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On the Solution of the Solitary Wave Problem

by

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Abstract

The solitary wave problem has been considered and a new approach is proposed which may give some further insight concerning the series solution of this problem, especially whether it is a convergent or an asymptotic series. The velocity potential is expressed as a surface integral, and it is shown how the series solution emerges from this integral. A procedure is given to calculate to any order the coefficients in the series of the surface elevation and the velocity potential, and it is shown *analytically* that these series must necessarily be asymptotic. Some numerical results are presented in order to compare with previously known results; the series of the surface elevation and the velocity potential are obtained to ninth order, and since the computer algebra system Maple has been used it is possible to give the exact values of the coefficients in these series. The series solutions to a given order of the surface elevation and the velocity potential are introduced into the surface integral to obtain another approximate value of the velocity potential which becomes different from the one given by the series solution; these to values being compared to estimate how good the series solution to a given order is.

1 Introduction

The steady finite solitary wave was first reported by Russel (1844), and since the first approximate solutions given by Boussinesq (1871) and Rayleigh (1876) there have been several attempts to improve upon the solution of this problem. The first shallow-water theory of periodic waves was given by Korteveg & de Vries (1895) who showed that the first approximation to the surface profile of steadily progressing waves in shallow water was cnoidal, and that the solitary wave was a particular limiting case, that of infinite wave-length. So, the shallow-water wave problem has a long history with a number of participants and with a comprehensive list of references, see for instance the review articles by Miles (1980) and Schwartz and Fenton (1982). In the papers of Fenton (1972, 1979) high order solitary- and cnoidal wave solutions have been presented well suited for practical use. The solutions have been obtained to ninth order; however, by comparing with "exact" numerical results, Cokelet (1977), and with experimental investigations, Le Méhauté, Divoky & Lin (1968) and Iwagaki & Sakai (1970) , it is found that there is no gain in accuracy to be had by including terms after the fifth, suggesting that the series involved are asymptotic rather than convergent.

In this paper we present a new approach to obtain the series solution of the solitary wave problem, and which enable us to show *analytically* that the series must necessarily be asymptotic. The velocity potential is given as a surface integral, Engevik (1986, 1991). In principal this integral can be evaluated by using the residue theorem, and it is found that only two of the poles of the integrand contribute to the series solution of the solitary wave problem. It is shown how the series of the surface elevation and the velocity potential can be obtained to any order, and we carry out the calculations to ninth order to compare our solution of the surface profile with that of Fenton (1972). Since we have used the computer algebra system Maple when doing our numerical calculations, the coefficients in the series can be given their exact values, which have not been given before. Moreover, our analysis reveals *analytically* that the series must be asymptotic rather than convergent. Another approximate value of the velocity potential is obtained from the surface integral by introducing into it the series to a given order of the surface elevation and the velocity potential; this value being different from the one given by the series solution. The difference between these two values of the velocity potential can not be expected to tend to zero when the number of terms in the series tends to infinity since, as will be shown, the series do not converge. On the contrary we would expect this difference to reveal the asymptotic nature of the series, i.e. for a given wave amplitude we would expect that there is a limited number of terms which must be included in the series to obtain a minimum value of this difference, with no improvement by further increasing the number of terms in the series, rather the opposite. Some numerical results are presented which seem to support this behaviour.

2 Formulation and solution

We consider waves on the free surface of a homogeneous, incompressible and irrotational fluid of infinite horizontal extent and of finite and uniform depth. The wave motion is assumed to be two-dimensional and to take place in the $(x^\star, z^\star)$-plane, with the $x^\star$-axis in the horizontal direction and the $z^\star$-axis directed vertically upwards. Furthermore, the wave motion is assumed to be stationary in this frame of reference, i.e. the velocity potential $\Phi = \Phi(x^\star, z^\star)$ and the surface elevation $z^\star = \eta^\star(x^\star)$ are independent of time. The equations governing this problem are,

$$\nabla^{\star 2}\Phi = 0\,, \qquad -d^\star < z^\star < \eta^\star(x^\star) \tag{1}$$

$$\left.\begin{aligned} \tfrac{1}{2}(\nabla^\star\Phi)^2 + g\eta^\star &= C \\ \underline{n}\cdot\nabla^\star\Phi &= 0 \end{aligned}\right\} \qquad \text{at } z^\star = \eta^\star(x^\star) \tag{2}$$

$$\frac{\partial\Phi}{\partial z^\star} = 0 \qquad \text{at } z^\star = -d^\star\,, \tag{3}$$

where $\nabla^\star \equiv \underline{i}\frac{\partial}{\partial x^\star} + \underline{k}\frac{\partial}{\partial z^\star}$, $\underline{n} = (\underline{k} - \eta^{\star\prime}\underline{i})/\sqrt{1+\eta^{\star\prime 2}}$, where $\underline{i}$ and $\underline{k}$ are the unit vectors in the $x^\star$- and the $z^\star$-direction respectively, and the prime denotes differentiation with respect to $x^\star$, $\nabla^{\star 2} \equiv \frac{\partial^2}{\partial x^{\star 2}} + \frac{\partial^2}{\partial z^{\star 2}}$, and C is a constant.
(2) are the dynamic and the kinematic boundary conditions at the free surface, and (3) is the boundary condition at the bottom.
We introduce the dimensionless quantities $x = x^\star/d^\star$, $z = z^\star/d^\star$, $\zeta = \eta^\star/d^\star$, $\phi = \Phi/cd^\star$ where $c = \sqrt{gd^\star}$ into the equations (1) – (3) and get,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0\,, \qquad -1 < z < \zeta(x) \tag{4}$$

$$\left.\begin{aligned} &(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial z})^2 + 2\zeta = R \\ &\frac{\partial\phi}{\partial z} - \zeta'(x)\frac{\partial\phi}{\partial x} = 0 \end{aligned}\right\} \quad \text{at } z = \zeta(x) \tag{5}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = -1\,, \tag{6}$$

where R is a constant, and here the prime denotes differentiation with respect to x.

In this paper we will study the solitary wave problem, i.e. the surface wave problem in the shallow-water limit, when the characteristic length $L^\star$ in the $x^\star$-direction is much greater than the depth $d^\star$ of the water; $d^\star/L^\star$ being of order ϵ. The solitary wave problem has a formal series solution of the form,

$$\zeta(x) = \sum_{j=1}^{\infty} \epsilon^{2j} A_j sech^{2j}(\epsilon x)\,. \tag{7}$$

The velocity potential at the free surface, $\phi_s(x) = \phi(x, \zeta(x))$, can be expressed as,

$$\phi_s(x) = \sqrt{R}\left(x + \tanh(\epsilon x)\sum_{j=1}^{\infty}\epsilon^{2j-1}B_j sech^{2(j-1)}(\epsilon x)\right)\,. \tag{8}$$

The connection between the A_j's and the B_j's is given by the equation,

$$(\frac{d\phi_s}{dx})^2 = (R - 2\zeta)(1 + \zeta'^2)\,, \tag{9}$$

which is obtained from the boundary conditions at the free surface. The solution of (4) can be expressed as a surface integral, Engevik (1986, 1991), i.e.,

$$\phi(x, z) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{S_r}\phi_s\frac{\partial G}{\partial n}ds\,, \tag{10}$$

where S_r is the surface lying between $x = -r$ and $x = r$, $\frac{\partial}{\partial n}$ denotes the derivative normal to the surface, and $G(\mu, \tau; x, z)$ is the Green function as given by Engevik (1991), i.e.

$$G(\mu, \tau; x, z) = \frac{1}{2}\ln\left\{[(\mu - x)^2 + (\tau - z)^2][(\mu - x)^2 + (\tau + z + 2)^2]\right\}. \tag{11}$$

(11) introduced into (10) yields,

$$\left.\begin{aligned}\phi(x,z) &= \frac{1}{2\pi}\lim_{r\to\infty}\left[\int_{-r}^{r}\phi_s(\mu)\frac{\zeta(\mu)-z-\zeta'(\mu)(\mu-x)}{(\mu-x)^2+(\zeta(\mu)-z)^2}d\mu\right.\\ &\quad\left.+\int_{-r}^{r}\phi_s(\mu)\frac{\zeta(\mu)+z+2-\zeta'(\mu)(\mu-x)}{(\mu-x)^2+(\zeta(\mu)+z+2)^2}d\mu\right].\end{aligned}\right\} \tag{12}$$

If $\zeta(x)$ and $\phi_s(x)$ are known, then $\phi(x,z)$ can be calculated, and if the series solutions of $\zeta(x)$ and $\phi_s(x)$ converge, then we would expect $\phi(x,z)$ to converge to its exact value when the number of terms in the series is increased, which means that the value of $\phi_s(x)$ calculated from the integral expression in (12) should approach the value obtained from the series solution when the number of terms in the series is increased.
The integrals in (12) can be considered as integrals along the real axis in the complex σ-plane, where $\sigma = \mu + i\nu$; the μ-axis being the real axis and the ν-axis the imaginary axis, and can in principal be evaluated by using the residue theorem, assuming that the integrands can be continued analytically into the complex σ-plane. The integrands have poles where the following functions have zeros,

$$\left.\begin{aligned}F_1(\sigma) &= \zeta(\sigma) - z + i(\sigma - x)\\ F_2(\sigma) &= \zeta(\sigma) - z - i(\sigma - x)\\ F_3(\sigma) &= \zeta(\sigma) + z + 2 + i(\sigma - x)\\ F_4(\sigma) &= \zeta(\sigma) + z + 2 - i(\sigma - x)\,.\end{aligned}\right\} \tag{13}$$

