Adaptive Learning in Mathematics

Situating Multi Smart Øving in the Landscape of Digital Technologies for Mathematics Education

SLATE Research Report 2019-3

SLATE
Centre for the Science
of Learning & Technology

SLATE Research Report 2019-3

Kynigos, C. (2019)

Contact information:
Kjetil Egelandsdahl (kjetil.egelandsdahl@uib.no)
Centre for the Science of Learning & Technology (SLATE)
University of Bergen
Post Box 7807
5020 Bergen
Norway

ISBN: 978-82-93789-00-0 (PDF)
ISBN: 978-82-93789-01-7 (PRINT)

When referring to this report, use:
Kynigos, C. (2019). Adaptive Learning in Mathematics: Situating Multi Smart Øving in the Landscape of Digital Technologies for Mathematics Education. *SLATE Research Report 2019-3*, Bergen, Norway: Centre for the Science of Learning & Technology (SLATE). ISBN: 978-82-93789-00-0

Preface

From the autumn of 2018 to the autumn of 2019, the Centre for the Science of Learning and Technology (SLATE), University of Bergen studied the adaptive teaching program “Multi Smart Øving” (MSØ). The Adaptive Learning in Mathematics (ALMATH) project was lead by Researcher Kjetil Egelandsdal and was an independent research project fully financed by SLATE. ALMATH has resulted in two research reports: 1) an evaluation report by Professor Chronis Kynigos at the University of Athens and Professor II at SLATE, and 2) an empirical report by SLATE researchers.

This is the evaluation report where Kynigos situates MSØ in the international landscape of digital teaching tools. The report includes a review of digital technologies for mathematics education.

About Multi Smart Øving

Multi Smart Øving (MSØ) is a Norwegian adaptive learning system, developed by Gyldendal Norsk Forlag AS, for Norwegian mathematics education that is built on a platform developed by the New York–based company Knewton. The system is created for primary schools, first to seventh grade, where the age of the pupils ranges from 5–13 years old. According to Gyldendal, 188,000 students and 9,822 teachers have licenses for the program. As this represents over 40% of all Norwegian pupils in primary school, it can be said that the program is widely distributed.

MSØ is built around individual task solving and is considered by Gyldendal as a digital workbook for practicing mathematics. Each pupil works with one task at a time and is given three attempts to answer correctly. If the pupil answers correctly, or incorrectly three times, she or he is presented with a new task.

As an adaptive learning program, the intention behind MSØ is that the tasks are constantly adapted to the individual pupil’s abilities and needs. A pupil who shows mastery of a task receives a new task that is increasingly advanced, while a pupil who struggles with a certain type of task will receive other tasks adapted to his or her competence. In this way, the intention is that the tasks are aligned to the individual pupil’s level of performance.

This individual customisation is based on a competence profile developed for each pupil. The profile changes continuously in line with how the pupil solves the tasks, and the teacher is presented with an overview of the competency of each pupil (and the class) and how the pupils are progressing. In addition, the teacher can see with which learning objectives the pupil is working, the number of correct and wrong answers, and how long the pupil has worked with MSØ. The pupil can send tasks, along with comments, to the teacher if she or he wishes, for instance, to discuss a particular task with the teacher.

Table of Contents

1. How should a specific medium be used for mathematics in school?

It has now been around 50 years since digital technology was seriously considered as a tool for mathematical activity for learners (Papert, 1972, 1980). In this time, there has been a great diversity of digital media and approaches to the ways in which they can be used for learning mathematics. Indeed, these approaches have been influenced by varying standing points on the nature of mathematics and of the learning and teaching of mathematics, as well as on the role digital media can play regarding the influence on and intervention in educational practice.

In reviews on the uses of digital media for mathematics education over the last decade, this complexity in the adopted approaches has been addressed in different ways. With this context in mind, the current report constitutes a meta-review that is aimed at illuminating and supporting decisions on whether and how to use a specific medium or configuration of diverse media in a school or classroom. Who should decide which medium/media to employ in mathematics courses in a particular classroom? Based on what criteria? How should they be supported to make such decisions?

To address these questions, a careful selection was made of a group of important, but divergent reviews over the last decade (see the 'Primary References' section). These were used to organize this report primarily in relation to a key dichotomy:

> Is a medium going to be used to change or enhance exiting educational practices in a classroom?

Subsequently, the current report analyses the diverse affordances of media designed for mathematics education in connection with the above dichotomy. This is followed by a different perspective, that of the temporal evolution of the ways in which technologies have been used in mathematics education, specifically in connection to the technological developments, and in education in general. The most popular productions are listed and characterized in relation to the above issues. A comprehensive, overall suggestion is finally made as to how to make decisions on the available avenues to integrate technology use in the mathematics classroom.

In educational systems around the world, mathematics education is a practice that is mostly associated with a discontent that is expressed by all kinds of stakeholders and, most importantly, the students themselves. The frustration pertains to the diversity of issues associated with the relevance of the mathematics education paradigm in the digital era. Symptomatically, specific issues are being questioned, ranging from the content and structure of mathematics curricula (Goldenberg, 1999) to the practice of teachers and their available initial training and in-service professional development and support, the understanding and the assessment performance of students, and, finally, to the validity of the assessment methods themselves (Pimm &

Johnston-Wilder, 2005; Trouche, Drijvers, Gueudet & Sacristan, 2013). To most students, mathematics provokes stress, anxiety, a sense of helplessness, and a lack of relevance to their social and intellectual lives, doing little to spark their curiosity and creativity (Papert, 1980; Klein, 2002). Doing mathematics is painstaking and is associated with learned helplessness, reprimand avoidance, provoking judgment, and a general sense of failure and low self-esteem (Dreyfus, 1999; Topcu, 2011). However, most of all, there is a growing concern regarding the epistemology of mathematics, of what it means to engage in mathematical thinking and of what type of mathematical education might be of value and relevance in the current era (Chevallard, 2012; Schoenfeld, 1992). The educational paradigm in mathematics education is openly under question, with Alan Schoenfeld calling it “the math wars” (Schoenfeld, 2004, p. 253).

Digital technologies play an important role in this reflective process, providing the tools for potential solutions both to the proponents of making traditional processes more effective and to the proponents of paradigm transformation (Ainley et al., 2011; Conole, 2008; Olive et al., 2010; Puentedura, 2006; Yelland, 2005). Therefore, no school should just go ahead and use a particular type of technology over others, ignoring the pedagogy associated with its designs and the kinds of pedagogies that might be compatible with its use. Furthermore, pedagogy should come first: a district, a school, or a teacher should have a clear idea of the kind of education they wish to provide their students with and then make an educated choice regarding the diverse technologies and other resources appropriate for that pedagogy.

As will be shown, Multi Smart Øving (MSØ) is clearly a tool for the proponents of enhancing traditional approaches to mathematics education. Designed to be used for the augmentation of students’ rote learning of mathematical routines, it is coupled with an automated and traditional type of assessment that also embeds adaptivity based on a generalized recommender system applied in mathematics education.

This report attempts to place MSØ within the context of the diverse digital media for mathematics education, providing useful classifications that are connected to pedagogy and examples of diverse, digital solutions. It also aims to provide a picture of the potential best practices in classrooms today, aiming for reform, but also maintaining relevance to current norms, especially regarding traditional assessments. In a nutshell, the current report concludes that MSØ could have a role as an integrated solution involving a range of digital tools used to make rote learning more effective so as to leave time for transformative experiences for students to engage with meaningful mathematics and mathematical thinking.

There are several initiatives and institutions claiming to have approaches that can address the problem of mathematics education and even solutions to specific aspects, such as the ubiquitous availability of high-quality video explanations of distinct mathematical concepts (see https://www.khanacademy.org/).

This problematic area of education, however, has inspired the generation and development of a scientific field studying mathematical teaching and learning and a multiplicity of aspects of mathematics education. This field is young, compared with, for instance, the field of mathematics itself. As a field of scientific study it can be said that it was initiated either by

- the theory suggested in the 1950s by Jean Piaget (e.g. Piaget, 1955) regarding the nature and process of rational logical thinking and understanding in humans or
- in 1976, the date of the first Conference in the Psychology of Mathematics Education (http://www.igpme.org/), a community of researchers engaged in scientific study that is focused on the teaching and learning of mathematics.

Despite its relatively recent emergence, the field of mathematics education has been developing and generating an impressive volume of work, either with a diagnostic or an illuminative approach, or a design or interventionist approach (Von Glaserfeld, 1989; Walkerdine, 1988; Cobb, 1994; Noss & Hoyles, 1996). The diagnostic and illuminative approaches attempted to help us understand learners' understanding of mathematics either by identifying misconceptions or by studying the mathematical thinking that learners actually engage with, respectively. The design approach involves the study of educational practice put into effect as a result of a carefully designed intervention to transform traditional everyday practices. Most of this work, however, perceives mathematics education as a dynamic body of values, institutions processes, and practices warranting fundamental change. Mathematics education should be a service to young generations that are under pressure to transform, rather than to enhance the existing features and practices.

Inevitably, digital solutions—including artifacts, services, and tools—have been an important part of the development of efforts to address the problem found by mathematics education stakeholders, such as education system policy bodies and the corporate sector who identify potential markets through perceived user needs. Often, however, these efforts by-pass or simply ignore the knowledge emerging from the scientific field of mathematics education.

