Geometric reconstruction and persistence methods

Belén García Pascual

Institute of Mathematics
University of Bergen

Master Thesis

Supervisor: Morten Brun

June 2020

Abstract

In the present work we reconstruct the homotopy type of an unknown Euclidean subspace from a known sample of data. We carry out such reconstruction through generalized Čech complexes, by choosing radii which are less or equal than the reach of the subspace and by applying the Nerve Lemma. We also approach the reconstruction of a geodesic subspace through its convexity radius and a dense enough sample. Afterwards, we obtain homology and homotopy groups in terms of persistences, together with interleavings and isomorphisms between them. We conclude studying the reconstruction of a particular subspace that has reach equal to zero, where our results cannot be applied.

Contents

0.1 Introduction and historical context.

This thesis is a piece of work in algebraic topology, particularly in homotopy theory, and in differential geometry, with strong connections to topological data analysis.

The main aim is to reconstruct the geometry of an unknown Euclidean subspace from a known sample of data. We approach such task for a subspace $X \subseteq \mathbb{R}^d$ with positive reach, by covering X with balls whose centers lie in a sample $A \subseteq \mathbb{R}^d$, and by studying how large the radii of these balls can be in order to not lose information of the subspace of interest (since if the radii are too small or too large, it is not possible to capture all the geometric properties). Then, using topological, algebraic and analytical tools, we determine when the non-empty finite intersections of such collection of balls are contractible, and hence forms a good cover. Finally, we take the nerve of such good cover and we identify it with a particular $\check{C}$ech complex, and by the Nerve Lemma we obtain that the geometric realization of such a $\check{C}$ech complex has the same homotopy type as the subspace X.

Simplicial complexes that have the same homotopy type as the subspace of interest carry a lot of the geometric information of such subspace, hence by geometric reconstruction we mean finding an abstract simplicial complex, in our case a $\check{C}$ech complex, whose geometric realization has the same homotopy type.

For $X, A \subseteq (M, d)$, where (M, d) denotes a metric space M with a metric d, the $\check{C}$ech complex $\mathscr{C}_X^d(A, r)$ is an abstract simplicial complex with vertex set A and such that a finite set of points in A is a simplex if and only if balls with centers these points and given radii have non-empty intersection.

The main results here are Corollary 2.14 ([12] Corollary 6) which gives a successful geometric reconstruction when the radii are less or equal than the reach:

Corollary (2.14). *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, $A \subseteq \mathbb{R}^d$ and suppose that $\{B_X(a_j, r_j)\}_{a_j \in A}$ is a cover of X, where $r = \{r_j \mid a_j \in A\}$ is a set of radii. If $sup_{a_j \in A}\ r_j \leq \tau$, then X is homotopy equivalent to the geometric realization of $\mathscr{C}_X(A, r)$.*

And a new theorem, Theorem 2.16, which also gives a homotopy equivalence, but using the directed Hausdorff distance from the subspace to the sample:

Theorem (2.16). *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, and let $A \subseteq \mathbb{R}^d$ be compact.*

If $\alpha \in \left(\overrightarrow{d_H}(X, A), \tau\right]$, then the geometric realization of $\mathscr{C}_X(A, \alpha)$ is homotopy equivalent to X.

We highlight that we also recover the homotopy type of a geodesic subspace through its convexity radius and a dense enough sample in Lemma 3.18, with the length metric on $X \subseteq \mathbb{R}^d$ defined by $d_L(x, y) := inf_\gamma\ L(\gamma)$ for all $x, y \in X$, where $L(\gamma)$ denotes the length of a continuous path $\gamma : I \to X$ connecting x and y.

Lemma (3.18). *Let $X \subseteq \mathbb{R}^d$ be a geodesic subspace with the length metric d_L and positive convexity radius ρ. Let A be an s-dense subset of X, where $0 < s \leq \rho$. Then, $|\mathscr{C}_X^{d_L}(A, s)|$ is homotopy equivalent to X.*

Another aim of this thesis is to study how we can approximate the homology of a subspace if we do not achieve a homotopy equivalence, since if we have a homotopy equivalence, then the induced maps on their homotopy and homology groups in any dimension are isomorphisms. This problem is also relevant if we obtain a homotopy equivalence with a Čech complex that involves points in the subspace X, since it is unknown and therefore it is not possible to give exact computations. In order to do that, we understand homology and homotopy groups as persistence groups, so that we can form interleavings between them.

An important result here is Proposition 3.25, which for X a geodesic subspace, $A \subseteq X$ an s-dense subset and $B \subseteq X$, gives a $(0, s)$-interleaving between $\{\pi_n(|\mathscr{C}_B^{d_L}(A, p)|, \bullet)\}_{p>0}$ and $\{\pi_n(|\mathscr{C}_B^{d_L}(X, p)|, \bullet)\}_{p>0}$, and a $(0, s)$-interleaving between $\{H_n(|\mathscr{C}_B^{d_L}(A, p)|)\}_{p>0}$ and $\{H_n(|\mathscr{C}_B^{d_L}(X, p)|)\}_{p>0}$, for any $n \geq 1$.

We also have Theorem 3.31 and Corollary 3.34, which under certain conditions, for an abelian group G give the following isomorphisms:

$$\pi_1(|\mathscr{C}_X^{d_L}(A, r)|, \bullet) \cong \pi_1(|\mathscr{C}_X^{d_L}(X, r)|, \bullet)$$
$$H_1(|\mathscr{C}_X^{d_L}(A, r)|; G) \cong H_1(|\mathscr{C}_X^{d_L}(X, r)|; G)$$

where $H_1(\_; G)$ denotes the first homology group with coefficients in G.

In addition, we work with two metrics on $X \subseteq \mathbb{R}^d$, the length metric d_L and the restriction of the Euclidean metric, obtaining an important parameter which is the distortion of X, and by Dowker's Theorem 3.10, in the new Corollary 3.11 we also get homotopy equivalences and commutative diagrams at the level of geometric realizations of Čech complexes:

Corollary (3.11). *Let $X \subseteq \mathbb{R}^d$ with both the length metric d_L and the restriction of the Euclidean distance d_E, δ be the distortion of X, $A \subseteq X$ and $\alpha \in \mathbb{R}_{>0}$. Then, we obtain the following homotopy equivalences and commutative diagram, up to homotopy:*

$$\begin{array}{ccccc}
|\mathscr{C}_X^{d_L}(A, \alpha)| & \hookrightarrow & |\mathscr{C}_X^{d_E}(A, \alpha)| & \hookrightarrow & |\mathscr{C}_X^{d_L}(A, \delta\alpha)| \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
|\mathscr{C}_A^{d_L}(X, \alpha)| & \hookrightarrow & |\mathscr{C}_A^{d_E}(X, \alpha)| & \hookrightarrow & |\mathscr{C}_A^{d_L}(X, \delta\alpha)|.
\end{array}$$

Regarding some relevant prior work and what we have improved from before, we give an overview of the bibliography that we have used, pointing out our contributions.

Our main reference for the geometric reconstruction is the newly published paper [12], which provides sufficient conditions for finite intersections of balls of the form $B_X(a_j, r_j)$ to be contractible, for $a_j \in A \subseteq \mathbb{R}^d$ and $X \subseteq \mathbb{R}^d$ with positive reach ([12], Theorem 5). It also presents an application of the Nerve Lemma in Corollary 2.14 as above ([12], Corollary 6), which we have slightly changed by taking an infinite sample A rather than a finite one to be more consistent with the Nerve Lemma 1.20, and we give a new proof of it. In addition, we apply such Corollary 2.14 in the proof of our new Theorem 2.16, as above, and this new theorem provides a stronger result based on this paper, since it does not require a cover of X because we can construct one along its proof.

We also use [17] (Proposition 3.1) to show a well-known result that follows a similar approach as our Theorem 2.16, and which states the following: a compact submanifold $X \subseteq \mathbb{R}^d$ with positive reach τ is homotopy equivalent to $\bigcup_{a\in A} B_{\mathbb{R}^d}(a, \alpha)$, where $\alpha \in (2\epsilon, \sqrt{\frac{3}{5}}\,\tau)$ for $\epsilon := \overrightarrow{d_H}(X, A)$. Hence, we improve such result by giving a larger interval from where to pick the values of the radius and by generalizing compact manifolds to Euclidean subspaces.

We study [6] in order to give detailed proofs of the results from [12], and we show in this thesis the most important results from [6] for our work, which are quite analytical and more oriented to geometric measure theory. We slightly change some statements so that they fit better with the ones in [12].

We mention the recent paper [5] to introduce the length metric d_L, the concept of distortion, geodesic subspaces for path connected Euclidean subspaces and Lemma 3.18 as above.

We continue with persistence methods and the reconstruction of homotopy and homology groups for a geodesic subspace, where our main references are [23] and [25]. From there we get the idea of considering homology and homotopy groups as persistence groups, we generalize results for the first homology and homotopy groups to higher dimension, and we change several definitions and results for Rips complexes into new versions for Čech complexes, like the definition of r-sample or our Proposition 3.25 described above ([23] Definition 2.4 and Proposition 3.3).

In such proposition we also give a different proof by defining the desired maps at the level of Čech complexes, and by obtaining induced maps on their homology and homotopy groups in any dimension (rather than just for $n = 1$), which lead to commutative diagrams and the desired $(0, s)$-interleaving. We also present several new results in Section 3.1 describing simplicial maps and commutative diagrams of Čech complexes, like Corollary 3.11 presented above.

Theorem 3.31, presented above, is based on Theorem 4.2 in [23] (which includes both an isomorphism for the fundamental groups and another one for the first homology groups with coefficients in an abelian group, only proving the case of the fundamental group), but we did not quite understand the original statement, so we reformulated it with some different conditions for the case of the fundamental group, achieving a satisfactory result. We treat the case of the first homology group with coefficients in an abelian group in our Corollary 3.34 showed above, as a consequence of Theorem 3.31, and we present a detailed proof of it. We also improve the formalization of how loops in simplicial complexes are determined by their sequence of vertices.

For general topological aspects, we reference [9], [16], [19] and [22], and for some details about the Universal Coefficient Theorem A.25 we use [8]. Such theorem shows the existence of a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X;G) \longrightarrow Tor(H_{n-1}(X), G) \longrightarrow 0$$

and from it we obtain a specific corollary, Corollary A.26, that says that for an abstract simplicial complex K, if $|K|$ is path-connected, then $H_1(|K|;G) \cong$ $\cong H_1(|K|) \otimes G$, and this result is exactly what we need to conclude our proof of Corollary 3.34, showed above.

Therefore, this thesis is framed in a field of active research and interest, and the work presented here is based on newly published papers (like [12]), connecting them to more classical results (like [6]), and hence providing new points of view and a broad understanding of the theory of reconstruction of the topological and geometric features of an underlying Euclidean subspace.

We proceed giving an overview of the chapters:

In Chapter 1 we present definitions and fundamental results from algebraic topology in order to give the theoretical foundation of our work, like the Nerve Lemma 1.20 or the reach.

In Chapter 2 we develop the geometric reconstruction of an unknown Euclidean subspace X with positive reach, that is, we find a $\check{C}$ech complex constructed from a sample of data A and from X, and we carefully choose the radii of the balls forming such $\check{C}$ech complex, so that its geometric realization is homotopy equivalent to X. Here we have Corollary 2.14 and the new theorem, Theorem 2.16 introduced above.

In chapter 3 we consider the topological reconstruction of a geodesic subspace of $\mathbb{R}^d$, that is, the reconstruction of its homology and homotopy groups using persistence methods. Here we find Proposition 3.25, Theorem 3.31, Corollary 3.34 and Corollary 3.11 introduced above. In addition, we have Lemma 3.18, as above, where we reconstruct its homotopy type by the convexity radius and a dense enough sample.

In Chapter 4 we work with the square as a particular example of an Euclidean subspace that has reach equal to zero, where our previous results cannot be applied. We explain several counterexamples and we give a successful reconstruction for a very basic case and for when we consider it as a geodesic subspace.

In the Appendix we explain concepts used throughout the thesis. In A1 we present basic definitions in algebraic topology, in A2 we explain homotopy groups, simplicial and singular homology and we conclude with the Universal Coefficient Theorem A.25, introduced above. In A3 we define convergence, continuity and gradient of a function in a metric space. In A4 we briefly introduce Rips complexes and we relate them to $\check{C}$ech complexes.

Brief historical context.

Differential geometry was named as a concept by L.Bianchi (1856–1928) in 1894, meaning manifolds equipped with a Riemannian or a more general metric. Geodesic paths were previously introduced by Johann Bernoulli (1667-1748), in 1697, and in our thesis we generalize manifolds into geodesic subspaces in an Euclidean space, as Nash embedding theorem says that every Riemannian manifold can be isometrically embedded into some Euclidean space. Most of the contributions to differential geometry were made by C.F.Gauss (1777–1855) and B.Riemann (1826–1866).

In the early 20th century, H.Weyl (1885–1955) characterized a manifold as a topological space, and together with F.Hausdorff (1868-1942) and H.Poincaré (1854–1912), among other mathematicians, topology was developed as the study of the properties of spaces that are invariant under continuous deformations. H.Poincaré also developed algebraic topology, by introducing homology and the fundamental group.

It was W.Hurewicz (1904-1956) who introduced the concept of homotopy type, which is of great relevance since many algebraic invariants depend only upon the homotopy type of the space.