Let the zeros of $F_n(\sigma)$, $n = 1,2,3,4$ in the upper half of the σ-plane be denoted by σ_{nj}, $n = 1,2,3,4$, and assuming the zeros to be simple we get,

$$\phi(x,z) = \frac{1}{2}\sum_{n=1}^{4}(-1)^{n+1}\sum_j \phi_s(\sigma_{nj}) + \frac{1}{2\pi}\lim_{r\to\infty}\int_\Gamma(\cdots)d\sigma\,, \tag{14}$$

where Γ is the semicircle with radius r lying in the upper half plane. (In obtaining (14) we have assumed that the only singularities of the integrands in the upper half plane are poles, which is the case if the series of $\zeta(x)$ and $\phi_s(x)$ to a given order are used in the integrands). It is easily verified that

$\phi_s(\sigma_{nj})$ as a function of x and z satisfies the Laplace equation.
We want to obtain the series solution of the solitary wave problem, and in this connection one of the zeros of $F_1(\sigma)$, which is denoted by σ_{10}, and one of the zeros of $F_3(\sigma)$, denoted by σ_{30}, turn out to be of special interest. The zero σ_{10} is defined as follows: Let (x, z) be a point in the fluid just below the free surface. We put $z = \zeta(x) - \epsilon_1$, where $0 < \epsilon_1 \ll 1$, into the equation $F_1(\sigma) = 0$, and find that,

$$\left.\begin{array}{l} \sigma_{10} = x + \alpha_1 + i\beta_1, \\ \\ \text{where} \\ \\ \alpha_1 = -\epsilon_1\zeta'(x)/\{1 + (\zeta'(x))^2\} \ + O(\epsilon_1^2) \\ \\ \beta_1 = \epsilon_1/\{1 + (\zeta'(x))^2\} \ + O(\epsilon_1^2)\,, \end{array}\right\} \qquad (15)$$

which shows that σ_{10} lies in the upper half of the σ-plane near the point $(x, 0)$. Furthermore, $\sigma_{10} \rightarrow x$ when $\epsilon_1 \rightarrow 0$, i.e. when the free surface is approached from below. $F_2(\sigma)$ has a zero, $\hat{\sigma_{20}}$, near the point $(x, 0)$ as well, but this zero lies in the lower half plane and is therefore not among the zeros σ_{2j}. σ_{30} is defined to be that zero of $F_3(\sigma)$ which is equal to σ_{10} at the bottom where $z = -1$, i.e. $\sigma_{30}(x, -1) = \sigma_{10}(x, -1)$.
We may also write $\phi(x, z)$ as,

$$\phi(x, z) = \frac{1}{2}[\phi_s(\sigma_{10}) \ + \ \phi_s(\sigma_{30})] \ + \ \frac{1}{2\pi}\int_{\Gamma_1}(\cdots)d\sigma\,, \qquad (16)$$

where Γ_1 is a contour in the upper half plane which together with the real axis constitutes a closed contour, and where we have assumed that the integrands are analytic within and on this closed contour except for the poles at σ_{10} and σ_{30}. (If all the the singularities of the integrands in the upper half plane are simple poles, (16) can be evaluated to obtain (14), but (16) also allows the integrands to have singularties of other kinds than poles, lying outside the closed contour). We notice that the expression $\frac{1}{2}[\phi_s(\sigma_{10}) \ + \ \phi_s(\sigma_{30})]$, as a function of x and z, satisfies the Laplace equation *and* the boundary condition at the bottom.

3 The series solution of the solitary wave problem

The function

$$\phi(x,z) = \frac{1}{2} Re\{\phi_s(\sigma_{10}) + \phi_s(\sigma_{30})\}\,, \tag{17}$$

where $Re\{\cdots\}$ denotes the real part of the expression within the brackets, satisfies the Laplace equation and the boundary condition at the bottom, which follows from the remark at the end of §**2**. If it is to be a solution of our problem it has to satisfy the boundary conditions at the free surface as well. At the free surface $z = \zeta(x)$, $\sigma_{10} = x$ and $\sigma_{30} = \sigma_{30}(x, \zeta(x)) = \sigma_{30}^s$, which introduce into (17) yields,

$$Re\{\phi_s(x) - \phi_s(\sigma_{30}^s)\} = 0\,, \tag{18}$$

where σ_{30}^s is given by the equation,

$$\zeta(\sigma) + \zeta(x) + 2 + i(\sigma - x) = 0\,. \tag{19}$$

From (7) and (8) it follows that, in the outskirt $x > 0$, $\zeta(x)$ and $\phi_s(x)$ can be written as,

$$\left.\begin{aligned} \zeta(x) &= \epsilon^2 \sum_{j=1}^{\infty} a_j \exp(-2j\epsilon x) \\ \phi_s(x) &= \sqrt{R}\left(x + \epsilon \sum_{j=1}^{\infty} b_j \exp(-2j\epsilon x)\right)\,,\ x > 0\,. \end{aligned}\right\} \tag{20}$$

It is easily found that a_k can be expressed by A_j, $j = 1, 2, \cdots, k$ as,

$$a_k = \sum_{j=1}^{k} \epsilon^{2(j-1)} 2^{2j} \begin{pmatrix} -2j \\ k-j \end{pmatrix} A_j\,, \tag{21}$$

where $\begin{pmatrix} -2j \\ k-j \end{pmatrix} = \dfrac{(-1)^{k-j}(j+k-1)!}{(2j-1)!(k-j)!}$ is the binomial coefficient.
The relations between the coefficients a_j and b_j, $j = 1, 2, \cdots$ are obtained

from (9),

$$\left.\begin{aligned}
b_1 &= \frac{a_1}{2R} \\
b_2 &= \frac{a_2}{4R} + \frac{\epsilon^2 {b_1}^2}{2} - \frac{\epsilon^4 a_1^2}{2} \\
b_{n+2} &= \frac{1}{n+2}\left[\frac{a_{n+2}}{2R} + \epsilon^2 \sum_{j=1}^{n+1} j(n+2-j) b_j b_{n+2-j} - \epsilon^4 \sum_{j=1}^{n+1} j(n+2-j) a_j a_{n+2-j}\right. \\
&\qquad \left. + \frac{2\epsilon^6}{R} \sum_{k=1}^{n} \sum_{j=1}^{k} j(k+1-j) a_j a_{k+1-j} a_{n+1-k}\right], \quad n = 1, 2, \cdots .
\end{aligned}\right\} \quad (22)$$

The equations (18) and (19) are to be satisfied. We introduce the expressions for $\zeta(x)$ and $\phi_s(x)$ given by (20) into these equations, and in addition introduce the new variables, X_0, X and Y, defined by,

$$X_0 = \exp(-2\epsilon x), \text{ and } X + iY = \exp(-2\epsilon \sigma_{30}^s), \quad (23)$$

which yields,

$$\left.\begin{aligned}
x &= -\frac{1}{2\epsilon} \ln X_0 \\
\sigma_{30}^s &= -\frac{1}{4\epsilon} \ln(X^2 + Y^2) - \frac{i\theta}{2\epsilon}, \text{ where } \tan\theta = \frac{Y}{X}.
\end{aligned}\right\} \quad (24)$$

Then (18) becomes after having used (19),

$$-\epsilon \sum_{n=1}^{\infty} a_n Im\{(X+iY)^n\} + \sum_{n=1}^{\infty} b_n Re\{(X+iY)^n\} - \sum_{n=1}^{\infty} b_n X_0^n = 0, \quad (25)$$

where $Im\{\cdots\}$ denotes the imaginary part of the expression within the brackets.

The real- and imaginary part of (19) become,

$$\epsilon^2 \sum_{n=1}^{\infty} a_n Re\{(X+iY)^n\} + \epsilon^2 \sum_{n=1}^{\infty} a_n X_0^n + 2 + \frac{\theta}{2\epsilon} = 0, \quad (26)$$

and

$$\epsilon^2 \sum_{n=1}^{\infty} a_n Im\{(X+iY)^n\} - \frac{1}{4\epsilon} \ln\left(\frac{X^2+Y^2}{X_0^2}\right) = 0\,, \tag{27}$$

respectively.
(26) indicates that θ should be written as,

$$\theta = -4\epsilon + \epsilon^3 \sum_{n=1}^{\infty} \theta_n X_0^n\,, \tag{28}$$

which introduced into (26) yields,

$$\sum_{n=1}^{\infty} a_n Re\{(X+iY)^n\} + \sum_{n=1}^{\infty} a_n X_0^n + \frac{1}{2} \sum_{n=1}^{\infty} \theta_n X_0^n = 0\,. \tag{29}$$

Moreover, it seems reasonable to express X and Y as,

$$\left.\begin{aligned} X &= X_0 \cos 4\epsilon \left[1 + \epsilon^2 \sum_{n=1}^{\infty} \alpha_n X_0^n\right] \\ Y &= -X_0 \sin 4\epsilon \left[1 + \epsilon^2 \sum_{n=1}^{\infty} \beta_n X_0^n\right] \end{aligned}\right\} \tag{30}$$

since then $\tan\theta = \dfrac{Y}{X}$ becomes,

$$-\tan 4\epsilon \left[\frac{1 + \epsilon^2 \sum_{n=1}^{\infty} \beta_n X_0^n}{1 + \epsilon^2 \sum_{n=1}^{\infty} \alpha_n X_0^n}\right] = \tan\left(-4\epsilon + \epsilon^3 \sum_{n=1}^{\infty} \theta_n X_0^n\right), \tag{31}$$

which is obviously correct in the limit $\epsilon \to 0$.
If we introduce the expressions for X and Y into the ln-term in (27), we obtain,

$$\left.\begin{aligned} &4\epsilon^3 \sum_{n=1}^{\infty} a_n Im\{(X+iY)^n\} - \ln\left[1 + 2\epsilon^2 \sum_{n=1}^{\infty} \left(\cos^2(4\epsilon)\alpha_n + \sin^2(4\epsilon)\beta_n\right) X_0^n \right. \\ &\qquad \left. + \epsilon^4 \sum_{n=1}^{\infty}\sum_{j=1}^{n} \left(\cos^2(4\epsilon)\alpha_j\alpha_{n+1-j} + \sin^2(4\epsilon)\beta_j\beta_{n+1-j}\right) X_0^{n+1}\right] = 0\,. \end{aligned}\right\} \tag{32}$$

