morphed control law (5.2.26) all the terms involving $\boldsymbol{M}_d$ dropped out. This controller presents the question, if the desired mass matrix successfully morphs to a positive definite matrix, will a simulation of the morphed controller with morphed equations produce a stable result?

5.4 Conclusions

In this chapter, the equations of motion of the pendubot fulfilled the sufficient conditions for morphing to the inverted pendulum cart. An IDA-PBC controller from [24] was applied to the pendubot and rendered dimensionless. Simulations of the dimensionless equations of motion and controller were performed, demonstrating that the process of rendering a system dimensionless did not alter the simulations results. The process of morphing the controller did not lead to stable simulation results for the morphed system. This chapter has shown another underactuated system which supports the sufficient conditions for equations of motion to morph.

Chapter 6 - Fully Actuated Two Link Manipulator

The examples that have been presented in the prior chapters have all been underactuated systems. In this chapter, a fully actuated system will be investigated to illustrate that the sufficient conditions for morphing equations of motions applies to fully actuated systems. A controller will also be presented that will successfully morph.

In [28], the author presents a two-link planar robot manipulating an unknown load, shown in Figure 6.1. This robot is fully actuated and as the length of the first link grows large this system would resemble a fully actuated inverted pendulum cart. To test this idea, first the equations of motion will be presented for the model in Figure 6.1. Then the controller from [28] will be presented. Next, the equations of motion and the control law will be converted to dimensionless form, and then morphed. Lastly, simulation results will be presented that verify the process of rendering the system dimensionless does not alter the response and the successful morphing of the two-link manipulator to a fully actuated inverted pendulum cart.

6.1 Equations of Motion

For the two-link robot manipulator of Figure 6.1, the kinetic energy is

$$T = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} m_1 \left(\dot{x}_{1c}^2 + \dot{y}_{1c}^2 \right) + \frac{1}{2} m_2 \left(\dot{x}_{2c}^2 + \dot{y}_{2c}^2 \right) + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2 \right)^2 \tag{6.1.1}$$

while the gravitational potential energy is

$$V = m_1 g y_{1c} + m_2 g y_{2c}. \tag{6.1.2}$$

The global position and velocity for the center of mass of the first link, the link attached to the origin, are

$$x_{1c} = L_{c1} \cos\left(q_1 \right), \tag{6.1.3}$$

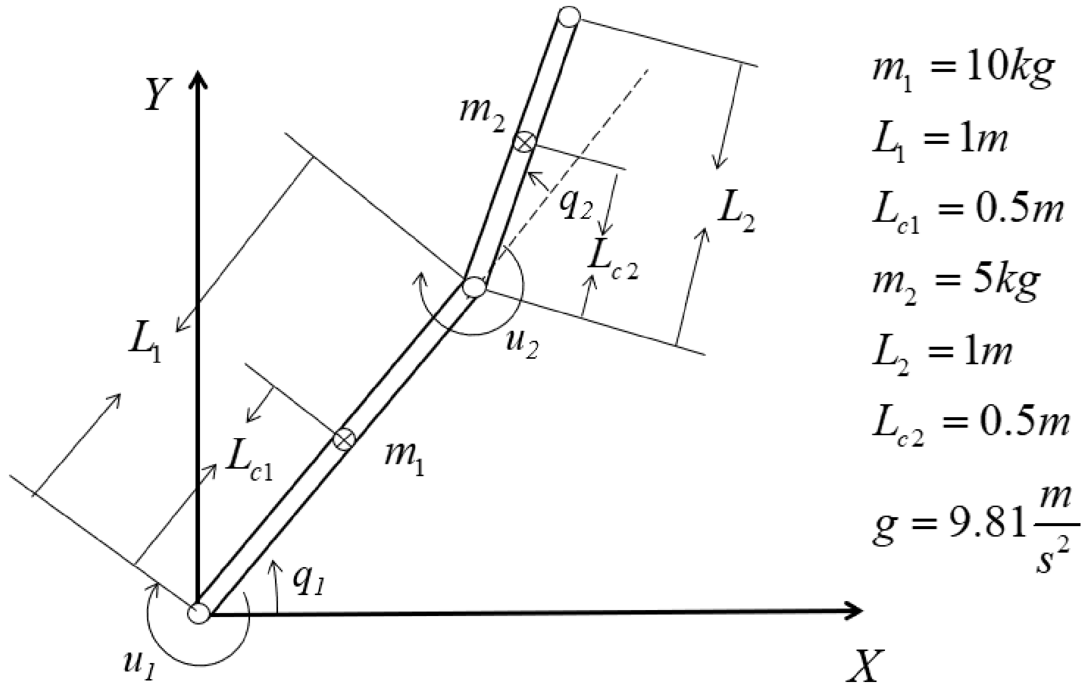


Figure 6.1: Two Link Robot Manipulator

$$y_{1c} = L_{c1}\sin(q_1), \tag{6.1.4}$$

$$\dot{x}_{1c} = -L_{c1}\sin(q_1)\dot{q}_1, \tag{6.1.5}$$

and

$$\dot{y}_{1c} = L_{c1}\cos(q_1)\dot{q}_1. \tag{6.1.6}$$

The global position and velocity for the center of mass of the second link are

$$x_{2c} = L_1\cos(q_1) + L_{c2}\cos(q_1 + q_2), \tag{6.1.7}$$

$$y_{2c} = L_1\sin(q_1) + L_{c2}\sin(q_1 + q_2), \tag{6.1.8}$$

$$\dot{x}_{2c} = -L_1\sin(q_1)\dot{q}_1 - L_{c2}\sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2), \tag{6.1.9}$$

and

$$\dot{y}_{2c} = L_1 \cos(q_1)\dot{q}_1 + L_{c2}\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2). \tag{6.1.10}$$

The equation of motion for the first link, found using Lagrange's equation, (2.2), is

$$\begin{aligned}&\left(2L_{c2}m_2L_1\cos(q_2)+\left(L_1^2+L_{c2}^2\right)m_2+L_{c1}^2m_1+I_1+I_2\right)\ddot{q}_1+\\&\left(L_{c2}m_2L_1\cos(q_2)+m_2L_{c2}^2+I_2\right)\ddot{q}_2-2m_2L_1L_{c2}\sin(q_2)\dot{q}_1\dot{q}_2-\\&m_2L_1L_{c2}\sin(q_2)\dot{q}_2^2+g\left(m_2L_{c2}\cos(q_1+q_2)+(L_1m_2+L_{c1}m_1)\cos(q_1)\right)=u_1\end{aligned} \tag{6.1.11}$$

and the second link is

$$\begin{aligned}&\left(m_2L_{c2}L_1cos(q_2)+m_2L_{c2}^2+I_2\right)\ddot{q}_1+\left(L_{c2}^2m_2+I_2\right)\ddot{q}_2\\&+m_2L_1L_{c2}sin(q_2)\dot{q}_1^2+gm_2L_{c2}\cos(q_1+q_2)=u_2.\end{aligned} \tag{6.1.12}$$

The matrices of (3.1.21) for this system are

$$\boldsymbol{M}(\boldsymbol{q})=\begin{bmatrix}\left(2L_{c2}L_1\cos(q_2)+L_1^2+L_{c2}^2\right)m_2+L_{c1}^2m_1+I_1+I_2 & \left(L_1\cos(q_2)+L_{c2}\right)m_2L_{c2}+I_2\\ \left(L_1\cos(q_2)+L_{c2}\right)m_2L_{c2}+I_2 & L_{c2}^2m_2+I_2\end{bmatrix}, \tag{6.1.13}$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{bmatrix}-m_2L_1L_{c2}\sin(q_2)\dot{q}_2 & -m_2L_1L_{c2}\sin(q_2)(\dot{q}_1+\dot{q}_2)\\ m_2L_1L_{c2}\sin(q_2)\dot{q}_1 & 0\end{bmatrix}, \tag{6.1.14}$$

$$\boldsymbol{G}(\boldsymbol{q})=\begin{bmatrix}g\left(m_2L_{c2}\cos(q_1+q_2)+(L_1m_2+L_{c1}m_1)\cos(q_1)\right)\\ gm_2L_{c2}\cos(q_1+q_2)\end{bmatrix}, \tag{6.1.15}$$

and

$$\boldsymbol{\tau}=\begin{bmatrix}u_1\\u_2\end{bmatrix}. \tag{6.1.16}$$

Lastly, if the second link is modified to be modeled as a long, massless, slender rod with a point mass a distance of L_2 from the end of the first link, then $I_2 = 0$ and $L_{c2} = L_2$. The matrices for (3.1.21) then become

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} \left(2L_2L_1\cos(q_2) + L_1^2 + L_2^2\right)m_2 + L_{c1}^2 m_1 + I_1 & \left(L_1\cos(q_2) + L_2\right)m_2L_2 \\ \left(L_1\cos(q_2) + L_2\right)m_2L_2 & L_2^2 m_2 \end{bmatrix}, \quad (6.1.17)$$

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} -m_2L_1L_2\sin(q_2)\dot{q}_2 & -m_2L_1L_2\sin(q_2)(\dot{q}_1 + \dot{q}_2) \\ m_2L_1L_2\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}, \quad (6.1.18)$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} g\left(m_2L_2\cos(q_1+q_2) + (L_1m_2 + L_{c1}m_1)\cos(q_1)\right) \\ gm_2L_2\cos(q_1+q_2) \end{bmatrix}, \quad (6.1.19)$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (6.1.20)$$

6.2 Controller

In [28], the first step in designing the controller is to recast the equations of motion into the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{\sigma} = \boldsymbol{\tau} \quad (6.2.1)$$

where $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$, for the two-link manipulator, is a 2x6 matrix of functions of the generalized coordinates and its derivatives, $\boldsymbol{\sigma}$ is a column vector of inertia parameters. Using equations (6.1.11) and (6.1.12), $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ and $\boldsymbol{\sigma}$ are

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \begin{bmatrix} \ddot{q}_1 & \ddot{q}_1 + \ddot{q}_2 & Y1 & g\cos(q_1) & g\cos(q_1) & g\cos(q_1+q_2) \\ 0 & \ddot{q}_1 + \ddot{q}_2 & Y2 & 0 & 0 & g\cos(q_1+q_2) \end{bmatrix} \quad (6.2.2)$$

$$Y1 = \cos(q_2)(2\ddot{q}_1 + \ddot{q}_2) - \sin(q_2)\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right), \quad (6.2.3)$$

$$Y2 = \cos(q_2)\ddot{q}_1 + \sin(q_2)\dot{q}_1^2 \quad (6.2.4)$$

and

$$\boldsymbol{\sigma} = \begin{bmatrix} L_1^2 m_2 + L_{c1}^2 m_1 + I_1 & L_2^2 m_2 & m_2 L_2 L_1 & m_1 L_{c1} & m_2 L_1 & m_2 L_2 \end{bmatrix}^T . \tag{6.2.5}$$

Then the control law is

$$\boldsymbol{\tau} = \boldsymbol{Y}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}, \boldsymbol{a}\right)\left(\boldsymbol{\sigma}_0 + \boldsymbol{u}_{2L}\right) - \boldsymbol{K}\boldsymbol{r} \tag{6.2.6}$$

where $\boldsymbol{Y}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}, \boldsymbol{a}\right)$ is a 2x6 matrix of nominal functions of the generalized coordinates, velocities, and errors relative to a reference trajectory, $\boldsymbol{\sigma}_0$ is a column vector of nominal inertia parameters, $\boldsymbol{u}_{2L}$ is designed to achieve robustness to the uncertainty of $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0)$, and $\boldsymbol{K}$ is a positive definite diagonal gain matrix. The new variables are defined as

$$\boldsymbol{v} = \dot{\boldsymbol{q}}^d - \boldsymbol{\lambda}\tilde{\boldsymbol{q}}, \tag{6.2.7}$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}}, \tag{6.2.8}$$

$$\boldsymbol{r} = \dot{\tilde{\boldsymbol{q}}} + \boldsymbol{\lambda}\tilde{\boldsymbol{q}}, \tag{6.2.9}$$

and

$$\tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}^d \tag{6.2.10}$$

where $\boldsymbol{\lambda}$ is a positive definite diagonal gain matrix and $\boldsymbol{q}^d$ is a reference trajectory. For the two-link manipulator of Figure 6.1,

$$\boldsymbol{Y}\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}, \boldsymbol{a}\right) = \begin{bmatrix} a_1 & a_1 + a_2 & Ya1 & g\cos\left(q_1\right) & g\cos\left(q_1\right) & g\cos\left(q_1 + q_2\right) \\ 0 & a_1 + a_2 & Ya2 & 0 & 0 & g\cos\left(q_1 + q_2\right) \end{bmatrix} \tag{6.2.11}$$

$$Ya1 = \cos\left(q_2\right)\left(2a_1 + a_2\right) - \sin\left(q_2\right)\left(\dot{q}_2 v_2 + \dot{q}_1 v_2 + \dot{q}_2 v_1\right), \tag{6.2.12}$$

$$Ya2 = \cos\left(q_2\right) a_1 + \sin\left(q_2\right)\dot{q}_1 v_1, \tag{6.2.13}$$

$$\boldsymbol{u}_{2L} = \left\{ \begin{array}{ll} -\psi_i \left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i \Big/ \left|\left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i\right| & \text{if } \left|\left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i\right| > \varepsilon_{2Li} \\ -\psi_i \left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i \Big/ \varepsilon_{2Li} & \text{if } \left|\left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i\right| \le \varepsilon_{2Li} \end{array} \right\}, \tag{6.2.14}$$

$$\boldsymbol{\sigma}_0 = \begin{bmatrix} L_1^2\left(m_2+\Delta m_2\right)+L_{c1}^2 m_1 + I_1 \\ \left(L_2+\Delta L_2\right)^2\left(m_2+\Delta m_2\right) \\ L_1\left(m_2+\Delta m_2\right)\left(L_2+\Delta L_2\right) \\ L_{c1}m_1 \\ L_1\left(m_2+\Delta m_2\right) \\ \left(m_2+\Delta m_2\right)\left(L_2+\Delta L_2\right) \end{bmatrix}, \tag{6.2.15}$$

and $\boldsymbol{\Psi} = \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}$. Then, after substituting and simplifying, the control law is

$$\begin{aligned} &a_1\left(\left(\boldsymbol{\sigma}_0\right)_1+\left(u_{2L}\right)_1\right)+\left(a_1+a_2\right)\left(\left(\boldsymbol{\sigma}_0\right)_2+\left(u_{2L}\right)_2\right)+Ya1\left(\left(\boldsymbol{\sigma}_0\right)_3+\left(u_{2L}\right)_3\right)+ \\ &\quad g\cos\left(q_1\right)\left(\left(\boldsymbol{\sigma}_0\right)_4+\left(u_{2L}\right)_4\right)+g\cos\left(q_1\right)\left(\left(\boldsymbol{\sigma}_0\right)_5+\left(u_{2L}\right)_5\right) \\ &\quad +g\cos\left(q_1+q_2\right)\left(\left(\boldsymbol{\sigma}_0\right)_6+\left(u_{2L}\right)_6\right)-K\left(1,1\right)r_1 = u_1 \end{aligned} \tag{6.2.16}$$

and

$$\begin{aligned} &\left(a_1+a_2\right)\left(\left(\boldsymbol{\sigma}_0\right)_2+\left(u_{2L}\right)_2\right)+Ya2\left(\left(\boldsymbol{\sigma}_0\right)_3+\left(u_{2L}\right)_3\right)+g\cos\left(q_1+q_2\right)\left(\left(\boldsymbol{\sigma}_0\right)_6+\left(u_{2L}\right)_6\right) \\ &-K\left(2,2\right)r_2 = u_2. \end{aligned} \tag{6.2.17}$$

6.3 Dimensionless System

6.3.1 Equations of Motion

As the length of the first link grows large, the first link can be morphed to a cart with a straight-line displacement measured relative to the positive y-axis. For this purpose, redefine q_1 as $\pi/2+\delta q_1$, then the derivatives of q_1 are the derivatives of δq_1. Next, convert the first coordinate to measuring the circumferential displacement of the link by factoring out L_1 from terms multiplying the derivatives of δq_1. The matrices for (3.1.21) become

$$M(q)\ddot{q} =$$

$$\begin{bmatrix} \left(2L_2\cos(q_2)+L_1+\frac{L_2^2}{L_1}\right)m_2+\frac{\left(L_{c1}^2 m_1+I_1\right)}{L_1} & \left(L_1\cos(q_2)+L_2\right)m_2L_2 \\ \left(\cos(q_2)+\frac{L_2}{L_1}\right)m_2L_2 & L_2^2 m_2 \end{bmatrix}\begin{bmatrix} \delta\ddot{q}_1 L_1 \\ \ddot{q}_2 \end{bmatrix}, \tag{6.3.1}$$

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} -m_2L_2\sin(q_2)\dot{q}_2 & -m_2L_2\sin(q_2)\left(\left(\delta\dot{q}_1L_1\right)+L_1\dot{q}_2\right) \\ m_2\frac{L_2}{L_1}\sin(q_2)\left(\delta\dot{q}_1L_1\right) & 0 \end{bmatrix}\begin{bmatrix} \delta\dot{q}_1 L_1 \\ \dot{q}_2 \end{bmatrix}, \tag{6.3.2}$$

$$G(q) = \begin{bmatrix} g\left(m_2L_2\cos\left(\frac{\pi}{2}+\delta q_1+q_2\right)+\left(L_1m_2+L_{c1}m_1\right)\cos\left(\frac{\pi}{2}+\delta q_1\right)\right) \\ gm_2L_2\cos\left(\frac{\pi}{2}+\delta q_1+q_2\right) \end{bmatrix}, \tag{6.3.3}$$

and

$$\tau = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.3.4}$$

For a symmetric mass matrix, divide the first link equation, top row, by L_1. After simplifying the cosine functions in the **G** matrix, (6.3.1) through (6.3.4) are

$$M(q)\ddot{q} = \begin{bmatrix} \left(2\frac{L_2}{L_1}\cos(q_2)+1+\frac{L_2^2}{L_1^2}\right)m_2+\frac{\left(L_{c1}^2 m_1+I_1\right)}{L_1^2} & \left(\cos(q_2)+\frac{L_2}{L_1}\right)m_2L_2 \\ \left(\cos(q_2)+\frac{L_2}{L_1}\right)m_2L_2 & L_2^2 m_2 \end{bmatrix}\begin{bmatrix} \delta\ddot{q}_1 L_1 \\ \ddot{q}_2 \end{bmatrix}, \tag{6.3.5}$$

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} -m_2\frac{L_2}{L_1}\sin(q_2)\dot{q}_2 & -m_2\frac{L_2}{L_1}\sin(q_2)\left(\left(\delta\dot{q}_1L_1\right)+L_1\dot{q}_2\right) \\ m_2\frac{L_2}{L_1}\sin(q_2)\left(\delta\dot{q}_1L_1\right) & 0 \end{bmatrix}\begin{bmatrix} \delta\dot{q}_1 L_1 \\ \dot{q}_2 \end{bmatrix}, \tag{6.3.6}$$

$$
\boldsymbol{G}(\boldsymbol{q})=\begin{bmatrix} -g\left(m_2\dfrac{L_2}{L_1}\sin\left(\delta q_1+q_2\right)+\left(m_2+\dfrac{L_{c1}}{L_1}m_1\right)\sin\left(\delta q_1\right)\right) \\ -gm_2L_2\sin\left(\delta q_1+q_2\right) \end{bmatrix}, \tag{6.3.7}
$$

and

$$
\boldsymbol{\tau}=\begin{bmatrix} \dfrac{u_1}{L_1} \\ u_2 \end{bmatrix}. \tag{6.3.8}
$$

To render the equations dimensionless, divide the first link equation by m_2L_2 and the second link equation by $m_2L_2^2$. For dimensionless coordinates the first coordinate, δq_1L_1, needs to be divided by L_2. The dimensionless matrices for (3.1.21) then are

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}=\begin{bmatrix} \left(2\dfrac{L_2}{L_1}\cos\left(q_2\right)+1+\dfrac{L_2^2}{L_1^2}\right)+\dfrac{\left(L_{c1}^2m_1+I_1\right)}{m_2L_1^2} & \cos\left(q_2\right)+\dfrac{L_2}{L_1} \\ \cos\left(q_2\right)+\dfrac{L_2}{L_1} & 1 \end{bmatrix}\begin{bmatrix} \dfrac{\delta\ddot{q}_1L_1}{L_2} \\ \ddot{q}_2 \end{bmatrix}, \tag{6.3.9}
$$

$$
\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}=\begin{bmatrix} -\dfrac{L_2}{L_1}\sin\left(q_2\right)\dot{q}_2 & -\sin\left(q_2\right)\left(\dfrac{L_2}{L_1}\left(\dfrac{\delta\dot{q}_1L_1}{L_2}\right)+\dot{q}_2\right) \\ \dfrac{L_2}{L_1}\sin\left(q_2\right)\left(\dfrac{\delta\dot{q}_1L_1}{L_2}\right) & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\delta\dot{q}_1L_1}{L_2} \\ \dot{q}_2 \end{bmatrix}, \tag{6.3.10}
$$

$$
\boldsymbol{G}(\boldsymbol{q})=\begin{bmatrix} -\dfrac{g}{L_2}\left(\dfrac{L_2}{L_1}\sin\left(\dfrac{L_2}{L_1}\left(\dfrac{\delta q_1L_1}{L_2}\right)+q_2\right)+\left(1+\dfrac{L_{c1}}{L_1}\dfrac{m_1}{m_2}\right)\sin\left(\dfrac{L_2}{L_1}\left(\dfrac{\delta q_1L_1}{L_2}\right)\right)\right) \\ -\dfrac{g}{L_2}\sin\left(\dfrac{L_2}{L_1}\left(\dfrac{\delta q_1L_1}{L_2}\right)+q_2\right) \end{bmatrix}, \tag{6.3.11}
$$

$$
\boldsymbol{\tau}=\begin{bmatrix} \dfrac{u_1}{m_2L_1L_2} \\ \dfrac{u_2}{m_2L_2^2} \end{bmatrix}. \tag{6.3.12}
$$

Multiplying by $L_2/g = \gamma^2$, transforms time to unitless time, $\underline{t}$, and then using the dimensionless parameters from Table 6.1, equations (6.3.9) - (6.3.12) become

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{q}'' = \begin{bmatrix} 2k_2\cos(q_2)+1+k_2^2+k_9^2k_1+k_8 & \cos(q_2)+k_2 \\ \cos(q_2)+k_2 & 1 \end{bmatrix}\begin{bmatrix} \rho'' \\ q_2'' \end{bmatrix}, \tag{6.3.13}$$

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}')\boldsymbol{q}' = \begin{bmatrix} -k_2\sin(q_2)q_2' & -\sin(q_2)(k_2\rho'+q_2') \\ k_2\sin(q_2)\rho' & 0 \end{bmatrix}\begin{bmatrix} \rho' \\ q_2' \end{bmatrix}, \tag{6.3.14}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} -k_2\sin(k_2\rho+q_2)-(1+k_9k_1)\sin(k_2\rho) \\ -\sin(k_2\rho+q_2) \end{bmatrix}, \tag{6.3.15}$$

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ \tau_{N2} \end{bmatrix}. \tag{6.3.16}$$

Table 6.1: Dimensionless Two-Link Manipulator Parameters

Dimensionless Parameter	k_1	k_2	k_8	k_9	ρ	γ^2	τ_N	τ_{N2}
Two-Link Manipulator	$\frac{m_1}{m_2}$	$\frac{L_2}{L_1}$	$\frac{I_1}{m_2 L_1^2}$	$\frac{L_{c1}}{L_1}$	$\frac{\delta q_1 L_1}{L_2}$	$\frac{L_2}{g}$	$\frac{\gamma^2 u_1}{m_2 L_2 L_1}$	$\frac{\gamma^2 u_2}{m_2 L_2^2}$

6.3.2 Controller

Supporting the coordinate change for the q_1 to be measured relative to the positive y-axis $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{v},\boldsymbol{a})$ becomes

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{v},\boldsymbol{a}) = \begin{bmatrix} \frac{L_2}{L_1}a_1 & \frac{L_2}{L_1}a_1+a_2 & Ya1 & -g\sin(k_2\rho) & -g\sin(k_2\rho) & -g\sin(k_2\rho+q_2) \\ 0 & \frac{L_2}{L_1}a_1+a_2 & Ya2 & 0 & 0 & -g\sin(k_2\rho+q_2) \end{bmatrix} \tag{6.3.17}$$

where

$$Ya1=\cos\left(q_2\right)\left(2\frac{L_2}{L_1}a_1+a_2\right)-\sin\left(q_2\right)\left(\dot{q}_2v_2+\frac{L_2}{L_1}\dot{\rho}v_2+\dot{q}_2\frac{L_2}{L_1}v_1\right), \tag{6.3.18}$$

and

$$Ya2=\cos\left(q_2\right)\frac{L_2}{L_1}a_1+\sin\left(q_2\right)\frac{L_2}{L_1}\dot{\rho}\frac{L_2}{L_1}v_1. \tag{6.3.19}$$

To convert $\boldsymbol{\tau}$ to dimensionless form, as see in Table 6.1, (6.2.16) is multiplied by $\gamma^2/(m_2L_2L_1)$ and (6.2.17) by $\gamma^2/(m_2L_2^2)$. Performing this operation and simplifying slightly obtains

$$\begin{aligned}&a_1\frac{\left(\left(\boldsymbol{\sigma}_0\right)_1+\left(u_{2L}\right)_1\right)}{m_2L_1^2}+\left(\frac{L_2}{L_1}a_1+a_2\right)\frac{\left(\left(\boldsymbol{\sigma}_0\right)_2+\left(u_{2L}\right)_2\right)}{m_2L_1L_2}+Ya1\frac{\left(\left(\boldsymbol{\sigma}_0\right)_3+\left(u_{2L}\right)_3\right)}{m_2L_1L_2}\\&-\sin\left(k_2\rho\right)\left(\frac{\left(\left(\boldsymbol{\sigma}_0\right)_4+\left(u_{2L}\right)_4\right)}{m_2L_1}+\frac{\left(\left(\boldsymbol{\sigma}_0\right)_5+\left(u_{2L}\right)_5\right)}{m_2L_1}\right)-\sin\left(k_2\rho+q_2\right)\frac{\left(\left(\boldsymbol{\sigma}_0\right)_6+\left(u_{2L}\right)_6\right)}{m_2L_1}\\&-r_1\left[\frac{K\left(1,1\right)}{m_2L_1L_2}\sqrt{\frac{L_2}{g}}\right]=\tau_N\end{aligned} \tag{6.3.20}$$

and

$$\begin{aligned}&\left(\frac{L_2}{L_1}a_1+a_2\right)\frac{\left(\left(\boldsymbol{\sigma}_0\right)_2+\left(u_{2L}\right)_2\right)}{m_2L_2^2}+Ya2\frac{\left(\left(\boldsymbol{\sigma}_0\right)_3+\left(u_{2L}\right)_3\right)}{m_2L_2^2}\\&-\sin\left(k_2\rho+q_2\right)\frac{\left(\left(\boldsymbol{\sigma}_0\right)_6+\left(u_{2L}\right)_6\right)}{m_2L_2}-\left[\frac{K\left(2,2\right)}{m_2L_2^2}\sqrt{\frac{L_2}{g}}\right]r_2=\tau_{N2},\end{aligned} \tag{6.3.21}$$

where

$$Ya1=\cos\left(q_2\right)\left(2\frac{L_2}{L_1}a_1+a_2\right)-\sin\left(q_2\right)\left(q_2'v_2+\frac{L_2}{L_1}\rho'v_2+q_2'\frac{L_2}{L_1}v_1\right), \tag{6.3.22}$$

$$Ya2=\cos\left(q_2\right)\frac{L_2}{L_1}a_1+\sin\left(q_2\right)\frac{L_2}{L_1}\rho'\frac{L_2}{L_1}v_1. \tag{6.3.23}$$

The variables a_i and v_i are functions of unitless time and $\boldsymbol{\lambda}$ has been multiplied by $\sqrt{L_2/g}$. Let K_{1n} and K_{2n} be the terms in brackets in (6.3.20) and (6.3.21). Examining (6.2.14), ε_{2Li} has the same units as $(\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{r})_i$, then to render $\boldsymbol{u}_{2L}$ dimensionless, Ψ_i needs to be made dimensionless. Let $\left(\boldsymbol{u}_{2L}\right)_i = -\Psi_i\left(\boldsymbol{u}_{2Ld}\right)_i$, where

$$\left(\boldsymbol{u}_{2Ld}\right)_i = \begin{Bmatrix} \left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i \Big/ \left|\left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i\right| & \text{if } \left|\left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i\right| > \varepsilon_{2Li} \\ \left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i \Big/ \varepsilon_{2Li} & \text{if } \left|\left(\boldsymbol{Y}^T\boldsymbol{r}\right)_i\right| \le \varepsilon_{2Li} \end{Bmatrix}, \tag{6.3.24}$$

and $\Psi_i = (\boldsymbol{\sigma}_0)_i - \boldsymbol{\sigma}_i$, then (6.3.20) and (6.3.21) simplify to

$$\begin{aligned} &a_1\frac{\left(\boldsymbol{\sigma}_0\right)_1 - \Psi_1\left(u_{2Ld}\right)_1}{m_2L_1^2} + \left(\frac{L_2}{L_1}a_1 + a_2\right)\frac{\left(\boldsymbol{\sigma}_0\right)_2 - \Psi_2\left(u_{2Ld}\right)_2}{m_2L_1L_2} + Ya1\frac{\left(\boldsymbol{\sigma}_0\right)_3 - \Psi_3\left(u_{2Ld}\right)_3}{m_2L_1L_2} \\ &-\sin\left(k_2\rho\right)\left(\frac{\left(\boldsymbol{\sigma}_0\right)_4 - \Psi_4\left(u_{2Ld}\right)_4}{m_2L_1} + \frac{\left(\boldsymbol{\sigma}_0\right)_5 - \Psi_5\left(u_{2Ld}\right)_5}{m_2L_1}\right) \\ &-\sin\left(k_2\rho + q_2\right)\frac{\left(\boldsymbol{\sigma}_0\right)_6 - \Psi_6\left(u_{2Ld}\right)_6}{m_2L_1} - r_1K_{1n} = \tau_N \end{aligned} \tag{6.3.25}$$

and

$$\begin{aligned} &\left(\frac{L_2}{L_1}a_1 + a_2\right)\frac{\left(\boldsymbol{\sigma}_0\right)_2 - \Psi_2\left(u_{2Ld}\right)_2}{m_2L_2^2} + Ya2\frac{\left(\boldsymbol{\sigma}_0\right)_3 - \Psi_2\left(u_{2Ld}\right)_3}{m_2L_2^2} \\ &-\sin\left(k_2\rho + q_2\right)\frac{\left(\boldsymbol{\sigma}_0\right)_6 - \Psi_2\left(u_{2Ld}\right)_6}{m_2L_2} - K_{2n}r_2 = \tau_{N2}. \end{aligned} \tag{6.3.26}$$

Substituting $\boldsymbol{\sigma}_0$ from (6.2.15) and $\boldsymbol{\sigma}$ from (6.2.5) into (6.3.25) and (6.3.26) simplifies τ_N to

$$\begin{aligned} &a_1\left(1 + k_9^2k_1 + k_8 + dm_2\left(1 - \left(u_{2Ld}\right)_1\right)\right) + Ya1d\left(1 + \left(dm_2 + dm_2dL_2 + dL_2\right)\left(1 - \left(u_{2Ld}\right)_3\right)\right) \\ &+\left(k_2a_1 + a_2\right)k_2\left(\left(1 + dL_2\right)^2\left(1 + dm_2\right)\left(1 - \left(u_{2Ld}\right)_2\right) + \left(u_{2Ld}\right)_2\right) \\ &-\sin\left(k_2\rho\right)\left(k_9k_1 + 1 + dm_2\left(1 - \left(u_{2Ld}\right)_5\right)\right) \\ &-\sin\left(k_2\rho + q_2\right)k_2\left(1 + \left(dm_2 + dL_2 + dm_2dL_2\right)\left(1 - \left(u_{2Ld}\right)_6\right)\right) - r_1K_{1n} = \tau_N \end{aligned} \tag{6.3.27}$$

and τ_{N2} to

$$
\begin{aligned}
&\left(k_2 a_1 + a_2\right)\left(\left(1+dL_2\right)^2\left(1+dm_2\right)\left(1-\left(u_{2Ld}\right)_2\right)+\left(u_{2Ld}\right)_2\right) \\
&+Ya2d\left(1+\left(dm_2+dm_2 dL_2+dL_2\right)\left(1-\left(u_{2Ld}\right)_3\right)\right) \\
&-\sin\left(k_2\rho+q_2\right)\left(1+\left(dm_2+dm_2 dL_2+dL_2\right)\left(1-\left(u_{2Ld}\right)_6\right)\right)-K_{2n}r_2=\tau_{N2},
\end{aligned}
\tag{6.3.28}
$$

where

$$
Ya1d=\cos\left(q_2\right)\left(2k_2 a_1+a_2\right)-\sin\left(q_2\right)\left(q_2' v_2+k_2\rho' v_2+q_2' k_2 v_1\right), \tag{6.3.29}
$$

$$
Ya2d=\cos\left(q_2\right)a_1+\sin\left(q_2\right)\rho' k_2 v_1, \tag{6.3.30}
$$

$dL_2 = \Delta L_2/L_2$, and $dm_2 = \Delta m_2/m_2$.

6.4 Morphing

To satisfy the sufficient conditions necessary for the equations of motion of the two-link manipulator to morph to a fully actuated inverted pendulum cart, the generalized coordinates must morph into those of the inverted pendulum cart. The second generalized coordinate for both systems measures the angular displacement in a counterclockwise direction of the second link (pendulum) with respect to the vertical. The first generalized coordinate for the dimensionless two-link manipulator is $\delta q_1 L_1/L_2$. To maintain the same circumferential displacement as L_1 grows large δq_1 becomes small, then $\delta q_1 L_1$ becomes a straight-line displacement and $\delta q_1 L_1/L_2$ morphs to the ρ of the fully actuated inverted pendulum cart. Similarly, the generalized velocities of the two-link manipulator morph to those of the fully actuated inverted pendulum cart. The mass matrix of the dimensionless two-link manipulator is

$$
\boldsymbol{M}\left(\boldsymbol{q}\right)=\begin{bmatrix} 2k_2\cos\left(q_2\right)+1+k_2^2+k_9^2 k_1+k_8 & \cos\left(q_2\right)+k_2 \\ \cos\left(q_2\right)+k_2 & 1 \end{bmatrix}, \tag{6.4.1}
$$

and as L_1 grows large, k_2 and k_8 goes to zero, k_9 goes to one, showing (6.4.1) matches the mass matrix of the inverted pendulum cart, (4.1.13), provided k_1 is the same ratio of the cart mass to the two-link manipulator mass. Lastly, the gradient of the potential energy of the dimensionless two-link manipulator is (6.3.15) which after letting k_2 go to zero would result in

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ -\sin(q_2) \end{bmatrix} \tag{6.4.2}$$

the same gradient of the potential energy function of the dimensionless fully actuated inverted pendulum cart. Therefore, the sufficient conditions for morphing the two-link manipulator dynamics to that of the fully actuated inverted pendulum cart are met.

After letting L_1 grow large, k_2 and k_8 go to zero, k_9 go to one, and ρ go to zero, the matrices for the equations of motion of (3.1.21) become

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} 1+k_1 & \cos(q_2) \\ \cos(q_2) & 1 \end{bmatrix}, \tag{6.4.3}$$

$$\boldsymbol{C}(\boldsymbol{q},\boldsymbol{q}') = \begin{bmatrix} 0 & -\sin(q_2)q_2' \\ 0 & 0 \end{bmatrix}, \tag{6.4.4}$$

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ -\sin(q_2) \end{bmatrix}, \tag{6.4.5}$$

and

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_N \\ \tau_{N2} \end{bmatrix}. \tag{6.4.6}$$

Equations (6.4.3)-(6.4.6) are the same as the equations of motion for an inverted pendulum cart that is fully actuated. The control laws of (6.3.27) and (6.3.28) simply to

$$\begin{aligned} &\left(\cos(q_2)a_2 - \sin(q_2)q_2'v_2\right)\left(1+\left(dm_2 + dm_2 dL_2 + dL_2\right)\left(1-\left(u_{2Ld}\right)_3\right)\right) \\ &+a_1\left(1+k_1+dm_2\left(1-\left(u_{2Ld}\right)_1\right)\right) - r_1 K_{1n} = \tau_N \end{aligned} \tag{6.4.7}$$

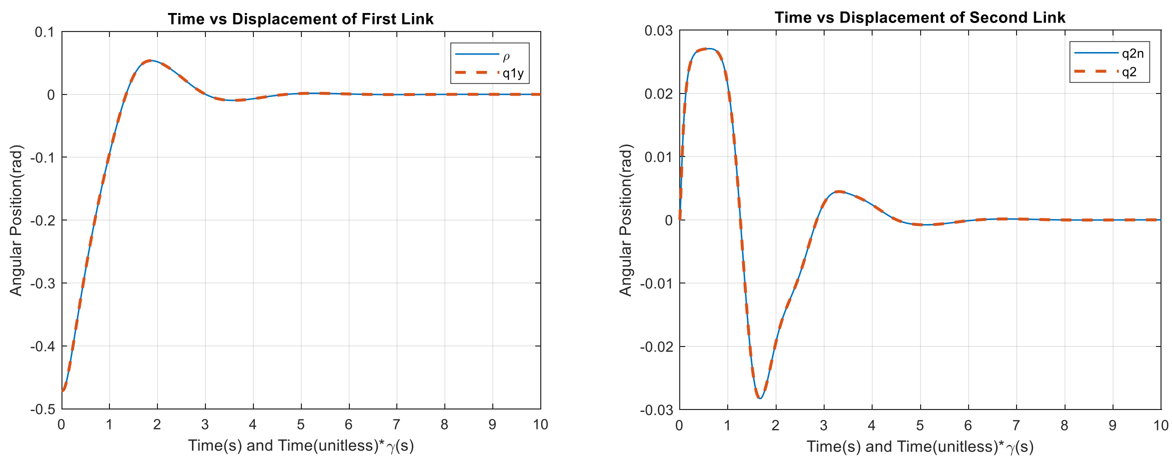
and

$$
\begin{aligned}
& a_2\left(\left(1+dL_2\right)^2\left(1+dm_2\right)\left(1-\left(u_{2Ld}\right)_2\right)+\left(u_{2Ld}\right)_2\right) \\
& +\cos\left(q_2\right)a_1\left(1+\left(dm_2+dm_2dL_2+dL_2\right)\left(1-\left(u_{2Ld}\right)_3\right)\right) \\
& -\sin\left(q_2\right)\left(1+\left(dm_2+dm_2dL_2+dL_2\right)\left(1-\left(u_{2Ld}\right)_6\right)\right)-K_{2n}r_2=\tau_{N2}.
\end{aligned}
\tag{6.4.8}
$$

To check that the morphed control law produces a stable response, simulations were performed.

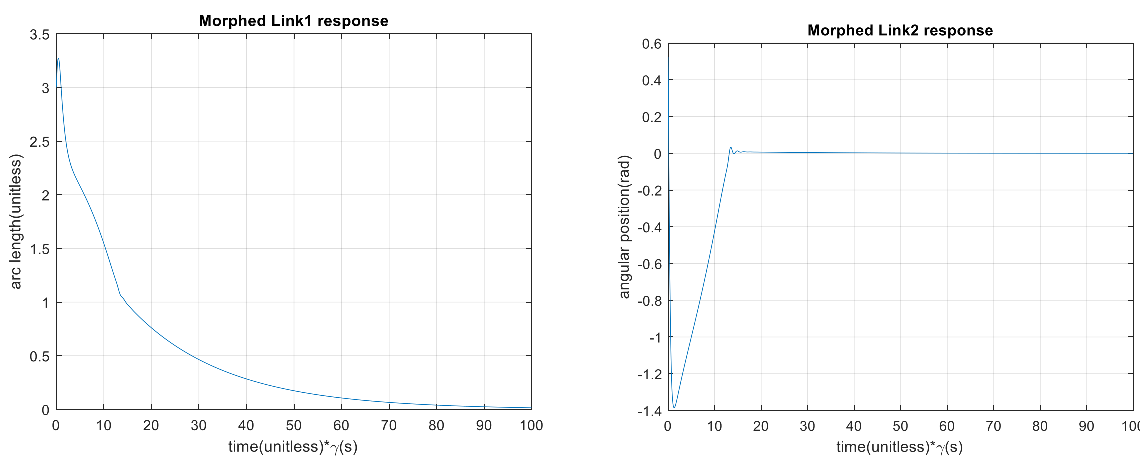
6.5 Simulations

After the equations of motion and controller had been rendered dimensionless and morphed, simulations were performed. The controller matrices $\boldsymbol{K}$ and $\boldsymbol{\lambda}$ were defined to be diagonal matrices with ones on the diagonals and zeros for off-diagonal terms and $\varepsilon_{2Li} = 1$. The unknown load parameters were defined as $\Delta L_2 = 0.125$m and $\Delta m_2 = 2$kg. For Figure 6.2, the initial conditions of $q_1 = (1.1\text{-}\pi/2)$, $\dot{q}_1 = 0$, $\theta = 0$, and $\dot{\theta} = 0$ were used for the two-link manipulator comprised of (6.1.17)-(6.1.20) with (6.2.6). The dimensionless equations of motion of (6.3.13)-(6.3.16) with the controllers of (6.3.27) and (6.3.28), had initial conditions of $\rho = (1.1\text{-}\pi/2)L_1/L_2$, $\dot{\rho} = 0$, $\theta = 0$, and $\dot{\theta} = 0$. For both systems, and the morphed system, the desired trajectory was $\boldsymbol{q}^d = [0;0]$, both links stabilized straight up, and $\dot{\boldsymbol{q}}^d = [0;0]$. To easily compare the plots in Figure 6.2, the x-axis for the dimensionless system was multiplied by γ to have units of seconds. Figure 6.2 demonstrates that converting to dimensionless form did not change the response of the system and was performed to validate that the dimensionless process was performed correctly. The simulation of the morphed system is shown in Figure 6.3. Figure 6.3 uses initial conditions of $\rho = (1.1\text{-}\pi/2)$, $\dot{\rho} = 0$, $\theta = 0$, and $\dot{\theta} = 0$, the same desired trajectory, the morphed control law of (6.4.7) and (6.4.8) applied to the system of (6.4.3)-(6.4.6). Similar to Figure 6.2, the x-axis was scaled by γ to have units of seconds. To improve the response for the morphed system, the controller matrices $\boldsymbol{K}$ and $\boldsymbol{\lambda}$ could be modified.