The abstract definition of a complex was given in 1907 by M.Dehn (1878-1952) and P.Heegaard (1871-1948), and such concept plays a fundamental role, since the spaces that can be decomposed into cells are easier to work with. In particular, the $\check{C}$ech complex is a type of simplicial complex that captures the homotopy type of the cover of a space by balls around its points, and it was named after E.$\check{C}$ech (1893–1960), a Czechoslovak mathematician who greatly contributed to the fields of differential geometry and combinatorial topology. He also worked in detail with the idea of the nerve of a finite open cover of a compact space, which was originally introduced by P.S. Alexandrov (1896-1982).

Nowadays, these notions are very popular in topological data analysis, which consists of analyzing sets of data using tools from algebraic topology and other fields of pure mathematics, with the goal of studying shape of data.

Some good references are [26], [10], [11] and Wikipedia.

Chapter 1

Preliminaries.

We present the fundamental notions for building up the theory of this thesis, like the reach of a Euclidean subspace or the Nerve Lemma 1.20.

Let $A, X \subseteq (M, d)$, where (M, d) denotes a metric space M with a distance d.

We mainly work with A and X as Euclidean subspaces, that is, $M = \mathbb{R}^d$; or with $M = X$.

The sample (or point cloud, or set of data) A is known, and the subspace X is the unknown object whose topology and geometry we are interested in.

Let $r = \{r_j \mid a_j \in A\}$ be a set of pre-specified radii, where $r_j \in \mathbb{R}_{>0}$.

Definition 1.1. *We define a **subspace ball** of centre $a_j \in A$ and radius r_j by:*

$$B_X^d(a_j, r_j) := \{x \in X \quad | \quad d(a_j, x) < r_j\}.$$

Remark 1.2. *We notice that the metric d on M can be any metric. Moreover, if $M = \mathbb{R}^d$, then we have the restriction onto X of the **Euclidean metric** in $\mathbb{R}^d$, defined by:*

$$d_E(x, y) :=\| x - y \|= \sqrt{\sum_{i=1}^{d} (x_i - y_i)^2}$$

for all $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. In this case, the subspace ball can also be defined by $B_X(a_j, r_j) = B_{\mathbb{R}^d}(a_j, r_j) \cap X$, where $B_{\mathbb{R}^d}(a_j, r_j)$ is the Euclidean open ball.

Clearly, subspace balls are open in X.

For $X \subseteq \mathbb{R}^d$, the reach of X is intuitively a real number that describes how curved its boundary is. To define it, first we take the points in $\mathbb{R}^d$ that have more than one nearest point in X, and afterwards we establish that the distance from X to the clousure of such points is the reach of X. Its formal definition is:

Definition 1.3. *Let $X \subseteq \mathbb{R}^d$, we define the set*

$$Y := \{y \in \mathbb{R}^d \mid \exists x_1, x_2 \in X \quad \textit{with} \quad x_1 \neq x_2 \quad \textit{and} \quad d_E(x_1, y) = d_E(x_2, y) = d(X, y)\}$$

where the distance from a point to a set is defined by $d(y, X) = d(X, y) = = inf_{x \in X}\ d(x, y)$.

*The closure of Y, denoted as $\overline{Y}$, is called the medial axis of Y, and the **reach** of X, denoted by τ, is defined by:*

$$\tau := d(X, \overline{Y}) = inf_{x \in X}\ d(x, \overline{Y}).$$

For our convention, if Y is the empty set, then the reach of X is infinite.

Remark 1.4. *For a convex subspace $X \subseteq \mathbb{R}^d$, the reach of X is $\tau = \infty$.*

That is because the set Y is always empty, since for $y \in \mathbb{R}^d \backslash X$, we can pick two points $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $d_E(y, x_1) = d_E(y, x_2) =: l$, and since X is convex, we take the straight line segment connecting x_1 to x_2, which lies entirely in X. So, we can project y to the midpoint x of such segment (forming an angle of $90°$), and get that $d(y, X) \leq d_E(y, x) < l$.

Hence, it is said that sets with positive reach are a generalization of convex sets, as $\{$convex sets$\} \subset \{$sets with positive reach$\}$.

Example 1.5. *Let X be a circle of center y_0 and radius r, then $\overline{Y} = \{y_0\}$ and $\tau = r$.*

Definition 1.6. *For $X \subseteq \mathbb{R}^d$ and $\alpha > 0$, we define the **tubular set** as*

$$Tub_\alpha := \{y \in \mathbb{R}^d \quad | \quad \delta_X(y) < \alpha\}$$

where δ_X is a distance function defined as follows:

$$\begin{aligned} \delta_X : \quad & \mathbb{R}^d \quad \longrightarrow \quad \mathbb{R} \\ & y \quad \mapsto \quad \delta_X(y) := inf\{\|y - x\| \quad | \quad x \in X\} = d(y, X). \end{aligned}$$

Now we want to define the projection of a point in $\mathbb{R}^d$ to its unique nearest point in the subspace X with reach $\tau > 0$. We define

$$U(X) := \{y \in \mathbb{R}^d \mid \exists! \text{ point of } X \text{ nearest to y}\} \tag{1.1}$$

and the projection map

$$\pi_X : U(X) \to X; \quad y \mapsto \pi_X(y) \in X \text{ such that } \delta_X(y) = \|y - \pi_X(y)\|.$$

To guarantee that a point $y \in \mathbb{R}^d$ has a unique nearest point in the subspace X of reach $\tau > 0$, we must take values $\alpha \leq \tau$ so that $\delta_X(y) < \alpha \leq \tau$.

Therefore, we have that $Tub_\tau := \{y \in \mathbb{R}^d \quad | \quad \delta_X(y) < \tau\} \subseteq U(X)$, and we restrict π_X to such subspace, giving the following definition that is used throughout this thesis:

Definition 1.7. *Let $X \subseteq \mathbb{R}^d$ with reach $\tau > 0$. We define the **projection map** to the subspace X by:*

$$\begin{array}{cccc} \pi_X : & Tub_\tau & \longrightarrow & X \\ & y & \mapsto & \pi_X(y) \textit{ unique nearest point of } y \textit{ in } X \end{array}$$

so that

$$\delta_X(y) = \|y - \pi_X(y)\|.$$

We can also give another equivalent definition of the reach of an Euclidean subspace, which is the first one given by Federer in [6]. First, the local reach at a point is defined, and then the reach of the whole space consist of the infimum local reach taken over each point in the space. We proceed with its formal definition:

Definition 1.8. *(Alternative definition of reach.)*

*The **local reach at a point** $x \in X$ is defined by*

$$\tau(x) := sup\{r \in \mathbb{R} \quad | \quad \forall y \in \mathbb{R}^d \textit{ such that } \|y - x\| < r, \exists! \textit{ nearest point of } y \textit{ in } X\}.$$

*And the **reach of the subspace** X is defined by*

$$\tau := inf\{\tau(x) \quad | \quad x \in X\}.$$

Claim 1.9. *If a subspace $X \subseteq \mathbb{R}^d$ has positive reach τ, then X is closed in $\mathbb{R}^d$.*

Proof. The idea of the proof is to assume that X is not closed, so that we get a contradiction.

If X is not closed, then $X \subsetneq \overline{X}$. We also assume that if X is not closed and there exists $y \in \overline{X} \backslash X$, then there exists $x \in X$ such that x is a nearest point to y in X.

Therefore, either $d_E(y, x) = 0$ or $d_E(y, x) > 0$.

1. If $d_E(y, x) = 0$, then $y = x$, so $y \in X$, which is a contradiction with $y \in \overline{X} \backslash X$.

2. If $d_E(y, x) > 0$, then we can take the Euclidean ball $B_{\mathbb{R}^d}(y, d_E(y, x))$, which is open in $\mathbb{R}^d$.

   We observe that X is dense in $\overline{X}$ (since the closure of X is exactly the whole space $\overline{X}$), and that $B_{\mathbb{R}^d}(y, d_E(y, x)) \bigcap \overline{X}$ is open in $\overline{X}$.

   Since the intersection of a dense set with any open set is non-empty, and $X \subseteq \overline{X}$, we get that

$$X \bigcap \left(B_{\mathbb{R}^d}(y, d_E(y, x)) \bigcap \overline{X}\right) = X \bigcap B_{\mathbb{R}^d}(y, d_E(y, x)) \neq \emptyset.$$

   Hence, we can take $p \in X \bigcap B_{\mathbb{R}^d}(y, d_E(y, x))$ so that $d_E(p, y) < d_E(y, x)$, and since $p \in X$, we conclude that x is not the nearest point to y in X. Moreover, we can take the ball $B_{\mathbb{R}^d}(y, d_E(y, p))$ and find another point closer to y in X by the same procedure, over and over again.

Therefore, we have shown that $y \in \overline{X} \backslash X$ does not have a nearest point in X, since we reached a contradiction in both cases (1) and (2). That means that

$$y \notin U(X) := \{y \in \mathbb{R}^d \mid \exists! \text{ point of } X \text{ nearest to y}\}. \tag{1.2}$$

Now, since for all $x \in X$ there exists $p \in X$ such that $d_E(y, p) < d_E(y, x)$, we can always pick a closer point to y in X, so we obtain that $d(y, X) = 0$. However, by hypothesis we had that $\tau > 0$, therefore $\delta_X(y) = d(y, X) = 0 < \tau$, and by definition of tubular set,

$$y \in Tub_\tau \subseteq U(X)$$

which contradicts (1.2).

In conclusion, such arbitrary $y \in \overline{X} \backslash X$ cannot exist. Hence, $\overline{X} \backslash X = \emptyset$, which implies that $\overline{X} = X$, which is the definition of closed subspace.

□

Proposition 1.10. *The distance function $\delta_X : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz and continuous in all $\mathbb{R}^d$, for any subspace $X \subseteq \mathbb{R}^d$.*

Proof. The idea of the proof is to show that such function is Lipschitz, and that being Lipschitz implies being continuous.

We have that for any $x, y \in \mathbb{R}^d$,
$\|\delta_X(y) - \delta_X(x)\| = |\delta_X(y) - \|x - \pi_X(x)\|| \leq \|y - \pi_X(x)\| - \|x - \pi_X(x)\| \leq \|y - x\|$, hence $\|\delta_X(y) - \delta_X(x)\| \leq \|y - x\|$ which is the definition of Lipschitz function (with constant $K = 1$).

Now (as explained in Appendix A3), δ_X is continuous at a point $x \in \mathbb{R}^d$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|y - x\| < \delta$ for all $y \in \mathbb{R}^d$, then $\|\delta_X(y) - \delta_X(x)\| < \varepsilon$.

By setting $\delta = \varepsilon$ and using that δ_X is Lipschitz (with constant $K = 1$), we have that $\|\delta_X(y) - \delta_X(x)\| \leq \|y - x\| < \varepsilon$.

We can do this same procedure for any $x \in \mathbb{R}^d$, therefore δ_X is a continuous function in all its domain $\mathbb{R}^d$.

□

Corollary 1.11. *The tubular set* $Tub_\tau = \{y \in \mathbb{R}^d \quad | \quad \delta_X(y) < \tau\}$*, for* $\tau > 0$*, is open in* $\mathbb{R}^d$*.*

Proof. Such set can be written as $Tub_\tau = \delta_X^{-1}((-\infty, \tau))$, where $(-\infty, \tau)$ is open. By Proposition 1.10, δ_X is continuous, and hence the preimage of $(-\infty, \tau)$ by such continuous function is open. □

Now it comes a central tool in algebraic topology, which are the simplicial complexes. The idea of these objects, roughly speaking, consists of a combinatorial structure that facilitates the use of algebra in topology.

We present the definitions of simplicial complex and of simplicial map from [15].

Let V be a set, and let $\mathfrak{P}(V)$ be the set of all finite, non-empty subsets of V. Such $\mathfrak{P}(V)$ is named the power set of V.

Definition 1.12. *An* ***(abstract) simplicial complex*** *is a set* V *and a subset* $K \subseteq \mathfrak{P}(V)$ *such that if* $\sigma \in K$ *and* $\varrho \subseteq \sigma$*, then* $\varrho \in K$*.*

We denote a simplicial complex just by K*.*

Such V is called the vertex set, whose elements $v \in V$ are vertices; and such $\sigma \in K$ is a simplex, represented by $\sigma = [v_0, \ldots, v_q]$, where $v_i \in V$ for $i = 0, \ldots, q$.

If a simplex contains exactly $q + 1$ vertices, it is named a q-simplex and its dimension is $dim(\sigma) := |\sigma| - 1 = q$. The dimension of K is defined by $dim(K) :=$ $= sup\{dim(\sigma) \mid \sigma \in K\}$.

Definition 1.13. *Given two simplicial complexes* K_1 *and* K_2*, a* ***simplicial map*** $F : K_1 \to K_2$ *is a function* $F : V_1 \to V_2$ *on the respective vertex sets of* K_1 *and* K_2*, such that if* $\sigma = [v_0, \ldots, v_q]$ *is a simplex in* K_1*, then* $F(\sigma) := [F(v_0), \ldots, F(v_q)]$ *is a simplex in* K_2*.*

The following theorem is from [19] (Theorem 2.5 in the introduction).

Theorem 1.14. *Let* X *be a set,* $\{A_i\}_{i \in I}$ *a collection of topological spaces with* $A_i \subseteq X$ *for every* i*, such that* $A_i \cap A_{i'}$ *is a closed (or open) subset of* A_i *and of* $A_{i'}$*, and the topology induced on* $A_i \cap A_{i'}$ *from* A_i *equals the topology induced from* $A_{i'}$*.*

Then, the finest topology on X *such that the inclusion maps* $A_i \hookrightarrow X$ *are continuous, is called the* ***coherent topology*** *with respect to* $\{A_i\}_{i \in I}$*.*

We proceed with the construction of the geometric realization of a non-empty abstract simplicial complex K, guided by [19] (if $K = \emptyset$, we simply set its geometric realization to be the empty set).