Furthermore, if we introduce the expressions for X and Y into (25) and (29) we see that they can be written respectively as,

$$\sum_{n=0}^{\infty} E_n X_0^{n+1} = 0\,, \tag{33}$$

where
$E_0 = E_0(a_1, b_1, \epsilon)$ is a function of a_1, b_1, and ϵ, and
$E_n = E_n(a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+1}, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \epsilon)$, $n = 1, 2, \cdots$ is a function of $a_1, \cdots, a_{n+1}$, $b_1, \cdots, b_{n+1}$, $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ and ϵ,
and

$$\sum_{n=1}^{\infty} F_n X_0^{n} = 0\,, \tag{34}$$

where
$F_1 = F_1(a_1, \theta_1, \epsilon)$ is a function of a_1, θ_1 and ϵ, and
$F_n = F_n(a_1, \cdots, a_n, \theta_n, \alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-2}, \epsilon)$, $n = 2, 3, \cdots$ is a function of $a_1, \cdots, a_n$, θ_n, $\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-2}$ and ϵ.
If we make the appropriate expansions of the terms in (31) this equation can be written as,

$$\sum_{n=1}^{\infty} G_n X_0^{n} = 0\,, \tag{35}$$

where
$G_n = G_n(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \theta_1, \cdots, \theta_n, \epsilon)$, $n = 1, 2 \cdots$ is a function of $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$, $\theta_1, \cdots, \theta_n$ and ϵ.
Likewise, in (32) the expansion of the ln-term is carried out and then this equation can be written as,

$$\sum_{n=1}^{\infty} H_n X_0^{n} = 0\,, \tag{36}$$

where
$H_n = H_n(a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \epsilon)$, $n = 1, 2, \cdots$ is a function of $a_1, \cdots, a_n$, $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ and ϵ.
In order for (33)$-$(36) to be satisfied, then,

$$\left.\begin{array}{l} E_n = 0\,,\ n = 0, 1, \cdots \\[2ex] F_n = 0\,,\ G_n = 0\,,\ H_n = 0\,,\ n = 1, 2, \cdots\,, \end{array}\right\} \tag{37}$$

which can be solved successively in the following way:
First we solve the equation,

$$E_0(a_1, b_1, \epsilon) = 0\,, \tag{38}$$

into which we introduce b_1 given by (22) and then we find that this equation is satisfied if,

$$R = \frac{\tan 2\epsilon}{2\epsilon}\,, \tag{39}$$

which is equal to Stokes' result (see Lamb, p. 425).
Secondly we solve the equations,

$$\left.\begin{aligned} &E_1(a_1, a_2, b_1, b_2, \alpha_1, \beta_1, \epsilon) = 0 \\ &F_1(a_1, \theta_1, \epsilon) = 0 \\ &G_1(\alpha_1, \beta_1, \theta_1, \epsilon) = 0 \\ &H_1(a_1, \alpha_1, \beta_1, \epsilon) = 0\,. \end{aligned}\right\} \tag{40}$$

The last three of these equations can be solved to give θ_1, α_1 and β_1 in terms of a_1 and ϵ. These expressions for α_1 and β_1 together with b_1 and b_2 given by (22) are introduced into the first of the equations in (40), which then yields a_2 in terms of a_1 and ϵ.
After n steps we end up with the following set of equations to be solved,

$$\left.\begin{aligned} &E_n(a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+1}, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \epsilon) = 0 \\ &F_n(a_1, \cdots, a_n, \theta_n, \alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-2}, \epsilon) = 0 \\ &G_n(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \theta_1, \cdots, \theta_n, \epsilon) = 0 \\ &H_n(a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \epsilon) = 0\,, \end{aligned}\right\} \tag{41}$$

from which a_{n+1} is obtained in terms of a_1 and ϵ, after having used the expressions for b_j, $j = 1, \cdots, n+1$ given by (22).
We can use this to calculate the coefficients of the series in (7). We know that

the b_j's can be expressed by the a_j's, see (22), and moreover, the a_j's can be expressed by the A_j's, see (21). Therefore (40) and (41) can be written as,

$$\left.\begin{aligned}
&E_1(A_1, A_2, \alpha_1, \beta_1, \epsilon) = 0 \\
&F_1(A_1, \theta_1, \epsilon) = 0, \\
&G_1(\alpha_1, \beta_1, \theta_1, \epsilon) = 0 \\
&H_1(A_1, \alpha_1, \beta_1, \epsilon) = 0\,,
\end{aligned}\right\} \tag{42}$$

and

$$\left.\begin{aligned}
&E_n(A_1, \cdots, A_{n+1}, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \epsilon) = 0 \\
&F_n(A_1, \cdots, A_n, \theta_n, \alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-2}, \epsilon) = 0 \\
&G_n(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \theta_1, \cdots, \theta_n, \epsilon) = 0 \\
&H_n(A_1, \cdots, A_n, \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \epsilon) = 0\,.
\end{aligned}\right\} \tag{43}$$

Into these equations we introduce,

$$\left.\begin{aligned}
&A_n = \sum_{j=1}^{\infty} A_{nj}\epsilon^{2(j-1)}, \quad \theta_n = \sum_{j=1}^{\infty} \theta_{nj}\epsilon^{2(j-1)} \\
&\alpha_n = \sum_{j=1}^{\infty} \alpha_{nj}\epsilon^{2j}, \quad \beta_n = \sum_{j=1}^{\infty} \beta_{nj}\epsilon^{2(j-1)}\,,
\end{aligned}\right\} \tag{44}$$

and make a series expansion in powers of ϵ to obtain,
$E_n = \sum_{j=1}^{\infty} e_{nj}\epsilon^{2j+2}$, $F_n = \sum_{j=1}^{\infty} f_{nj}\epsilon^{2j-2}$, $G_n = \sum_{j=1}^{\infty} g_{nj}\epsilon^{2j+1}$ and
$H_n = \sum_{j=1}^{\infty} h_{nj}\epsilon^{2j+2}$, $n = 1, 2, \cdots$.
In order for the equations in (42) and (43) to be satisfied, then,

$$e_{1j} = 0\,, \quad f_{1j} = 0\,, \quad g_{1j} = 0\,, \quad h_{1j} = 0\,, \quad j = 1, 2, \cdots \tag{45}$$

$$e_{nj} = 0\,, \quad f_{nj} = 0\,, \quad g_{nj} = 0\,, \quad h_{nj} = 0\,, \quad n = 2, 3, \cdots, \quad j = 1, 2, \cdots\,. \tag{46}$$

The first step is to solve,

$$\left.\begin{array}{l} f_{11} = 0\,, \quad g_{11} = 0\,, \quad h_{11} = 0 \\ \\ e_{11} = 0\,, \end{array}\right\} \tag{47}$$

which gives A_{11}.
The next step is to solve,

$$\left.\begin{array}{lll} f_{12} = 0\,, & g_{12} = 0\,, & h_{12} = 0 \\ \\ f_{21} = 0\,, & g_{21} = 0\,, & h_{21} = 0 \\ \\ f_{22} = 0\,, & g_{22} = 0\,, & h_{22} = 0 \\ \\ e_{12} = 0\,, & e_{21} = 0\,, & \end{array}\right\} \tag{48}$$

which yields A_{12} and A_{21}.
After n steps the following set of equations has to be solved,

$$\left.\begin{array}{lllll} f_{1n} = 0\,, & g_{1n} = 0\,, & h_{1n} = 0 & & \\ \\ f_{2n} = 0\,, & g_{2n} = 0\,, & h_{2n} = 0 & & \\ \vdots & & & & \\ f_{jn} = 0\,, & g_{jn} = 0\,, & h_{jn} = 0 & & \\ \vdots & & & & \\ f_{(n-1)n} = 0\,, & g_{(n-1)n} = 0\,, & h_{(n-1)n} = 0 & & \\ \\ f_{n1} = 0\,, & g_{n1} = 0\,, & h_{n1} = 0 & & \\ \\ f_{n2} = 0\,, & g_{n2} = 0\,, & h_{n2} = 0 & & \\ \vdots & & & & \\ f_{nn} = 0\,, & g_{nn} = 0\,, & h_{nn} = 0 & & \\ \\ e_{1n} = 0\,, & e_{2(n-1)} = 0\,, & \cdots,\ e_{j(n+1-j)} = 0\,, & \cdots,\ e_{n1} = 0\,, & \end{array}\right\} \tag{49}$$

and then A_{1n}, $A_{2(n-1)}$, $\cdots$, $A_{(n-1)2}$ and A_{n1} are found.
In the next section we present the results of our calculations carried out to

ninth order, i.e. $n = 9$. Using Maple the exact values of the coefficients in the series of the surface elevation and the velocity potential are found which have not been given before.

