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# A POLYNOMIAL-TIME ALGORITHM FOR LO BASED ON GENERALIZED LOGARITHMIC BARRIER FUNCTIONS

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**Abstract:** We present a polynomial-time primal-dual interior-point algorithm for solving linear optimization (LO) problems, based on generalized logarithmic barrier function. The growth term depends on a parameter $p \in [0, 1]$. The kernel functions are neither self-regular nor strongly convex. The classical logarithmic barrier function occurs if $p = 1$. The goal of this paper is to investigate such class of kernel functions and to show that the interior-point methods based on these functions have favorable complexity results. In order to achieve these complexity results, several new techniques had to be used for the analysis. Complexity issues are discussed and they are $O\left(n \log \frac{n}{\epsilon}\right)$, and $O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$, for large-update and small-update methods, respectively. Numerical tests show that the iteration bounds are influenced by $p$. We conclude that a gap still exists between the theoretical complexity and practical behavior of the algorithm.

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## 1. Introduction

In 1984 Karmarkar [5] proposed a new polynomial-time method for solving linear optimization problems. This method, and its variants that were developed subsequently, are now called interior-point methods (IPMs). For a survey we refer to recent books on the subject, as Hertog [3], Peng et al [7], Roos at al [8], Wright [10], and Ye [11]. In order to describe the idea of this paper we need to recall some ideas underlying new primal-dual IPM's. The purpose of this work is to present primal-dual interior-point methods (IPM's) based on generalized logarithmic barrier function for solving the standard linear optimization problem

$$(P) \qquad \min\left\{c^T x \;:\; Ax = b,\, x \geq 0\right\},$$

where $A \in R^{m\times n}$ is a real $m \times n$ matrix with rank $m$, and $c, x \in R^n$, $b \in R^m$. The dual problem of $(P)$ is given by

$$(D) \qquad \max\left\{b^T y \;:\; A^T y + s = c,\, s \geq 0\right\},$$

with $y \in R^m$ and $s \in R^n$.

### 1.1. Barrier Function

The generalized logarithmic barrier function considered in this paper is defined as follows:

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i), \tag{1}$$

where

$$\psi_p(t) = \psi(t) = \frac{t^{1+p} - 1}{1+p} - \log t, \quad 0 < t < \infty, \quad p \in [0, 1]. \tag{2}$$

Obviously, when $p = 1$, the growth term is quadratic and $\Psi(v)$ is the classical logarithmic barrier function, which has been studied extensively in the literature, see e.g. Roos at al [8]. When $p = 0$, $\psi(t)$ has a linear growth term. Following the terminology introduced in El Ghami [4] and Peng at al [7], we call $\psi(t)$ the *kernel function* of the barrier function $\Psi(v)$. In this paper, we derive iteration bounds for the primal-dual interior-point algorithm for solving LO problems based on the above function. We use new analysis tools developed in Bai at al [2] and El Ghami [4], yielding sharp estimates for the generalized logarithmic barrier function and its corresponding norm based proximity measure, which results in a relatively simple analysis, both for large- and small-update methods. The resulting iteration bounds for these methods are $O\left(n \log \frac{n}{\epsilon}\right)$, and $O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$, respectively. Although the theoretical iteration bound of the algorithm is not influenced by parameter $p$, we implement the algorithm and present some numerical experiments, which show that the practical behavior of the iteration bound of the algorithm relates closely to the parameter $p$ introduced in the generalized logarithmic barrier function.

The paper is organized as follows. In Section 2 we introduce the basic concepts of primal-dual interior-point algorithms for LO problems. In Section 3 we drive some properties of the kernel function $\psi(t)$, as well as the corresponding properties of the generalized logarithmic barrier function $\Psi(v)$. The estimate of the step size and the decrease behavior of the barrier function are discussed in Section 4. The inner iteration bound and the total iteration bound of the algorithm are derived in Section 5. In Section 6 we present some numerical tests. Finally, some concluding remarks follow in Section 7.