(a) (b)

Figure 6.2: Original vs Dimensionless Two-Link Manipulator Simulations



(a) (b)

Figure 6.3: Morphed Two-Link Manipulator Simulation

6.6 Conclusions

The two-link manipulator of Figure 6.1, is an example of a fully actuated mechanical system. The equations of motion for the two-link manipulator met the sufficient conditions for morphing to a fully actuated inverted pendulum cart. The control law presented in section 6.2, after being converted to dimensionless form morphed to a stable controller, as demonstrated by Figure 6.3 showing that this approach could work for fully actuated systems.

A real-world application that could benefit from the morphing of the two-link robot manipulator would be an overhead crane or a Segway. As the radius of the first link grows large, with some potential control gains modifications and coordinates changes, the resulting control law could be to be applied to stabilize the crane or Segway.

Chapter 7 - Conclusions

7.1 Summary

The sufficient conditions for when equations of motion will morph as size, lengths, or shapes grow large or shrink to zero was presented. The sufficient conditions are:

1) The successful morphing of the generalized coordinates and velocities of the original system to the target system.
2) The morphing of the original mass matrix as a function of the original generalized coordinates to the mass matrix of the target system where dependency is now on the target system's generalized coordinates.
3) The original potential energy expressed in terms of the original system's generalized coordinates morphs into the potential energy of the target system expressed in terms of the target system's generalized coordinates.

These sufficient conditions were applied to, and met by, three systems that were underactuated and one fully actuated system.

To aid in morphing, dimensionless parameters were utilized. To accomplish this, each equations of motion was divided by a term comprising the common units of that equation. For each system, the choice of dimensionless parameters for the original system was chosen with the end goal of matching the dimensionless target system. Then the process was applied to a chosen controller.

In chapter 4, a controller utilizing the method of Controlled Lagrangians was successfully morphed from the rotary pendulum to the inverted pendulum cart. Chapter 6, had a robust, sliding-mode controller successfully morphed from the two-link robot manipulator to a fully actuated inverted pendulum cart. The IDA-PBC controller presented in chapter 5, where $\boldsymbol{J}_2(\boldsymbol{q},\boldsymbol{p})$

was arbitrarily chosen, once morphed, did not produce a stable controller for the morphed equations of motion for the pendubot.

7.2 Future Work

The process of converting to a dimensionless system may seem ad hoc in the examples presented earlier in this work, to remedy this a general process for rendering a system dimensionless should be formalized.

The further investigation of morphing equations of motion and controllers is needed for systems that are underactuated to a greater extent than the systems presented here, or have more degrees of freedom than the systems used here.

Controller morphing needs more investigation and two major questions that result from the preceding analysis are: Under what circumstances will a control law morph and produce desirable results? What effect does the morphing process have on non-energy-based controller?

For energy-based controllers, one sufficient condition could be that the new (desired) mass matrix of the original system morphs to one that is positive definite in the target system. Another sufficient condition could be that the new potential energy function for the original system morphs to one that is positive definite in the neighborhood of the desired equilibrium. Are there other sufficient conditions for energy-based controller to morph?

To increase the applicability of the morphing process, the process for morphing from a simple to more complex system should be investigated for the equations of motion and control laws. As well as the scaling for when the length or radius is considered large enough for the system to morph.

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Appendix A - Ball and Beam

This Appendix is organized in three major parts. These are:

A.1 Derivations of Equations of Motion using Newton-Euler

A.2 Derivations of Equations of Motion using Lagrangian-Euler

A.3 Comparison of the Equations of Motion to those in the literature

A.1 Derivations of Equations of Motion using Newton-Euler

Ball and beam offset NE method.mw

> $restart:$

> $with(LinearAlgebra):$

Beam equations:

> $eq3 := J_{beam} \cdot dd\theta - \left(-N \cdot r + \tau_{BB} + F_f \cdot H + g \cdot m_B \cdot l_{beam} \cdot \sin(\theta) - J_B \cdot dd\theta_B + R_o \cdot F_f\right):$

Ball Equations:

> $eq4 := F_f - m \cdot g \cdot \sin(\theta) - m \cdot ddx:$

> $eq5 := -m \cdot g \cdot \cos(\theta) + N - m \cdot ddy:$

> $eq6 := R_o \cdot F_f - J_B \cdot dd\theta_B:$

Kinematics

> $X_c := r \cdot \cos(\theta) - (R_o + H) \cdot \sin(\theta):$

> $dX_c := dr \cdot \cos(\theta) - r \cdot \sin(\theta) \cdot d\theta - (R_o + H) \cdot \cos(\theta) \cdot d\theta:$

> $ddX_c := ddr \cdot \cos(\theta) - 2 \cdot dr \cdot \sin(\theta) \cdot d\theta - r \cdot \cos(\theta) \cdot d\theta^2 - r \cdot \sin(\theta) \cdot dd\theta + (R_o + H) \cdot \sin(\theta) \cdot d\theta^2 - (R_o + H) \cdot \cos(\theta) \cdot dd\theta:$

> $Y_c := r \cdot \sin(\theta) + (R_o + H) \cdot \cos(\theta):$

> $dY_c := dr \cdot \sin(\theta) + r \cdot \cos(\theta) \cdot d\theta - (R_o + H) \cdot \sin(\theta) \cdot d\theta:$

> $ddY_c := ddr \cdot \sin(\theta) + 2 \cdot dr \cdot \cos(\theta) \cdot d\theta - r \cdot \sin(\theta) \cdot d\theta^2 + r \cdot \cos(\theta) \cdot dd\theta - (R_o + H) \cdot \cos(\theta) \cdot d\theta^2 - (R_o + H) \cdot \sin(\theta) \cdot dd\theta:$

> $ddx := simplify(ddX_c \cdot \cos(\theta) + ddY_c \cdot \sin(\theta), 'trig')$

$$ddx := (-H - R_o)\, dd\theta - r\, d\theta^2 + ddr \qquad (1)$$

> $ddy := simplify(ddY_c \cdot \cos(\theta) - ddX_c \cdot \sin(\theta), 'trig')$

$$ddy := (-H - R_o)\, d\theta^2 + 2\, dr\, d\theta + r\, dd\theta \qquad (2)$$

> $dd\theta_B := dd\theta - \frac{ddr}{R_o}:$

Start substituting

can solve eq5 for N and eq4 for Ff

> $F_f := solve(eq4, F_f)$

$$F_f := -m\, r\, d\theta^2 - H m\, dd\theta + m\, g \sin(\theta) - m\, R_o\, dd\theta + ddr\, m \qquad (3)$$

> $N := solve(eq5, N)$

$$N := -H m\, d\theta^2 - m\, R_o\, d\theta^2 + m\, g \cos(\theta) + 2\, dr\, m\, d\theta + m\, r\, dd\theta \qquad (4)$$

simplify eq3 and eq6

> $expand(eq3)$

$$J_{beam}\, dd\theta + r\, m\, g \cos(\theta) + 2\, r\, dr\, m\, d\theta + m\, r^2\, dd\theta - \tau_{BB} + H^2\, m\, dd\theta - H m\, g \sin(\theta) + 2\, H m\, R_o\, dd\theta - H\, ddr\, m - g\, m_B\, l_{beam} \sin(\theta) + J_B\, dd\theta - \frac{J_B\, ddr}{R_o} - R_o\, m\, g \sin(\theta) + m\, R_o^2\, dd\theta - R_o\, ddr\, m \qquad (5)$$

> eq6

$$R_o\left(-m\,r\,d\theta^2 - H\,m\,dd\theta + m\,g\sin(\theta) - m\,R_o\,dd\theta + ddr\,m\right) - J_B\left(dd\theta - \frac{ddr}{R_o}\right) \tag{6}$$

> variables := [ddθ, ddr, dθ, dr] :

> Eq_{ball} := collect(eq6, variables)

$$Eq_{ball} := \left(R_o\left(-H\,m - R_o\,m\right) - J_B\right)dd\theta + \left(R_o\,m + \frac{J_B}{R_o}\right)ddr - r\,m\,R_o\,d\theta^2 + R_o\,m\,g\sin(\theta) \tag{7}$$

> Eq_{ball} := collect$\left(\text{expand}\left(\frac{Eq_{ball}}{R_o}\right), \text{variables}\right)$

$$Eq_{ball} := \left(-H\,m - R_o\,m - \frac{J_B}{R_o}\right)dd\theta + \left(m + \frac{J_B}{R_o^2}\right)ddr - m\,r\,d\theta^2 + m\,g\sin(\theta) \tag{8}$$

Substitute this into eq3 to get beam equation

> Eq_{beam} := collect(eq3, variables)

$$Eq_{beam} := \left(J_{beam} + m\,r^2 - \left(-H\,m - R_o\,m\right)H + J_B - R_o\left(-H\,m - R_o\,m\right)\right)dd\theta + \left(-H\,m - R_o\,m - \frac{J_B}{R_o}\right)ddr + \left(\left(-H\,m - R_o\,m\right)r + m\,r\,H + R_o\,m\,r\right)d\theta^2 + 2\,r\,dr\,m\,d\theta - H\,m\,g\sin(\theta) + r\,m\,g\cos(\theta) - R_o\,m\,g\sin(\theta) - g\,m_B\,l_{beam}\sin(\theta) - \tau_{BB} \tag{9}$$

Group terms for M, C and G matrices

> $M := \text{simplify}\left(\begin{bmatrix} \text{coeff}(Eq_{beam}, dd\theta) & \text{coeff}(Eq_{beam}, ddr) \\ \text{coeff}(Eq_{ball}, dd\theta) & \text{coeff}(Eq_{ball}, ddr) \end{bmatrix}, \text{'size'}\right)$

$$M := \begin{bmatrix} \left(H^2 + 2\,H\,R_o + r^2 + R_o^2\right)m + J_B + J_{beam} & -H\,m - R_o\,m - \frac{J_B}{R_o} \\ -H\,m - R_o\,m - \frac{J_B}{R_o} & m + \frac{J_B}{R_o^2} \end{bmatrix} \tag{10}$$

> $C := \begin{bmatrix} \frac{1}{2}\cdot\text{coeff}\left(Eq_{beam}, d\theta\right) & \frac{1}{2}\cdot\text{coeff}\left(Eq_{beam}, dr\right) \\ \text{coeff}\left(Eq_{ball}, d\theta^2\right)\cdot d\theta & \text{coeff}\left(Eq_{ball}, dr\right) \end{bmatrix}$

$$C := \begin{bmatrix} r\,dr\,m & r\,m\,d\theta \\ -r\,m\,d\theta & 0 \end{bmatrix} \tag{11}$$

> $G := \begin{bmatrix} \text{coeff}(Eq_{beam}, g)\cdot g \\ \text{coeff}(Eq_{ball}, g)\cdot g \end{bmatrix}$

$$G := \begin{bmatrix} \left(-H\,m\sin(\theta) + r\,m\cos(\theta) - R_o\,m\sin(\theta) - m_B\,l_{beam}\sin(\theta)\right)g \\ m\,g\sin(\theta) \end{bmatrix} \tag{12}$$

> $F := \begin{bmatrix} -\text{coeff}(Eq_{beam}, \tau_{BB})\cdot\tau_{BB} \\ 0 \end{bmatrix}$

$$F := \begin{bmatrix} \tau_{BB} \\ 0 \end{bmatrix} \tag{13}$$

A.2 Derivations of Equations of Motion using Lagrangian-Euler

Ball and beam offset lagrange.mw

> restart :

> with(LinearAlgebra) :

Lagrangian Derivation

Kinetic Energy

> $T := \frac{1}{2} \cdot J_{beam} \cdot d\theta^2 + \frac{1}{2} \cdot (J_b) \cdot (d\Theta)^2 + \frac{1}{2} \cdot m \cdot (dx^2 + dy^2)$

$$T := \frac{J_{beam}\, d\theta^2}{2} + \frac{J_b\, d\Theta^2}{2} + \frac{m\,(dx^2 + dy^2)}{2} \tag{1.1}$$

> $X_c := r \cdot \cos(\theta) - (R_o + H) \cdot \sin(\theta)$:

> $dX_c := dr \cdot \cos(\theta) - r \cdot \sin(\theta) \cdot d\theta - (R_o + H) \cdot \cos(\theta) \cdot d\theta$:

> $Y_c := r \cdot \sin(\theta) + (R_o + H) \cdot \cos(\theta)$:

> $dY_c := dr \cdot \sin(\theta) + \mathrm{r} \cdot \cos(\theta) \cdot \mathrm{d}\theta - (R_o + H) \cdot \sin(\theta) \cdot d\theta$:

> $dx := simplify(dX_c \cdot \cos(\theta) + dY_c \cdot \sin(\theta), 'trig')$

$$dx := (-H - R_o)\, d\theta + dr \tag{1.2}$$

> $dy := simplify(dY_c \cdot \cos(\theta) - dX_c \cdot \sin(\theta), 'trig')$

$$dy := r\, d\theta \tag{1.3}$$

> $d\Theta := d\theta - \frac{dr}{R_o}$:

> $expand(T)$

$$\frac{J_{beam}\, d\theta^2}{2} + \frac{J_b\, d\theta^2}{2} - \frac{J_b\, d\theta\, dr}{R_o} + \frac{J_b\, dr^2}{2 R_o^2} + \frac{H^2\, m\, d\theta^2}{2} + H\, m\, d\theta^2\, R_o + \frac{m\, r^2\, d\theta^2}{2} + \frac{m\, d\theta^2\, R_o^2}{2} - H\, dr\, m\, d\theta - dr\, m\, d\theta\, R_o + \frac{dr^2\, m}{2} \tag{1.4}$$

> $simplify(T, 'size')$

$$\frac{J_{beam}\, d\theta^2}{2} + \frac{J_b \left(d\theta - \frac{dr}{R_o}\right)^2}{2} + \frac{m\left(\left(H^2 + 2 H R_o + r^2 + R_o^2\right) d\theta^2 - 2\, dr\, (R_o + H)\, d\theta + dr^2\right)}{2} \tag{1.5}$$

Potential Energy

> $V := m \cdot g \cdot r \cdot \sin(\theta) + m \cdot g \cdot (R_o + H) \cdot \cos(\theta) + m_B \cdot g \cdot l_{beam} \cdot \cos(\theta)$:

Lagrange

> $L := expand(T - V)$

$$L := \frac{J_{beam}\, d\theta^2}{2} + \frac{J_b\, d\theta^2}{2} - \frac{J_b\, d\theta\, dr}{R_o} + \frac{J_b\, dr^2}{2 R_o^2} + \frac{H^2\, m\, d\theta^2}{2} + H\, m\, d\theta^2\, R_o + \frac{m\, r^2\, d\theta^2}{2} + \frac{m\, d\theta^2\, R_o^2}{2} - H\, dr\, m\, d\theta - dr\, m\, d\theta\, R_o + \frac{dr^2\, m}{2} - m\, g\, r \sin(\theta) - m\, g \cos(\theta)\, R_o - m\, g \cos(\theta)\, H - m_B\, g\, l_{beam} \cos(\theta) \tag{1.6}$$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $e1 := diff(L, dr)$:

> $e1 := subs(\{\theta = \theta(t), r = r(t)\}, e1)$:

> $e1 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e1)$:

> $e2 := diff(e1, t)$:

> $e2 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e2)$:

> $e2 := subs(\{\theta(t) = \theta, r(t) = r\}, e2)$:

> $variables := [dd\theta, ddr, d\theta, dr]$:

> $Eq_{ball} := collect(simplify(e2 - diff(L, r), 'trig'), variables)$

$$Eq_{ball} := \frac{\left(-m H R_o^2 - m R_o^3 - J_b R_o\right) dd\theta}{R_o^2} + \frac{\left(m R_o^2 + J_b\right) ddr}{R_o^2} - m\, r\, d\theta^2 + m\, g \sin(\theta) \tag{1.7}$$

> $e3 := diff(L, d\theta) :$

> $e3 := subs(\{\theta = \theta(t), r = r(t)\}, e3) :$

> $e3 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e3) :$

> $e4 := diff(e3, t) :$

> $e4 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e4) :$

> $e4 := subs(\{diff(\theta(t), t) = d\theta, diff(r(t), t) = dr\}, e4) :$

> $e4 := subs(\{\theta(t) = \theta, r(t) = r\}, e4) :$

> $Eq_{beam} := collect(simplify(e4 - diff(L, \theta) - \tau, 'trig'), variables)$

$$Eq_{beam} := \frac{\left(m R_o^3 + 2 H m R_o^2 + \left((H^2 + r^2) m + J_{beam} + J_b\right) R_o\right) dd\theta}{R_o} + \frac{\left(-H m R_o - m R_o^2 - J_b\right) ddr}{R_o} + 2 m r\, d\theta\, dr + \frac{-R_o g \left(H m + m R_o + l_{beam} m_B\right) \sin(\theta) + m g r \cos(\theta) R_o - \tau R_o}{R_o} \tag{1.8}$$

Group terms for M, C and G matrices

> $M := simplify\left(\begin{bmatrix} coeff(Eq_{beam}, dd\theta) & coeff(Eq_{beam}, ddr) \\ coeff(Eq_{ball}, dd\theta) & coeff(Eq_{ball}, ddr) \end{bmatrix}\right)$

$$M := \begin{bmatrix} (H^2 + 2 H R_o + r^2 + R_o^2) m + J_b + J_{beam} & \frac{-H m R_o - m R_o^2 - J_b}{R_o} \\ \frac{-H m R_o - m R_o^2 - J_b}{R_o} & \frac{m R_o^2 + J_b}{R_o^2} \end{bmatrix} \tag{1.9}$$

> $C := \begin{bmatrix} \frac{1}{2} \cdot coeff\left(Eq_{beam}, d\theta\right) & \frac{1}{2} \cdot coeff\left(Eq_{beam}, dr\right) + coeff\left(Eq_{beam}, d\theta^2\right) \cdot d\theta \\ coeff\left(Eq_{ball}, d\theta^2\right) \cdot d\theta & coeff\left(Eq_{ball}, dr^2\right) \cdot dr \end{bmatrix}$

$$C := \begin{bmatrix} m r\, dr & m r\, d\theta \\ -m r\, d\theta & 0 \end{bmatrix} \tag{1.10}$$

> $G := \begin{bmatrix} coeff(Eq_{beam}, g) \cdot g \\ coeff(Eq_{ball}, g) \cdot g \end{bmatrix}$

$$G := \begin{bmatrix} \frac{\left(-R_o \left(H m + m R_o + l_{beam} m_B\right) \sin(\theta) + m r \cos(\theta) R_o\right) g}{R_o} \\ m g \sin(\theta) \end{bmatrix} \tag{1.11}$$

> $F := \begin{bmatrix} -coeff(Eq_{beam}, \tau) \cdot \tau \\ 0 \end{bmatrix}$

$$F := \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{1.12}$$

A.3 Comparison of the Equations of Motion to those in the literature

Comparison to Models in Literature

To start let the offset be equal to the radius of the ball

> $H := -R_o :$

> X_c

$$r \cos(\theta) \tag{2.1}$$

> Y_c

$$r\sin(\theta) \tag{2.2}$$

> dX_c

$$dr\cos(\theta) - r\sin(\theta)\, d\theta \tag{2.3}$$

> dY_c

$$dr\sin(\theta) + r\cos(\theta)\, d\theta \tag{2.4}$$

> $d\Theta := d\theta:$

> T

$$\frac{J_{beam}\, d\theta^2}{2} + \frac{J_b\, d\theta^2}{2} + \frac{m\,(r^2\, d\theta^2 + dr^2)}{2} \tag{2.5}$$

> $simplify(dX_c^2 + dY_c^2)$

$$r^2\, d\theta^2 + dr^2 \tag{2.6}$$

> V

$$m\,g\,r\sin(\theta) + m_B\, g\, l_{beam}\cos(\theta) \tag{2.7}$$

> $L := expand(T - V)$

$$L := \frac{J_{beam}\, d\theta^2}{2} + \frac{J_b\, d\theta^2}{2} + \frac{m\, r^2\, d\theta^2}{2} + \frac{dr^2\, m}{2} - m\,g\,r\sin(\theta) - m_B\, g\, l_{beam}\cos(\theta) \tag{2.8}$$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $e1 := diff(L, dr):$

> $e1 := subs(\{\theta = \theta(t), r = r(t)\}, e1):$

> $e1 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e1):$

> $e2 := diff(e1, t):$

> $e2 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e2):$

> $e2 := subs(\{\theta(t) = \theta, r(t) = r\}, e2):$

> $variables := [dd\theta, ddr, d\theta, dr]:$

> $Eq_{ball} := collect(simplify(e2 - diff(L, r), 'trig'), variables)$

$$Eq_{ball} := ddr\, m - m\, r\, d\theta^2 + m\, g\sin(\theta) \tag{2.9}$$

> $e3 := diff(L, d\theta):$

> $e3 := subs(\{\theta = \theta(t), r = r(t)\}, e3):$

> $e3 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e3):$

> $e4 := diff(e3, t):$

> $e4 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e4):$

> $e4 := subs(\{diff(\theta(t), t) = d\theta, diff(r(t), t) = dr\}, e4):$

> $e4 := subs(\{\theta(t) = \theta, r(t) = r\}, e4):$

> $Eq_{beam} := collect(simplify(e4 - diff(L, \theta) - \tau, 'trig'), variables)$

$$Eq_{beam} := m\,g\,r\cos(\theta) - m_B\, g\, l_{beam}\sin(\theta) + (m\, r^2 + J_b + J_{beam})\, dd\theta + 2\, m\, r\, d\theta\, dr - \tau \tag{2.10}$$

> $M := simplify\left(\begin{bmatrix} coeff(Eq_{beam}, dd\theta) & coeff(Eq_{beam}, ddr) \\ coeff(Eq_{ball}, dd\theta) & coeff(Eq_{ball}, ddr) \end{bmatrix}\right)$

$$M := \begin{bmatrix} m\, r^2 + J_b + J_{beam} & 0 \\ 0 & m \end{bmatrix} \tag{2.11}$$

repeat above but roll without slip

> $d\Theta := d\theta - \frac{dr}{R_o}:$

> $expand(T)$

$$\frac{J_{beam}\, d\theta^2}{2} + \frac{J_b\, d\theta^2}{2} - \frac{J_b\, d\theta\, dr}{R_o} + \frac{J_b\, dr^2}{2\, R_o^2} + \frac{m\, r^2\, d\theta^2}{2} + \frac{dr^2\, m}{2} \tag{2.12}$$

> $simplify(dX_c^2 + dY_c^2)$

$$r^2\, d\theta^2 + dr^2 \tag{2.13}$$

> V

$$m\,g\,r\sin(\theta) + m_B\,g\,l_{beam}\cos(\theta) \tag{2.14}$$

> $L := expand(T - V)$

$$L := \frac{J_{beam}\,d\theta^2}{2} + \frac{J_b\,d\theta^2}{2} - \frac{J_b\,d\theta\,dr}{R_o} + \frac{J_b\,dr^2}{2\,R_o^2} + \frac{m\,r^2\,d\theta^2}{2} + \frac{dr^2\,m}{2} - m\,g\,r\sin(\theta) - m_B\,g\,l_{beam}\cos(\theta) \tag{2.15}$$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $e1 := diff(L, dr)$:

> $e1 := subs(\{\theta = \theta(t), r = r(t)\}, e1)$:

> $e1 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e1)$:

> $e2 := diff(e1, t)$:

> $e2 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e2)$:

> $e2 := subs(\{\theta(t) = \theta, r(t) = r\}, e2)$:

> $variables := [dd\theta, ddr, d\theta, dr]$:

> $Eq_{ball} := collect(simplify(e2 - diff(L, r), 'trig'), variables)$

$$Eq_{ball} := -\frac{J_b\,dd\theta}{R_o} + \frac{(m\,R_o^2 + J_b)\,ddr}{R_o^2} - m\,r\,d\theta^2 + m\,g\sin(\theta) \tag{2.16}$$

> $e3 := diff(L, d\theta)$:

> $e3 := subs(\{\theta = \theta(t), r = r(t)\}, e3)$:

> $e3 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e3)$:

> $e4 := diff(e3, t)$:

> $e4 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e4)$:

> $e4 := subs(\{diff(\theta(t), t) = d\theta, diff(r(t), t) = dr\}, e4)$:

> $e4 := subs(\{\theta(t) = \theta, r(t) = r\}, e4)$:

> $Eq_{beam} := collect(simplify(e4 - diff(L, \theta) - \tau, 'trig'), variables)$

$$Eq_{beam} := (m\,r^2 + J_b + J_{beam})\,dd\theta - \frac{J_b\,ddr}{R_o} + 2\,m\,r\,d\theta\,dr + \frac{-m_B\,g\,l_{beam}\sin(\theta)\,R_o + m\,g\,r\cos(\theta)\,R_o - \tau\,R_o}{R_o} \tag{2.17}$$

> $M := simplify\left(\begin{bmatrix} coeff(Eq_{beam}, dd\theta) & coeff(Eq_{beam}, ddr) \\ coeff(Eq_{ball}, dd\theta) & coeff(Eq_{ball}, ddr) \end{bmatrix}\right)$

$$M := \begin{bmatrix} m\,r^2 + J_b + J_{beam} & -\frac{J_b}{R_o} \\ -\frac{J_b}{R_o} & \frac{m\,R_o^2 + J_b}{R_o^2} \end{bmatrix} \tag{2.18}$$

repeat above but with M(1,2)=0

> $H := -\frac{(m \cdot R_o \cdot R_o + J_b)}{m \cdot R_o}$:

> $T := \frac{1}{2} \cdot J_{beam} \cdot d\theta^2 + \frac{1}{2} \cdot (J_b) \cdot (d\Theta)^2 + \frac{1}{2} \cdot m \cdot (dx^2 + dy^2)$

$$T := \frac{J_{beam}\,d\theta^2}{2} + \frac{J_b\left(d\theta - \frac{dr}{R_o}\right)^2}{2} + \frac{m\left(\frac{(dr\,m\,R_o + d\theta\,J_b)^2}{m^2\,R_o^2} + r^2\,d\theta^2\right)}{2} \tag{2.19}$$

> $X_c := r \cdot \cos(\theta) - (R_o + H) \cdot \sin(\theta)$:

> $dX_c := dr \cdot \cos(\theta) - r \cdot \sin(\theta) \cdot d\theta - (R_o + H) \cdot \cos(\theta) \cdot d\theta$:

> $Y_c := r \cdot \sin(\theta) + (R_o + H) \cdot \cos(\theta)$:

> $dY_c := dr \cdot \sin(\theta) + r \cdot \cos(\theta) \cdot d\theta - (R_o + H) \cdot \sin(\theta) \cdot d\theta$:

> $dx := simplify(dX_c \cdot \cos(\theta) + dY_c \cdot \sin(\theta), 'trig')$

$$dx := \frac{dr\,m\,R_o + d\theta\,J_b}{m\,R_o} \tag{2.20}$$

> $dy := simplify(dY_c \cdot \cos(\theta) - dX_c \cdot \sin(\theta), 'trig')$

$$dy := r\,d\theta \tag{2.21}$$

> $d\Theta := d\theta - \frac{dr}{R_o}$:

> $expand(T)$:

> $simplify(T, 'size')$

$$\frac{R_o^2\,(r^2\,d\theta^2 + dr^2)\,m^2 + \left(d\theta^2\,(J_b + J_{beam})\,R_o^2 + dr^2\,J_b\right)m + d\theta^2\,J_b^2}{2\,m\,R_o^2} \tag{2.22}$$

Potential Energy

> $V := m \cdot g \cdot r \cdot \sin(\theta) + m \cdot g \cdot (R_o + H) \cdot \cos(\theta) + m_B \cdot g \cdot l_{beam} \cdot \cos(\theta)$:

Lagrange

> $L := expand(T - V)$

$$L := \frac{J_{beam}\,d\theta^2}{2} + \frac{J_b\,d\theta^2}{2} + \frac{J_b\,dr^2}{2\,R_o^2} + \frac{dr^2\,m}{2} + \frac{d\theta^2\,J_b^2}{2\,m\,R_o^2} + \frac{m\,r^2\,d\theta^2}{2} - m\,g\,r\sin(\theta) + \frac{g\cos(\theta)\,J_b}{R_o} - m_B\,g\,l_{beam}\cos(\theta) \tag{2.23}$$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $e1 := diff(L, dr)$:

> $e1 := subs(\{\theta = \theta(t), r = r(t)\}, e1)$:

> $e1 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e1)$:

> $e2 := diff(e1, t)$:

> $e2 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e2)$:

> $e2 := subs(\{\theta(t) = \theta, r(t) = r\}, e2)$:

> $variables := [dd\theta, ddr, d\theta, dr]$:

> $Eq_{ball} := collect(simplify(e2 - diff(L, r), 'trig'), variables)$

$$Eq_{ball} := \frac{(R_o^2\,m + J_b)\,ddr}{R_o^2} - m\,r\,d\theta^2 + m\,g\sin(\theta) \tag{2.24}$$

> $e3 := diff(L, d\theta)$:

> $e3 := subs(\{\theta = \theta(t), r = r(t)\}, e3)$:

> $e3 := subs(\{d\theta = diff(\theta(t), t), dr = diff(r(t), t)\}, e3)$:

> $e4 := diff(e3, t)$:

> $e4 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(r(t), t, t) = ddr\}, e4)$:

> $e4 := subs(\{diff(\theta(t), t) = d\theta, diff(r(t), t) = dr\}, e4)$:

> $e4 := subs(\{\theta(t) = \theta, r(t) = r\}, e4)$:

> $Eq_{beam} := collect(simplify(e4 - diff(L, \theta) - \tau, 'trig'), variables)$

$$Eq_{beam} := \frac{\left(2\left(\frac{m\,r^2}{2} + \frac{J_{beam}}{2} + \frac{J_b}{2}\right)m\,R_o^2 + J_b^2\right)dd\theta}{R_o^2\,m} + 2\,m\,r\,d\theta\,dr + \frac{-g\,m\,R_o\,(R_o\,l_{beam}\,m_B - J_b)\sin(\theta) + m^2\,g\,r\cos(\theta)\,R_o^2 - \tau R_o^2\,m}{R_o^2\,m} \tag{2.25}$$

> $M := \begin{bmatrix} coeff(Eq_{beam}, dd\theta) & coeff(Eq_{beam}, ddr) \\ coeff(Eq_{ball}, dd\theta) & coeff(Eq_{ball}, ddr) \end{bmatrix}$

$$M := \begin{bmatrix} \frac{2\left(\frac{m\,r^2}{2} + \frac{J_{beam}}{2} + \frac{J_b}{2}\right)m\,R_o^2 + J_b^2}{R_o^2\,m} & 0 \\ 0 & \frac{R_o^2\,m + J_b}{R_o^2} \end{bmatrix} \tag{2.26}$$

Appendix B - Ball and Arc

This Appendix is organized in three major parts. These are:

B.1 Derivations of Equations of Motion using Newton-Euler

B.2 Derivations of Equations of Motion using Lagrangian-Euler

B.3 Derivation of the Dimensionless Equations of Motion

B.1 Derivations of Equations of Motion using Newton-Euler

ball and arc NE method.mw

> $restart:$

> $with(LinearAlgebra):$

Ball Equations:

> $eq2 := eval(expand(Ff \cdot \cos(\phi-\theta) + N \cdot \sin(\phi-\theta) - m \cdot ddx)):$

> $eq3 := eval(expand(-m \cdot g - Ff \cdot \sin(\phi-\theta) + N \cdot \cos(\phi-\theta) - m \cdot ddy)):$

> $eq4 := R_o \cdot Ff - J_b \cdot ddA:$

Arc equations:

> $eq1 := eval(expand(J_{arc} \cdot dd\theta - (-(R-H) \cdot N \cdot \sin(\phi) + \tau + g \cdot m_a \cdot l_{arc} \cdot \sin(\theta) + Ff \cdot R - Ff \cdot (R-H) \cdot \cos(\phi)))):$

Kinematics

> $x := (R+R_o) \cdot \sin(\phi-\theta) + (R-H) \cdot \sin(\theta):$

> $dx := diff(x, \phi) \cdot d\phi + diff(x, \theta) \cdot d\theta:$

> $ddx := simplify(simplify(diff(dx, \phi) \cdot d\phi + diff(dx, \theta) \cdot d\theta + diff(dx, d\phi) \cdot dd\phi + diff(dx, d\theta) \cdot dd\theta, 'trig'), 'size')$

$$ddx := -(d\phi - d\theta)^2 (R+R_o) \sin(\phi-\theta) + (dd\phi - dd\theta)(R+R_o)\cos(\phi-\theta) - \left(-d\theta^2 \sin(\theta) + dd\theta \cos(\theta)\right)(H-R) \quad (1)$$

> $y := (R+R_o) \cdot \cos(\phi-\theta) - (R-H) \cdot \cos(\theta):$

> $dy := diff(y, \phi) \cdot d\phi + diff(y, \theta) \cdot d\theta:$

> $ddy := simplify(simplify(diff(dy, \phi) \cdot d\phi + diff(dy, \theta) \cdot d\theta + diff(dy, d\phi) \cdot dd\phi + diff(dy, d\theta) \cdot dd\theta, 'trig'), 'size')$

$$ddy := -(d\phi - d\theta)^2 (R+R_o)\cos(\phi-\theta) - (dd\phi - dd\theta)(R+R_o)\sin(\phi-\theta) - \left(d\theta^2 \cos(\theta) + dd\theta \sin(\theta)\right)(H-R) \quad (2)$$

> $rdo := rso + rds:$

> $rso := (R-H) \cdot \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix}:$

> $rds := \begin{bmatrix} R \cdot \sin(\phi-\theta) \\ R \cdot \cos(\phi-\theta) \\ 0 \end{bmatrix}:$

> $normrds := R:$

> $er := \left(\frac{rds}{normrds}\right):$

> $dv_d := CrossProduct\left(-d\theta \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, rdo\right):$

> $dv_c := dx \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + dy \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}:$

> $et := CrossProduct\left(\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, er\right)$

$$et := \begin{bmatrix} \cos(\phi - \theta) \\ -\sin(\phi - \theta) \\ 0 \end{bmatrix} \qquad (3)$$

> $\#dA := -simplify\left(DotProduct\left(\frac{(dv_c - dv_d)}{R_o}, et\right)\right):$

> $dA := -simplify\left(\frac{(dv_c - dv_d)_1}{R_o} \cdot et_1 + \frac{(dv_c - dv_d)_2}{R_o} \cdot et_2, 'trig'\right)$

$$dA := -\frac{d\phi R + d\phi R_o - d\theta R_o}{R_o} \qquad (4)$$

> $ddA := simplify(simplify(diff(dA, \phi) \cdot d\phi + diff(dA, \theta) \cdot d\theta + diff(dA, d\phi) \cdot dd\phi + diff(dA, d\theta) \cdot dd\theta, 'trig'), 'size')$

$$ddA := -\frac{(R + R_o)\, dd\phi}{R_o} + dd\theta \qquad (5)$$

Start substituting

solve eq3 for N and eq2 for Ff

> $Ff := simplify(solve(eq2, Ff), 'size'):$

> $N := simplify(solve(eq3, N), 'size'):$

simplify eq1 and eq4

> $Eq_{ball} := simplify(simplify(expand(eval(eq4)), 'trig'), 'size'):$

> $variables := [dd\theta, dd\phi, d\theta, d\phi]:$

> $Eq_{ball} := collect(simplify(Eq_{ball}, 'size'), variables)$

$$Eq_{ball} := \frac{\left(-m R_o^3 - m R R_o^2 - J_b R_o - R_o^2 (H - R)\, m \cos(\phi)\right) dd\theta}{R_o} + \frac{\left(m R R_o^2 + m R_o^3 + J_b R + J_b R_o\right) dd\phi}{R_o} + R_o (H - R)\, m \sin(\phi)\, d\theta^2$$
$$+ \frac{R_o^2 m g \cos(\phi) \sin(\theta) - \sin(\phi) R_o^2 \cos(\theta)\, m g}{R_o} \qquad (6)$$

> $Eq_{arc} := simplify(simplify(expand(eq1 - eq4), 'trig'), 'size'):$

> $Eq_{arc} := collect(Eq_{arc}, variables)$

$$Eq_{arc} := \frac{\left(2 m R_o (R + R_o)(H - R) \cos(\phi) + m R_o^3 + 2 m R R_o^2 + \left((2 R^2 - 2 R H + H^2)\, m + J_{arc} + J_b\right) R_o\right) dd\theta}{R_o} \qquad (7)$$
$$+ \frac{\left(2 m R_o (R + R_o)\left(-\frac{1}{2} H + \frac{1}{2} R\right) \cos(\phi) - m R_o^3 - 2 m R R_o^2 + \left(-R^2 m - J_b\right) R_o - J_b R\right) dd\phi}{R_o} - 2\, d\phi\, (H - R)\, m\, (R$$
$$+ R_o) \sin(\phi)\, d\theta - 2 m (R + R_o)\left(-\frac{1}{2} H + \frac{1}{2} R\right) \sin(\phi)\, d\phi^2$$
$$+ \frac{-m R_o (R + R_o)\, g \sin(\theta) \cos(\phi) + g \cos(\theta)\, m R_o (R + R_o) \sin(\phi) - R_o g \left((H - R)\, m + l_{arc} m_a\right) \sin(\theta) - \tau R_o}{R_o}$$

> $Eq_{ball} := \frac{Eq_{ball} \cdot (R + R_o)}{R_o}$

$$Eq_{ball} := \frac{1}{R_o}\left(\left(\frac{\left(-m R_o^3 - m R R_o^2 - J_b R_o - R_o^2 (H - R)\, m \cos(\phi)\right) dd\theta}{R_o} + \frac{\left(m R R_o^2 + m R_o^3 + J_b R + J_b R_o\right) dd\phi}{R_o} + R_o (H - R)\, m \sin(\phi)\, d\theta^2\right.\right. \qquad (8)$$
$$\left.\left.+ \frac{R_o^2 m g \cos(\phi) \sin(\theta) - \sin(\phi) R_o^2 \cos(\theta)\, m g}{R_o}\right) (R + R_o)\right)$$

Group terms for M, C and G matrices

> $G := simplify\left(simplify\left(\begin{bmatrix} coeff(Eq_{arc}, g) \cdot g \\ coeff(Eq_{ball}, g) \cdot g \end{bmatrix}, 'trig'\right), 'size'\right)$

$$G := \begin{bmatrix} -\left(\left(m (R + R_o) \cos(\phi) + (H - R)\, m + l_{arc} m_a\right) \sin(\theta) - \cos(\theta) \sin(\phi)\, m (R + R_o)\right) g \\ (R + R_o)\, g m \left(-\sin(\phi) \cos(\theta) + \cos(\phi) \sin(\theta)\right) \end{bmatrix} \qquad (9)$$

> $M1 := simplify(simplify(coeff(Eq_{arc}, dd\theta),'trig'),'size')$

$$M1 := 2\,m\,(R+R_o)\,(H-R)\cos(\phi) + \left(2R^2 + (2R_o - 2H)\,R + R_o^2 + H^2\right)m + J_{arc} + J_b \qquad (10)$$

> $M2 := simplify(simplify(coeff(Eq_{arc}, dd\phi),'trig'),'size')$

$$M2 := -\frac{(R+R_o)\left(m\,R_o\,(H-R)\cos(\phi) + J_b + m\,R\,R_o + R_o^2\,m\right)}{R_o} \qquad (11)$$