However, the main question here is whether the series given by (7) converges for some values of $\epsilon < \epsilon_0$, where $\epsilon_0 > 0$, or whether it is an asymptotic series. Our procedure to obtain the A_{ij}'s enable us to give an answer to this question, and it is shown that the series expansion in (7) must necessarily be asymptotic. The reason is as follows:
The coefficient of X_0^n in (25), which has previously been denoted by E_{n-1}, is found to be,

$$\left.\begin{aligned} E_{n-1} = & - \epsilon a_n Im\{\exp(-4in\epsilon)\} + b_n\left[Re\{\exp(-4in\epsilon)\} - 1\right] \\ & + I(a_1, \cdots, a_{n-1}, b_1, \cdots, b_{n-1}, \alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-1}, \epsilon)\,, \end{aligned}\right\} \tag{50}$$

where I is a function of a_j, b_j, α_j, β_j, $j = 1, 2, \cdots, n-1$, and ϵ.
E_{n-1} must be put equal to zero as remarked previously, and introducing b_n and R, given by (22) and (39) respectively, into E_{n-1} we find that,

$$a_n = -\frac{\sin(2\epsilon)}{4\epsilon^2 \sin(2n\epsilon) K(\epsilon, n)} I_1(a_1, \cdots, a_{n-1}, b_1, \cdots, b_{n-1}, \alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-1}, \epsilon)\,, \tag{51}$$

where
$K(\epsilon, n) = \dfrac{\sin(2\epsilon)}{2\epsilon}\cos(2n\epsilon) - \dfrac{\sin(2n\epsilon)}{2n\epsilon}\cos(2\epsilon)\,,$
and I_1 is a function of a_j, b_j, α_j, β_j, $j = 1, 2, \cdots, n-1$, and ϵ. Moreover, the a_j's, the b_j's, the α_j's and the β_j's can be expressed in terms of a_1 and ϵ, as shown previously, so I_1 can be given in terms of a_1 and ϵ.
It is found that I_1 is of order ϵ^4 when ϵ tends to zero, so a_n has no singularity at $\epsilon = 0$, see Appendix. However, a_n has singularities where $\sin(2n\epsilon)$ and $K(\epsilon, n)$ have zeros, with the exception for $\epsilon = 0$, unless I_1 is proportional to both $\sin(2n\epsilon)$ and $K(\epsilon, n)$, which is not likely to be the case as far as we can see, see Appendix. The first zero of $\sin(2n\epsilon)$, after the zero at $\epsilon = 0$, is given by $\dfrac{\pi}{2n}$. Also, when $n \gg 1$ the zeros of $K(\epsilon, n)$ are given approximately by the equation,

$$\tan\gamma = \gamma\,, \quad \text{where } \gamma = 2n\epsilon\,, \tag{52}$$

the first zero of which, after the zero at $\gamma=0$, is $\gamma_1 = 4.49340\cdots$, i.e. the first zero of $K(\epsilon, n)$, after $\epsilon=0$, is therefore $\frac{\gamma_1}{2n}$ in the limit when $n \rightarrow \infty$. Now, if we make a series expansion of a_n given by (51) in powers of ϵ, which we have to do in order to find the A_j's of the series in (7), there are the two possibilities: either the series expansion of I_1 in power of ϵ is an asymptotic series, and then the series of a_n is an asymptotic series as well, or the series expansion of I_1 converges for $\epsilon < \epsilon_0$, where $\epsilon_0 > 0$, and then the power series of a_n will converge within a circle of radius equal to the distance from $\epsilon = 0$ to the nearest singular point of a_n, which will tend to zero in the limit when $n \rightarrow \infty$ as shown above. Consequently, in any case, the series expansion of $\zeta(x)$ in powers of ϵ as given by (7) is an asymptotic series.

4 Numerical results

We have solved the equations (47), (48) and so on up to $n=9$ to obtain the A_{ij}'s, i.e. we have calculated the coefficients of the series,

$$\left.\begin{aligned} \zeta(x) &= \sum_{j=1}^{9} \epsilon^{2j} A_j sech^{2j}(\epsilon x)\,, \text{ where} \\ A_j &= \sum_{k=1}^{10-j} \epsilon^{2(k-1)} A_{jk}\,,\quad j=1,\cdots,9\,. \end{aligned}\right\} \tag{53}$$

In our calculations we have used Maple and then the coefficients can be given their exact values,

$$\begin{aligned} A_1 = \;& \frac{4}{3} + \frac{8}{9}\epsilon^2 + \frac{8}{9}\epsilon^4 - \frac{286768}{212625}\epsilon^6 - \frac{40845656}{4465125}\epsilon^8 - \frac{115645861712}{3683728125}\epsilon^{10} \\ & - \frac{50138662476592}{558698765625}\epsilon^{12} - \frac{17985650041273504}{75424333359375}\epsilon^{14} \\ & - \frac{392860880150492515192}{634695765219140625}\epsilon^{16} \\ A_2 = \;& \frac{4}{3} - \frac{4}{135}\epsilon^2 + \frac{39512}{30375}\epsilon^4 + \frac{4864852}{4465125}\epsilon^6 + \frac{19627352}{111628125}\epsilon^8 \\ & - \frac{6290407694632}{1160374359375}\epsilon^{10} - \frac{5999593092750064}{226273000078125}\epsilon^{12} - \frac{249100218146915803888}{261345315090234375}\epsilon^{14} \end{aligned}$$

$$
\begin{aligned}
A_3 = {} & \frac{404}{135} + \frac{4928}{30375}\epsilon^2 + \frac{5348372}{893025}\epsilon^4 + \frac{109335056}{37209375}\epsilon^6 - \frac{12817076066392}{1160374359375}\epsilon^8 \\
& - \frac{16571007563968576}{226273000078125}\epsilon^{10} - \frac{6948708874218934950 8}{261345315090234375}\epsilon^{12} \\
A_4 = {} & \frac{23156}{3375} - \frac{3546716}{1488375}\epsilon^2 + \frac{6012292052}{334884375}\epsilon^4 + \frac{4768292900528}{386791453125}\epsilon^6 \\
& + \frac{38494939178392}{2793493828125}\epsilon^8 - \frac{468262544797273532}{9679456114453125}\epsilon^{10} \\
A_5 = {} & \frac{1775756}{99225} - \frac{709078936}{66976875}\epsilon^2 + \frac{14168470980848}{232074871875}\epsilon^4 + \frac{807343086171872}{45254600015625}\epsilon^6 \\
& - \frac{1140547457786321636}{52269063018046875}\epsilon^8 \\
A_6 = {} & \frac{3284536868}{66976875} - \frac{6950577231308}{165767765625}\epsilon^2 + \frac{2077562845996192}{9050920003125}\epsilon^4 \\
& + \frac{7388466006999452828}{37335045012890625}\epsilon^6 \\
A_7 = {} & \frac{2567593068748}{18418640625} - \frac{74878609460176}{430996190625}\epsilon^2 + \frac{21474508214603549948}{37335045012890625}\epsilon^4 \\
A_8 = {} & \frac{59427891023108}{143665396875} - \frac{762981724372609244}{1583911000546875}\epsilon^2 \\
A_9 = {} & \frac{70534877118487724}{58663370390625} .
\end{aligned}
$$

Instead of ϵ as the expansion parameter we may use the wave amplitude a. Then we have to expand ϵ in a series in powers of a, i.e.,

$$
\epsilon = \frac{\sqrt{3a}}{2}\left(1 + \sum_{j=1}^{8} a^j d_j\right), \tag{54}
$$

where

$$
d_1 = -\frac{5}{8}, \quad d_2 = \frac{71}{128}, \quad d_3 = -\frac{100627}{179200}, \quad d_4 = \frac{16259737}{28672000}, \quad d_5 = -\frac{7606868327}{12615680000}
$$

$$
d_6 = \frac{2295736286537}{3673686016000}, \quad d_7 = -\frac{352070152840157}{524812288000000}, \quad d_8 = \frac{979776092478366957 59}{1398939634892800000 00},
$$

which introduced into the series for $\zeta(x)$ given by (53) yields,

$$\left.\begin{aligned}\zeta(x) &= \sum_{j=1}^{9} a^j \tilde{A}_j sech^{2j}(\epsilon x)\,, \text{ where}\\ \tilde{A}_j &= \sum_{k=1}^{10-j} a^{k-1}\tilde{A_{jk}}\,,\ \ j=1,2,\cdots,9\,,\end{aligned}\right\} \tag{55}$$

where,

$$\begin{aligned}
\tilde{A}_1 =&\ 1-\frac{3}{4}a+\frac{5}{8}a^2-\frac{8209}{6000}a^3+\frac{364671}{196000}a^4-\frac{75679523}{29400000}a^5\\
&+\frac{78263417033}{22638000000}a^6-\frac{4595761996453}{980980000000}a^7+\frac{6012057610748687}{971170200000000}a^8\\
\tilde{A}_2 =&\ \frac{3}{4}-\frac{151}{80}a+\frac{11641}{3000}a^2-\frac{2920931}{392000}a^3+\frac{48824563}{3675000}a^4\\
&-\frac{3094446826693}{135828000000}a^5+\frac{15837237746581}{420420000000}a^6-\frac{823567885217539153}{13596382800000000}a^7\\
\tilde{A}_3 =&\ \frac{101}{80}-\frac{112393}{24000}a+\frac{2001361}{156800}a^2-\frac{130700377}{4200000}a^3+\frac{4635672338551}{67914000000}a^4\\
&-\frac{1639571505368813}{11771760000000}a^5+\frac{73377624107427019999}{27192765600000000}a^6\\
\tilde{A}_4 =&\ \frac{17367}{8000}-\frac{17906339}{1568000}a+\frac{2358279061}{58800000}a^2-\frac{10592199978011}{90552000000}a^3\\
&+\frac{3548497975278001}{11771760000000}a^4-\frac{129097663708320929947}{181285104000000000}a^5\\
\tilde{A}_5 =&\ \frac{1331817}{313600}-\frac{2674426609}{94080000}a+\frac{26185456824781}{217324800000}a^2\\
&-\frac{3874470304190711}{9417408000000}a^3+\frac{264961359540834528070}{217542124800000000}a^4
\end{aligned}$$

