BOOLEAN NEGATION AND NON-CONSERVATIVITY I
RELEVANT MODAL LOGICS

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ABSTRACT: Many relevant logics can be conservatively extended by Boolean negation. Mares showed, however, that **E** is a notable exception. Mares' proof is by and large a rather involved model-theoretic one. This paper presents a much easier proof-theoretic proof which not only covers **E**, but generalizes so as to also cover relevant logics with a primitive modal operator added. It is shown that from even very weak relevant logics augmented by a weak **K**-ish modal operator, and up to the strong relevant logic **R** with a **S5** modal operator, all fail to be conservatively extended by Boolean negation. The proof, therefore, also covers Meyer and Mares' proof that **NR**—**R** with a primitive **S4**-modality added—also fails to be conservatively extended by Boolean negation.

KEYWORDS: *Boolean negation, non-conservative extension, entailment, modality, relevant logics*

1. INTRODUCTION

Modern modal logic came about as an attempt at augmenting classical logic with a connective which more plausibly than the material conditional could be read "implies" or "entails". Both C.I. Lewis and Hugh MacColl before him objected to reading the material conditional as "implies" which, for instance, Whitehead and Russell did in *Principia Mathematica*. The most provoking was the readings of the formulas $A \supset (B \supset A)$ and $\sim A \supset (A \supset B)$ as, respectively, "a true proposition is implied by any proposition", and "a

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false proposition implies any proposition".[1] These readings of these formulas are known as two of the *paradoxes of material implication*. To remedy this, MacColl and Lewis introduced a strict implication, commonly symbolized by '$\strictif$', where $A \strictif B$ is regarded as true when it is impossible that A is true, but B is false. However, $\strictif$, read as logical implication, has paradoxes of its own: if A is necessarily true—such as a logical truth of classical propositional logic—then $B \strictif A$ will be true too, and if A is necessarily false, then $A \strictif B$ will be true. Thus both $B \strictif A \vee {\sim}A$ and $A \wedge {\sim}A \strictif B$ are true. That this is so is sometimes levied against the identification of implication with the strict conditional and labeled *paradoxes of strict implication*.

Willhelm Ackermann began the study of logics free of both the paradoxes of the material and the strict conditional and introduced the concept of a *rigorous* implication with his 1956 essay *Begründung einer strengen Implikation* ([1]). Like MacColl and Lewis, Ackermann wanted his logic to be capable of expressing modal notions, but unlike them he didn't make any attempt at defining his rigorous implication in terms of modal and extensional notions.

The two paradoxes of the strict implication above express the key feature of a *Boolean* negation, that the conjunction of any contradictory pair of propositions implies every proposition which in turn imply any instance of a Boolean excluded middle. To avoid the paradoxes of the strict conditional, therefore, the logic of negation must in some sense be weaker than the Boolean negation. Relevant logics have since Ackermann's essay therefore adopted the weaker De Morgan negation. Like Lewis and MacColl, Anderson and Belnap also thought of entailment as a modal notion, and although they found Ackermann's logic intriguing, they took issue with some minor features of it. These were found to be easily amendable, however: by simply replacing Ackermann's δ-rule $A \to (B \to C), B \vdash A \to C$ with the axiom $(((A \to A) \wedge (B \to B)) \to C) \to C$, and deleting the γ-rule $A, {\sim}A \vee B \vdash B$, they arrived at the logic called **E** for *entailment*.[2] **E** does not have implicational paradoxes, yet can express a **S4**-modality through a definable modal operator, namely $\Box A =_{df} (A \to A) \to A$.

[1]For instance, Whitehead and Russell write with regards to the first:

> The most important propositions proved in the present number are the following:
> $*\mathbf{2{\cdot}02}.\ \vdash : q . \supset . p \supset q$
> *I.e.* q implies that p implies q, *i.e.* a true proposition is implied by any proposition. ([23, p. 103])

[2]Ackermann explicitly demanded the δ-rule to be restricted to cases where the second premise was a logical truth. δ taken this way turns out to be derivable also in **E**, and so, arguably, the only difference between the logics is that the latter does not have γ as a primitive rule.

Although the standard relevant negation has ever been the De Morgan one, Robert Meyer and Richard Routley started in the seventies to investigate so-called *classical relevant logics*—relevant logics with a Boolean negation added as an additional primitive negation. One of the hallmarks of relevant logics is the variable sharing property, that if $A \to B$ is a logical truth, then A and B share a propositional variable. Thus since, to quote Belnap, "commonality of meaning is carried by identity of propositional variables" ([3, p. 144]), for an object-language entailment statement to be true, the relata need to be content-wise related, which was one of the original requirements of Ackermann. The key feature of a Boolean negation explicitly violates this, but it turned out that one can preserve the variable sharing property for the Boolean free fragment if such a negation is added to just about any relevant logic.[3] Not only does such an extension preserve one of the key ideas of relevant logics, but it turned out that the extension is in many cases also *conservative*. But then again, not always. This is the case with **E**. As a consequence of Meyer and Dunn's proof of the admissibility of γ for **E** ([11]) it follows that if A is a classical tautology, then any substitutional instance of it is an **E**-theorem. Thus **E** extends classical logic theorem-vise. Since its modal operator expresses a **S4**-modality, the question arises whether this extends to classical modal logic **S4** as well. This, however, is not the case as Mares showed in [8] that the extensional **K**-sentence $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is not a theorem of **E**, although it is a theorem of **E** extended by Boolean negation. The extension is therefore non-conservative.

Mares' proof is by and large a rather involved model-theoretic one. This paper presents a much easier proof-theoretic proof which also holds for significantly weaker logic, as well as the **S5**-extension of **E**. Although it is easy to extend **E** so as to make the definable $\Box$ into a **S5** modality, it is not clear how to weaken **E** and **E5** so as to make $\Box$ into the other standard modalities **K**, **D**, **T** and **B**.[4] The most interesting feature of the proof, however, is that it generalizes to the case where the modal operator is taken as primitive. Meyer and Mares showed in [12] that $\Box(A \supset B) \supset (\Box A \supset \Box B)$ fails to be a theorem also of the relevant logic **R** extended by a primitive **S4** modality, but that it is a theorem of its Boolean extension. Thus also this logic, **NR**, fails to be conservatively extended by Boolean negation. Surely there most be a common explanation, and the proof to be given shows forth three common denominators:

(1) the meta-rule of reasoning by cases,
(2) the derivability of γ in the Boolean extension,

[3]There are no exceptions for sublogics of **R**. There are, however, exceptions. One such is exhibited in [15].

[4]A very interesting first step in this direction is Mares and Standefer's [7]. So far, however, their approach only pertains to the negation-free fragment, and since negation is the heart of the topic at hand, I will not discuss that paper further.

(3) that the modal operator is such that it distributes over true implications, that is if B is derivable from A, then $\Box B$ is derivable from $\Box A$.

We shall see that this latter property holds not only of **E**'s $\Box$, but of any primitive **K**-ish modal operator over quite weak relevant logics.

The structure of the proof is as follows:

(1) $A, A \supset B \vdash B$, where $A \supset B =_{df} {\sim}A \vee B$, is a derivable rule in the Boolean extension.
(2) The properties of $\Box$ are such that the meta-rule

$$(^{\Box}/_{\vdash}\text{-dist}) \quad \frac{A \vdash B}{\Box A \vdash \Box B},$$

is provable for in the Boolean extension.
(3) From (1) and (2) we get that $\Box A, \Box(A \supset B) \vdash \Box B$ is a derivable rule.
(4) The meta-rule of reasoning by cases,

$$(\text{RbC}) \quad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C},$$

is available in the Boolean extension.
(5) Using (3) and (4), then, we get that $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is a theorem (excluded middle is assumed).
(6) $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is not a theorem of in the Boolean-free logic.
(7) Hence: the Boolean extension is non-conservative.

The plan for the paper is as follows: Sect. 2 gives some common definitions before Sect. 3 gives a quick presentation of Mares' proof. Sect. 4 provides a standard proof of reasoning by cases using the pseudo modus ponens axiom $A \wedge (A \rightarrow B) \rightarrow B$. Sect. 4.1 gives a brief walk-through of ways to enthymematically weaken the logic without loosing reasoning by cases. Sect. 5 shows that $\Box$ distributes over true implications and the short Sect. 5 discusses possible weakenings. Sect. 6 shows that γ is a derivable rule in any Boolean extension of a logic for which (De Morgan) excluded middle is a theorem. Sect. 6.1 then gives two new proofs of reasoning by cases pertaining only to the Boolean extension. A discussion regarding the Boolean axiom $(A \rightarrow B) \sqsupset (A \sqsupset B)$, where $\sqsupset$ is the Boolean material implication, is given in light of this. Sect. 6.2 then summarizes the possible enthymematic weakenings discussed before Sect. 7 gives the main result with regards to **E**. Sect. 7.1 then introduces the notion of a *disjunctive rule* and shows how to further weaken the requirements in light of this.

Sect. 8 and Sect. 8.1 shows that the main result and its weakening using disjunctive rules can be generalized to the case where the modal operator is taken as primitive. Sect. 9 gives a slight variation in order to show that the logics $\mathbf{DK}^d$ and $\mathbf{TK}^d$ are not conservatively extended by Boolean negation, either, and thus solving a problem posed by Restall in [17]. Sect. 10 then finally summarizes.