## 2. Preliminaries

We consider the LO problem (P) and its dual (D) as mentioned above and we assume that both satisfy the interior-point condition (IPC), i.e., there exists $(x^0, s^0, y^0)$ such that

$$Ax^0 = b, \;\; x^0 > 0 \quad A^T y^0 + s^0 = c, \;\; s^0 > 0.$$

It is well known that the IPC can be assumed without loss of generality, in fact we may, and will assume that $x^0 = s^0 = \mathbf{e}$, where $\mathbf{e}$ denotes the all-one vector. For this and some other properties mentioned below, see, e.g., Roos at al [8]. Finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$\begin{array}{rcl} Ax & = & b, \;\; x \geq 0, \\ A^T y + s & = & c, \;\; s \geq 0, \\ xs & = & 0, \end{array} \tag{3}$$

where $xs$ denotes the coordinatewise product of $x$ and $s$.

### 2.1. The Central Path

The basic idea of the primal-dual IPM's is to replace the third equation in (3), the so-called *complementarity condition* for (P) and (D), by the parameterized equation $xs = \mu \mathbf{e}$, with $\mu > 0$. Thus we consider the system

$$\begin{aligned} Ax &= b, \quad x \geq 0, \\ A^T y + s &= c, \quad s \geq 0, \\ xs &= \mu \mathbf{e}. \end{aligned} \tag{4}$$

Since $rank(A) = m$, and the IPC holds, the parameterized system (4) has a unique solution, for each $\mu > 0$. This solution is denoted as $(x(\mu), y(\mu), s(\mu))$ and we call $x(\mu)$ the $\mu$-center of $(P)$ and $(y(\mu), s(\mu))$ the $\mu$-center of $(D)$. The set of $\mu$-centers (with $\mu$ running through all positive real numbers) gives a homotopy path, which is called *the central path* of $(P)$ and $(D)$, see Megiddo [6] and Sonnevend [9]. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for $(P)$ and $(D)$.

### 2.2. New Search Direction

All existing primal-dual IPM's follow the central path approximately. We describe briefly how this can be done. Suppose that $x$ is primal feasible and $(y, s)$ dual feasible and $\mu > 0$. Then (4) will be satisfied if and only if the vector

$$v := \sqrt{\frac{xs}{\mu}} \tag{5}$$

equals the all-one vector $\mathbf{e}$. Now let $\Psi(v)$, $v \in \mathbf{R}^n_{++}$ be a strictly convex function such that $\Psi(v)$ is minimal at $v = \mathbf{e}$ and $\Psi(\mathbf{e}) = 0$. Then we have

$$\nabla \Psi(v) = 0 \Leftrightarrow v = \mathbf{e} \Leftrightarrow \Psi(v) = 0. \tag{6}$$

Hence, the value of $\Psi(v)$ can be considered as a measure for the distance of the given primal-dual pair to the $\mu$-center. Now consider the following system of linear equations in the variables $\triangle x$, $\triangle y$, and $\triangle s$ :

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ s\Delta x + x\Delta s &= -\mu v \nabla \Psi(v). \end{aligned} \tag{7}$$

Because $A$ has full row rank, this system has a unique solution. It follows from (7) that the right hand side in the system vanishes if and only if $v = \mathbf{e}$. Thus we conclude that $\Delta x$, $\Delta y$ and $\Delta s$ all vanish if and only if $v = \mathbf{e}$, i.e., if and only

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**Generic Primal-Dual Algorithm for LO**

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**Input:**
  A proximity function $\Psi(v)$;
  a threshold parameter $\tau > 0$;
  an accuracy parameter $\epsilon > 0$;
  a fixed barrier update parameter $\theta$, $0 < \theta < 1$;
**begin**
  $x := \mathbf{1}$; $s := \mathbf{1}$; $\mu := 1$;
  **while** $n\mu \geq \epsilon$ **do**
  **begin**
    $\mu := (1-\theta)\mu$;
    $v := \sqrt{\frac{xs}{\mu}}$;
    **while** $\Psi(v) > \tau$ **do**
    **begin**
      $x := x + \alpha\Delta x$;
      $s := s + \alpha\Delta s$;
      $y := y + \alpha\Delta y$;
    **end**
  **end**
**end**