> $M3 := simplify(simplify(coeff(Eq_{ball}, dd\theta),'trig'),'size')$

$$M3 := -\frac{(R+R_o)\left(m\,R_o\,(H-R)\cos(\phi) + J_b + m\,R\,R_o + R_o^2\,m\right)}{R_o} \qquad (12)$$

> $M4 := simplify(simplify(coeff(Eq_{ball}, dd\phi),'trig'),'size')$

$$M4 := \frac{(R_o^2\,m + J_b)\,(R+R_o)^2}{R_o^2} \qquad (13)$$

> $M := \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix}:$

> $C1 := simplify\left(simplify\left(\frac{1}{2}\cdot coeff(Eq_{arc}, d\theta) + coeff(Eq_{arc}, d\theta^2)\cdot d\theta,'trig'\right),'size'\right)$

$$C1 := -d\phi\,(H-R)\,m\,(R+R_o)\sin(\phi) \qquad (14)$$

> $C2 := simplify\left(simplify\left(\frac{1}{2}\cdot coeff(Eq_{arc}, d\phi) + coeff(Eq_{arc}, d\phi^2)\cdot d\phi,'trig'\right),'size'\right)$

$$C2 := m\,(R+R_o)\sin(\phi)\,(d\phi - d\theta)\,(H-R) \qquad (15)$$

> $C3 := simplify(simplify(coeff(Eq_{ball}, d\theta^2)\cdot d\theta,'trig'),'size')$

$$C3 := (H-R)\,m\,(R+R_o)\sin(\phi)\,d\theta \qquad (16)$$

> $C4 := simplify(simplify(coeff(Eq_{ball}, d\phi^2)\cdot d\phi,'trig'),'size')$

$$C4 := 0 \qquad (17)$$

> $C := \begin{bmatrix} C1 & C2 \\ C3 & C4 \end{bmatrix}:$

B.2 Derivations of Equations of Motion using Lagrangian-Euler

ball and arc lagrange.mw

> $restart:$

> $with(LinearAlgebra):$

Kinetic Energy

> $T := expand\left(\frac{1}{2}\cdot m\cdot(dx_c^2 + dy_c^2) + \frac{1}{2}\cdot J_{arc}\cdot d\theta^2 + simplify\left(\frac{1}{2}\,J_b\cdot(dA\cdot dA)\right)\right):$

> $x_c := (R+R_o)\cdot\sin(\phi-\theta) + (R-H)\cdot\sin(\theta):$

> $dx_c := (R+R_o)\cdot\cos(\phi-\theta)\cdot(d\phi - d\theta) + (R-H)\cdot\cos(\theta)\cdot d\theta:$

> $y_c := (R+R_o)\cdot\cos(\phi-\theta) - (R\text{-}H)\cdot\cos(\theta):$

> $dy_c := -(R+R_o)\cdot\sin(\phi-\theta)\cdot(d\phi - d\theta) + (R\text{-}H)\cdot\sin(\theta)\cdot d\theta:$

> $rdo := rso + rds:$

> $rso := (R\text{-}H)\cdot\begin{bmatrix}\sin(\theta)\\ -\cos(\theta)\\ 0\end{bmatrix}:$

$$> rds := \begin{bmatrix} R\cdot\sin(\phi-\theta) \\ R\cdot\cos(\phi-\theta) \\ 0 \end{bmatrix}:$$

$$> normrds := \mathrm{abs}\left(simplify\left(\mathrm{sqrt}\left(rds_1^2+rds_2^2+rds_3^2\right)\right)\right):$$

$$> normrds := \frac{simplify\left(rds_1^2+rds_2^2+rds_3^2\right)}{R}:$$

$$> er := \left(\frac{rds}{normrds}\right):$$

$$> v_d := CrossProduct\left(-d\theta\cdot\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, rdo\right):$$

$$> v_c := dx_c\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + dy_c\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}:$$

$$> et := CrossProduct\left(-\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, er\right):$$

$$> dA := -simplify\left(\frac{(v_c-v_d)_1}{R_o}\cdot et_1 + \frac{(v_c-v_d)_2}{R_o}\cdot et_2, 'size'\right)$$

$$dA := -\frac{\left((d\phi-d\theta)\,R_o + R\,d\phi\right)\left(\sin(\phi-\theta)^2+\cos(\phi-\theta)^2\right)}{R_o} \quad \textbf{(1)}$$

$$> simplify(simplify(expand(eval(T)),'trig'),'size')$$

$$\frac{1}{2R_o^2}\Big(-2\,m\,R_o^2\,d\theta\,(d\phi-d\theta)\,(R+R_o)\,(H-R)\cos(\phi) + m\,(d\phi-d\theta)^2 R_o^4 + 2\,R\,m\,(d\phi-d\theta)^2 R_o^3 + \Big(\big((H^2-2\,H\,R+2\,R^2)\,m + J_b + J_{arc}\big)\,d\theta^2 - 2\,d\phi\,(R^2\,m+J_b)\,d\theta + d\phi^2\,(R^2\,m+J_b)\Big)R_o^2 + 2\,R\,d\phi\,J_b\,(d\phi-d\theta)\,R_o + J_b\,R^2\,d\phi^2\Big) \quad \textbf{(2)}$$

Potential Energy

$$> V := m\cdot g\cdot y_c + m_a\cdot g\cdot l_{arc}\cdot\cos(\theta):$$

Lagrange

$$> L := expand(T-V):$$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

$$> dLdphi := diff(L, d\phi):$$

$$> dLdphi := subs(\{\theta=\theta(t), \phi=\phi(t)\}, dLdphi):$$

$$> e1 := \mathrm{subs}\left(\left\{d\theta = diff(\theta(t), t), d\phi = diff(\phi(t), t),\right\}, dLdphi\right):$$

$$> dLdphidt := diff(e1, t):$$

$$> e2 := subs(\{diff(\phi(t), t, t) = dd\phi, diff(\phi(t), t) = d\phi, diff(\theta(t), t, t) = dd\theta, diff(\theta(t), t) = d\theta\}, dLdphidt):$$

$$> e2 := subs(\{\phi(t)=\phi, \theta(t)=\theta\}, e2):$$

$$> variables := [dd\theta, dd\phi, d\theta, d\phi]:$$

$$> Eq_{ball} := collect(simplify(simplify(e2 - diff(L, \phi),'trig'),'size'), variables)$$

$$Eq_{ball} := \frac{\left(-m\,R_o^2\,(H-R)\cos(\phi) - m\,R_o^3 - R\,m\,R_o^2 - J_b\,R_o\right)(R+R_o)\,dd\theta}{R_o^2} + \frac{\left(R\,m\,R_o^2 + m\,R_o^3 + R\,J_b + J_b\,R_o\right)(R+R_o)\,dd\phi}{R_o^2} + m\,(H-R)\sin(\phi)\,(R+R_o)\,d\theta^2 + \frac{\left(-\sin(\phi)\cos(\theta)\,g\,m\,R_o^2 + \sin(\theta)\cos(\phi)\,g\,m\,R_o^2\right)(R+R_o)}{R_o^2} \quad \textbf{(3)}$$

$$> dLdtheta := diff(L, d\theta):$$

$$> dLdtheta := subs(\{\theta=\theta(t), \phi=\phi(t)\}, dLdtheta):$$

$$> e3 := \mathrm{subs}\left(\left\{d\theta = diff(\theta(t), t), d\phi = diff(\phi(t), t),\right\}, dLdtheta\right):$$

$$> dLdthetadt := diff(e3, t):$$

$$> e4 := subs(\{diff(\phi(t), t, t) = dd\phi, diff(\phi(t), t) = d\phi, diff(\theta(t), t, t) = dd\theta, diff(\theta(t), t) = d\theta\}, dLdthetadt):$$

$$> e4 := subs(\{\phi(t)=\phi, \theta(t)=\theta\}, e4):$$

> $Eq_{arc} := collect(simplify(simplify(e4 - diff(L, \theta), 'trig'), 'size'), variables)$

$$Eq_{arc} := \frac{\left(2\, m\, (R + R_o)\, (H - R)\, R_o \cos(\phi) + m\, R_o^3 + 2\, R\, m\, R_o^2 + \left(\left(H^2 - 2\, H R + 2\, R^2\right) m + J_b + J_{arc}\right) R_o\right) dd\theta}{R_o} + \frac{\left(2\, m\, (R + R_o) \left(-\frac{H}{2} + \frac{R}{2}\right) R_o \cos(\phi) - m\, R_o^3 - 2\, R\, m\, R_o^2 + \left(-R^2 m - J_b\right) R_o - J_b R\right) dd\phi}{R_o} - 2\, m\, d\phi\, (H - R)\, (R + R_o) \sin(\phi)\, d\theta - 2\, m\, (R + R_o) \left(-\frac{H}{2} + \frac{R}{2}\right) \sin(\phi)\, d\phi^2 + \frac{-m\, (R + R_o) \sin(\theta)\, g\, R_o \cos(\phi) + m \cos(\theta)\, g\, (R + R_o)\, R_o \sin(\phi) - \left((H - R)\, m + m_a l_{arc}\right) g\, R_o \sin(\theta)}{R_o} \quad \textbf{(4)}$$

Group terms for M, C and G matrices

> $G := simplify\left(simplify\left(\begin{bmatrix} coeff(Eq_{arc}, g) \cdot g \\ coeff(Eq_{ball}, g) \cdot g \end{bmatrix}, 'trig'\right), 'size'\right)$

$$G := \begin{bmatrix} -\left(\left(m\, (R + R_o) \cos(\phi) + (H - R)\, m + m_a l_{arc}\right) \sin(\theta) - \sin(\phi) \cos(\theta)\, m\, (R + R_o)\right) g \\ (R + R_o)\, g\, m\, (\sin(\theta) \cos(\phi) - \sin(\phi) \cos(\theta)) \end{bmatrix} \quad \textbf{(5)}$$

> $M1 := simplify(simplify(coeff(Eq_{arc}, dd\theta), 'trig'), 'size')$

$$M1 := 2\, m\, (R + R_o)\, (H - R) \cos(\phi) + \left(2\, R^2 + \left(-2\, H + 2\, R_o\right) R + H^2 + R_o^2\right) m + J_b + J_{arc} \quad \textbf{(6)}$$

> $M2 := simplify(simplify(coeff(Eq_{arc}, dd\phi), 'trig'), 'size')$

$$M2 := -\frac{(R + R_o)\left(m\, R_o\, (H - R) \cos(\phi) + R\, m\, R_o + m\, R_o^2 + J_b\right)}{R_o} \quad \textbf{(7)}$$

> $M3 := simplify(simplify(coeff(Eq_{ball}, dd\theta), 'trig'), 'size')$

$$M3 := -\frac{(R + R_o)\left(m\, R_o\, (H - R) \cos(\phi) + R\, m\, R_o + m\, R_o^2 + J_b\right)}{R_o} \quad \textbf{(8)}$$

> $M4 := simplify(simplify(coeff(Eq_{ball}, dd\phi), 'trig'), 'size')$

$$M4 := \frac{\left(m\, R_o^2 + J_b\right)(R + R_o)^2}{R_o^2} \quad \textbf{(9)}$$

> $M := \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix}:$

> $C1 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff(Eq_{arc}, d\theta) + coeff(Eq_{arc}, d\theta^2) \cdot d\theta, 'trig'\right), 'size'\right)$

$$C1 := -m\, d\phi\, (H - R)\, (R + R_o) \sin(\phi) \quad \textbf{(10)}$$

> $C2 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff(Eq_{arc}, d\phi) + coeff(Eq_{arc}, d\phi^2) \cdot d\phi, 'trig'\right), 'size'\right)$

> $C2 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff(Eq_{arc}, d\phi) + coeff(Eq_{arc}, d\phi^2) \cdot d\phi, 'trig'\right), 'size'\right)$

$$C2 := m\, (R + R_o) \sin(\phi)\, (d\phi - d\theta)\, (H - R) \quad \textbf{(11)}$$

> $C3 := simplify(simplify(coeff(Eq_{ball}, d\theta^2) \cdot d\theta, 'trig'), 'size')$

$$C3 := m\, (H - R)\, (R + R_o) \sin(\phi)\, d\theta \quad \textbf{(12)}$$

> $C4 := simplify(simplify(coeff(Eq_{ball}, d\phi^2) \cdot d\phi, 'trig'), 'size')$

$$C4 := 0 \quad \textbf{(13)}$$

> $C := \begin{bmatrix} C1 & C2 \\ C3 & C4 \end{bmatrix}$

$$C := \begin{bmatrix} -m\, d\phi\, (H - R)\, (R + R_o) \sin(\phi) & m\, (R + R_o) \sin(\phi)\, (d\phi - d\theta)\, (H - R) \\ m\, (H - R)\, (R + R_o) \sin(\phi)\, d\theta & 0 \end{bmatrix} \quad \textbf{(14)}$$

comparing to [23]

> $l_{arc} := 0 : J_b := 0 : R_o := 0 :$

> $x_c := R \cdot \sin(\phi - \theta) + l \cdot \sin(\theta) :$

> $y_c := R\cdot\cos(\phi - \theta) - l\cdot\cos(\theta)$:

> $dx_c := R\cdot\cos(\phi - \theta)\cdot(d\phi - d\theta) + l\cdot\cos(\theta)\cdot d\theta$

$$dx_c := R\cos(\phi - \theta)\,(d\phi - d\theta) + l\cos(\theta)\,d\theta \quad (15)$$

> $dy_c := -R\sin(\phi - \theta)\,(d\phi - d\theta) + l\cdot\sin(\theta)\,d\theta$

$$dy_c := -R\sin(\phi - \theta)\,(d\phi - d\theta) + l\sin(\theta)\,d\theta \quad (16)$$

> $T := simplify\left(\frac{1}{2}\cdot m\cdot\left(dx_c^2 + dy_c^2\right) + \frac{1}{2}\cdot J_{arc}\cdot d\theta^2, 'size'\right)$

$$T := \frac{1}{2}\Big(m\,\big(R^2\,(d\phi - d\theta)^2\cos(\phi-\theta)^2 + 2\,R\,l\,d\theta\cos(\theta)\,(d\phi - d\theta)\cos(\phi-\theta) + R^2\,(d\phi-d\theta)^2\sin(\phi-\theta)^2 - 2\,R\,l\,d\theta\sin(\theta)\,(d\phi - d\theta)\sin(\phi-\theta) + l^2\,d\theta^2\,(\cos(\theta)^2 + \sin(\theta)^2)\big)\Big) + \frac{J_{arc}\,d\theta^2}{2} \quad (17)$$

> $V := m\cdot g\cdot y_c$:

> $L := expand(T - V)$:

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $dLdphi := diff(L, d\phi)$:

> $dLdphi := subs(\{\theta = \theta(t), \phi = \phi(t)\}, dLdphi)$:

> $e1 := subs\left(\left\{d\theta = diff(\theta(t), t), d\phi = diff(\phi(t), t), \right\}, dLdphi\right)$:

> $dLdphidt := diff(e1, t)$:

> $e2 := subs(\{diff(\phi(t), t, t) = dd\phi, diff(\phi(t), t) = d\phi, diff(\theta(t), t, t) = dd\theta, diff(\theta(t), t) = d\theta\}, dLdphidt)$:

> $e2 := subs(\{\phi(t) = \phi, \theta(t) = \theta\}, e2)$:

> $variables := [dd\theta, dd\phi, d\theta, d\phi]$:

> $Eq_{ball} := collect(simplify(simplify(e2 - diff(L, \phi), 'trig'), 'size'), variables)$

$$Eq_{ball} := -R\,m\,(-l\cos(\phi) + R)\,dd\theta + R^2\,m\,dd\phi - R\,m\,l\sin(\phi)\,d\theta^2 - R\,m\,(\sin(\phi)\cos(\theta)\,g - \sin(\theta)\cos(\phi)\,g) \quad (18)$$

> $dLdtheta := diff(L, d\theta)$:

> $dLdtheta := subs(\{\theta = \theta(t), \phi = \phi(t)\}, dLdtheta)$:

> $e3 := subs\left(\left\{d\theta = diff(\theta(t), t), d\phi = diff(\phi(t), t), \right\}, dLdtheta\right)$:

> $dLdthetadt := diff(e3, t)$:

> $e4 := subs(\{diff(\phi(t), t, t) = dd\phi, diff(\phi(t), t) = d\phi, diff(\theta(t), t, t) = dd\theta, diff(\theta(t), t) = d\theta\}, dLdthetadt)$:

> $e4 := subs(\{\phi(t) = \phi, \theta(t) = \theta\}, e4)$:

> $Eq_{arc} := collect(simplify(simplify(e4 - diff(L, \theta), 'trig'), 'size'), variables)$

$$Eq_{arc} := \left(-2\,R\,m\,l\cos(\phi) + (R^2 + l^2)\,m + J_{arc}\right)dd\theta + \left(R\,m\,l\cos(\phi) - R^2\,m\right)dd\phi + 2\,m\,R\,l\,d\theta\,d\phi\sin(\phi) - m\,R\,l\,d\phi^2\sin(\phi) + m\,g\,R\sin(\phi)\cos(\theta) - m\,g\,R\cos(\phi)\sin(\theta) + m\,g\,l\sin(\theta) \quad (19)$$

Group terms for M, C and G matrices

> $G := simplify\left(simplify\left(\begin{bmatrix} coeff(Eq_{arc}, g)\cdot g \\ coeff(Eq_{ball}, g)\cdot g \end{bmatrix}, 'trig'\right), 'size'\right)$

$$G := \begin{bmatrix} ((-R\cos(\phi) + l)\sin(\theta) + \cos(\theta)\sin(\phi)\,R)\,m\,g \\ m\,(\cos(\phi)\sin(\theta) - \sin(\phi)\cos(\theta))\,R\,g \end{bmatrix} \quad (20)$$

> $M := \begin{bmatrix} simplify(simplify(coeff(Eq_{arc}, dd\theta), 'trig'), 'size') & simplify(simplify(coeff(Eq_{arc}, dd\phi), 'trig'), 'size') \\ simplify(simplify(coeff(Eq_{ball}, dd\theta), 'trig'), 'size') & simplify(simplify(coeff(Eq_{ball}, dd\phi), 'trig'), 'size') \end{bmatrix}$

$$M := \begin{bmatrix} -2\,m\,l\,R\cos(\phi) + (R^2 + l^2)\,m + J_{arc} & -R\,m\,(-l\cos(\phi) + R) \\ -R\,m\,(-l\cos(\phi) + R) & R^2\,m \end{bmatrix} \quad (21)$$

> $C := \Big[\Big[simplify\Big(simplify\Big(\frac{1}{2}\cdot coeff(Eq_{arc}, d\theta) + coeff(Eq_{arc}, d\theta^2)\cdot d\theta, 'trig'\Big), 'size'\Big), simplify\Big(simplify\Big(\frac{1}{2}\cdot coeff(Eq_{arc}, d\phi) + coeff(Eq_{arc}, d\phi^2)\cdot d\phi, 'trig'\Big), 'size'\Big)\Big],$
$\Big[simplify(simplify(coeff(Eq_{ball}, d\theta^2)\cdot d\theta, 'trig'), 'size'), simplify(simplify(coeff(Eq_{ball}, d\phi^2)\cdot d\phi, 'trig'), 'size')\Big]\Big]$

$$C := \begin{bmatrix} m\,R\,l\,d\phi\sin(\phi) & -\sin(\phi)\,R\,l\,m\,(d\phi - d\theta) \\ -m\,R\,l\,d\theta\sin(\phi) & 0 \end{bmatrix} \quad (22)$$

B.3 Derivation of the Dimensionless Equations of Motion

BA dimensionless.mw

> restart :

> with(LinearAlgebra) :

> $Eq_{ball} := \frac{(R+R_o)\left(-m R_o^3 - R m R_o^2 - J_b R_o - (-R+H) R_o^2 m \cos(\phi)\right) dd\theta}{R_o^2} + \frac{(R+R_o)\left(R m R_o^2 + m R_o^3 + J_b R + J_b R_o\right) dd\phi}{R_o^2} + (R + R_o)(-R+H) m \sin(\phi) d\theta^2 + \frac{(R+R_o)\left(\sin(\theta) g R_o^2 m \cos(\phi) - \cos(\theta) g R_o^2 m \sin(\phi)\right)}{R_o^2}$

$$Eq_{ball} := \frac{(R+R_o)\left(-m R_o^3 - R m R_o^2 - J_b R_o - (-R+H) R_o^2 m \cos(\phi)\right) dd\theta}{R_o^2} + \frac{(R+R_o)\left(R m R_o^2 + m R_o^3 + J_b R + J_b R_o\right) dd\phi}{R_o^2} + (R + R_o)(-R+H) m \sin(\phi) d\theta^2 + \frac{(R+R_o)\left(\sin(\theta) g R_o^2 m \cos(\phi) - \cos(\theta) g R_o^2 m \sin(\phi)\right)}{R_o^2} \quad (1)$$

> $Eq_{arc} := \frac{\left(2(-R+H) R_o (R+R_o) m \cos(\phi) + m R_o^3 + 2 R m R_o^2 + \left((H^2 - 2RH + 2R^2) m + J_b + J_{arc}\right) R_o\right) dd\theta}{R_o} + \frac{\left(2\left(\frac{1}{2}R - \frac{1}{2}H\right) R_o (R+R_o) m \cos(\phi) - m R_o^3 - 2 R m R_o^2 + \left(-m R^2 - J_b\right) R_o - J_b R\right) dd\phi}{R_o} - 2 d\phi (-R+H)(R + R_o) m \sin(\phi) d\theta - 2\left(\frac{1}{2}R - \frac{1}{2}H\right)(R+R_o) m \sin(\phi) d\phi^2 + \frac{-\sin(\theta) g R_o (R+R_o) m \cos(\phi) + \cos(\theta) g R_o (R+R_o) m \sin(\phi) - g\left((-R+H) m + l_{arc} m_a\right) R_o \sin(\theta)}{R_o}$

$$Eq_{arc} := \frac{\left(2(-R+H) R_o (R+R_o) m \cos(\phi) + m R_o^3 + 2 R m R_o^2 + \left((H^2 - 2RH + 2R^2) m + J_b + J_{arc}\right) R_o\right) dd\theta}{R_o} + \frac{\left(2\left(\frac{R}{2} - \frac{H}{2}\right) R_o (R+R_o) m \cos(\phi) - m R_o^3 - 2 R m R_o^2 + \left(-m R^2 - J_b\right) R_o - J_b R\right) dd\phi}{R_o} - 2 d\phi (-R+H)(R + R_o) m \sin(\phi) d\theta - 2\left(\frac{R}{2} - \frac{H}{2}\right)(R+R_o) m \sin(\phi) d\phi^2 + \frac{-\sin(\theta) g R_o (R+R_o) m \cos(\phi) + \cos(\theta) g R_o (R+R_o) m \sin(\phi) - g\left((-R+H) m + l_{arc} m_a\right) R_o \sin(\theta)}{R_o} \quad (2)$$

Group terms for M, C and G matrices

> $G := simplify\left(simplify\left(\begin{bmatrix} coeff(Eq_{arc}, g) \cdot g \\ coeff(Eq_{ball}, g) \cdot g \end{bmatrix}, 'trig'\right), 'size'\right)$

$$G := \begin{bmatrix} -\left(\left(m (R+R_o) \cos(\phi) + (-R+H) m + l_{arc} m_a\right) \sin(\theta) - \sin(\phi)\cos(\theta) m (R+R_o)\right) g \\ (R+R_o) g m \left(\sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi)\right) \end{bmatrix} \quad (3)$$

> $M1 := simplify(simplify(coeff(Eq_{arc}, dd\theta), 'trig'), 'size')$

$$M1 := 2 m (R+R_o)(-R+H)\cos(\phi) + \left(2R^2 + (-2H + 2R_o) R + H^2 + R_o^2\right) m + J_b + J_{arc} \quad (4)$$

> $M2 := simplify(simplify(coeff(Eq_{arc}, dd\phi), 'trig'), 'size')$

$$M2 := -\frac{\left(m R_o (-R+H)\cos(\phi) + R m R_o + m R_o^2 + J_b\right)(R+R_o)}{R_o} \quad (5)$$

> $M3 := simplify(simplify(coeff(Eq_{ball}, dd\theta), 'trig'), 'size')$

$$M3 := -\frac{\left(m R_o (-R+H)\cos(\phi) + R m R_o + m R_o^2 + J_b\right)(R+R_o)}{R_o} \quad (6)$$

$> M4 := simplify(simplify(coeff(Eq_{ball}, dd\phi), 'trig'), 'size')$

$$M4 := \frac{(R+R_o)^2 (m R_o^2 + J_b)}{R_o^2} \tag{7}$$

$> M := \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix}:$

$> C1 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff(Eq_{arc}, d\theta) + coeff(Eq_{arc}, d\theta^2) \cdot d\theta, 'trig'\right), 'size'\right)$

$$C1 := -d\phi\,(-R+H)\,(R+R_o)\,m \sin(\phi) \tag{8}$$

$> C2 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff(Eq_{arc}, d\phi) + coeff(Eq_{arc}, d\phi^2) \cdot d\phi, 'trig'\right), 'size'\right)$

$$C2 := (R+R_o)\,m \sin(\phi)\,(d\phi - d\theta)\,(-R+H) \tag{9}$$

$> C3 := simplify(simplify(coeff(Eq_{ball}, d\theta^2) \cdot d\theta, 'trig'), 'size')$

$$C3 := (-R+H)\,(R+R_o)\,m \sin(\phi)\,d\theta \tag{10}$$

$> C4 := simplify(simplify(coeff(Eq_{ball}, d\phi^2) \cdot d\phi, 'trig'), 'size')$

$$C4 := 0 \tag{11}$$

$> C := \begin{bmatrix} C1 & C2 \\ C3 & C4 \end{bmatrix}$

$$C := \begin{bmatrix} -d\phi\,(-R+H)\,(R+R_o)\,m \sin(\phi) & (R+R_o)\,m \sin(\phi)\,(d\phi - d\theta)\,(-R+H) \\ (-R+H)\,(R+R_o)\,m \sin(\phi)\,d\theta & 0 \end{bmatrix} \tag{12}$$

conversion to dimensionless

$> Eq_{arc} := \frac{Eq_{arc}}{m \cdot R_o^2}:$

$> Eq_{ball} := \frac{Eq_{ball}}{m \cdot R_o \cdot (R+R_o)}:$

$> G := simplify\left(simplify\left(\begin{bmatrix} coeff(Eq_{arc}, g) \cdot g \\ coeff(Eq_{ball}, g) \cdot g \end{bmatrix}, 'trig'\right), 'size'\right)$

$$G := \begin{bmatrix} -\dfrac{\left(\left(m (R+R_o) \cos(\phi) + (-R+H) m + l_{arc} m_a\right) \sin(\theta) - \sin(\phi) \cos(\theta)\, m\, (R+R_o)\right) g}{R_o^2\, m} \\ \dfrac{g\,(\sin(\theta) \cos(\phi) - \cos(\theta) \sin(\phi))}{R_o} \end{bmatrix} \tag{13}$$

$> M1 := simplify(simplify(coeff(Eq_{arc}, dd\theta), 'trig'), 'size')$

$$M1 := \frac{2 m (R+R_o)(-R+H) \cos(\phi) + \left(2 R^2 + (-2H + 2R_o) R + H^2 + R_o^2\right) m + J_b + J_{arc}}{m R_o^2} \tag{14}$$

$> M2 := simplify(simplify(coeff(Eq_{arc}, dd\phi), 'trig'), 'size')$

$$M2 := -\frac{\left(m R_o (-R+H) \cos(\phi) + R m R_o + m R_o^2 + J_b\right)(R+R_o)}{R_o^3\, m} \tag{15}$$

$> M3 := simplify(simplify(coeff(Eq_{ball}, dd\theta), 'trig'), 'size')$

$$M3 := \frac{-m R_o (-R+H) \cos(\phi) - R m R_o - m R_o^2 - J_b}{m R_o^2} \tag{16}$$

$> M4 := simplify(simplify(coeff(Eq_{ball}, dd\phi), 'trig'), 'size')$

$$M4 := \frac{(m R_o^2 + J_b)(R+R_o)}{m R_o^3} \tag{17}$$

> $M := \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix}$:

> $C1 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff\left(Eq_{arc}, d\theta\right) + coeff\left(Eq_{arc}, d\theta^2\right) \cdot d\theta, 'trig'\right), 'size'\right)$

$$C1 := -\frac{d\phi\,(-R+H)\,(R+R_o)\,\sin(\phi)}{R_o^2} \tag{18}$$

> $C2 := simplify\left(simplify\left(\frac{1}{2} \cdot coeff\left(Eq_{arc}, d\phi\right) + coeff\left(Eq_{arc}, d\phi^2\right) \cdot d\phi, 'trig'\right), 'size'\right)$

$$C2 := \frac{(-R+H)\,(R+R_o)\,\sin(\phi)\,(d\phi - d\theta)}{R_o^2} \tag{19}$$

> $C3 := simplify\left(simplify\left(coeff\left(Eq_{ball}, d\theta^2\right) \cdot d\theta, 'trig'\right), 'size'\right)$

$$C3 := \frac{(-R+H)\,\sin(\phi)\,d\theta}{R_o} \tag{20}$$

> $C4 := simplify\left(simplify\left(coeff\left(Eq_{ball}, d\phi^2\right) \cdot d\phi, 'trig'\right), 'size'\right)$

$$C4 := 0 \tag{21}$$

> $C := \begin{bmatrix} C1 & C2 \\ C3 & C4 \end{bmatrix}$:

substitute in dimensionless parameters

> $M := \begin{bmatrix} -2\cdot(k_7 - k_6)\cdot(k_7+1)\cdot\cos(\phi) + k_3 + 2\cdot k_7^2 - 2\cdot k_7\cdot k_6 + 2\cdot k_7 + 1 + k_6^2 + k_4 & -((-k_7+k_6)\cdot\cos(\phi) + k_7 + 1 + k_4) \\ -((-k_7+k_6)\cdot\cos(\phi) + k_7 + 1 + k_4) & 1 + k_4 \end{bmatrix}$:

> $C := \begin{bmatrix} d\phi\cdot(k_7 - k_6)\cdot(k_7+1)\sin(\phi) & (-k_7+k_6)\sin(\phi)\cdot(d\phi - d\theta) \\ (-k_7+k_6)\cdot\sin(\phi)\cdot d\theta & 0 \end{bmatrix}$:

> $G := \begin{bmatrix} (k_7+1)\cdot\sin(\phi-\theta) + (k_7 - k_6 - k_1\cdot k_5)\cdot\sin(\theta) \\ -\sin(\phi-\theta) \end{bmatrix}$:

> $F := \begin{bmatrix} \tau_N \\ 0 \end{bmatrix}$:

>

> $M1 := expand\left(simplify\left(subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, M_{1,1}\right), 'size'\right)\right)$

$$M1 := -\phi^2 k_6 k_7 + \phi^2 k_7^2 - \phi^2 k_6 + \phi^2 k_7 + k_6^2 + k_3 + k_4 + 2\,k_6 + 1 \tag{22}$$

> $M2 := simplify\left(subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, M_{1,2}\right)\right)$

$$M2 := \frac{(-k_7+k_6)\,\phi^2}{2} - k_4 - k_6 - 1 \tag{23}$$

> $M3 := subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, M_{2,2}\right)$

$$M3 := 1 + k_4 \tag{24}$$

> $C1 := expand\left(subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, C_{1,1}\right)\right)$

$$C1 := -d\phi\,\phi\,k_6 k_7 + d\phi\,\phi\,k_7^2 - d\phi\,\phi\,k_6 + d\phi\,\phi\,k_7 \tag{25}$$

> $C2 := expand\left(subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, C_{1,2}\right)\right)$

$$C2 := d\phi\,\phi\,k_6 - d\phi\,\phi\,k_7 - \phi\,d\theta\,k_6 + \phi\,d\theta\,k_7 \tag{26}$$

> $C3 := expand\left(subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, C_{2,1}\right)\right)$

$$C3 := \phi\,d\theta\,k_6 - \phi\,d\theta\,k_7 \tag{27}$$

> $G2 := simplify\left(subs\left(\left\{\cos(\phi) = 1 - \frac{1}{2}\cdot\phi^2, \sin(\phi) = \phi\right\}, G_{2,1}\right)\right)$

$$G2 := -\sin(-\theta + \phi) \quad \textbf{(29)}$$

> $M1 := subs\left(\left\{k_7^2\cdot\phi^2 = \rho^2, \phi^2 = 0, \phi = 0\right\}, M1\right)$

$$M1 := \rho^2 + k_6^2 + k_3 + k_4 + 2\,k_6 + 1 \quad \textbf{(30)}$$

> $M2 := subs\left(\left\{\phi^2 = 0, \phi = 0\right\}, M2\right)$

$$M2 := -1 - k_4 - k_6 \quad \textbf{(31)}$$

> $M3 := subs\left(\left\{\phi^2 = 0, \phi = 0\right\}, M3\right)$

$$M3 := 1 + k_4 \quad \textbf{(32)}$$

> $C1 := subs\left(\left\{d\phi\,\phi\,k_7^2 = \rho\cdot d\rho, \phi = 0, \phi^2 = 0\right\}, C1\right)$

$$C1 := \rho\,d\rho \quad \textbf{(33)}$$

> $C2 := subs\left(\left\{\phi\,k_7\,d\theta = \rho\cdot d\theta, \phi = 0, \phi^2 = 0\right\}, C2\right)$

$$C2 := \rho\,d\theta \quad \textbf{(34)}$$

> $C3 := subs\left(\left\{\phi\,k_7\,d\theta = \rho\cdot d\theta, \phi = 0, \phi^2 = 0\right\}, C3\right)$

$$C3 := -\rho\,d\theta \quad \textbf{(35)}$$

> $G1 := simplify\left(subs\left(\left\{\phi^2 = 0, \phi = 0\right\}, simplify\left(subs\left(\left\{k_7\,\phi\cos(\theta) = \rho\cos(\theta)\right\}, expand(G1)\right), 'trig'\right)\right), 'size'\right)$

$$G1 := \left(-k_1\,k_5 - k_6 - 1\right)\sin(\theta) + \rho\cos(\theta) \quad \textbf{(36)}$$

> $G2 := simplify\left(subs\left(\left\{k_7\cdot\phi = \rho, \phi^2 = 0, \phi = 0\right\}, G2\right), 'trig'\right)$

$$G2 := \sin(\theta) \quad \textbf{(37)}$$

Appendix C - Inverted Pendulum Cart

This Appendix is organized in four major parts. These are:

C.1 Derivations of Equations of Motion and Controller

C.2 Simulink file and MATLAB code for the simulation of the Full System

C.3 Simulink file and MATLAB code for the simulation of the Dimensionless System

C.4 MATLAB code to produce the plot for Chapter 4

C.1 Derivations of Equations of Motion and Controller

IPC.mw

```
> restart :
> with(LinearAlgebra) :
```

Equations of Motion

Kinetic Energy

$> T := \frac{1}{2}\cdot m_p \cdot l^2 \cdot d\theta^2 + \frac{1}{2}\cdot(m_c + m_p)\cdot dx^2 + m_p \cdot l \cdot \cos(\theta)\cdot dx \cdot d\theta :$

Potential Energy

$> V := m_p \cdot g \cdot l \cdot \cos(\theta) :$

Lagrange

$> L := expand(T - V) :$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

$> e1 := diff(L, dx)$

$$e1 := dx\, m_c + dx\, m_p + m_p\, l \cos(\theta)\, d\theta \quad \textbf{(1.1)}$$

$> e1 := subs(\{\theta = \theta(t), x = x(t)\}, e1)$

$$e1 := dx\, m_c + dx\, m_p + m_p\, l \cos(\theta(t))\, d\theta \quad \textbf{(1.2)}$$

$> e1 := subs(\{d\theta = diff(\theta(t), t), dx = diff(x(t), t)\}, e1)$

$$e1 := \left(\frac{d}{dt} x(t)\right) m_c + \left(\frac{d}{dt} x(t)\right) m_p + m_p\, l \cos(\theta(t)) \left(\frac{d}{dt} \theta(t)\right) \quad \textbf{(1.3)}$$

$> e2 := diff(e1, t)$

$$e2 := \left(\frac{d^2}{dt^2} x(t)\right) m_c + \left(\frac{d^2}{dt^2} x(t)\right) m_p - m_p\, l \sin(\theta(t)) \left(\frac{d}{dt} \theta(t)\right)^2 + m_p\, l \cos(\theta(t)) \left(\frac{d^2}{dt^2} \theta(t)\right) \quad \textbf{(1.4)}$$

$> e2 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(x(t), t, t) = ddx\}, e2)$

$$e2 := ddx\, m_c + ddx\, m_p - m_p\, l \sin(\theta(t)) \left(\frac{d}{dt} \theta(t)\right)^2 + m_p\, l \cos(\theta(t))\, dd\theta \quad \textbf{(1.5)}$$

$> e2 := subs(\{diff(\theta(t), t) = d\theta, diff(x(t), t) = dx\}, e2)$

$$e2 := ddx\, m_c + ddx\, m_p - m_p\, l \sin(\theta(t))\, d\theta^2 + m_p\, l \cos(\theta(t))\, dd\theta \quad \textbf{(1.6)}$$

$> e2 := subs(\{\theta(t) = \theta, x(t) = x\}, e2)$

$$e2 := ddx\, m_c + ddx\, m_p - m_p\, l \sin(\theta)\, d\theta^2 + m_p\, l \cos(\theta)\, dd\theta \quad \textbf{(1.7)}$$

$> variables := [dd\theta, ddx, d\theta, dx] :$

$> Eq_{cart} := collect(simplify(e2 - diff(L, x) - \tau_{IPC}, 'trig'), variables)$

$$Eq_{cart} := m_p\, l \cos(\theta)\, dd\theta + (m_c + m_p)\, ddx - m_p\, l \sin(\theta)\, d\theta^2 - \tau_{IPC} \quad \textbf{(1.8)}$$

> $e3 := diff(L, d\theta)$

$$e3 := m_p\, l^2\, d\theta + m_p\, l \cos(\theta)\, dx \tag{1.9}$$

> $e3 := subs(\{\theta = \theta(t), x = x(t)\}, e3)$

$$e3 := m_p\, l^2\, d\theta + m_p\, l \cos(\theta(t))\, dx \tag{1.10}$$

> $e3 := subs(\{d\theta = diff(\theta(t), t), dx = diff(x(t), t)\}, e3)$

$$e3 := m_p\, l^2 \left(\frac{d}{dt}\theta(t)\right) + m_p\, l \cos(\theta(t)) \left(\frac{d}{dt}x(t)\right) \tag{1.11}$$

> $e4 := diff(e3, t)$

$$e4 := m_p\, l^2 \left(\frac{d^2}{dt^2}\theta(t)\right) - m_p\, l \sin(\theta(t)) \left(\frac{d}{dt}\theta(t)\right)\left(\frac{d}{dt}x(t)\right) + m_p\, l \cos(\theta(t)) \left(\frac{d^2}{dt^2}x(t)\right) \tag{1.12}$$

> $e4 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(x(t), t, t) = ddx\}, e4)$

$$e4 := m_p\, l^2\, dd\theta - m_p\, l \sin(\theta(t)) \left(\frac{d}{dt}\theta(t)\right)\left(\frac{d}{dt}x(t)\right) + m_p\, l \cos(\theta(t))\, ddx \tag{1.13}$$

> $e4 := subs(\{diff(\theta(t), t) = d\theta, diff(x(t), t) = dx\}, e4)$

$$e4 := m_p\, l^2\, dd\theta - m_p\, l \sin(\theta(t))\, d\theta\, dx + m_p\, l \cos(\theta(t))\, ddx \tag{1.14}$$

> $e4 := subs(\{\theta(t) = \theta, x(t) = x\}, e4)$

$$e4 := m_p\, l^2\, dd\theta - m_p\, l \sin(\theta)\, d\theta\, dx + m_p\, l \cos(\theta)\, ddx \tag{1.15}$$

> $Eq_{ball} := collect(simplify(e4 - diff(L, \theta), 'trig'), variables)$

$$Eq_{ball} := m_p\, l^2\, dd\theta + m_p\, l \cos(\theta)\, ddx - m_p\, g\, l \sin(\theta) \tag{1.16}$$

> $M := \begin{bmatrix} coeff(Eq_{cart}, ddx) & coeff(Eq_{cart}, dd\theta) \\ coeff(Eq_{ball}, ddx) & coeff(Eq_{ball}, dd\theta) \end{bmatrix}$

$$M := \begin{bmatrix} m_c + m_p & m_p\, l \cos(\theta) \\ m_p\, l \cos(\theta) & m_p\, l^2 \end{bmatrix} \tag{1.17}$$