The scientific field of mathematics education has developed theories pertaining to the learning of mathematics through the use of digital media and has designed original pedagogically principled solutions. Furthermore, a new research method termed design research (Cobb et al, 2003) was developed with the aim of producing “frameworks for action,” that is, theoretical constructs and lenses with which to design innovations that take part in their implementation; another goal of design research is studying mathematical learning taking place within the new dynamics of the ensuing educational practices (DiSessa & Cobb, 2004). Thus, the current report is based on the knowledge emerging from the academic field of mathematics education and the growing number of empirical studies of mathematical educational practice within and out of the classroom.

From a realistic and practical point of view, it is worth considering problems faced in real classrooms, where the school or individual teachers have taken initiative to use a particular type of digital technology in mathematics courses. For any given school or group of students, the typical approaches that have proven ineffective are the following:

- Employing a single digital tool designed for one particular approach to mathematics education, that is, to enhance traditional practices or to pursue a transformative intervention
- Employing a tool designed for a particular approach and then assessing students’ learning with methods and principles belonging to a different approach

- Employing a tool without much thought as to the kind of pedagogy for which it has been designed
- Allowing the emergence of a “quick fix” type of expectation in parents and educational administration and policy makers, such as “use this tool and student performance will improve”
- Expecting a clear, quick, and sustainable result from the use of one tool and assigning a causal frame to the tool itself without paying much notice to the ways it is being used and adopted

In the current report, MSØ is evaluated with respect to its potentially beneficial use in a transformative intervention; indeed, MSØ is looked upon as part of a diversity of digital tools and a pedagogy attempting to make rote learning more effective. In this way leaving space for deeper, more relevant kinds of learning that can help with the growth of a sustainable rationality, rigor, economy, proof, and mathematical justification; generalization and problem posing; and solving given, predefined problems.

2. The nature of transformation sought in mathematics education

2.1 Epistemology of mathematics and of the learning of mathematics

Over the past 40 years, there have been a growing number of voices questioning traditional assumptions about the nature of mathematics and of doing mathematics. This is having a fundamental influence on the academic community in mathematics education and, in turn, on educational reform initiatives in policy by stakeholders. Until the beginning of the twentieth century, mathematics was considered to be a positivist discipline, a collection of irrefutable truths based on axiomatic systems embedding robust sets of rules that are mediated by an arbitrary but agreed set of representations. In school, it was inevitably considered and taught as such, that is, as a set of arbitrary, fixed truths to be understood, even at the rote level, by students so that they can give correct answers on standardized tests and assessment procedures.

The argument on the nature of mathematics began with scholars such as Foucault (1984), Kitcher (1985), Papert (1972), and Lakatos, (1976). Synthetically, they argued that is makes more sense to think of mathematics as a human intellectual construct, one that is essentially fallible (in the sense of Ernest, 1991; that is, each mathematical definition, lemma, theorem, and proof has the status of a proposition for others to try to refute. Even in cases where theorems are proven and problems are solved, these problems wait for other mathematicians to question the process, the context, the point of view of articulating the question, and the assumptions. This process is essentially part of doing mathematics, and it does not matter whether and to what extent an axiomatic system remains robust or a theorem is proven to have a shaky foundation (see also Kynigos, 2015). Engaging in mathematical activity necessarily involves the process of refutation together with logical thought, deduction, generalization, and proof. At the time, Lakatos (1976) was provoking in his book proofs and refutations, where he analysed mathematicians' activity as a process of conjecture, showing it to be a public expression of thought and subsequent engagement with a cycle of refutation, redrafting–drafting, and new proofs:

> …deductivist style tears the proof generated definitions off their "proof-ancestors" presents them out of the blue in an artificial and authoritarian way. It hides the global counter examples that led to their discovery. Heuristic style, on the contrary, highlights these factors. It emphasizes the problem situation: it emphasizes the "logic" which gave birth to the new concept (Lakatos, 1976, p. 144)

This reflection and questioning of the nature of mathematics and doing mathematics fundamentally influenced thinkers in the academic field of mathematics education. In the ensuing years, it has permeated reform agendas and even practices in schools, including the use of digital media.

The kind of mathematical activity described by Lakatos and almost concurrently by Davis and Hersch (1981) was not only addressed by mathematicians' activities, but was also connected to the essence of learning mathematics. Papert (1972) argued that this kind of activity cultivates learning not only in established mathematicians, but for all, even for young children. He went further, saying that this activity is natural and that traditional schooling denies students the opportunity and encouragement to engage in the logic that gives birth to mathematical concepts (Papert, 1980). As Lakatos put it, traditional perceptions of the nature of mathematics imposes an artificial picture of mathematics to be the practice of trying to understand the abstract and irrelevant products of mathematical activity rather than the activity itself.

Like Papert, Foucault (1984) and Kitcher (1985) maintained that mathematical activity and mathematical learning emerge as a result of generating mathematical meaning and engaging in negotiation regarding mathematical meaning. In this framework, mathematics itself has the status of transparency and negotiability, rather than that of an arbitrary and irrefutable truth.

Negotiating mathematical meaning requires a means of externalizing thought through semiotic mediation (Vygotsky, 1978; Bartolini-Busi & Mariotti, 1997; Berger, 2005). Digital media are perceived of as a new set of semiotic registers, a means to negotiate mathematical meanings with (Marriotti, 2006). Varying the expressive medium results in a restructuration of mathematical meaning-making and in alternative ways of knowing (Willensky & Papert, 2010; Noss & Hoyles, 1996, Noss et al, 2007). Brown (1994) proposed that mathematical learning should be perceived as a kind of immersion into a mathematical culture of people using media to negotiate mathematical meanings and that the teacher should facilitate this process rather than try to control it with the rote memorization of mathematical fact.

Regarding the uses of digital media for generating and negotiating meanings, apart from Papert (1980) and Noss and Hoyles (1996), there are other important approaches, that of instrumental genesis (Star & Griesmeyer, 1985; Artigue, 2010; Trouche, 2005; Verillon & Rabardel, 1995) and humans-with-media (Borba & Villareal, 2005). Instrumental genesis allows us to understand and illuminate how a learner uses a digital medium. Once learners pick up a specific digital artifact they develop a scheme of use; a conceptual construct of the purpose, meanings, and situations this artifact is useful for. Thus, each individual develops his or her own schema, turning the artifact into a unique instrument that is associated with the individual. In this process, the artifact does not remain unaltered; it is also shaped reciprocally with the generation of the instrument in a process called instrumentalisation. Therefore, each individual makes his or her own sense of a tool, moulding the tool itself. Good digital media designs when it comes to learning mathematics are thus those that usefully support this process: they are malleable, afford deep structural access to users, and yet have embedded powerful mathematical ideas. Humans-with-media is an approach of perceiving mathematical meaning-making as residing in the ways in which collectives of people interacting with each other and with digital media reorganize their reasoning and thinking in an on-going process. Meanings are negotiated through discourse, communication, and the interactions with media (i.e., by creating or tinkering with mathematical models and their embedded operational rules and properties by handling data, by generating experiments with models, by outsourcing computational process, and so forth (Steele, 2001). To generate dense mathematical learning, we need pedagogical interventions that involve collectives of students, teachers, and experts in mathematical application domains, all of whom discuss a problem and mutually re-organize their understandings as a reciprocal process of communication and tool use.

2.2 Approaches toward mathematical learning with digital media

Mathematics education research has compellingly showed that mathematical thinking and the use of mathematical concepts as tools and as objects (Douady, 1985) are hard for most learners. The main aspects of mathematics, generalization, rigor, economy, mathematical proofs, and the sense of certainty focus on the process of doing mathematics rather than the ability to reproduce routines or memorize factual information (Radford, Bardini & Sabena, 2007; Sfard, 1991; Hoyles, 2016). Without personal experience in engaging with mathematical thinking and being able to appreciate the practical and cultural communicational empowerment that these aspects generate, it is impossible for learners to conceptualize abstract concepts and the process of abstraction itself.

Therefore, an education system promoting rote learning in effect hinders engagement with mathematical thinking. As Bray and Tangney pointed out (2017), from the point of view of the mathematics education community, the most important problem hindering students' understanding of the mathematics subject is the widespread belief that mathematics in school should be a stably structured collection of abstract concepts and rules to be memorized, including solutions to predefined exercises involving procedures leading to one correct and unquestioned answer (Hoyles, 2016; Noss & Hoyles, 1996; Ernest, 1998 Maas & Artigue, 2013). Regarding teaching and learning, in many countries, mathematics has been emphasized as a scientific production, a formal and abstract body of knowledge to be understood and digested by students. The teacher is viewed as the authority on the subject, and the teacher's primary mission is to deliver this knowledge to the students by explaining and transmitting information. To fit in large-scale traditional assessment methods and tools (Sinclair et al., 2010), mathematics is seen as a disjointed set of rules and procedures, rather than a complex grid of concepts that can be structured in a variety of ways. Recent curriculum reforms, however, recognize that a view of mathematical competence as solely related to procedures and concepts that can be accumulated with practice is naïve and incomplete (Contreras, 2014). Students do not generate experiential meaning related to mathematical concepts; they are not able to use math in realistic problem situations, hence becoming disenchanted and losing motivation.

Although the importance of embedding mathematics in meaningful contexts so that it can be experienced and used by students has been recognized (Kolb, 1984; Boaler, 2008), the related task designs to achieve this have fallen short of meaning-making. One of the reasons is that the concepts in their abstract and formal form still remain a key objective, resulting in the production of pseudo-real-world problems that are hardly motivating for the students.

Hence, strategically, the mathematics education community has turned to the question of whether—and if so how—digital technologies can provide added value to the design of contexts for mathematical meaning-making in situations where mathematical concepts and mathematical, rational thinking can be experienced and put to use for relevant and realistic purposes (Clements, 2000). This strategic challenge is becoming more and more clear thanks to research in the field, which has been changing the paradigm. However, developing tools to observe the results is proving to be a difficult, nonlinear, and lengthy process, and digital media have been addressed as tools that have the potential to facilitate this strategic change.