Let $|K|$ be the space of all functions $\alpha : V \to I$, also called barycentric coordinates, with $I = [0,1] \subset \mathbb{R}$, such that:

1. for any α, its support $supp(\alpha) := \{v \in V \quad | \quad \alpha(v) \neq 0\}$ is a simplex of K. In particular, $\alpha(v) \neq 0$ for only a finite set of vertices.

2. $\sum_{v \in V} \ \alpha(v) = 1$, for any α.

We want to define a topology on $|K|$, which is going to be a coherent topology. In order to do that, we define the closed simplex $|\sigma|$, for $\sigma \in K$, by

$$|\sigma| := \{\alpha \in |K| \quad | \quad \alpha(v) \neq 0 \Rightarrow v \in \sigma\}.$$

If σ is a q-simplex, then $|\sigma|$ is in one to one correspondence with the set

$$\{x = (x_0, \ldots, x_q) \in \mathbb{R}^{q+1} \quad | \quad 0 \leq x_i \leq 1, \, \Sigma \, x_i = 1\} \subseteq \mathbb{R}^{q+1} \tag{1.3}$$

given by $x_i = \alpha(v_i)$ for $v_i \in \{v_0, \ldots, v_q\} = \sigma$.

Moreover, we give to $|\sigma|$ the subspace Euclidean topology from such one to one correspondence.

To make $|K|$ a topological space with the coherent topology, we need to check the following condition: if $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2$ is either empty (in which case $|\sigma_1| \cap |\sigma_2| = \emptyset$) or a face of σ_1 and of σ_2 (in which case $|\sigma_1| \cap |\sigma_2| = |\sigma_1 \cap \sigma_2|$).

Therefore, in either case $|\sigma_1| \cap |\sigma_2|$ is a closed set in $|\sigma_1|$ and in $|\sigma_2|$, and the topology induced on this intersection from the first closed simplex is equal to the topology induced from the second one. Hence, we can apply Theorem 1.14, and get that there is a topology on $|K|$ coherent with $\{|\sigma| \quad | \quad \sigma \in K\}$.

This also means that

$$|K| = \bigcup_{\sigma \in K} |\sigma|.$$

Now we have all the ingredients to formally define geometric realizations:

Definition 1.15. *The **geometric realization** of an abstract simplicial complex K is a topological space consisting of the set $|K|$, constructed as above, together with the coherent topology. We also denote it by $|K|$.*

We have seen that we can construct topological spaces from abstract simplicial complexes, and it also makes sense to construct continuous maps between such topological spaces from given simplicial maps ([19], page 113).

We start with a simplicial map $\varphi : K_1 \to K_2$, then we can define a continuous map $|\varphi| : |K_1| \to |K_2|$ by

$$|\varphi|(\alpha)(v') = \Sigma_{\varphi(v)=v'}\ \alpha(v) \quad \text{for } v' \in K_2.$$

The definition of such continuous map can be understood in an easier way if we present the following diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle |\varphi|(\alpha)} & \\ I & & \end{array}$$

where for a function $\alpha \in |K_1|$ and a simplex $\sigma = \{v \mid \alpha(v) \neq 0\} \in K_1$, we have that $\varphi(\sigma) = \{v' \mid |\varphi|(\alpha)(v') \neq 0\} \in K_2$, if $|\varphi|(\alpha)(v') = \Sigma_{\varphi(v)=v'}\ \alpha(v)$.

Definition 1.16. *Let T be a topological space. An* ***open cover*** *of $X \subseteq T$ is a collection of subspaces $\mathcal{U} = \{U_i\}_{i \in I}$ with $U_i \subseteq T$ open, for every $i \in I$, such that $X \subseteq \bigcup_{i \in I} U_i$.*

We notice that if $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X with $U_i \subseteq X$ open, for every $i \in I$, then $X = \bigcup_{i \in I} U_i$.

Definition 1.17. *An open cover of X, $\mathcal{U} = \{U_i\}_{i \in I}$ with $U_i \subseteq X$ for all $i \in I$, is a* ***good (open) cover*** *if all its open sets and all intersections of finitely many open sets $U_{\alpha_1} \cap \ldots \cap U_{\alpha_n}$, are either contractible or empty.*

Definition 1.18. *The* ***nerve of a cover*** *$\mathcal{U} = \{U_i\}_{i \in I}$, also named nerve complex, is defined by:*

$$N(\mathcal{U}) := \{\sigma = [U_{i_0}, \cdots, U_{i_k}] \subseteq \mathcal{U} \mid \bigcap_{j=0}^{k} U_{i_j} \neq \emptyset\}.$$

Remark 1.19. *We notice that the nerve of a cover is an abstract simplicial complex, where $\mathcal{U}$ is the vertex set and for every non-empty finite intersection $\bigcap_{j=0}^{k} U_{i_j}$, the set $\{U_{i_0}, \cdots, U_{i_k}\}$ is a simplex.*

The homotopy type of a space X can be recovered from the nerve complex under certain conditions. This is shown in the Nerve Lemma, a central result which for instance can be found in [9] (Corollary 4G.3, page 459).

Lemma 1.20 (Nerve Lemma)**.** *If $\mathcal{U} = \{U_i\}_{i \in I}$ is a good (open) cover of a paracompact space X, then the geometric realization of its nerve is homotopy equivalent to X, i.e, $|N(\mathcal{U})| \simeq X$.*

We proceed to prove that convex subspaces in $\mathbb{R}^d$ are contractible. A convex subspace $X \subseteq \mathbb{R}^d$ is a space such that for every two points in X, the straight line segment connecting them lies entirely in X. This property is useful in order to construct good covers.

Proposition 1.21. *Let $X \subseteq \mathbb{R}^d$ be a convex subspace, then X is contractible.*

Proof. We fix a point $x_0 \in X$ and we define a homotopy $H : X \times I \to X$ as the straight line segment joining the point x_0 to any other generic point in X, that is

$$H(x,t) = tx_0 + (1-t)x$$

where H is continuous and well-defined, since such line segment is contained in X by definition of convex set. We have $H(x,0) = x = id_X(x)$ and $H(x,1) = x_0 =$ $= c_{x_0}(x)$ for all $x \in X$, where $c_{x_0} : X \to \{x_0\} \subset X$ is the constant map, so $id_X \simeq c_{x_0}$, and as we define in Appendix A1, X is contractible. □

Moreover, finite intersections of convex sets in $\mathbb{R}^d$ which are non-empty, are again convex. Therefore, in [14] (Lemma 2.7.2) we can find the statement of the following version of the Nerve Lemma for covers consisting of convex sets in $\mathbb{R}^d$, since they are good covers.

Corollary 1.22 (Nerve Lemma: Convex Version)**.** *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a collection of open convex sets of $\mathbb{R}^d$, then $|N(\mathcal{U})| \simeq \bigcup_{i\in I} U_i$.*

Proof. First we remark that every metric space is paracompact ([16], Theorem 41.4).

We define the subspace $X := \bigcup_{i\in I} U_i$, where $U_i \subseteq X$ open for every $i \in I$, so that X is paracompact since it is a Euclidean subspace.

The collection $\mathcal{U} = \{U_i\}_{i\in I}$ consist of convex sets in $\mathbb{R}^d$ whose finite, non-empty intersections are again convex, so by Proposition 1.21, such sets are contractible, and hence they form a good cover of X.

Therefore, by the Nerve Lemma 1.20, $|N(\mathcal{U})| \simeq X = \bigcup_{i\in I} U_i$. □

Now we introduce the Čech complex ([12]), which is going to play a fundamental role throughout this whole thesis in recovering the homotopy type of an underlying topological space. This type of abstract simplicial complex can be viewed as the nerve of a collection of subspace balls.

Definition 1.23. *Let $X, A \subseteq (M,d)$, and let $r = \{r_j \mid a_j \in A\}$ be a pre-specified radii. We define the **Čech complex** by:*

$$\mathscr{C}^d_X(A,r) := \{\sigma = [a_0, \cdots, a_k] \subseteq A \mid \bigcap_{j=0}^{k} B^d_X(a_j, r_j) \neq \emptyset\}$$

or equivalently,

$$\mathscr{C}^d_X(A,r) := \{\sigma = [a_0, \cdots, a_k] \subseteq A \mid \exists x \in X \text{ such that } d(a_j, x) < r_j \quad \forall j = 0, \cdots, k\}.$$

In the case where $M = \mathbb{R}^d$ with the Euclidean metric d_E, we usually drop it from our notation and just write $\mathscr{C}_X(A,r)$, except if we want to stress that we are using d_E.

If we take just one value to construct the radius of the balls, we set it as $r = r_j$ for all $a_j \in A$ and we define the Čech complex as

$$\mathscr{C}^d_X(A,r) := \{\sigma = [a_0, \cdots, a_k] \subseteq A \mid \bigcap_{j=0}^{k} B^d_X(a_j, r) \neq \emptyset\}.$$

A nice and classic application of the Nerve Lemma 1.20 consist of recovering the homotopy type of an underlying space X from Čech complexes.

We can illustrate this idea by the following basic example, and in the next chapters of this thesis, we will be using this application of the Nerve Lemma for Euclidean subspaces (since metric spaces are paracompact by [16], Theorem 41.4) with different properties (like having positive reach).

Example 1.24 (Classic application of the Nerve Lemma)**.** *Let $X, A \subseteq (M,d)$ and suppose that $\mathcal{U} = \{B^d_X(a_j, r_j)\}_{a_j \in A}$ is a good cover of X.*

Then, we can identify $N(\mathcal{U})$ with $\mathscr{C}^d_X(A,r)$ by noticing that the vertex set of the nerve complex is $\mathcal{U} = \{B^d_X(a_j, r_j)\}_{a_j \in A}$ and the vertex set of the Čech complex is A, so that we rename each $B^d_X(a_j, r_j)$ as its center $a_j \in A$, and we get by the definitions exactly the same simplices in both simplicial complexes. Finally, the Nerve Lemma 1.20 gives us $|\mathscr{C}^d_X(A,r)| \simeq X$.

We also stress that since the Čech complex $\mathscr{C}^d_X(A,r)$ is the nerve of the cover $\{B^d_X(a_j, r_j)\}_{a_j \in A}$, if the subspace X is convex, then each ball $B^d_X(a_j, r_j)$ is contractible, and their finite intersections are also contractible, so we can automatically apply the Nerve Lemma 1.20. However, if $X \subseteq \mathbb{R}^d$ is more general, without the property of being convex, then we have to study if finite intersections of such balls are contractible, so that the Nerve Lemma can be applied. This is done in the next chapter through analytical and technical results for subspaces with positive reach, since as we point out in Remark 1.4, subspaces with positive reach are a generalization of convex subspaces.

Chapter 2

Geometric reconstruction.

The aim of this chapter is to reconstruct geometrically a Euclidean subspace, that is, to reconstruct its homotopy type by finding a simplicial complex constructed from a sample of data, whose geometric realization is homotopy equivalent to the subspace. We use Čech complexes as our simplicial complexes, and we see that such reconstruction is satisfactory when the reach of the Euclidean subspace is positive and we pick radii less or equal than the reach.

We divide this chapter in three different sections:

In Section 2.1 we present the necessary results used to develop our theory, which are from [6] (Theorem 4.8 parts 4, 2, 3, 5 and 6; pages 434-438) and from [12] (Claim 9, Lemma 10 and Claim 11), along with more detailed proofs.

In Section 2.2 we continue with a theorem from [12] (Theorem 5), that establish the machinery for subspace balls in order to form a good cover, by determining when finite intersections of subspace balls are contractible; and the main reconstruction result, its corollary ([12]. Corollary 6), taking an infinite set $A \subseteq \mathbb{R}^d$ rather than exclusively a finite one, in order to be more consistent with the Nerve Lemma, and giving a new proof for this version.

In Section 2.3 we present a new theorem, Theorem 2.16, which reconstructs the homotopy type of an underlying Euclidean subspace using the directed Hausdorff distance, and we compare this result with an already known result, Proposition 2.18 ([17] Proposition 3.1), that follows a similar approach. We also mention some future work.

We first begin with the following simple result to motivate all the work we develop along this chapter.

Proposition 2.1. *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, A be a sample consisting of the whole Euclidean space $\mathbb{R}^d$, and $\alpha \leq \tau$. Then, the geometric realization of*

$\mathscr{C}_{\mathbb{R}^d}(X, \alpha)$ *is homotopy equivalent to* X.

Proof. We approach this result by showing that X is a deformation retract of $\bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha)$. For this, we find a homotopy relative to X between the identity map of $\bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha)$ and a retract from such union to X.

We define the retract $\pi_X : \bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha) \to X$ as the projection which sends a point $p \in \bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha)$ to its unique nearest point in X. Such map is well-defined, since by hypothesis $\alpha \leq \tau$, and therefore $\bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha) \subseteq Tub_\tau$. This map is also continuous by the following Proposition 2.2, and verifies that $\pi_X|_X = = id_X$.

We see that both $id_{\bigcup_{x\in X} B_{\mathbb{R}^d}(x,\alpha)}$ and π_X agree on X, so we can define a homotopy relative to X (defined in Appendix A1) between them:

$$H : \bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha) \times [0, 1] \longrightarrow \bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha); \quad H(p, t) = t\pi_X(p) + (1 - t)p$$

where H is well-defined and continuous, since if a point p is in such union, then $p \in B_{\mathbb{R}^d}(x_0, \alpha)$ for some $x_0 \in X$, which means that $\|p - x_0\| < \alpha$. By definition of π_X, we have that $\|p-\pi_X(p)\| \leq \|p-x_0\| < \alpha$. So, $p \in B_{\mathbb{R}^d}(\pi_X(p), \alpha)$, and of course $\pi_X(p) \in B_{\mathbb{R}^d}(\pi_X(p), \alpha)$. Moreover, $B_{\mathbb{R}^d}(\pi_X(p), \alpha) \subseteq \bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha)$ because such union is taken over all points in X. So, we can take the straight line segment joining p and $\pi_X(p)$ since they are in one same convex ball of such union.