This is the first of in all three essays on Boolean negation and non-conservativeness pertaining to relevant logics. The second essay, [15], deals with the question whether the variable sharing property is always preserved when extending a logic with Boolean negation, whereas the third essay, [16], deals with the question whether relevant logics with the truth-constant known as the Ackermann constant can be conservatively extended by Boolean negation. Together the three essays paint a picture of relevant logics being quite often non-conservatively extended by Boolean negation. It should therefore be noted that many relevant logics in fact are conservatively extended by Boolean negation. Neither of the three papers make any effort to survey such proofs, however. The interested reader should consult [4], [6], [13] and [17].

2. Definitions and common lemmas

This section simply gives some definitions of various relevant logics and then proves some useful lemmas. As a start, all proofs in this essay will be standard Hilbert-style proofs:

Definition 1 (Hilbert proof)**.** *A Hilbert proof of a formula A from a set of formulas Γ in the logic* **L** *is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of* **L***, or there is a set $\Delta \subseteq \{A_j | j < i\}$ such that $\Delta \vdash A_i$ is an instance of a rule of* **L***. The existential claim that there is such a proof is is written $\Gamma \vdash_{\mathbf{L}} A$.*

BB	A1–A5, R1–R7	**T**	**TW** +A12, +A13
B	**BB** +A6, +A7, −R5, −R6	**E**	**T** +A14, +A15
DW	**B** +A8, −R7	**R**	**T** (or **E**) +A11
TW	**DW** +A9, +A10, −R3, −R4	**LX**	**L** +A12$^\flat$
EW	**TW** +R8	C**L**	**L** +B1–B2
RW	**TW** +A11	$C^\sharp$**L**	C**L** +B3

Table 1. Definitions of various relevant logics

Definition 2.

$$\begin{array}{ll} \Box A =_{df} (A \to A) \to A & \\ A \supset B =_{df} {\sim}A \vee B & \textit{De Morgan material implication} \\ A \sqsupset B =_{df} \neg A \vee B & \textit{Boolean material implication} \\ e_A =_{df} A \to A & \end{array}$$

(A1)	$A \to A$	identity
(A2)	$A \to A \vee B$ and $B \to A \vee B$	$\vee$-introduction
(A3)	$A \wedge B \to A$ and $A \wedge B \to B$	$\wedge$-elimination
(A4)	$\sim\sim A \to A$	double negation elimination
(A5)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	distribution
(A6)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	strong lattice $\wedge$
(A7)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	strong lattice $\vee$
(A8)	$(A \to \sim B) \to (B \to \sim A)$	contraposition axiom
(A9)	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
(A10)	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
(A11)	$A \to ((A \to B) \to B)$	assertion axiom
(A12$^\flat$)	$A \vee \sim A$	excluded middle
(A12)	$(A \to \sim A) \to \sim A$	reductio
(A13$^\flat$)	$A \wedge (A \to B) \to B$	pseudo modus ponens
(A13)	$(A \to (A \to B)) \to (A \to B)$	contraction axiom
(A14)	$((A \to A) \to B) \to B$	1. **E**-distinctive axiom
(A15)	$\Box A \wedge \Box B \to \Box(A \wedge B)$	2. **E**-distinctive axiom
(B1)	$A \wedge \neg A \to B$	Boolean explosion axiom
(B2)	$A \to B \vee \neg B$	Boolean excl. middle axiom
(B3)	$(A \to B) \sqsupset (A \sqsupset B)$	Boolean interaction axiom
(R1)	$A, B \vdash A \wedge B$	adjunction
(R2)	$A, A \to B \vdash B$	modus ponens
(R3)	$A \to B \vdash (B \to C) \to (A \to C)$	suffixing rule
(R4)	$A \to B \vdash (C \to A) \to (C \to B)$	prefixing rule
(R5)	$A \to B, A \to C \vdash A \to B \wedge C$	lattice $\wedge$-rule
(R6)	$A \to C, B \to C \vdash A \vee B \to C$	lattice $\vee$-rule
(R7)	$A \to \sim B \vdash B \to \sim A$	contraposition rule
(R8)	$A \to (B \to C), B \vdash A \to C$	δ
(R9)	$A, \sim A \vee B \vdash B$	γ

Axiom A12$^\flat$, $A \vee \sim A$ is in fact interderivable in **BB** with the rule version of reductio, axiom A12. Similarly, axiom A13$^\flat$ is interderivable with the rule version of the contraction axiom, A13. This, then, is the reason behind the superscripted '$\flat$'.

Definition 3 (Admissible rule)**.** *A rule* $\Gamma \vdash A$ *is an* admissible rule *in* **L** *if it is the case that* $\varnothing \vdash_{\mathbf{L}} A$ *when* $\varnothing \vdash_{\mathbf{L}} B$ *for all* $B \in \Gamma$.

Definition 4 (Boolean extension)**.** C**L** *is called the* Boolean extension *of* **L**.

Definition 5 (Strong Boolean extension)**.** $C^\sharp$**L** *is called the* strong Boolean extension *of* **L**.

Definition 6 (Conservative Extension)**.** *If* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are logics over, respectively, languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *such that* $\mathcal{L}_1 \subseteq \mathcal{L}_2$*, then* $\mathbf{L_2}$ *conservatively extends* $\mathbf{L_1}$ *if* $\varnothing \vdash_{\mathbf{L}_1} A$ *for every* $\mathcal{L}_1$*-formula* A *such that* $\varnothing \vdash_{\mathbf{L}_2} A$.

Lemma 1 (Basic logical properties)**.** *Any logic extending* **BB** *has the following derived rules:*

$$\begin{array}{ll}(DR1) & A \to B \vdash A \vee C \to B \vee C \\ (DR2) & A \to B \vdash A \wedge C \to B \wedge C \\ (DR3) & (A \vee C), (B \vee C) \vdash (A \wedge B) \vee C\end{array}$$

Proof. Left for the reader. □

3. Mares' proof and an E-counter-model to the extensional K-sentence

Mares showed in [8] that the extensional **K**-sentence $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is derivable in the Boolean extension of **E** and therefore—since the sentence is not a theorem of **E**—that the extension is non-conservative. The most interesting fact and main feature of Mares' proof is that every model for C**E** contains a model for the classical modal logic **S4**. By letting τ be the simple translation from the **S4**-language to the **E**-language which translates $\wedge_{\mathbf{S4}}$ to $\wedge_{\mathbf{E}}$, $\sim_{\mathbf{S4}}$ to $\sim_{\mathbf{E}}$ and $\Box_{\mathbf{S4}}$ to $\Box_{\mathbf{E}}$. Mares shows that every C**E**-model will validate the τ-translation of every logical theorem of **S4**, and therefore $\Box(A \supset B) \supset (\Box A \supset \Box B)$ for every A and B got from any **S4**-formula via τ. By the soundness theorem of **S4** and the completeness theorem of C**E** (supplied by Mares), it follows that C**E** is a non-conservative extension of **E**.

There are two things that should be noted concerning Mares' proof. First of all it should be noted that his proof does not show that $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is a general theorem of C**E**, only that it is for assorted A's and B's. This is reflected in Mares' statement of the theorem which uses propositional variables and not metalinguistic variables over formulas as in the other results of his paper. The proof to be given in this paper shows that the extensional **K**-sentence is a general C**E**-theorem.

The second thing to note is that Mares defines **E** and C**E** rather differently than what I have done in this paper. The Boolean extension is got by adding the rule $A \sqsupset B \vdash A \to B$ provided $A \sqsupset B$ is a substitution instance of a classical tautology. This obviously suffices for yielding B1 and B2 as logical theorems. Mares defines **E** slightly differently as well. Most importantly he defines it to include both the Ackermann constant **t** as well an intensional conjunction, $\circ$, called *fusion*. Note, then, that the Ackermann constant can be add conservatively to **E** as here defined.[5] The addition of the fusion connective is a bit trickier: using the Boolean rule it yields $A \circ B \to C \vee \neg C$ as a theorem, from which the rules for $\circ$ yield $A \to (B \to C \vee \neg C)$. This is not a theorem of C**E** as here defined.[6] However, Mares' version of **E** is known to be a conservative extension of **E** as defined here, and so his proof does suffice for establishing his result.[7] That C**E**, as here defined, suffices,

[5] See [2, lem. 2] for a proof which obviously also covers **E**.

[6] The B2 axiom is sometimes taken to be this stronger axiom $A \to (B \to C \vee \neg C)$ (cf. [6] and [17]).

[7] See the first appendix of [20] for a proof that $\circ$ can be added conservatively to **E**.

then, goes to show that Mares' result does not require a stronger Boolean extension than that got by adding only B1 and B2.

I will show forth a much simpler proof of Mares' result which also holds for logics slightly weaker and slightly stronger than **E**. We will also see that it can be modified to cover different logics as well.

Mares showed forth a model for **E** which invalidates $\Box(A \supset B) \supset (\Box A \supset \Box B)$. There are three points that I wish to make regarding it.

(1) First of all note that **E**'s $\Box$ is a **S4**-modality. **E** can, however, easily be extended to yield **S5**-modality; as in classical modal logic, one can simply add the Brouwerian axiom $A \to \Box\Diamond A$. Mares' model also validates this extended logic, **E5**, and so also this logic fails to be conservatively extended by Boolean negation.