> $C := \begin{bmatrix} coeff(Eq_{cart}, dx^2) \cdot dx & coeff(Eq_{cart}, d\theta^2) \cdot d\theta \\ coeff(Eq_{ball}, dx^2) \cdot dx & coeff(Eq_{ball}, d\theta) \end{bmatrix}$

$$C := \begin{bmatrix} 0 & -m_p\, l \sin(\theta)\, d\theta \\ 0 & 0 \end{bmatrix} \tag{1.18}$$

> $G := \begin{bmatrix} coeff(Eq_{cart}, g) \cdot g \\ coeff(Eq_{ball}, g) \cdot g \end{bmatrix}$

$$G := \begin{bmatrix} 0 \\ -m_p\, g\, l \sin(\theta) \end{bmatrix} \tag{1.19}$$

> $F := -\begin{bmatrix} coeff(Eq_{cart}, \tau_{IPC}) \cdot \tau_{IPC} \\ coeff(Eq_{ball}, \tau_{IPC}) \cdot \tau_{IPC} \end{bmatrix}$

$$F := \begin{bmatrix} \tau_{IPC} \\ 0 \end{bmatrix} \tag{1.20}$$

> $q := \begin{bmatrix} x \\ \theta \end{bmatrix}$

$$q := \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{1.21}$$

Dimensionless Equations of Motion

divide cart equation by mp*l and pendulum equation by mp*l^2

$> Eqnorm_{cart} := \frac{Eq_{cart}}{m_p \cdot l}:$

$> Eqnorm_{ball} := \frac{Eq_{ball}}{m_p \cdot l^2}:$

$> Mn := \begin{bmatrix} coeff(Eqnorm_{cart}, ddx) & coeff(Eqnorm_{cart}, dd\theta) \\ coeff(Eqnorm_{ball}, ddx) & coeff(Eqnorm_{ball}, dd\theta) \end{bmatrix}$

$$Mn := \begin{bmatrix} \frac{m_c + m_p}{m_p\, l} & \cos(\theta) \\ \frac{\cos(\theta)}{l} & 1 \end{bmatrix} \quad \textbf{(2.1)}$$

$> Cn := \begin{bmatrix} coeff(Eqnorm_{cart}, dx^2) \cdot dx & coeff(Eqnorm_{cart}, d\theta^2) \cdot d\theta \\ coeff(Eqnorm_{ball}, dx^2) \cdot dx & coeff(Eqnorm_{ball}, d\theta) \end{bmatrix}$

$$Cn := \begin{bmatrix} 0 & -\sin(\theta)\, d\theta \\ 0 & 0 \end{bmatrix} \quad \textbf{(2.2)}$$

$> Gn := \begin{bmatrix} coeff(Eqnorm_{cart}, g) \cdot g \\ coeff(Eqnorm_{ball}, g) \cdot g \end{bmatrix}$

$$Gn := \begin{bmatrix} 0 \\ -\frac{\sin(\theta)\, g}{l} \end{bmatrix} \quad \textbf{(2.3)}$$

$> Fn := -\begin{bmatrix} coeff(Eqnorm_{cart}, \tau_{IPC}) \cdot \tau_{IPC} \\ coeff(Eqnorm_{ball}, \tau_{IPC}) \cdot \tau_{IPC} \end{bmatrix}$

$$Fn := \begin{bmatrix} \frac{\tau_{IPC}}{m_p\, l} \\ 0 \end{bmatrix} \quad \textbf{(2.4)}$$

$> qn := \begin{bmatrix} \frac{x}{l} \\ \theta \end{bmatrix}:$

$> Mn := \begin{bmatrix} coeff(Eqnorm_{cart}, ddx) \cdot l & coeff(Eqnorm_{cart}, dd\theta) \\ coeff(Eqnorm_{ball}, ddx) \cdot l & coeff(Eqnorm_{ball}, dd\theta) \end{bmatrix}$

$$Mn := \begin{bmatrix} \frac{m_c + m_p}{m_p} & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix} \quad \textbf{(2.5)}$$

change of variables:
J/mp/l^2=k1, mc/mp=k6, g/l=1/gamma^2 multiply both equations by gamma^2 create rho'(ι) and θ'(ι)
q(original)=[x;θ] q(nn)=[rho;θ] q(normalized)=[rho'(ι), θ'(ι)] have to change sin(t) to sin(ι)
τ_{IPC}/mp/l=tn

$> Mn := \begin{bmatrix} k_6 + 1 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix}:$

> Cn

$$\begin{bmatrix} 0 & -\sin(\theta)\, d\theta \\ 0 & 0 \end{bmatrix} \quad (2.6)$$

> $Gn := \begin{bmatrix} 0 \\ -\sin(\theta) \end{bmatrix}:$

> $Fn := \begin{bmatrix} 0 \\ \tau_N \end{bmatrix}:$

>

Controller from "A Normal Form For Energy Shaping: Application to the Furuta Pendulum"

> $\tau := \dfrac{\left(\kappa\cdot\beta\cdot\sin(\theta)\cdot(\alpha\cdot d\theta^2 + d\cdot\cos(\theta)) - \dfrac{B\cdot\varepsilon\cdot d\cdot(m_c+m_p)^2\cdot x}{\beta^2} + B\cdot u_{diss}\right)}{\left(\alpha - \dfrac{\beta^2\cdot(\kappa+1)\cdot(\cos(\theta))^2}{(m_c+m_p)}\right)}$

$$\tau := \frac{\kappa\beta\sin(\theta)\,(\alpha\, d\theta^2 + d\cos(\theta)) - \dfrac{B\,\varepsilon\, d\,(m_c+m_p)^2 x}{\beta^2} + B\, u_{diss}}{\alpha - \dfrac{\beta^2(\kappa+1)\cos(\theta)^2}{m_c+m_p}} \quad (3.1)$$

> $u_{diss} := c\cdot(m_c+m_p)\cdot(dx + p\cdot\cos(\theta)\cdot d\theta)$

$$u_{diss} := c\,(m_c+m_p)\,(dx + p\cos(\theta)\, d\theta) \quad (3.2)$$

> $B := \dfrac{1}{\zeta}\cdot\left(\alpha - \dfrac{\beta^2\cdot(\cos(\theta))^2}{(m_c+m_p)}\right)$

$$B := \frac{\alpha - \dfrac{\beta^2\cos(\theta)^2}{m_c+m_p}}{\zeta} \quad (3.3)$$

> $P := \dfrac{\beta}{(m_c+m_p)}\cdot\left(\kappa + \left(\dfrac{(\zeta-1)}{\zeta}\right)\right)$

$$P := \frac{\beta\left(\kappa + \dfrac{\zeta-1}{\zeta}\right)}{m_c+m_p} \quad (3.4)$$

> $simplify(\tau,'size')$

$$\frac{\kappa\beta\sin(\theta)\,(\alpha\, d\theta^2 + d\cos(\theta)) - \dfrac{\left(\alpha - \dfrac{\beta^2\cos(\theta)^2}{m_c+m_p}\right)\varepsilon\, d\,(m_c+m_p)^2 x}{\zeta\beta^2} + \dfrac{\left(\alpha - \dfrac{\beta^2\cos(\theta)^2}{m_c+m_p}\right) c\,(m_c+m_p)\,(dx + p\cos(\theta)\, d\theta)}{\zeta}}{\alpha - \dfrac{\beta^2(\kappa+1)\cos(\theta)^2}{m_c+m_p}} \quad (3.5)$$

Dimensionless Controller

> $\alpha := m_p\cdot l^2:$

> $\beta := m_p\cdot l:$

> $\#\lambda := (m_c+m_p):$

> $d := -m_p\cdot g\cdot l:$

> $\lambda_d := 1 + k_1$:

> $d_d := -\frac{1}{\gamma^2}$:

> P

$$\frac{m_p\, l \left(\kappa + \frac{\zeta - 1}{\zeta}\right)}{m_c + m_p} \tag{4.1}$$

> $Pn := l \cdot P_d$:

> $P_d := \frac{\left(\kappa + \frac{\zeta - 1}{\zeta}\right)}{\lambda_d}$:

> B

$$\frac{m_p\, l^2 - \frac{m_p^2\, l^2 \cos(\theta)^2}{m_c + m_p}}{\zeta} \tag{4.2}$$

> $Bn := m_p\, l^2 \cdot B_d$:

> $B_d := \frac{1 - \frac{\cos(\theta)^2}{\lambda_d}}{\zeta}$:

> u_{diss}

$$c\,(m_c + m_p)\,(dx + p\cos(\theta)\, d\theta) \tag{4.3}$$

> $un := m_p \cdot l \cdot u_{dd}$

$$un := m_p\, l\, u_{dd} \tag{4.4}$$

> $u_{dd} := c \cdot \lambda_d \cdot (\rho + P_d \cdot \cos(\theta)\, d\theta)$:

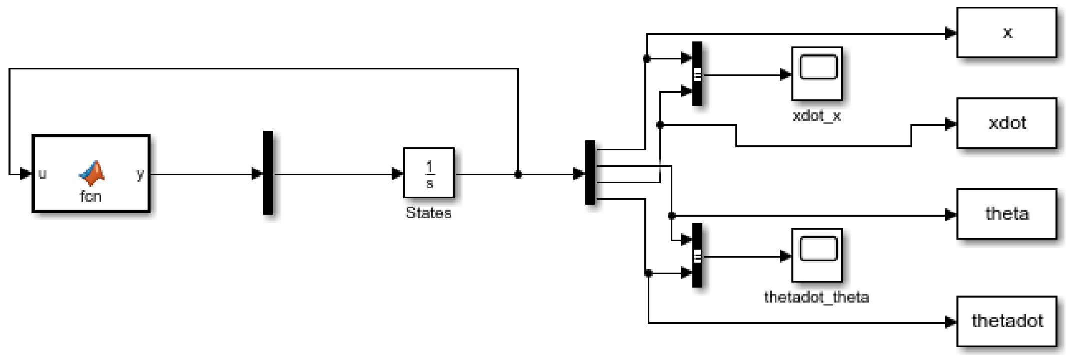
> $vn := m_p \cdot l \cdot \frac{\left(\kappa \cdot \sin(\theta) \cdot \left(d\theta^2 + d_d \cdot \cos(\theta)\right) - B_d \cdot \varepsilon \cdot d_d \cdot (\lambda_d)^2 \cdot \left(\frac{x}{l}\right) + B_d \cdot u_{dd}\right)}{\left(1 - \frac{(\kappa + 1) \cdot (\cos(\theta))^2}{\lambda_d}\right)}$:

> $vd := simplify\left(\frac{\left(\kappa \cdot \sin(\theta) \cdot \left(d\theta^2 + d_d \cdot \cos(\theta)\right) - B_d \cdot \varepsilon \cdot d_d \cdot (\lambda_d)^2 \cdot (\rho) + B_d \cdot u_{dd}\right)}{\left(1 - \frac{(\kappa + 1) \cdot (\cos(\theta))^2}{\lambda_d}\right)}, 'size'\right)$:

multiply by gamma^2

> $vd := simplify\left(\frac{\left(\kappa \cdot \sin(\theta) \cdot \left(d\theta^2 - \cos(\theta)\right) + B_d \cdot \varepsilon \cdot (\lambda_d)^2 \cdot (\rho) + \gamma \cdot B_d \cdot u_{dd}\right)}{\left(1 - \frac{(\kappa + 1) \cdot (\cos(\theta))^2}{\lambda_d}\right)}, 'size'\right)$:

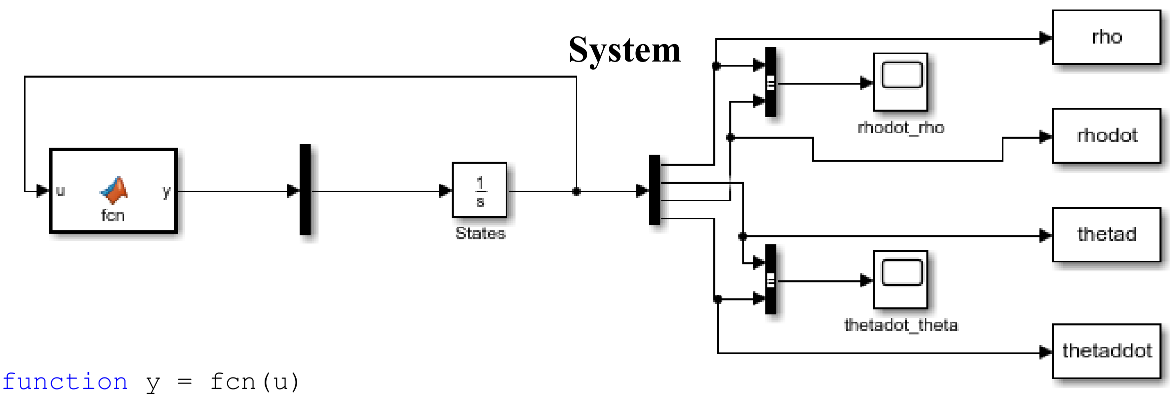
C.2 Simulink file and MATLAB code for the simulation of the Full System




```matlab
function y = fcn(u)
%% Main Vectors
x             = u(1);               % feedback array
theta         = u(2);
xdot          = u(3);
tdot          = u(4);
%% Generalized quantities
q             = [x theta]';         % Generalized coordinates
qdot          = [xdot tdot]'; % Generalized velocities
%% Physical parameter values
Mc            = .44;                  % kg       - cart mass
Mp            = .14;                  % kg       - pendulum mass
g             = 9.81;                % m/s^2    - gravity
l             = 0.215;                % m        - pendulum length
alpha         = Mp*l^2;
beta          = Mp*l;
gamma         = Mc+Mp;
d             = -Mp*g*l;
%% Linear model parameters
c             = 0.015;%0.001;
kappa         = 20;%50;
epsilon       = 0.00001;
rho           = -0.02;%1;
%psi            = 0;
%% The G,M,C,  P and KD matrices
G             = [0; d*sin(theta)]; %gravity terms
mass          = [gamma, beta*cos(theta); beta*cos(theta), alpha];%mass matrix
C             = [0, -beta*sin(theta)*tdot; 0, 0];    %Centripetal and coriolis
matrix
%inclined plane
% G             = [-sin(psi)*(Mp+Mc)*g; -Mp*l*g*sin(theta)]; %gravity terms
% mass          = [Mc+Mp, Mp*l*cos(-theta+psi); Mp*l*cos(-theta+psi),
Mp*l^2];%mass matrix
% C             = [0, -Mp*l*sin(-theta+psi)*tdot; 0, 0];    %Centripetal and
coriolis matrix
%% Evaluate the control law
B              = 1/rho*(alpha-beta^2/gamma*(cos(theta))^2);
p              = (kappa+(rho-1)/rho)*(beta/gamma);
ud             = c*gamma*(xdot+p*cos(theta)*tdot);
tau            = (kappa*beta*sin(theta)*(alpha*tdot^2+d*cos(theta))-B*...
               epsilon*d*gamma^2*x/beta^2+B*ud)/...
               (alpha-beta^2/gamma*(kappa+1)*(cos(theta))^2);
```

```
%inclined plane
% tau        = (kappa*Mp*l*(-Mp*l^2*sin(-theta+psi)*tdot^2-cos(-theta+...
%              psi)*Mp*g*l*sin(theta))+(Mp*l^2-Mp^2*l^2*cos(-theta+psi)^2/...
%              (Mc+Mp))*epsilon*g*(Mc+Mp)^2*(x+(kappa+(rho-1)/rho)*Mp*l*(-...
%              sin(-theta+psi)+sin(psi))/(Mc+Mp))/(rho*Mp*l)+(Mp*l^2-Mp^2*...
%              l^2*cos(-theta+psi)^2/(Mc+Mp))*c*(Mc+Mp)*(xdot+(kappa+(rho-...
%              1)/rho)*Mp*l*cos(-theta+psi)*tdot/(Mc+Mp))/rho)/(Mp*...
%              l^2-Mp^2*l^2*(kappa+1)*cos(-theta+psi)^2/(Mc+Mp))-...
%              (Mc+Mp)*g*sin(psi);
%% Evaluate the Dynamic
qdotdot      = inv(mass)*([tau;0]-C*qdot-G);
xdotdot      = qdotdot(1);
ddtheta      = qdotdot(2);
%% M-File output
y            = [xdot;tdot;xdotdot;ddtheta];
%% End of
```

C.3 Simulink file and MATLAB code for the simulation of the Dimensionless System




```
function y = fcn(u)
%% Main Vectors
x            = u(1);              % feedback array
theta        = u(2);
xdot         = u(3);
tdot         = u(4);
%% Generalized quantities
q            = [x theta]';        % Generalized coordinates
qdot         = [xdot tdot]'; % Generalized velocities
%% Physical parameter values
Mc           = .44;                 % kg      - cart mass
Mp           = .14;                 % kg      - pendulum mass
g            = 9.81;               % m/s^2   - gravity
l            = 0.215;               % m       - pendulum length
k1           = Mc/Mp;
dn           = -1;
%d             = g/l;
gn           = 1+k1;
%% Linear model parameters
c            = 0.015;
kappa        = 20;
epsilon      = 0.00001;
rho          = -0.02;
%% The G,M,C,  P and KD matrices
```

```
G           = [0; dn*sin(theta)];
mass        = [gn, cos(theta); cos(theta),1];
C           = [0, -sin(theta)*tdot; 0, 0];
%% Evaluate the control law
Bn           = 1/rho*(1-(cos(theta))^2/gn);
pn           = (kappa+(rho-1)/rho)/gn;
udn          = sqrt(l/g)*c*gn*(xdot+pn*cos(theta)*tdot);
tau          = (kappa*sin(theta)*(tdot^2+dn*cos(theta))-Bn*epsilon*...
                dn*gn^2*x+Bn*udn)*gn/(gn-(kappa+1)*(cos(theta))^2);
%% Evaluate the Dynamic
qdotdot     = inv(mass)*([tau;0]-C*qdot-G);
xdotdot     = qdotdot(1);
ddtheta     = qdotdot(2);
%% M-File output
y           = [xdot;tdot;xdotdot;ddtheta];
%% End of
```

C.4 MATLAB code to produce the plot for Chapter 4

```
l=0.215;
g=9.8;
figure(1);
plot(x.time,x.signals.values,'-k',rho.time*sqrt(l/g),l*rho.signals.values,'--
k','LineWidth',1)
grid on
legend('x','\rho')
title('Position Response of the Cart for the Full vs Dimensionless Systems');
xlabel('time(s) and time(dimensionless)*\gamma');
ylabel('x(m), \rho(dimensionless)*l(m)');

figure(2);
plot(theta.time,theta.signals.values,'-k',thetad.time*sqrt(l/g),
thetad.signals.values,'--k','LineWidth',1);
grid on
legend('\theta','\theta_{N}')
title('Theta Responses for Full vs Dimensionless Equations of Motion');
xlabel('time(s) and time(dimensionless)*\gamma');
ylabel('angle (rad)');
```

Appendix D - Rotary Pendulum

This Appendix is organized in six major parts. These are:

D.1 Derivations of Equations of Motion and Controller

D.2 Simulink file and MATLAB code for the simulation of the Full System

D.3 Simulink file and MATLAB code for the simulation of the Dimensionless System

D.4 Simulink file and MATLAB code for the simulation of the Morphed System

D.5 MATLAB code to produce the dimensionless plot for Chapter 4

D.6 MATLAB code to produce the morphed plot for Chapter 4

D.1 Derivations of Equations of Motion and Controller

RP.mw

```
> restart :
> with(LinearAlgebra) :
```

Rotary Pendulum

Lagrangian derivation point mass for arc point mass for pendulum

> $T := \frac{1}{2} \cdot m_p \cdot l^2 \cdot d\theta^2 + \frac{1}{2} \cdot (m_a + m_p) \cdot R^2 \cdot d\phi^2 + m_p \cdot l \cdot R \cdot \cos(\theta) \cdot d\phi \cdot d\theta + \frac{1}{2} \cdot m_p \cdot l^2 \cdot (\sin(\theta))^2 \cdot d\phi^2$:

Potential Energy

> $V := m_p \cdot g \cdot l \cdot \cos(\theta)$:

Lagrange

> $L := expand(T - V)$:

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $e1 := diff(L, d\phi)$:

> $e1 := subs(\{\theta = \theta(t), \phi = \phi(t)\}, e1)$:

> $e1 := subs(\{d\theta = diff(\theta(t), t), d\phi = diff(\phi(t), t)\}, e1)$:

> $e2 := diff(e1, t)$:

> $e2 := subs(\{diff(\theta(t), t, t) = dd\theta, diff(\phi(t), t, t) = dd\phi\}, e2)$:

> $e2 := subs(\{diff(\theta(t), t) = d\theta, diff(\phi(t), t) = d\phi\}, e2)$:

> $e2 := subs(\{\theta(t) = \theta, \phi(t) = \phi\}, e2)$:

> $variables := [dd\theta, dd\phi, d\theta, d\phi]$:

> $Eq_{arm} := collect(simplify(e2 - diff(L, \phi) - \tau, 'trig'), variables)$

$$Eq_{arm} := m_p\, l\, R \cos(\theta)\, dd\theta + \left(-m_p\, l^2 \cos(\theta)^2 + m_a R^2 + R^2 m_p + m_p\, l^2\right) dd\phi + 2\, m_p\, l^2 \sin(\theta)\, d\phi \cos(\theta)\, d\theta - m_p\, l\, R \sin(\theta)\, d\theta^2 - \tau \qquad \textbf{(1.1.1)}$$

> $e3 := diff(L, d\theta) :$

> $e3 := subs(\{\theta = \theta(t),\ \phi = \phi(t)\}, e3) :$

> $e3 := subs(\{d\theta = diff(\theta(t), t),\ d\phi = diff(\phi(t), t)\}, e3) :$

> $e4 := diff(e3, t) :$

> $e4 := subs(\{diff(\theta(t), t, t) = dd\theta,\ diff(\phi(t), t, t) = dd\phi\}, e4) :$

> $e4 := subs(\{diff(\theta(t), t) = d\theta,\ diff(\phi(t), t) = d\phi\}, e4) :$

> $e4 := subs(\{\theta(t) = \theta,\ \phi(t) = \phi\}, e4) :$

> $Eq_{ball} := collect(simplify(e4 - diff(L, \theta), 'trig'), variables)$

$$Eq_{ball} := m_p\, l^2\, dd\theta + m_p\, l\, R\cos(\theta)\, dd\phi - m_p\, l^2 \sin(\theta)\, d\phi^2 \cos(\theta) - m_p\, g\, l \sin(\theta) \tag{1.1.2}$$

> $M := \begin{bmatrix} coeff(Eq_{arm}, dd\phi) & coeff(Eq_{arm}, dd\theta) \\ coeff(Eq_{ball}, dd\phi) & coeff(Eq_{ball}, dd\theta) \end{bmatrix}$

$$M := \begin{bmatrix} -m_p\, l^2 \cos(\theta)^2 + m_a R^2 + R^2 m_p + m_p\, l^2 & m_p\, l\, R\cos(\theta) \\ m_p\, l\, R\cos(\theta) & m_p\, l^2 \end{bmatrix} \tag{1.1.3}$$

> $M := \begin{bmatrix} m_p\, l^2 \sin(\theta)^2 + (m_a + m_p)\cdot R^2 & m_p \cdot l \cdot R\cos(\theta) \\ m_p \cdot l \cdot R\cos(\theta) & m_p\, l^2 \end{bmatrix} :$

> $C := \left[\left[coeff\left(Eq_{arm}, d\phi^2\right)\cdot d\phi + \frac{1}{2}\cdot coeff\left(Eq_{arm}, d\phi\right), coeff\left(Eq_{arm}, d\theta^2\right)\cdot d\theta + \frac{1}{2}\cdot coeff\left(Eq_{arm}, d\theta\right)\right], \left[coeff\left(Eq_{ball}, d\phi^2\right)\cdot d\phi, coeff\left(Eq_{ball}, d\theta\right)\right]\right]$

$$C := \begin{bmatrix} m_p\, l^2 \sin(\theta)\cos(\theta)\, d\theta & -m_p\, l\, R\sin(\theta)\, d\theta + m_p\, l^2 \sin(\theta)\, d\phi\cos(\theta) \\ -m_p\, l^2 \sin(\theta)\, d\phi\cos(\theta) & 0 \end{bmatrix} \tag{1.1.4}$$

> $G := \begin{bmatrix} coeff(Eq_{arm}, g)\cdot g \\ coeff(Eq_{ball}, g)\cdot g \end{bmatrix}$

$$G := \begin{bmatrix} 0 \\ -m_p\, g\, l \sin(\theta) \end{bmatrix} \tag{1.1.5}$$

> $F := -\begin{bmatrix} coeff(Eq_{arm}, \tau)\cdot \tau \\ coeff(Eq_{ball}, \tau)\cdot \tau \end{bmatrix}$

$$F := \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{1.1.6}$$

> $dq := \begin{bmatrix} d\phi \\ d\theta \end{bmatrix} : ddq := \begin{bmatrix} dd\phi \\ dd\theta \end{bmatrix} :$

> $\#\tau{:=}u1+u2:$

> $eq := Multiply(M, ddq) + Multiply(C, dq) + G - \begin{bmatrix} \tau \\ 0 \end{bmatrix}$

$$eq := \Big[\Big[\big(m_p\, l^2 \sin(\theta)^2 + (m_a + m_p)\, R^2\big)\, dd\phi + m_p\, l\, R \cos(\theta)\, dd\theta + m_p\, l^2 \sin(\theta)\, d\phi \cos(\theta)\, d\theta + \big(-m_p\, l\, R \sin(\theta)\, d\theta + m_p\, l^2 \sin(\theta)\, d\phi \cos(\theta)\big)\, d\theta - \tau\Big], \Big[m_p\, l^2\, dd\theta + m_p\, l\, R \cos(\theta)\, dd\phi - m_p\, l^2 \sin(\theta)\, d\phi^2 \cos(\theta) - m_p\, g\, l \sin(\theta)\Big]\Big] \quad \textbf{(1.1.7)}$$

controller for rotary pendulum

> $\#\tau{:=}u1+u2:$

> $\#arp{:=}m_p \cdot l^2:$

> $\#drp{:=}-m_p \cdot g \cdot l:$

> $\#srp{:=}(m_a + m_p) \cdot R^2:$

> $\#brp{:=}m_p \cdot l \cdot R:$

> $Falpha := arp \cdot \sin(\theta) \cdot \cos(\theta) \cdot d\phi^2 :$

> $Brp := simplify\left(\frac{1}{\rho} \cdot \left(arp - \frac{brp^2 \cdot (\cos(\theta))^2}{srp}\right), 'size'\right) :$

> $prp := simplify\left(\left(\kappa + \frac{(\rho - 1)}{\rho}\right) \cdot \left(\frac{brp}{srp}\right), 'size'\right) :$

> $udrp := simplify(c \cdot srp \cdot (d\phi + prp \cdot \cos(\theta) \cdot d\theta), 'size') :$

> $$u2 := simplify\left(\frac{\kappa \cdot brp \cdot \sin(\theta) \cdot \left(arp \cdot d\theta^2 + drp \cdot \cos(\theta)\right) - \frac{Brp \cdot \varepsilon \cdot drp \cdot srp^2 \cdot \phi}{brp^2} + Brp\, udrp}{\left(arp - \left(\frac{brp^2 \cdot (\kappa + 1) \cdot \cos(\theta)^2}{srp}\right)\right)} - \frac{\kappa \cdot brp \cdot Falpha \cdot \cos(\theta)}{\left(arp - \left(\frac{brp^2 \cdot (\kappa + 1) \cdot \cos(\theta)^2}{srp}\right)\right)}, 'size'\right) :$$

> $u1 := arp \cdot \sin(\theta)^2 \cdot dd\phi + arp \cdot \sin(\theta) \cdot \cos(\theta) \cdot d\phi \cdot d\theta :$

> $urp := simplify(u1 + u2, 'size') :$

Dimensionless equations

> $eq1 := \dfrac{eq_1}{m_p\, l\, R}$:

> $eq2 := \dfrac{eq_2}{m_p\, l^2}$:

> $Md := \begin{bmatrix} coeff(eq1, dd\phi) & coeff(eq1, dd\theta) \\ coeff(eq2, dd\phi) & coeff(eq2, dd\theta) \end{bmatrix}$

$$Md := \begin{bmatrix} \dfrac{m_p\, l^2 \sin(\theta)^2 + (m_a + m_p)\, R^2}{m_p\, l\, R} & \cos(\theta) \\ \dfrac{R\cos(\theta)}{l} & 1 \end{bmatrix} \quad \textbf{(1.3.1)}$$

> $Md := \begin{bmatrix} \dfrac{(m_a + m_p)}{m_p}\cdot\dfrac{R}{l} + \dfrac{l}{R}\cdot\left(\sin\left(\theta\right)\right)^2 & \cos(\theta) \\ \dfrac{R}{l}\cdot\cos\left(\theta\right) & 1 \end{bmatrix}$:

> $Md := \begin{bmatrix} (1 + k_1) + k_2\cdot(\sin(\theta))^2 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix}$:

> $Cd := \begin{bmatrix} coeff\left(eq1, d\phi^2\right)\cdot d\phi + \dfrac{1}{2}\cdot coeff\left(eq1, d\phi\right) & coeff\left(eq1, d\theta^2\right)\cdot d\theta + \dfrac{1}{2}\cdot coeff\left(eq1, d\theta\right) \\ coeff\left(eq2, d\phi^2\right)\cdot d\phi & coeff(eq2, d\theta) \end{bmatrix}$

$$Cd := \begin{bmatrix} \dfrac{l\sin(\theta)\cos(\theta)\, d\theta}{R} & -\sin(\theta)\, d\theta + \dfrac{l\sin(\theta)\, d\phi\cos(\theta)}{R} \\ -\sin(\theta)\cos(\theta)\, d\phi & 0 \end{bmatrix} \quad \textbf{(1.3.2)}$$

> $Cd :=$
$$\begin{bmatrix} \left(\dfrac{l}{R}\right)^2 \sin(\theta)\cos(\theta)\, d\theta\left(\dfrac{R}{l}\right) & -\sin(\theta)\, d\theta + \left(\dfrac{l}{R}\right)^2 \sin(\theta)\left(\dfrac{R}{l} d\phi\right)\cos(\theta) \\ -\left(\dfrac{l}{R}\right)^2 \sin(\theta)\cos(\theta)\left(\dfrac{R}{l} d\phi\right)\cdot\left(\dfrac{R}{l}\right) & 0 \end{bmatrix}$$:

> $Cd := \begin{bmatrix} k_2^2\cdot\sin(\theta)\cos(\theta)\, d\theta & -\sin(\theta)\, d\theta + k_2^2\cdot\sin(\theta)\cos(\theta)\, d\phi \\ -\sin(\theta)\cos(\theta)\, k_2^2\cdot d\psi & 0 \end{bmatrix}$:

> $Gd := \begin{bmatrix} coeff(eq1, g)\cdot g \\ coeff(eq2, g)\cdot g \end{bmatrix}$

$$Gd := \begin{bmatrix} 0 \\ -\dfrac{\sin(\theta)\, g}{l} \end{bmatrix} \quad \textbf{(1.3.3)}$$

> $Gd := \begin{bmatrix} 0 \\ -\sin(\theta) \end{bmatrix}$:

> $Fd := -\begin{bmatrix} coeff(eq1, \tau) \cdot \tau \\ coeff(eq2, \tau) \cdot \tau \end{bmatrix}$

$$Fd := \begin{bmatrix} \dfrac{\tau}{m_p \, l \, R} \\ 0 \end{bmatrix} \tag{1.3.4}$$

> $Fd := \begin{bmatrix} \dfrac{\tau \cdot \gamma^2}{m_p \, l \, R} \\ 0 \end{bmatrix}$:

Dimensionless controller

> $\#arp := m_p \cdot l^2$:

> $\#arpn := m_p \cdot l^2 \cdot arpd$:

> $\#arpd := 1$:

> $\#drp := -m_p \cdot g \cdot l$:

> $\#drpn := -m_p \cdot g \cdot l \cdot drp$:

> $\#drpd := -1$:

> $\#srp := (m_a + m_p) \cdot R^2$:

> $\#srpn := m_p \cdot R^2 \cdot sprd$:

> $\#srpd := (k_6 + 1)$:

> $\#brp := m_p \cdot l \cdot R$:

> $\#brpn := m_p \cdot l \cdot R \cdot brpd$

> $\#brpd := 1$:

> $\#Falpha := arp \cdot sin(\theta) \cdot \cos(\theta) \cdot d\phi^2$:

> $\#Falphan := m_p \cdot l^2 \cdot \left(\frac{l}{R}\right)^2 \cdot Falphad$:

> $\#Falphad := sin(\theta) \cdot \cos(\theta) d\psi^2$:

> $\#Brp := simplify\left(\frac{1}{\rho} \cdot \left(arp - \frac{brp^2 \cdot (\cos(\theta))^2}{srp}\right), 'size'\right)$:

> $\#Brpn := m_p \cdot l^2 \cdot Brpd$:

> $\#Brpd := \left(\frac{1}{\rho} \cdot \left(1 - \frac{(\cos(\theta))^2}{srpd}\right)\right)$:

> $\#prp := simplify\left(\left(\kappa + \frac{(\rho - 1)}{\rho}\right) \cdot \left(\frac{brp}{srp}\right), 'size'\right)$:

> $\#udrp := simplify\left(c \cdot srp \cdot \left(d\phi + prp \cdot \cos(\theta) \cdot d\theta\right), 'size'\right):$

> $\#udrpn := m_p \cdot l \cdot R \cdot udrpd:$

> $\#udrpd := \left(c \cdot srpd \cdot \left(d\psi + prpd \cdot \cos(\theta) \cdot d\theta\right)\right):$

> $$\#u2rp := simplify\left(\frac{\kappa \cdot \sin(\theta) \cdot \left(arp \cdot d\theta^2 + drp \cdot \cos(\theta)\right) - \frac{Brp \cdot \varepsilon \cdot drp \cdot srp^2 \cdot \phi}{brp^2} + Brp\, udrp}{\left(arp - \left(\frac{brp^2 \cdot (\kappa + 1) \cdot \cos(\theta)^2}{srp}\right)\right)} - \frac{\kappa \cdot brp \cdot Falpha \cdot \cos(\theta)}{\left(arp - \left(\frac{brp^2 \cdot (\kappa + 1) \cdot \cos(\theta)^2}{srp}\right)\right)}, 'size'\right):$$

> $\#u2rpn := m_p \cdot l \cdot R \cdot u2rpd:$

> $$u2rpd := \left(\frac{1}{\left(1 - \left(\frac{(\kappa + 1) \cdot \cos(\theta)^2}{srpd}\right)\right)}\left(\left(\kappa \cdot \sin(\theta) \cdot \left(d\theta^2 + drpd \cdot \cos(\theta)\right) - Brpd \cdot \varepsilon \cdot drpd \cdot srpd^2 \cdot \psi + Brpd\, udrpd - k_2^2\, \kappa \cdot Falphad \cdot \cos(\theta)\right)\right)\right):$$

> $\#u1 := arp \cdot \left(\sin(\theta)^2 \cdot dd\phi + \sin(\theta) \cdot \cos(\theta) \cdot d\phi \cdot d\theta\right):$

> $\#u1n := m_p \cdot l \cdot R \cdot k_2^2 \cdot u1d:$

> $u1d := k_2^2 \cdot \left(\sin(\theta)^2 \cdot dd\psi + \sin(\theta) \cdot \cos(\theta) \cdot d\psi \cdot d\theta\right):$

morphing

> $k_2 := 0:$

> $u1d$

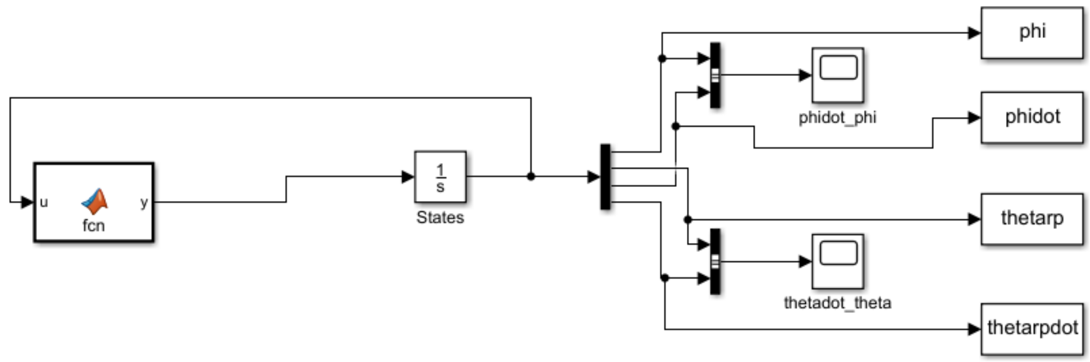
$$0 \qquad \textbf{(1.5.1)}$$

> $drpd := -1:$

> $u2rpd$

$$\frac{\kappa \sin(\theta)\, \left(d\theta^2 - \cos(\theta)\right) + Brpd\, \varepsilon\, srpd^2\, \psi + Brpd\, udrpd}{1 - \frac{(\kappa + 1)\, \cos(\theta)^2}{srpd}} \qquad \textbf{(1.5.2)}$$

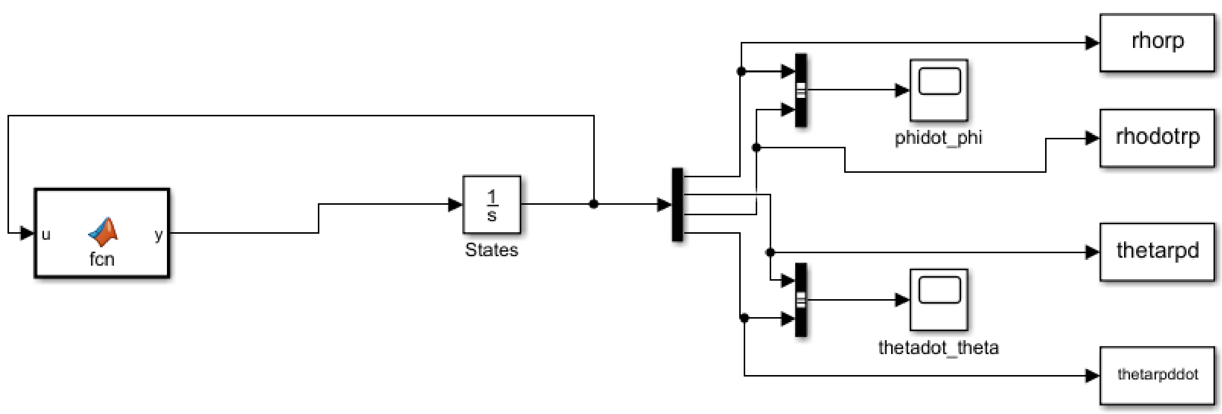
D.2 Simulink file and MATLAB code for the simulation of the Full System




```
function y = fcn(u)
%% Main Vectors
phi             = u(1);             % feedback array
theta           = u(2);
phidot          = u(3);
thetadot        = u(4);
%% Generalized quantities
q             = [phi theta]';       % Generalized coordinates
qdot          = [phidot thetadot]'; % Generalized velocities
%% Physical parameter values
Md            = 0.44;                % kg       - cart mass
Mp            = 0.14;                % kg       - pendulum mass
g             = 9.8;                 % m/s^2    - gravity
l             = 0.215;               % m        - pendulum length
R             = 1;
alpha         = Mp*l^2;
beta          = Mp*l*R;
gamma         = (Mp+Md)*R^2;
D             = -Mp*g*l;
%% Linear model parameters
kappa         = 25;
psi           = -0.02;
c             = 0.015;
epsilon       = 0.00001;
%% The G,M,C,  P and KD matrices
G             = [0; D*sin(theta)]; %gravity terms
mass          = [gamma+alpha*(sin(theta))^2, beta*cos(theta);...
               beta*cos(theta), alpha ];%mass matrix
C             =  [alpha*cos(theta)*sin(theta)*thetadot,...
               alpha*cos(theta)*sin(theta)*phidot-
beta*sin(theta)*thetadot;...
               -alpha*cos(theta)*sin(theta)*phidot, 0];     %Centripetal and
coriolis matrix
%% Evaluate the control law
Falpha       = alpha*sin(theta)*cos(theta)*phidot^2;
Fa           = -alpha*sin(theta)*cos(theta)*phidot*thetadot;
 p            = (kappa+(psi-1)/psi)*(beta/gamma);
 ud           = c*gamma*(phidot+p*cos(theta)*thetadot);
```

```
 B            = (alpha-beta^2*(cos(theta))^2/gamma)/psi;
 u2           = (kappa*beta*sin(theta)*(alpha*thetadot^2+D*cos(theta))-(B*...
                epsilon*D*gamma^2*phi)/beta^2+B*ud-kappa*beta*Falpha*cos(...
                theta))/(alpha-(beta^2*(kappa+1)*(cos(theta))^2)/gamma);
 qdd           = inv(mass)*([u2;0]-C*qdot-G);
 phidd         = qdd(1);
 u1            = alpha*sin(theta)^2*phidd+Fa;
 tau           = u2+u1;
%% Evaluate the Dynamic
qdotdot      = inv(mass)*([tau;0]-C*qdot-G);
ddphi        = qdotdot(1);
ddtheta      = qdotdot(2);
%% M-File output
y            = [phidot;thetadot;ddphi;ddtheta];
%% End of
```