We have $H(p, 0) = p = id_{\bigcup_{x\in X} B_{\mathbb{R}^d}(x,\alpha)}(p)$, $H(p, 1) = \pi_X(p)$ for all p, and $H(x, t) = x$ for all $x \in X$ and $t \in [0, 1]$. Hence, $\pi_X \simeq_X id_{\bigcup_{x\in X} B_{\mathbb{R}^d}(x,\alpha)}$.

By Proposition A.7,

$$\bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha) \simeq X$$

By the convex version of the Nerve Lemma 1.22,

$$|N(\{B_{\mathbb{R}^d}(x, \alpha)\}_{x\in X})| \simeq \bigcup_{x\in X} B_{\mathbb{R}^d}(x, \alpha)$$

and by identifying $N(\{B_{\mathbb{R}^d}(x, \alpha)\}_{x\in X})$ with $\mathscr{C}_{\mathbb{R}^d}(X, \alpha)$, as we explain in Example 1.24, we conclude the proof.

□

2.1 Previous results.

The following results correspond to part 4, 2, 3, 5 and 6 of Theorem 4.8 in [6].

Proposition 2.2 (Continuity of the projection map). *Let* $X \subseteq \mathbb{R}^d$ *with positive reach* τ*, then the projection* $\pi_X : Tub_\tau \to X$ *is continuous, where* $\pi_X(y)$ *is the unique point in* X *such that* $\delta_X(y) = \|y - \pi_X(y)\|$.

Proof. Since we are in a metric space, we take as the definition of continuous map the following (as explained in Appendix A3): a map $f : X \to Y$, with X, Y metric spaces, is continuous if for any convergent sequence $\{x_n\}_{n\in\mathbb{N}} \to x$ in X, then $\{f(x_n)\}_{n\in\mathbb{N}} \to f(x)$ in Y.

Suppose π_X is not continuous. Hence, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points in Tub_τ convergent to a point $y \in Tub_\tau$ such that the sequence $\{\pi_X(y_n)\}_{n\in\mathbb{N}}$ does not converge to $\pi_X(y)$.

That is, for every $\varepsilon_1 > 0$, there exists $N_1 \in \mathbb{N}$ such that $\|y_n - y\| < \varepsilon_1$ for every $n \geq N_1$; and there exists $\varepsilon_2 > 0$ such that for all $n \in \mathbb{N}$, there exists $N \geq n$ such that

$$\|\pi_X(y_N) - \pi_X(y)\| \geq \varepsilon_2. \tag{2.1}$$

We construct such non-convergent sequence using the different N's, so that we have the sequence $\{\pi_X(y_n)\}_{n\in\mathscr{N}}$, where $\mathscr{N}$ is the set of the natural numbers N's.

Since $\|\pi_X(y_n) - y_n\| = \delta_X(y_n)$ for each n, by definition of the projection map and by applying the Triangle Inequality twice, we get:

$\|\pi_X(y_n) - y\| \leq \|\pi_X(y_n) - y_n\| + \|y_n - y\| \leq \|\pi_X(y) - y_n\| + \|y_n - y\| \leq$
$\leq \|\pi_X(y) - y\| + \|y - y_n\| + \|y_n - y\| = \delta_X(y) + 2\|y_n - y\| < \tau + 2\varepsilon_1$, for every $n \geq N_1$.

Hence, the sequence $\{\pi_X(y_n)\}_{n\in\mathscr{N}, n\geq N_1}$ lies in a bounded subset of X. By Bolzano-Weierstrass Theorem ([20], Corollary in page 458), every bounded sequence has a convergent subsequence, so we can take a subsequence $\{\pi_X(y_{n_j})\}_{n_j\in\mathbb{N}}$ such that it converges to a point $x \in X$.

Also, if a sequence converges to a point, then every subsequence of such sequence converges to the same point. So, since $\{y_n\}_{n\in\mathbb{N}} \to y$, the subsequence $\{y_{n_j}\}_{n_j\in\mathbb{N}}$ also converges to y.

But then, since the distance function is continuous,

$\delta_X(y) = \lim_{n_j\to\infty} \delta_X(y_{n_j}) = \lim_{n_j\to\infty} \|\pi_X(y_{n_j}) - y_{n_j}\| = \|\lim_{n_j\to\infty} \pi_X(y_{n_j}) -$
$- \lim_{n_j\to\infty} y_{n_j}\| = \|x - y\|$

which means that $\pi_X(y) = x$, because $y \in Tub_\tau$, so $\pi_X(y)$ is the unique point such that $\delta_X(y) = \|y - \pi_X(y)\|$.

We get the following contradiction with (2.1):

$$0 = \|x - \pi_X(y)\| = \lim_{n_j\to\infty} \|\pi_X(y_{n_j}) - \pi_X(y)\| \geq \varepsilon_2 > 0$$

(since if (2.1) holds, then for any subsequence $\{\pi_X(y_{n_j})\}_{n_j\in\mathbb{N}}$ of $\{\pi_X(y_n)\}_{n\in\mathscr{N}}$ we also have that $\|\pi_X(y_{n_j}) - \pi_X(y)\| \geq \varepsilon_2$).

Therefore, the projection map π_X has to be continuous in Tub_τ.

□

Proposition 2.3. *Let $X \subseteq \mathbb{R}^d$ and let v be a vector in $\mathbb{R}^d$.*

If $a \in X$ and $P := \{v \mid \pi_X(a+v) = a\}$, $Q := \{v \mid \delta_X(a+v) = \|v\|\}$, then the sets P and Q are convex.

Proof. It is enough to do the proof for $a = 0$, since we can just translate for any other point in X. We notice that

- (i) $v \in P$ if and only if $\|b - v\| > \|v\|$ for all $b \in X \backslash \{a\}$.
- (ii) $v \in Q$ if and only if $\|b - v\| \geq \|v\|$ for all $b \in X$.

To prove the implication to the right in (i), we assume that $v \in P$. Hence by definition of P, a is the nearest point of $a+v$ in X and $\delta_X(a+v) = \|a+v-a\| = \|v\|$, therefore for any other point $b \in X\backslash\{a\}$ we have that $d_E(a+v, b) > \delta_X(a+v)$, and since $a = 0$ by assumption, $d_E(a+v, b) = \|b - v\|$. So, $\|b - v\| > \|v\|$.

To prove the implication to the left in (i), we assume that $\|b - v\| > \|v\|$ for $b \neq a$. Then this means that a is the closest point to $v + a$ in X, with $a = 0$, so $v \in P$.

To prove the implication to the right in (ii), we assume that $v \in Q$ and $b \in X$, hence $\delta_X(a+v) = \|v\|$ and $\|b - a - v\| = d_E(a+v, b) \geq \delta_X(a+v) = \|v\|$. Giving the value $a = 0$, we finish this part of the proof.

To prove the implication to the left in (ii), we assume that $\|b - v\| \geq \|v\|$ with $a = 0$. Then, by definition of δ_X, we have $d_E(a, a+v) \geq \delta_X(a+v) \geq \|v\|$. But $d_E(a, a+v) = \|v\|$, hence $\delta_X(a+v) = \|v\|$.

We proceed showing that P and Q are convex. Let

$$\|b - v\|^2 - \|v\|^2 = b \bullet (b - 2v) \tag{2.2}$$

for $b, v \in \mathbb{R}^d$, since we can represent the scalar products in the following way and get the equality (2.2):

$b \bullet (b - 2v) = \langle b, b - 2v \rangle = \langle b, b \rangle - 2\langle b, v \rangle$,

$\|b - v\|^2 - \|v\|^2 = \langle b - v, b - v \rangle - \langle v, v \rangle = \langle b, b \rangle - 2\langle b, v \rangle + \langle v, v \rangle - \langle v, v \rangle$.

To check that P is convex, we take $v, w \in P$ and prove that the convex combination $sv + tw$ is also contained in P, for $s, t \geq 0$ such that $s + t = 1$.

To show this, we take $b \in X\backslash\{a\}$ and we apply the equality (2.2) to the following expression:

$\|b-(sv+tw)\|^2 - \|sv+tw\|^2 = b \bullet (b-2sv-2tw) = b \bullet (s(b-2v)+t(b-2w)) =$

$$= sb \bullet (b-2v) + tb \bullet (b-2w) = s(\|b-v\|^2 - \|v\|^2) + t(\|b-w\|^2 - \|w\|^2) \quad (2.3)$$

The expression (2.3) is strictly positive by (i) since $v, w \in P$, and hence we get that $sv + tw \in P$ also by (i).

We proceed in an analogous way to check that Q is convex, taking $v, w \in Q$ and showing that the expression (2.3) is non-negative, with $b \in X$ allowed to be $b = a$, and hence by (ii), $sv + tw \in Q$.

□

The following result stresses that the projection of a point y in $\mathbb{R}^d$ to a space $X \subseteq \mathbb{R}^d$ is unique, and moreover there exists a formula for calculating it, which is

$$\pi_X(y) = y - \delta_X(y) grad\delta_X(y)$$

where $grad\delta_X(y)$ denotes the gradient of $\delta_X(y)$, defined in Appendix A3.

Proposition 2.4. *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, $y \in \mathbb{R}^d \backslash X$ and δ_X differentiable at y. Then $y \in U(X)$ and*

$$grad\delta_X(y) = \frac{y - \pi_X(y)}{\delta_X(y)}.$$

Proof. We first prove the equality and later that $y \in U(X)$.

We take $a \in X$ such that $\delta_X(y) = \|y - a\|$.

We consider the line segment $(1-t)y + ta$, for $0 \leq t \leq 1$. Hence,

$\delta_X((1-t)y+ta) := inf_{x \in X} \ \|(1-t)y + ta - x\| = inf_{x \in X} \ \|y - t(y-a) - x\| \geq$
$\geq inf_{x \in X} \ (\|y - x\| - t\|y - a\|) = \delta_X(y) - t\delta_X(y) = (1-t)\delta_X(y)$

and we also have

$\delta_X((1-t)y+ta) := inf_{x \in X} \ \|(1-t)y + ta - x\| = inf_{x \in X} \ \|(1-t)y + ta + tx -$
$- tx - x\| = inf_{x \in X} \ \|(1-t)y + ta - tx - (1-t)x\| \leq (1-t) inf_{x \in X} \ \|y - x\| +$
$+ t \, inf_{x \in X} \ \|a - x\| = (1-t)\delta_X(y)$

(since $\delta_X(a) = 0$ as $a \in X$).

Therefore, by the two previous inequalities, we get

$$\delta_X((1-t)y + ta) = (1-t)\delta_X(y). \quad (2.4)$$

Now we derivate (2.4) with respect to t:

$grad\delta_X((1-t)y+ta) \bullet (-y+a) = -\delta_X(y)$, equivalently, $\|grad\delta_X((1-t)y+ta)\|\|a-y\|\cos\theta(t) = -\delta_X(y)$, and since $\|a-y\| = \delta_X(y)$, the two terms cancel and we get

$$\|grad\delta_X((1-t)y+ta)\|\cos\theta(t) = -1. \tag{2.5}$$

For $t=0$, we have the vectors $grad\delta_X(y)$ and $(a-y)$ over the same line (which is the straight line through y and a). The vector $(a-y)$ points towards a and $grad\delta_X(y)$ (which is well-defined as δ_X is differentiable at y by hypothesis), points towards y. That is because the gradient points to the direction of increase, and at the point y, the distance $\delta_X(y)$ is bigger than for any other value $\delta_X((1-t)y+ta)$ (since $y \notin X$ and $a \in X$, so δ_X decreases as long as it gets closer to a).

Therefore, such two vectors can be written as $a - y = -\lambda grad\delta_X(y)$ for some constant $\lambda \in \mathbb{R}$, and they form an angle $\theta(0) = 180°$. So, $\cos\theta(0) = -1$ and the expression (2.5) (for $t=0$) becomes

$$\|grad\delta_X(y)\| = 1.$$

We return now to the first expression of the derivative setting $t=0$, $grad\delta_X(y) \bullet (-y+a) = -\delta_X(y)$. Hence,

$$grad\delta_X(y) \bullet \frac{y-a}{\delta_X(y)} = 1. \tag{2.6}$$

In addition, it is known that for any two unit vectors $v, w \in \mathbb{R}^d$ in the same direction such that $v \bullet w = \|v\|\|w\|\cos\theta = 1$ (since $\theta = 0$), then $v = w$.

So, since we have proven that $grad\delta_X(y)$ is a unit vector, and so is $\frac{y-a}{\delta_X(y)}$ as $\delta_X(y) = \|y-a\|$, by (2.6) it follows that

$$grad\delta_X(y) = \frac{y-a}{\delta_X(y)}$$

(since we are changing the direction of the vector $a-y$, and now $y-a$ and $grad\delta_X(y)$ form an angle of $\theta = 0$).

We now prove that $y \in U(X) := \{y \in \mathbb{R}^d \mid \exists!$ point of X nearest to y$\}$, by showing that $y \in \mathbb{R}^d \backslash X$ has a unique nearest point in X.

Let $a \in X$ such that $\delta_X(y) = \|y-a\|$, and let $a' \in X$ such that $\delta_X(y) = \|y - -a'\|$. By the formula of the statement of this proposition that we have just proven, we have $grad\delta_X(y) = \frac{y-a}{\delta_X(y)}$ and $grad\delta_X(y) = \frac{y-a'}{\delta_X(y)}$, therefore $a = a'$, and we can denote such point as $\pi_X(y)$, hence $y \in U(X)$.