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Figure 1: The algorithm

if $x = x(\mu)$, $y = y(\mu)$ and $s = s(\mu)$. Otherwise, at least one of $\Delta x$, $\Delta y$ and $\Delta s$ will be nonzero. We will use these vectors as our search directions. For future use we introduce scaled versions of the search directions $\Delta x$ and $\Delta s$ as follows:

$$d_x := \frac{v\Delta x}{x}, \qquad d_s := \frac{v\Delta s}{s}. \tag{8}$$

The system (7) can then be rewritten as

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y + d_s &= 0, \\ d_x + d_s &= \nabla\Psi(v), \end{aligned} \tag{9}$$

where

$$\bar{A} := AV^{-1}X = \mu AS^{-1}V,$$

and

$$V := \operatorname{diag}(v),\ X := \operatorname{diag}(x),\ S := \operatorname{diag}(s).$$

Note that $d_x$ and $d_s$ are orthogonal vectors, since $d_x$ belongs to the null space and $d_s$ to the row space of the matrix $\bar{A}$; moreover, their sum is the steepest descent direction for the barrier function $\Psi(v)$.

### 2.3. Generic Primal-Dual Algorithm for LO

In principle each barrier function $\Psi(v)$ gives rise to a primal-dual algorithm. Without loss of generality we assume that a point $(x(\mu), y(\mu), s(\mu))$ is known for some positive parameter $\mu$. For example, due the above assumption we may assume this for $\mu = 1$, with $x(1) = s(1) = \mathbf{e}$. We then decrease $\mu$ to $\mu := (1-\theta)\mu$, for some $\theta \in (0,1)$ and we solve (9). By choosing an appropriate step size $\alpha$, we move along the search direction, and construct a new triple $(x_+, y_+, s_+)$ with

$$x_+ = x + \alpha\Delta x \quad y_+ = y + \alpha\Delta y \quad s_+ = s + \alpha\Delta s. \tag{10}$$

If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), y(\mu), s(\mu))$. Then $\mu$ is again reduced by the factor $1-\theta$ and we apply the same procedure targeting at the new $\mu$-centers. This process is repeated until $\mu$ is small enough, say until $n\mu \le \epsilon$, at this stage we have found an $\epsilon$-solution of (P) and (D).

The generic form of this algorithm is given in Figure 1.

The parameters $\tau$, $\theta$ and the step size $\alpha$ in the algorithm should be chosen in such a way that the algorithm is 'optimized' in the sense that the number of iterations required by the algorithm is as small as possible. Obviously, the resulting iteration bound will depend on the kernel function, and on the choice of $\tau$, $\theta$ and $\alpha$.

## 3. Properties of the Barrier Function

In this section, we focus studying the properties of $\psi(t)$. The first, second and third derivative of $\psi(t)$ with respect to $t$ are given by

$$\psi'(t) = t^p - \frac{1}{t}, \quad \psi''(t) = pt^{p-1} + \frac{1}{t^2} \tag{11}$$

$$\psi'''(t) = -p(1-p)t^{p-2} - 2\frac{1}{t^3}. \tag{12}$$

The analysis of the IPM depends on the next technical lemmas.

**Lemma 1.** *$\psi$ has the following properties*

$$\begin{aligned} t\psi''(t)+\psi'(t) &> 0, t<1, && \text{(13-a)}\\ \psi'''(t) &< 0, && \text{(13-b)}\\ \psi''(t)\psi'(\beta t)-\beta\psi'(t)\psi''(\beta t) &> 0, t>1,\ \beta>1. && \text{(13-c)}\end{aligned}$$

*Proof.* Using (11) we have

$$\psi'(t)+t\psi^{''}(t)=(1+p)\,t^{p}.$$

Thus (13-a) follows. (13-b) is an immediate consequence of (12). Finally, (13-b) follows since

$$\psi''(t)\psi'(\beta t)-\beta\psi'(t)\psi''(\beta t)=\frac{t^{p}\,(1+p)\left(\beta^{1+p}-1\right)}{\beta t^{2}}>0.$$

Thus the lemma follows. □

Due to Lemma 2.1.2 in Peng at al [7], (13-a) implies that $\psi$ is $e$-convex, i.e., if and only if $\psi(\sqrt{t_1t_2})\leq\frac{1}{2}\left(\psi(t1)+\psi(t_2)\right)$ for all $t_1,t_2>0$.