The affordances of such tools are an important factor, and these are analysed further in a subsequent section. Indeed, they allow and encourage students to engage in modelling (Confrey & Maloney, 2007), in the visualization of abstract concepts (Confrey, Maloney, Ford & Nguyen, 2006), in the dynamic manipulation of mathematically constructed representations (Sacristan & Noss, 2008, Artigue, 2002), in testing out the students' ideas, in handling large amounts of data (spreadsheets), and in seeing the mathematical in models of complex socio-scientific issues such as environmental sustainability. They also afford collaborative problem solving and posing (Conneely, Lawlor & Tangney, 2015), mathematical argumentation, and the expression of ideas (Wegerif, 2007). In effect, they are tools for expression, experimentation, model construction, and modification and are designed to allow for open-ended investigations and student choice of alternative routes to a solution. They not only allow, but also rather encourage, students to take some control over their activity and their progress, inspiring agency and meaning in their work. Therefore, students can make practical use of concepts and processes to achieve results, acquiring a sense of purpose and utility of mathematical thinking. These experiences have been found to deepen the students' understanding and boost their confidence and enjoyment of thinking mathematically (Chance, Garfield & delMas, 2000; Clements, Sarama, Yelland & Glass, 2008; Roschelle, Kaput & Stroup, 2000; Sacristan & Noss 2008; Villarreal, 2000).

This kind of medium, however, does not produce a mathematical experience on its own. The right kinds of tasks need to be designed in the form of authentic and interesting challenges, where mathematics provides a solution or a better understanding of the issues. Hence, a paradigm shift toward mathematical inquiry, problem solving and posing, curiosity, creativity, and imagination, can be accompanied by taking risks and perceiving errors and false avenues as opportunities for understanding and reflection (Battista, 1999; Hyde & Jones, 2013; Laborde, Kynigos, Hollebrands & Strasser, 2006; Lesh & Doer, 2003; Watson & Mason, 2005)

Furthermore, the teachers' role and pedagogical interactions with the students needs to change and become resonant with this kind of mathematical activity (Walshaw, 2010; Boaler, 2003). Teachers need to act as facilitators in the mathematical experience by inspiring students to enjoy a challenge. All of these transformations are hard to design and even harder to implement. One important way of doing this is empowering teachers to engage in the design of tools for their students and take part in professional communities reflecting on these designs and classroom practice (Ruthven, 2011; Yackel, 2002, Koeler and Mishra, 2009).

It is inevitable that these dynamic constructionist tools that allow for design at many levels have often been used not to transform the classroom experience but, on the contrary, to augment traditional teaching (Ruthven, Hennessy & Deaney, 2008). Therefore, professional support, carefully prepared analytical resources, and the design of alternative assessment methods, including student portfolios, are a key strategy for any organization or group developing digital media for transformational mathematics education.

Not all digital media for mathematics education are designed for paradigm transformations. There are two main categories of such media outside this realm. One can be defined as media intended to help augment the creation of traditional instruction. The other involves media built in the context of other research fields or agendas, such as AI, telecommunications, digital games (Kebritchi, Hirumi & Bai, 2010), or perceiving mathematics education as a field of application (Sinclair & Jackiw, 2005; Voogt & Roblin, 2012).

The typical types of such tools are adaptive exercisers such as MSØ and computer-assisted instruction modules either embedding a model of "tutor" and "student" or that are based on recommender systems such as MSØ itself. These applications, however, are more implementations of other research fields, such as AI, with mathematics education being cast in the role of one such application domain. In this sense, they do not incorporate the reflections and aims of the mathematics education research field in the sense that has been analysed in the present report. Another important category of applications is large explanatory video portals such as Kahn Academy, which is an exemplary case of using technology to augment and make more effective the traditional approaches toward mathematical teaching and learning (Hampton, 2014).

3. The uses of digital media for mathematics education

As mentioned in the introductory section, a large diversity of digital media has been developed over the years, addressing both transformative approaches toward mathematics education and, conversely, approaches aiming to enhance traditional schooling, epistemology, and pedagogy. This section provides the taxonomies regarding the diverse uses for which these media have been designed.

In an excellent review of research using digital media for mathematics education, Bray and Tangney (2017) offered an overall hierarchy (see figure 1) that seems particularly suited to the problematic nature of the field; it is called "SAMR - Substitution, Augmentation, Modification, Redefinition" (Puentendura, 2006). They then developed an amalgamated classification based on SAMR, learning theory, technology, and purpose (see table 1).

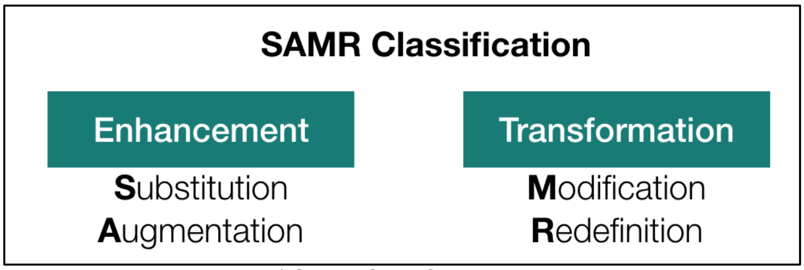

Figure 1 SAMR Classification Hierarchy

Table 1 Intended uses of technology (summary of Bray & Tangney, 2017)

INTENDED USES OF DIGITAL MEDIA
TECHNOLOGY
collaborative by design
dynamic geometry environment
multiple linked representations
outsourcing computation
outsourcing content
programming tools
toolkit
LEARNING THEORIES
behaviorist
cognitive
constructivist
constructionist
PURPOSE
change in attitude
improve performance
improve conceptual understanding
skills-focused
support teachers
collaboration and discussion

This hierarchy (see figure 1) refers to the intended use of a piece of software or a configuration of digital media in mathematics courses. The first category addresses media designed to facilitate transformative approaches, as analysed in the previous sections; this splits the category into two subcategories: one referring to the modification of activities, teaching methods, and assessment tools and the other addressing the diverse structures of mathematics curricula and the types of mathematical activity. The second category addresses the use of digital media to enhance existing practices and epistemologies, and these can be split into media designed to augment human roles, for example, the teacher, and into media designed to substitute for such roles (i.e., digital tutors).

Having this hierarchy in mind, Bray and Tangney (2017) suggested alternative classifications regarding the specific views and interests of the intended uses of digital media (see Figure 2). One such classification has to do with the kinds of uses the technology affords. Another is the classification in relation to diverse learning theories and the intention or purpose behind the design of the digital medium, respectively. In fact, Bray and Tangney (2017) classified the behaviourist theory as standing on its own and connected the other three theories, i.e. cognitive, constructivist and constructionist, into one family.

Thus, the two classifications of technology and learning theory provide a lens for the kinds of affordances (we elaborate on this notion later) of a digital medium and how these affordances connect to how learning mathematics is perceived. Bray and Tangney (2017) offered yet another classification in connection to the purpose of the intervention and the context in which the digital medium is used. This classification assumes that a digital medium is not just inserted seamlessly into an established, unchanging practice. Rather, it perceives the use of a medium as an integral aspect of an intended pedagogical intervention, focusing on some change, improvement, or emphasis not given before. Hence, digital media can be classified according to the purpose of the intervention for which it was designed.

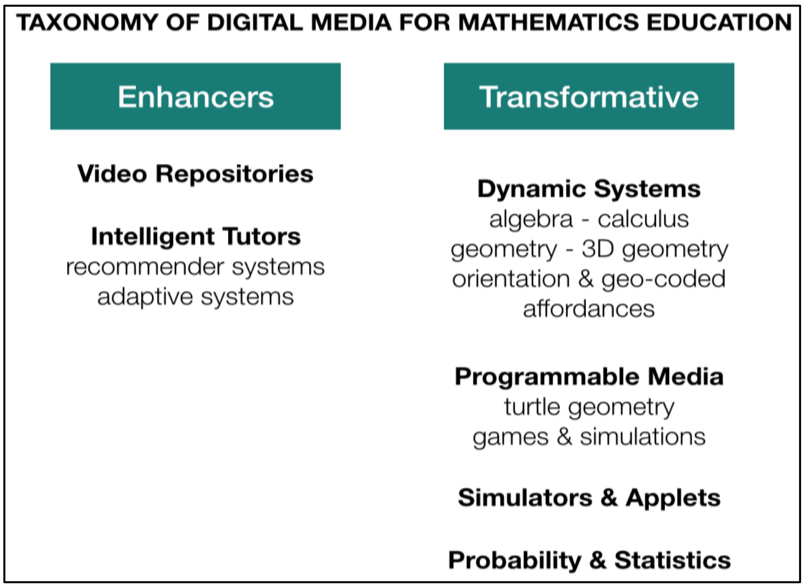

Figure 2 Taxonomy of Digital Media for Mathematics education

Bray and Tangney (2017) offered a set of two-way classifications, for example, the SAMR level and learning theory, as well as an approach taking all of these classifications into consideration. A further suggestion here would be to consider a wider taxonomy of digital media for mathematics education based on the SAMR classification, as in Figure 2. This classification allows for a consideration of which medium or configuration of media to use when planning an intervention according to the intent of the SAMR. For example, a school may wish to place emphasis on the use of an enhancer to make time for the use of a transformative medium, such as a programmable tool.