□

Definition 2.5. *A subspace* $N \subseteq \mathbb{R}^d$ *is a* ***set of Lebesgue measure zero*** *if for every* $\varepsilon > 0$*, it can be covered by countably many products of d-dimensional unit cubes and the volume of the union of such d-cubes is less than* ε*.*

The next theorem is from [7] (Theorem 3.1.6).

Theorem 2.6 (Rademacher's Theorem)**.** *If* $U \subseteq \mathbb{R}^d$ *is open and* $f : U \to \mathbb{R}^m$ *is Lipschitz, then* f *is differentiable almost everywhere in* U*, that is, the points in* U *at which* f *is not differentiable form a set of Lebesgue measure zero.*

Remark 2.7. *We have already seen in Corollary 1.11 that for* $X \subseteq \mathbb{R}^d$ *with positive reach* τ*, the subspace* Tub_τ *is open in* $\mathbb{R}^d$*. Also, by definition of such tubular set, we always have that* $X \subseteq Tub_\tau$*.*

By Claim 1.9, if $X \subseteq \mathbb{R}^d$ *has positive reach, then* X *is closed. Hence,*

$$Tub_\tau \backslash X = Tub_\tau \bigcap (\mathbb{R}^d \backslash X)$$

is open in $\mathbb{R}^d$*, since a finite intersection of open sets is open.*

We slightly modify the original statements of [6] (Theorem 4.8, parts 5 and 6), because there it is taken the interior of the sets $U(X)$ and $U(X)\backslash X$, but since $Tub_\tau \subseteq U(X)$, we restrict such results to the sets Tub_τ and $Tub_\tau\backslash X$ because these are the ones that we need for the next results, and by Corollary 1.11 and Remark 2.7, they are open in $\mathbb{R}^d$, so their interiors are the sets themselves. We give the corresponding next two results:

Proposition 2.8. *The distance function* $\delta_X : \mathbb{R}^d \to \mathbb{R}$ *is differentiable on* $Tub_\tau\backslash X$*.*

Proof. By Proposition 1.10, δ_X is Lipschitz on all $\mathbb{R}^d$, and we can take its restriction to the subspace $Tub_\tau\backslash X \subseteq \mathbb{R}^d$, which is open by Remark 2.7. Hence, by Theorem 2.6, δ_X is differentiable almost everywhere in $Tub_\tau\backslash X$.

We take $y \in Tub_\tau\backslash X$ such that δ_X is differentiable at y. Then, by Proposition 2.4,

$$\left(\frac{\partial \delta_X}{\partial x_1}(y), \dots, \frac{\partial \delta_X}{\partial x_d}(y)\right) = grad\delta_X(y) = \frac{y - \pi_X(y)}{\delta_X(y)}$$

and by Proposition 2.2, π_X is continuous on Tub_τ. So,

$$\frac{y - \pi_X(y)}{\delta_X(y)} : Tub_\tau\backslash X \to \mathbb{R}^d$$

is a continuous map.

Hence, since the partial derivatives of δ_X at y can be defined almost everywhere in the open subset $Tub_\tau\backslash X$, and we can extend them by the continuous function

$\dfrac{y-\pi_X(y)}{\delta_X(y)}$, then by Lemma 4.7 in [6], we precisely get that $\dfrac{\partial\delta_X}{\partial x_1}(y),\ldots,\dfrac{\partial\delta_X}{\partial x_d}(y)$ exist for all $y \in Tub_\tau\backslash X$, which is the definition of δ_X differentiable on $Tub_\tau\backslash X$. Moreover, such partial derivatives are continuous, which is the definition of being continuously differentiable.

□

Proposition 2.9. *If $X \subseteq \mathbb{R}^d$ has positive reach τ and there is a point $a \in X$, a vector $v \in \mathbb{R}^d$ and $0 < \varepsilon = sup\{t \in \mathbb{R} \mid \pi_X(a+tv) = a\} < \infty$, then $a + \varepsilon v \notin Tub_\tau$.*

Proof. Let $v \in \mathbb{R}^d$ such that $\|v\| = 1$, and suppose that $y := a+\varepsilon v \in Tub_\tau$, where $y \notin X$ (since $\pi_X(y) = a$ by how we have defined y, and so $\delta_X(y) = \|y - \pi_X(y)\| =$ $= \|a + \varepsilon v - a\| = \varepsilon > 0$, which means that $y \notin X$ as the distance from a point in a set to such set, is always zero).

By Proposition 2.2, π_X is continuous, and by Proposition 2.8, δ_X is differentiable at $y \in Tub_\tau\backslash X$. Hence, we can apply Proposition 2.4 and get

$$grad\delta_X(y) = \frac{y-\pi_X(y)}{\delta_X(y)} = \frac{a+\varepsilon v - a}{\varepsilon} = v.$$

Let $C : (-r, r) \longrightarrow Tub_\tau\backslash X$, for a real number $r > 0$, be any continuous map solving the differential equation

$$C' = (grad\delta_X) \circ C \quad \text{with initial condition } C(0) = y.$$

By Peano Existence Theorem (for a reference, see Wikipedia), such function C exists for such initial value problem, giving a local solution, where $(-r, r)$ is a neighborhood of 0 in $\mathbb{R}$.

For $s \in (-r, r)$, $\|s\| < r$ and

$$\|C'(s)\| = \|grad\delta_X(C(s))\| = 1 \tag{2.7}$$

so, by the Chain Rule and by (2.7),

$$(\delta_X \circ C)'(s) = grad\delta_X(C(s)) \bullet C'(s) = C'(s) \bullet C'(s) = \|C'(s)\|\|C'(s)\|cos(0) = 1. \tag{2.8}$$

Now, for $-r < p < q < r$, we take an arbitrary point $x \in X$ and by definition of π_X, (2.7), (2.8) and the Fundamental Theorem of Calculus ([20], Corollary in page 287), we get:

$$\int_p^q \|C'(s)\|ds = \int_p^q 1ds = \int_p^q (\delta_X \circ C)'(s)ds = (\delta_X \circ C)(q) - (\delta_X \circ C)(p) \leq \|C(q) - x\| -$$

$-\|C(p) - x\| \leq \|C(q) - C(p)\|$.

It follows that the curve C parameterizes a straight line segment between $C(-r)$ and $C(r)$, in the direction $C'(0) = grad\delta_X(y) = v$.

We take $0 < s < r$, and we define $t := \varepsilon + s > \varepsilon$, hence $C(s)$ is a point between $C(0) = y$ and $C(r)$, and since $C'(0) = v$, we have that $C(s) = y + sv = a + \varepsilon v + sv = a + tv$.

Since $\|v\| = 1$ and $\delta_X(y) = \varepsilon$,

$$\delta_X(C(s)) = \delta_X(y + sv) = \delta_X(y) + s = \varepsilon + s = t = \|a + tv - a\| = \|C(s) - a\|.$$

We defined the map C such that $C(s)$ belongs to Tub_τ for any $s \in (-r, r)$, so $\delta_X(C(s)) = \|C(s) - \pi_X(C(s))\|$, and therefore $a = \pi_X(C(s))$, where $C(s) = a + tv$.

This implies that $t \in \{t' \in \mathbb{R} \mid \pi_X(a + t'v) = a\}$, but $t > \varepsilon$ where ε is the supremum of such set. Hence, we have a contradiction and we conclude with $a + \varepsilon v \notin Tub_\tau$. □

The following three results correspond to Claim 9, Lemma 10 and Claim 11 in [12]. The first one presents a simple calculation of the distance from one vertex of a triangle to another point lying on the edge formed by the other two vertices:

Claim 2.10. *Let $x, y, z \in \mathbb{R}^d$ and $\lambda \in [0, 1]$, then*

$$\|(\lambda y + (1 - \lambda)z) - x\| = \sqrt{\lambda\|y - x\|^2 + (1 - \lambda)\|z - x\|^2 - \lambda(1 - \lambda)\|y - z\|^2}.$$

Proof. The distance from $\lambda y + (1 - \lambda)z$ to x can be expanded as

$\|(\lambda y + (1 - \lambda)z) - x\|^2 = \|(\lambda(y - x) + (1 - \lambda)(z - x)\|^2 = \lambda^2\|y - x\|^2 + (1 - \lambda)^2\|z - x\|^2 + 2\lambda(1 - \lambda)\langle y - x, z - x\rangle$.

Then, applying the identity $2\langle y - x, z - x\rangle = \|y - x\|^2 + \|z - x\|^2 - \|y - z\|^2$ to this last expansion and after some calculations, we get

$\|(\lambda y + (1 - \lambda)z) - x\|^2 = \lambda\|y - x\|^2 + (1 - \lambda)\|z - x\|^2 - \lambda(1 - \lambda)\|y - z\|^2$.

And the claim directly follows. □

Given a straight line segment whose endpoints are in $X \subseteq \mathbb{R}^d$, the following lemma gives a bound on the distance from any point on that segment to its projection on X.

Lemma 2.11. *Let $X \subseteq \mathbb{R}^d$ with positive reach τ and $y, z \in X$. Let $u := \lambda y + (1 - \lambda)z$, for $\lambda \in [0, 1]$, be such that $\delta_X(u) < \tau$. Then*

$$\delta_X(u) = \|\pi_X(u) - u\| \leq \tau - \sqrt{max\{\tau^2 - \lambda(1 - \lambda)\|y - z\|^2, 0\}}.$$

Proof. If $\pi_X(u) = u$, then there is nothing to prove.

Suppose $\pi_X(u) \neq u$, and let

$$w := \pi_X(u) + \tau \frac{u - \pi_X(u)}{\|u - \pi_X(u)\|}.$$

Then, we have that $\|w - \pi_X(u)\| = \tau$ and $w - u = \left(\frac{\tau - \|\pi_X(u) - u\|}{\|\pi_X(u) - u\|} \right) (u - - \pi_X(u))$. Therefore,

$$\|u - \pi_X(u)\| = \|w - \pi_X(u)\| - \|w - u\| \tag{2.9}$$

since geometrically, the points w, u and $\pi_X(u)$ are over the same line, with $d_E(w, \pi_X(u)) = \tau$, $d_E(w, u) := \epsilon > 0$ and $d_E(u, \pi_X(u)) = \tau - \epsilon < \tau$ by the hypothesis $\delta_X(u) < \tau$.

We have the following equality

$$\pi_X \left(\pi_X(u) + \|u - \pi_X(u)\| \frac{u - \pi_X(u)}{\|u - \pi_X(u)\|} \right) = \pi_X(u).$$

Now we define a point

$$w_r := \pi_X(u) + r \frac{u - \pi_X(u)}{\|u - \pi_X(u)\|} \quad \text{with } r < \tau.$$

We see that $\delta_X(w_r) = \|w_r - \pi_X(w_r)\| \leq \|w_r - \pi_X(u)\| = r < \tau$, which by definition of tubular set means that

$$w_r = \pi_X(u) + r \frac{u - \pi_X(u)}{\|u - \pi_X(u)\|} \in Tub_\tau. \tag{2.10}$$

We apply by contraposition Proposition 2.9 to (2.10), hence, $r < sup\{t \mid \pi_X(\pi_X(u) + + tv) = \pi_X(u)\}$ with

$$v = \frac{u - \pi_X(u)}{\|u - \pi_X(u)\|}.$$

We pick $s \in \{t \mid \pi_X(\pi_X(u) + tv) = \pi_X(u)\}$ such that $s > r$.

Now we apply Proposition 2.3, with such a as our $\pi_X(u)$, to $\pi_X(\pi_X(u) + sv) = = \pi_X(u)$, getting that $sv \in P$ where P is a convex set. We observe that also $0 \in P$. Hence, for $r \in [0, s]$, we have the convex combination $rv = \alpha(sv) + (1 - \alpha)0$, for $\alpha \in [0, 1]$. We conclude that $rv \in P$, so again by Proposition 2.3,

$$\pi_X(w_r) = \pi_X \left(\pi_X(u) + r \frac{u - \pi_X(u)}{\|u - \pi_X(u)\|} \right) = \pi_X(u)$$

for all $r < \tau$.

We also have $B_{\mathbb{R}^d}(w, \tau) \cap X = \emptyset$.

To prove this empty intersection, we notice that it holds if and only if $d(w, X) \geq \tau$. We see that

$$\lim_{r \to \tau_-} d(w_r, X) = \lim_{r \to \tau_-} \|w_r - \pi_X(w_r)\| = \lim_{r \to \tau_-} \|w_r - \pi_X(u)\| = \lim_{r \to \tau_-} r =$$

$$= \tau.$$

So, $d(w, X) = \lim_{r \to \tau_-} d(w_r, X) = \tau$.

We can conclude that y and z must not be in $B_{\mathbb{R}^d}(w, \tau)$ because they belong to X by hyphotesis, hence $\|w - y\| \geq \tau$ and $\|w - z\| \geq \tau$.