(2) In the sketch of the proof to be given, the only property needed from the Boolean extension is that it yields γ, that is modus ponens for $\supset$, as a derivable rule. However, simply adding this rule to **E** does not suffice for deriving $\Box(A \supset B) \supset (\Box A \supset \Box B)$, although Mares' model does not show this since it does not validate the rule.[8]

(3) Mares uses a version of **E** which is endowed with the fusion connective. Although the primitive rules for $\circ$ are the residuation rules

$$\begin{array}{ll} (\circ I) & A \to (B \to C) \vdash A \circ B \to C \\ (\circ E) & A \circ B \to C \vdash A \to (B \to C), \end{array}$$

$(A \circ B \to C) \to (A \to (B \to C))$ is a theorem of **E** augmented with these rules, whereas $(A \to (B \to C)) \to (A \circ B \to C)$ is not. Mares' model does not validate this latter axiom form of the $(\circ I)$-rule. The algebraic counter-model displayed in Fig. 1 validates $\mathbf{E5}^{\mathbf{t}\circ^+}[\gamma]$—**E** augmented with γ as a new primitive rule, the Brouwerian axiom, the axiomatic versions of the $(\circ I)$- and $(\circ E)$-axioms, as well as $\mathbf{t} \to (A \to A)$ and $(\mathbf{t} \to A) \to A$—the axioms for **t**.[9]

The counter-model to $\Box(A \supset B) \supset (\Box A \supset \Box B)$ in Fig. 1 consists, like all models to be displayed in this paper, of a displyed partial ordering over which conjunction and disjunction are to be interpreted as, respectively,

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[8]That the proof does not, therefore, extend to Ackermann's logic is further explained in the parenthetical in Sect. 7. Mares' model is a Routley-Meyer model and consists of three "worlds"—0, 1 and 2. The first two are the "regular" worlds which the rules need to be truth-preserving over. 0 is a consistent and complete world—it validates either A or $\sim A$ for every A and never both; 2 is a gappy world—it fails to validate either **t** or $\sim$**t**, where **t** is the Ackermann constant; whereas 1 is an inconsistent world in that it validates both **t** and $\sim$**t**. Since 1 validates both **t** and $\mathbf{t} \supset A$ for every A, but does not validate every such A, it follows that Mares' model fails to validate γ, that is, modus ponens for $\supset$.

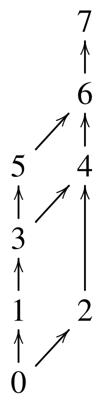
[9]Note that Mares states the rules of **E** to be logically theorem-preserving. Adjunction is, for instance, stated as $\vdash A \& \vdash B \Rightarrow \vdash A \wedge B$. Mares axiomatizes **t** using the axiom $(\mathbf{t} \to A) \to A$ and the logical theorem-preserving rule $\vdash A \Rightarrow \vdash \mathbf{t} \to A$. Note, then, that the meta-rule that $\mathbf{t} \to A$ is a logical theorem if A is, is provable using the axioms specified here, and $\mathbf{t} \to (A \to A)$ is obviously a theorem using Mares' **t**-rule. The two axiomatizations of **t** are, therefore, equivalent.

infimum and supremum. Alongside there will be a matrix which shows how the conditional and the negation(s) are to be interpreted and possible other connective. A subset $\mathcal{T}$ of the algebra—a *filter* to be precise—is selected as the set of *designated elements*. A rule holds in an algebra just in case the conclusion is assigned a value in $\mathcal{T}$ when all its premises are. I also list how to interpret the relevant formulas so as to make the model a counter-model to the intended formula.[10]

Theorem 1. *$\Box(A \supset B) \supset (\Box A \supset \Box B)$ is not a theorem of* $\mathbf{E5}^{\mathbf{t}\circ^+}[\gamma]$.

Proof. A counter-model is displayed in Fig. 1.

$\mathcal{T} = \{2, 4, 6, 7\}$
$[\![\mathbf{t}]\!] = 4$
$[\![A]\!] = 4$
$[\![B]\!] = 2$



$\to$	0	1	2	3	4	5	6	7	$\sim$	$\Box$	$\Diamond$
0	7	7	7	7	7	7	7	7	7	0	0
1	0	4	4	4	4	4	4	7	6	0	3
2	0	3	4	3	4	3	4	7	5	0	4
3	0	0	0	4	4	4	4	7	4	3	3
4	0	0	0	3	4	3	4	7	3	4	4
5	0	0	0	0	0	4	4	7	2	3	7
6	0	0	0	0	0	3	4	7	1	4	7
7	0	0	0	0	0	0	0	7	0	7	7

$\supset$	0	1	2	3	4	5	6	7
0	7	7	7	7	7	7	7	7
1	6	6	6	6	6	6	6	7
2	5	5	6	5	6	5	6	7
3	4	4	4	4	4	6	6	7
4	3	3	4	3	4	5	6	7
5	2	2	2	4	4	6	6	7
6	1	1	2	3	4	5	6	7
7	0	1	2	3	4	5	6	7

$\circ$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	1	3	3	5	5	7
2	0	1	2	3	4	5	6	7
3	0	1	1	3	3	5	5	7
4	0	1	2	3	4	5	6	7
5	0	7	7	7	7	7	7	7
6	0	7	7	7	7	7	7	7
7	0	7	7	7	7	7	7	7

FIGURE 1. $\mathbf{E5}^{\mathbf{t}_4\circ^+}[\gamma]$-counter-model to $\Box(A \supset B) \supset (\Box A \supset \Box B)$

$\Box$

4. REASONING BY CASES AND PSEUDO MODUS PONENS

The outline of the proof given mentioned reasoning by cases as one ingredient. Reasoning by cases, in the simplest case, can be stated as

$$\text{(RbC)} \quad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C},$$

[10]All models depicted in this paper have been found with the help of `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney ([22]). I have made heavy use of both it as well as William McCune's theorem prover/model generator package *Prover9/Mace4* ([9]) in arriving at the results reported in this essay.

and reads: *if there is a proof of C from the assumption of A and also a proof of C from the assumption of B, then there is a proof of C from assumption of* $A \vee B$. Reasoning by cases is thus not a rule, but a *meta-rule*; a statement about the derivability relation $\vdash$.

I will provide three different proofs of reasoning by cases; one which utilizes pseudo modus ponens, the axiom $A \wedge (A \to B) \to B$, one which uses the reductio axiom $(A \to {\sim}A) \to {\sim}A$ in the Boolean extension, and one which makes use of the third Boolean axiom $(A \to B) \sqsupset (A \sqsupset B)$. To avoid having to repeat too much, I will now show forth a well-known and simple lemma which show that the question whether RbC holds, can be reduced to the simpler meta-rule that $B \vee C$ is provable from $A \vee C$ provided B is provable from A.

Lemma 2.

$$(^{\vee}/_{\vdash}\text{-}dist)\ \frac{A \vdash B}{A \vee C \vdash B \vee C} \implies (RbC)\ \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}$$

Proof. Assume that $A \vdash C$ and $B \vdash C$. By the assumption of the meta-rule $^{\vee}/_{\vdash}$-dist together with the commutativity of $\vee$, we get that $A \vee B \vdash B \vee C$ and $B \vee C \vdash C \vee C$. Since $C \vee C \vdash C$ we get $A \vee B \vdash C$ by the transitivity of $\vdash$. □

Theorem 2.

$$\frac{A \vdash B}{A \vee C \vdash B \vee C}$$

holds for any logic **L** *which extends* **BB**[A13$^\flat$]—**BB** *with the pseudo modus ponens axiom* $A \wedge (A \to B) \to B$ *added—provided* **L** *has no more primitive rules than modus ponens and adjunction.*

Proof. Assume that $D_1, \ldots, D_n$ is a proof of B from A. The goal is to show that $A \vee C \vdash D_i \vee C$ for all $i \leq n$.

First if D_i is either an axiom or is identical to A, then obviously we have that $A \vee C \vdash D_i \vee C$. For induction hypothesis assume that $A \vee C \vdash D_i \vee C$ and $A \vee C \vdash D_j \vee C$ and D_k is got from D_i and D_j using adjunction. Using Lem. 1 (DR3) we get $A \vee C \vdash (D_i \wedge D_j) \vee C$. Assume now that D_k is got from D_i and $D_j = D_i \to D_k$ using modus ponens. Since A13$^\flat$ is an axiom of **L**, we get $(D_i \wedge (D_i \to D_k)) \vee C \to D_k \vee C$ from $D_i \wedge (D_i \to D_k) \to D_k$ by using Lem. 1 (DR1). As before we get that $A \vee C \vdash (D_i \wedge (D_i \to D_k)) \vee C$, and therefore $A \vee C \vdash D_k \vee C$. This ends the proof for logics with A13$^\flat$.[11] □

Corollary 1. *RbC is a provable meta-rule of* **E** *and C***E***, as well as of* **E5** *and C***E5***.*

4.1. **Possible weakenings.** Note the restriction on not having further primitive rules than adjunction and modus ponens. Beyond this, however, it is only the assumption of pseudo modus ponens, i.e. A13$^\flat$, which is a substantial assumption. We will later see two other proofs which do away with

[11]The proof given here is essentially that given by Meyer and Dunn in [11, p. 462f].

this assumption and instead require properties of the Boolean extension to go through.