$$\tilde{A}_6 = \frac{821134217}{94080000} - \frac{22060716178577}{310464000000}a + \frac{334383022700383}{941740800000}a^2$$

$$-\frac{6146113078456594781}{4439635200000000}a^3$$

$$\tilde{A}_7 = \frac{641898267187}{34496000000} - \frac{38791575861419}{215255040000}a + \frac{635889504168604077731}{62154892800000000}a^2$$

$$\tilde{A}_8 = \frac{14856972755777}{358758400000} - \frac{2374720426192371311}{5273748480000000}a$$

$$\tilde{A}_9 = \frac{158703473516597379}{1757916160000000}.$$

The exact values of the coefficients in the series above have not been given before. If the coefficients in (54) and (55) are evaluated we find them to be equal to those obtained by Fenton (1972), so our procedure leads to the same series solutions as does Fenton's procedure.
In the investigation to follow we need the expression for the velocity potential at the free surface as well. We write,

$$\left.\begin{aligned} \frac{d\phi_s}{dx} &= \sqrt{R}\left(1 + \sum_{j=1}^{9}\epsilon^{2j}D_j sech^{2j}(\epsilon x)\right), \quad \text{where} \\ D_j &= \sum_{k=1}^{10-j}\epsilon^{2(k-1)}D_{jk}, \quad j = 1, \cdots, 9, \end{aligned}\right\} \tag{56}$$

and when the A_{ik}'s are known then the D_{ik}'s can be obtained from (9). From (56) it follows that,

$$\phi_s(x) = \sqrt{R}\left(x + \sum_{j=1}^{9}\epsilon^{2j-1}D_j J_{2j}(\epsilon x)\right), \tag{57}$$

where $J_{2j}(\epsilon x)$, $j = 1, \cdots, 9$, can be obtained from the recurrence formula,

$$J_{2j}(u) = \frac{1}{2j-1}tanh(u)sech^{2j-2}(u) + \frac{2j-2}{2j-1}J_{2j-2}(u). \tag{58}$$

Then,

$$\left.\begin{aligned}
&\phi_s(x) = \sqrt{R}\left(x + tanh(\epsilon x)\sum_{j=1}^{9}\epsilon^{2j-1}B_j sech^{2(j-1)}(\epsilon x)\right)\ ,\ \text{where} \\
&B_j = \sum_{k=1}^{10-j}\epsilon^{2(k-1)}B_{jk}\ ,\quad j = 1,\cdots,9\,.
\end{aligned}\right\}(59)$$

We find that,

$$\begin{aligned}
B_1 = \;& -\frac{4}{3} - \frac{16}{27}\epsilon^2 + \frac{128}{2025}\epsilon^4 + \frac{110336}{70875}\epsilon^6 + \frac{350881792}{66976875}\epsilon^8 + \frac{2902269952}{200930625}\epsilon^{10} \\
& + \frac{190772707131392}{5028288890625}\epsilon^{12} + \frac{7518671353348096}{75424333359375}\epsilon^{14} \\
& + \frac{1187778155316291043328}{4442870356533984375}\epsilon^{16} \\
B_2 = \;& -\frac{20}{27} - \frac{716}{2025}\epsilon^2 - \frac{31336}{42525}\epsilon^4 - \frac{87715444}{66976875}\epsilon^6 - \frac{24703972168}{11051184375}\epsilon^8 \\
& - \frac{2366146649864}{718326984375}\epsilon^{10} - \frac{251787271080944}{75424333359375}\epsilon^{12} + \frac{6638309874823850012}{4442870356533984375}\epsilon^{14} \\
B_3 = \;& -\frac{428}{225} - \frac{357064}{212625}\epsilon^2 - \frac{80488316}{22325625}\epsilon^4 - \frac{408218824}{136434375}\epsilon^6 + \frac{75326553748568}{15084866671875}\epsilon^8 \\
& + \frac{15434929402096}{430996190625}\epsilon^{10} + \frac{111430712923522805356}{888574071306796875}\epsilon^{12} \\
B_4 = \;& -\frac{270868}{70875} - \frac{26212316}{13395375}\epsilon^2 - \frac{103830490228}{11051184375}\epsilon^4 - \frac{78163942611152}{5028288890625}\epsilon^6 \\
& - \frac{458224766353432}{17405615390625}\epsilon^8 - \frac{134991807220145080276}{4442870356533984375}\epsilon^{10} \\
B_5 = \;& -\frac{133058068}{13395375} - \frac{10062684448}{2210236875}\epsilon^2 - \frac{331188255301 84}{1005657778125}\epsilon^4 - \frac{20356549591328}{430996190625}\epsilon^6 \\
& - \frac{418736389254523244}{6582030157828125}\epsilon^8
\end{aligned}$$

$$B_6 = -\frac{19438910108}{736745625} - \frac{2248320278236}{430996190625}\epsilon^2 - \frac{530362195822912}{5028288890625}\epsilon^4$$
$$-\frac{19874124467019102484}{126939153043828125}\epsilon^6$$

$$B_7 = -\frac{53346962630644}{718326984375} + \frac{248453358808}{83960296875}\epsilon^2 - \frac{217223196572709871972}{634695765219140625}\epsilon^4$$

$$B_8 = -\frac{465528827570284}{2154980953125} + \frac{617652508221145588}{11539923003984375}\epsilon^2$$

$$B_9 = -\frac{17237031652480284388}{26926487009296875}.$$

If the amplitude a is used as the expansion parameter instead of ϵ, we can introduce ϵ given by (54) into (59) to obtain,

$$\left.\begin{aligned} &\phi_s(x) = \sqrt{R}\left(x + \frac{tanh(\epsilon x)}{\epsilon}\sum_{j=1}^{9} a^j \tilde{B}_j sech^{2(j-1)}(\epsilon x)\right), \text{ where} \\ &\tilde{B}_j = \sum_{k=1}^{10-j} a^{k-1}\tilde{B_{jk}}, \quad j = 1, \cdots, 9, \end{aligned}\right\} \quad (60)$$

and where ϵ is given by (54).
We get,

$$\tilde{B}_1 = -1 + \frac{11}{12}a - \frac{16}{25}a^2 + \frac{4129}{6000}a^3 - \frac{2899517}{4410000}a^4 + \frac{6666571}{9240000}a^5 - \frac{1545131983}{2122312500}a^6$$
$$+\frac{542258478911}{679140000000}a^7 - \frac{3653264197023601}{4444971300000000}a^8$$

$$\tilde{B}_2 = -\frac{5}{12} + \frac{357}{400}a - \frac{8819}{5600}a^2 + \frac{22585483}{8820000}a^3 - \frac{44744683}{11550000}a^4 + \frac{205051074301}{36036000000}a^5$$
$$-\frac{143469677983249}{17657640000000}a^6 + \frac{329244359157695959}{28892313450000000}a^7$$

$$\tilde{B}_3 = -\frac{321}{400} + \frac{416309}{168000}a - \frac{16381013}{2940000}a^2 + \frac{1877930689}{161700000}a^3 - \frac{39394059988483}{1765764000000}a^4$$
$$+\frac{41432045299}{1027353600}a^5 - \frac{805933747166634521}{11556925380000000}a^6$$
$$\tilde{B}_4 = -\frac{67717}{56000} + \frac{78770341}{14112000}a - \frac{14972870699}{862400000}a^2 + \frac{319029708033769}{7063056000000}a^3$$
$$-\frac{821068054363797}{7847840000000}a^4 + \frac{7959611893440750281}{35559770400000000}a^5$$
$$\tilde{B}_5 = -\frac{33264517}{14112000} + \frac{14407679921}{1034880000}a - \frac{9328271484409}{176576400000}a^2 + \frac{922680256799297}{5650444800000}a^3$$
$$-\frac{1361212578419884793}{3081846768000000}a^4$$
$$\tilde{B}_6 = -\frac{4859727527}{1034880000} + \frac{27867325963391}{807206400000}a - \frac{8859386104171501}{56504448000000}a^2$$
$$+\frac{3714327357805899041}{6603957360000000}a^3$$
$$\tilde{B}_7 = -\frac{13336740657661}{1345344000000} + \frac{200734356601637}{2306304000000}a - \frac{40330633634975339407}{88052764800000000}a^2$$
$$\tilde{B}_8 = -\frac{116382206892571}{5381376000000} + \frac{8464102699184855797}{38423024640000000}a$$
$$\tilde{B}_9 = -\frac{4309257913120071097}{89653724160000000}.$$

We may obtain an approximate value of the velocity potential $\phi(x, z)$ from the integral in (12) if we introduce into it the series solutions of $\zeta(x)$ and $\phi_s(x)$ to a given order n. In our calculations we let n vary from 1 to 9. Now, letting $z \to \zeta^-(x)$ the value of the velocity potential at the surface given by the integral is found and it can be compared with the value of the velocity potential given by the series expansion, (59) or (60). We define the quantity,

$$\Delta = \frac{\phi_s^{ser}(x) - \phi_s^{int}(x)}{\phi_s^{ser}}, \tag{61}$$

where $\phi_s^{ser}(x)$ and $\phi_s^{int}(x)$ are the velocity potentials given by the series solution and the integral respectively,

$$\left.\begin{aligned}\phi_s^{int}(x) = \frac{1}{2\pi}\left[\lim_{r\to\infty}\left\{\lim_{z\to\zeta^-(x)}\int_{-r}^{r}\phi_s^{ser}(\mu)\frac{\zeta(\mu)-z-\zeta'(\mu)(\mu-x)}{(\mu-x)^2+(\zeta(\mu)-z)^2}d\mu\right.\right.\\ \left.\left.+\int_{-r}^{r}\phi_s^{ser}(\mu)\frac{\zeta(\mu)+\zeta(x)+2-\zeta'(\mu)(\mu-x)}{(\mu-x)^2+(\zeta(\mu)+\zeta(x)+2)^2}d\mu\right\}\right].\end{aligned}\right\}\quad(62)$$

As indicated in (62) care must be taken when going to the limit $z \longrightarrow \zeta^-(x)$ in the first integral. As pointed out earlier, when z is below and close to the surface, σ_{10} is in the upper half of the σ-plane and close to x, and $\hat{\sigma_{20}}$ is in the lower half plane close to x, and both tend to x when $z \longrightarrow \zeta^-(x)$. With this in mind and by applying Plemelj's formula (see Muskhelishvili (1946)) we find that,

$$\left.\begin{aligned}\frac{1}{2\pi}\lim_{z\to\zeta^-(x)}\left[\int_{-r}^{r}\phi_s^{ser}(\mu)\frac{\zeta(\mu)-z-\zeta'(\mu)(\mu-x)}{(\mu-x)^2+(\zeta(\mu)-z)^2}d\mu\right] =\\ \frac{1}{2}\phi_s^{ser}(x)+\frac{1}{2\pi}\int_{-r}^{r}\phi_s^{ser}(\mu)\frac{\zeta(\mu)-\zeta(x)-\zeta'(\mu)(\mu-x)}{(\mu-x)^2+(\zeta(\mu)-\zeta(x))^2}d\mu\,,\end{aligned}\right\}\quad(63)$$

which is used in (62) to obtain $\phi_s^{int}(x)$.