4. The affordances of digital media for mathematics education

The classifications in the previous section were broad and organized according to a diversity of interests and viewpoints; however, this was done in a way that considered the overall descriptions of the design features of digital media. In this section, we take a deeper look into the affordances of a medium, that is, the technical features pertinent to its intended use. First, the notion of affordance is addressed. To do this, we draw inspiration from Calder (2011) as a primary reference. Affordances are a "potential for action, the capacity of an environment or object to enable the intentions of the student within a particular problem situation" (Tanner & Jones, 2000, p. 78). We might consider them as the perceived opportunities offered through the pedagogical medium in relationship to the propensities and intentions of the user. Indeed, affordance implies the complementarity of the learner and the environment (Gibson, 1977).

Calder (2011) elaborated that, as for Gibson (1977), affordances are a precondition for activity; they define potential, allowable actions between an environment or object and the person or animal acting. Affordances are not just abstract physical properties, but rather are the potential relationships between the user and the "artefact" (Brown, 1996). Indeed, affordances have been considered in general educational situations and are applicable wherever there is a relationship between an environment and a living entity. However, the existence of an affordance does not necessarily imply that an activity will occur. In a digital environment, affordances are the opportunities that the environment offers the learning process. Although the digital medium "exerts influence on the student's approach, and hence the understanding that evolves, it is his/her existing knowledge that guides the way the technology is used and, in a sense, shapes the technology. The student's engagement is influenced by the medium but also influences the medium (Hoyles & Noss, 2003)." (Calder & Campbell 2016, p 52).

Given these considerations, it is illuminating to identify some key affordances in terms of the technological characteristics playing a pertinent role in the use of a medium. This approach is useful mainly for transformative media because enhancers are not designed to emphasize new types of interactions with a technology in connection to learning processes. For example, a traditional exerciser would not normally put emphasis on the ways in which an affordance is used by the learner; instead, it would try to minimize the importance of any special interaction by considering it a "noise" with respect to the purpose of the activity at hand, which is to solve the mathematical problem. Table 2 offers a set of example affordances, direct affordances (i.e., in terms of or in reference to technical specifications and indirect affordances (i.e., learning activities that are connected to technical features without the need to specify which ones in particular).

Table 2 Affordances of Digital Media

AFFORDANCES OF DIGITAL MEDIA	
Direct	**Indirect**
• multiple interdependent representations (Sacristan & Noss, 2008) • dynamic manipulation (Granberg & Olsoon, 2015; Jackiw & Sinclair, 2006; Ruthven et al., 2008) • computer feedback • visualization (Villarreal, 2000) • exploratory and constructionist (Geraniou & Mavrikis, 2015) • offering deep structural access to students and teachers • low threshold, high ceiling • management of large amounts of realistic data • modeling of mathematical objects (geometrical–algebraic graphical representations) • modeling of phenomena embedding mathematical properties, behaviors, and field properties (Gravemeijer, 2002) • modeling of socio-scientific issues, larger issues where mathematical ideas may reside	• offering the possibility for multiple strategies • encouraging collaborative approaches, discourse, and argumentation around mathematical ideas • promoting accuracy, rigor, and economy of expression • promoting generalization • promoting conjecture building and proof • promoting constructionism (i.e., tinkering and changing of models by manipulating or redefining their constituent properties) • posing of problems • creating tension when learners get an unexpected output • fostering risk taking and experimentation (Calder, 2011), allowing space for students to explore • imagining possibilities • integrating analytical thinking with more intuitive, creative approaches • formation of new teacher and learner communities • learning from feedback • observing patterns • seeing connections • working with dynamic images • exploring data • “teaching” the computer

5. The uses of digital media for enhancement and substitution: ITS for procedural knowledge in mathematics education

The argumentation and framing of the potential uses of digital media for transformative education, has mainly been articulated by mathematics education researchers and theorists on mathematical teaching and learning. Other scientific communities, however, took educational practice at face value and as an unchanging given, focusing on the potential for digital technologies to enhance and augment traditional schooling. One community bases its approach on the behaviourist theorists Skinner (1953) and Thorndike (1898) and has done so since as early as the beginning of the twentieth century; another community has focused on the artificial intelligence community in the field of computer science. In both cases, the view of learning respectively perceives humans as dependent on responding to stimuli and as teaching as the intentional provision of stimuli designed to provide the right response. This is the view originally adopted by the field of artificial intelligence (AI). A founding application in the 1980s was called the PLATO system (see Hansen et al, 2013), which then developed into a variety of what were called intelligent tutoring systems (ITS). In mathematics education, this approach toward teaching and learning is best explained through its difference with transition-style approaches. Mavrikis and Holmes (2018) made a very clarifying distinction between ITS supporting procedural knowledge and transition-oriented exploratory learning environments supporting conceptual knowledge.

5.1 Procedural vs. Conceptual knowledge and ITS

Drawn from Hansen, Mavrikis, Mazziotti, and Rummel's (2013) procedural knowledge in mathematics is the ability to apply procedures, that is, sequences of actions such as steps to solve a problem, the application of rules, and the routines leading to a solution (Rittle-Johnson & Alibali, 1999). Procedural knowledge becomes implicit with practice, and according to Skemp (1976), it does not require a deep understanding of the underlying concepts and is hence easy to apply in routine tasks; indeed, answers are quickly and reliably found, and students feel a sense of success when the answers are correct. Hansen et al. (2013) contrasted this to conceptual knowledge, that is, the deep understanding of principles and concepts underlying a mathematical domain (see also Fullan & Langworthy, 2014). Procedural knowledge is the type of mathematical knowledge that has traditionally been taught, supported, and assessed in schools all over the world.

Procedural knowledge is thus acquired through repeated structured practice and a deepening of problem-solving procedures (Polya, 1945). With the use of ITS, it is facilitated by the drill and practice of routine problems, with hints and feedback given in support of students' procedural knowledge. Some ITS systems adapt task selection and feedback to the individual learners' cognitive skills by employing models about the learners themselves.

Hansen et al.'s (2013) summary of how ITS supports procedural knowledge in juxtaposition to how exploratory and constructionist learning environments support conceptual knowledge clearly sets the picture, as seen in figure 3.

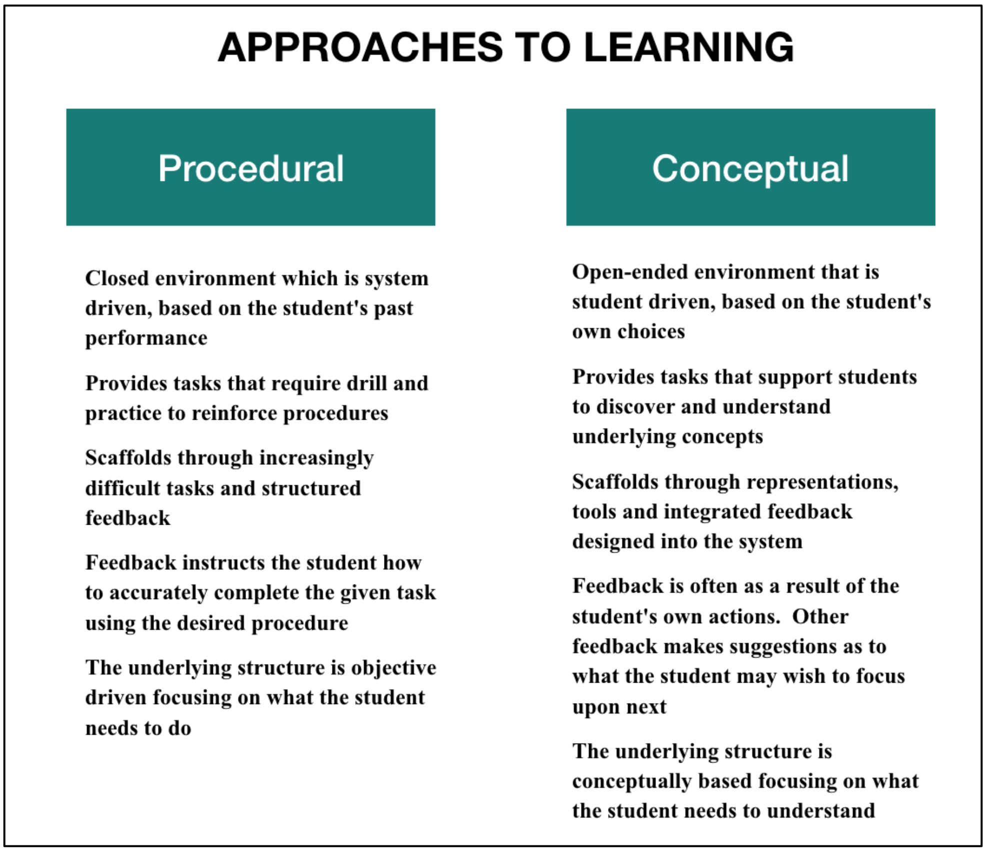

Figure 3 ITS support of procedural and conceptual knowledge

According to this summary, MSØ is a system supporting procedural mathematical knowledge in a variety of mathematical domains, ranging from arithmetic and algebra to geometry.

5.2 Data driven vs. knowledge engineering ITS

There are various ways to implement an ITS. The two most common are a) based on knowledge engineering or b) driven by data. Mavrikis and Holmes (2019) reported that in knowledge engineering methods, a detailed analysis of the task is required, and this must be combined with an understanding of students' common misconceptions, difficulties on a particular task, and so forth to design the hints and other feedback messages. This often requires involving experts, teachers, or other domain experts in what is commonly referred to as knowledge elicitation. There are several approaches to knowledge elicitation that can be combined with the general design approaches in the field, here recognizing the need for iterative, agile processes (e.g., Good & Robertson, 2006; Sharples et al., 2002 Conlon & Pain, 1996) and iterative theory development that can explain the phenomena of interest (DiSessa & Cobb, 2004).