By applying Claim 2.10 to $\|w - u\|$, we get

$$\|w - u\| = \sqrt{\lambda\|w - y\|^2 + (1 - \lambda)\|w - z\|^2 - \lambda(1 - \lambda)\|y - z\|^2} \geq \sqrt{\tau^2 - \lambda(1 - \lambda)\|y - z\|^2}. \tag{2.11}$$

Then, from (2.11) and since $\|w - \pi_X(u)\| = \tau$, the expression (2.9) is bounded as follows:

$$\|u - \pi_X(u)\| \leq \tau - \sqrt{max\{\tau^2 - \lambda(1 - \lambda)\|y - z\|^2, 0\}}.$$

□

Claim 2.12. *Let $X \subseteq \mathbb{R}^d$ be a set with positive reach τ, $y, z \in X$, $\lambda \in [0, 1]$ and $u := \lambda y + (1 - \lambda)z$. Let $x \in \mathbb{R}^d$ with $\|x - y\| < \tau$ and $\|x - z\| < \tau$. Then*

$$\|x - \pi_X(u)\| \leq \sqrt{\lambda\|y - x\|^2 + (1 - \lambda)\|z - x\|^2}.$$

Proof. Let $r := \sqrt{\lambda\|y - x\|^2 + (1 - \lambda)\|z - x\|^2} < \sqrt{\lambda\tau^2 + (1 - \lambda)\tau^2} = \tau$.

From Claim 2.10 we have the following equality

$$\|x - u\| = \sqrt{\lambda\|y - x\|^2 + (1 - \lambda)\|z - x\|^2 - \lambda(1 - \lambda)\|y - z\|^2} = \sqrt{r^2 - \lambda(1 - \lambda)\|y - z\|^2} \tag{2.12}$$

We also notice that by the Triangle Inequality and by hypothesis:

$$\|y - z\| \leq \|y - x\| + \|x - z\| < 2\tau$$

and as illustrated in the following figure, by definition of δ_X,

$$\delta_X(u) \leq min\{\|u - y\|, \|u - z\|\} < \tau.$$

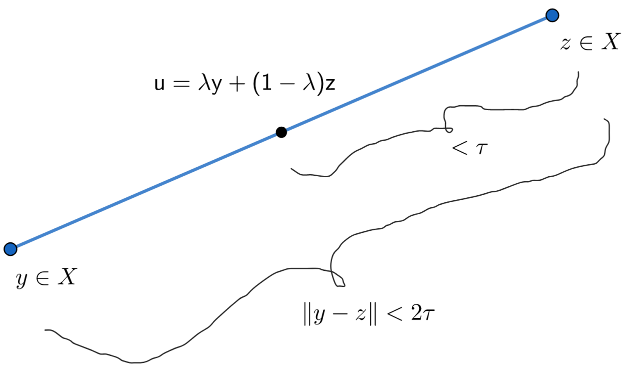


Figure 2.1: (Every point in the line segment between two points in X such that they are at a distance less than the reach from a chosen point in $\mathbb{R}^d$, belongs to Tub_τ and therefore has a unique nearest point in X).

Hence, we can apply Lemma 2.11 and obtain the following inequality:

$$\|u - \pi_X(u)\| \leq \tau - \sqrt{max\{\tau^2 - \lambda(1-\lambda)\|y-z\|^2, 0\}}. \tag{2.13}$$

Then, from (2.12), (2.13) and by applying the Triangle Inequality to $\|x - \pi_X(u)\|$, we get:

$$\|x-\pi_X(u)\| \leq \|x-u\|+\|u-\pi_X(u)\| \leq \sqrt{r^2 - \lambda(1-\lambda)\|y-z\|^2}+\tau-\sqrt{\tau^2 - \lambda(1-\lambda)\|y-z\|^2} =$$

$$= r-\lambda(1-\lambda)\|y-z\|^2 \left(\frac{1}{r+\sqrt{r^2 - \lambda(1-\lambda)\|y-z\|^2}} - \frac{1}{\tau+\sqrt{\tau^2 - \lambda(1-\lambda)\|y-z\|^2}}\right).$$

The product $\lambda(1-\lambda)\|y-z\|^2$ is always non-negative, and since $r < \tau$, the subtraction $\left(\frac{1}{r+\sqrt{r^2 - \lambda(1-\lambda)\|y-z\|^2}} - \frac{1}{\tau+\sqrt{\tau^2 - \lambda(1-\lambda)\|y-z\|^2}}\right)$ is always positive. Hence,

$$\|x - \pi_X(u)\| \leq r.$$

□

2.2 Reconstruction of the homotopy type of a Euclidean subspace.

In this section we show a theorem from [12] (Theorem 5) that establish when finite intersections of Euclidean balls intersected with X are contractible (it is in fact when the radii of the balls are less or equal than the reach of X); and the main reconstruction result, its corollary ([12]. Corollary 6), taking an infinite set $A \subseteq \mathbb{R}^d$ rather than exclusively a finite one, and giving a new proof for this version.

Theorem 2.13. *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, $A \subseteq \mathbb{R}^d$ be a finite subspace and $\{r_j \mid a_j \in A\}$ be a set of radii. If $max_{a_j \in A}\, r_j \leq \tau$, then $\bigcap_{a_j \in A} B_X(a_j, r_j)$ is either contractible or empty.*

Proof. Let fix $a_\alpha \in A$ and $y_1, y_2 \in B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \cap X$.

By definition of subspace ball for $X \subseteq \mathbb{R}^d$, $B_X(a_\alpha, r_\alpha) = B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \cap X$.

We define the straight line segment connecting the points y_1 and y_2 in such Euclidean ball, since it is convex:

$$\begin{array}{cccc} l: & [0,1] & \longrightarrow & B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \\ & & & \\ & t & \mapsto & l(t) = ty_1 + (1-t)y_2 \end{array}$$

Now, we want to define the curve between the projections to X of the points lying on the line segment $l(t)$, where such projections are the images of the map $\pi_X : Tub_\tau \to X$.

In order to do that, first we need to check that $l(t) \in Tub_\tau = \{x \in \mathbb{R}^d \mid \delta_X(x) <$

$< \tau\}$, so that $\pi_X(l(t))$ is well-defined.

Since $y_1, y_2 \in B_{\mathbb{R}^d}(a_\alpha, r_\alpha)$ with $r_\alpha \leq \tau$, and by the Triangle Inequality, we get that

$$\|y_1 - y_2\| \leq \|y_1 - a_\alpha\| + \|a_\alpha - y_2\| < 2\tau.$$

We have that $l(t)$ is a point in the line segment connecting y_1 and y_2, for any $t \in [0,1]$, so as illustrated in Figure 2.1 from Claim 2.12, either $\|l(t) - y_1\| < \tau$ or $\|l(t) - y_2\| < \tau$.

So, since $y_1, y_2 \in X$,

$$\delta_X(l(t)) \leq min\{\|l(t) - y_1\|, \|l(t) - y_2\|\} < \tau$$

and we have by definition that $l(t) \in Tub_\tau$.

Therefore, let

$$\begin{array}{cccc} \gamma_{y_1,y_2}: & [0,1] & \longrightarrow & X \\ & t & \mapsto & \pi_X(l(t)) \end{array}$$

where $\pi_X : Tub_\tau \to X$ is the projection from a point $x \in Tub_\tau$ to its nearest point in X. This projection is continuous by Proposition 2.2, hence γ_{y_1,y_2} is also continuous. Moreover, since $l(t) \in Tub_\tau$ with τ positive, the projection of $l(t)$ for all $t \in [0,1]$ is well-defined, and so is γ_{y_1,y_2}.

We want to check that γ_{y_1,y_2} is contained in $B_{\mathbb{R}^d}(a_\alpha, r_\alpha)$ for all $t \in [0,1]$.

In order to do this, we apply Claim 2.12, where $a_\alpha \in A \subseteq \mathbb{R}^d$ and $y_1, y_2 \in$ $\in X \cap B_{\mathbb{R}^d}(a_\alpha, r_\alpha)$ with $\|a_\alpha - y_1\| < r_\alpha \leq \tau$, $\|a_\alpha - y_2\| < r_\alpha \leq \tau$, and we get:

$\|a_\alpha - \gamma_{y_1,y_2}(t)\| = \|a_\alpha - \pi_X(l(t))\| \leq \sqrt{\lambda\|a_\alpha - y_1\|^2 + (1-\lambda)\|a_\alpha - y_2\|^2} <$ $< \sqrt{\lambda r_\alpha^2 + (1-\lambda) r_\alpha^2} = r_\alpha$.

Hence, $\gamma_{y_1,y_2}(t) \in B_{\mathbb{R}^d}(a_\alpha, r_\alpha)$ for all $t \in [0,1]$, so $\gamma_{y_1,y_2}(t) \in X \cap B_{\mathbb{R}^d}(a_\alpha, r_\alpha)$ for all $t \in [0,1]$.

Now we fix $y_0 \in \bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \bigcap X$ and we give the following continuous and well defined map:

$$\begin{array}{cccc} H: & (\bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \bigcap X) \times [0,1] & \longrightarrow & \bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \bigcap X \\ & (y,t) & \mapsto & \gamma_{y_0,y}(t) = \pi_X(ty_0 + (1-t)y) \end{array}$$

We see that this is a homotopy, since $H(y,0) = y$ for all $y \in \bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \bigcap$ $\bigcap X$, which is the identity map in $\bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \bigcap X$; and $H(y,1) = y_0$ for all $y \in \bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \bigcap X$, which is the constant map c_{y_0}.

Hence, $id_{\bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \cap X} \simeq c_{y_0}$.

So by definition (Appendix A1), $\bigcap_{a_\alpha \in A} B_{\mathbb{R}^d}(a_\alpha, r_\alpha) \cap X$ is contractible.

□

In this theorem 2.13 we have shown that a finite intersection of subspace balls is either empty, or contractible, in the case of being non-empty.

Note that we have not stated that we have a good cover, because we did not suppose at any moment that such collection of subspace balls covers our space $X \subseteq \mathbb{R}^d$. Such assumption is only included in the next corollary 2.14, and there we apply this theorem to such cover, getting a good cover from where to reconstruct the homotopy type of X through the Nerve Lemma 1.20.

Corollary 2.14. *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, $A \subseteq \mathbb{R}^d$ and suppose that $\{B_X(a_j, r_j)\}_{a_j \in A}$ is a cover of X, where $r = \{r_j \mid a_j \in A\}$ is a set of radii. If $sup_{a_j \in A}\, r_j \leq \tau$, then X is homotopy equivalent to the geometric realization of $\mathscr{C}_X(A, r)$.*

Proof. We need to show that $\{B_X(a_j, r_j)\}_{a_j \in A}$ is a good cover of X. In order to do that, we take any finite subset $\sigma \subseteq A$, and since $max_{a_j \in \sigma}\, r_j \leq \tau$, we apply Theorem 2.13, setting that A as our finite set σ. Therefore, we get that $\bigcap_{a_j \in \sigma} B_X(a_j, r_j)$ is either contractible or empty.

Also, $x \in \bigcap_{a_j \in \sigma} B_X(a_j, r_j)$ if and only if $x \in B_X(a_j, r_j)$ for every $a_j \in \sigma$.

To check that each of the subspace balls forming such cover is contractible, we take a finite subset of A consisting of just one point, that is, $\sigma = \{a_j\} \subseteq A$. Then, $x \in \bigcap_{a_j \in \{a_j\}} B_X(a_j, r_j)$ if and only if $x \in B_X(a_j, r_j)$.

Hence, $B_X(a_j, r_j) = \bigcap_{a_j \in \{a_j\}} B_X(a_j, r_j)$, where $r_j \leq \tau$, so by Theorem 2.13, such subspace ball is contractible.

Therefore, we have by definition that $\{B_X(a_j, r_j)\}_{a_j \in A}$ is a good cover of X, so by the Nerve Lemma 1.20,

$$|N(\{B_X(a_j, r_j)\}_{a_j \in A})| \simeq X.$$

Then, by identifying $N(\{B_X(a_j, r_j)\}_{a_j \in A})$ with the Čech complex $\mathscr{C}_X(A, r)$ (as explained in Example 1.24), we get that

$$|\mathscr{C}_X(A, r)| \simeq X.$$

□

2.3 Reconstruction of the homotopy type of a Euclidean subspace using the directed Hausdorff distance.

Here we present a new theorem, Theorem 2.16, which reconstructs the homotopy type of an underlying Euclidean subspace using the directed Hausdorff distance and a Čech complex of an appropriate radius. It is a strong result, because it recovers the homotopy type without having to include in its statement that we have a cover of such subspace, since we find one along the proof which is a good cover. We also mention some future work and a discussion regarding the well-known result Proposition 2.18 ([17] Proposition 3.1).

The intuitive idea of the directed Hausdorff distance is that two sets are close in the Hausdorff distance if every point of either set is near to some point of the other set. Hence, the Hausdorff distance can be defined as the supreme distance of all the distances from a point in one set X to the nearest point in the other set Y.

Definition 2.15. *Let $X, Y \subseteq (M, d)$ be two metric subspaces. We define the* ***directed Hausdorff distance*** *between X and Y by:*

$$\overrightarrow{d_H}(X, Y) := sup_{x \in X}\, inf_{y \in Y}\ d(x, y),$$

Where $\overrightarrow{d_H}(X, Y) = 0$ if $X \subseteq Y$.

Theorem 2.16. *Let $X \subseteq \mathbb{R}^d$ with positive reach τ, and let $A \subseteq \mathbb{R}^d$ be compact. If $\alpha \in \left(\overrightarrow{d_H}(X, A), \tau\right]$, then the geometric realization of $\mathscr{C}_X(A, \alpha)$ is homotopy equivalent to X.*

Proof. We notice that $\left(\overrightarrow{d_H}(X, A), \tau\right] \neq \emptyset$ if and only if $\overrightarrow{d_H}(X, A) \leq \tau$.

We first check that the set of subspace balls $\{B_X(a_j, \alpha)\}_{a_j \in A}$ forms a cover of X, i.e, $X \subseteq \bigcup_{a_j \in A} B_X(a_j, \alpha)$.