As we have shown the series of $\zeta(x)$ and $\phi_s(x)$ are not convergent, and therefore it can not be expected that Δ will tend to zero when n tends to infinity. For a given wave amplitude there is likely to be some limited number of terms to be included in the series to obtain the minimum value of Δ, and increasing the number of terms beyond this limit will give no further improvement, rather the opposite.

In the tables $1-5$, Δ has been calculated for different values of x letting the order n and the wave amplitude a vary.

The tables $1-5$ show that for the amplitudes 0.2, 0.3 and 0.4, Δ is decreasing essentially when the order of the series is increased, indicating that the series solution is improved when the number of terms of the series is increased. Also, when $a = 0.5$ and 0.6 there seems to be an improvement of the series solution if more terms are included in the series, although not so marked as when $a = 0.2$, 0.3 and 0.4. In all these cases there seems to be a general feature that there is only a small gain to be had in the accuracy going beyond the sixth order. When the amplitude is 0.7 or 0.8 we see that there is hardly any improvement at all when including more than four terms in the series. These results seem to reflect the asymptotic nature of the series.

5 Conclusion

The velocity potential has been expressed as a surface integral, and can be obtained if the surface elevation and the velocity potential at the free surface are known. In principal the integral can be evaluated by using the residue theorem, and it is shown that only two of the poles of the integrand contribute to the well-known series solution of the solitary wave problem, i.e. the velocity potential can be expressed as simple as $\phi(x,z) = \frac{1}{2}Re\{\phi_s(\sigma_{10}) + \phi_s(\sigma_{30})\}$, where $\phi_s(x)$ is the expression of the velocity potential at the free surface, and σ_{10} and σ_{30} are specified zeros of the functions $F_1(\sigma)$ and $F_3(\sigma)$ given by (13). A procedure is given to obtain the series solution to any order, and which enable us to show *analytically* that the series is asymptotic rather than convergent. The coefficients in the series of the surface elevation and the velocity potential have been calculated to ninth order, and their exact values are given. If these expressions for the surface elevation and the velocity potential are introduced into the surface integral, another approximate value of the velocity potential can be calculated which can be compared with the value obtained from the series expression of the velocity potential. This has been done for different values of x and by varying the wave amplitude a and the order n of the series. It is found that if $a \leq 0.6$, then the series solution is improved by increasing the order of the series, although the general feature is that there seems to be little gain to be had in the accuracy including more than six terms in the series. When $a = 0.7$ or 0.8 there is hardly any improvement going beyond the fourth order.

$n\backslash a$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
1	$-3.8\text{x}10^{-3}$	$-1.1\text{x}10^{-2}$	$-2.5\text{x}10^{-2}$	$-5.0\text{x}10^{-2}$	$-9.6\text{x}10^{-2}$	$-1.9\text{x}10^{-1}$	$-3.8\text{x}10^{-1}$
2	$8.0\text{x}10^{-4}$	$2.5\text{x}10^{-3}$	$5.6\text{x}10^{-3}$	$1.0\text{x}10^{-2}$	$1.6\text{x}10^{-2}$	$2.4\text{x}10^{-2}$	$3.2\text{x}10^{-2}$
3	$-1.7\text{x}10^{-4}$	$-9.1\text{x}10^{-4}$	$-3.1\text{x}10^{-3}$	$-8.5\text{x}10^{-3}$	$-2.1\text{x}10^{-2}$	$-5.2\text{x}10^{-2}$	$-1.3\text{x}10^{-1}$
4	$2.8\text{x}10^{-5}$	$1.7\text{x}10^{-4}$	$5.8\text{x}10^{-4}$	$1.3\text{x}10^{-3}$	$2.3\text{x}10^{-3}$	$3.0\text{x}10^{-3}$	$1.1\text{x}10^{-3}$
5	$-6.2\text{x}10^{-6}$	$-6.8\text{x}10^{-5}$	$-3.7\text{x}10^{-4}$	$-1.4\text{x}10^{-3}$	$-4.6\text{x}10^{-3}$	$-1.4\text{x}10^{-2}$	$-4.7\text{x}10^{-2}$
6	$6.4\text{x}10^{-7}$	$7.0\text{x}10^{-6}$	$1.7\text{x}10^{-5}$	$-9.6\text{x}10^{-5}$	$-9.9\text{x}10^{-4}$	$-5.0\text{x}10^{-3}$	$-1.9\text{x}10^{-2}$
7	$-3.2\text{x}10^{-7}$	$-5.0\text{x}10^{-6}$	$-3.7\text{x}10^{-5}$	$-1.7\text{x}10^{-4}$	$-6.2\text{x}10^{-4}$	$-2.3\text{x}10^{-3}$	$-1.2\text{x}10^{-2}$
8	$-1.1\text{x}10^{-7}$	$-1.0\text{x}10^{-6}$	$-1.6\text{x}10^{-5}$	$-1.8\text{x}10^{-4}$	$-1.3\text{x}10^{-3}$	$-6.8\text{x}10^{-3}$	$-2.9\text{x}10^{-2}$
9	$-1.3\text{x}10^{-7}$	$-6.8\text{x}10^{-7}$	$2.0\text{x}10^{-6}$	$6.3\text{x}10^{-5}$	$5.7\text{x}10^{-4}$	$3.1\text{x}10^{-3}$	$8.3\text{x}10^{-3}$

Table 1: Δ calculated for $x = 0.3$ and for different values of the order n and the amplitude a.

$n\backslash a$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
1	$-3.8\text{x}10^{-3}$	$-1.1\text{x}10^{-2}$	$-2.5\text{x}10^{-2}$	$-5.0\text{x}10^{-2}$	$-9.4\text{x}10^{-2}$	$-1.8\text{x}10^{-1}$	$-3.4\text{x}10^{-1}$
2	$7.8\text{x}10^{-4}$	$2.5\text{x}10^{-3}$	$5.5\text{x}10^{-3}$	$1.0\text{x}10^{-2}$	$1.6\text{x}10^{-2}$	$2.4\text{x}10^{-2}$	$3.2\text{x}10^{-2}$
3	$-1.6\text{x}10^{-4}$	$-8.7\text{x}10^{-4}$	$-2.9\text{x}10^{-3}$	$-7.8\text{x}10^{-3}$	$-1.9\text{x}10^{-2}$	$-4.3\text{x}10^{-2}$	$-9.8\text{x}10^{-2}$
4	$2.6\text{x}10^{-5}$	$1.6\text{x}10^{-4}$	$5.5\text{x}10^{-4}$	$1.3\text{x}10^{-3}$	$2.6\text{x}10^{-3}$	$4.0\text{x}10^{-3}$	$4.3\text{x}10^{-3}$
5	$-5.9\text{x}10^{-6}$	$-6.3\text{x}10^{-5}$	$-3.3\text{x}10^{-4}$	$-1.2\text{x}10^{-3}$	$-3.7\text{x}10^{-3}$	$-1.0\text{x}10^{-2}$	$-2.9\text{x}10^{-2}$
6	$6.2\text{x}10^{-7}$	$7.2\text{x}10^{-6}$	$2.6\text{x}10^{-5}$	$-1.2\text{x}10^{-6}$	$-4.1\text{x}10^{-4}$	$-2.4\text{x}10^{-3}$	$-9.6\text{x}10^{-3}$
7	$-3.0\text{x}10^{-7}$	$-4.5\text{x}10^{-6}$	$-3.1\text{x}10^{-5}$	$-1.3\text{x}10^{-4}$	$-4.1\text{x}10^{-4}$	$-1.2\text{x}10^{-3}$	$-5.1\text{x}10^{-3}$
8	$-9.7\text{x}10^{-8}$	$-6.6\text{x}10^{-7}$	$-9.1\text{x}10^{-6}$	$-1.0\text{x}10^{-4}$	$-7.2\text{x}10^{-4}$	$-3.6\text{x}10^{-3}$	$-1.4\text{x}10^{-2}$
9	$-1.1\text{x}10^{-7}$	$-6.6\text{x}10^{-7}$	$-2.2\text{x}10^{-7}$	$3.1\text{x}10^{-5}$	$3.0\text{x}10^{-4}$	$1.6\text{x}10^{-3}$	$3.0\text{x}10^{-3}$

Table 2: Δ calculated for $x = 0.5$ and for different values of the order n and the amplitude a.