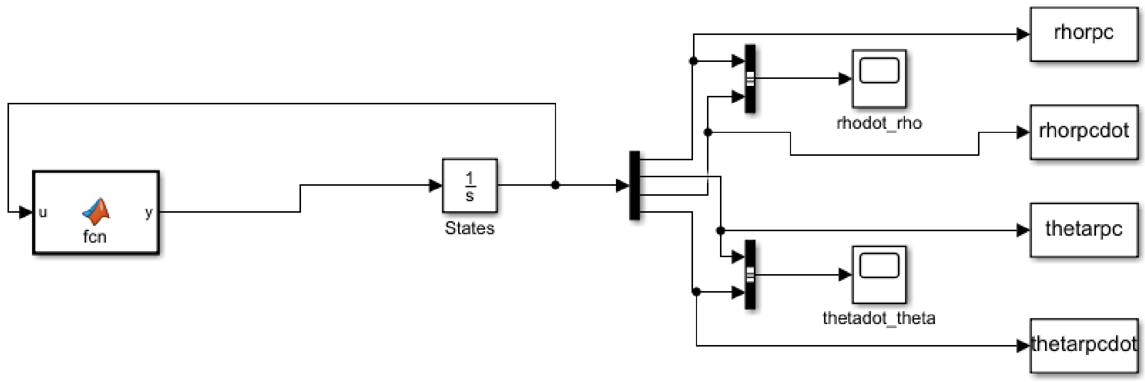
D.3 Simulink file and MATLAB code for the simulation of the Dimensionless System




```
function y = fcn(u)
%% Main Vectors
rho            = u(1);            % feedback array
theta          = u(2);
rhodot         = u(3);
thetadot       = u(4);
%% Generalized quantities
q            = [rho theta]';      % Generalized coordinates
qdot         = [rhodot thetadot]'; % Generalized velocities
%% Physical parameter values
Ma           = 0.44;               % kg       - cart mass
Mp           = 0.14;               % kg       - pendulum mass
g            = 9.81;               % m/s^2    - gravity
l            = 0.215;              % m        - pendulum length
R            = 1;

k6           = Ma/Mp;
```

```
k4          = 1/R;
gn          = 1+k6;
%% Linear model parameters
kappa       = 25;
psi         = -0.02;
c           = 0.015;
epsilon     = 0.00001;
%% The G,M,C,  P and KD matrices
G           = [0; -sin(theta)]; %gravity terms
mass        = [1+k6+k4^2*(sin(theta))^2 cos(theta); cos(theta) 1];%mass
matrix
C           =  [k4^2*cos(theta)*sin(theta)*thetadot...
                k4^2*cos(theta)*sin(theta)*rhodot-sin(theta)*thetadot;...
               -k4^2*cos(theta)*sin(theta)*rhodot 0];    %Centripetal and
coriolis matrix
%% Evaluate the control law
Falpha     = sin(theta)*cos(theta)*rhodot^2;
Fa         = sin(theta)*cos(theta)*rhodot*thetadot;
p          = (kappa+(psi-1)/psi)/gn;
ud         = c*gn*(rhodot+p*cos(theta)*thetadot);
B          = (1-(cos(theta))^2/gn)/psi;

u2         = (kappa*sin(theta)*(thetadot^2-cos(theta))+...
             B*epsilon*((1+k6)^2)*rho+B*ud*sqrt(l/g)-...
             kappa*k4^2*Falpha*cos(theta))/...
             (1-((kappa+1)*(cos(theta))^2)/gn);

qdd         = inv(mass)*([u2;0]-C*qdot-G);
rhodd       = qdd(1);
u1          = k4^2*(sin(theta)^2*rhodd-Fa);
tau         = u2+u1;
%% Evaluate the Dynamic
qdotdot     = inv(mass)*([tau;0]-C*qdot-G);
ddrho       = qdotdot(1);
ddtheta     = qdotdot(2);
%% M-File output
y           = [rhodot;thetadot;ddrho;ddtheta];
%% End of
```

D.4 Simulink file and MATLAB code for the simulation of the Morphed System




```
function y = fcn(u)
%% Main Vectors
rho            = u(1);               % feedback array
theta          = u(2);
rhodot         = u(3);
thetadot       = u(4);
%% Generalized quantities
q              = [rho theta]';        % Generalized coordinates
qdot           = [rhodot thetadot]'; % Generalized velocities
%% Physical parameter values
Mc             = 0.44;                 % kg       - cart mass
Mp             = 0.14;                 % kg       - pendulum mass
k6             = Mc/Mp;
l              = 0.215;
R              = 1;
g              = 9.81;
k4             = l/R;
gn             = 1+k6;
dn             = -1;
%% Linear model parameters
kappa          = 20;
psi            = -0.02;
c              = 0.015;
epsilon        = 0.00001;
%% The G,M,C,  P and KD matrices
G              = [0; -sin(theta)];
mass           = [1+k6, cos(theta); cos(theta),1];
C              = [0, -sin(theta)*thetadot; 0, 0];
%% Evaluate the control law
p             = (kappa+(psi-1)/psi)/gn;
ud            = c*gn*(rhodot+p*cos(theta)*thetadot);
B             = (1-(cos(theta))^2/gn)/psi;
```

```
tau          = (kappa*sin(theta)*(thetadot^2-cos(theta))+...
                B*epsilon*((1+k6)^2)*rho+B*ud*sqrt(l/g))/...
                (1-((kappa+1)*(cos(theta))^2)/gn);
%% Evaluate the Dynamic
qdothetadot      = inv(mass)*([tau;0]-C*qdot-G);
ddrho        = qdothetadot(1);
ddtheta      = qdothetadot(2);
%% M-File output
y            = [rhodot;thetadot;ddrho;ddtheta];
%% End of
```

D.5 MATLAB code to produce the dimensionless plot for Chapter 4

```
l=0.215;
g=9.81;
R=1;
%T=t*sqrt(g/l);
figure(3);
plot(phi.time,phi.signals.values,'-
k',rhorp.time*sqrt(l/g),l/R*rhorp.signals.values,'--k','LineWidth',1)
grid on
legend('\phi','\rho')
title('Position of Arm Responses to Full vs Dimensionless Controllers');
xlabel('time(s) and time(dimensionless)*\gamma');
ylabel('Arm Position(rad)');

figure(4);
plot(thetarp.time,thetarp.signals.values,'-k',thetarpd.time*sqrt(l/g),
thetarpd.signals.values,'--k','LineWidth',1);
grid on
legend('\theta','\theta_{n}')
title('Theta Responses to Full vs Dimensionless Controllers');
xlabel('time(s) and time(dimensionless)*\gamma');
ylabel('Pendulum Position(rad)');
```

D.6 MATLAB code to produce the morphed plot for Chapter 4

```
l= 0.215;
g=9.81;
figure(5);
plot(x.time,x.signals.values,'-
k',rhorpc.time*sqrt(l/g),rhorpc.signals.values*l,'--k','LineWidth',1)
grid on
legend('IPC','RP')
title('Cart Position Response for Inverted Pendulum Cart vs Morphed Rotary
Pendulum');
xlabel('time(s) and time(dimensionless)*\gamma');
ylabel('x(m), \rho(dimensionless)*l(m)');

figure(6);
plot(theta.time,theta.signals.values,'-k',thetarpc.time*sqrt(l/g),
thetarpc.signals.values,'--k','LineWidth',1);
grid on
legend('IPC','RP')
```

```
title('Pendulum Angle Response for Inverted Pendulum Cart vs Morphed Rotary
Pendulum');
xlabel('time(s) and time(dimensionless)*\gamma');
ylabel('Pendulum Angle(rad)');
```

Appendix E - Pendubot

This Appendix is organized in six major parts. These are:

E.1 Derivations of Equations of Motion and Controller

E.2 Simulink file and MATLAB code for the simulation of the Full System

E.3 Simulink file and MATLAB code for the simulation of the Dimensionless System

E.4 Simulink file and MATLAB code for the simulation of the Morphed System

E.5 MATLAB code to produce the plots for Chapter 5

E.1 Derivations of Equations of Motion and Controller

Pendubot_IDAPBC.mw

> $e3 := diff(L, dq2)$:

> $e3 := subs(\{q1 = q1(t), q2 = q2(t)\}, e3)$:

> $e3 := subs(\{dq1 = diff(q1(t), t), dq2 = diff(q2(t), t)\}, e3)$:

> $e4 := diff(e3, t)$:

> $e4 := subs(\{diff(q1(t), t, t) = ddq1, diff(q2(t), t, t) = ddq2\}, e4)$:

> $e4 := subs(\{diff(q1(t), t) = dq1, diff(q2(t), t) = dq2\}, e4)$:

> $e4 := subs(\{q1(t) = q1, q2(t) = q2\}, e4)$:

> $Eq_{link2} := collect(simplify(e4 - diff(L, q2), 'trig'), variables)$

$$Eq_{link2} := \left(m2\,Lc2\,L1\cos(q2) + m2\,Lc2^2 + I2\right) ddq1 + \left(m2\,Lc2^2 + I2\right) ddq2 + m2\,L1\,dq1^2\,Lc2\sin(q2) + m2\,g\,Lc2\sin(q1)\cos(q2) + m2\,g\,Lc2\cos(q1)\sin(q2) \tag{1.3}$$

> $M1 := simplify\left(coeff\left(Eq_{link1}, ddq1\right), 'size'\right)$

$$M1 := 2\,m2\,Lc2\,L1\cos(q2) + \left(L1^2 + Lc2^2\right) m2 + Lc1^2\,m1 + I1 + I2 \tag{1.4}$$

> $M2 := simplify\left(coeff\left(Eq_{link1}, ddq2\right), 'size'\right)$

$$M2 := m2\,Lc2\,L1\cos(q2) + m2\,Lc2^2 + I2 \tag{1.5}$$

> $M3 := simplify\left(coeff\left(Eq_{link2}, ddq1\right), 'size'\right)$

$$M3 := m2\,Lc2\,L1\cos(q2) + m2\,Lc2^2 + I2 \tag{1.6}$$

> $M4 := simplify\left(coeff\left(Eq_{link2}, ddq2\right), 'size'\right)$

$$M4 := m2\,Lc2^2 + I2 \tag{1.7}$$

> $\#c1 := m1 \cdot Lc1^2 + m2 \cdot L1^2 + I1$:

> $\#c2 := m2 \cdot Lc2^2 + I2$:

> $\#c3 := m2 \cdot L1 \cdot Lc2$:

> $M1 := 2 \cdot c3 \cdot \cos(q2) + c1 + c2$:

> $M2 := c3 \cdot \cos(q2) + c2$:

> $M3 := c3 \cdot \cos(q2) + c2$:

> $M4 := c2$:

> $C1 := simplify\left(coeff\left(Eq_{link1}, dq1^2\right) \cdot dq1 + \frac{1}{2} \cdot coeff\left(Eq_{link1}, dq1\right), 'size'\right)$

$$C1 := -m2\,L1\,Lc2\,dq2\sin(q2) \tag{1.8}$$

> $C2 := simplify\left(coeff\left(Eq_{link1}, dq2^2\right) \cdot dq2 + \frac{1}{2} \cdot coeff\left(Eq_{link1}, dq2\right), 'size'\right)$

$$C2 := -L1\,Lc2\sin(q2)\,m2\,(dq1 + dq2) \tag{1.9}$$

> $C3 := simplify\left(coeff\left(Eq_{link2}, dq1^2\right) \cdot dq1 + \frac{1}{2} \cdot coeff\left(Eq_{link2}, dq1\right), 'size'\right)$

$$C3 := m2\,L1\,dq1\,Lc2\sin(q2) \tag{1.10}$$

> $C4 := simplify\left(coeff\left(Eq_{link2}, dq2^2\right) \cdot dq2 + \frac{1}{2} \cdot coeff\left(Eq_{link2}, dq2\right), 'size'\right)$

$$C4 := 0 \tag{1.11}$$

> $C1 := -c3 \cdot \sin(q2) \cdot dq2$:

> $C2 := -c3 \cdot \sin(q2) \cdot (dq1 + dq2)$:

> $C3 := c3 \cdot \sin(q2) \cdot dq1$:

> $C := \begin{bmatrix} C1 & C2 \\ C3 & C4 \end{bmatrix}$:

> $G1 := simplify\left(coeff\left(Eq_{link1}, g\right) \cdot g, 'size'\right)$

$$G1 := g\left((\cos(q2)\,m2\,Lc2 + m2\,L1 + Lc1\,m1)\sin(q1) + m2\,Lc2\cos(q1)\sin(q2)\right) \tag{1.12}$$

> $G2 := simplify\left(coeff\left(Eq_{link2}, g\right) \cdot g, 'size'\right)$

$$G2 := Lc2\,m2\,(\sin(q1)\cos(q2) + \cos(q1)\sin(q2))\,g \tag{1.13}$$

> $G := \begin{bmatrix} G1 \\ G2 \end{bmatrix}$:

> tau := - $\begin{bmatrix} coeff(Eq_{link1}, u) \cdot u \\ coeff(Eq_{lin21}, u) \cdot u \end{bmatrix}$

$$\tau := \begin{bmatrix} u \\ 0 \end{bmatrix} \qquad \textbf{(1.14)}$$

> #c4:=m1·Lc1+m2·L1:

> #c5:=m2·Lc2:

> expand(V)

$$-g\,m1\,Lc1\cos(q1) - m2\,g\,L1\cos(q1) - m2\,g\,Lc2\cos(q1)\cos(q2) + m2\,g\,Lc2\sin(q1)\sin(q2) \qquad \textbf{(1.15)}$$

> $V := -c4 \cdot g \cdot \cos(q1) - c5 \cdot g \cdot \cos(q1 + q2)$:

> $M := \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix}$

$$M := \begin{bmatrix} 2\,c3\cos(q2) + c1 + c2 & c3\cos(q2) + c2 \\ c3\cos(q2) + c2 & c2 \end{bmatrix} \qquad \textbf{(1.16)}$$

> $H := simplify\left(\frac{1}{2} \cdot Multiply(Multiply(Transpose(p), MatrixInverse(M)), p) + V, 'size'\right)$:

> dHq1 := simplify(diff(H, q1),'size')

$$dHq1 := g\,(\sin(q1)\,c4 + c5\sin(q1 + q2)) \qquad \textbf{(1.17)}$$

> dHq2 := simplify(diff(H, q2),'size')

$$dHq2 := \frac{1}{(\cos(q2)^2 c3^2 - c1\,c2)^2}\Big(g\,c5\,(\cos(q2)^2 c3^2 - c1\,c2)^2 \sin(q1 + q2) + (c3\,p2\cos(q2) - c2\,(p1 - p2))\,c3\sin(q2)\,(c3\,(p1 - p2)\cos(q2) - c1\,p2)\Big) \qquad \textbf{(1.18)}$$

> dHp1 := simplify(diff(H, p1),'size')

$$dHp1 := \frac{-c3\,p2\cos(q2) + c2\,(p1 - p2)}{-\cos(q2)^2 c3^2 + c1\,c2} \qquad \textbf{(1.19)}$$

> dHp2 := simplify(diff(H, p2),'size')

$$dHp2 := \frac{-c3\,(p1 - 2\,p2)\cos(q2) + (-p1 + p2)\,c2 + c1\,p2}{-\cos(q2)^2 c3^2 + c1\,c2} \qquad \textbf{(1.20)}$$

Controller

> V

$$-c4\,g\cos(q1) - c5\,g\cos(q1 + q2) \qquad \textbf{(2.1)}$$

> $G := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

> $Gt := \begin{bmatrix} 0 & 1 \end{bmatrix}$:

> Multiply(Gt, G)

$$0 \qquad \textbf{(2.2)}$$

> $p := \begin{bmatrix} p1 \\ p2 \end{bmatrix}$:

> M

$$\begin{bmatrix} 2\,c3\cos(q2) + c1 + c2 & c3\cos(q2) + c2 \\ c3\cos(q2) + c2 & c2 \end{bmatrix} \qquad \textbf{(2.3)}$$

> $Md := \begin{bmatrix} k \cdot \text{phi} & k \cdot (c1 - c2) \\ k \cdot (c1 - c2) & k \cdot (-c2 + c3 \cdot \cos(q2)) \end{bmatrix}$:

$$> J2 := \begin{bmatrix} 0 & j2 \\ -j2 & 0 \end{bmatrix}:$$

$$> pMdp := simplify(Multiply(Multiply(Transpose(p), MatrixInverse(Md)), p),'size')$$

$$pMdp := \frac{-\cos(q2)\, c3\, p1^2 + p1\,(p1 - 2\,p2)\, c2 + 2\, c1\, p1\, p2 - p2^2\, \phi}{\left(-\cos(q2)\, c3\, \phi + c2^2 + (-2\, c1 + \phi)\, c2 + c1^2\right) k} \tag{2.4}$$

$$> pMp := Multiply(Multiply(Transpose(p), MatrixInverse(M)), p):$$

$$> dpMdpq2 := simplify(diff(pMdp, q2))$$

$$dpMdpq2 := \frac{c3\, \left((c1 - c2)\, p1 - p2\, \phi\right)^2 \sin(q2)}{\left(-\cos(q2)\, c3\, \phi + c2^2 + (-2\, c1 + \phi)\, c2 + c1^2\right)^2 k} \tag{2.5}$$

Md is a function of q2 only similar to M

$$> eq1 := Multiply\left(Gt, \begin{bmatrix} simplify(diff(pMp, q1)) \\ simplify(diff(pMp, q2)) \end{bmatrix}\right) - simplify\left(Multiply\left(Multiply(Md, MatrixInverse(M)), \begin{bmatrix} 0 \\ dpMdpq2 \end{bmatrix}\right)\right) + simplify(2 \cdot Multiply(Multiply(J2, MatrixInverse(Md)), p))$$

$$eq1 := \frac{2\,(-c3\, p2 \cos(q2) + c2\,(p1 - p2))\,(-c3\,(p1 - p2) \cos(q2) + c1\, p2) \sin(q2)\, c3}{\left(-\cos(q2)^2\, c3^2 + c1\, c2\right)^2} + \frac{2 \sin(q2)\, \left((c1 - c2)\, p1 - p2\, \phi\right)^2 c3}{\left(-\cos(q2)\, c3\, \phi + c2^2 + (-2\, c1 + \phi)\, c2 + c1^2\right)^2} - \frac{2\, j2\, (p1 \cos(q2)\, c3 + (-p1 + p2)\, c2 - c1\, p2)}{\left(\cos(q2)\, c3\, \phi - c2^2 + (2\, c1 - \phi)\, c2 - c1^2\right) k} \tag{2.6}$$

$$> l3 := k:$$

$$> l4 := -2 \cdot k:$$

$$> eq2 := simplify\left(Multiply\left(Gt, \begin{bmatrix} diff(V, q1) \\ diff(V, q2) \end{bmatrix}\right) - Multiply\left(\begin{bmatrix} l1 & l2 \\ l3 & l4 \end{bmatrix}, \begin{bmatrix} diff(Vd(q1, q2), q1) \\ diff(Vd(q1, q2), q2) \end{bmatrix}\right)\right)$$

$$eq2 := c5\, g \sin(q1 + q2) - k\left(\frac{\partial}{\partial q1} Vd(q1, q2)\right) + 2\, k\left(\frac{\partial}{\partial q2} Vd(q1, q2)\right) \tag{2.7}$$

$$> pdsolve(eq2 = 0,\ Vd(q1, q2))$$

$$Vd(q1, q2) = \frac{c5\, g \cos(q1 + q2) + \_F1(q2 + 2\, q1)\, k}{k} \tag{2.8}$$

$$> Vd(q1, q2) := \frac{c5\, g}{k} \cdot (\cos(q1 + q2) + 1) + \frac{kp}{2} \cdot (q2 + 2 \cdot q1 - (\mathrm{pi} + qd1))^2 :$$

$$> simplify(eq2)$$

$$0 \tag{2.9}$$

$$> L := simplify(Multiply(Md, MatrixInverse(M)),'size')$$

$$L := \begin{bmatrix} -\dfrac{(c3\,(c1 - c2) \cos(q2) + c2\,(c1 - c2 - \phi))\, k}{-\cos(q2)^2\, c3^2 + c1\, c2} & \dfrac{\left(2\, c3\left(c1 - c2 - \frac{\phi}{2}\right) \cos(q2) + c1^2 - c2^2 - c2\, \phi\right) k}{-\cos(q2)^2\, c3^2 + c1\, c2} \\ k & -2\, k \end{bmatrix} \tag{2.10}$$

$$> l1 := L(1, 1):$$

$$> l2 := L(1, 2):$$

$$> alpha1 := c3 \cdot \sin(q2) \cdot (l1^2 + l1 \cdot l2):$$

$$> j2 := simplify\left(Multiply\left(Transpose\left(Multiply\left(\frac{1}{dMd} \cdot \left(\begin{bmatrix} d4 & -d2 \\ -d3 & d1 \end{bmatrix}\right), p\right)\right), \begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix}\right),'size'\right)$$

$$j2 := \frac{1}{dMd\left(\cos(q2)^2\, c3^2 - c1\, c2\right)^2}\Big(\big((c3\,(\phi - c1 + c2) \cos(q2) - c1\,(c1 - c2))\, c3\,(-c3\,(c1 - c2) \cos(q2) + c2\,(\phi - c1 + c2))\big)\, k^2\,(d2\, p2 - d4\, p1) \sin(q2) + \alpha 2\left(\cos(q2)^2\, c3^2 - c1\, c2\right)^2 (d1\, p2 - d3\, p1)\Big) \tag{2.11}$$

$$> j2 := simplify\left(Multiply\left(Transpose(Multiply(MatrixInverse(Md), p)), \begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix}\right),'size'\right)$$

$$j2 := \left(-\left(p1\cos(q2)\,c3 + (-p1+p2)\,c2 - c1\,p2\right)\left(c3\,(c1-c2-\phi)\cos(q2) + c1\,(c1-c2)\right)c3\left(c3\,(c1-c2)\cos(q2) + c2\,(c1-c2-\phi)\right)k^2\sin(q2) - \alpha 2\left(\cos(q2)^2 c3^2 - c1\,c2\right)^2\left(c1\,p1 - p1\,c2 - p2\,\phi\right)\right) \Big/ \left(\left(\cos(q2)\,c3\,\phi - c2^2 + (2\,c1-\phi)\,c2 - c1^2\right)k\left(\cos(q2)^2 c3^2 - c1\,c2\right)^2\right) \quad \textbf{(2.12)}$$

> simplify(eq1,'size')

$$\left(4\left(\frac{\cos(q2)^2 c3^2\,\alpha 2}{2} + k^2\sin(q2)\,c3^2\left(c1 - c2 - \frac{\phi}{2}\right)\cos(q2) + \frac{k^2 c3\left(c1^2 - c2^2 - c2\,\phi\right)\sin(q2)}{2} - \frac{c1\,c2\,\alpha 2}{2}\right)\left(p1\cos(q2)\,c3 + (-p1+p2)\,c2 - c1\,p2\right)\left(c1\,p1 - p1\,c2 - p2\,\phi\right)\right) \Big/ \left(\left(\cos(q2)^2 c3^2 - c1\,c2\right)\left(\cos(q2)\,c3\,\phi - c2^2 + (2\,c1-\phi)\,c2 - c1^2\right)^2 k^2\right) \quad \textbf{(2.13)}$$

> alpha2 := simplify(solve(eq1 = 0, alpha2),'size')

$$\alpha 2 := \frac{\left(2\,c3\left(c1 - c2 - \frac{\phi}{2}\right)\cos(q2) + c1^2 - c2^2 - c2\,\phi\right)c3\sin(q2)\,k^2}{-\cos(q2)^2 c3^2 + c1\,c2} \quad \textbf{(2.14)}$$

> dVq1 := diff(V, q1)

$$dVq1 := c4\,g\sin(q1) + c5\,g\sin(q1+q2) \quad \textbf{(2.15)}$$

> dVdd1 := diff(Vd(q1, q2), q1)

$$dVdd1 := -\frac{c5\,g\sin(q1+q2)}{k} + 2\,kp\,(q2 + 2\,q1 - \pi - qd1) \quad \textbf{(2.16)}$$

> dVdd2 := diff(Vd(q1, q2), q2)

$$dVdd2 := -\frac{c5\,g\sin(q1+q2)}{k} + kp\,(q2 + 2\,q1 - \pi - qd1) \quad \textbf{(2.17)}$$

> simplify(j2,'size')

$$\left(-\left(p1\cos(q2)\,c3 + (-p1+p2)\,c2 - c1\,p2\right)\left(c3\,(c1-c2-\phi)\cos(q2) + c1\,(c1-c2)\right)c3\left(c3\,(c1-c2)\cos(q2) + c2\,(c1-c2-\phi)\right)k^2\sin(q2) - \frac{\left(2\,c3\left(c1-c2-\frac{\phi}{2}\right)\cos(q2) + c1^2 - c2^2 - c2\,\phi\right)c3\sin(q2)\,k^2\left(\cos(q2)^2 c3^2 - c1\,c2\right)^2\left(c1\,p1 - p1\,c2 - p2\,\phi\right)}{-\cos(q2)^2 c3^2 + c1\,c2}\right) \Big/ \left(\left(\cos(q2)\,c3\,\phi - c2^2 + (2\,c1-\phi)\,c2 - c1^2\right)k\left(\cos(q2)^2 c3^2 - c1\,c2\right)^2\right) \quad \textbf{(2.18)}$$

> u1 := simplify(dVq1,'size')

$$u1 := g\,(\sin(q1)\,c4 + c5\sin(q1+q2)) \quad \textbf{(2.19)}$$

> u2 := -lam1·dVdq1

$$u2 := -lam1\,dVdq1 \quad \textbf{(2.20)}$$

> u3 := -lam2·dVdq2

$$u3 := -lam2\,dVdq2 \quad \textbf{(2.21)}$$

> u4 := $-lam2\cdot\frac{1}{2}\,pMdpdq2$

$$u4 := -\frac{lam2\,pMdpdq2}{2} \quad \textbf{(2.22)}$$

> u5 := Multiply$\left(\text{Transpose}(G), \text{simplify}\left(\text{Multiply}\left(\text{Multiply}\left(\begin{bmatrix} 0 & n2 \\ -n2 & 0 \end{bmatrix}, \frac{1}{detMd}\begin{bmatrix} d4 & -d2 \\ -d3 & d1 \end{bmatrix}\right), p\right), \text{'size'}\right)\right)$

$$u5 := \frac{n2\,(d1\,p2 - d3\,p1)}{detMd} \quad \textbf{(2.23)}$$

> $u6 := -simplify\left(Kv \cdot Multiply\left(Multiply\left(Transpose(G), \frac{1}{detMd}\begin{bmatrix} d4 & -d2 \\ -d3 & d1 \end{bmatrix}\right), p\right), 'size'\right)$

$$u6 := \frac{Kv\,(d2\,p2 - d4\,p1)}{detMd} \quad (2.24)$$

> #u:=u1+u2+u3+u4:

Change of Coordinates

Modify q1 to measure the circumferential displacement clockwise relative to the positive vertical axis, factor out -L1 from terms mutltiply the time derivatives of q1

> $q1 := \mathrm{pi} + \delta q1$

$$q1 := \pi + \delta q1 \quad (3.1)$$

> $M := \begin{bmatrix} \frac{M1}{-L1} & M2 \\ \frac{M3}{-L1} & M4 \end{bmatrix}$

$$M := \begin{bmatrix} -\frac{2\,c3\cos(q2) + c1 + c2}{L1} & c3\cos(q2) + c2 \\ -\frac{c3\cos(q2) + c2}{L1} & c2 \end{bmatrix} \quad (3.2)$$

> $C := \begin{bmatrix} \frac{C1}{-L1} & C2 \\ \frac{C3}{-L1} & C4 \end{bmatrix}$

$$C := \begin{bmatrix} \frac{c3\sin(q2)\,dq2}{L1} & -c3\sin(q2)\,(dq1 + dq2) \\ -\frac{c3\sin(q2)\,dq1}{L1} & 0 \end{bmatrix} \quad (3.3)$$

for a symmetric mass matrix divide top rows by -L1

> $M := \begin{bmatrix} \frac{M1}{(-L1)^2} & \frac{M2}{-L1} \\ \frac{M3}{-L1} & M4 \end{bmatrix}$

$$M := \begin{bmatrix} \frac{2\,c3\cos(q2) + c1 + c2}{L1^2} & -\frac{c3\cos(q2) + c2}{L1} \\ -\frac{c3\cos(q2) + c2}{L1} & c2 \end{bmatrix} \quad (3.4)$$

> $C := \begin{bmatrix} \frac{C1}{(-L1)^2} & \frac{C2}{-L1} \\ \frac{C3}{-L1} & C4 \end{bmatrix}$

$$C := \begin{bmatrix} -\frac{c3\sin(q2)\,dq2}{L1^2} & \frac{c3\sin(q2)\,(dq1 + dq2)}{L1} \\ -\frac{c3\sin(q2)\,dq1}{L1} & 0 \end{bmatrix} \quad (3.5)$$

> $G := \begin{bmatrix} \dfrac{G1}{-L1} \\ G2 \end{bmatrix}$

$$G := \begin{bmatrix} -\dfrac{g\left(\left(\cos(q2)\, m2\, Lc2 + m2\, L1 + Lc1\, m1\right)\sin(\pi + \delta q1) + m2\, Lc2 \cos(\pi + \delta q1)\sin(q2)\right)}{L1} \\ Lc2\, m2\left(\sin(\pi + \delta q1)\cos(q2) + \cos(\pi + \delta q1)\sin(q2)\right) g \end{bmatrix} \quad \mathbf{(3.6)}$$

> $G := \begin{bmatrix} \dfrac{\left(c4 \cdot \sin(\delta q1) + c5 \sin(\delta q1 + q2)\right) g}{L1} \\ -c5 \sin(\delta q1 + q2)\, g \end{bmatrix}$:

> $\mathrm{tau} := \begin{bmatrix} \dfrac{u}{-L1} \\ 0 \end{bmatrix}$

$$\tau := \begin{bmatrix} -\dfrac{u}{L1} \\ 0 \end{bmatrix} \quad \mathbf{(3.7)}$$

Dimensionless Equations of Motion

model the 2nd link as a massless rod with a point mass of m2 located at distance Lc2 from pivot point, this will make I2 = 0. Then divide the first link by m2L2 and the second link by m2L2^2 to convert to dimensionless form. Note to convert to rho need -deltaq1*L1/L2

> $M := \begin{bmatrix} \dfrac{M1}{m2 \cdot (-L1)^2} & \dfrac{M2}{-L1 \cdot m2 \cdot L2} \\ \dfrac{M3}{-L1 \cdot m2 \cdot L2} & \dfrac{M4}{m2 \cdot L2^2} \end{bmatrix}$

$$M := \begin{bmatrix} \dfrac{2\, c3 \cos(q2) + c1 + c2}{m2\, L1^2} & -\dfrac{c3 \cos(q2) + c2}{m2\, L1\, L2} \\ -\dfrac{c3 \cos(q2) + c2}{m2\, L1\, L2} & \dfrac{c2}{m2\, L2^2} \end{bmatrix} \quad \mathbf{(4.1)}$$

> $C := \begin{bmatrix} -\dfrac{c3 \sin(q2)\, dq2}{m2 \cdot L1^2} & \dfrac{c3 \sin(q2)\left(-\dfrac{L2}{L1} \cdot \left(-\dfrac{L1}{L2} \cdot ddeltaq1\right) + dq2\right)}{L1 \cdot m2 \cdot L2} \\ -\dfrac{c3 \sin(q2) - \dfrac{L2}{L1} \cdot \left(-\dfrac{L1}{L2} \cdot ddeltaq1\right)}{L1 \cdot m2} & 0 \end{bmatrix}$:

> $C := \begin{bmatrix} -\dfrac{c3 \sin(q2)\, dq2}{m2 \cdot L1^2} & \dfrac{c3 \sin(q2)\left(-\dfrac{L2}{L1} \cdot (drho) + dq2\right)}{L1 \cdot m2 \cdot L2} \\ -\dfrac{c3 \sin(q2) - \dfrac{L2}{L1} \cdot (drho)}{L1 \cdot m2} & 0 \end{bmatrix}$

$$C := \begin{bmatrix} -\dfrac{c3 \sin(q2)\, dq2}{m2\, L1^2} & \dfrac{c3 \sin(q2)\left(-\dfrac{L2\, drho}{L1} + dq2\right)}{m2\, L1\, L2} \\ -\dfrac{c3 \sin(q2) - \dfrac{L2\, drho}{L1}}{m2\, L1} & 0 \end{bmatrix} \quad \mathbf{(4.2)}$$

$$> G := \begin{bmatrix} \dfrac{\left(c4 \cdot \sin\left(-\frac{L2}{L1}\left(-\frac{L1}{L2}\cdot \delta q1\right)\right) + c5 \sin\left(-\frac{L2}{L1}\left(-\frac{L1}{L2}\cdot \delta q1\right) + q2\right)\right) g}{L1 \cdot m2 \cdot L2} \\ -\dfrac{c5 \sin\left(-\frac{L2}{L1}\left(-\frac{L1}{L2}\cdot \delta q1\right) + q2\right) g}{m2 \cdot L2^2} \end{bmatrix}$$

$$G := \begin{bmatrix} \dfrac{(c4 \sin(\delta q1) + c5 \sin(\delta q1 + q2))\, g}{m2\, L1\, L2} \\ -\dfrac{c5 \sin(\delta q1 + q2)\, g}{m2\, L2^2} \end{bmatrix} \tag{4.3}$$

$$> G := \begin{bmatrix} \dfrac{\left(c4 \cdot \sin\left(-\frac{L2}{L1}\cdot(\mathrm{rho})\right) + c5 \sin\left(-\frac{L2}{L1}\cdot(\mathrm{rho}) + q2\right)\right) g}{L1 \cdot m2 \cdot L2} \\ -\dfrac{c5 \sin\left(-\frac{L2}{L1}\cdot(\mathrm{rho}) + q2\right) g}{m2 \cdot L2^2} \end{bmatrix}$$

$$G := \begin{bmatrix} \dfrac{\left(-c4 \sin\left(\frac{L2\,\rho}{L1}\right) + c5 \sin\left(-\frac{L2\,\rho}{L1} + q2\right)\right) g}{m2\, L1\, L2} \\ -\dfrac{c5 \sin\left(-\frac{L2\,\rho}{L1} + q2\right) g}{m2\, L2^2} \end{bmatrix} \tag{4.4}$$

$$> \mathrm{tau} := \begin{bmatrix} \dfrac{u}{-L1 \cdot m2 \cdot L2} \\ 0 \end{bmatrix}$$

$$\tau := \begin{bmatrix} -\dfrac{u}{m2\, L1\, L2} \\ 0 \end{bmatrix} \tag{4.5}$$

switch to dimensionless parameters and mutliply through by gamma^2 to convert all to unitless time

$> c1 := m1 \cdot Lc1^2 + m2 \cdot L1^2 + I1 :$

$> c2 := m2 \cdot L2^2 :$

$> c3 := m2 \cdot L1 \cdot L2 :$

$> c4 := m1 \cdot Lc1 + m2 \cdot L1 :$

$> c5 := m2 \cdot L2 :$

$> map(eval, M)$

$$\begin{bmatrix} \dfrac{2\, m2\, L1\, L2 \cos(q2) + m2\, L1^2 + Lc1^2\, m1 + I1 + m2\, L2^2}{m2\, L1^2} & -\dfrac{m2\, L1\, L2 \cos(q2) + m2\, L2^2}{m2\, L1\, L2} \\ -\dfrac{m2\, L1\, L2 \cos(q2) + m2\, L2^2}{m2\, L1\, L2} & 1 \end{bmatrix} \tag{4.6}$$

$> map(eval, C)$

$$\begin{bmatrix} -\dfrac{L2 \sin(q2)\, dq2}{L1} & \sin(q2)\left(-\dfrac{L2\, drho}{L1} + dq2\right) \\ -\dfrac{m2\, L1\, L2 \sin(q2) - \frac{L2\, drho}{L1}}{m2\, L1} & 0 \end{bmatrix} \tag{4.7}$$

> map(eval, G)

$$\begin{bmatrix} \frac{\left(-(m2\,L1+Lc1\,m1)\sin\left(\frac{L2\,\rho}{L1}\right)+m2\,L2\sin\left(-\frac{L2\,\rho}{L1}+q2\right)\right)g}{m2\,L1\,L2} \\ -\frac{\sin\left(-\frac{L2\,\rho}{L1}+q2\right)g}{L2} \end{bmatrix} \tag{4.8}$$

> map(eval, tau)

$$\begin{bmatrix} -\frac{u}{m2\,L1\,L2} \\ 0 \end{bmatrix} \tag{4.9}$$

> $c1 := c1n \cdot m2 \cdot L2^2$:

> $c2 := c2n \cdot m2 \cdot L2^2$:

> $c3 := c3n \cdot m2 \cdot L2^2$:

> $c1n := k1 \cdot k9^2 + 1 + k8$:

> $c2n := k2^2$:

> $c3n := k2$:

> $M := \begin{bmatrix} 2\,c3n\cos(q2)+c1n+c2n & -(\cos(q2)+k2) \\ -(\cos(q2)+k2) & 1 \end{bmatrix}$

$$M := \begin{bmatrix} 2\,k2\cos(q2)+k1\,k9^2+k8+1+k2^2 & -\cos(q2)-k2 \\ -\cos(q2)-k2 & 1 \end{bmatrix} \tag{4.10}$$

> $C := \begin{bmatrix} -k2\sin(q2)\,dq2 & \sin(q2)\,(-k2\,drho+dq2) \\ -k2\sin(q2)\,drho & 0 \end{bmatrix}$

$$C := \begin{bmatrix} -k2\sin(q2)\,dq2 & \sin(q2)\,(-k2\,drho+dq2) \\ -k2\sin(q2)\,drho & 0 \end{bmatrix} \tag{4.11}$$

> $G := \begin{bmatrix} (k1 \cdot k9+1)\cdot\sin(-k2\,\rho)+k2\cdot\sin(-k2\,\rho+q2) \\ -\sin(-k2\,\rho+q2) \end{bmatrix}$

$$G := \begin{bmatrix} -(k1\,k9+1)\sin(k2\,\rho)-k2\sin(k2\,\rho-q2) \\ \sin(k2\,\rho-q2) \end{bmatrix} \tag{4.12}$$

> $\tau := \begin{bmatrix} \tau_n \\ 0 \end{bmatrix}$:

> $Vn := int(G(1,1), \rho)$

$$Vn := \cos(k2\,\rho-q2)-\frac{(-k1\,k9-1)\cos(k2\,\rho)}{k2} \tag{4.13}$$

> $Vn := int(G(2,1), q2)$

$$Vn := \cos(k2\,\rho-q2) \tag{4.14}$$

> $Vn := \cos(k2\,\rho-q2)+\frac{(k1\,k9+1)}{k2}\cos(k2\,\rho)$:

Dimensionless Controller

utlizing the dimensionless parameter un to convert u to dimensionless form means dividing u by (m2L1L2) and multiply by gamma^(2);

> $q1 := \pi - k2 \cdot \rho$:

> $dq1 := -k2 \cdot drho$:

> diff(Vn, rho)

$$-(k1\,k9+1)\sin(k2\,\rho)-k2\sin(k2\,\rho-q2) \tag{5.1}$$

> $u1n := \dfrac{u1}{m2\cdot L1\cdot g}$

$$u1n := \frac{-\sin(k2\,\rho-\pi)\,(m2\,L1+Lc1\,m1)-m2\,L2\sin(k2\,\rho-\pi-q2)}{m2\,L1} \tag{5.2}$$

> $u1n := \sin(k2\,\rho)\cdot(k1\,k9+1)-k2\cdot\sin(-k2\,\rho+q2):$

> $u2n := \dfrac{u2}{m2\cdot L1\cdot g}$

$$u2n := -\frac{lam1\,dVdq1}{m2\,L1\,g} \tag{5.3}$$

> $I1 := k8\cdot m2\cdot L1^2:$

> $m1 := k1\cdot m2:$

> $Lc1 := k9\cdot L1:$

> $L2 := k2\cdot L1:$

> $lam1 := l1:$

> dVdd1

$$\frac{m2\,k2\,L1\,g\sin(k2\,\rho-\pi-q2)}{k}+2\,kp\,(-2\,k2\,\rho+\pi+q2-qd1) \tag{5.4}$$

> $dVdq1 := \dfrac{m2\,k2\,L1\,g\sin(-k2\,\rho+q2)}{k}+2\,kp\,(-2\,k2\,\rho+\pi+q2-qd1):$