For every $x \in X$, we have the following inequalities by definition of the directed Hausdorff distance and by hypothesis:

$$inf_{a_j \in A}\ d_E(x, a_j) \leq \overrightarrow{d_H}(X, A) < \alpha.$$

We take the restriction of the Euclidean distance to $\{x\} \times A \subseteq \mathbb{R}^d \times \mathbb{R}^d$, $d_E : \{x\} \times A \to \mathbb{R}$, which is a continuous function defined on a compact subspace, since the topological product of two spaces is compact if and only if each of them is compact ([21], Theorem 13.21).

A continuous function defined on a compact reaches its infimum ([21], Corollary 13.18), hence, there exists an element $a_0 \in A$ such that

$$d_E(x, a_0) = inf_{a_j \in A}\ d_E(x, a_j) < \alpha.$$

So, by definition of subspace ball, $x \in B_X(a_0, \alpha)$ and $x \in \bigcup_{a_j \in A} B_X(a_j, \alpha)$.

Since $\{B_X(a_j, \alpha)\}_{a_j \in A}$ is a cover of X and $\alpha \leq \tau$, we apply Corollary 2.14 and get that $|\mathscr{C}_X(A, \alpha)| \simeq X$. Moreover, we implicitly obtain that $\{B_X(a_j, \alpha)\}_{a_j \in A}$ is a good cover of X.

□

Remark 2.17. *Theorem 2.16 also holds if we take different radii, that is, if $r = \{r_j \mid a_j \in A\}$ such that $inf_{a_j \in A}\ r_j > \overrightarrow{d_H}(X, A)$ and $sup_{a_j \in A}\ r_j \leq \tau$, instead of taking a unique value $\alpha \in \left(\overrightarrow{d_H}(X, A), \tau\right]$.*

The only change in the proof is that there exists $a_0 \in A$ such that $d_E(x, a_0) = = inf_{a_j \in A}\ d_E(x, a_j) < inf_{a_j \in A}\ r_j \leq r_0$, so $x \in B_X(a_0, r_0)$ and $x \in \bigcup_{a_j \in A} B_X(a_j, r_j)$. Hence, $\{B_X(a_j, r_j)\}_{a_j \in A}$ is a cover of X and since $sup_{a_j \in A}\ r_j \leq \tau$, we apply Corollary 2.14 and get that $|\mathscr{C}_X(A, r)| \simeq X$.

Some future work can be to find new bounds for the radii that are larger than the reach of the subspace, so that we still recover its homotopy type. For that, in the original paper [12] (Theorem 5 and Corollary 6), our main reference in Chapter 2, it is also discussed the case when $A \subseteq X$, since for this particular case, Theorem 2.13 and Corollary 2.14 also hold for a bound of the radii slightly bigger than the reach $\tau > 0$ of the subspace X, which is $sup_{a_j \in A}\ r_j \leq \sqrt{2}\tau$.

We can also look at the new paper [13] from the same authors of [12], where they give the bound $r_j \leq \sqrt{\tau^2 + (\tau - \delta_X(a_j))^2}$ for every $a_j \in A$, providing a new version for Theorem 2.13 and Corollary 2.14 ([13] Theorem 9 and Corollary 10).

2.3.1 Discussion and contributions.

We now compare our Theorem 2.16 with the well-known result from [17] (Proposition 3.1), which states the following:

Proposition 2.18. *For X a compact submanifold in $\mathbb{R}^d$ with positive reach τ, and A a finite subspace in $\mathbb{R}^d$ such that $\overrightarrow{d_H}(X, A) = \epsilon < \sqrt{\frac{3}{20}}\ \tau$, then for all $\alpha \in (2\epsilon, \sqrt{\frac{3}{5}}\ \tau)$, the open set $U := \bigcup_{a_j \in A} B_{\mathbb{R}^d}(a_j, \alpha)$ deformation retracts into X.*

Therefore, $X \simeq \bigcup_{a_j \in A} B_{\mathbb{R}^d}(a_j, \alpha)$.

First of all, we notice that $\bigcup_{a_j \in A} B_{\mathbb{R}^d}(a_j, \alpha)$ is homotopy equivalent to $|\mathscr{C}_{\mathbb{R}^d}(A, \alpha)|$ by the convex version of the Nerve lemma 1.22.

Hence, this result is showing us that under such conditions, the ambient Čech complex $\mathscr{C}_{\mathbb{R}^d}(A, \alpha)$ recovers the homotopy type of a compact submanifold $X \subseteq \mathbb{R}^d$.

Let us proceed with our discussion.

1. We have found in our theorem 2.16 a larger interval $\left(\overrightarrow{d_H}(X, A), \tau\right]$ which contains $(2\epsilon, \sqrt{\frac{3}{5}}\ \tau)$ from Proposition 2.18, so we have a wider range of choice from where to pick the values α that will determine the radius of the subspace balls which form the Čech complexes we reconstruct the homotopy type of X from.

2. Proposition 2.18 works with a somehow trivial good cover of X, $\{B_{\mathbb{R}^d}(a_j, \alpha)\}_{a_j \in A}$, since it consists of Euclidean open balls which are convex, and therefore contractible (as we have shown in Proposition 1.21). We instead construct a

cover $\{B_X(a_j,\alpha)\}_{a_j\in A}$ in the proof of Theorem 2.16, where X does not need to be convex, and by the previous results presented along this chapter, we show that it is indeed a good cover.

3. We work with a more general Euclidean subspace X, instead of just compact submanifolds in a Euclidean space, even though in both cases we require the reach to be positive.

4. We proved that the sample A can be infinite, as long as it is compact. This might only be relevant in more theoretical contexts, since it is more convenient to stick to finite samples in applied topological problems, as these are easier to compute. Also, recovering the homotopy type from $\mathscr{C}_{\mathbb{R}^d}(A,\alpha)$ by Proposition 2.18 makes it possible to do computations, since we know the spaces A and $\mathbb{R}^d$; whereas recovering the homotopy type from $\mathscr{C}_X(A,\alpha)$ does not allow us to compute the topological space $|\mathscr{C}_X(A,\alpha)|$, since the subspace X is unknown.

   However, it might be more interesting to work with this last simplicial complex theoretically, and approximate its computation through inclusions with other Čech complexes that are possible to compute. For example, if $A\subseteq X\subseteq\mathbb{R}^d$, then we have the following inclusions

$$\mathscr{C}_A(A,\alpha)\hookrightarrow\mathscr{C}_X(A,\alpha)\hookrightarrow\mathscr{C}_{\mathbb{R}^d}(A,\alpha)$$

   where $\mathscr{C}_A(A,\alpha)$ and $\mathscr{C}_{\mathbb{R}^d}(A,\alpha)$ are possible to compute. We will study this in the next chapter.

Chapter 3

Topological reconstruction. Geometric reconstruction of geodesic subspaces.

In this chapter we consider the topological reconstruction of a Euclidean subspace, that is, the reconstruction of its homology and homotopy groups.

In Section 3.1 we present inclusions of filtered Čech complexes, which form commutative diagrams (Proposition 3.3). We work with the length metric and the restriction of the Euclidean metric on $X \subseteq \mathbb{R}^d$, and by the distortion of X and Dowker's Theorem 3.10, we find homotopy equivalences and new commutative diagrams (Corollay 3.11).

In Section 3.2 we define geodesic subspace and we reconstruct the homotopy type of such subspace using its convexity radius and a dense enough sample (Lemma 3.18).

In Section 3.3 we define persistence modules and groups, and interleavings between them. We show interleavings between homology and homotopy groups constructed from Čech complexes by a geodesic subspace and a dense enough sample, understanding such groups as persistences (Proposition 3.25), and we conclude giving an isomorphism between fundamental groups of the geometric realizations of different Čech complexes, paying attention to the critical values of such persistences (Theorem 3.31), and an analogous isomorphism for the first homology groups with coefficients in an abelian group (Corollary 3.34) applying Hurewicz Theorem 3.33 and the Universal Coefficient Theorem for Homology A.25.

3.1 Filtrations via Čech complexes.

Now we study how different Čech complexes form a filtration, that is, a nested sequence of increasing subsets. We recall that the definition of Čech complex was given in 1.23.

Definition 3.1. *The* ***Čech filtration*** *of a metric space* (X, d) *is the collection of Čech complexes* $\{\mathscr{C}_X^d(X, p)\}_{p>0}$ *with the inclusions* $i_{p,q} : \mathscr{C}_X^d(X, p) \hookrightarrow \mathscr{C}_X^d(X, q)$, *for all* $0 < p < q$, *that are identities on the vertex sets.*

We say that $K := \bigcup_p \mathscr{C}_X^d(X, p)$ *is a filtered simplicial complex, where* $K_p := = \mathscr{C}_X^d(X, p)$ *for each* $p > 0$.

We also see in the next results other inclusions different from this previous definition, where the Čech complexes are for example of the form $\{\mathscr{C}_X^d(A, p)\}_{p>0}$ and $\{\mathscr{C}_X^d(X, p)\}_{p>0}$ for $A \subseteq (X, d)$, and therefore we can get inclusions of filtered complexes $\mathscr{C}_X^d(A, p) \hookrightarrow \mathscr{C}_X^d(X, p)$ for each $p > 0$. On the other hand, we work with two metrics on an Euclidean subspace X where one is greater than the other, obtaining an important parameter, called the distortion of X, which leads to some particular inclusions between Čech complexes, as shown in Corollary 3.11.

Lemma 3.2. *Let* $\alpha \in \mathbb{R}_{>0}$. *For* $A \subseteq X \subseteq (M, d)$ *and* $B \subseteq (M, d)$, *where* M *is a metric space with a distance* d, *we have the following two inclusions:*

$$\mathscr{C}_B^d(A, \alpha) \hookrightarrow \mathscr{C}_B^d(X, \alpha) \qquad \mathscr{C}_A^d(B, \alpha) \hookrightarrow \mathscr{C}_X^d(B, \alpha).$$

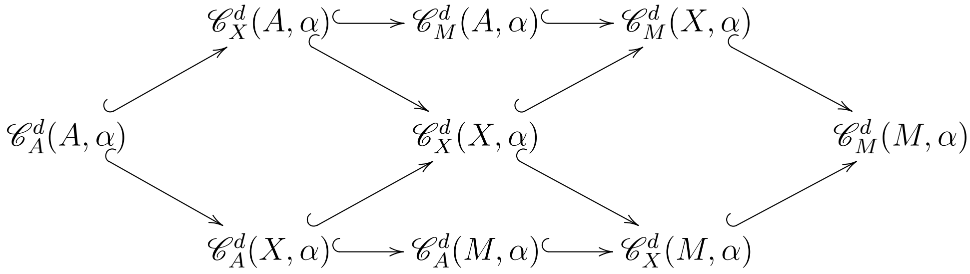
Proof. On the one hand, $\mathscr{C}_B^d(A, \alpha) = \{\sigma \subseteq A \mid \exists b \in B \text{ such that } d(b, a) < < \alpha \quad \forall a \in \sigma\}$ and $\mathscr{C}_B^d(X, \alpha) = \{\sigma \subseteq X \mid \exists b \in B \text{ such that } d(b, x) < \alpha \quad \forall x \in \sigma\}$, so since $A \subseteq X$, then every simplex $\sigma \subseteq A$ is a simplex in X.

Hence, $\mathscr{C}_B^d(A, \alpha) \hookrightarrow \mathscr{C}_B^d(X, \alpha)$.

On the other hand, $\mathscr{C}_A^d(B, \alpha) = \{\sigma \subseteq B \mid \exists a \in A \text{ such that } d(a, b) < \alpha \quad \forall b \in \in \sigma\}$ and $\mathscr{C}_X^d(B, \alpha) = \{\sigma \subseteq B \mid \exists x \in X \text{ such that } d(x, b) < \alpha \quad \forall b \in \sigma\}$, so since $A \subseteq X$, then for every element $a \in A$, we have that $a \in X$.

Hence, $\mathscr{C}_A^d(B, \alpha) \hookrightarrow \mathscr{C}_X^d(B, \alpha)$. □

Proposition 3.3. *Let* $\alpha \in \mathbb{R}_{>0}$. *For* $A \subseteq X \subseteq (M, d)$, *we obtain the following inclusions of Čech complexes, and therefore we have commutative diagrams.*



Proof. We just replace the spaces X, A and M in Lemma 3.2 accordingly, so that we have the same inclusions as in this proposition.

□

We proceed giving definitions related to paths, since we will be working in this chapter with a particular type of paths, which are the geodesics.

Definition 3.4. *Let X be a topological space and $I = [0, 1]$. A **path** in X is a continuous map $\gamma : I \to X$, whose endpoints are $\gamma(0)$ and $\gamma(1)$.*

*A **loop** in X is a path $\gamma : I \to X$ such that $\gamma(0) = \gamma(1)$.*

*The **length of a path** γ is defined by*

$$L(\gamma) := sup\, \Sigma_{i=1}^{k} \|\gamma(t_{i-1}) - \gamma(t_i)\|$$

where the supremum is taken over all partitions $\mathfrak{P} = \{0 = t_0, \ldots, t_i, \ldots, t_k = 1\}$ of I.

*A path γ is called **rectifiable** if $L(\gamma) < \infty$.*

*The space X is **path-connected** if for any two points $x, y \in X$, there exists a path $\gamma : I \to X$ connecting them, i.e, $\gamma(0) = x$ and $\gamma(1) = y$.*

From now on we follow the terminology of [5].

Definition 3.5. *For a path-connected subspace $X \subseteq \mathbb{R}^d$, we define the **length metric**, or **geodesic metric**, on X by*

$$d_L(x, y) := inf_{\gamma}\ L(\gamma) \quad \text{for all } x, y \in X$$

where the infimum is taken over all continuous paths $\gamma : I \to X$ connecting x and y.