In data-driven methods, generalizable applicable techniques such as machine learning are used to help the system "learn" from past behaviours (e.g., to predict the score of a student based on past data that can help decide which task to provide next, for example, by giving one of the appropriate difficulties). For detailed reviews of recommender systems that support learning, the reader is referred to a recent review of the field (Drachsler et al., 2015). MSØ falls clearly under this category because it uses a generalized recommender system that provides feedback to the user and also adapts the sequence of tasks according to user answers to questions.

5.3 Examples of ITS in Mathematics Education

Some examples of ITS in mathematics education that are drawn from the iTalk2Learn project review deliverable (Hansen et al., 2013) provide the area of affordances of this kind of tool, including language recognition, hints, tutorials, and, in some cases, embedding an overall theme of mathematical application such as "Animal Watch." In most cases, the ITS system is focused on a specific domain, especially when it is based on knowledge engineering, which requires deep expertise in mathematics education for its development.

- Active Math[1]: "Rechnen mit Brüchen: Addition und Subtraktion", Adding & Subtracting fractions. The LeActiveMath project introduced tutorial dialogues to some structured problems. Immediate feedback, hints, worked examples, and mostly structured tasks. Some exploratory applets are provides and it uses an Open Student Model
- RM Reasoning Mind[2]: Fractions and Mixed Numbers. Allows problem selection within a topic. Tasks are just one part of a richer learning environment.
- Animal Watch[3]: Fractions; Partly spoken feedback, spoken worked examples (2 avatars), Multimedia tutorial resources (from text explanations to worked examples and interactive solutions). Problem solving in context of endangered species and tracing of their life. Includes some interactive games. Problem selection and difficulty according to skills observed. Basic problem solving skills development for pre- algebra topics. Motivation through understanding how mathematics could be applied in the real world (Cohen, Beal, & Adams, 2008)
- Cognitive Tutors®[4] (in general): Fractions Tutor and Math Tutor include relevant content. Some versions provide spoken task description. Uses a cognitive model to provide feedback to students as they are working through problems and uses 'model tracing' to monitor progress and provide immediate hints and solution when needed (Anderson, et al., 1990). Structured questions broken down into steps.
- Ms. Lindquist[5]: Focuses on writing expressions for algebraic word problems using typed natural language interaction. Uses concrete example, explanations, worked examples, decomposition substitution (Heffernan, 2003) and word problems to algebra expressions.

[1] http://leactivemath.org/index_id_1435.html
[2] http://www.reasoningmind.org/syllabus/syllabus.php?n=2
[3] http://animalwatch.arizona.edu/an imalwatch_learning_objectives
[4] http://ctat.pact.cs.cmu.edu/
[5] http://www.cs.cmu.edu/~neil/examplePage.html

6. The most popular pedagogically principled solutions so far

This section contains a selection of important examples of digital media that are mainly used for transformational learning (except for Kahn Academy). The list is by no means exclusive, and there are no doubt other important that have been media developed and are used around the world. The list is meant to provide the reader with a scope of the diversity and quality of such media and, at the same time, instances of media that can be then considered against the classifications and considerations outlined earlier. Details of each medium can be found on their respective site. Some of them have obtained accreditations of popularity nationally and internationally, while others are exemplary cases used in academic research.

To begin with, there are three open source solutions that stand out because they have hitherto gained outstanding popularity, having millions of active users. All three have been developed under the open access sustainability model.

- **Kahn Academy** (augmentation): Video tutorial explanations with exercisers embedding analytics. This is developed by one central team under Zalman Kahn, a mathematician at MIT who began by making tutorials for his niece. It has over 100 million users.
- **Scratch** (redefinition): Programmable constructionist medium for building models and games. Based at the Media Lab at MIT, it was launched in 2007 and is a successor of LCSI microworlds pro. It has been developed and is supported by a development team at the media lab led by Mitch Resnick, the successor of S. Papert's chair. It has over 30 million users and is meant to attract everyone into programming activity, offering users the potential to develop their own simulations and games. In recent years, it has specifically been considered and used for mathematical learning integrated with the learning of programming via a curriculum designed and implemented in more than 100 schools in the UK (Benton, Hoyles, Kalas & Noss, 2017).
- **Geogebra** (modification): A dynamic manipulation medium for geometry, algebra, and models. It originated from students at the University of Linz and has now over 100 million users. It has a distributed development team.

Other prominent pieces of digital media for mathematics education are as follows:

- **NETLOGO**, https://ccl.northwestern.edu/netlogo/, a development team lead by Uri Willensky, focusing on the modelling of complex phenomena and based on the idea of parallel programming (redefinition).
- **Fathom**, https://fathom.concord.org/, https://www.chartwellyorke.com/fathom.html. It was made by William Finzer; it is the best known tool for statistical investigations and large databases (modification).
- **Derive**, https://www.chartwellyorke.com/derive.html. A well-established Computer Algebra Systems (CAS) from Canada (augmentation).

- **Maple**, https://www.maplesoft.com/. Another well-established CAS advertised to provide affordances for system simulation, calculation management, and systems engineering (augmentation).
- **Stella**, https://www.iseesystems.com/store/books/lessons-in-mathematics/. A CAS for algebra and calculus (augmentation).
- **Mathematica**, https://www.wolframalpha.com/. A well-known programmable medium for algebra and calculus, mainly for undergraduate and postgraduate mathematics (modification).
- **Function Probe**, https://dl.acm.org/citation.cfm?id=232036&dl=ACM&coll=DL. A lightweight tool for investigations around functions (modification); it was developed by a constructivist theoretician (J. Confrey) and A. Maloney.
- **Modellus**, http://www.modellus.pt. A modeler of physical phenomena based on mathematical equations developed by V. Duarte Teodoro at the University of Lisbon (modification).
- **MaLT2**, http://etl.ppp.uoa.gr/malt2. 3d Logo - Turtle Geometry tool integrating dynamic manipulation of variable values, made by C. Kynigos at the Educational Technology Lab, NK University of Athens (redefinition).
- **Tinkerplots**, https://www.tinkerplots.com/. It is based on a proprietary sustainability model, affording investigations into data handling and statistics (modification).
- **Aplusix**, https://www.chartwellyorke.com/aplusix/index.html. It is based on a proprietary sustainability model, a factorization speller based at the University of Grenoble; it was made by J.F. Nicaud (substitution).
- **Cabri Geometre**, https://cabri.com/en/. It is based on a proprietary sustainability model, a very popular Dynamic Geometry Systems (DGS) system based at the University of Grenoble and originating from Collette Laborde and Jean Maria Laborde (modification).
- **Geometry Sketchpad**, http://www.dynamicgeometry.com/. It is based on a proprietary sustainability model, an equally popular DGS originating from Sketchpad Technologies and Key Curriculum Press by Nicholas Jakiw (modification).
- **Autograph**, http://www.autograph-maths.com. It is based on a proprietary sustainability model, a well-known CAS by Douglas Butler (augmentation).

7. Major temporal phases of digital solutions for mathematics education

This section discusses a distinctly different angle to the question of what digital solutions to use for school mathematics and to what end, transformative or preservationist. It contains a review of the life and resilience of digital solutions in mathematics education given the wider temporal context involving changing sustainability models for such media and changing infrastructures for digital citizenship. An interesting review taking this angle can be found in Borba et al. (2017).

Chronologically, the digital technologies designed to bring about added value in mathematics education first appeared early on. The first report of such use of technology was the 'Brookline Logo Project' in 1979, while the first application of the Logo programming language for engaging in mathematical constructionist thinking was announced earlier in 1969 and involved a computer giving commands to a robot vehicle; a screen version later appeared in 1972 (see, http://el.media.mit.edu/logo-foundation/what_is_logo/history.html).

In the early 1980s, with the advent of personal computers, two widely different kinds of applications appeared. One was Logo-like constructionist software (see Kynigos et al., 1993; Jonassen, 1999) and the other was behaviourist computer-assisted instruction exercisers under the general characterization of computer-assisted instruction. Very soon after, a next important phase came about with respect both to affordances and a pedagogical perspective. It was based on a representational affordance that acquired central kudos in mathematics education: dynamic manipulation (Hollebrands, 2003). This was first made available in the early 1980s from two distinct sources, the Geometers' Sketchpad from Nick Jakiw and Key Curriculum Press in California and the Cabri Geometre from the University of Grenoble, France. Graphic calculators were used for investigating functions. The affordances were quickly ported to computers in the form of digital media and have been maintained up until today (e.g., the "Desmos" web application, https://www.desmos.com/). All of the above solutions were perceived as products for sale accompanied by manuals and educational material. The sustainability model was a permutation of the classic revenue generation for the purchase of the product itself, while distribution was typically done through the shop, and the product itself was occasionally locked with a license for the number of devices it could be installed on (Desmos is an exception to this because it appeared very recently).

The next phases were not necessarily based on affordances for mathematical expression per se but on more general advances in digital media affordances. The only exception was the advent of digital libraries consisting of learning objects, typically simulations in the form of interactive visualizations of a mathematical or physical object, or that have a setting with an embedded and narrowly defined mathematical concept or property. The main idea here came from the field of portal technology and was based on the accessibility, metadata, tagging, and user evaluation of such objects, including properties such as reusability. Typical examples of such libraries are the Multimedia Educational Resource for Learning and Online Teaching

(MERLOT[6],), the DME portal[7] based at the Freudenthal Institute (Boon, 2006), the "Photodendro" portal[8] of the Greek Ministry of Education, CTI-Diophantus , and the Khan Academy portal[9] of short videos explaining mathematical concepts with exercisers and a student and teacher dashboard.