> simplify(u2n,'size')

$$-\frac{1}{k2^2\,m2^2\,L1^3\,(-k1\,k9^2+\cos(q2)^2-k8-1)\,g}\Big((m2\,k2\,L1\,g\sin(k2\,\rho-q2)-2\,k\,kp\,(-2\,k2\,\rho+\pi+q2-qd1))\,\big(-L1^2\,k2\,m2\,(k1\,k9^2-k2^2+k8+1)\cos(q2)+k2^4\,m2\,L1^2-(k1\,k9^2+k8+1)\,m2\,k2^2\,L1^2+\phi\big)\Big) \tag{5.5}$$

> $u2n := -\dfrac{1}{(-k1\,k9^2+\cos(q2)^2-k8-1)}\left(\left(-\dfrac{1}{k2}\,(k1\,k9^2-k2^2+k8+1)\cos(q2)+k2^2-(k1\,k9^2+k8+1)+\dfrac{\phi}{m2\,L1^2\,k2^2}\right)\left(k2\,\sin(k2\,\rho-q2)-2\,k\cdot\dfrac{kp\,k2}{m2\,g\,L2}\,(-2\,k2\,\rho+\pi+q2-qd1)\right)\right):$

> $u2n := -\dfrac{(-(k1\,k9^2-k2^2+k8+1)\cos(q2))\left(\sin(k2\,\rho-q2)-2\,k\cdot\dfrac{kp}{m2\,g\,L2}\,(-2\,k2\,\rho+\mathrm{pi}+q2-qd1)\right)}{(-k1\,k9^2+\cos(q2)^2-k8-1)}-\dfrac{\left(k2^2-(k1\,k9^2+k8+1)+\dfrac{\phi}{m2\,L1^2\,k2^2}\right)\left(k2\,\sin(k2\,\rho-q2)-2\,k\cdot\dfrac{kp\,k2}{m2\,g\,L2}(-2\,k2\,\rho+\mathrm{pi}+q2-qd1)\right)}{(-k1\,k9^2+\cos(q2)^2-k8-1)}:$

> $\mathrm{phi} := phin\cdot m2\cdot L2^2:$

> simplify(u2n,'size')

$$-\frac{1}{m2\,g\,k2\,L1\,(-k1\,k9^2+\cos(q2)^2-k8-1)}\Big(\big((-k1\,k9^2+k2^2-k8-1)\cos(q2)+k2\,(-k1\,k9^2+k2^2-k8+phin-1)\big)\,(m2\,k2\,L1\,g\sin(k2\,\rho-q2)-2\,k\,kp\,(-2\,k2\,\rho+\pi+q2-qd1))\Big) \tag{5.6}$$

> $u3n := \dfrac{u3}{m2\cdot L1\cdot g}$

$$u3n := -\frac{lam2\,dVdq2}{m2\,L1\,g} \tag{5.7}$$

> $lam2 := l2:$

> dVdd2

$$\frac{m2\,k2\,L1\,g\sin(k2\,\rho-\pi-q2)}{k}+kp\,(-2\,k2\,\rho+\pi+q2-qd1) \tag{5.8}$$

> $dVdq2 := \dfrac{m2\,k2\,L1\,g\sin(k2\,\rho-\pi-q2)}{k}+kp\,(-2\,k2\,\rho+\pi+q2-qd1):$

> simplify(u3n,'size')

$$-\frac{1}{k2^2\, m2\, L1\left(-k1\, k9^2+\cos(q2)^2-k8-1\right) g}\Big(\big(m2\, k2\, L1\, g \sin(k2\,\rho-\pi-q2)+k\, kp\left(-2\, k2\,\rho+\pi+q2-qd1\right)\big)\big(k2\left(-2\, k1\, k9^2+2\, k2^2-2\, k8+phin-2\right)\cos(q2)+k2^4+k2^2\, phin-\left(k1\, k9^2+k8+1\right)^2\big)\Big) \quad (5.9)$$

> $u3n := \frac{1}{k2\left(-k1\, k9^2+\cos(q2)^2-k8-1\right)}\left(\left(k2\left(-2\, k1\, k9^2+2\, k2^2-2\, k8+phin-2\right)\cos(q2)+k2^4+k2^2\, phin-\left(k1\, k9^2+k8+1\right)^2\right)\left(\sin(k2\,\rho-q2)-\frac{k\, kp}{m2\, g\, L2}\left(-2\, k2\,\rho+\mathrm{pi}+q2-qd1\right)\right)\right):$

> $kpn := \frac{kp\, k2}{m2\, g\, L2}$

$$kpn := \frac{kp}{m2\, g\, L1} \quad (5.10)$$

> $kp := kpn \cdot m2 \cdot g \cdot L1:$

> simplify(u3n,'size')

$$\frac{1}{k2^2\, m2\, L1\left(-k1\, k9^2+\cos(q2)^2-k8-1\right) g}\Big(\big(k2\left(-2\, k1\, k9^2+2\, k2^2-2\, k8+phin-2\right)\cos(q2)+k2^4+k2^2\, phin-\left(k1\, k9^2+k8+1\right)^2\big)\big(m2\, k2\, L1\, g \sin(k2\,\rho-q2)-k\, kp\left(-2\, k2\,\rho+\pi+q2-qd1\right)\big)\Big) \quad (5.11)$$

> simplify(u2n,'size')

$$-\frac{1}{m2\, g\, k2\, L1\left(-k1\, k9^2+\cos(q2)^2-k8-1\right)}\Big(\big(\left(-k1\, k9^2+k2^2-k8-1\right)\cos(q2)+k2\left(-k1\, k9^2+k2^2-k8+phin-1\right)\big)\big(m2\, k2\, L1\, g\sin(k2\,\rho-q2)-2\, k\, kp\left(-2\, k2\,\rho+\pi+q2-qd1\right)\big)\Big) \quad (5.12)$$

> $d1 := k \cdot \mathrm{phi}$

$$d1 := k phin m2\, k2^2\, L1^2 \quad (5.13)$$

> $d2 := k\cdot(c1-c2)$

$$d2 := k\left(\left(k1\, k9^2+k8+1\right) m2\, k2^2\, L1^2-k2^4\, m2\, L1^2\right) \quad (5.14)$$

> $d3 := d2$

$$d3 := k\left(\left(k1\, k9^2+k8+1\right) m2\, k2^2\, L1^2-k2^4\, m2\, L1^2\right) \quad (5.15)$$

> $d4 := k\cdot(c3\cdot\cos(q2)-c2)$

$$d4 := k\left(-k2^4\, m2\, L1^2+k2^3\, m2\, L1^2\cos(q2)\right) \quad (5.16)$$

> detMd := simplify(d1·d4 − d2·d3,'size')

$$detMd := m2^2\left(\cos(q2)\, k2\, phin-k2^4+\left(2\, k1\, k9^2+2\, k8-phin+2\right) k2^2-\left(k1\, k9^2+k8+1\right)^2\right) L1^4\, k2^4\, k^2 \quad (5.17)$$

> $M1 := \mathrm{simplify}(\mathrm{coeff}(Eq_{link1}, ddq1),\text{'size'})$

$$M1 := 2\, m2\, Lc2\, L1\cos(q2)+\left(\left(k1\, k9^2+k8+1\right) L1^2+Lc2^2\right) m2+I2 \quad (5.18)$$

> $M2 := \mathrm{simplify}(\mathrm{coeff}(Eq_{link1}, ddq2),\text{'size'})$

$$M2 := m2\, Lc2\, L1\cos(q2)+m2\, Lc2^2+I2 \quad (5.19)$$

> $M3 := \mathrm{simplify}(\mathrm{coeff}(Eq_{link2}, ddq1),\text{'size'})$

$$M3 := m2\, Lc2\, L1\cos(q2)+m2\, Lc2^2+I2 \quad (5.20)$$

> $M4 := \mathrm{simplify}(\mathrm{coeff}(Eq_{link2}, ddq2),\text{'size'})$

$$M4 := m2\, Lc2^2+I2 \quad (5.21)$$

> $p := \mathrm{Multiply}\left(m2\cdot L1^2\cdot M, \begin{bmatrix} -drho\cdot\sqrt{\frac{g}{L2}} \\ -dq2\cdot\sqrt{\frac{g}{L2}} \end{bmatrix}\right)$

$$p := \begin{bmatrix} -m2\, L1^2\left(2\, k2\cos(q2)+k1\, k9^2+k8+1+k2^2\right) drho\sqrt{\frac{g}{k2\, L1}}-m2\, L1^2\left(-\cos(q2)-k2\right) dq2\sqrt{\frac{g}{k2\, L1}} \\ -m2\, L1^2\left(-\cos(q2)-k2\right) drho\sqrt{\frac{g}{k2\, L1}}-m2\, L1^2\, dq2\sqrt{\frac{g}{k2\, L1}} \end{bmatrix} \quad (5.22)$$

> $p1 := p(1,1)$

$$p1 := -m2\,L1^2\left(2\,k2\cos(q2) + k1\,k9^2 + k8 + 1 + k2^2\right) drho\sqrt{\frac{g}{k2\,L1}} - m2\,L1^2\left(-\cos(q2) - k2\right) dq2\sqrt{\frac{g}{k2\,L1}} \tag{5.23}$$

> $p2 := p(2,1)$

$$p2 := -m2\,L1^2\left(-\cos(q2) - k2\right) drho\sqrt{\frac{g}{k2\,L1}} - m2\,L1^2\,dq2\sqrt{\frac{g}{k2\,L1}} \tag{5.24}$$

> $u6n := simplify\left(\frac{u6}{m2\cdot L1\cdot g}, 'size'\right)$

$$u6n := -\Bigg(\Big(k2\left(-2\,k2\,drho + dq2\right)\cos(q2)^2 - drho\left(k2+1\right)\left(k1\,k9^2 - k2^2 + k8 + 1\right)\cos(q2) + drho\,k2^4 + \left(-dq2 + drho\right)k2^3 + \left(\left(k1\,k9^2 + k8 + 1\right)drho - dq2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)drho\,k2 + dq2\left(k1\,k9^2 + k8 + 1\right)\Big)\sqrt{\frac{g}{k2\,L1}}\,Kv\Bigg)\Big/\Big(\left(\cos(q2)\,k2\,phin - k2^4 + \left(2\,k1\,k9^2 + 2\,k8 - phin + 2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)^2\right)m2\,k\,L1\,k2^2\,g\Big) \tag{5.25}$$

> $Kv := Kvn\cdot \mathrm{sqrt}\left(\frac{g}{L2}\right):$

> $simplify(u6n, 'size')$

$$-\Big(\Big(k2\left(-2\,k2\,drho + dq2\right)\cos(q2)^2 - drho\left(k2+1\right)\left(k1\,k9^2 - k2^2 + k8 + 1\right)\cos(q2) + drho\,k2^4 + \left(-dq2 + drho\right)k2^3 + \left(\left(k1\,k9^2 + k8 + 1\right)drho - dq2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)drho\,k2 + dq2\left(k1\,k9^2 + k8 + 1\right)\Big)Kvn\Big)\Big/\Big(k2^3\,L1^2\left(\cos(q2)\,k2\,phin - k2^4 + \left(2\,k1\,k9^2 + 2\,k8 - phin + 2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)^2\right)m2\,k\Big) \tag{5.26}$$

> $Kv := \frac{Kvn\cdot \mathrm{sqrt}\left(\frac{g}{L2}\right)\cdot m2\cdot L2^3}{L1}:$

> $simplify(u6n, 'size')$

$$-\Big(\Big(k2\left(-2\,k2\,drho + dq2\right)\cos(q2)^2 - drho\left(k2+1\right)\left(k1\,k9^2 - k2^2 + k8 + 1\right)\cos(q2) + drho\,k2^4 + \left(-dq2 + drho\right)k2^3 + \left(\left(k1\,k9^2 + k8 + 1\right)drho - dq2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)drho\,k2 + dq2\left(k1\,k9^2 + k8 + 1\right)\Big)Kvn\Big)\Big/\Big(\Big(\cos(q2)\,k2\,phin - k2^4 + \left(2\,k1\,k9^2 + 2\,k8 - phin + 2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)^2\Big)k\Big) \tag{5.27}$$

> $u5n := simplify\left(\frac{u5}{m2\cdot L1\cdot g}, 'size'\right)$

$$u5n := -\Bigg(\sqrt{\frac{g}{k2\,L1}}\,n2\Big(\Big(\left(2\,k2^3 + \left(-2\,k1\,k9^2 - 2\,k8 - 2\right)k2 - phin\right)drho + dq2\left(k1\,k9^2 - k2^2 + k8 + 1\right)\Big)\cos(q2) + \Big(k2^4 - phin\,k2 - \left(k1\,k9^2 + k8 + 1\right)^2\Big)drho + \left(-k2^3 + \left(k1\,k9^2 + k8 + 1\right)k2 + phin\right)dq2\Big)\Bigg)\Big/\Big(k2^2\Big(\cos(q2)\,k2\,phin - k2^4 + \left(2\,k1\,k9^2 + 2\,k8 - phin + 2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)^2\Big)k\,m2\,g\,L1\Big) \tag{5.28}$$

> $simplify(alpha1, 'size')$

$$-\frac{1}{\left(-k1\,k9^2 + \cos(q2)^2 - k8 - 1\right)^2}\Big(m2\,L1^2\sin(q2)\Big(k2\left(k1\,k9^2 - k2^2 + k8 - phin + 1\right)\cos(q2) + \left(k1\,k9^2 + k8 + 1\right)\left(k1\,k9^2 - k2^2 + k8 + 1\right)\Big)\Big(\left(k1\,k9^2 - k2^2 + k8 + 1\right)\cos(q2) + k2\left(k1\,k9^2 - k2^2 + k8 - phin + 1\right)\Big)k^2\Big) \tag{5.29}$$

> $simplify(alpha2, 'size')$

$$-\frac{2\,m2\left(k2\left(k1\,k9^2 - k2^2 + k8 - \frac{phin}{2} + 1\right)\cos(q2) - \frac{k2^4}{2} - \frac{k2^2\,phin}{2} + \frac{\left(k1\,k9^2 + k8 + 1\right)^2}{2}\right)L1^2\sin(q2)\,k2\,k^2}{-k1\,k9^2 + \cos(q2)^2 - k8 - 1} \tag{5.30}$$

> $n2 := simplify\left(Multiply\left(Transpose\left(Multiply\left(\frac{1}{detMd}\cdot\left(\begin{bmatrix} d4 & -d2 \\ -d3 & d1 \end{bmatrix}\right), p\right)\right), \begin{bmatrix} alpha1 \\ alpha2 \end{bmatrix}\right), 'size'\right)$

$$n2 := \left(\sqrt{\frac{g}{k2\,L1}}\; m2\left(\left(k1\,k9^2 - k2^2 + k8 + 1\right)\cos(q2) + k2\left(k1\,k9^2 - k2^2 + k8 - phin + 1\right)\right)\left(\left(-2\,drho\,k2^2 + dq2\,k2\right)\cos(q2)^2 - drho\,(k2+1)\left(k1\,k9^2 - k2^2 + k8 + 1\right)\cos(q2) + drho\,k2^4 + (-dq2 + drho)\,k2^3 + \left(\left(k1\,k9^2 + k8 + 1\right)drho - dq2\right)k2^2 - \left(k1\,k9^2 + k8 + 1\right)drho\,k2 + dq2\left(k1\,k9^2 + k8 + 1\right)\right)k\left(k2\left(k1\,k9^2 - k2^2 + k8 - phin + 1\right)\cos(q2) + \left(k1\,k9^2 + k8 + 1\right)\left(k1\,k9^2 - k2^2 + k8 + 1\right)\right)L1^2\sin(q2)\right) \Big/ \left(\left(k1\,k9^2 - \cos(q2)^2 + k8 + 1\right)^2\left(-\cos(q2)\,k2\,phin + k2^4 + \left(-2\,k1\,k9^2 - 2\,k8 + phin - 2\right)k2^2 + \left(k1\,k9^2 + k8 + 1\right)^2\right)k2^2\right)$$

$$- \left(\sqrt{\frac{g}{k2\,L1}}\; m2\left(2\,k2\left(k1\,k9^2 - k2^2 + k8 - \frac{phin}{2} + 1\right)\cos(q2) - k2^4 - k2^2\,phin + \left(k1\,k9^2 + k8 + 1\right)^2\right)k\left(\left(-2\,drho\,k2^3 + \ldots - dq2\,phin\right)L1^2\sin(q2)\right)\right) \Big/ \left(\left(k1\,k9^2 - \cos(q2)^2 + k8 + 1\right)\left(-\cos(q2)\,k2\,phin + k2^4 + \left(-2\,k1\,k9^2 - 2\,k8 + phin - 2\right)k2^2 + \left(k1\,k9^2 + k8 + 1\right)^2\right)k2\right)$$

> simplify(expand(u5n),'size')

$$\left(\left(\left(-2\,drho\,k2^3 + dq2\,k2^2 + 2\left(k1\,k9^2 + k8 + 1\right)drho\,k2 + phin\,drho - dq2\left(k1\,k9^2 + k8 + 1\right)\right)\cos(q2) - drho\,k2^4 + dq2\,k2^3 + \left(phin\,drho - dq2\left(k1\,k9^2 + k8 + 1\right)\right)k2 + \left(k1\,k9^2 + k8 + 1\right)^2 drho - dq2\,phin\right)\left(-2\,k2^2\left(k2^5\,drho - \frac{k2^4\,dq2}{2} - 2\,drho\left(k1\,k9^2 + k8 + 1\right)k2^3 + \left(-phin\,drho + dq2\left(k1\,k9^2 + k8 + 1\right)\right)k2^2 + \left(k1\,k9^2 + k8 + 1\right)^2 drho\,k2 + phin\left(1 + k8 - \frac{phin}{2} + k1\,k9^2\right)drho - \frac{dq2\left(k1\,k9^2 + k8 + 1\right)^2}{2}\right)\cos(q2)^4 - \left(3\,drho\,k2^7 + (-2\,dq2 + drho)\,k2^6 - 5\,drho\left(k1\,k9^2 + k8 + 1\right)k2^5 + \left(\left(-3\,k1\,k9^2 - 3\,k8 - 2\,phin - 3\right)drho + 4\,dq2\left(k1\,k9^2 + k8 + 1\right)\right)k2^4 + \left(\left(k1^2\,k9^4 + 2\,k9^2\,(k8+1)\,k1 + k8^2 - 2\,phin^2 + 2\,k8 + 1\right)drho + 2\,dq2\,phin\right)k2^3 + \left(\left(3\,k1^2\,k9^4 + 6\,k9^2\,(k8+1)\,k1 + 3\,k8^2 - 2\,phin^2 + 6\,k8 + 3\right)drho - 2\left(k1^2\,k9^4 + 2\,k9^2\,(k8+1)\,k1 + k8^2 - \frac{phin^2}{2} + 2\,k8 + 1\right)dq2\right)k2^2 + \left(\left(k1\,k9^2 + k8 + 1\right)^3 drho - 2\left(1 + k8 - \frac{phin}{2} + k1\,k9^2\right)dq2\,phin\right)k2 - drho\left(k1\,k9^2 + k8 + 1\right)^2\left(k1\,k9^2 + k8 - 2\,phin + 1\right)\right)k2\cos(q2)^3 + \left(-3\,drho\,k2^9 + (2\,dq2 - 2\,drho)\,k2^8 + \left(\left(9\,k1\,k9^2 + 9\,k8 - 4\,phin + 9\right)drho + dq2\right)k2^7 + \left(\left(5\,k1\,k9^2 + 5\,k8 - 2\,phin + 5\right)drho - 6\left(k1\,k9^2 + k8 - \frac{phin}{3} + 1\right)dq2\right)k2^6 + \left(\left(-9\,k1^2\,k9^4 - 18\left(k8 - \frac{4\,phin}{9} + 1\right)k9^2\,k1 - 9\,k8^2 + (8\,phin - 18)\,k8 - phin^2 + 8\,phin - 9\right)drho - 3\,dq2\left(k1\,k9^2 + k8 + 1\right)\right)k2^5 + 6\left(k1\,k9^2 + k8 + 1\right)\left(\left(-\frac{k1\,k9^2}{2} - \frac{k8}{2} + \frac{2\,phin}{3} - \frac{1}{2}\right)drho + dq2\left(k1\,k9^2 + k8 - \frac{2\,phin}{3} + 1\right)\right)k2^4 + 3\left(k1\,k9^2 + k8 - phin + 1\right)\left(\left(k1\,k9^2 + k8 + 1\right)\left(k1\,k9^2 + k8 - \frac{phin}{3} + 1\right)drho + dq2\left(k1\,k9^2 + k8 + \frac{phin}{3} + 1\right)\right)k2^3 - 2\left(k1\,k9^2 + k8 + 1\right)^2\left(\left(\frac{k1\,k9^2}{2} + \frac{k8}{2} + phin + \frac{1}{2}\right)drho + dq2\left(k1\,k9^2 + k8 - phin + 1\right)\right)k2^2 - dq2\left(k1\,k9^2 + k8 + 1\right)^2\left(k1\,k9^2 + k8 - 2\,phin + 1\right)k2 + drho\left(k1\,k9^2 + k8 + 1\right)^4\right)\cos(q2)^2 + \left(-drho\,k2^{10} + (dq2 - drho)\,k2^9 + \left(\left(3\,k1\,k9^2 + 3\,k8 - 2\,phin + 3\right)drho + dq2\right)k2^8 + \left(\left(k1\,k9^2 + k8 - 2\,phin + 1\right)drho - 4\left(1 + k8 - \frac{phin}{2} + k1\,k9^2\right)dq2\right)k2^7 + \left(\left(k1^2\,k9^4 + 2\,k9^2\,(k8 + phin + 1)\,k1 + k8^2 + (2\,phin + 2)\,k8 - phin^2 + 2\,phin + 1\right)drho - 2\,dq2\left(k1\,k9^2 + k8 - phin + 1\right)\right)k2^6$$ **(5.32)**

$$+\left(\left(3\,k1^2\,k9^4+6\,k9^2\,(k8+1)\,k1+3\,k8^2-phin^2+6\,k8+3\right)drho+4\left(1+k8-\frac{phin}{2}+k1\,k9^2\right)^2 dq2\right)k2^5+\Bigg($$

$$-7\left(k1\,k9^2+k8+1\right)\left(k1^2\,k9^4+2\,k9^2\left(k8-\frac{phin}{7}+1\right)k1+k8^2+\left(-\frac{2\,phin}{7}+2\right)k8+\frac{phin^2}{7}-\frac{2\,phin}{7}+1\right)drho$$

$$-2\left(1+k8-\frac{phin}{2}+k1\,k9^2\right)dq2\,phin\Bigg)k2^4-5\left(k1\,k9^2+k8+1\right)\Bigg(\Bigg(k1^2\,k9^4+2\left(k8-\frac{phin}{5}+1\right)k9^2\,k1+k8^2$$

$$+\left(-\frac{2\,phin}{5}+2\right)k8+\frac{phin^2}{5}-\frac{2\,phin}{5}+1\Bigg)drho-\frac{2\,dq2\,phin\left(k1\,k9^2+k8+1\right)}{5}\Bigg)k2^3+4\left(k1\,k9^2+k8+1\right)^3\Bigg(\Bigg(1$$

$$+k8-\frac{phin}{2}+k1\,k9^2\Bigg)drho+\frac{dq2}{2}\Bigg)k2^2-\left(k1\,k9^2+k8+1\right)^4\left(dq2-2\,drho\right)k2-dq2\left(k1\,k9^2+k8+1\right)^4\Bigg)\cos(q2)$$

$$+\left(k1\,k9^2+k8+1\right)\Bigg(-drho\,k2^7+\left(\left(k1\,k9^2+k8+1\right)drho+dq2\right)k2^6+\Big(\left(3\,k1\,k9^2+3\,k8-2\,phin+3\right)drho$$

$$-dq2\left(k1\,k9^2+k8+1\right)\Big)k2^5+\left(-\left(k1\,k9^2+k8+1\right)^2 drho-3\,dq2\left(k1\,k9^2+k8-\frac{2\,phin}{3}+1\right)\right)k2^4+\Bigg(\Bigg(-3\,k1^2\,k9^4$$

$$-6\left(k8-\frac{phin}{3}+1\right)k9^2\,k1-3\,k8^2+(2\,phin-6)\,k8-phin^2+2\,phin-3\Bigg)drho+2\,dq2\left(k1\,k9^2+k8+1\right)^2\Bigg)k2^3$$

$$+\Bigg(-\left(k1\,k9^2+k8+1\right)^3 drho+3\Bigg(k1^2\,k9^4+2\left(k8-\frac{phin}{3}+1\right)k9^2\,k1+k8^2+\left(-\frac{2\,phin}{3}+2\right)k8+\frac{phin^2}{3}$$

$$-\frac{2\,phin}{3}+1\Bigg)dq2\Bigg)k2^2-\left(k1\,k9^2+k8+1\right)^3\left(dq2-drho\right)k2+\left(k1\,k9^2+k8+1\right)^3\Big(\left(k1\,k9^2+k8+1\right)drho$$

$$-dq2\Big)\Bigg)k2\Bigg)\sin(q2)\Bigg)\Bigg/\Big(\left(k1\,k9^2-\cos(q2)^2+k8+1\right)^2 k2^5\Big(-\cos(q2)\,k2\,phin+k2^4+\big(-2\,k1\,k9^2-2\,k8+phin$$

$$-2\big)\,k2^2+\left(k1\,k9^2+k8+1\right)^2\Big)^2\Big)$$

> $u4n := \frac{u4}{m2\cdot L1\cdot g}$

$$u4n := -\Bigg(\Bigg(2\,k2^3\,m2\,L1^2\left(\left(k1\,k9^2+k8+1\right)m2\,k2^2\,L1^2-k2^4\,m2\,L1^2-\frac{phin\,m2\,k2^2\,L1^2}{2}\right)\cos(q2)+\left(k1\,k9^2+k8+1\right)^2 m2^2\,k2^4\,L1^4$$

$$-k2^8\,m2^2\,L1^4-k2^6\,m2^2\,L1^4\,phin\Bigg)kpMdpdq2\Bigg)\Bigg/\Big(2\left(-\cos(q2)^2\,k2^6\,m2^2\,L1^4+\big(k1\,k9^2+k8\right.$$

$$\left.+1\big)\,m2^2\,k2^6\,L1^4\right)m2\,L1\,g\Big)$$

> pMdpdq2 := simplify(dpMdpq2,'size')

$$pMdpdq2 := \Big(\sin(q2)\Big(\Big(\left(2\,k2^3+\left(-2\,k1\,k9^2-2\,k8-2\right)k2-phin\right)drho+dq2\left(k1\,k9^2-k2^2+k8+1\right)\Big)\cos(q2)+\Big(k2^4-phin\,k2-\big(k1\,k9^2$$

$$+k8+1\big)^2\Big)drho+\left(-k2^3+\left(k1\,k9^2+k8+1\right)k2+phin\right)dq2\Big)^2\,m2\,g\,L1\Big)\Big/\Big(k2^2\,k\Big(\cos(q2)\,k2\,phin-k2^4+\left(2\,k1\,k9^2+2\,k8\right.$$

$$\left.-phin+2\right)k2^2-\left(k1\,k9^2+k8+1\right)^2\Big)^2\Big)$$

> simplify(u4n,'size')

$$-\Big(\sin(q2)\Big(\left(-2\,drho\,k2^3+dq2\,k2^2+2\left(k1\,k9^2+k8+1\right)drho\,k2-dq2\,k1\,k9^2+phin\,drho-dq2\,k8-dq2\right)\cos(q2)-drho\,k2^4 \quad (5.35)$$

$$+dq2\,k2^3+\left(-dq2\,k1\,k9^2-dq2\,k8+phin\,drho-dq2\right)k2+drho\,k1^2\,k9^4+2\,k9^2\,drho\,(k8+1)\,k1+drho\,k8^2-dq2\,phin$$

$$+2\,drho\,k8+drho\Big)^2\Big(k2\left(-2\,k1\,k9^2+2\,k2^2-2\,k8+phin-2\right)\cos(q2)+k2^4+k2^2\,phin-\left(k1\,k9^2+k8+1\right)^2\Big)\Big)\Big/$$

$$\Big(2\Big(\cos(q2)\,k2\,phin-k2^4+\left(2\,k1\,k9^2+2\,k8-phin+2\right)k2^2-\left(k1\,k9^2+k8+1\right)^2\Big)^2\left(-k1\,k9^2+\cos(q2)^2-k8-1\right)k2^4\Big)$$

Morphed Equations of Motion

> $Mn := subs(\{k2 = 0, k9 = 1, k8 = 0\}, map(eval, M))$

$$Mn := \begin{bmatrix} 1+k1 & -\cos(q2) \\ -\cos(q2) & 1 \end{bmatrix} \quad (6.1)$$

> $Cm := subs(\{k2 = 0, k9 = 1, k8 = 0\}, map(eval, C))$

$$Cm := \begin{bmatrix} 0 & \sin(q2)\,dq2 \\ 0 & 0 \end{bmatrix} \quad (6.2)$$

> $Gm := subs(\{k2 = 0, k9 = 1, k8 = 0\}, map(eval, G))$

$$Gm := \begin{bmatrix} -(1 + k1)\sin(0) \\ \sin(-q2) \end{bmatrix} \quad (6.3)$$

> $taum := subs(\{k2 = 0, k9 = 1, k8 = 0\}, map(eval, \tau))$

$$taum := \begin{bmatrix} \tau_n \\ 0 \end{bmatrix} \quad (6.4)$$

> $q2 := -\theta:$

> $dq2 := -d\theta:$

> $ddq2 := -dd\theta:$

> $Mn := \begin{bmatrix} 1 + k1 & \cos(\theta) \\ -\cos(\theta) & -1 \end{bmatrix}:$

> $Cm := \begin{bmatrix} 0 & \sin(q2)\, d\theta \\ 0 & 0 \end{bmatrix}:$

> $Gm := \begin{bmatrix} 0 \\ \sin(\theta) \end{bmatrix}:$

for symmetric mass matrix, multiply the bottom row by -1

> $Mn := \begin{bmatrix} 1 + k1 & \cos(\theta) \\ \cos(\theta) & 1 \end{bmatrix}:$

> $Cm := \begin{bmatrix} 0 & \sin(q2)\, d\theta \\ 0 & 0 \end{bmatrix}:$

> $Gm := \begin{bmatrix} 0 \\ -\sin(\theta) \end{bmatrix}:$

>

>

Morphed Controller

> $subs(\{k2 = 0, k9 = 1, k8 = 0, \phi = 0\}, u1n)$

$$(1 + k1)\sin(0) \quad (7.1)$$

> $subs(\{k2 = 0, k9 = 1, k8 = 0, \phi = 0\}, u2n)$

Error, numeric exception: division by zero

> $subs(\{k2 = 0, k9 = 1, k8 = 0, \phi = 0\}, u3n)$

Error, numeric exception: division by zero

> $subs(\{k2 = 0, k9 = 1, k8 = 0, \phi = 0\}, u4n)$

Error, numeric exception: division by zero

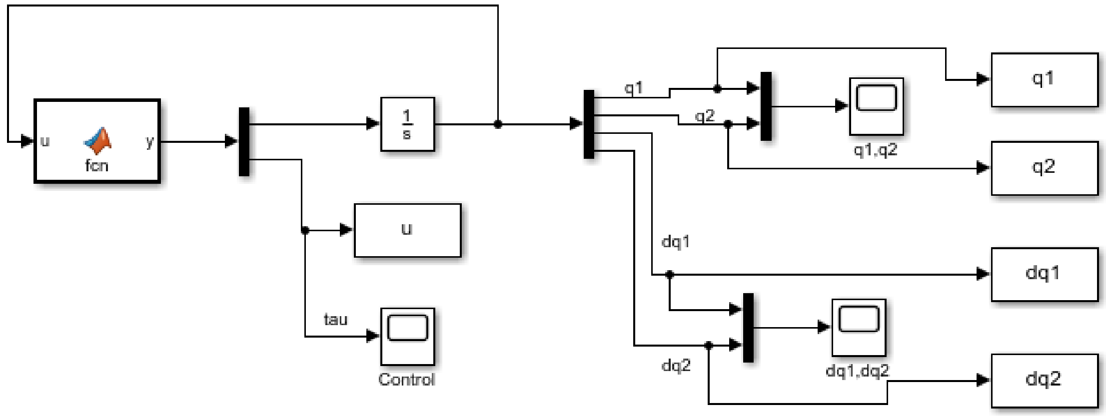
> $subs(\{k2 = 0, k9 = 1, k8 = 0, \phi = 0\}, u5n)$

Error, numeric exception: division by zero

> $subs(\{k2 = 0, k9 = 1, k8 = 0, \phi = 0\}, u6n)$

$$\frac{(-drho\,(1 + k1)\cos(\theta) - d\theta\,(1 + k1))\,Kvn}{(1 + k1)^2\,k} \quad (7.2)$$

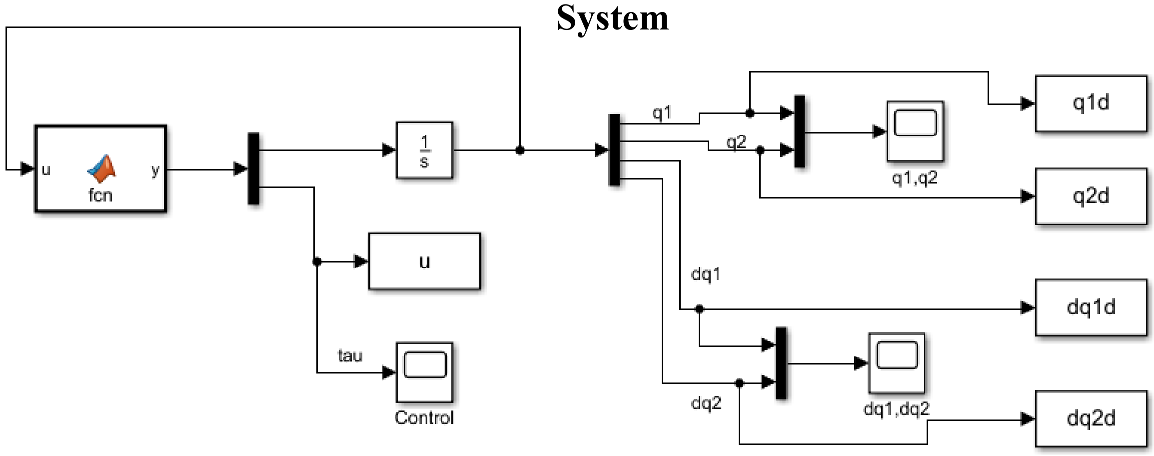
E.2 Simulink file and MATLAB code for the simulation of the Full System




```matlab
function y = fcn(u)
%% Main Vectors
q1   = u(1);              % feedback array
q2   = u(2);
dq1  = u(3);
dq2  = u(4);
%% Generalized quantities
q    = [q1 q2]';         % Generalized coordinates
qdot = [dq1 dq2]';        % Generalized velocities
%% Parameter values
m1   = 2;
L1   = 2;
Lc1  = 1;
I1   = 0.667;
m2   = 1;
%L2   = 1;
Lc2  = 0.5;
I2   = 0;% 0.083;
g    = 9.81;
c1   = m1*Lc1^2+m2*L1^2+I1;
c2   = m2*Lc2^2+I2;
c3   = m2*L1*Lc2;
c4   = m1*Lc1+m2*L1;
c5   = m2*Lc2;
a1   = c1+c2+2*c3*cos(q2);
a2   = c2+c3*cos(q2);
a3   = c2;
phi  = 500;
k    = 0.0033;
kp   = 30;
kv   = 20;
qd1  = pi;
%% Equation of motion pieces
M    = [a1 a2; a2 a3];
C    = [-c3*sin(q2)*dq2 -c3*sin(q2)*(dq1+dq2); c3*sin(q2)*dq1 0];
G    = [c4*g*sin(q1)+c5*g*sin(q1+q2);c5*g*sin(q1+q2)];
%% Evaluate the control law
d1   = k*phi;
d2   = k*(c1-c2);
d3   = k*(c1-c2);
d4   = k*(-c2+c3*cos(q2));
```

```
Md    = [d1 d2; d3 d4];
dMd   = det(Md);
dVd1  = -c5*g*sin(q1+q2)/k+2*kp*(q2+2*q1-pi-qd1);
dVd2  = -c5*g*sin(q1+q2)/k+kp*(q2+2*q1-pi-qd1);

p = M*qdot;
p1 = p(1);
p2 = p(2);
dq1H    = c4*g*sin(q1)+c5*g*sin(q1+q2);
dpMdp2  = (p1*(c1-c2)-p2*phi)^2*c3*sin(q2)/...
           ((-cos(q2)*c3*phi+c2^2+(-2*c1+phi)*c2+c1^2)^2*k);

lambda1 = -(c3*cos(q2)*(c1-c2)+c2*(c1-c2-phi))*k/(-cos(q2)^2*c3^2+c1*c2);
lambda2 = k*(2*c3*(-1/2*phi+c1-c2)*cos(q2)-c2*phi+c1^2-c2^2)/...
             (-cos(q2)^2*c3^2+c1*c2);

alpha1 = c3*sin(q2)*(lambda1^2+lambda1*lambda2);
alpha2 = c3*sin(q2)*k^2*(2*c3*(c1-c2-phi/2)*cos(q2)+c1^2-c2^2-c2*phi)/...
         (-cos(q2)^2*c3^2+c1*c2);

j2 = (p1*(d4*alpha1-d3*alpha2)+p2*(-d2*alpha1+d1*alpha2))/dMd;

u = dq1H-(lambda1*dVd1+lambda2*(dVd2+1/2*dpMdp2))+j2*(-d3*p1+d1*p2)/dMd...
    -kv*(d4*p1-d2*p2)/dMd;
%% Evaluate the Dynamic
qdotdot      = inv(M)*([u;0]-C*qdot-G);
xdotdot      = qdotdot(1);
ddtheta      = qdotdot(2);
%% M-File output
y            = [dq1;dq2;xdotdot;ddtheta;u];
%% End of
```

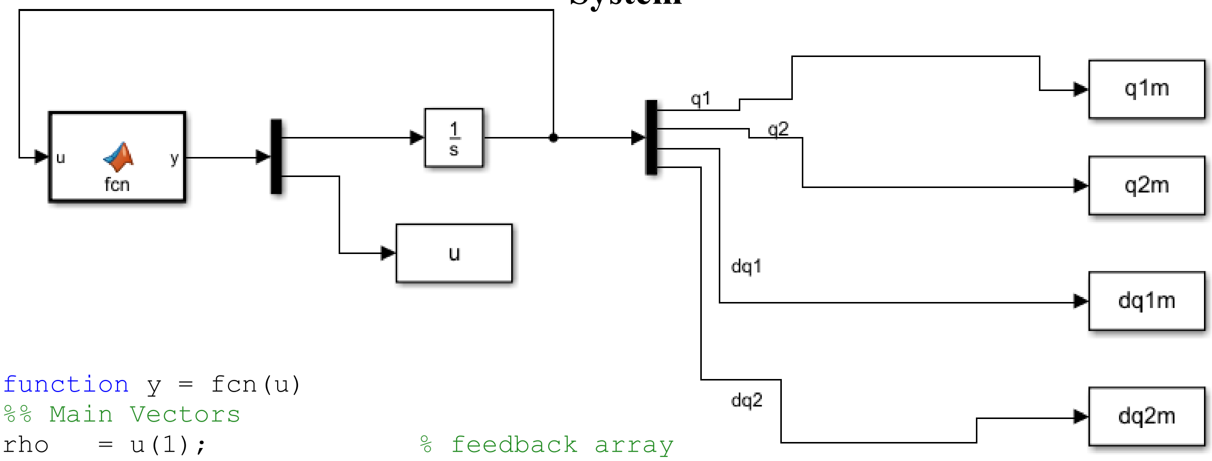
E.3 Simulink file and MATLAB code for the simulation of the Dimensionless System



```
function y = fcn(u)
%% Main Vectors
rho   = u(1);              % feedback array
q2    = u(2);
drho  = u(3);
dq2   = u(4);
%% Generalized quantities
q    = [rho q2]';        % Generalized coordinates
qdot = [drho dq2]';        % Generalized velocities
%% Parameter values
m1   = 2;
L1   = 2;
Lc1  = 1;
I1   = 0.667;
m2   = 1;
% L2   = 1;
Lc2  = 0.5;
% I2   = 0.083;
g    = 9.81;
qd1  = pi;
%% Dimensionless Parameters
k1 = m1/m2;
k2 = Lc2/L1;
gamma = sqrt(Lc2/g);
k8 = I1/(m2*L1^2);
k9 = Lc1/L1;
%% Equation of motion pieces
c1n   = k1*k9^2+1+k8;
c2n   = k2^2;
c3n   = k2;
c4n   = k1*k9+1;
c5n   = k2;

a1n   = c1n+c2n+2*c3n*cos(q2);
a2n   = k2+cos(q2);
a3n   = 1;

M    = [a1n -a2n; -a2n a3n];
C    = [-c3n*sin(q2)*dq2 sin(q2)*(-drho*k2+dq2); c3n*sin(q2)*(drho) 0];
G    = [(c4n*sin(-k2*rho)+c5n*sin(q2-k2*rho));-sin(q2-k2*rho)];
%% Evaluate the control law
phi  = 500;
phin = phi/L1^2/m2;
k    = 0.0033;

d1n   = k*phin;
d2n   = k*(c1n-c2n);
d3n   = k*(c1n-c2n);
d4n   = k*(c3n*cos(q2)-c2n);
Mdn   = [d1n d2n; d3n d4n];

kp   = 30;
kpn  = kp/(g*L1*m2);

dVd1n = c5n/k*sin(q2-k2*rho)+2*kpn*(q2+2*(-k2*rho)+pi-qd1);
```

```
dVd2n = c5n/k*sin(q2-k2*rho)+kpn*(q2+2*(-k2*rho)+pi-qd1);

%p = M*qdot;
p1n = (c1n+c2n+2*c3n*cos(q2))*(-drho)+(k2+cos(q2))*dq2;
p2n = (c2n+c3n*cos(q2))*(-drho)+k2*dq2;

dq1Hn = (c4n*sin(k2*rho)-c5n*sin(q2-k2*rho));

dpMdp2n = k2*(p1n*(c1n-c2n)-p2n*phin)^2*c3n*sin(q2)/...
          ((-cos(q2)*c3n*phin+c2n^2+(-2*c1n+phin)*c2n+c1n^2)^2*k);

lambda1n = (c3n*cos(q2)*(c1n-c2n)+c2n*(c1n-c2n-phin))*k/...
               (cos(q2)^2*c3n^2-c1n*c2n);

lambda2n = -((cos(q2)*c3n+c2n)*lambda1n-k*(c1n-c2n))/c2n;

alpha1n = c3n*sin(q2)*(lambda1n^2+lambda1n*lambda2n);

alpha2n = c3n*sin(q2)*k^2*(2*c3n*(c1n-c2n-phin/2)*cos(q2)+c1n^2-c2n^2-...
           c2n*phin)/(-cos(q2)^2*c3n^2+c1n*c2n);

j2n = (p1n*(d4n*alpha1n-d3n*alpha2n)+p2n*(-d2n*alpha1n+d1n*alpha2n))/...
          det(Mdn);

kv   = 20;
kvn   = gamma*kv/L1^2/m2;

un = -dq1Hn+lambda1n*dVd1n+lambda2n*dVd2n+lambda2n*1/2*dpMdp2n+...
         (-j2n*k2*(-d3n*p1n+d1n*p2n)/det(Mdn))+...
          kvn*(d4n*p1n-d2n*p2n)/det(Mdn);
%% Evaluate the Dynamic
qdotdot      = inv(M)*([un;0]-C*qdot-G);
rhodotdot    = qdotdot(1);
ddtheta      = qdotdot(2);
%% M-File output
y            = [drho;dq2;rhodotdot;ddtheta;un];
%% End of
```