**Lemma 2.** *One has*

$$\frac{(t-1)^2}{2t}\leq t-1-\log t\leq\psi(t),\quad \forall p\in[0,1]\quad \textit{and}\quad t\geq 1.$$

*Proof.* Setting $g(t)=\frac{(t-1)^2}{2t}-t+1+\log t$, we have $g(1)=0$, and $g'(t)=-\frac{(t-1)^2}{2t^2}<0$, for all $t>1$. This implies that $g(t)\leq 0$, for all $t>1$. From this the first inequality of the lemma follows. The second inequality follows in a similar way. □

**Lemma 3.** *One has*

$$\psi(t)<\frac{1}{2}\psi''(1)\,(t-1)^2\,,\quad \textit{if}\quad t>1\,.$$

*Proof.* By using Taylor's theorem and $\psi(1)=\psi'(1)=0$, we obtain

$$\psi(t)=\frac{1}{2}\psi''(1)\,(t-1)^2+\frac{1}{6}\psi'''(\xi)\,(\xi-1)^3\,,$$

where $1<\xi<t$ if $t>1$. Since $\psi'''(\xi)<0$. Thus the lemma follows. □

**Lemma 4.** *One has*

$$t\psi'(t)\geq\psi(t),\ \ \textit{if}\ \ t\geq 1.$$

*Proof.* Defining $g(t) := t\psi'(t) - \psi(t)$, one has $g(1) = 0$ and $g'(t) = t\psi''(t) \geq 0$. Hence $g(t) \geq 0$ and the lemma follows. $\square$

We introduce a norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}\left(v_i^p - \frac{1}{v_i}\right)^2}, \quad v \in \mathbf{R}_{++}^n. \tag{14}$$

### 3.1. Relation between $\Psi(v)$ and $\delta(v)$

For the analysis of the algorithm in Section 4 we need to establish the relation between $\Psi(v)$ and $\delta(v)$. A curial observation is that the inverse function of $\psi(t)$, for $t \geq 1$, plays an important role in this relation.

The next theorem, which is one of main results in Bai at al [2], gives a lower bound on $\delta(v)$ in terms of $\Psi(v)$. This is due to the fact that $\psi(t)$ satisfies (13-b).

**Theorem 5** (Theorem 4.9 in Bai at al [2])**.** *Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of $\psi$ on $[0,\infty)$. One has*

$$\delta(v) \geq \tfrac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right).$$

**Lemma 6.** *If $\Psi(v) \geq 1$, then*

$$\delta(v) \geq \frac{1}{12}. \tag{15}$$

*Proof.* The proof of this lemma uses Lemma 4 and Theorem 5. So we have to estimate the inverse function $\varrho$ of $\psi$ for $t \in [1,\infty)$. This is obtained by solving $t$ from the following equation:

$$\psi(t) = \frac{t^{1+p} - 1}{1+p} - \log t = s, \qquad t \geq 1.$$

Since it is hard to solve this equation explicitly, we derive an upper bound for $t$, as this suffices for our goal. From Lemma 2 we have

$$\frac{(t-1)^2}{2t} \leq \psi(t) = s,$$

hence we have the following inequality

$$t^2 - 2t\left(1+s\right) + 1 \leq 0,$$

which can be solved by

$$1+s-\sqrt{s^2+2s} \le t=\varrho(s) \le 1+s+\sqrt{s^2+2s}.$$

Therefore, the upper bound of $t=\varrho(s)$ is

$$t=\varrho(s) \le 1+s+\sqrt{s^2+2s}. \tag{16}$$

Assuming $s \ge 1$, we get

$$t=\varrho(s) \le 6s.$$

Note that if $\Psi(v) \ge 1$, then

$$\varrho(\Psi(v)) \le 6\Psi(v).$$

Now, using Theorem 5, and Lemma 4 we obtain

$$\delta(v) \ge \frac{1}{2}\psi'(\varrho(\Psi(v))) \ge \frac{\psi\left(\varrho(\Psi(v))\right)}{2\varrho\left(\Psi(v)\right)} = \frac{\Psi(v)}{2\varrho\left(\Psi(v)\right)} \ge \frac{1}{12}.$$