Coming back to the phases following the advent of the Internet, enthusiastic attention was given to the idea of computer-assisted collaborative learning, which in the mathematics education community gave rise to theory on the reorganization of the ways a learner can communicate and perceive of engagement with mathematical thinking through communication and collaboration with others (Albert & Kim, 2013; Borba & Villareal, 2005; Hunter, 2010; Sinclair, 2005; Kynigos & Moustaki, 2013). This was followed by the social media revolution. In addition, mathematics education was also highly influenced by the advent of blended learning, distance learning, and then MOOCs (Massive Open Online Courses).

[6] https://www.merlot.org

[7] https://www.numworx.nl/

[8] http://dschool.edu.gr

[9] http://www.kahnacademy.org

8. Suggestions for comprehensive interventions in schooling

MSØ is a system designed to support procedural mathematical knowledge in arithmetic, early algebra, and geometry. It is based on a general data-driven recommender system for feedback and adaptivity and has diverse, linked representations for mathematical meanings that are not always cohesive or carefully thought through from a mathematics education science point of view. It has the technical readiness of a commercial product and can operate robustly in schools and from home.

Decisions on how to use this system in a real school situation and in a sustainable way, that is, not as a one-off trial or short-term intervention, should not be made by the vendor, nor as a result of some marketing campaign, but rather should be done through a process generating a comprehensive school pedagogy for transformation in mathematics education. In each school and education system, this can be different depending on a large variety of factors. Also, a particular school operates in the continuity of time, so the process of transformation is not linear, and the contextual factors may continually change. Therefore, schools need primarily to be supported by educational policy to have the required mechanisms to generate and maintain their own agenda for mathematics education.

To change the balance between transformative and preservationist educational practice, a double emphasis is needed a) on the quality of transformative practice, including the tools used and the social orchestration between students and teacher and b) on the improvement of the effectiveness of support for procedural knowledge to make time and space for transformational practices supporting conceptual knowledge and meaning-making. MSØ can be effectively used in b), provided this is a part of a wider mathematics education policy in each school.

Overall, in each school, the hardest thing to consider changing is the set mathematics curriculum, the time slots in the week for dedicated mathematics work, and the assessment system, which mainly focuses on a diagnosis of the level of procedural knowledge in students. To meet with these restrictions on the transformative process, a number of alternative solutions are available and are broadly termed as “blended learning” (Christensen, Horn & Staker, 2013; Horn & Staker, 2011). Digital media can be used to support engagement with either procedural or meaning-making discursive activity at home and in hours outside the school day. This can make homework much more effective and free up face-to-face time in the classroom for activities that are an alternative to frontal teaching and exercising.

Drawn from Christensen et al. (2013, p. 26), see table 3, there are a number of models for applying blended learning in a school to implement a hybrid innovation, that is, an innovation that may be initially disruptive but leading to sustainability. These models are the Rotation Model, the Flex Model, the A La Carte Model, and the Enriched Virtual Model.

Table 3 Models for applying blended learning (adapted from Christensen et al., 2013, p. 26)

ROTATION MODEL The rotation model is one in which within a given course or subject (e.g., math), the students rotate on a fixed schedule or at the teacher's discretion between learning modalities, at least one of which is online learning. Other modalities might include activities such as small-group or full-class instruction, group projects, individual tutoring, and pencil-and-paper assignments. The rotation model has four submodels— station rotation, lab rotation, flipped classroom, and individual rotation—as follows: The station rotation model is one in which students rotate within a contained classroom. The lab rotation model is one in which the rotation occurs between a classroom and a learning lab for online learning. The flipped classroom model is one in which the rotation occurs between the school for face-to-face teacher-guided practice (or projects) and the home or another off-site location for online content and instruction (see also Eisenhut & Taylor, 2015; Triantagyllou & Timcenko, 2015). The individual rotation model differs from the other rotation models because each student has an individualized playlist and does not necessarily rotate to each available station or modality.
FLEX MODEL The flex model is one in which online learning is the backbone of student learning, even if it directs students to offline activities at times. Students move on an individually customized, fluid schedule among learning modalities, and the teacher of record is on-site.
A LA CARTE MODEL The a la carte model is one in which students take one or more courses entirely online with an online teacher of record and, at the same time, continue to have brick-and-mortar educational experiences. Students may take the online courses either on the brick-and-mortar campus or off-site.
ENRICHED VIRTUAL MODEL The enriched virtual model is a whole-school experience in which within each course (e.g., math), students divide their time between attending a brick-and-mortar campus and learning remotely using online delivery of content and instruction.

Hence, within each school, a model needs to be built and sustained on how to use technology to make procedural learning more effective so as to make time for what's more difficult to achieve yet is more important: meaning-making and exploratory and discursive learning. In this context, time will be made for students to work with expressive mathematical media such as Scratch, Geogebra, and MaLT2, and to gain experience in discursive activities, including the elaboration of a mathematical language to communicate ideas and concepts. This is crucial and will be achieved by means of the in-service education of teachers in the pedagogy of employing such media for transformative mathematics. The time for students to engage in this activity can be greatly enhanced by making the necessary procedural knowledge acquisition more effective through using video tutorials such as the Kahn Academy infrastructure and exercisers such as MSØ.

Importantly, the corporate sector can contribute by developing technically robust solutions for both transformative and procedural approaches. However, in each case, silo solutions should be avoided, i.e. marketing the use of one specific vendor product. Instead, vendor products should be marketed as part of more comprehensive solutions that will inevitably comprise digital artifacts and tools that are either freely or commercially available not only by the company at hand, but also by other companies or research centres, such as the MIT Media Lab. New collaborations between companies and between companies and research institutions are necessary to provide schools with comprehensive avenues for transformative mathematics education. Otherwise, schools will be exposed to diverse chaotic marketing that conveys a fragmented picture of how technology can assist the teaching and learning of mathematics. In effect, this will be ultimately unsustainable, and schools will soon drop the use of such tools or engage in unwanted skewing of their educational services toward partial or ineffective solutions.

Primary references

Borba, M., Askar, P., Engelbrecht, J., Gadanidis, G., Llinares, S. & Aguilar, M. S. (2017). Digital technology in mathematics education: Research over the last decade. In *Proceedings of the 13th International Congress on Mathematical Education* (pp. 221-233). Springer.

Bray & Tangney (2017). Technology usage in mathematics education research – A systematic review of recent trends. *Computers & Education, 114,* 255-273, Elsevier Pubs.

Calder, N. (2011). *Processing mathematics through digital technologies - The primary years*. Utrecht: Sense Publishers.

Christensen, C. M., Horn, M. B. & Staker, H. (2013). *Is K–12 blended learning disruptive? – An introduction of the theory of hybrids*. San Mateo, CA: Clayton Christensen Institute. Retrieved from http://www.christenseninstitute.org/wp-content/uploads/2013/05/Is-K-12-Blended-LearningDisruptive.pdf

Hoyles, C. & Lagrange, J. B. (2010). *Mathematics education and technology: Rethinking the terrain: The 17th ICMI study*. New York, NY: Springer.

Kynigos, C. (2015). Constructionism: Theory of learning or theory of design? In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 417-438). New York, London: Springer International Publishing. doi:10.1007/978-3-319-17187-6

References

Ainley, J., Button, T., Clark-Wilson, A., Hewson, S., Johnston-Wilder, S., Martin, D., … Sutherland, R. (2011). *Digital technologies and mathematics education*. Retrieved from https://www.ncetm.org.uk/files/9793653/JMC_Digital_Technologies_Report_2011.pdf

Albert, L. R. & Kim, R. (2013). Developing creativity through collaborative problem solving. *Journal of Mathematics Education at Teachers College, 4*(2), 32-38.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245-274.

Artigue, M. (2010). The future of teaching and learning mathematics with digital technologies. In C. Hoyles & J-B Lagrange (Eds.), *Mathematics education and technology – Rethinking the terrain* (pp. 463–475). New York, NY: Springer.

Bartolini Bussi,M. G. & Mariotti,M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh & D. Tirosh (Eds.) Handbook of international research in mathematics education, second revised edition (pp. 746–805). Mahwah: Lawrence Erlbaum.

Battista, M. T. (1999). Fifth graders enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education, 30*(4), 417-448.

Benton, L., Hoyles, C., Kalas, I. & Noss, R. (2017). Bridging primary programming and mathematics: Some findings of design research in England. *Digital Experiences in Mathematics Education*, 1-24, MIT Press, N.Y.

Berger, M. (2005). Vygotsky's theory of concept formation and mathematics education. In H.L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 153-160). Melbourne: PME.

Boaler, J. (2003). Studying and capturing the complexity of process: The case of the dance of agency. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th PME International Conference* (Vol. 1, pp. 3–16), Hawai.

Boaler, J. (2008). *Bridging the gap between research and practice: International examples of success the first century of the International Commission on Mathematics Instruction (1908-2008): Reflecting and shaping the world of mathematics education*. Roma: Instituto della Enciclopedia Italiana foundata da Giovanni Treccani.

Boon, P. (2006). Designing didactic tools and micro worlds for mathematics Educations. In C. Hoyles, J-B. Lagrange, L.H. Son & N. Sinclair (Eds.), *Proceedings of 17th ICMI Study conference, Technology Revisited*. Hanoi: Hanoi University of Technology.

Borba, M.C. & Villarreal, M.E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. New York, NY: Springer.