E.4 Simulink file and MATLAB code for the simulation of the Morphed System




```
function y = fcn(u)
%% Main Vectors
rho    = u(1);                % feedback array
theta     = u(2);
drho   = u(3);
dtheta    = u(4);
%% Generalized quantities
q     = [rho theta]';          % Generalized coordinates
qdot = [drho dtheta]';          % Generalized velocities
%% Parameter values
m1    = 2;
L1    = 2;
Lc1   = 1;
I1    = 0.667;
m2    = 1;
% L2    = 1;
Lc2   = 0.5;
% I2    = 0.083;
g     = 9.81;

%% Dimensionless Parameters
k1 = m1/m2;
gamma = sqrt(Lc2/g);

k    = 0.0033;
kp    = 30;
kpn = kp/(g*L1*m2);

kv    = 20;
kvn   = gamma*kv/(m2*L1^2);

%% Equation of motion pieces
Mm     = [1+k1 cos(theta); cos(theta) 1];
Cm     = [0 sin(theta)*dtheta; 0 0];
Gm     = [0;-sin(theta)];
%% Evaluate the control law
um = 2*k*kpn*theta+kvn*cos(theta)*drho/(k*(k1+1));
%% Evaluate the Dynamic
qdotdot       = inv(Mm)*([um;0]-Cm*qdot-Gm);
rhodotdot     = qdotdot(1);
```

```
ddtheta      = qdotdot(2);
%% M-File output
y            = [drho;dtheta;rhodotdot;ddtheta;um];
%% End of
```

E.5 MATLAB code to produce the plots for Chapter 5

```
g = 9.81;
L2 = 0.5;
L1 = 2;
figure(1)
plot1 = plot(q1.time,q1.signals.values,...
            q1d.time*sqrt(L2/g),-q1d.signals.values/2*0.5+pi,'--');

plot1(2).LineWidth = 2;
title("First Link Displacement for full vs dimensionless systems")
xlabel("Time(s) and Time(unitless)*\gamma(s)")
ylabel("Link Position (rad)")
legend("q_1","\rho")

figure(2)
plot1 = plot(q2.time,q2.signals.values,...
             q2d.time*sqrt(L2/g),q2d.signals.values,'--');
plot1(2).LineWidth = 2;
title("Second Link Displacement for full vs dimensionless systems")
xlabel("Time(s) and Time(unitless)*\gamma(s)")
ylabel("Link Position (rad)")
legend("q_2","q_2d")
```

Appendix F - Two-Link Manipulator

This Appendix is organized in six major parts. These are:

F.1 Derivations of Equations of Motion and Controller

F.2 Simulink file and MATLAB code for the simulation of the Full System

F.3 Simulink file and MATLAB code for the simulation of the Coordinated-Changed System

F.4 Simulink file and MATLAB code for the simulation of the Dimensionless System

F.5 Simulink file and MATLAB code for the simulation of the Morphed System

F.6 MATLAB code to produce the plots for Chapter 6

F.1 Derivations of Equations of Motion and Controller

2linkmanipulator.mw

> $restart:$

> $with(LinearAlgebra):$

Equations of Motion from "On the Robust Control of Robot Manipulators"

> $x1c := Lc1 \cdot \cos(q1):$

> $y1c := Lc1 \cdot \sin(q1):$

> $x2c := L1 \cdot \cos(q1) + Lc2 \cdot \cos(q1 + q2):$

> $y2c := L1 \cdot \sin(q1) + Lc2 \cdot \sin(q1 + q2):$

> $dx1c := -Lc1 \cdot \sin(q1) \cdot dq1:$

> $dy1c := Lc1 \cdot \cos(q1) \cdot dq1:$

> $dx2c := -L1 \cdot \sin(q1) \cdot dq1 - Lc2 \cdot \sin(q1 + q2) \cdot (dq1 + dq2):$

> $dy2c := L1 \cdot \cos(q1) \cdot dq1 + Lc2 \cdot \cos(q1 + q2) \cdot (dq1 + dq2):$

> $q := \begin{bmatrix} q1 \\ q2 \end{bmatrix} : dq := \begin{bmatrix} dq1 \\ dq2 \end{bmatrix} : ddq := \begin{bmatrix} ddq1 \\ ddq2 \end{bmatrix} : p := \begin{bmatrix} p1 \\ p2 \end{bmatrix} :$

> $T := \frac{1}{2} \cdot I1 \cdot dq1^2 + \frac{1}{2} \cdot I2 \cdot (dq2 + dq1)^2 + \frac{1}{2} \cdot m1 \cdot (dx1c^2 + dy1c^2) + \frac{1}{2} \cdot m2 \cdot (dx2c^2 + dy2c^2):$

> $V := m1 \cdot g \cdot y1c + \mathrm{m2} \cdot g \cdot y2c$

$$V := g\, m1\, Lc1 \sin(q1) + g\, m2\, (L1 \sin(q1) + Lc2 \sin(q1 + q2)) \quad \mathbf{(1.1)}$$

Lagrange

> $L := expand(T - V):$

diff(diff(L,q_dot),t)-diff(L,q)=Qi

> $e1 := diff(L, dq1):$

> $e1 := subs(\{q1 = q1(t), q2 = q2(t)\}, e1):$

> $e1 := subs(\{dq1 = diff(q1(t), t), dq2 = diff(q2(t), t)\}, e1):$

> $e2 := diff(e1, t):$

> $e2 := subs(\{diff(q1(t), t, t) = ddq1, diff(q2(t), t, t) = ddq2\}, e2):$

> $e2 := subs(\{diff(q1(t), t) = dq1, diff(q2(t), t) = dq2\}, e2):$

> $e2 := subs(\{q1(t) = q1, q2(t) = q2\}, e2):$

> $variables := [ddq1, ddq2, dq1, dq2]:$

> $Eq_{link1} := collect(simplify(e2 - diff(L, q1) - tau1, 'trig'), variables)$

$$Eq_{link1} := \left(2\,Lc2\,L1\,m2\cos(q2) + \left(L1^2 + Lc2^2\right)m2 + Lc1^2\,m1 + I1 + I2\right)ddq1 + \left(Lc2\,L1\,m2\cos(q2) + m2\,Lc2^2 + I2\right)ddq2 - 2\,m2\,L1\,dq1\,Lc2\,dq2\sin(q2) - m2\,L1\,Lc2\,dq2^2\sin(q2) + g\,(Lc2\cos(q2)\,m2 + L1\,m2 + m1\,Lc1)\cos(q1) - m2\,g\,Lc2\sin(q1)\sin(q2) - \tau 1 \quad \textbf{(1.2)}$$

> $e3 := diff(L, dq2):$
> $e3 := subs(\{q1 = q1(t), q2 = q2(t)\}, e3):$
> $e3 := subs(\{dq1 = diff(q1(t), t), dq2 = diff(q2(t), t)\}, e3):$
> $e4 := diff(e3, t):$
> $e4 := subs(\{diff(q1(t), t, t) = ddq1, diff(q2(t), t, t) = ddq2\}, e4):$
> $e4 := subs(\{diff(q1(t), t) = dq1, diff(q2(t), t) = dq2\}, e4):$
> $e4 := subs(\{q1(t) = q1, q2(t) = q2\}, e4):$
> $Eq_{link2} := collect(simplify(e4 - diff(L, q2) - tau2, 'trig'), variables)$

$$Eq_{link2} := \left(Lc2\,L1\,m2\cos(q2) + m2\,Lc2^2 + I2\right)ddq1 + \left(m2\,Lc2^2 + I2\right)ddq2 + m2\,L1\,dq1^2\,Lc2\sin(q2) + m2\,g\,Lc2\cos(q1)\cos(q2) - m2\,g\,Lc2\sin(q1)\sin(q2) - \tau 2 \quad \textbf{(1.3)}$$

> $M1 := simplify\left(coeff\left(Eq_{link1}, ddq1\right), 'size'\right)$

$$M1 := 2\,Lc2\,L1\,m2\cos(q2) + \left(L1^2 + Lc2^2\right)m2 + Lc1^2\,m1 + I1 + I2 \quad \textbf{(1.4)}$$

> $M2 := simplify\left(coeff\left(Eq_{link1}, ddq2\right), 'size'\right)$

$$M2 := Lc2\,L1\,m2\cos(q2) + m2\,Lc2^2 + I2 \quad \textbf{(1.5)}$$

> $M3 := simplify\left(coeff\left(Eq_{link2}, ddq1\right), 'size'\right)$

$$M3 := Lc2\,L1\,m2\cos(q2) + m2\,Lc2^2 + I2 \quad \textbf{(1.6)}$$

> $M4 := simplify\left(coeff\left(Eq_{link2}, ddq2\right), 'size'\right)$

$$M4 := m2\,Lc2^2 + I2 \quad \textbf{(1.7)}$$

> $c1 := m1 \cdot Lc1^2 + m2 \cdot L1^2 + I1:$
> $c2 := m2 \cdot Lc2^2 + I2:$
> $c3 := m2 \cdot L1 \cdot Lc2:$
> $M1 := 2 \cdot c3 \cdot \cos(q2) + c1 + c2:$
> $M2 := c3 \cdot \cos(q2) + c2:$
> $M3 := c3 \cdot \cos(q2) + c2:$
> $M4 := c2:$
> $M := \begin{bmatrix} M1 & M2 \\ M3 & M4 \end{bmatrix}:$
> $C1 := simplify\left(coeff\left(Eq_{link1}, dq1^2\right) \cdot dq1 + \frac{1}{2} \cdot coeff\left(Eq_{link1}, dq1\right), 'size'\right)$

$$C1 := -m2\,L1\,Lc2\,dq2\sin(q2) \quad \textbf{(1.8)}$$

> $C2 := simplify\left(coeff\left(Eq_{link1}, dq2^2\right) \cdot dq2 + \frac{1}{2} \cdot coeff\left(Eq_{link1}, dq2\right), 'size'\right)$

$$C2 := -L1\,Lc2\sin(q2)\,m2\,(dq1 + dq2) \quad \textbf{(1.9)}$$

> $C3 := simplify\left(coeff\left(Eq_{link2}, dq1^2\right) \cdot dq1 + \frac{1}{2} \cdot coeff\left(Eq_{link2}, dq1\right), 'size'\right)$

$$C3 := m2\,L1\,dq1\,Lc2\sin(q2) \quad \textbf{(1.10)}$$

> $C4 := simplify\left(coeff\left(Eq_{link2}, dq2^2\right) \cdot dq2 + \frac{1}{2} \cdot coeff\left(Eq_{link2}, dq2\right), 'size'\right)$

$$C4 := 0 \quad \textbf{(1.11)}$$

> $C1 := -c3 \cdot \sin(q2) \cdot dq2:$
> $C2 := -c3 \cdot \sin(q2) \cdot (dq1 + dq2):$
> $C3 := c3 \cdot \sin(q2) \cdot dq1:$
> $C := \begin{bmatrix} C1 & C2 \\ C3 & C4 \end{bmatrix}:$
> $G1 := simplify\left(coeff\left(Eq_{link1}, g\right) \cdot g, 'size'\right)$

$$G1 := ((Lc2\cos(q2)\,m2 + L1\,m2 + m1\,Lc1)\cos(q1) - m2\,Lc2\sin(q1)\sin(q2))\,g \quad \textbf{(1.12)}$$

> $G2 := simplify\left(coeff\left(Eq_{link2}, g\right) \cdot g, 'size'\right)$

$$G2 := Lc2\,m2\,(\cos(q1)\cos(q2) - \sin(q1)\sin(q2))\,g \quad \textbf{(1.13)}$$

> $c4 := m1 \cdot Lc1:$
> $c5 := m2 \cdot L1:$
> $c6 := m2 \cdot Lc2:$

> $G1 := (c4 + c5) \cdot g \cdot \cos(q1) + c6 \cdot g \cdot \cos(q1 + q2)$:

> $G2 := c6 \cdot g \cdot \cos(q1 + q2)$:

> $G := \begin{bmatrix} G1 \\ G2 \end{bmatrix}$:

> $\text{tau} := -\begin{bmatrix} coeff(Eq_{linkl1}, tau1) \cdot tau1 \\ coeff(Eq_{lin21}, tau2) \cdot tau2 \end{bmatrix}$

$$\tau := \begin{bmatrix} \tau 1 \\ 0 \end{bmatrix} \quad \textbf{(1.14)}$$

Dimensionless

For figure 1 in "On the Robust Control of Robot Manipulators" need to change q1 to being measured from y axis so q1y = q1-pi/2, dq1y = dq1, ddq1y = ddq1
and then convert to arc length: s = q1y*L1 ds = dq1y*L1 dds = ddq1y*L1

> $q1 := \frac{\text{pi}}{2} + q1y$:

> $dq1 := dq1y$:

> $ddq1 := ddq1y$:

to convert to arc length by dividing the terms multiplying r1 and it's derivative

> $Mr := \begin{bmatrix} M(1,1) \cdot \frac{1}{L1} & M(1,2) \\ M(2,1) \cdot \frac{1}{L1} & M(2,2) \end{bmatrix}$

$$Mr := \begin{bmatrix} \frac{2\, Lc2\, L1\, m2 \cos(q2) + L1^2\, m2 + Lc1^2\, m1 + m2\, Lc2^2 + I1 + I2}{L1} & Lc2\, L1\, m2 \cos(q2) + m2\, Lc2^2 + I2 \\ \frac{Lc2\, L1\, m2 \cos(q2) + m2\, Lc2^2 + I2}{L1} & m2\, Lc2^2 + I2 \end{bmatrix} \quad \textbf{(2.1)}$$

> $Cr := map\left(eval, \begin{bmatrix} C(1,1) \cdot \frac{1}{L1} & C(1,2) \\ C(2,1) \cdot \frac{1}{L1} & C(2,2) \end{bmatrix}\right)$

$$Cr := \begin{bmatrix} -m2\, Lc2\, dq2 \sin(q2) & -L1\, Lc2 \sin(q2)\, m2\, (dq1y + dq2) \\ m2\, dq1y\, Lc2 \sin(q2) & 0 \end{bmatrix} \quad \textbf{(2.2)}$$

for symmetric Mr divide top row by L1

> $Mr := \begin{bmatrix} M(1,1) \cdot \frac{1}{L1^2} & M(1,2) \cdot \frac{1}{L1} \\ M(2,1) \cdot \frac{1}{L1} & M(2,2) \end{bmatrix}$

$$Mr := \begin{bmatrix} \frac{2\, Lc2\, L1\, m2 \cos(q2) + L1^2\, m2 + Lc1^2\, m1 + m2\, Lc2^2 + I1 + I2}{L1^2} & \frac{Lc2\, L1\, m2 \cos(q2) + m2\, Lc2^2 + I2}{L1} \\ \frac{Lc2\, L1\, m2 \cos(q2) + m2\, Lc2^2 + I2}{L1} & m2\, Lc2^2 + I2 \end{bmatrix} \quad \textbf{(2.3)}$$

> $Cr := map\left(eval, \begin{bmatrix} C(1,1) \cdot \frac{1}{L1^2} & C(1,2) \cdot \frac{1}{L1} \\ C(2,1) \cdot \frac{1}{L1} & C(2,2) \end{bmatrix}\right)$

$$Cr := \begin{bmatrix} -\frac{m2\, Lc2\, dq2 \sin(q2)}{L1} & -Lc2 \sin(q2)\, m2\, (dq1y + dq2) \\ m2\, dq1y\, Lc2 \sin(q2) & 0 \end{bmatrix} \quad \textbf{(2.4)}$$

> $Gr := map\left(eval, \begin{bmatrix} \frac{G(1)}{L1} \\ G(2) \end{bmatrix}\right)$

$$Gr := \begin{bmatrix} \frac{(L1\,m2 + m1\,Lc1)\,g\cos\left(\frac{\pi}{2} + q1y\right) + Lc2\,m2\,g\cos\left(\frac{\pi}{2} + q1y + q2\right)}{L1} \\ Lc2\,m2\,g\cos\left(\frac{\pi}{2} + q1y + q2\right) \end{bmatrix} \qquad (2.5)$$

> $taur := \begin{bmatrix} \frac{\mathrm{tau}(1)}{L1} \\ \mathrm{tau}(2) \end{bmatrix}$

$$taur := \begin{bmatrix} \frac{\tau 1}{L1} \\ 0 \end{bmatrix} \qquad (2.6)$$

for dimensionless equations divide top row by M2*Lc2 and bottom row by M2*Lc2^2, note terms multiplying r1 and dr1 will need an Lc2 pulled out to make the coordinate dimensionless

> $\#qd := \begin{bmatrix} \frac{dq1y \cdot L1}{Lc2} \\ dq2 \end{bmatrix}$:

> $\#qdd := \begin{bmatrix} \frac{ddq1y \cdot L1}{Lc2} \\ dq2 \end{bmatrix}$:

> $Md := \begin{bmatrix} Mr(1,1) \cdot \frac{1}{m2 \cdot Lc2} & Mr(1,2) \cdot \frac{1}{m2 \cdot Lc2} \\ Mr(2,1) \cdot \frac{1}{m2 \cdot Lc2^2} & Mr(2,2) \cdot \frac{1}{m2 \cdot Lc2^2} \end{bmatrix}$

$$Md := \begin{bmatrix} \frac{2\,Lc2\,L1\,m2\cos(q2) + L1^2\,m2 + Lc1^2\,m1 + m2\,Lc2^2 + I1 + I2}{L1^2\,Lc2\,m2} & \frac{Lc2\,L1\,m2\cos(q2) + m2\,Lc2^2 + I2}{L1\,Lc2\,m2} \\ \frac{Lc2\,L1\,m2\cos(q2) + m2\,Lc2^2 + I2}{L1\,m2\,Lc2^2} & \frac{m2\,Lc2^2 + I2}{m2\,Lc2^2} \end{bmatrix} \qquad (2.7)$$

> $Md := \begin{bmatrix} \frac{2\,Lc2}{L1}\cos(q2) + 1 + \frac{Lc2^2}{L1^2} + \frac{Lc1^2\,m1 + I1 + I2}{L1^2\,m2} & \cos(q2) + \frac{Lc2}{L1} + \frac{I2}{L1\,Lc2\,m2} \\ \cos(q2) + \frac{Lc2}{L1} + \frac{I2}{L1\,m2\,Lc2} & 1 + \frac{I2}{m2\,Lc2^2} \end{bmatrix}$:

> $Cd := map\left(eval, \begin{bmatrix} Cr(1,1) \cdot \frac{1}{m2 \cdot Lc2} & Cr(1,2) \cdot \frac{1}{m2 \cdot Lc2} \\ Cr(2,1) \cdot \frac{1}{m2 \cdot Lc2^2} & Cr(2,2) \cdot \frac{1}{m2 \cdot Lc2^2} \end{bmatrix}\right)$

$$Cd := \begin{bmatrix} -\frac{dq2\sin(q2)}{L1} & -\sin(q2)\,(dq1y + dq2) \\ \frac{dq1y\sin(q2)}{Lc2} & 0 \end{bmatrix} \qquad (2.8)$$

> $Cd := \begin{bmatrix} -\frac{Lc2}{L1}\,dq2\sin(q2) & -\sin(q2)\left(\frac{Lc2}{L1} \cdot \left(\frac{L1}{Lc2} \cdot dq1y\right) + dq2\right) \\ \frac{Lc2}{L1} \cdot \left(\frac{L1}{Lc2} \cdot dq1y\right)\sin(q2) & 0 \end{bmatrix}$:

> $Gd := map\left(eval, \begin{bmatrix} \frac{Gr(1)}{m2 \cdot Lc2} \\ \frac{Gr(2)}{m2 \cdot Lc2^2} \end{bmatrix}\right)$

$$Gd := \begin{bmatrix} \dfrac{(L1\,m2 + m1\,Lc1)\,g\cos\left(\frac{\pi}{2} + q1y\right) + Lc2\,m2\,g\cos\left(\frac{\pi}{2} + q1y + q2\right)}{L1\,Lc2\,m2} \\ \dfrac{g\cos\left(\frac{\pi}{2} + q1y + q2\right)}{Lc2} \end{bmatrix} \quad (2.9)$$

> $Gd := \begin{bmatrix} \dfrac{-g}{Lc2} \cdot \dfrac{((L1\,m2 + Lc1\,m1)\sin(q1y) + Lc2\,m2\,\sin(q1y + q2))}{L1\,m2} \\ -\dfrac{g}{Lc2} \cdot \sin(q1y + q2) \end{bmatrix}:$

> $taud := \begin{bmatrix} \dfrac{taur(1)}{m2 \cdot Lc2} \\ \dfrac{taur(2)}{m2 \cdot Lc2^2} \end{bmatrix}$

$$taud := \begin{bmatrix} \dfrac{\tau 1}{L1\,Lc2\,m2} \\ 0 \end{bmatrix} \quad (2.10)$$

> $I2 := 0:$

> $Lc2 := L2:$

Use dimensionless parameters where mu = L1*q1y/L2, k2 = L2/L1, k9 = Lc1/L1, k8 = I1/m2/L1^2, k1= m1/m2

> $Md := \begin{bmatrix} 2\,k2\cos(q2) + 1 + k2^2 + k9^2\,k1 + k8 & \cos(q2) + k2 \\ \cos(q2) + k2 & 1 \end{bmatrix}:$

> $Cd := \begin{bmatrix} -k2\,dq2d\sin(q2) & -\sin(q2)\,(k2 \cdot dmud + dq2d) \\ k2 \cdot dmud\sin(q2) & 0 \end{bmatrix}:$

> $Gd := \begin{bmatrix} -\dfrac{g}{L2} \cdot \left((1 + k9\,k1)\sin(k2 \cdot \mathrm{mu}) + k2\sin(k2 \cdot \mathrm{mu} + q2)\right) \\ -\dfrac{g}{L2} \cdot \sin(k2 \cdot \mathrm{mu} + q2) \end{bmatrix}:$

> $taud := \begin{bmatrix} \dfrac{tau1}{L1\,L2\,m2} \\ \dfrac{tau2}{L2^2\,m2} \end{bmatrix}:$

now need to render time dimensionless by multiplying by Lc2/g this changes derivatives to functions of unitless time so Md and Cd are unchanged

> $Gd := \begin{bmatrix} -((1 + k9\,k1)\sin(k2 \cdot \mathrm{mu}) + k2\sin(k2 \cdot \mathrm{mu} + q2)) \\ -\sin(k2 \cdot \mathrm{mu} + q2) \end{bmatrix}:$

> $taud := \begin{bmatrix} \dfrac{tau1}{L1\,L2\,m2} \cdot \dfrac{L2}{g} \\ \dfrac{tau2}{L2^2\,m2} \cdot \dfrac{L2}{g} \end{bmatrix}:$

Controller

> $tau0 = Ya \cdot theta0 - Kr$

$$\tau 0 = Ya\,\theta 0 - Kr \quad (3.1)$$

> $\mathrm{tau} = tau0 + Ya \cdot u = Ya(\theta 0 + u) - Kr$

$$false \quad (3.2)$$

simplifies EoM to

> $\#M(q)dr + C(q, qdot)r + Kr = Ya(theta0 - \mathrm{theta} + u)$

> $\#tau := Multiply(Ya, (\mathrm{psi} + u)):$

> $Ya := \begin{bmatrix} y1 & y2 & y3 & y4 & y5 & y6 \\ 0 & y7 & y8 & 0 & 0 & y9 \end{bmatrix}:$

> theta0 := $\begin{bmatrix} L1^2 \cdot (m2 + \Delta m2) + Lc1^2\, m1 + I1 \\ (L2 + \Delta L2)^2\, (m2 + \Delta m2) \\ (m2 + \Delta m2)\, L1\, (L2 + \Delta L2) \\ Lc1\, m1 \\ L1\, (m2 + \Delta m2) \\ (L2 + \Delta L2)\, (m2 + \Delta m2) \end{bmatrix}$:

> theta := $\begin{bmatrix} L1^2\, m2 + Lc1^2\, m1 + I1 \\ L2^2\, m2 \\ m2\, L2\, L1 \\ Lc1\, m1 \\ L1\, m2 \\ L2\, m2 \end{bmatrix}$:

> psi := simplify(theta0 − theta,'size')

$$\psi := \begin{bmatrix} L1^2\, \Delta m2 \\ (L2 + \Delta L2)^2\, (m2 + \Delta m2) - L2^2\, m2 \\ L1\, ((m2 + \Delta m2)\, \Delta L2 + L2\, \Delta m2) \\ 0 \\ L1\, \Delta m2 \\ (m2 + \Delta m2)\, \Delta L2 + L2\, \Delta m2 \end{bmatrix} \quad \textbf{(3.3)}$$

> psi := $\begin{bmatrix} psi1 \\ psi2 \\ psi3 \\ 0 \\ psi5 \\ psi6 \end{bmatrix}$:

> r := $\begin{bmatrix} r1 \\ r2 \end{bmatrix}$:

> zeta := Multiply(Transpose(Ya), r)

$$\zeta := \begin{bmatrix} y1\, r1 \\ y2\, r1 + y7\, r2 \\ y3\, r1 + y8\, r2 \\ y4\, r1 \\ y5\, r1 \\ y6\, r1 + y9\, r2 \end{bmatrix} \quad \textbf{(3.4)}$$

if abs(zeta(i))>epsilon(i) then u(i)=-(psi(i)*zeta(i))/(abs(zeta(i))), else u(i)-=(psi(i))/(epsilon(i))*zeta(i); so call the part that varies based on value of zeta(i) beta then u = psi(i)*zeta(i)/beta(i)

> u := $\begin{bmatrix} -\frac{\text{psi}(1)}{b1} \cdot \text{zeta}(1) \\ -\frac{\text{psi}(2)}{b2} \cdot \text{zeta}(2) \\ -\frac{\text{psi}(3)}{b3} \cdot \text{zeta}(3) \\ -\frac{\text{psi}(4)}{b4} \cdot \text{zeta}(4) \\ -\frac{\text{psi}(5)}{b5} \cdot \text{zeta}(5) \\ -\frac{\text{psi}(6)}{b6} \cdot \text{zeta}(6) \end{bmatrix}$

$$u := \begin{bmatrix} -\frac{\psi 1\, y1\, r1}{b1} \\ -\frac{\psi 2\,(y2\, r1 + y7\, r2)}{b2} \\ -\frac{\psi 3\,(y3\, r1 + y8\, r2)}{b3} \\ 0 \\ -\frac{\psi 5\, y5\, r1}{b5} \\ -\frac{\psi 6\,(y6\, r1 + y9\, r2)}{b6} \end{bmatrix} \tag{3.5}$$

> $tu := \text{psi} + u$:

> $K := \begin{bmatrix} K1 & 0 \\ 0 & K2 \end{bmatrix}$:

> tau := Multiply(Ya, tu) − Multiply(K, r) :

> simplify(tau(1),'size')

$$y1\left(\psi 1 - \frac{\psi 1\, y1\, r1}{b1}\right) + \frac{y2\,\psi 2\,(-y2\, r1 - y7\, r2 + b2)}{b2} + \frac{y3\,\psi 3\,(-y3\, r1 - y8\, r2 + b3)}{b3} + y5\left(\psi 5 - \frac{\psi 5\, y5\, r1}{b5}\right) + \frac{y6\,\psi 6\,(-y6\, r1 - y9\, r2 + b6)}{b6} - K1\, r1 \tag{3.6}$$

> simplify(tau(2),'size')

$$\frac{y7\,\psi 2\,(-y2\, r1 - y7\, r2 + b2)}{b2} + \frac{y8\,\psi 3\,(-y3\, r1 - y8\, r2 + b3)}{b3} + \frac{y9\,\psi 6\,(-y6\, r1 - y9\, r2 + b6)}{b6} - K2\, r2 \tag{3.7}$$

now let beta be zeta/abs(zeta) or zeta/epsilon since they have to have the same units that would cancel!

> $u := \begin{bmatrix} -\text{psi}(1)\cdot b1 \\ -\text{psi}(2)\cdot b2 \\ -\text{psi}(3)\cdot b3 \\ -\text{psi}(4)\cdot b4 \\ -\text{psi}(5)\cdot b5 \\ -\text{psi}(6)\cdot b6 \end{bmatrix}$:

> $tu := \text{psi} + u$:

> tau := Multiply(Ya, tu) − Multiply(K, r) :

> simplify(tau(1),'size')

$$-y1\,(b1 - 1)\,\psi 1 - y2\,(b2 - 1)\,\psi 2 - y3\,(b3 - 1)\,\psi 3 - y5\,(b5 - 1)\,\psi 5 - y6\,(b6 - 1)\,\psi 6 - K1\, r1 \tag{3.8}$$

> simplify(tau(2),'size')

$$-y7\,(b2 - 1)\,\psi 2 - y8\,(b3 - 1)\,\psi 3 - y9\,(b6 - 1)\,\psi 6 - K2\, r2 \tag{3.9}$$

> $Ya := \begin{bmatrix} a1 & a1 + a2 & \cos(q2)\cdot(2\cdot a1 + a2) - \sin(q2)\cdot(dq2\cdot v2 + dq1\cdot v2 + dq2\cdot v1) & g\cdot\cos(q1) & g\cdot\cos(q1) & g\cdot\cos(q1 + q2) \\ 0 & a1 + a2 & \cos(q2)\cdot a1 + \sin(q2)\cdot dq1\cdot v1 & 0 & 0 & g\cdot\cos(q1 + q2) \end{bmatrix}$:

> tau0 := Multiply(Ya, theta0) − Multiply(K, r)

$$\tau 0 := \Big[\Big[a1\,(L1^2\,(m2 + \Delta m2) + Lc1^2\, m1 + I1) + (a1 + a2)\,(L2 + \Delta L2)^2\,(m2 + \Delta m2) + (\cos(q2)\,(2\, a1 + a2) - \sin(q2)\,(dq1y\, v2 + dq2\, v1 + dq2\, v2))\,(m2 + \Delta m2)\, L1\,(L2 + \Delta L2) + g\cos\left(\frac{\pi}{2} + q1y\right) m1\, Lc1 + g\cos\left(\frac{\pi}{2} + q1y\right)(m2 + \Delta m2)\, L1 + g\cos\left(\frac{\pi}{2} + q1y + q2\right)(L2 + \Delta L2)\,(m2 + \Delta m2) - K1\, r1\Big],$$
$$\Big[(a1 + a2)\,(L2 + \Delta L2)^2\,(m2 + \Delta m2) + (\cos(q2)\, a1 + \sin(q2)\, dq1y\, v1)\,(m2 + \Delta m2)\, L1\,(L2 + \Delta L2) + g\cos\left(\frac{\pi}{2} + q1y + q2\right)(L2 + \Delta L2)\,(m2 + \Delta m2) - K2\, r2\Big]\Big] \tag{3.10}$$

$$> \ collect\left(\left(Multiply\left(M, \begin{bmatrix} ddq1 \\ ddq2 \end{bmatrix}\right) + Multiply\left(C, \begin{bmatrix} dq1 \\ dq2 \end{bmatrix}\right) + G - subs(\{a1 = -lam1\cdot dq1, a2 = -lam2\cdot dq2, v1 = -lam1\cdot q1, v2 = -lam2\cdot q2\}, tau0)\right)\left(1\right), [ddq1, ddq2, dq1, dq2]\right)$$

$$\begin{aligned}&\left(2\,L2\,L1\,m2\cos(q2) + L1^2\,m2 + Lc1^2\,m1 + L2^2\,m2 + I1\right) ddq1y + \left(L2\,L1\,m2\cos(q2) + L2^2\,m2\right) ddq2 + \left(-2\,m2\,L1\,L2\,dq2\sin(q2)\right.\\&\quad + lam1\left(L1^2\left(m2 + \Delta m2\right) + Lc1^2\,m1 + I1\right) + lam1\left(L2 + \Delta L2\right)^2\left(m2 + \Delta m2\right) - \left(\sin(q2)\,lam2\,q2 - 2\cos(q2)\,lam1\right)\left(m2\right.\\&\quad \left.\left.+ \Delta m2\right) L1\left(L2 + \Delta L2\right)\right) dq1y - L1\,L2\sin(q2)\,m2\,dq2^2 + \left(lam2\left(L2 + \Delta L2\right)^2\left(m2 + \Delta m2\right) - \left(-\cos(q2)\,lam2 - \sin(q2)\left(\right.\right.\right.\\&\quad \left.\left.\left.-lam1\left(\frac{\pi}{2} + q1y\right) - lam2\,q2\right)\right)\left(m2 + \Delta m2\right) L1\left(L2 + \Delta L2\right)\right) dq2 + \left(L1\,m2 + m1\,Lc1\right) g\cos\left(\frac{\pi}{2} + q1y\right) + L2\,m2\,g\cos\left(\frac{\pi}{2} + q1y\right.\\&\quad \left.+ q2\right) - g\cos\left(\frac{\pi}{2} + q1y\right) m1\,Lc1 - g\cos\left(\frac{\pi}{2} + q1y\right)\left(m2 + \Delta m2\right) L1 - g\cos\left(\frac{\pi}{2} + q1y + q2\right)\left(L2 + \Delta L2\right)\left(m2 + \Delta m2\right) + K1\,r1\end{aligned} \quad \textbf{(3.11)}$$

Coordinate Change Controller

$$> \ Ya := \begin{bmatrix} a1 & a1 + a2 & \cos(q2)\cdot(2\cdot a1 + a2) - \sin(q2)\cdot(dq2\cdot v2 + dq1\cdot v2 + dq2\cdot v1) & g\cdot\cos(q1) & g\cdot\cos(q1) & g\cdot\cos(q1 + q2) \\ 0 & a1 + a2 & \cos(q2)\cdot a1 + \sin(q2)\cdot dq1\cdot v1 & 0 & 0 & g\cdot\cos(q1 + q2) \end{bmatrix}:$$

For figure 1 in "On the Robust Control of Robot Manipulators" need to change q1 to being measured from y axis so q1y = q1-pi/2, dq1y = dq1, ddq1y = ddq1
and then convert to arc length: s = q1y*L1 ds = dq1y*L1 dds = ddq1y*L1
mu = s/L2, dmu = ds/L2, ddmu = dds/L2

$$\begin{aligned}> \ Ya := \Bigg[&\left[\frac{L2}{L1}\cdot a1, \frac{L2}{L1}\cdot a1 + a2, \cos\left(q2\right)\cdot\left(2\cdot\frac{L2}{L1}\cdot a1 + a2\right) - \sin\left(q2\right)\cdot\left(dq2\cdot v2 + \frac{L2}{L1}\cdot dmu\cdot v2 + dq2\cdot\frac{L2}{L1}\cdot v1\right), -g\cdot\sin(\mathrm{mu}), -g\cdot\sin(\mathrm{mu}), -g\cdot\sin(\mathrm{mu} + q2)\right],\\&\left[0, \frac{L2}{L1}\cdot a1 + a2, \cos\left(q2\right)\cdot\frac{L2}{L1}\cdot a1 + \sin\left(q2\right)\cdot\frac{L2}{L1}\cdot dmu\cdot\frac{L2}{L1}\cdot v1, 0, 0, -g\cdot\sin(\mathrm{mu} + q2)\right]\Bigg]:\end{aligned}$$

in converting to a point mass link 2 have to modify the rho(2) term to subtract off the mass moment of inertia
7.29-m2*L2^2/12

$$> \ \mathrm{psi} := \begin{bmatrix} L1^2\,\Delta m2 \\ \left(L2 + \Delta L2\right)^2\left(m2 + \Delta m2\right) - L2^2\,m2 \\ \left(\left(m2 + \Delta m2\right)\Delta L2 + L2\,\Delta m2\right) L1 \\ 0 \\ L1\,\Delta m2 \\ \left(m2 + \Delta m2\right)\Delta L2 + L2\,\Delta m2 \end{bmatrix}:$$

$$> \ r := \begin{bmatrix} r1 \\ r2 \end{bmatrix}:$$

$$> \ \#zeta := Multiply(Transpose(Ya), r)$$

$$> \ u := \begin{bmatrix} -\mathrm{psi}(1)\cdot b1 \\ -\mathrm{psi}(2)\cdot b2 \\ -\mathrm{psi}(3)\cdot b3 \\ -\mathrm{psi}(4)\cdot b4 \\ -\mathrm{psi}(5)\cdot b5 \\ -\mathrm{psi}(6)\cdot b6 \end{bmatrix}:$$

$$> \ tu := \mathrm{psi} + u:$$

> tau := Multiply(Ya, tu) − Multiply(K, r)

$$\tau := \left[\left[\frac{L2\,a1\left(-L1^2\,\Delta m2\,b1 + L1^2\,\Delta m2\right)}{L1} + \left(\frac{L2\,a1}{L1} + a2\right)\left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2 - \left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2\right)b2\right) + \left(\cos(q2)\left(\frac{2\,L2\,a1}{L1} + a2\right) - \sin(q2)\left(dq2\,v2 + \frac{L2\,dmuv2}{L1} + \frac{dq2\,L2\,v1}{L1}\right)\right)\left(L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2) - L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b3\right) - g\sin(\mu)\,(-L1\,\Delta m2\,b5 + L1\,\Delta m2) - g\sin(\mu+q2)\left((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - ((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b6\right) - K1\,r1\right], \left[\left(\frac{L2\,a1}{L1} + a2\right)\left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2 - \left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2\right)b2\right) + \left(\frac{\cos(q2)\,L2\,a1}{L1} + \frac{\sin(q2)\,L2^2\,dmuv1}{L1^2}\right)\left(L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2) - L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b3\right) - g\sin(\mu+q2)\left((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - ((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b6\right) - K2\,r2\right]\right] \tag{4.1}$$

> tau1 := Multiply(Ya, tu)(1)

$$\tau1 := \frac{L2\,a1\left(-L1^2\,\Delta m2\,b1 + L1^2\,\Delta m2\right)}{L1} + \left(\frac{L2\,a1}{L1} + a2\right)\left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2 - \left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2\right)b2\right) + \left(\cos(q2)\left(\frac{2\,L2\,a1}{L1} + a2\right) - \sin(q2)\left(dq2\,v2 + \frac{L2\,dmuv2}{L1} + \frac{dq2\,L2\,v1}{L1}\right)\right)\left(L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2) - L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b3\right) - g\sin(\mu)\,(-L1\,\Delta m2\,b5 + L1\,\Delta m2) - g\sin(\mu+q2)\left((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - ((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b6\right) \tag{4.2}$$