This proves the lemma. □

### 3.2. Growth Behavior of the Barrier Function

Note that at the start of each outer iteration of the algorithm, just before the update of $\mu$ with the factor $1-\theta$, we have $\Psi(v) \le \tau$. Due to the update of $\mu$ the vector $v$ is divided by the factor $\sqrt{1-\theta}$, with $0<\theta<1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold $\tau$ again. Hence, during the course of the algorithm the largest values of $\Psi(v)$ occur just after the updates of $\mu$. That is why in this section we derive an estimate for the effect of a $\mu$-update on the value of $\Psi(v)$. We start with an important theorem. This is due to the fact that $\psi(t)$ satisfies (13-c).

**Theorem 7** (Theorem 3.2 in Bai at al [2]). *Let $\varrho:[0,\infty)\to[1,\infty)$ be the inverse function of $\psi$ on $[0,\infty)$. Then for any positive vector $v$ and any $\beta \ge 1$ we have:*

$$\Psi(\beta v) \le n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right).$$

**Corollary 8.** *One has*

$$\Psi(\beta v) \le \frac{n\left(1+p\right)}{2}\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)-1\right)^2. \tag{17}$$

*Proof.* Since $\beta \geq 1$ and $\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$, the corollary follows from Lemma 3, Theorem 7 and $\psi''(1) = 1 + p$. □

**Corollary 9.** *Let $0 \leq \theta \leq 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq \frac{n\,(1+p)}{2}\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2. \tag{18}$$

*Proof.* By using (17), and $\beta = \frac{1}{\sqrt{1-\theta}}$, the corollary is proved. □

Suppose that the barrier update parameter $\theta$ and threshold value $\tau$ are given. According to the algorithm, at the start of each outer iteration we have $\Psi(v) \leq \tau$. By Corollary 9, after each $\mu$-update the growth of $\Psi(v)$ is limited by (18). Therefore we define

$$L(n, \theta, \tau) := \frac{n\,(1+p)}{2}\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2. \tag{19}$$

Obviously, $L(n, \theta, \tau)$ is an upper bound of $\Psi(v_+)$, the value of $\Psi(v)$ after the $\mu$-update.

## 4. Analysis of the Algorithm

In this section, we determine a default step size which not only keeps the iterations feasible but also gives rise to a sufficiently large decrease of the barrier function $\Psi(v)$ in each inner iteration. Apart from the necessary adaptations to the present context and some simplifications, the analysis below follows the same line of arguments that was used first in Peng at al [7], and later in Bai at al [1] and Bai at al [2].

### 4.1. Decrease of the Proximity

After a damped step, with step size $\alpha$, using (8) we have

$$x_+ = x + \alpha\Delta x = \frac{x}{v}\left(v + \alpha d_x\right), \quad y_+ = y + \alpha\Delta y,$$

and

$$s_+ = s + \alpha\Delta s = \frac{s}{v}\left(v + \alpha d_s\right).$$

Thus from (5) we obtain

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s). \tag{20}$$

Now $\psi$ satisfies the (13-a). Hence, $\psi(t)$ is $e$-convex, see [1]. This implies

$$\Psi(v_+) = \Psi\left(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\right) \leq \tfrac{1}{2}\left[\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right].$$

Thus we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \tfrac{1}{2}\left[\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right] - \Psi(v),$$

is a convex function of $\alpha$, since $\Psi(v)$ is convex. Obviously, $f(0) = f_1(0) = 0$. Taking the derivative to $\alpha$, we get

$$f_1'(\alpha) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\psi'(v_i + \alpha d_{xi})\, d_{xi} + \psi'(v_i + \alpha d_{si})\, d_{si}\right).$$