$n \backslash a$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
1	$-4.0 \text{x} 10^{-3}$	$-1.1 \text{x} 10^{-2}$	$-2.5 \text{x} 10^{-2}$	$-4.8 \text{x} 10^{-2}$	$-8.5 \text{x} 10^{-2}$	$-1.4 \text{x} 10^{-1}$	$-2.4 \text{x} 10^{-1}$
2	$6.7 \text{x} 10^{-4}$	$2.2 \text{x} 10^{-3}$	$4.9 \text{x} 10^{-3}$	$9.4 \text{x} 10^{-3}$	$1.6 \text{x} 10^{-2}$	$2.4 \text{x} 10^{-2}$	$3.3 \text{x} 10^{-2}$
3	$-1.4 \text{x} 10^{-4}$	$-6.9 \text{x} 10^{-4}$	$-2.2 \text{x} 10^{-3}$	$-5.4 \text{x} 10^{-3}$	$-1.2 \text{x} 10^{-2}$	$-2.3 \text{x} 10^{-2}$	$-4.3 \text{x} 10^{-2}$
4	$2.0 \text{x} 10^{-5}$	$1.2 \text{x} 10^{-4}$	$4.4 \text{x} 10^{-4}$	$1.2 \text{x} 10^{-3}$	$2.9 \text{x} 10^{-3}$	$6.0 \text{x} 10^{-3}$	$1.1 \text{x} 10^{-2}$
5	$-4.6 \text{x} 10^{-6}$	$-4.2 \text{x} 10^{-5}$	$-1.9 \text{x} 10^{-4}$	$-6.1 \text{x} 10^{-4}$	$-1.6 \text{x} 10^{-3}$	$-4.1 \text{x} 10^{-3}$	$-1.0 \text{x} 10^{-2}$
6	$5.8 \text{x} 10^{-7}$	$7.9 \text{x} 10^{-6}$	$4.7 \text{x} 10^{-5}$	$1.9 \text{x} 10^{-4}$	$6.2 \text{x} 10^{-4}$	$1.8 \text{x} 10^{-3}$	$4.1 \text{x} 10^{-3}$
7	$-2.0 \text{x} 10^{-7}$	$-2.9 \text{x} 10^{-6}$	$-1.9 \text{x} 10^{-5}$	$-8.2 \text{x} 10^{-5}$	$-3.1 \text{x} 10^{-4}$	$-1.3 \text{x} 10^{-3}$	$-5.7 \text{x} 10^{-3}$
8	$-3.4 \text{x} 10^{-8}$	$4.4 \text{x} 10^{-7}$	$6.7 \text{x} 10^{-6}$	$4.7 \text{x} 10^{-5}$	$2.4 \text{x} 10^{-4}$	$9.7 \text{x} 10^{-4}$	$3.0 \text{x} 10^{-3}$
9	$-6.2 \text{x} 10^{-8}$	$-6.0 \text{x} 10^{-7}$	$-7.2 \text{x} 10^{-6}$	$-6.7 \text{x} 10^{-5}$	$-4.6 \text{x} 10^{-4}$	$-2.5 \text{x} 10^{-3}$	$-1.2 \text{x} 10^{-2}$

Table 3: Δ calculated for $x = 1.0$ and for different values of the order n and the amplitude a.

$n \backslash a$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
1	$-4.9 \text{x} 10^{-4}$	$-8.9 \text{x} 10^{-4}$	$-1.4 \text{x} 10^{-3}$	$-1.9 \text{x} 10^{-3}$	$-2.5 \text{x} 10^{-3}$	$-3.2 \text{x} 10^{-3}$	$-3.9 \text{x} 10^{-3}$
2	$1.0 \text{x} 10^{-4}$	$3.0 \text{x} 10^{-4}$	$6.4 \text{x} 10^{-4}$	$1.2 \text{x} 10^{-3}$	$2.0 \text{x} 10^{-3}$	$3.3 \text{x} 10^{-3}$	$5.0 \text{x} 10^{-3}$
3	$-1.4 \text{x} 10^{-5}$	$-5.8 \text{x} 10^{-5}$	$-1.6 \text{x} 10^{-4}$	$-3.4 \text{x} 10^{-4}$	$-6.3 \text{x} 10^{-4}$	$-1.0 \text{x} 10^{-3}$	$-1.6 \text{x} 10^{-3}$
4	$3.2 \text{x} 10^{-6}$	$2.0 \text{x} 10^{-5}$	$7.1 \text{x} 10^{-5}$	$1.9 \text{x} 10^{-4}$	$4.5 \text{x} 10^{-4}$	$9.4 \text{x} 10^{-4}$	$1.9 \text{x} 10^{-3}$
5	$-6.3 \text{x} 10^{-7}$	$-5.7 \text{x} 10^{-6}$	$-2.7 \text{x} 10^{-5}$	$-8.9 \text{x} 10^{-5}$	$-2.4 \text{x} 10^{-4}$	$-5.3 \text{x} 10^{-4}$	$-1.0 \text{x} 10^{-3}$
6	$1.4 \text{x} 10^{-7}$	$1.9 \text{x} 10^{-6}$	$1.2 \text{x} 10^{-5}$	$5.0 \text{x} 10^{-5}$	$1.6 \text{x} 10^{-4}$	$4.5 \text{x} 10^{-4}$	$1.1 \text{x} 10^{-3}$
7	$-3.0 \text{x} 10^{-8}$	$-6.0 \text{x} 10^{-7}$	$-4.9 \text{x} 10^{-6}$	$-2.5 \text{x} 10^{-5}$	$-9.7 \text{x} 10^{-5}$	$-2.9 \text{x} 10^{-4}$	$-7.4 \text{x} 10^{-4}$
8	$6.8 \text{x} 10^{-9}$	$2.0 \text{x} 10^{-7}$	$2.2 \text{x} 10^{-6}$	$1.4 \text{x} 10^{-5}$	$6.5 \text{x} 10^{-5}$	$2.4 \text{x} 10^{-4}$	$7.8 \text{x} 10^{-4}$
9	$-1.4 \text{x} 10^{-9}$	$-6.5 \text{x} 10^{-8}$	$-9.5 \text{x} 10^{-7}$	$-7.6 \text{x} 10^{-6}$	$-4.1 \text{x} 10^{-5}$	$-1.7 \text{x} 10^{-4}$	$-5.6 \text{x} 10^{-4}$

Table 4: Δ calculated for $x = 10.0$ and for different values of the order n and the amplitude a.

$n\backslash a$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
1	$-4.4\text{x}10^{-6}$	$-8.1\text{x}10^{-6}$	$-1.2\text{x}10^{-5}$	$-1.7\text{x}10^{-5}$	$-2.3\text{x}10^{-5}$	$-2.9\text{x}10^{-5}$	$-3.5\text{x}10^{-5}$
2	$8.8\text{x}10^{-7}$	$2.6\text{x}10^{-6}$	$5.6\text{x}10^{-6}$	$1.0\text{x}10^{-5}$	$1.8\text{x}10^{-5}$	$2.8\text{x}10^{-5}$	$4.4\text{x}10^{-5}$
3	$-1.2\text{x}10^{-7}$	$-4.9\text{x}10^{-7}$	$-1.4\text{x}10^{-6}$	$-3.0\text{x}10^{-6}$	$-5.6\text{x}10^{-6}$	$-9.3\text{x}10^{-6}$	$-1.4\text{x}10^{-5}$
4	$2.5\text{x}10^{-8}$	$1.6\text{x}10^{-7}$	$5.8\text{x}10^{-7}$	$1.6\text{x}10^{-6}$	$3.8\text{x}10^{-6}$	$7.9\text{x}10^{-6}$	$1.5\text{x}10^{-5}$
5	$-4.6\text{x}10^{-9}$	$-4.3\text{x}10^{-8}$	$-2.1\text{x}10^{-7}$	$-7.3\text{x}10^{-7}$	$-2.0\text{x}10^{-6}$	$-4.5\text{x}10^{-6}$	$-8.8\text{x}10^{-6}$
6	$1.0\text{x}10^{-9}$	$1.4\text{x}10^{-8}$	$9.1\text{x}10^{-8}$	$3.9\text{x}10^{-7}$	$1.3\text{x}10^{-6}$	$3.6\text{x}10^{-6}$	$8.9\text{x}10^{-6}$
7	$-2.0\text{x}10^{-10}$	$-4.2\text{x}10^{-9}$	$-3.6\text{x}10^{-8}$	$-1.9\text{x}10^{-7}$	$-7.5\text{x}10^{-7}$	$-2.3\text{x}10^{-6}$	$-6.1\text{x}10^{-6}$
8	$4.4\text{x}10^{-11}$	$1.4\text{x}10^{-9}$	$1.6\text{x}10^{-8}$	$1.0\text{x}10^{-7}$	$4.9\text{x}10^{-7}$	$1.8\text{x}10^{-6}$	$5.8\text{x}10^{-6}$
9	$-8.9\text{x}10^{-12}$	$-4.2\text{x}10^{-10}$	$-6.4\text{x}10^{-9}$	$-5.3\text{x}10^{-8}$	$-3.0\text{x}10^{-7}$	$-1.3\text{x}10^{-6}$	$-4.4\text{x}10^{-6}$

Table 5: Δ calculated for $x = 100.0$ and for different values of the order n and the amplitude a.