TW is the weakest of the commonly refered to relevant logics which has only adjunction and modus ponens as primitive rules, and so **TW**[$A13^\flat$] is the weakest such logic which is covered by the above result on RbC. However, **TW**[$A13^\flat$] can be weakened significantly. The following *enthymemized* versions of the **TW**[$A13^\flat$]-axioms would, as the reader can easily check, do just as well for the above RbC-proof, and, unless otherwise stated, just as well also in any other proof in this paper. For readability I'll use e_A as short for $A \to A$.

(eA6)	$((A \to B) \wedge (A \to C)) \wedge e_A \to (A \to B \wedge C)$
(eA7)	$((A \to C) \wedge (B \to C)) \wedge e_A \to (A \vee B \to C)$
(eA8)	$(A \to {\sim}B) \wedge e_A \to (B \to {\sim}A)$
(eA9)	$(A \to B) \wedge e_C \to ((B \to C) \to (A \to C))$
(eA10)	$(A \to B) \wedge e_C \to ((C \to A) \to (C \to B))$
(eA13$^\flat$)	$(A \wedge (A \to B)) \wedge e_B \to B$

5. Modal properties

The purpose of this section is simply to show that the meta-rule

$$(\Box/_\vdash\text{-dist}) \quad \frac{A \vdash B}{\Box A \vdash \Box B},$$

which states that the necessity operator distributes over true implications, holds for **E** and *C***E** and to show what logical properties are needed for it to be provable.

Lemma 3. $A \to B \vdash_{\mathbf{BB}[A14]} \Box(A \to B)$

Proof.

(1)	$A \to B$	assumption
(2)	$(A \to A) \to (A \to B)$	1, R4
(3)	$((A \to B) \to (A \to B)) \to ((A \to A) \to (A \to B))$	2, R3
(4)	$((A \to A) \to (A \to B)) \to (A \to B)$	A14
(5)	$((A \to B) \to (A \to B)) \to (A \to B)$	3, 4, transitivity
(6)	$\Box(A \to B)$	5, def. of $\Box$

□

Lemma 4. $A \to B, \Box A \vdash_{\mathbf{BB}[A14]} \Box B$

Proof.

(1)	$A \to B$	assumption
(2)	$\Box A$	assumption
(3)	$(A \to A) \to B$	1, 2, transitivity
(4)	$(B \to B) \to ((A \to A) \to B)$	3, R3
(5)	$((A \to A) \to B) \to B$	A14
(6)	$\Box B$	4, 5, transitivity & def. of $\Box$

$\square$

Theorem 3.

$$(\square/\vdash\text{-}dist)\quad \frac{A \vdash B}{\square A \vdash \square B},$$

holds for any logic **L** *which extends* **BB**$[A14|A15]$, **BB** *extended by the two* ***E****-distinctive axioms*

(A14)	$((A \to A) \to B) \to B$	*1.* ***E****-distinctive axiom*
(A15)	$\square A \wedge \square B \to \square(A \wedge B)$	*2.* ***E****-distinctive axiom,*

by rules the main connective of the conclusion of which is $\to$*, or axioms the main connective of which is* $\to$*.*

Proof. Assume that $D_1, \ldots, D_n$ is a proof of B from A. The proof is an induction to the effect that $\square A \vdash \square D_i$ for every i. If D_i is A, then obviously $\square A \vdash \square D_i$. If D_i is an axiom, then as all axioms of **L** are $\to$-formulas, Lem. 3 ensures that $\square D_i$ is a theorem, and so $\square A \vdash \square D_i$.

Assume for inductive hypothesis that both $\square A \vdash \square D_i$ and $\square A \vdash \square D_j$ and D_k is got from D_i and D_j by adjunction. Using adjunction we can infer that $\square A \vdash \square D_i \wedge \square D_j$. Using the A15 axiom $\square D_i \wedge \square D_j \to \square(D_i \wedge D_j)$ and modus ponens we then get that $\square A \vdash \square(D_i \wedge D_j)$. Assume now that D_k is got from D_i and $D_j = D_i \to D_k$ by modus ponens. From the induction hypothesis we get that $\square A \vdash \square D_i$ and $\square A \vdash \square(D_i \to D_k)$ and therefore $\square A \vdash D_i \to D_k$. Lem. 4 then suffices for $\square A \vdash \square D_k$.

Now assume that D_k is got by any of the other primitive rules of **BB**. Since $\square A \vdash A$ is derivable we can infer the hypothesis of rule used to derive D_k from their necessitated hypothesis in the induction proof. Using this one then derives D_k itself using the rule in question. Since this is a $\to$-formula, we can then use Lem. 3 to conclude that $\square A \vdash \square D_k$ which then ends the proof. Thus, for instance, from the inductive hypothesis that $\square A \vdash \square D_i$, where $D_i = E \to B$, and $D_k = (B \to C) \to (E \to C)$ is got by using the R3 rule, simply infer $\square A \vdash E \to B$ from $\square A \vdash \square(E \to B)$, then infer $\square A \vdash (B \to C) \to (E \to C)$ using R3 and lastly $\square A \vdash \square((B \to C) \to (E \to C))$ using Lem. 3. $\square$

Corollary 2.

$$(\square/\vdash\text{-}dist)\quad \frac{A \vdash B}{\square A \vdash \square B}$$

holds for both **E** *and* **CE***.*

If we abstract away from the particular definition of $\square$ and considers instead a primitive modal operator $\boxdot$ it is worth mentioning the following:

Corollary 3.

$$(\boxdot/\vdash\text{-}dist)\quad \frac{A \vdash B}{\boxdot A \vdash \boxdot B}$$

is derivable in any logic extending **TW** *augmented by necessitation rule for* $\boxdot$ *as well as both the* $\mathbf{K}^{\boxdot}$ *and the* ${}^{\boxdot}/_{\wedge}$*-distribution axioms:*

$$\begin{array}{ll} (\boxdot\text{-}Nec) & \dfrac{\varnothing \vdash A}{\varnothing \vdash \boxdot A} \\ (\mathbf{K}^{\boxdot}) & \boxdot(A \to B) \to (\boxdot A \to \boxdot B) \\ ({}^{\boxdot}/_{\wedge}\text{-}dist) & \boxdot A \wedge \boxdot B \to \boxdot(A \wedge B) \end{array}$$

Proof. Simple induction similar to that of Thm. 3. □

I will get back to this result later on.

5.1. **Possible weakenings.** The proof that $\Box$ distributes over true implications relied on three essential assumptions beyond that necessarily supplied by **BB**:

(1) Every axiom needs to be a $\to$ formula: needed to ensure that $\varnothing \vdash \Box A$ for every axiom A.
(2) A14: used in both Lem. 3 and Lem. 4 which Thm. 3 relies on.
(3) A15: used in Thm. 3 to ensure that $\Box A, \Box B \vdash \Box(A \wedge B)$ is a derivable rule.

I will get back to the first requirement after the next section on the Boolean lemmas. With regards to A14 and A15 it is easy to verify that the proof of the (${}^{\Box}/_{\vdash}$-dist) meta-rule would still go through if A14 and A15 were enthymemized to the following versions:

$$\begin{array}{ll} (\text{eA14}) & ((A \to A) \to B) \wedge e_B \to B \\ (\text{eA15}) & (\Box A \wedge \Box B) \wedge e_A \to \Box(A \wedge B) \end{array}$$

6. Boolean lemmas

Lemma 5. $A \to \neg B \vdash_{\mathbf{CBB}} B \to \neg A$

Proof.

(1)	$A \to \neg B$	assumption
(2)	$B \wedge A \to B \wedge \neg B$	1, fiddling
(3)	$B \wedge \neg B \to \neg A$	B1
(4)	$B \wedge A \to \neg A$	2, 3, transitivity of $\to$
(5)	$B \wedge \neg A \to \neg A$	A3
(6)	$(B \wedge A) \vee (B \wedge \neg A) \to \neg A$	4, 5, R6
(7)	$B \wedge (A \vee \neg A) \to (B \wedge A) \vee (B \wedge \neg A)$	A5
(8)	$B \wedge (A \vee \neg A) \to \neg A$	6, 7, transitivity of $\to$
(9)	$B \to A \vee \neg A$	B2
(10)	$B \to B \wedge (A \vee \neg A)$	9, A1 + R6
(11)	$B \to \neg A$	8, 10, transitivity of $\to$

□

Corollary 4 (Boolean facts)**.** *The following are all theorems of C**BB**:*

$$\begin{array}{ll} (BF1) & \neg\neg A \leftrightarrow A \\ (BF2) & \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B) \\ (BF3) & \neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B) \\ (BF4) & A \wedge (A \sqsupset B) \to B \\ (BF5) & (A \sqsupset B) \wedge (B \sqsupset C) \sqsupset (A \sqsupset C) \\ (BF6) & (A \sqsupset B) \sqsupset (A \vee C \sqsupset B \vee C) \end{array}$$

Proof. Left for the reader. □

It can also be shown that $\neg{\sim}A \leftrightarrow {\sim}\neg A$ is a theorem of C**BB**. The Boolean negation ($\neg$) and the De Morgan negation ($\sim$) are importantly different, however. For instance, $(A \to \neg B) \to (B \to \neg A)$, and the other versions of the contraposition axiom fail to be theorems even in C**R**, although every such corresponding rule is derivable in C**BB**.