This gives, using the third equation in (9) and (14),

$$f_1'(0) = \tfrac{1}{2}\nabla\Psi(v)^T (d_x + d_s) = -\tfrac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{21}$$

Differentiating once more, we obtain

$$f_1''(\alpha) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i + \alpha d_{xi})\, {d_{x}}_i^2 + \psi''(v_i + \alpha d_{si})\, {d_{s}}_i^2\right). \tag{22}$$

Below we recall without proof three lemmas from Bai at al [2], and we use the following notation:

$$v_1 := \min(v), \qquad \delta := \delta(v).$$

**Lemma 10** (Lemma 4.1 in Bai at al [2])**.** *One has*

$$f_1''(\alpha) \leq 2\delta^2\, \psi''(v_1 - 2\alpha\delta).$$

Since $f_1(\alpha)$ is convex, we will have $f_1'(\alpha) \leq 0$ for all $\alpha$ less than or equal to the value where $f_1(\alpha)$ is minimal, and vice versa. In this respect the next result is important.

**Lemma 11** (Lemma 4.2 in Bai at al [2])**.** *One has $f_1'(\alpha) \leq 0$ if $\alpha$ satisfies the inequality*

$$-\psi'(v_1 - 2\alpha\delta) + \psi'(v_1) \leq 2\delta. \tag{23}$$

**Lemma 12** (Lemma 4.4 in Bai at al [2])**.** *Let* $\rho : [0, \infty) \to (0, 1]$, *denote the inverse function of* $-\frac{1}{2}\psi'(t)$ *restricted to the interval* $(0, 1]$. *Then*

$$\bar{\alpha} \geq \frac{1}{\psi''\left(\rho\left(2\delta\right)\right)}. \tag{24}$$

In the sequel we use the notation

$$\tilde{\alpha} = \frac{1}{\psi''\left(\rho(2\delta)\right)}. \tag{25}$$

By Lemma 12 we have $\tilde{\alpha} \leq \bar{\alpha}$.

**Lemma 13** (Lemma 1.3.3 in Peng at al [7])**.** *Let $h$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$, which attains its minimum at $t^* > 0$. If $h''$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \tfrac{1}{2}th'(0), \quad 0 \leq t \leq t^*.$$

**Lemma 14.** *If the step size $\alpha$ is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\,\delta^2. \tag{26}$$

*Proof.* Let the univariate function $h$ be such that

$$h(0) = f_1(0) = 0, \quad h'(0) = f_1'(0) = -2\delta^2,$$

and

$$h''(\alpha) = 2\delta^2\,\psi''\left(v_1 - 2\alpha\delta\right).$$

Due to Lemma 10, $f_1''(\alpha) \leq h''(\alpha)$. As a consequence, $f_1'(\alpha) \leq h'(\alpha)$ and $f_1(\alpha) \leq h(\alpha)$. We may write

$$\begin{aligned} h'(\alpha) &= -2\delta^2 + 2\delta^2 \int_0^\alpha \psi''\left(v_1 - 2\xi\delta\right)\,d\xi \\ &= -2\delta^2 - \delta\left(\psi'\left(v_1 - 2\alpha\delta\right) - \psi'\left(v_1\right)\right). \end{aligned}$$

Since $\alpha \leq \bar{\alpha}$, inequality (23) is certainly satisfied. Thus it follows that $h'(\alpha) \leq 0$, for all $\alpha \leq \bar{\alpha}$. Since $\psi''$ is decreasing, as a function of $t$, $h''$ is increasing in $\alpha$. Hence Lemma 13 applies and we obtain

$$f(\alpha) \leq f_1(\alpha) \leq h(\alpha) \leq \tfrac{1}{2}\alpha h'(0) = -\alpha\delta^2.$$

This proves the lemma. □

**Theorem 15.** *Let $\rho$ be as defined in Lemma 12 and $\tilde{\alpha}$ as in (25). Then*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\frac{1}{512}. \tag{27}$$