> tau2 := Multiply(Ya, tu)(2)

$$\tau2 := \left(\frac{L2\,a1}{L1} + a2\right)\left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2 - \left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2\right)b2\right) + \left(\frac{\cos(q2)\,L2\,a1}{L1} + \frac{\sin(q2)\,L2^2\,dmuv1}{L1^2}\right)\left(L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2) - L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b3\right) - g\sin(\mu+q2)\left((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - ((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b6\right) \tag{4.3}$$

> tauk1 := −Multiply(K, r)(1)

$$tauk1 := -K1\,r1 \tag{4.4}$$

> tauk2 := −Multiply(K, r)(2)

$$tauk2 := -K2\,r2 \tag{4.5}$$

Dimensionless Controller

to convert to dimensionless do it in to two steps: first eliminate all units except time

> $td := \begin{bmatrix} \dfrac{tau1}{m2 \cdot L1 \cdot L2} \\ \dfrac{tau2}{m2 \cdot L2^2} \end{bmatrix}$

$$td := \left[\left[\frac{1}{m2\,L2\,L1}\left(\frac{L2\,a1\left(-L1^2\,\Delta m2\,b1 + L1^2\,\Delta m2\right)}{L1} + \left(\frac{L2\,a1}{L1} + a2\right)\left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2 - \left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2\right)b2\right) + \left(\cos(q2)\left(\frac{2\,L2\,a1}{L1} + a2\right) - \sin(q2)\left(dq2\,v2 + \frac{L2\,dmuv2}{L1} + \frac{dq2\,L2\,v1}{L1}\right)\right)\left(L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2) - L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b3\right) - g\sin(\mu)\,(-L1\,\Delta m2\,b5 + L1\,\Delta m2) - g\sin(\mu+q2)\left((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - ((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b6\right)\right)\right], \left[\frac{1}{L2^2\,m2}\left(\left(\frac{L2\,a1}{L1} + a2\right)\left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2 - \left((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\,m2\right)b2\right) + \left(\frac{\cos(q2)\,L2\,a1}{L1} + \frac{\sin(q2)\,L2^2\,dmuv1}{L1^2}\right)\left(L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2) - L1\,((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b3\right) - g\sin(\mu+q2)\left((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - ((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2)\,b6\right)\right)\right]\right] \tag{5.1}$$

> $tkd := \begin{bmatrix} \frac{tauk1}{m2 \cdot L1 \cdot L2} \\ \frac{tauk2}{m2 \cdot L2^2} \end{bmatrix}$

$$tkd := \begin{bmatrix} -\frac{K1\, r1}{m2\, L2\, L1} \\ -\frac{K2\, r2}{L2^2\, m2} \end{bmatrix} \tag{5.2}$$

> $\#K1d := \frac{K1}{m2 \cdot L2 \cdot L1}$:

> $K1 := K1d \cdot m2 \cdot L2 \cdot L1$:

> $\#K2d := \frac{K2}{m2 \cdot L2^2}$:

> $K2 := K2d \cdot m2 \cdot L2^2$:

> $simplify(tkd)$

$$\begin{bmatrix} -K1d\, r1 \\ -K2d\, r2 \end{bmatrix} \tag{5.3}$$

> $td(1)$

$$\frac{1}{m2\, L2\, L1}\Bigg(\frac{L2\, a1\left(-L1^2\, \Delta m2\, b1 + L1^2\, \Delta m2\right)}{L1} + \left(\frac{L2\, a1}{L1} + a2\right)\Big((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\, m2 - \big((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\, m2\big)\, b2\Big) + \left(\cos(q2)\left(\frac{2\, L2\, a1}{L1} + a2\right) - \sin(q2)\left(dq2\, v2 + \frac{L2\, dmuv2}{L1} + \frac{dq2\, L2\, v1}{L1}\right)\right)\Big(L1\big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2\big) - L1\big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2\big)\, b3\Big) - g\sin(\mu)\left(-L1\,\Delta m2\, b5 + L1\,\Delta m2\right) - g\sin(\mu+q2)\Big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - \big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2\big)\, b6\Big)\Bigg) \tag{5.4}$$

> $tau1d := \Big(a1\, dm2\,(-b1+1) + (k2\, a1 + a2)\cdot\big((k2+dk2)\cdot(1+dL2)\,(1+dm2) - k2\big)\cdot(1-b2) + \big(\cos(q2)\,(2\, k2\, a1 + a2) - \sin(q2)\,(dq2\, v2 + k2\, dmuv2 + dq2\, k2\, v1)\big)\cdot\big(((1+dm2)\, dL2 + dm2)\big)\cdot(1-b3) - \frac{g}{L2}\sin(\mu)\,(-b5+1)\cdot dm2 - \frac{g}{L2}\sin(\mu+q2)\cdot\big((1+dm2)\, dk2 + k2\, dm2\big)\cdot(1-b6)\Big)$:

> $td(2)$

$$\frac{1}{L2^2\, m2}\Bigg(\left(\frac{L2\, a1}{L1} + a2\right)\Big((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\, m2 - \big((L2+\Delta L2)^2\,(m2+\Delta m2) - L2^2\, m2\big)\, b2\Big) + \left(\frac{\cos(q2)\, L2\, a1}{L1} + \frac{\sin(q2)\, L2^2\, dmuv1}{L1^2}\right)\Big(L1\big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2\big) - L1\big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2\big)\, b3\Big) - g\sin(\mu+q2)\Big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2 - \big((m2+\Delta m2)\,\Delta L2 + L2\,\Delta m2\big)\, b6\Big)\Bigg) \tag{5.5}$$

> $tau2d := (k2\, a1 + a2)\cdot\big((1+dL2)^2\,(1+dm2) - 1\big)\cdot(1-b2) + \big(\cos(q2)\, a1 + \sin(q2)\, k2\, dmuv1\big)\cdot\big(((1+dm2)\, dL2 + dm2)\big)\cdot(1-b3) - \frac{g}{L2}\sin(\mu+q2)\cdot\big((1+dm2)\, dL2 + dm2\big)\cdot(1-b6)$:

> $Yad := \Big[\Big[a1,\ k2\cdot a1 + a2,\ \cos(q2)\,(2\, k2\, a1 + a2) - \sin(q2)\,(dq2\, v2 + k2\, dmuv2 + dq2\, k2\, v1),\ -\frac{g}{L2}\sin(\mu),\ -\frac{g}{L2}\sin(\mu),\ -\frac{g}{L2}\sin(\mu+q2)\Big],$
> $\Big[0,\ k2\cdot a1 + a2,\ \cos(q2)\, a1 + \sin(q2)\, k2\, dmuv1,\ 0,\ 0,\ -\frac{g}{L2}\sin(\mu+q2)\Big]\Big]$:

$> Yatheta0d := \Big[\Big[a1\cdot\big((1+dm2)+k9^2\,k1+k8\big)+\Big(\big(k2\cdot a1+a2\big)\cdot\big((1+dL2)\,(k2+dk2)\,(1+dm2)-k2\big)\Big)+\Big(\cos(q2)\,(2\,k2\,a1 + a2)-\sin(q2)\,\big(dq2\,v2+k2\,drho\,v2+dq2\,k2^2\,v1\big)\Big)\cdot(1+dm2)\,(1+dL2)-\frac{g}{L2}\sin(k2\cdot\text{rho})\cdot k9\,k1-\frac{g}{L2}\sin(k2\cdot\text{rho})\cdot(1+dm2) - \frac{g}{L2}\sin(k2\cdot\text{rho}+q2)(1+dm2)\Big],$

$\Big[\big(k2\cdot a1+a2\big)\cdot(1+dL2)^2\,(1+dm2)+\big(\cos(q2)\,a1+\sin(q2)\;drho\,k2\,v1\big)\,(1+dm2)\,(1+dL2)-\frac{g}{L2}\sin(k2\cdot\text{rho}+q2)\cdot\big((1 + dL2)\,(1+dm2)\big)\Big]\Big]:$

now to render time unitless first multiply velocity terms by sqrt(g/L2) and acceleration terms by g/L2 then multiply taud through by L2/g

$> v1 := v1d\cdot\text{sqrt}\left(\frac{g}{L2}\right):$

$> v2 := v2d\cdot\text{sqrt}\left(\frac{g}{L2}\right):$

$> r1 := r1d\cdot\text{sqrt}\left(\frac{g}{L2}\right):$

$> r2 := r2d\cdot\text{sqrt}\left(\frac{g}{L2}\right):$

$> dq2 := dq2d\cdot\text{sqrt}\left(\frac{g}{L2}\right):$

$> drho := drhod\cdot\text{sqrt}\left(\frac{g}{L2}\right):$

$> a1 := a1d\cdot\frac{g}{L2}:$

$> a2 := a2d\cdot\frac{g}{L2}:$

$> tau1d$

$$\frac{a1d\,g\,dm2\,(-b1+1)}{L2}+\left(\frac{a1d\,g\,k2}{L2}+\frac{a2d\,g}{L2}\right)\big((k2+dk2)\,(1+dL2)\,(1+dm2)-k2\big)\,(1-b2)+\left(\cos(q2)\left(\frac{2\,a1d\,g\,k2}{L2}+\frac{a2d\,g}{L2}\right) - \sin(q2)\left(dmu\,k2\,v2d\sqrt{\frac{g}{L2}}+\frac{dq2d\,g\,k2\,v1d}{L2}+\frac{dq2d\,g\,v2d}{L2}\right)\right)\big((1+dm2)\,dL2+dm2\big)\,(1-b3)-\frac{g\sin(\mu)\,(-b5+1)\,dm2}{L2} - \frac{g\sin(\mu+q2)\,\big((1+dm2)\,dk2+k2\,dm2\big)\,(1-b6)}{L2} \quad (5.6)$$

$> tau2d$

$$\left(\frac{a1d\,g\,k2}{L2}+\frac{a2d\,g}{L2}\right)\big((1+dL2)^2\,(1+dm2)-1\big)\,(1-b2)+\left(\frac{\cos(q2)\,a1d\,g}{L2}+\sin(q2)\,k2\,dmu\,v1d\sqrt{\frac{g}{L2}}\right)\big((1+dm2)\,dL2 + dm2\big)\,(1-b3)-\frac{g\sin(\mu+q2)\,\big((1+dm2)\,dL2+dm2\big)\,(1-b6)}{L2} \quad (5.7)$$

$> tau1n := simplify\left(\frac{tau1d\cdot L2}{g},\text{'size'}\right)$

$$tau1n := \frac{1}{g}\Bigg(\Bigg(\frac{a1d\,g\,dm2\,(-b1+1)}{L2}+\left(\frac{a1d\,g\,k2}{L2}+\frac{a2d\,g}{L2}\right)\big((k2+dk2)\,(1+dL2)\,(1+dm2)-k2\big)\,(1-b2) + \left(\cos(q2)\left(\frac{2\,a1d\,g\,k2}{L2}+\frac{a2d\,g}{L2}\right)-\sin(q2)\left(dmu\,k2\,v2d\sqrt{\frac{g}{L2}}+\frac{dq2d\,g\,k2\,v1d}{L2}+\frac{dq2d\,g\,v2d}{L2}\right)\right)\big((1+dm2)\,dL2+dm2\big)\,(1 - b3)-\frac{g\sin(\mu)\,(-b5+1)\,dm2}{L2}-\frac{g\sin(\mu+q2)\,\big((1+dm2)\,dk2+k2\,dm2\big)\,(1-b6)}{L2}\Bigg)L2\Bigg) \quad (5.8)$$

$> tau2n := simplify\left(\frac{tau2d\cdot L2}{g},\text{'size'}\right)$

$$tau2n := \frac{1}{g}\Bigg(-dmu\sin(q2)\,v1d\,\big((1+dm2)\,dL2+dm2\big)\,(b3-1)\,L2\,k2\sqrt{\frac{g}{L2}}-g\Big(-(b6-1)\,\big((1+dm2)\,dL2+dm2\big)\sin(\mu+q2) + a1d\,\big((1+dm2)\,dL2+dm2\big)\,(b3-1)\cos(q2)+(a1d\,k2+a2d)\,(b2-1)\,\big((1+dm2)\,dL2^2+(2\,dm2+2)\,dL2+dm2\big)\Big)\Bigg) \quad (5.9)$$

$> Yad := [[a1d,\ k2\cdot a1d+a2d,\ \cos(q2)\,(2\,k2\,a1d+a2d)-\sin(q2)\,(dq2d\,v2d+k1\,drhod\,v2d+dq2d\,k2\,v1d),\ -\sin(k2\cdot\text{rho}),\ -\sin(k2\cdot\text{rho}),\ -\sin(k2\cdot\text{rho}+q2)],$
$[0,\ k2\cdot a1d+a2d,\ \cos(q2)\,a1d+\sin(q2)\,k2\,drhod\,v1d,\ 0,\ 0,\ -\sin(k2\cdot\text{rho}+q2)]]:$

> $Yatheta0d := [[\,a1d\cdot((1+dm2)+k9^2\,k1+k8)+((k2\cdot a1d+a2d)\cdot((1+dL2)\,(k2+dk2)\,(1+dm2)-k2)+(\cos(q2)\,(2\,k2\,a1d+a2d)-\sin(q2)\,(dq2d\,v2d+k2\,drhod\,v2d+dq2d\,k2^2\,v1d)))\cdot(1+dm2)\,(1+dL2)-\sin(k2\cdot\text{rho})\cdot k9\,k1-\sin(k2\cdot\text{rho})\cdot(1+dm2)-\sin(k2\cdot\text{rho}+q2)\,(1+dm2)],$
> $[(k2\cdot a1d+a2d)\cdot((1+dL2)^2\,(1+dm2))+(\cos(q2)\,a1d+\sin(q2)\,drhod\,k2\,v1d)\,((1+dm2)\,(1+dL2))-\sin(k2\cdot\text{rho}+q2)\cdot((1+dL2)\,(1+dm2))]]:$

> $tkn := simplify\left(\frac{tkd\cdot L2}{g}\right)$

$$tkn := \begin{bmatrix} -\dfrac{L2\,K1d\,r1d\sqrt{\dfrac{g}{L2}}}{g} \\ -\dfrac{L2\,K2d\,r2d\sqrt{\dfrac{g}{L2}}}{g} \end{bmatrix} \quad (5.10)$$

morphed

as length L1 grows k2->0, k9->1, k8->0 and mu ->0 to maintain same circumferential displacement

> $k2 := 0:$

> $k9 := 1:$

> $k8 := 0:$

> $dk42 := 0:$

> $Mm := map(eval, Md)$

$$Mm := \begin{bmatrix} 1+k1 & \cos(q2) \\ \cos(q2) & 1 \end{bmatrix} \quad (6.1)$$

> $Cm := map(eval, Cd)$

$$Cm := \begin{bmatrix} 0 & -\sin(q2)\,dq2d \\ 0 & 0 \end{bmatrix} \quad (6.2)$$

> $Gm := map(eval, Gd)$

$$Gm := \begin{bmatrix} 0 \\ -\sin(q2) \end{bmatrix} \quad (6.3)$$

> $t1m := eval(subs(\{\text{rho}=0\}, tau1n))$

$$t1m := \frac{1}{g}\left(\left(\frac{a1d\,g\,dm2\,(-b1+1)}{L2}+\frac{a2d\,g\,dk2\,(1+dL2)\,(1+dm2)\,(1-b2)}{L2}+\left(\frac{\cos(q2)\,a2d\,g}{L2}-\frac{\sin(q2)\,dq2d\,g\,v2d}{L2}\right)((1+dm2)\,dL2+dm2)\,(1-b3)-\frac{g\sin(\mu)\,(-b5+1)\,dm2}{L2}-\frac{g\sin(\mu+q2)\,(1+dm2)\,dk2\,(1-b6)}{L2}\right)L2\right) \quad (6.4)$$

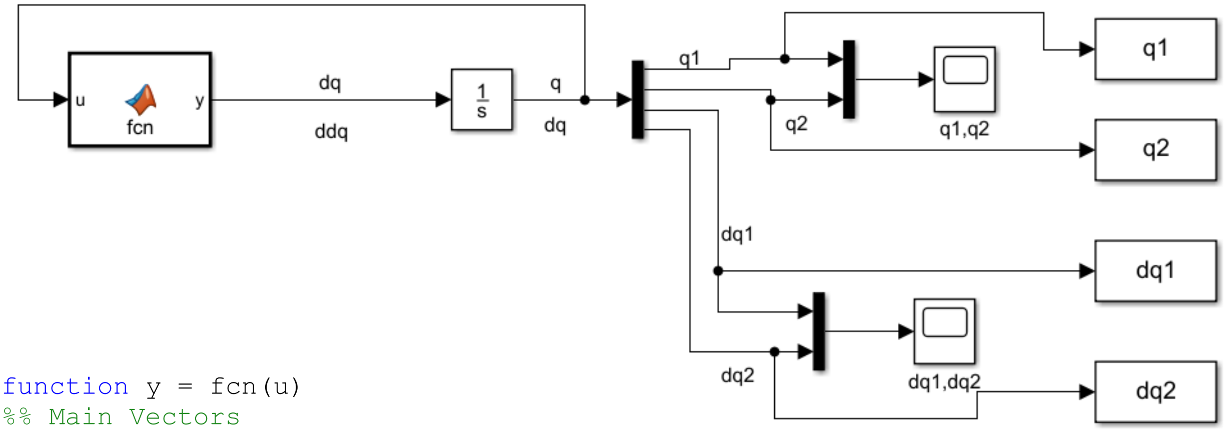
> $t2m := eval(subs(\{\text{rho}=0\}, tau2n))$

$$t2m := (b6-1)\,((1+dm2)\,dL2+dm2)\sin(\mu+q2)-a1d\,((1+dm2)\,dL2+dm2)\,(b3-1)\cos(q2)-a2d\,(b2-1)\,((1+dm2)\,dL2^2+(2\,dm2+2)\,dL2+dm2) \quad (6.5)$$

> $taum := \begin{bmatrix} t1m-K1d\cdot r1d \\ t2m-K2d\cdot r2d \end{bmatrix}$

$$taum := \left[\left[-K1d\,r1d+\frac{1}{g}\left(\left(\frac{a1d\,g\,dm2\,(-b1+1)}{L2}+\frac{a2d\,g\,dk2\,(1+dL2)\,(1+dm2)\,(1-b2)}{L2}+\left(\frac{\cos(q2)\,a2d\,g}{L2}-\frac{\sin(q2)\,dq2d\,g\,v2d}{L2}\right)((1+dm2)\,dL2+dm2)\,(1-b3)-\frac{g\sin(\mu)\,(-b5+1)\,dm2}{L2}-\frac{g\sin(\mu+q2)\,(1+dm2)\,dk2\,(1-b6)}{L2}\right)L2\right)\right],\right.$$
$$\left.\left[-K2d\,r2d+(b6-1)\,((1+dm2)\,dL2+dm2)\sin(\mu+q2)-a1d\,((1+dm2)\,dL2+dm2)\,(b3-1)\cos(q2)-a2d\,(b2-1)\,((1+dm2)\,dL2^2+(2\,dm2+2)\,dL2+dm2)\right]\right] \quad (6.6)$$

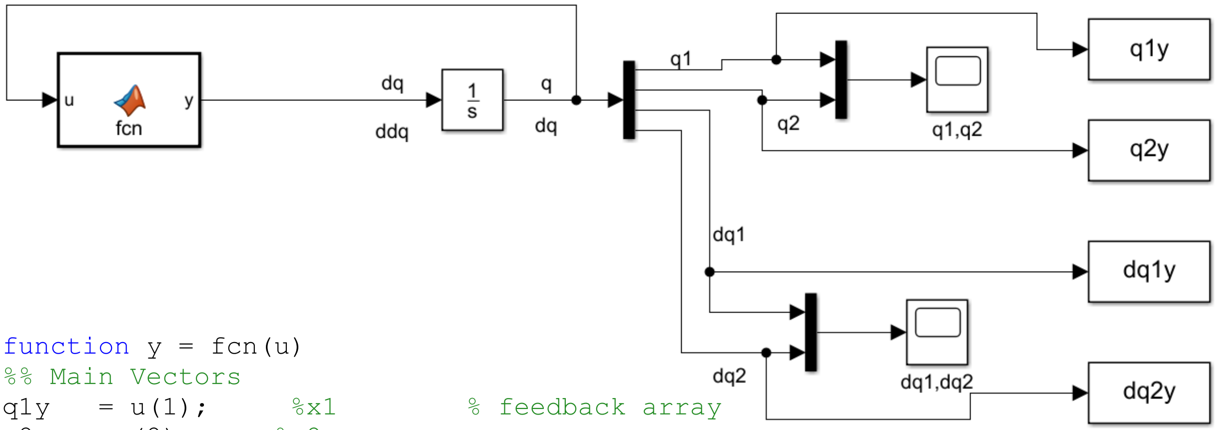
F.2 Simulink file and MATLAB code for the simulation of the Full System




```
function y = fcn(u)
%% Main Vectors
q1    = u(1);      %x1         % feedback array
q2    = u(2);      %x2
dq1   = u(3);      %x3
dq2   = u(4);      %x4
%% Generalized quantities
q     = [q1 q2]';         % Generalized coordinates
dq    = [dq1 dq2]';         % Generalized velocities
%% Generalized quantities
m1    = 10;
L1    = 1;
Lc1   = 0.5;
I1    = m1*L1^2/12;
m2    = 5;
L2    = 1;
Lc2   = 0.5;
I2    = m2*L2^2/12;% 0.083;
g     = 9.81;

c1    = m1*Lc1^2+m2*L1^2+I1;
c2    = m2*Lc2^2+I2;
c3    = m2*L1*Lc2;
c4    = m1*Lc1;
c5    = m2*L1;
c6    = m2*Lc2;

a1    = c1+c2+2*c3*cos(q2);
a2    = c2+c3*cos(q2);
a3    = c2;
%% Equation of motion pieces
M     = [a1 a2; a2 a3];
C     = c3*sin(q2)*[-dq2 -dq1-dq2; dq1 0];
G     = [(c4+c5)*g*cos(q1)+c6*g*cos(q1+q2);c6*g*cos(q1+q2)];
%% Trajectory Tracking variables
q1d   = pi/2;
q2d   = 0;
qd    = [q1d; q2d];

dq1d = 0;
dq2d = 0;
```

```
dqd  = [dq1d; dq2d];

ddq1d = 0;
ddq2d = 0;
ddqd  = [ddq1d; ddq2d];

qt   = q-qd;
dqt  = dq-dqd;

la1 = 1;
la2 = 1;
lambda = [la1 0;0 la2];
v   = dqd - lambda*qt;
a   = ddqd - lambda*dqt;
r   = dqt + lambda*qt;
%% Evaluate the control law
ac1 = a(1);
ac2 = a(2);
v1  = v(1);
v2  = v(2);

Y = [ac1, ac1+ac2, cos(q2)*(2*ac1+ac2)-sin(q2)*(dq2*v2+dq1*v2+dq2*v1), ...
      g*cos(q1), g*cos(q1), g*cos(q1+q2);...
      0, ac1+ac2, cos(q2)*ac1+sin(q2)*dq1*v1, 0, 0, g*cos(q1+q2)];
rho = [5; 7.29; 6.25; 0; 5; 6.25];
zeta = Y'*r;

epsilon = 1;
beta = [0;0;0;0;0;0];
u = [0;0;0;0;0;0];

for i = 1:6
    if zeta(i)>epsilon
        beta(i) = abs(zeta(i));
    else
        beta(i) = epsilon;
    end
    u(i)=-rho(i)*zeta(i)/beta(i);
end

k1 = 1;
k2 = 1;
K = [k1 0; 0 k2];
%% Evaluate the dynamics
ddq = inv(M)*(Y*(rho+u)-K*r-C*r)-lambda*dqt+ddqd;
ddq1 = ddq(1);
ddq2 = ddq(2);
%% M-File output
y   = [dq1;dq2;ddq1;ddq2];
%% End of
```

F.3 Simulink file and MATLAB code for the simulation of the Coordinated-Changed System

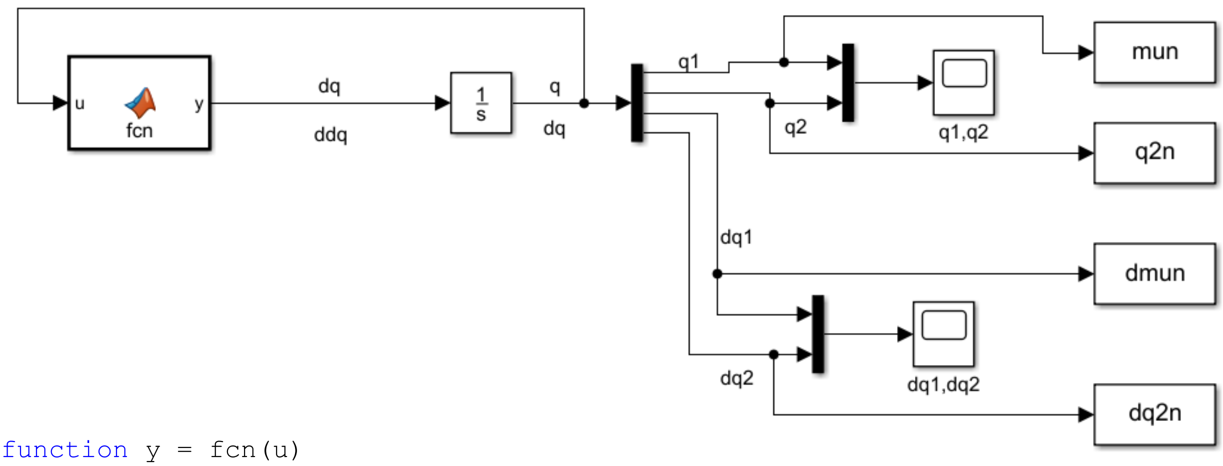



```
function y = fcn(u)
%% Main Vectors
q1y   = u(1);     %x1        % feedback array
q2    = u(2);     %x2
dq1   = u(3);     %x3
dq2   = u(4);     %x4
%% Generalized quantities
q     = [q1y q2]';        % Generalized coordinates
dq    = [dq1 dq2]';        % Generalized velocities
%% Generalized quantities
m1    = 10;
L1    = 1;
Lc1   = 0.5;
I1    = m1*L1^2/12;
m2    = 5;
L2    = 1;
%Lc2   = 0.5;
I2    = 0; %m2*L2^2/12;% 0.083;
g     = 9.81;

c1    = m1*Lc1^2+m2*L1^2+I1;
c2    = m2*L2^2+I2;
c3    = m2*L1*L2;
c4    = m1*Lc1;
c5    = m2*L1;
c6    = m2*L2;

a1    = c1+c2+2*c3*cos(q2);
a2    = c2+c3*cos(q2);
a3    = c2;
%% Equation of motion pieces
M     = [a1 a2; a2 a3];
C     = c3*sin(q2)*[-dq2 -dq1-dq2; dq1 0];
G     = [-g*((c4+c5)*sin(q1y)+c6*sin(q1y+q2));-c6*g*sin(q1y+q2)];
%% Trajectory Tracking variables
q1d   = 0;
q2d   = 0;
qd    = [q1d; q2d];

dq1d = 0;
dq2d = 0;
```

```
dqd  = [dq1d; dq2d];

ddq1d = 0;
ddq2d = 0;
ddqd  = [ddq1d; ddq2d];

qt   = q-qd;
dqt  = dq-dqd;

la1 = 1;
la2 = 1;
lambda = [la1 0;0 la2];
v   = dqd - lambda*qt;
a   = ddqd - lambda*dqt;
r   = dqt + lambda*qt;
%% Evaluate the control law
ac1 = a(1);
ac2 = a(2);
v1  = v(1);
v2  = v(2);

Y = [ac1, ac1+ac2, cos(q2)*(2*ac1+ac2)-sin(q2)*(dq2*v2+dq1*v2+dq2*v1), ...
      -g*sin(q1y), -g*sin(q1y), -g*sin(q1y+q2);...
      0, ac1+ac2, cos(q2)*ac1+sin(q2)*dq1*v1, 0, 0, -g*sin(q1y+q2)];

%rho = [5; 7.29; 6.25; 0; 5; 6.25];
deltam2 = 5;
deltaL2 = 0.125;
rho1 = L1^2*deltam2;
rho2 = (L2+deltaL2)^2*(m2+deltam2)-L2^2*m2;
rho3 = ((m2+deltam2)*deltaL2+deltam2*L2)*L1;
rho4 = 0;
rho5 = L1*deltam2;
rho6 = (m2+deltam2)*deltaL2+deltam2*L2;
rho = [rho1; rho2; rho3; rho4; rho5; rho6];

zeta = Y'*r;

epsilon = 1;
beta = [0;0;0;0;0;0];
u = [0;0;0;0;0;0];

for i = 1:6
    if zeta(i)>epsilon
        beta(i) = abs(zeta(i));
    else
        beta(i) = epsilon;
    end
    u(i)=-rho(i)*zeta(i)/beta(i);
end

k1 = 1;
k2 = 1;
K = [k1 0; 0 k2];
%% Evaluate the dynamics
```

```
ddq = inv(M)*(Y*(rho+u)-K*r-C*r)-lambda*dqt+ddqd;
ddq1 = ddq(1);
ddq2 = ddq(2);
%% M-File output
y   = [dq1;dq2;ddq1;ddq2];
%% End of
```

F.4 Simulink file and MATLAB code for the simulation of the Dimensionless System

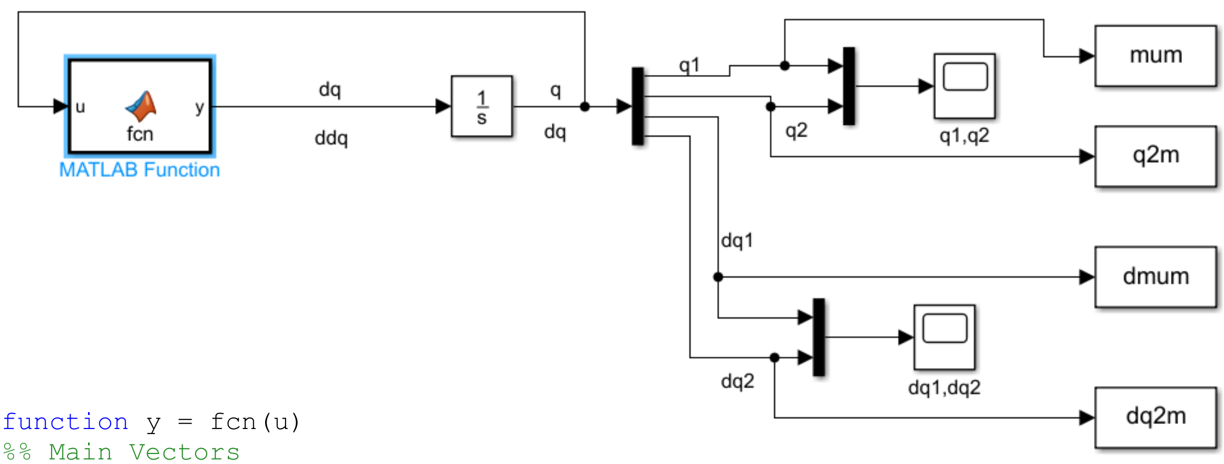



```
function y = fcn(u)
%% Main Vectors
mun    = u(1);      %x1         % feedback array
q2n    = u(2);      %x2
dmun   = u(3);      %x3
dq2n   = u(4);      %x4
%% Generalized quantities
q     = [mun q2n]';        % Generalized coordinates
dq    = [dmun dq2n]';        % Generalized velocities
%% Generalized quantities
m1    = 10;
L1    = 1;
Lc1   = 0.5;
I1    = m1*L1^2/12;
m2    = 5;
L2    = 1;
%Lc2   = 0.5;
%I2    = 0; %m2*L2^2/12;% 0.083;
g     = 9.81;

k3 = I1/(m2*L1^2);
k4 = L2/L1;
k5 = Lc1/L1;
k6 = m1/m2;

a1    = 2*k4*cos(q2n)+1+k4^2+k5^2*k6+k3;
a2    = k4+cos(q2n);
a3    = 1;
%% Equation of motion pieces
```

```
M     = [a1 a2; a2 a3];
C     = sin(q2n)*[-k4*dq2n -k4*dmun-dq2n; k4*dmun 0];
G     = [-g/L2*(k5*k6+1)*sin(mun)-g/L2*k4*sin(mun+q2n);-g/L2*sin(mun+q2n)];
%% Trajectory Tracking variables
q1d  = 0;
q2d  = 0;
qd   = [q1d; q2d];

dq1d = 0;
dq2d = 0;
dqd  = [dq1d; dq2d];

ddq1d = 0;
ddq2d = 0;
ddqd  = [ddq1d; ddq2d];

qt   = q-qd;
dqt  = dq-dqd;

la1 = 1;
la2 = 1;
lambda = [la1*sqrt(L2/g) 0;0 la2*sqrt(L2/g)];
v   = dqd - lambda*qt;
a   = ddqd - lambda*dqt;
r   = dqt + lambda*qt;
%% Evaluate the control law
ac1 = a(1);
ac2 = a(2);
v1  = v(1);
v2  = v(2);

Y = [k4*ac1, k4*ac1+ac2, cos(q2n)*(2*k4*ac1+ac2)-
sin(q2n)*(dq2n*v2+k4*dmun*v2+k4*dq2n*v1), ...
      -L2*sin(mun), -L2*sin(mun), -L2*sin(mun+q2n);...
      0, k4*ac1+ac2, cos(q2n)*k4*ac1+sin(q2n)*k4*dmun*k4*v1, 0, 0, -
L2*sin(mun+q2n)];

rho = [5/m2/L2; 7.29/m2/L2; 6.25/m2/L2; 0; 5/m2/L2; 6.25/m2/L2];

zeta = Y'*r;

epsilon = 1;
beta = [0;0;0;0;0;0];
u = [0;0;0;0;0;0];

for i = 1:6
    if zeta(i)>epsilon*sqrt(L2/g)*L2/g
        beta(i) = abs(zeta(i));
    else
        beta(i) = epsilon*sqrt(L2/g)*L2/g;
    end
    u(i)=-rho(i)*zeta(i)/beta(i);
end
```

```
k1 = 1/(m2*L1*L2)*sqrt(L2/g);
k2 = 1/(m2*L2^2)*sqrt(L2/g);
K = [k1 0; 0 k2];

tau = Y*(rho+u);
taud = [tau(1)/L1;tau(2)/L2];
%% Evaluate the dynamics
ddq = inv(M)*(taud-K*r-C*r)-lambda*dqt+ddqd;
ddmu = ddq(1);
ddq2 = ddq(2);
%% M-File output
y   = [dmun;dq2n;ddmu;ddq2];
%% End of
```

F.5 Simulink file and MATLAB code for the simulation of the Morphed System




```
function y = fcn(u)
%% Main Vectors
mun    = u(1);     %x1         % feedback array
q2n    = u(2);     %x2
dmun   = u(3);     %x3
dq2n   = u(4);     %x4
%% Generalized quantities
q     = [mun q2n]';        % Generalized coordinates
dq    = [dmun dq2n]';       % Generalized velocities
%% Generalized quantities
m1    = 10;
L1    = 1;
Lc1   = 0.5;
I1    = m1*L1^2/12;
m2    = 5;
L2    = 1;
%Lc2   = 0.5;
%I2    = 0; %m2*L2^2/12;% 0.083;
g     = 9.81;

k3 = 0; %I1/(m2*L1^2);
k4 = 0; %L2/L1;
k5 = 1; %Lc1/L1;
k6 = m1/m2;
```

```matlab
%% Equation of motion pieces
M    = [2*k4*cos(q2n)+1+k4^2+k5^2*k6+k3 k4+cos(q2n); k4+cos(q2n) 1];
C    = sin(q2n)*[-k4*dq2n -k4*dmun-dq2n; k4*dmun 0];
G    = [0;-sin(q2n)];
%% Trajectory Tracking variables
q1d  = 0;
q2d  = 0;
qd   = [q1d; q2d];

dq1d = 0;
dq2d = 0;
dqd  = [dq1d; dq2d];

ddq1d = 0;
ddq2d = 0;
ddqd  = [ddq1d; ddq2d];

qt   = q-qd;
dqt  = dq-dqd;

la1 = 1;
la2 = 1;
lambda = [la1*sqrt(L2/g) 0;0 la2*sqrt(L2/g)];
v   = dqd-lambda*qt;
a   = ddqd-lambda*dqt;
r   = dqt+lambda*qt;
%% Evaluate the control law
a1 = a(1);
a2 = a(2);
v1  = v(1);
v2  = v(2);
deltam2 = 5;
deltaL2 = 0.125;
dm2 = deltam2/m2;
dL2 = deltaL2/L2;

Y = [a1, k4*a1+a2, cos(q2n)*(2*k4*a1+a2)-sin(q2n)*(dq2n*v2+k4*dmun*...
                    v2+k4*dq2n*v1), 0, 0, -sin(q2n);...
      0, k4*a1+a2, cos(q2n)*a1+sin(q2n)*dmun*k4*v1,...
                                      0, 0, -sin(q2n)];

zeta = Y'*r;

epsilon = 1;
beta = [0;0;0;0;0;0];

for i = 1:6
    if zeta(i)>epsilon*sqrt(L2/g)*L2/g
        beta(i) = abs(zeta(i));
    else
        beta(i) = epsilon*sqrt(L2/g)*L2/g;
    end
end
u1 = zeta(1)/beta(1);
```

```
u2 = zeta(2)/beta(2);
u3 = zeta(3)/beta(3);
u6 = zeta(6)/beta(6);


tau1 = (u3-1)*((1+dL2)*dm2+dL2)*(v2*dq2n*sin(q2n)-a2*cos(q2n))+...
       (-u1+1)*a1*dm2;
tau2 = (u6-1)*((1+dm2)*dL2+dm2)*sin(q2n)-((1+dm2)*dL2+dm2)*(u3-1)*a1*...
        cos(q2n)-(u2-1)*a2*((1+dL2)^2*(1+dm2)-1);
taud = [tau1;tau2];


k1 = 1/(m2*L1*L2)*sqrt(L2/g);
k2 = 1/(m2*L2^2)*sqrt(L2/g);
K = [k1 0; 0 k2];
%% Evaluate the dynamics
ddq = inv(M)*(taud-K*r-C*r)-lambda*dqt+ddqd;
ddmu = ddq(1);
ddq2 = ddq(2);
%% M-File output
y   = [dmun;dq2n;ddmu;ddq2];
%% End of
```

F.6 MATLAB code to produce the plots for Chapter 6

```
g = 9.81;
L2 = 1;
%% full
figure(1);
plot(q1.time,q1.signals.values,q2.time,q2.signals.values);
grid on
title('time vs q link1 measured from x');
legend('q1','q2')
xlabel('time(s)')
ylabel('angular position(rad)')

%% coordinate change and point mass
figure(2);
plot(q1y.time,q1y.signals.values,q2y.time,q2y.signals.values);
grid on
title('time vs q link1 measured from y link2 point mass');
legend('q1y','q2')
xlabel('time(s)')
ylabel('angular position(rad)')

figure(3);
plot(q1y.time,q1y.signals.values+pi/2,q1.time,q1.signals.values,'--',...
     q2y.time,q2y.signals.values,q2.time,q2.signals.values,'--');
grid on
title('time vs q link1 measured from x link2 point mass compared to figure
1');
legend('q1new','q1','q12new','q2')
xlabel('time(s)')
ylabel('angular position(rad)')
```

```
%% dimensionless
figure(4);
plot(mun.time,mun.signals.values,q2n.time,q2n.signals.values);
grid on
title('time vs q link1 measured from y link2 point mass dimensionless');
legend('mu','q2')
xlabel('time(unitless)')
ylabel('angular position(rad)')

figure(5);
plot(mun.time*sqrt(L2/g),mun.signals.values,q1y.time,q1y.signals.values,'--
',...
    q2n.time*sqrt(L2/g),q2n.signals.values,q2y.time,q2y.signals.values,'--
');
grid on
title('time vs q link1 measured from y link2 point mass dimensionless');
legend('mu','q1y','q2n','q2')
xlabel('time(unitless)')
ylabel('angular position(rad)')

%% morphed
figure(6);
plot(mum.time,mum.signals.values,q2m.time,q2m.signals.values);
grid on
title('time vs q link1 measured from y link2 point mass morphed');
legend('\rho','\theta')
xlabel('time(unitless)')
ylabel('arc length(unitless) and angular position(rad)')

figure(7);
plot(mum.time*sqrt(L2/g),mum.signals.values);
grid on
title('Morphed Link1 response');
xlabel('time(unitless)*\gamma(s)')
ylabel('arc length(unitless)')

figure(8);
plot(q2m.time*sqrt(L2/g),q2m.signals.values);
grid on
title('Morphed Link2 response');
xlabel('time(unitless)*\gamma(s)')
ylabel('angular position(rad)')
```