Corollary 5. $A \to B \vdash_{C\mathbf{BB}} A \sqsupset B$

Proof.

$$\begin{array}{lll} (1) & A \to B & \text{assumption} \\ (2) & A \wedge \neg B \to B \wedge \neg B & \text{1, fiddling} \\ (3) & B \wedge \neg B \to \neg(A \to A) & \text{B1} \\ (4) & A \wedge \neg B \to \neg(A \to A) & \text{2, 3, transitivity} \\ (5) & (A \to A) \to \neg(A \wedge \neg B) & \text{4, Lem. 5} \\ (6) & (A \to A) \to (A \sqsupset B) & \text{5, Cor. 4 + fiddling} \\ (7) & A \sqsupset B & \text{6, A1 + R2} \end{array}$$

□

Lemma 6. $A \vdash_{C\mathbf{BBX}} {\sim}\neg A$

Proof.

$$\begin{array}{lll} (1) & A \wedge \neg A \to {\sim}\neg A & \text{B2} \\ (2) & A \wedge {\sim}\neg A \to {\sim}\neg A & \text{A3} \\ (3) & (A \wedge \neg A) \vee (A \wedge {\sim}\neg A) \to {\sim}\neg A & \text{1, 2, R6} \\ (4) & \neg A \vee {\sim}\neg A & \text{A12} \\ (5) & A & \text{assumption} \\ (6) & A \wedge (\neg A \vee {\sim}\neg A) & \text{4, 6, R1} \\ (7) & (A \wedge \neg A) \vee (A \wedge {\sim}\neg A) & \text{6, A5} \\ (8) & {\sim}\neg A & \text{3, 7, R1} \end{array}$$

□

Theorem 4. *γ, i.e. the rule $A, {\sim}A \vee B \vdash B$, is derivable in any logic extending C**BBX**.*

Proof.

(1)	A	assumption
(2)	$\sim A \vee B$	assumption
(3)	$\sim\neg A$	1, Lem. 6
(4)	$\sim\neg A \wedge (\sim A \vee B)$	2, 3, R1
(5)	$(\sim\neg A \wedge \sim A) \vee (\sim\neg A \wedge B)$	4, A5, R2
(6)	$\sim B \rightarrow A \vee \neg A$	B2
(7)	$\sim A \wedge \sim\neg A \rightarrow B$	6, fiddling
(8)	$\sim\neg A \wedge B \rightarrow B$	A3
(9)	$(\sim\neg A \wedge \sim A) \vee (\sim\neg A \wedge B) \rightarrow B$	7, 8, R6
(10)	B	5, 9, R2

□

Corollary 6. *γ is derivable in C***E**.

«*Parenthetical remark.* Relevant logics have traditionally been *paraconsistent* and therefore do not include γ.[12] However, there has been considerable interest in showing that the logical theorems of the various logics are closed under this rule; that B is a logical theorem if both A and $\sim A \vee B$ are. This is the problem of the *admissibility of* γ which was solved in the positive for both **R**, **E** and **T** by Meyer and Dunn in [11]. The above theorem shows that γ is in fact *derivable* in the Boolean extension of any logic with excluded middle. From this it follows that the admissibility of γ is a necessary condition for such a logic to be conservatively extended by Boolean negation:

Corollary 7. *γ is admissible in any logic* **L** *extending* **BBX***, provided C***L** *is a conservative extension of* **L**.

Proof. Assume that both $\vdash_{\mathbf{L}} A$ and $\vdash_{\mathbf{L}} A \supset B$. From the above lemma it now follows that $\vdash_{C\mathbf{L}} B$ and since B is Boolean-free and C**L** is a conservative extension of **L**, it then follows that $\vdash_{\mathbf{L}} B$. *End parenthetical.*» □

6.1. **Two Boolean proofs of reasoning by cases.** The proof of reasoning by cases above (Thm. 2) made use of A13$^\flat$—the pseudo modus ponens axiom. I will now show forth two other proof of reasoning by cases, one which makes use of axiom A12—the reductio axiom $(A \rightarrow \sim A) \rightarrow \sim A$—together with the fact that γ is a derivable rule in C**BBX**, and another one which makes use of the B3 axiom $(A \rightarrow B) \sqsupset (A \sqsupset B)$.

Lemma 7.

$$\frac{A \vdash B}{A \vee C \vdash B \vee C}$$

*holds in any logic C***L** *where* **L** *extends* **BB**[$A12$]*, and in any logic $C^\sharp$***L** *where* **L** *extends* **BB***, provided only adjunction and modus ponens are primitive rules* **L**.

[12] Ackermann's Π', as well as the logic $\Pi'_{\mathbf{E}}$ presented in [14] are worth mentioning as exceptions. See the latter paper for a discussion of why relevant logics ended up being wrongly viewed as inherently paraconsistent.

Proof. The proof is again an induction. The base case and the inductive case for adjunction is quite similar to that of Thm. 2, and so is left for the reader.

Assume for inductive hypothesis that $A \vee C \vdash D_i \vee C$ and $A \vee C \vdash (D_i \to D_j) \vee C$. The reductio axiom is in fact interderivable with $(A \to B) \to (A \supset B)$, and so yields $(D_i \to D_j) \to (D_i \supset D_j)$ and therewith $(D_i \to D_j) \vee C \to (D_i \supset D_j) \vee C$. We can then detatch and thus get that $A \vee C \vdash (D_i \supset D_j) \vee C$. From this and the other hypothesis we get $A \vee C \vdash (D_i \wedge (D_i \supset D_j)) \vee C$ by distribution fiddling. $(D_i \wedge (D_i \supset D_j)) \supset D_j$ is a theorem of **BB**[$A12$] from which we get $(D_i \wedge (D_i \supset D_j)) \vee C \supset D_j \vee C$ by fiddling. Since reductio is assumed, excluded middle is a theorem and so Thm. 4 entails that modus ponens holds for $\supset (\gamma)$. By using γ we can therefore detach to get $A \vee C \vdash D_j \vee C$.

The proof using the B3-axiom $(A \to B) \sqsupset (A \sqsupset B)$ of $C^\sharp$**L** is quite similar and is therefore left for the reader. ◻

From the above lemma we get the following corollary:

Corollary 8.

$$\frac{A \vdash B}{A \vee C \vdash B \vee C}$$

holds in both $C^\sharp$**TW**, $C^\sharp$**TWX** *and in* C**TW**[$A12$].

Note that this last result on reasoning by cases can by itself be used to show non-conservativity:

Theorem 5. $C^\sharp$**TWX** *is not a conservative extension of* **TWX**.

Proof. Since excluded middle is an axiom of **TWX**, and reasoning by cases is derivable in $C^\sharp$**TWX** (Cor. 8), we get that $\sim(A \wedge (A \to B)) \vee B$ is a theorem of $C^\sharp$**TWX**. This is, however, not a theorem of even C**TWX** (counter-model in Fig. 2[13]), and so the result follows.

$$\mathcal{T} = \{3\} \qquad [\![A]\!] = 1 \qquad [\![B]\!] = 0$$

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] & & 2 \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\to$	0	1	2	3	$\sim$	$\neg$
0	3	3	3	3	3	3
1	2	3	2	3	2	2
2	2	2	3	3	1	1
3	2	2	2	3	0	0

FIGURE 2. A C**TWX**-counter-model to $\sim(A \wedge (A \to B)) \vee B$

◻

The mere addition of reasoning by cases is not sufficient for non-conservativeness in the case of **TW** since this logic is itself prime[14] ([21, cor. 1]). Primeness,

[13]This is in fact the model used by Giambrone and Meyer to show that B3 is independent from B1 and B2 over **TW** ([6, pp. 2f]).

[14]So that $A \vee B$ is a theorem of **TW** if and only if at least one of the disjuncts are.

however, is a different property than that of reasoning by cases which is not provable for **TW**:

Corollary 9. *Reasoning by cases does not hold for* **TW**.