A Appendix

From the equations (25), (29), (31) and (32) we get,

$$\left.\begin{aligned}&\left[\epsilon\sin(4n\epsilon)+\frac{\cos(4n\epsilon)-1}{2Rn}\right]a_n+\epsilon^2\frac{(\cos(4n\epsilon)-1)}{4R^2n}\left[\sum_{j=1}^{n-1}a_ja_{n-j}+\cdots\right]\\&+P(\epsilon)+Q(\epsilon)=0\,,\end{aligned}\right\}\tag{64}$$

where,

$$P(\epsilon)=\frac{\epsilon^2}{2R}\left[\sum_{j=1}^{n-1}a_j\left[\cos(4\epsilon)\cos(4(j-1)\epsilon)\alpha_{n-j}-\sin(4\epsilon)\sin(4(j-1)\epsilon)\beta_{n-j}\right]+\cdots\right],$$

and,

$$Q(\epsilon)=\epsilon^3\left[\sum_{j=1}^{n-1}ja_j\left[\cos(4\epsilon)\sin(4(j-1)\epsilon)\alpha_{n-j}+\sin(4\epsilon)\cos(4(j-1)\epsilon)\beta_{n-j}\right]+\cdots\right],$$

$$\theta_n=-2(\cos(4n\epsilon)+1)a_n+\cdots\,,\tag{65}$$

$$\alpha_n-\beta_n=\frac{2\epsilon}{\sin(8\epsilon)}\theta_n+\cdots\,,\tag{66}$$

$$-2\epsilon\sin(4n\epsilon)a_n=\alpha_n\cos^2(4\epsilon)+\beta_n\sin^2(4\epsilon)+\cdots\,.\tag{67}$$

In these equations and in the following the dots $\cdots$ denote a sum of terms of the form $\epsilon^n q_n(\epsilon)$ where $n\geq 2$.
The equations (65) $-$ (67) can be solved with respect to α_n and β_n to give,

$$\alpha_n=-\frac{2\epsilon}{\cos(4\epsilon)}\left[\sin(4(n+1)\epsilon)+\sin(4\epsilon)\right]a_n+\cdots\,,\tag{68}$$

$$\beta_n=\frac{2\epsilon}{\sin(4\epsilon)}\left[\cos(4(n+1)\epsilon)+\cos(4\epsilon)\right]a_n+\cdots\,,\tag{69}$$

which introduced into the expressions for $P(\epsilon)$ and $Q(\epsilon)$ yields,

$$P(\epsilon)=-\frac{\epsilon^3}{R}\left[\sum_{j=1}^{n-1}\left[\sin(4n\epsilon)+\sin(4j\epsilon)\right]a_ja_{n-j}+\cdots\right],\tag{70}$$

$$Q(\epsilon) = 2\epsilon^4 \left[\sum_{j=1}^{n-1} [\cos(4n\epsilon) + \cos(4j\epsilon)]\, j a_j a_{n-j} + \cdots\right]. \tag{71}$$

Now, from (21) it follows that,

$$a_j = (-1)^{j-1} j a_1 + \cdots, \tag{72}$$

which yields,

$$\sum_{j=1}^{n-1} a_j a_{n-j} = (-1)^n a_1^2 \sum_{j=1}^{n-1} j(n-j) + \cdots = \frac{(-1)^n a_1^2}{6}\left(n^3 - n\right) + \cdots, \tag{73}$$

$$\sum_{j=1}^{n-1} j a_j a_{n-j} = (-1)^n a_1^2 \sum_{j=1}^{n-1} j^2(n-j) + \cdots = \frac{(-1)^n a_1^2}{12}\left(n^4 - n^2\right) + \cdots, \tag{74}$$

$$\left.\begin{aligned} &\sum_{j=1}^{n-1} a_j a_{n-j} \sin(4j\epsilon) = (-1)^n a_1^2 \sum_{j=1}^{n-1} j(n-j)\sin(4j\epsilon) + \cdots = \\ &\frac{(-1)^n a_1^2}{\sin^3(2\epsilon)}\left[\sin^2(2n\epsilon)/2 - \sin^2(2n\epsilon)\sin^2(\epsilon) - n\sin(4n\epsilon)\sin(2\epsilon)/4\right] + \cdots, \end{aligned}\right\} \tag{75}$$

$$\left.\begin{aligned} &\sum_{j=1}^{n-1} j a_j a_{n-j} \cos(4j\epsilon) = (-1)^n a_1^2 \sum_{j=1}^{n-1} j^2(n-j)\cos(4j\epsilon) + \cdots = \\ &\frac{(-1)^n a_1^2}{\sin^4(2\epsilon)}\Big[\sin^2(2n\epsilon)\sin^2(2\epsilon)/2 - 3\sin^2(2n\epsilon)/4 + n\sin(4n\epsilon)\sin(4\epsilon)/4 \\ &\qquad - n^2\cos(4n\epsilon)\sin^2(2\epsilon)/4\Big] + \cdots. \end{aligned}\right\} \tag{76}$$

It follows from (64) that I_1, which is defined by (51), can be written as,

$$I_1 = \epsilon^2 \frac{(\cos(4n\epsilon) - 1)}{4R^2 n}\left[\sum_{j=1}^{n-1} a_j a_{n-j} + \cdots\right] + P(\epsilon) + Q(\epsilon). \tag{77}$$

When $\epsilon = \frac{\pi}{2n}$ then the coefficients of ϵ^2 and ϵ^3 in the expression for I_1 are both zero, but the coefficient of ϵ^4 is not. This follows from the equations (70), (71), (75) and (76). Moreover, it follows that neither of these coefficients has

the same zeros as the function $K(\epsilon, n)$, except for the zero at $\epsilon = 0$.
When the expressions (73) – (76) are taken into account we can write,

$$\left.\begin{aligned} I_1 = (-1)^n a_1^2 \Bigg[\epsilon^2 \left\{ \frac{(n^2-1)(\cos(4n\epsilon)-1)}{24R^2} \right\} - \epsilon^3 \Bigg\{ \frac{(n^3-n)\sin(4n\epsilon)}{6R} \\ + \frac{\sin^2(2n\epsilon)/2 - \sin^2(2n\epsilon)\sin^2(\epsilon) - n\sin(4n\epsilon)\sin(2\epsilon)/4}{R\sin^3(2\epsilon)} \Bigg\} \\ + \epsilon^4 \Bigg\{ \frac{(n^4-n^2)\cos(4n\epsilon)}{6} + \frac{\sin^2(2n\epsilon)\sin^2(2\epsilon) - 3\sin^2(2n\epsilon)/2}{\sin^4(2\epsilon)} \\ + \frac{n\sin(4n\epsilon)\sin(4\epsilon)/2 - n^2\cos(4n\epsilon)\sin^2(2\epsilon)/2}{\sin^4(2\epsilon)} \Bigg\} \Bigg] + \cdots . \end{aligned}\right\} \quad (78)$$

As is seen from (78) the term which has been found in the expression for I_1 is proportional to a_1^2. This is the only term in the expression for I_1 which is proportional to a_1^2; the terms that have been left out are proportional to a_1^k, with the exponent k greater than two.
a_n is given by (51) and by introducing the expression for I_1 given by (78) and by making an expansion about $\epsilon = 0$ in powers of ϵ we get,

$$a_n = -\frac{\sin(2\epsilon)}{4\epsilon^2 \sin(2n\epsilon) K(\epsilon, n)} I_1 = (-1)^n a_1^2 \sum_{j=1}^{\infty} c_j \epsilon^j + \cdots , \quad (79)$$

where the terms that have been left out are proportional to a_1^k, with the exponent k greater than two, as mentioned previously. It follows from what has been said above that the series $\sum_{j=1}^{\infty} c_j \epsilon^j$ will converge for $\epsilon < \frac{\pi}{2n}$, and we see that the radius of convergence of this series tends to zero when $n \to \infty$.
In conclusion: the calculations carried out in this appendix show that it is reasonable to expect that the series of $\zeta(x)$ given by (7) will be an asymptotic series.

References

[1] Cokelet, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. A* **286**, 183-230.

[2] Boussinesq, M. J. 1871 Théorie de l'instrumescence liquide appelée onde solitaire ou translation se propageant dans un canal rectangulaire.*Acad. Sci. Paris, Comptes Rendues* **72**, 755-759.

[3] Engevik, L. E. 1987 A new approximate solution to the surface wave problem. *Applied Ocean Research* **9**, 104-113.

[4] Engevik, L. E. 1991 A note on the wave field in the surface zone. *Rep. No. 91, Dept. Appl. Maths, Univ. Bergen, Norway.*

[5] Fenton, J. D. 1972 A ninth-order solution for solitary wave. *J. Fluid Mech.* **53**, 257-271.

[6] Fenton, J. D. 1979 A high-order cnoidal wave theory. *J. Fluid Mech.* **94**, 129-161.

[7] Iwagaki, Y & Sakai, T. 1970 Horizontal water particle velocity of finite amplitude waves. *Proc. 12th Conf. Coastal Engng.* **1**, 309-325.

[8] Korteweg, D. J. & de Vries, G. 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave. *Phil. Mag.* **39**, 422-443.

[9] Lamb, H. 1932 *Hydrodynamics, 6th edn.* Cambridge University Press.

[10] Le Méhauté, B., Divoky, D. & Lin, A. 1968 Shallow water waves: a comparison of theories and experiments. *Proc. 11th Conf. Coastal Engng.* **1**, 86-107.

[11] Miles, J. W. 1980 Solitary waves. *Ann. Rev. Fluid Mech.* **12**, 11-43.

[12] Muskhelishvili, N. I. 1946 *Singular integral equations.* P. Noordhoff LTD, Groningen, Holland.

[13] Rayleigh, Lord 1876 On waves. *Phil. Mag.* **1**, 257-279. (Sci. Papers, **1** 251-271, Cambridge University Press.)

[14] Russel, J. S. 1844 *Report on Waves.* British Association Reports.

[15] Schwartz, L. W. and Fenton, J. D. 1982 Strongly nonlinear waves. *Ann. Rev. Fluid Mech.* **14**, 39-60.