*Proof.* The first inequality of theorem follows immediately if we apply Lemma 14 to the default step size (25). To obtain the inverse function $t = \rho(s)$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0,1]$, we need to solve $t$ from the equation

$$-\left(t^p - \frac{1}{t}\right) = 2s.$$

This gives $\frac{1}{t} = 2s + t^p \leq 2s + 1$, whence $\rho(s) = t \geq \frac{1}{2s+1}$. Hence

$$\tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))} = \frac{1}{p(\rho(2\delta))^{p-1} + \frac{1}{(\rho(2\delta))^2}} \geq \frac{1}{p\,(4\delta+1)^{1-p} + (4\delta+1)^2}.$$

Since $p\,(4\delta+1)^{1-p} \leq (4\delta+1)^2$, for $p \in [0,1]$, it follows that

$$\tilde{\alpha} \geq \frac{1}{2\,(4\delta+1)^2}.$$

Denote

$$\tilde{\alpha} := \frac{1}{2\,(4\delta+1)^2}, \tag{28}$$

this will be our default step size. Hence

$$f(\widetilde{\alpha}) \leq -\frac{\delta^2}{2\,(4\delta+1)^2}.$$

By using (15), that $12\delta \geq 1$, we have

$$f(\widetilde{\alpha}) \leq -\frac{1}{512}.$$

This proves the theorem. $\square$

## 5. Iteration Bounds

In this section we drive the complexity bounds for large-update methods and small-update methods. An upper bound for the total number of iterations is obtained by multiplying (the upper bound for) the number $K$ by the number of barrier parameter updates, which is bounded above by (cf. Roos at al [8], Lemma II.17, page 116)

$$\frac{1}{\theta}\log\frac{n}{\epsilon}.$$

**Lemma 16** (Proposition 1.3.2 in Peng at al [7])**.** *Let $t_0, t_1, \cdots, t_K$ be a sequence of numbers such that*

$$0 < t_{k+1} \le t_k - \kappa t_k^{1-\gamma}, \qquad k = 0, 1, \cdots, K-1, \tag{29}$$

*where $\kappa > 0$ and $0 < \gamma \le 1$. Then $K \le \left\lfloor \frac{t_0^\gamma}{\kappa\gamma} \right\rfloor$.*

**Lemma 17.** *If $K$ denotes the number of inner iterations, we have*

$$K \le 512\Psi_0.$$

*Proof.* The definition of $K$ implies $\Psi_{K-1} > \tau$ and $\Psi_K \le \tau$ and

$$\Psi_{k+1} \le \Psi_k - \kappa\left(\Psi_k\right)^{1-\gamma}, \qquad k = 0, 1, \cdots, K-1,$$

with $\kappa = \frac{1}{512}$ and $\gamma = 1$. Application of Lemma 16, with $t_k = \Psi_k$ yields the desired inequality. □

Let $L = L(n, \theta, \tau)$, as defined in (19). Using $\psi_0 \le L$, and Lemma 17 we obtain the following upper bound on the total number of iterations:

$$\frac{512L}{\theta}\log\frac{n}{\epsilon}. \tag{30}$$

We finally have to estimate $L$, i.e., to drive an upper bound for $\Psi(v)$ just after a $\mu$-update. Using (16), and Lemma (18), we obtain

$$L \le \frac{n\,(1+p)}{2}\left(\frac{1+\frac{\tau}{n}+\sqrt{\left(\frac{\tau}{n}\right)^2+\frac{2\tau}{n}}}{\sqrt{1-\theta}} - 1\right)^2.$$

Using also $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \le \theta$, this leads to

$$L \le \frac{n}{(1-\theta)}\left(\theta+\frac{\tau}{n}+\sqrt{\left(\frac{\tau}{n}\right)^2+\frac{2\tau}{n}}\right)^2 = \frac{\left(\theta\sqrt{n}+\frac{\tau}{\sqrt{n}}+\sqrt{\frac{\tau^2}{n}+2\tau}\right)^2}{(1-\theta)}.$$

We conclude that the total number of iterations is bounded by

$$\frac{K}{\theta}\log\frac{n}{\epsilon} \le 512\frac{\left(\theta\sqrt{n}+\frac{\tau}{\sqrt{n}}+\sqrt{\frac{\tau^2}{n}+2\tau}\right)^2}{\theta\,(1-\theta)}.$$

A large-update methods uses $\tau = O(n)$ and $\theta = \Theta(1)$. Then the right hand side expression is $O\left(n \log \frac{n}{\epsilon}\right)$.