Proof. Since both

$$A \wedge (A \to B) \vdash {\sim}(A \wedge (A \to B)) \vee B$$
$${\sim}(A \wedge (A \to B)) \vdash {\sim}(A \wedge (A \to B)) \vee B,$$

reasoning by cases would yield that

$$(A \wedge (A \to B)) \vee {\sim}(A \wedge (A \to B)) \vdash {\sim}(A \wedge (A \to B)) \vee B$$

which the model in Fig. 2 shows is not the case. □

The Boolean extension of a relevant logic is sometimes taken to include the B3-axiom $(A \to B) \sqsupset (A \sqsupset B)$. This is the case in both Giambrone and Meyer's joint paper [6], and Restall's paper [17]. Neither paper provide any argument for defining the Boolean extension to include B3, the reason for including it, it seems, is to get a so-called *reduced* semantics.[15] It should, however, be noted that B3 is rather different from the other two Boolean axioms: whereas B1 and B2 simply express that any instance of a Boolean excluded middle is entailed by every formula and a Boolean contradiction entails everything, B3 expresses that any relevant conditional either fails to be true, or the Boolean material conditional is true, or to put it equivalently; either the premises of any instance of modus ponens holds, or its conclusion does. Since B3 is equivalent to $A \wedge (A \to B) \sqsupset B$, and $A \to B \vdash A \sqsupset B$ is a derivable rule of even C**BB**, it follows that the axiom is derivable in any logic with the pseudo modus ponens axiom $A \wedge (A \to B) \to B$ which, again, is interderivable in **BB** with the rule of contraction, i.e. $A \to (A \to B) \vdash A \to B$. Furthermore, if reasoning by cases is available, then B3 will obviously also be a theorem. It seems, however, that for some contraction free logics for which reasoning by cases is not provable, the B3 axiom simply is too strong; the non-conservativity in the case of **TWX** is at least got rather easily if we were only to consider Boolean logics with the B3-axiom. Whether C**TWX** is a conservative extension of **TWX** is a matter that is, as far as I know, unresolved. To further emphasize how strong the B3-axiom is, note that it even makes it possible to re-axiomatize any logic so as to replace adjunction with the Boolean axiom $A \sqsupset (B \sqsupset A \wedge B)$ and $\to$-modus ponens with Boolean material modus ponens, i.e. $A, A \sqsupset B \vdash B$. Thus it seems that $C^\sharp$**L** is too strong in some cases where C**L** may not be, and so I have found it prudent to distinguish between $C^\sharp$**L** and C**L**.

6.2. **Summary over possible enthymematic weakenings.** The main result of this section, Thm. 4, was that γ, i.e. the rule $A, A \supset B \vdash B$ is derivable in C**BBX**. The main theorem of this paper makes use of this, and so relies on the presence of excluded middle. However, the proof that $\Box$ distributes

[15] Giambrone and Meyer's use of B3 is in their proof of *Fact 1* ([6, p. 4]) which states that any set of formulas containing all the axioms is also closed under modus ponens.

over true implications, Thm. 3, relied on every axiom of the logic being a $\to$-formula. For this reason we need to beef excluded middle up to such a $\to$ formula which suffices for deriving excluded middle. The reductio axiom, of course, would suffice, but so would any enthymemized version of it. As the reductio axiom is equivalent to the axiom $(A \to B) \to (A \supset B)$, one such version would be

$$\text{(eA12)} \quad (A \to B) \wedge e_A \to (A \supset B)$$

which also suffices for the proof of Lem. 7. Another possible way would be to simply weaken it to the necessitated excluded middle axiom

$$(\mathbf{X}_\Box) \quad \Box(A \vee {\sim}A),$$

which turns out to be properly weaker than the eA12-axiom.

To sum up some of the possible ways to enthymematically weaken **E** while still making sure reasoning by cases is provable, that $\Box$ distributes over true implications, and that γ is a derivable rule in the Boolean extension is then to augment **BB** by the following axioms and deleting any primitive rule of **BB** save adjunction and modus ponens:

$$\begin{array}{ll}
\text{(eA6)} & ((A \to B) \wedge (A \to C)) \wedge e_A \to (A \to B \wedge C) \\
\text{(eA7)} & ((A \to C) \wedge (B \to C)) \wedge e_A \to (A \vee B \to C) \\
\text{(eA8)} & (A \to {\sim}B) \wedge e_A \to (B \to {\sim}A) \\
\text{(eA9)} & (A \to B) \wedge e_C \to ((B \to C) \to (A \to C)) \\
\text{(eA10)} & (A \to B) \wedge e_C \to ((C \to A) \to (C \to B)) \\
\text{(eA12)} & (A \to B) \wedge e_A \to (A \supset B) \\
\text{(eA14)} & ((A \to A) \to B) \wedge e_B \to B \\
\text{(eA15)} & (\Box A \wedge \Box B) \wedge e_A \to \Box(A \wedge B)
\end{array}$$

If eA12 is weakened to $\mathbf{X}_\Box$, one needs to add modus ponens in some axiomatic form to ensure that the first proof of reasoning by cases (Thm. 2). Adding A13$^\flat$ suffices, but so does any enthymemized version such as $(A \wedge (A \to B)) \wedge e_A \to B$. Lastly, one could stick with excluded middle (A12$^\flat$), but then one would have to beef up the Boolean assumptions with the B3 axiom $(A \to B) \sqsupset (A \sqsupset B)$ in order for, again, reasoning by cases to hold.

7. Main theorem

It's now time to give the proof of $\Box(A \supset B) \supset (\Box A \supset \Box B)$.

Theorem 6. *Any sublogic* **L** *of* $\mathbf{E5}^{\mathbf{t}_4\circ^+}[\gamma]$ *for which*

(1) *excluded middle is a theorem of* **L***;*
(2) $\Box$ *distributes over true implications in* C**L***;*
(3) *reasoning by cases is provable for* C**L***,*

fails to be conservatively extended by Boolean negation.

Proof. $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is not a theorem of $\mathbf{E5}^{\mathbf{t}_4\circ^+}[\gamma]$ by Thm. 1. However, it is derivable in C**L** as the following shows:

By Thm. 4, $A \wedge (A \supset B) \vdash B$ is a derivable rule of $C\mathbf{L}$. Since $\Box$ distributes over true implications (Thm. 3), we get that $\Box(A \wedge (A \supset B) \vdash \Box B$, and since $\Box C \wedge \Box D \vdash \Box(C \wedge D)$ is a derivable rule of $C\mathbf{L}$, we can infer that $\Box A \wedge \Box(A \supset B) \vdash \Box B$. It then easily follows that both

$$\Box A \wedge \Box(A \supset B) \vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$$
$$\sim(\Box A \wedge \Box(A \supset B)) \vdash \Box(A \supset B) \supset (\Box A \supset \Box B).$$

From this, and the fact that excluded middle is a theorem and that reasoning by cases holds, one can finally infer that $\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$. ◻

Corollary 10. *$C\mathbf{E}$ is not a conservative extension of $\mathbf{E}$, and $C\mathbf{E5}$ is not a conservative extension of $\mathbf{E5}$.*

Proof. This follows from Thm. 6 since

(1) excluded middle is a theorem of $\mathbf{E}$ and $\mathbf{E5}$;
(2) by Thm. 3, $\Box$ distributes over true implications in $C\mathbf{E}$ and in $\mathbf{E5}$;
(3) by Cor. 1, reasoning by cases is provable for $C\mathbf{E}$ and $C\mathbf{E5}$.

◻

«*Parenthetical remark.* As mentioned introductory-wise, $\mathbf{E}$ was originally got by deleting both δ and γ from Ackermann's Π'. Both these rules turn out to be admissible in $\mathbf{E}$, and so the two logics are theorem-vise identical and so $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is not a theorem of Π' either. Since γ was the only factor needed from $C\mathbf{E}$ to prove this sentence and Π' has this rule as a primitive, one would be excused for thinking that something is amiss. The explanation is, however, simply that even though $\mathbf{E}$ and Π' are theorem-vise identical, they are not identical with regards to which meta-rules they validate. It can be shown that the γ-rule does not create problems in the inductive proof of reasoning by cases as in fact $A \vee C, (A \supset B) \vee C \vdash B \vee C$ is a derivable rule of Π'. However, it does for the ($\Box_\vdash$-dist)-rule as any proof of this meta-rule would need to show that $\Box A, \Box(A \supset B) \vdash \Box B$ is a derivable rule, which is not the case. However, since $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is provable in $C\mathbf{E}$ it is also provable in the stronger logic $C\Pi'$, and so Π' also fails to be conservatively extended by Boolean negation. *End parenthetical.*»

7.1. **Further possible weakening: disjunctive rules.** There is yet another possible way to significantly weaken the assumptions needed for the proof to go through: by adding the disjunctive version of every primitive rule, reasoning by cases automatically becomes provable as the next lemma shows.

Definition 7 (d-extension)**.** *$\mathbf{L}^d$ is got from $\mathbf{L}$ by adding the disjunctive version of every primitive rule of $\mathbf{L}$, that is, if $\{A_1, \ldots, A_n\} \vdash B$ is such a primitive rule, then $\mathbf{L}^d$ has $\{A_1 \vee C, \ldots, A_n \vee C\} \vdash B \vee C$ as an additional primitive rule.*

Lemma 8. *The meta-rule of* reasoning by cases*,*

$$(RbC)\quad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C},$$

holds for any logic $\mathbf{L}$ *if and only if* $\mathbf{L} = \mathbf{L}^d$.