Brown, T. (1994). Towards an hermeneutic understanding of mathematics and mathematical learning. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education* (pp. 141–149). London: Falmer Press.

Brown, T. (1996). The phenomenology of the mathematics classroom. *Education Studies in Mathematics, 31*, 115-150.

Calder, N.S. (2010). Using scratch: An integrated problem-solving approach to mathematical thinking. *Australian Primary Mathematics Classroom, 5*(4), 9-14.

Calder, N.S. & Brown, T. (2010). Learning through digital technologies. In M. Walshaw (Ed.), *Unpacking pedagogy: New perspectives for mathematics classrooms* (pp. 223-243). Charlotte, NC: Information Age Publishing.

Calder, N.S. & Campbell, A. (2016). Using Mathematical Apps with Reluctant Learners. Digital Experiences in Mathematical Education, 2, 50–69 DOI 10.1007/s40751-016-0011-y

Calder, N.S. & Taylor, M. (2010). Scratching below the surface: Mathematics through an alternative digital lens? In L. Sparrow, B. Kissane & C. Hurst (Eds.), *Shaping the future of mathematics education, Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia* (pp. 117-124). Fremantle: MERGA.

Chance, B., Garfield, J. & delMas, R. (2000). Developing simulation activities to improve students' statistical reasoning. In M.O.T. Thomas (Ed.), *Proceedings of TIME 2000*. Auckland: The University of Auckland and Auckland University of Technology.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique. Perspectives apportées par une approche anthropologique. Recherche en didactique des Mathématiques, 12(1), 73–112.

Christensen, C.M., Horn, M.B. & Staker, H. (2013). *Is K–12 blended learning disruptive? – An introduction of the theory of hybrids*. San Mateo, CA: Clayton Christensen Institute. Retrieved from http://www.christenseninstitute.org/wp-content/uploads/2013/05/Is-K-12-Blended-LearningDisruptive.pdf

Clements, D.H. (2000). Unique contributions of computers to innovative mathematics education. *The Journal of Mathematical Behavior, 19*(1), 9-47.

Clements, D.H., Sarama, J., Yelland, N.J. & Glass, B. (2008). Learning and teaching geometry with computers in the elementary and middle school. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Vol 1. Research synthesis* (pp. 109-154), N.C.T.M. Publications, North Carolina.

Cobb, P. (1994). Where is the mind? Constructivist and socio-cultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13-19.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R. & Schauble, L. (2003). Design experiment in educational research. Educational Researcher, 32(1), 9–13.

Confrey, J. & Maloney, A. (2007). A theory of mathematical modelling in technological settings. In W. Blum, P. L. Galbraith, H.-W. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 57-68). New York, NY: Springer.

Confrey, J., Maloney, A., Ford, L. & Nguyen, K. (2006). Graphs 'n glyphs as a means to teach animation and graphics to motivate proficiency in mathematics by middle grades urban students. In C. Hoyles, J-B Lagrange, L.H. Son & N. Sinclair (Eds.), *Proceedings of 17th ICMI Study Conference, Technology Revisited*. Hanoi: Hanoi University of Technology.

Conlon, T. and Pain, H. (1996). Persistent collaboration: a methodology for applied AIED, Journal of Artificial Intelligence in Education, 7, 219-252.

Conneely, C., Lawlor, J. & Tangney, B. (2015). Technology, teamwork and 21st century skills in the Irish classroom. In D. Butler, K. Marshall & M. Leahy (Eds.), *Shaping our future: How the lessons of the past can shape educational transformation*. Dublin: Liffey Press.

Conole, G. (2008). New schemas for mapping pedagogies and technologies. *Ariadne, 56*, 2.

Contreras, J. (2014). Where is the treasure? Ask interactive geometry software! *Journal of Mathematics Education at Teachers College, 5*(1), 35-40.

Dede, C. (2010). Technological supports for acquiring 21st century skills. In E. Baker, B. McGaaw & P. Peterson (Eds.), International Encyclopedia of Education, 3rd Edition. Oxford, England: Elsevier.

DiSessa, A. A. & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *Journal of the Learning Sciences*, *13*(1), 77-103. doi:10.1207/s15327809jls1301_4

Douady, R. 1985, 'The Interplay between Different Settings; Tool-Object Dialectic, *Proceedings of PME* 9, Noordwijkerhout, Vol. 2, 33–52.

Drachsler, H., Cooper, A., Hoel, T., Ferguson, R., Berg, A., Scheffel, M., Kismihók, G., Manderveld, J. & Chen, W. (2015). Ethical and privacy issues in the application of learning analytics. In: *5th International Learning Analytics & Knowledge Conference (LAK15): Scaling Up: Big Data to Big Impact*, 16-20 Mar 2015, Poughkeepsie, NY, USA.

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics, 38*, 85-109.

Eisenhut, L. & Taylor, C. (2015). In-class purposes of flipped mathematics educators. *Journal of Mathematics Education at Teachers College, 6*(2), 17-26.

Ernest, P. (1991). The philosophy of mathematics education. UK: Falmer Press.

Ernest, P. (1998) *Social Constructivism as a Philosophy of Mathematics*, Albany, New York: SUNY Press. In S. Restivo (Ed.) Science Technology and Society Series. Also in J. Sowder (Ed.) Reform in Mathematics Education Series, xiv+303pp. ISBN0791435873 (h/b) 0791435881 (p/b).

Foucault, M. (1984). Truth and power. In P. Rabinow (Ed.), *The Foucault reader* (pp. 51-75). Harmondsworth: Penguin.

Fullan, M. & Langworthy, M. (2014). *A rich seam: How new pedagogies find deep learning* (Vol. 100). London: Pearson.

Geraniou, E. & Mavrikis, M. (2015). *Crossing the bridge: From a constructionist learning environment to formal algebra*. Paper presented at the Ninth Congress of European Research in Mathematics Education. Prague, Czech Republic

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting, and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Lawrence Erlbaum.

Goldenberg, P. E. (1999). Principles, arts, and crafts in curriculum design: The case of connected geometry. *International Journal of Computers for Mathematical Learning, 4*(2–3), 191–224.

Good, J., Robertson, J. (2006) CARSS: A Framework for Learner-Centred Design with Children, International Journal of Artificial Intelligence in Education, vol. 16, no. 4, pp. 381-413, IOS Press, Amsterdam.

Granberg, C. & Olsson, J. (2015). ICT-supported problem solving and collaborative creative reasoning: Exploring linear functions using dynamic mathematics software. *The Journal of Mathematical Behavior, 37*, 48-62.

Gravemeijer, K. (2002). Preamble: From models to modelling. In K. Gravemeijer, R. Lehrer, B. van Oers & L. Verschaffel (Eds.), *Symbolising, modeling and tool use in mathematics education* (pp. 7-24). Dordrecht: Kluwer.

Hampton, C. E. (2014). Is there a difference in motivation and mathematics self-efficacy among online mathematics instructional video viewers. *Instructional Technology Education Specialist Research Paper*, 1-50.

Hansen, A., Mavrikis, M., Mazziotti, C. & Rummel, N. (2013). *State of the art for intelligent tutoring systems and exploratory learning environments* (iTalk2Learn project deliverable). Retrieved from http://www.italk2learn.eu/

Hollebrands, K. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *Journal of Mathematical Behaviour, 22*(1), 55-72.

Horn, B. M. B. & Staker, H. (2011), Blended learning. *Innosight Institute, 5*(3).

Hoyles, C. (2016). Engaging with mathematics in the digital age. In *Cuadernos de Investigacion y Formaci on en Educaci on Matem atica 15: Trabajos de la XIV CIAEM* Conference (pp. 225-236). Costa Rica: Universidad di Costa Rica.

Hoyles, C., Lagrange, J.B. (Eds.), (2010) *Mathematics education and technology: Rethinking the terrain: The 17th ICMI study* (Vol. 13). Us: Springerverlag.

Hoyles, C. & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 323-349). Netherlands: Springer.

Hunter, R. (2010). Changing roles and identities in the construction of a community of mathematical inquiry. *Journal of Mathematics Teacher Education, 13*(5), 397-409.

Hyde, R. & Jones, K. (2013). Developing pedagogical approaches to using digital technologies in mathematics. In R. Hyde & J.-A. Edwards (Eds.), *Mentoring in mathematics education* (p. 25). Routledge. London.

Jackiw, N. & Sinclair, N. (2006). Dynamic geometry design for elementary school mathematics. In C. Hoyles, J-B. Lagrange, L.H. Son & N. Sinclair (Eds.), *Proceedings of 17th ICMI Study Conference, Technology Revisited*. Hanoi: Hanoi University of Technology.

Jonassen, D. (1999). Designing constructivist learning environments. *Instructional Design Theories and Models: A New Paradigm of Instructional Theory, 2*, 215-239.

Kebritchi, M., Hirumi, A. & Bai, H. (2010). The effects of modern mathematics computer games on mathematics achievement and class motivation. *Computers & Education, 55*(2), 427-443.

Kitcher, P. (1985) The Nature of Mathematical Knowledge, Oxford Scholarship Online Print ISBN-13: 9780195035414, Published to Oxford Scholarship Online: November 2003 DOI:10.1093/0195035410.001.0001

Klein, M. (2002). Teaching mathematics in/for new times: A poststructuralist analysis of the productive quality of the pedagogic process. *Educational Studies in Mathematics, 50*, 63-78.

Koehler, M. J. & Mishra, P. (2009). What is technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education, 9*(1), 60-70.

Kolb, D. A. (1984). Experiential learning: Experience as the source of learning and development. Englewood Cliffs, N.J. Prentice Hall.