For Small-update methods use $\tau = \Theta(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Then the right hand sid expression is $O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$.

## 6. Numerical Tests

In this section we investigate the influence of the choice of the parameter $p$ on the computational behavior of the generic primal-dual algorithm for LO, as given in Figure 1. In our experiment we used the kernel function $\psi(t)$ with several values of parameters namely $p \in \{0; 0.25; 0.5; 0.75; 0.9; 1\}$.

For the test problems we used problems from the well-known library Netlib.[1] To limit the number of test problems we applied the algorithm only to a selection[2] of ten of these problems. We used a straightforward implementation of our algorithm in MATLAB.[3] We employed the self-dual embedding model Roos at al [8] to enable us to start the algorithm as indicated in Figure 1, namely with $x = s = \mathbf{1}$ and $\mu = 1$. Our experiments were performed on a standard PC with a Pentium 4 processor and with 1 GB internal memory.

Since we wanted to compare iteration numbers for several kernel functions, and since these numbers depend on the parameters $\tau$, $\theta$ and the accuracy parameter $\epsilon$, we fixed these parameters in our experiments to $\tau = 1$, $\theta = 0.99$ and $\epsilon = 10^{-8}$. In this way the iteration numbers depend only on the choice of the parameter $p$ and the problem instance.

For each of the ten problems we used **bold** font to highlight the best, i.e., the smallest, iteration number. From Table 1 we may draw the following conclusion: the numbers of iterations obtained by using $\psi_0$, which has a linear growth term, are the worst. For $\psi_p$, it becomes clear that smaller values of the parameter $p$ influence the iteration count negatively. Hence, $p = 1$ seems to be the best possible choice, which gives $\psi_1$, the kernel function of the logarithmic barrier function.

[1] http://www-fp.mcs.anl.gov/otc/Guide/TestProblems/index.html

[2] The selection was based en the problem instance: Small instance (Adlittle, Afiro, Sc105 and Sc205), Medium instance (Degen2, Grow15 and Shell) and Large instance (Degen3, Maros and Sctap2).

[3] http://www.matlab.com

| LO Problem | Number of iterations | | | | | |
|---|---|---|---|---|---|---|
| | $\psi_1$ | $\psi_{0.9}$ | $\psi_{0.75}$ | $\psi_{0.5}$ | $\psi_{0.25}$ | $\psi_0$ |
| ADLITTLE | **22** | 23 | 30 | 58 | 201 | $\geq 300$ |
| AFIRO | **16** | 18 | 26 | 58 | 137 | $\geq 300$ |
| DEGEN2 | **24** | 28 | 44 | 141 | $\geq 300$ | $\geq 300$ |
| DEGEN3 | **28** | 32 | 43 | 138 | $\geq 300$ | $\geq 300$ |
| GROW15 | **35** | 49 | 56 | 111 | $\geq 300$ | $\geq 300$ |
| MAROS | **67** | 69 | 81 | 171 | $\geq 300$ | $\geq 300$ |
| SC105 | **20** | 25 | 35 | 64 | 161 | $\geq 300$ |
| SC205 | **19** | 24 | 53 | 123 | $\geq 300$ | $\geq 300$ |
| SCTAP2 | **24** | 29 | 40 | 127 | $\geq 300$ | $\geq 300$ |
| SHELL | **55** | 59 | 71 | 175 | $\geq 300$ | $\geq 300$ |

Table 1: Iteration numbers for $\psi_p$.

## 7. Concluding Remarks

In this paper we prove that the iteration bound of a Large-update interior-point method based on this type of barrier function with $p \in [0,1]$, gives the classical iteration complexity $O\left(n \log \frac{n}{\epsilon}\right)$. For small-update methods gives the best known iteration complexity namely $O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$. Numerical tests demonstrate that in practice the iteration bound of the algorithm depends on the parameter $p \in [0,1]$, and that $p = 1$ seems to be the best choice in practice.

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