Proof. Left for the reader. $\square$

Note that the proof of Thm. 3— the provability of the meta-rule $\dfrac{A \vdash B}{\Box A \vdash \Box B}$ — holds for $\mathbf{BB}[eA14|eA15]$. However, it does not fully extend to $\mathbf{BB}^d[eA14|eA15]$, but, of course if there is a $\mathbf{BB}^d[eA14|eA15]$-proof of $A \vdash B$ which does not make use of the disjunctive rules, then it is also the case that $\Box A \vdash \Box B$:

Corollary 11. *Let* $\mathbf{L}$ *by any extension of* $\mathbf{BB}[eA14|eA15]$ *by rules the main connective of the conclusion being* $\to$*, or axioms the main connective of which are* $\to$*. Then it holds that if there is a* $\mathbf{L}^d$*-proof of* $A \vdash B$ *which does not make use of any of the disjunctive rules, then it is also the case that there is a* $\mathbf{L}^d$*-proof that* $\Box A \vdash \Box B$*.*

Since Thm. 4 states that $A, A \supset B \vdash B$ is a derivable rule of $C\mathbf{BBX}$ and the proof does not make use of reasoning by cases, it follows from the above corollary that $\Box(A \wedge (A \supset B) \vdash \Box B$ holds for $C\mathbf{BB}^d[eA12|eA14|eA15]$ and for $C\mathbf{BBX}^d_\Box[eA14|eA15]$. We can then use reasoning by cases to infer that $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is also a theorem of of these two logics. We therefore conclude with the following generalization of Mares' result:

Theorem 7. *All logics between* $\mathbf{BBX}^d_\Box[eA14|eA15]$ *and* $\mathbf{E5}^{\mathbf{t}_4\circ^+}[\gamma]$ *(=* $\mathbf{E5}^{d\mathbf{t}_4\circ^+}[\gamma]$*) fail to be conservatively extended by Boolean negation.*

This generalizes Mares' result quite a bit. That it also covers the **S5**-extension of **E** is interesting, but since neither of the commonly refered to relevant logics weaker than **E** validate A14 or A15, or their enthymemized versions, it seems that the generalization does not cover many new logics of interest. What is interesting, however, is that the result shows that it is not the **S4**-characteristics of **E** which begets the non-conservatively as all of the following principles fail in $\mathbf{BBX}^d_\Box[eA14|eA15]$:[16]

(**Kr**)	$\Box(A \to B) \vdash \Box A \to \Box B$
(**T**)	$\Box A \to A$
(**S4**)	$\Box A \to \Box\Box A$
($^\Box/_\wedge$-dist)	$\Box A \wedge \Box B \to \Box(A \wedge B)$

Modally the non-conservativeness proof presented in this paper only needed that the meta-rule $\dfrac{A \vdash B}{\Box A \vdash \Box B}$ be provable; the other requirements were either got from the underlying logic—reasoning by cases—or from the Boolean extension—modus ponens for $\supset$. The next section abstracts away from the particular definition of $\Box$ and shows that the proof also goes through for any relevant logic extending **BBX** provided reasoning by cases is available and the logic has a modal **K**-operator $\boxdot$. This, then, will generalize a result by Meyer and Mares who showed in [12] that **R** augmented with a **S4**-operator is not conservatively extended by Boolean negation.

[16]MaGIC will easily verify this, and so I leave it as a MaGICal exercise for the reader to do so.

8. Generalizing: non-conservativeness using a primitive modal operator

Meyer proposed in [10] to add a primitive **S4** modal operator to the logic **R**. That logic, **NR** is axiomatized by adding to **R** the following principles:

(**NEC**)	$\varnothing \vdash A \Rightarrow \varnothing \vdash \boxdot A$
(**K**)	$\boxdot(A \to B) \to (\boxdot A \to \boxdot B)$
$(^{\boxdot}/_{\wedge})$	$\boxdot A \wedge \boxdot B \to \boxdot(A \wedge B)$
(**T**)	$\boxdot A \to A$
(**4**)	$\boxdot A \to \boxdot\boxdot A$

By using the classical equivalents of (**NEC**) and (**K**) one can easily derive the classical equivalent of the $(^{\boxdot}/_{\wedge})$ axiom. This axiom, however, remains independent from the others in a relevant context. The proper definition of the relevant equivalent of the basic modal logic **K** is therefore as follows:

Definition 8. $\mathbf{K}_{\boxdot}\mathbf{L}$ *is the logic got by adding* (**NEC**)*,* (**K**) *and* $(^{\boxdot}/_{\wedge})$ *to* **L***.*

Thus $\mathbf{K}_{\boxdot}\mathbf{R}$ is a sublogic of Meyer's **NR**. Meyer and Mares showed in [12] that **NR** is not conservatively extended by Boolean negation. Their proof is quite similar to that of [8] and is also to the effect that $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is a theorem of C**NR**, but not of **NR**. Their counter-model to the extensional **K** sentence was, like Mares' counter-model in [8], a Routley-Meyer model. Unlike, Mares' model, however, the one presented in [12] does not validate the Brouwerian axiom. The purpose of this section is to show that Meyer and Mares' result can be generalized to cover any logic between $\mathbf{K}_{\boxdot}\mathbf{TW}[A12]$ and the **S5**-strengthening of Meyer's **NR**, which in this essay will be referred to as $\mathbf{5}_{\boxdot}\mathbf{R}$ and identified as the logic got by adding to **R** all of (**NEC**), (**K**), (**T**), (**4**) and $(^{\boxdot}/_{\wedge})$, as well as the **B**-axiom:

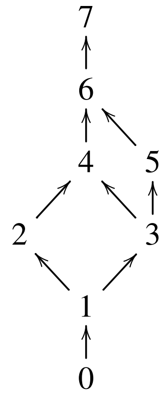
$$(\mathbf{B}) \quad A \to \boxdot\Diamond A.$$

Theorem 8. $\boxdot(A \supset B) \supset (\boxdot A \supset \boxdot B)$ *is not a theorem of* $\mathbf{5}_{\boxdot}\mathbf{R}$*.*

Proof. Counter-model displayed in Fig. 3.[17] □

[17] A few comments on the model in Fig. 3: First of all note that $\mathcal{T}$ is not a *prime* truth-filter: $2 \vee 3 \in \mathcal{T}$, although both $2 \notin \mathcal{T}$ and $3 \notin \mathcal{T}$. Furthermore, the model validates not only (**NEC**), but even the rule $A \vdash \boxdot A$ as $\mathcal{T}$ is closed under the $\boxdot$-operator. However, by slightly enlarging its truth-filter, both these shortcomings are rectified: simply enlarge the $\mathcal{T}$ to the prime filter $\mathcal{T}' = \mathcal{T} \cup \{2\}$. It is then easily checked that the model still validates both adjunction and modus ponens. Since $\boxdot 2 = 0 \notin \mathcal{T}'$, the rule $A \vdash \boxdot A$ is not validated any more. However, every axiom of $\mathbf{5}_{\boxdot}\mathbf{R}$ is assigned a value in $\mathcal{T}$ which *is* closed under the $\boxdot$-operator, and so the model still validates (**NEC**). Since $[\![\boxdot(A \supset B) \supset (\boxdot A \supset \boxdot B)]\!] = 3 \notin \mathcal{T}'$, the model with $\mathcal{T}$ replaced by $\mathcal{T}'$, then, still validates every axiom and rule of $\mathbf{5}_{\boxdot}\mathbf{R}$, while invalidates extensional **K**-sentence. Lastly, note that replacing $\mathcal{T}$ by $\mathcal{T}'$ results in an algebra which is not a model in the sense of MaGIC as $\mathcal{T}$ is, but $\mathcal{T}'$ is not an *implicational* filter, meaning that $x \to y \in \mathcal{T} \Leftrightarrow x \leq y$, where $\leq$ is the partial order of the algebra: for instance, $3 \to 2 = 2 \in \mathcal{T}'$, but $3 \nleq 2$.

$\mathcal{T} = \{4, 6, 7\}$
$[\![A]\!] = 4$
$[\![B]\!] = 2$



$\supset$	0	1	2	3	4	5	6	7	$\rightarrow$	0	1	2	3	4	5	6	7	$\sim$	$\boxdot$	$\Diamond$
0	7	7	7	7	7	7	7	7	0	7	7	7	7	7	7	7	7	7	0	0
1	6	6	6	6	6	6	6	7	1	0	6	6	6	6	6	6	7	6	0	3
2	5	5	6	5	6	5	6	7	2	0	5	6	5	6	5	6	7	5	0	4
3	4	4	4	4	4	6	6	7	3	0	2	2	4	4	6	6	7	4	3	3
4	3	3	4	3	4	5	6	7	4	0	1	2	3	4	5	6	7	3	4	4
5	2	2	2	4	4	6	6	7	5	0	2	2	2	2	6	6	7	2	3	7
6	1	1	2	3	4	5	6	7	6	0	1	2	1	2	5	6	7	1	4	7
7	0	1	2	3	4	5	6	7	7	0	0	0	0	0	0	0	7	0	7	7

FIGURE 3. $\mathbf{5}_{\boxdot}\mathbf{R}$-counter-model to $\boxdot(A \supset B) \supset (\boxdot A \supset \boxdot B)$

Lemma 9.

$$\frac{A \vdash B}{A \vee C \vdash B \vee C}$$

holds for $\mathbf{K}_{\boxdot}\mathbf{L}$ *provided it holds for* $\mathbf{L}$.

Proof. The proof is simply to confirm that the induction step for (**NEC**) also holds, so let $D_1 \ldots D_n$ be a proof of B from A and assume that $A \vee C \vdash D_i$ and D_j is got from D_i using (**NEC**). In that case, $\varnothing \vdash D_i$ and $\varnothing \vdash D_j$, and so rather trivially also $A \vee C \vdash D_j \vee C$. □

Corollary 12. *Reasoning by cases holds for* $C\mathbf{K}_{\boxdot}\mathbf{TW}[A12]$.