Kynigos C. , Kotsanis, Y., Georgiadis P. and Gyftodimos, G. (Eds), (1993) Logo-like Learning Environments: Reflection and Prospects, *Proceedings of the 4th European Logo Conference*, Doukas School Publication, 377-389.

Kynigos, C. (2015). Constructionism: Theory of learning or theory of design? In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 417-438). New York, London: Springer International Publishing.

Kynigos, C. & Moustaki, F. (2013). *Online discussions about emerging mathematical ideas* (Turkey). Paper presented at the Eighth Congress of European Research in Mathematics Education (CERME 8). Lyon France.

Laborde, C., Kynigos, C., Hollebrands, K. & Strasser, R, (2006). Teaching and learning geometry with technology. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 275-304) Rotterdam: Sense Publishers.

Lesh, R. & Doerr, H. (2003). *Beyond constructivism. Models and modelling perspectives on mathematics problem-solving, learning, and teaching*. Mahwah, NJ: Lawrence Erlbaum.

Maaß, K. & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: A synthesis. *ZDM, 45*(6), 779-795.

Marriotti, M. A. (2006). New artifacts and the mediation of mathematical meanings. In C. Hoyles, J-B. Lagrange, L. H. Son & N. Sinclair (Eds.), *Proceedings of 17th ICMI Study conference, Technology Revisited*. Hanoi: Hanoi University of Technology.

Mavrikis, M. & Holmes, W. (2018). Intelligent learning environments: Design, usage and analytics for future schools. In, *Shaping future schools with digital technology - An international handbook*. Springer.

Noss, R. & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht: Kluwer Academic Publishers.

Noss, R., Healy, L. & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics, 33*(2), 203-233.

Olive, J., Makar, K., Hoyos, V., Kor, L. K., Kosheleva, O. & Straber, R. (2010). Mathematical knowledge and practices resulting from access to digital technologies. In, *Mathematics education and technology - Rethinking the terrain: The 17th ICMI study* (Hoyles, C. and Lagrange, J.B. (Eds) (Vol. 13, pp. 133-177). Springer. The Netherlands.

Papert, S., Watt, D., diSessa, A., Weir , S. (1979) A.I. Memo No. 545 LOGO Memo No. 53 FINAL REPORT OF THE BROOKLINE LOGO PROJECT PART II: Project Summary and Des Analysis https://files.eric.ed.gov/fulltext/ED196423.pdf

Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. Journal of Mathematics in Science and Technology, 31, 249–262.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York, NY: Basic Books, Inc.

Papert, S. & Harel, I. (1991). Situating constructionism. In S. Papert & I. Harel (Eds.), *Constructionism*. New Jersey: Ablex Publishing Corporation.

Piaget, J. (1955). The construction of reality in the child. *Journal of Consulting Psychology, 19*(1), 77.

Pimm, D. & Johnston-Wilder, S. (2005). Technology, mathematics and secondary schools: A brief UK historical perspective. In S. Johnston-Wilder & D. Pimm (Eds.) *Teaching secondary mathematics with ICT* (pp. 3-17). McGraw - Hill, London.

Polya, G. (1945). *How to solve it*. Princetown: Princetown University Press.

Puentedura, R. (2006). Transformatiom, Technology, and Education. Presentation given on August 18, 2006 as part of the *Strengthening Your District Through Technology* workshops, coordinated by the Maine School Superintendents Association. Available at: http://hippasus.com/resources/tte/

Radford, L., Bardini, C. & Sabena, C. (2007). Perceiving the general: The multisemiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education, 38*(5), 507-530.

Rittle-Johnson, B. & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175-189.

Roschelle, J., Kaput, J. & Stroup, W. (2000). SimCalc: Accelerating students' engagement with the mathematics of change. In M. J. Jacobsen & R. B. Kozma (Eds.) *Learning the sciences of the 21st century: Research, design, and implementing advanced technology learning environments* (pp. 47-75). Hillsdale, NJ: Erlbaum.

Ruthven, K. (2001). Mathematics teaching, teacher education, and educational research: Developing "practical theorising" in initial teacher education. In F.-L. Lin & T. J. Cooney (Eds.) *Making sense of mathematics teacher education* (pp. 33-52). Dordrecht: Kluwer.

Ruthven, K., Hennessy, S. & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. Computers & Education, 51(1), 297-317.

Sacristan, A.I. & Noss, R. (2008). Computational construction as a means to coordinate representations of infinity. *International Journal of Computers for Mathematical Learning*, 13(1):47-70 doi:10.1007/ s10758-008-9127-5.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.) Handbook of research on mathematics teaching and learning (pp. 334-370). New York, NY: Macmillan Library Reference.

Schoenfeld, A. (2004). The math wars. *Educational Policy: An Interdisciplinary Journal Of Policy And Practice*, 18(1), 253-286.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sharples, IM, Jeffery, J.B., du Boulay, H. Teather, D. Teather, B. Socio-cognitive engineering: A methodology for the design of human-centred technology, *European Journal of Operational Research*, 136(2), 310-323.

Sinclair, N. (2005). Mathematics on the internet. In S. Johnston-Wilder & D. Pimm (Eds.), Teaching secondary mathematics with ICT (pp. 203-216). Berkshire, UK: Open University Press.

Sinclair, N. & Jackiw, N. (2005). Understanding and projecting ICT trends in mathematics education. In S. Johnston-Wilder & D. Pimm (Eds.), Teaching secondary mathematics with ICT (pp. 235-251). Berkshire, UK: Open University Press.

Sinclair, N., Arzarello, F., Gaisman, M. T., Lozano, M. D., Dagiene, V., Behrooz, E. & Jackiw, N. (2010). Implementing digital technologies at a national scale. In, Mathematics education and technology-rethinking the terrain: The 17th ICMI study (Vol. 13, pp. 61-78). Springer. The Netherlands.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching, 77, 20-26.

Skinner, B. F. (1938). The behavior of organisms: An experimental analysis. Oxford, England: Appleton-Century.

Skinner, B. F. (1953). Science and human behavior. SimonandSchuster.com.

Star, S. L. & Griesemer, J. R. (1989). Institutional ecology, 'translations' and boundary objects: Amateurs and professionals in Berkeley's Museum of Vertebrate Zoology, 1907–1939. Social Studies of Science, 19, 387–420.

Steele, D. (2001). Using sociocultural theory to teach mathematics: A Vygotskian perspective. School Science and Mathematics, 404-416.

Tanner, H. & Jones, S. (2000). Using ICT to support interactive teaching and learning on a secondary mathematics PGCE course. Proceedings of the 2000 Annual Conference of the Australian Association for Research in Education, Sydney. Retrieved from https://www.aare.edu.au/data/publications/2000/tan00226.pdf

Thorndike, E. L. (1898). Animal intelligence: An experimental study of the associative processes in animals. *Psychological Monographs: General and Applied,* 2(4), i-109.

Topcu, A. (2011). Effects of using spreadsheets on secondary school students' self-efficacy for algebra. *International Journal of Mathematical Education in Science and Technology, 42*(5), 605-613.

Triantagyllou, E. & Timcenko, O. (2015). *Student perceptions on learning with online resources in a flipped mathematics classroom* (Prague, Czech Republic). Paper presented at the Ninth Congress of European Research in Mathematics Education (CERME9). Prague, Check Republic.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven & L. Trouche (Eds.), *The didactic challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197-230). New York, NY: Springer.

Trouche, L., Drijvers, P., Gueudet, G. & Sacristan, A I. (2013). Technology-driven developments and policy implications for mathematics education. In M. A. Clements, A. J. Bishop, C. Keitel-Kreidt, J. Kilpatrick & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (1st ed, pp. 753-789). New York, NY: Springer-Verlag.

Verillon, P. & Rabardel P. (1995) Cognition and artifacts: A contribution to the study of thought in relation to instrument activity. *European Journal of Psychology in Education, 9*(3), 77-101.

Villarreal, M. (2000). Mathematical thinking and intellectual technologies: The visual and the algebraic. *For the Learning of Mathematics, 20*(2), 2-7.

Von Glaserfeld, E. (1989). Constructivism in education. In T. Huseln & T. N. Postlewaite (Eds.), *The international encyclopaedia of education: Research and studies* (Sup. Vol.). Oxford: Pergamon Press.

Voogt, J. & Roblin, N.P. (2012). A comparative analysis of international frameworks for 21st century competences: Implications for national curriculum policies. *Journal of Curriculum Studies, 44*(3), 299-321.

Vygotsky, L.S. (1978). Mind in society. The development of higher psychological processes. Cambridge: Harvard University Press.

Walkerdine, V. (1988). *The mastery of reason.* London: Routledge.

Walshaw, M. (2010). Learning to teach: Powerful practices at work during the practicum. In M. Walshaw (Ed.) *Unpacking pedagogy: New perspectives for mathematics classrooms*. Charlotte, NC: New Age Publishing.

Watson, A. & Mason, J. (2005). *Mathematics as a constructive activity*. Mahwah: Lawrence Erlbaum Associates.

Wegerif, R. (2007). *Dialogical education and technology*. New York, NY: Springer.

Wilensky, U. & Papert, S. (2010). Restructurations: Reformulations of knowledge disciplines through new representational forms. In J. Clayson (Ed) *Proceedings of the Constructionism 2010 Conference* (p. 97), Paris, France: The Americal University of Paris (Pubs).

Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *The Journal of Mathematical Behaviour, 21*(4), 423-440.

Yelland, N. (2005). Curriculum, pedagogies and practice with ICT in the information age. In N. Yelland (Ed.), *Critical issues in early childhood education* (pp. 224-242). Maidenhead: Open University.