Proof. Immediate from Cor. 8 and Lem. 9. □

Lemma 10. $A, A \supset B \vdash B$ *is a derivable rule of* $C\mathbf{K}_{\boxdot}\mathbf{TW}[A12]$.

Proof. Same proof as Thm. 4. □

Lemma 11.

$$(\boxdot/\vdash\text{-}dist) \quad \frac{A \vdash B}{\boxdot A \vdash \boxdot B}$$

holds in $C\mathbf{K}_{\boxdot}\mathbf{TW}[A12]$.

Proof. The proof simply extends that of Lem. 7; so the task is simply to confirm that the induction step for (**NEC**) also holds. Let $D_1 \ldots D_n$ be a proof of B from A and assume that $\boxdot A \vdash D_i$ and D_j is got from D_i using

(**NEC**). In that case, $\varnothing \vdash D_i$ and $\varnothing \vdash D_j$, and so trivially also $A \vee C \vdash D_j (= \boxdot D_i)$. □

Theorem 9. *Any logic between* $\mathbf{K}_{\boxdot}\mathbf{TW}[A12]$ *and* $\mathbf{5}_{\boxdot}\mathbf{R}$ *is non-conservatively extended by Boolean negation.*

Proof. Same as that of Thm. 6. □

8.1. **Possible weakenings.** Let's swiftly consider possible ways of weakening $\mathbf{K}_{\boxdot}\mathbf{TW}[A12]$. First of all one can replace the reductio axiom $(A \to {\sim}A) \to {\sim}A$ with simply excluded middle if the disjunctive rules are added. Note, though that the meta-rule $\boxdot/_{\vdash}$-dist does not hold for logics with more primitive rules than adjunction and modus ponens as, for instance, $\boxdot((B \to C) \to (A \to C))$ is not derivable from $\boxdot(A \to B)$ unless $A \to B$ is a logical theorem. If $\boxdot$ is to be added to logics with more primitive rules one will therefore need to add the modal variant of every such rule.

Definition 9. $\mathbf{L}^{\boxdot}$ *is got from* $\mathbf{L}$ *by adding for every primitive rule* $A, \ldots, A_n \vdash B$ *of* $\mathbf{L}$*, the rule* $\boxdot A, \ldots, \boxdot A_n \vdash \boxdot B$.

Definition 10. $\mathbf{L}^{d\boxdot}$ *is got from* $\mathbf{L}$ *by inductively adding* $\boxdot A, \ldots, \boxdot A_n \vdash \boxdot B$ *and* $A \vee C, \ldots, A_n \vee C \vdash B \vee C$ *for every primitive rule* $A, \ldots, A_n \vdash B$.

Theorem 10. $C\mathbf{L}$ *is not a conservative extension of* $\mathbf{L}$ *for any logic between* $\mathbf{BBX}^{d\boxdot}$ *and* $\mathbf{5}_{\boxdot}\mathbf{R}$.

Proof. It is trivial to verify that $\mathbf{BBX}^{d\boxdot}$ validates the meta-rules of reasoning by cases and $\boxdot/_{\vdash}$-distribution and has $A, A \supset B \vdash B$ as a derivable rule. The result therefore follows by the same proof as that given in Thm. 6 together with the fact that $\mathbf{BBX}^{d\boxdot}$ is a sublogic of $\mathbf{5}_{\boxdot}\mathbf{R}$. □

This last theorem, then, makes clear that it is only three essential ingredients needed to prove Meyer and Mares' result, namely

(1) the fact that modus ponens for $\supset$ becomes a derivable rule in the Boolean extension provided excluded middle is a logical theorem
(2) that reasoning by cases is an available meta-rule,
(3) that the modal operator is such as to distribute over every primitive rule of the logic so as to make the meta-rule $\dfrac{A \vdash B}{\boxdot A \vdash \boxdot B}$ hold true of the logic in question.

9. A variation

Restall ended his paper [17] by raising the question whether Routley's two favorite logics $\mathbf{DK}^d$ and $\mathbf{DL}^d$ are conservatively extended by Boolean negation. These logics are got by adding to $\mathbf{DW}^d$ the axiom $(A \to B) \wedge (B \to C) \to (A \to C)$ called *conjunctive syllogism*, as well as, respectively, excluded middle and reductio. Thus $\mathbf{DK}^d$ is a sublogic of $\mathbf{DL}^d$. Restall's question has, to my knowledge, remained an open one ever since.[18] It turns

[18]Restall ended his paper by promising an answer in [18], but no such answer is to be found therein.

out that the sentence $\Box(A \supset B) \supset (\Box A \supset B)$, where as before $\Box A =_{df} (A \to A) \to A$, is a theorem of $C\mathbf{DK}^d$, but not of $\mathbf{DK}^d$, nor of $\mathbf{TK}^d$ ($\mathbf{TW}^d$ extended by conjunctive syllogism and excluded middle).[19] The purpose of this section is to give a quick an easy proof of this. I have not been able to decide whether $\mathbf{DL}^d$ is conservatively extended by Boolean negation, and so this question remains an open one.

Lemma 12. *$\Box(A \supset B) \supset (\Box A \supset B)$ is a theorem of $C\mathbf{DK}^d$.*

Proof. Since both $A, A \supset B \vdash B$ (Thm. 4) and $\Box C \vdash C$ are derivable rules of $C\mathbf{DK}^d$, it follows that so is $\Box A, \Box(A \supset B) \vdash B$. Thus $\Box(A \supset B) \supset (\Box A \supset B)$ is derivable from both $\Box A \wedge \Box(A \supset B)$ and from $\sim(\Box A \wedge \Box(A \supset B))$. Reasoning by cases, then, one can infer that $\Box(A \supset B) \supset (\Box A \supset B)$ is a logical theorem since $(\Box A \wedge \Box(A \supset B)) \vee \sim(\Box A \wedge \Box(A \supset B))$ is a logical theorem of $C\mathbf{DK}^d$. □

Theorem 11. *$C\mathbf{DK}^d$ and $C\mathbf{TK}^d$ are not conservative extensions of, respectively, $\mathbf{DK}^d$ and $\mathbf{TK}^d$.*

Proof. This follows from the above lemma together with the fact that $\Box(A \supset B) \supset (\Box A \supset B)$ is not a theorem of $\mathbf{TK}^d$ as the $\mathbf{TK}^d$-model displayed in Fig. 4 falsifies it. □

$\mathcal{T} = \{1, 2, 3, 4\}$
$[\![A]\!] = 2$
$[\![B]\!] = 0$

$$0 \to 1 \to 2 \to 3 \to 4$$

$\to$	0	1	2	3	4	$\sim$	$\Box$
0	4	4	4	4	4	4	0
1	0	1	4	4	4	3	1
2	0	0	1	4	4	2	4
3	0	0	0	1	4	1	4
4	0	0	0	0	4	0	4

FIGURE 4. $\mathbf{TK}^d$-counter-model to $\Box(A \supset B) \supset (\Box A \supset B)$

Open Problem. *Is $C\mathbf{DL}^d$ a conservative extension of $\mathbf{DL}^d$?*

10. SUMMARY

I have in this essay shown that modal relevant logics often fail to be conservatively extended by Boolean negation. The first such proof was given by Mares in [8] who showed that the relevant logic **E** is not conservatively extended by Boolean negation. This paper provided a new and significantly easier proof of Mares' result. Mares' proof, as well as the one presented

[19] $\mathbf{TK}^d$ is an interesting logic as it is, as of yet, the strongest paraconsistent relevant logic for which naïve set theory—with both the generalized abstraction schema as well as extensionality in axiomatic form—is non-trivial (cf. [5, sect. 6.3]).

here, is to the effect that the extensional **K**-sentence $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is a theorem of C**E**—the Boolean extension of **E**—but not of **E** itself. In addition to simplifying the proof considerably, it was also shown that the nonconservativeness result holds for the **S5**-extension of **E** as well as to various ways of weakening **E**. This was then further generalized by showing that the proof also pertains to logics in which the modal operator is taken as primitive. In such a context it was shown that any sublogic of **R** augmented with a primitive **S5** modality, but for which excluded middle and reasoning by cases holds, and for which the modal operator distributes over true implications so that $\Box A \vdash \Box B$ holds if $A \vdash B$ does, will fail to be conservatively extended by Boolean negation. The proof, then, not only covers Mares' result, but also that of Meyer and Mares who gave a model-theoretic proof in [12] to the effect that **NR**—**R** augmented with a **S4** modality—fails to be conservatively extended by Boolean negation.

That the extensional **K**-sentence is not a theorem of either **E** or **NR** was first noted in [19] which then mentions a suggestion by Belnap of adding the axiom $\Box(A \vee B) \to (\Box A \vee \Diamond B)$ to **NR**. The Boolean extension of this logic—known as **R4**—conservatively extends both **R**, C**R** and **S4**, although it is not known whether it conservatively extends **R4** itself ([12]). I end this paper, therefore, noting that in light of the fact shown in this paper that the vast logical space spanning everything from the weak **K**-ish logic $\mathbf{BBX}^{d\boxdot}$ of Thm. 10 to the strong **S5** extension of **NR** is marred by non-conservativity when Boolean negation is added, it seems that the question whether logics with Belnap's axiom fare better ought to be more fully investigated.

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