Remark 3.6. *Since in the definition of length metric we take the length of a path, which is the supremum of Euclidean metrics, we have that*

$$d_L(x, y) \geq d_E(x, y) \quad \text{for all} \quad x, y \in X.$$

Now we detail how we must take samples of data when we are working with a subspace $X \subseteq \mathbb{R}^d$ that has both the length metric and the restriction of the Euclidean metric.

If we want to define the restriction of the length metric on the sample A, then A must be contained in X, since d_L is a metric specifically defined on X.

If we are working with the restriction of the Euclidean metric on $X \subseteq \mathbb{R}^d$, and we are just interested in a sample with the restriction of the Euclidean metric, then it suffices to have $A \subseteq \mathbb{R}^d$ (the same applies in a more general context where $X \subseteq (M, d)$).

Corollary 3.7. *Let $\alpha \in \mathbb{R}_{>0}$. For $A \subseteq (X, d)$, we have the following inclusions of Čech complexes (although we will be particularly interested in the length metric d_L of X, and its restriction to A).*

$$\begin{array}{ccccc} & & \mathscr{C}^d_X(A,\alpha) & & \\ & \nearrow & & \searrow & \\ \mathscr{C}^d_A(A,\alpha) & & & & \mathscr{C}^d_X(X,\alpha) \\ & \searrow & & \nearrow & \\ & & \mathscr{C}^d_A(X,\alpha) & & \end{array}$$

Proof. Follows from Lemma 3.2.

□

We now present an important sampling parameter, which is the distortion of a subspace $X \subseteq \mathbb{R}^d$ that has both the length metric and the restriction of the Euclidean one. Intuitively, this can be understood as the best Lipschitz constant $K \in \mathbb{R}_{>0}$ for the map $f : (X, d_E) \to (X, d_L)$ where $f(x) = x$, since by definition of f and by Remark 3.6, $d_E(f(x), f(y)) = d_E(x, y) \leq d_L(x, y)$ for all $x, y \in X$. So, there must exist $K > 0$ such that $d_L(x, y) \leq K d_E(x, y)$.

Definition 3.8. *The **distortion** of $X \subseteq \mathbb{R}^d$ is defined by*

$$\delta := sup_{x \neq y} \frac{d_L(x,y)}{d_E(x,y)} \quad \text{for } x, y \in X.$$

The distortion is bounded below by 1, and bounded above by $+\infty$, where both bounds can be achieved.

Remark 3.9. *We observe that by the definition of distortion being a supremum, we have that $\delta \geq \dfrac{d_L(x,y)}{d_E(x,y)}$ for any $x, y \in X$.*

We also have $d_L(x,y) \geq d_E(x,y)$ by Remark 3.6, so that we get the following inequalities:

$$d_E(x,y) \leq d_L(x,y) \leq \delta d_E(x,y) \qquad \textit{for all } x,y \in X. \tag{3.1}$$

Now we give the statement of Dowker's Theorem, which can be found together with its proof in [2] (Theorem 3, page 4).

For a subset $R \subseteq X \times Y$, the transpose of R, denoted by R^T, is defined by $R^T := \{(y,x) \in Y \times X \mid (x,y) \in R\} \subseteq Y \times X$.

Theorem 3.10 (Dowker's Theorem)**.** *Let $R \subseteq R' \subseteq X \times Y$ be subsets, and R^T, R'^T be the transpose subsets of R and R', respectively.*

Let $i : N(R) \hookrightarrow N(R')$ and $i^T : N(R^T) \hookrightarrow N(R'^T)$ be inclusions, where $N(R) := \{\sigma \in \mathfrak{P}(X) \mid \exists y \in Y$ such that $(x,y) \in R \quad \forall x \in \sigma\}$ is the Dowker simplicial complex of R.

Then, there exist the homotopy equivalences $|\Gamma_R| : |N(R)| \to |N(R^T)|$ and $|\Gamma_{R'}| : |N(R')| \to |N(R'^T)|$ such that the following diagram commutes up to homotopy (that is, $|\Gamma_{R'}| \circ |i| \simeq |i^T| \circ |\Gamma_R|$).

$$\begin{array}{ccc} |N(R)| & \xhookrightarrow{|i|} & |N(R')| \\ \downarrow{\scriptstyle |\Gamma_R|} & & \downarrow{\scriptstyle |\Gamma_{R'}|} \\ |N(R^T)| & \xhookrightarrow{|i^T|} & |N(R'^T)| \end{array}$$

Corollary 3.11. *Let $X \subseteq \mathbb{R}^d$ with both the length metric d_L and the restriction of the Euclidean distance d_E, δ be the distortion of X, $A \subseteq X$ and $\alpha \in \mathbb{R}_{>0}$. Then, there is the chain of inclusions*

$$\mathscr{C}_X^{d_L}(A,\alpha) \hookrightarrow \mathscr{C}_X^{d_E}(A,\alpha) \hookrightarrow \mathscr{C}_X^{d_L}(A,\delta\alpha)$$

and the following homotopy equivalences and commutative diagram, up to homotopy:

$$\begin{array}{ccccc} |\mathscr{C}_X^{d_L}(A,\alpha)| & \hookrightarrow & |\mathscr{C}_X^{d_E}(A,\alpha)| & \hookrightarrow & |\mathscr{C}_X^{d_L}(A,\delta\alpha)| \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ |\mathscr{C}_A^{d_L}(X,\alpha)| & \hookrightarrow & |\mathscr{C}_A^{d_E}(X,\alpha)| & \hookrightarrow & |\mathscr{C}_A^{d_L}(X,\delta\alpha)|. \end{array}$$

Proof. Since $d_E \leq d_L \leq \delta d_E$ by Remark 3.6, we directly obtain such chain of inclusions.

Then, to show the square on the left, let us define the subset $R_\alpha := \{(a,x) \in\, \in A \times X \mid d_L(a,x) < \alpha\}$, with transpose $R_\alpha^T := \{(x,a) \in X \times A \mid (a,x) \in R_\alpha\}$.

Similarly, $R'_\alpha := \{(a,x) \in A \times X \mid d_E(a,x) < \alpha\}$, with $R'^T_\alpha := \{(x,a) \in \in X \times A \mid (a,x) \in R'_\alpha\}$.

Taking the Dowker complexes of such subsets, we obtain directly the Čech complexes we need:
$N(R_\alpha) := \{\sigma \subseteq A \mid \exists x \in X \text{ such that } (a,x) \in R_\alpha \quad \forall a \in \sigma\} =$
$= \{\sigma \subseteq A \mid \exists x \in X \text{ such that } d_L(a,x) < \alpha \quad \forall a \in \sigma\} = \mathscr{C}^{d_L}_X(A,\alpha)$.

$N(R^T_\alpha) := \{\sigma \subseteq X \mid \exists a \in A \text{ such that } (x,a) \in R^T_\alpha \quad \forall x \in \sigma\} =$
$= \{\sigma \subseteq X \mid \exists a \in A \text{ such that } d_L(a,x) < \alpha \quad \forall x \in \sigma\} = \mathscr{C}^{d_L}_A(X,\alpha)$.

Similarly, $N(R'_\alpha) = \mathscr{C}^{d_E}_X(A,\alpha)$ and $N(R'^T_\alpha) = \mathscr{C}^{d_E}_A(X,\alpha)$.

Since $d_E \leq d_L$ by Remark 3.6, we have that $R_\alpha \subseteq R'_\alpha \subseteq A \times X$, together with the inclusions $i : N(R_\alpha) \hookrightarrow N(R'_\alpha)$ and $i^T : N(R^T_\alpha) \hookrightarrow N(R'^T_\alpha)$.

Therefore, we apply Dowker's Theorem 3.10 and get the commutative diagram corresponding to the left square, together with the homotopy equivalences

$$|\mathscr{C}^{d_L}_X(A,\alpha)| \simeq |\mathscr{C}^{d_L}_A(X,\alpha)| \text{ and } |\mathscr{C}^{d_E}_X(A,\alpha)| \simeq |\mathscr{C}^{d_E}_A(X,\alpha)|.$$

To show the square on the right, we can define the subset $R_{\delta\alpha} := \{(a,x) \in \in A \times X \mid d_L(a,x) < \delta\alpha\}$. Since $d_L \leq \delta d_E$, we have that $R'_\alpha \subseteq R_{\delta\alpha} \subseteq A \times X$ and by applying Dowker's Theorem 3.10 to these two subsets, we get the commutativity of the right square and the homotopy equivalence

$$|\mathscr{C}^{d_L}_X(A,\delta\alpha)| \simeq |\mathscr{C}^{d_L}_A(X,\delta\alpha)|.$$

□

3.2 Convexity radius: geometric reconstruction of geodesic subspaces.

Here we define geodesic subspace and we reconstruct the homotopy type of such subspace using its convexity radius and a dense enough sample (Lemma 3.18).

Definition 3.12. *A subspace $X \subseteq \mathbb{R}^d$ is a **geodesic subspace** if for any two points $x, y \in X$, there always exists a rectifiable path γ on X connecting them such that $L(\gamma) = d_L(x,y)$.*

*We call such path γ a **length-minimizing geodesic**, or simply a **geodesic**.*

Example 3.13. *Manifolds are a particular case of geodesic subspaces.*

We introduce now the concept of convexity radius in a geodesisc subspace. This is an intrinsic property of geodesic spaces, and determines which is the largest possible radius such that we have the property of being geodesically convex in a local neighborhood of such geodesic space (a space is geodesically convex if for every two points in such space, there exists a unique geodesic connecting them that lies entirely inside such space, and that is continuous with respect to its endpoints).

Definition 3.14. *The* ***convexity radius*** *of a geodesic subspace $X \subseteq \mathbb{R}^d$ is the supremum of all the real values $r > 0$ with the following two properties: for any $x \in X$ and $y_1, y_2 \in B_X^{d_L}(x, r)$,*

1. *there exists a unique length-minimizing geodesic γ joining y_1 and y_2, and γ lies entirely inside $B_X^{d_L}(x, r)$;*
2. *this unique geodesic is continuous with respect to its endpoints.*

Lemma 3.15. *Let $X \subseteq \mathbb{R}^d$ be a geodesic subspace and $\{B_X^{d_L}(x, r)\}_{x \in X}$, for $r > 0$, be a collection of geodesically convex subspace balls. Then, such collection forms a good cover of X.*

Proof. It is clear that $\{B_X^{d_L}(x, r)\}_{x \in X}$ forms a cover of X, and each $B_X^{d_L}(x, r) \subseteq X$ is open.

Now, for each subspace ball, we define a homotopy by the following way :

$$H : B_X^{d_L}(x_0, r) \times I \to B_X^{d_L}(x_0, r); \quad (x, t) \mapsto \gamma(t) \text{ geodesic from } x \text{ to } x_0$$

where H is continuous and well-defined, since $\gamma(t)$ is continuous with respect to its endpoints, and $\gamma(t)$ is contained in $B_X^{d_L}(x_0, r)$ as it is geodesically convex. We have that $H(x, 0) = \gamma(0) = x = id_{B_X^{d_L}(x_0, r)}(x)$ and $H(x, 1) = \gamma(1) = x_0 = c_{x_0}(x)$ for every $x \in B_X^{d_L}(x_0, r)$, where c_{x_0} denotes the constant map. Hence, $B_X^{d_L}(x_0, r)$ is contractible (as defined in Appendix A1).

We check that finite intersections of geodesically convex balls are contractible, in the case of being non-empty. We fix a point $y \in \bigcap_{i=0}^{k} B_X^{d_L}(x_i, r)$, and we define a homotopy

$$H : \bigcap_{i=0}^{k} B_X^{d_L}(x_i, r) \times I \to \bigcap_{i=0}^{k} B_X^{d_L}(x_i, r); \quad (x, t) \mapsto \gamma(t) \text{ geodesic from } x \text{ to } y$$

where H is continuous since $\gamma(t)$ is continuous with respect to its endpoints, and $\gamma(t)$ is contained in every ball that contains both x and y, since each ball is geodesically convex, hence $\gamma(t)$ is contained in a finite intersection of these balls (this implies that $\bigcap_{i=0}^{k} B_X^{d_L}(x_i, r)$ is geodesically convex), so H is also well-defined.

We have that $H(x,0) = \gamma(0) = x = id_{\cap_{i=0}^k B_X^{d_L}(x_i,r)}(x)$ and $H(x,1) = \gamma(1) = = y = c_y(x)$ for every $x \in \bigcap_{i=0}^k B_X^{d_L}(x_i, r)$.

Hence, $\bigcap_{i=0}^k B_X^{d_L}(x_i, r)$ is contractible, so $\{B_X^{d_L}(x,r)\}_{x\in X}$ forms a good cover of X.

□

Definition 3.16. *Let $X \subseteq \mathbb{R}^d$ be a geodesic subspace and $s \in \mathbb{R}_{>0}$, then $A \subseteq X$ is an **s-dense subset** if for every $x \in X$ there exists $a \in A$ such that $d_L(x,a) < s$.*

Or equivalently, if $\{B_X^{d_L}(a,s)\}_{a\in A}$ is a cover of X.

Remark 3.17. *It is interesting to notice that having an s-dense subset $A \subseteq X$, implies that the directed Hausdorff distance (Definition 2.15) is $\overrightarrow{d_H}(X,A) < s$.*

The following lemma consists of a geometric reconstruction result for geodesic subspaces ([5], Lemma 2.7):

Lemma 3.18. *Let $X \subseteq \mathbb{R}^d$ be a geodesic subspace with the length metric d_L and positive convexity radius ρ. Let A be an s-dense subset of X, where $0 < s \leq \rho$. Then, $|\mathscr{C}_X^{d_L}(A,s)|$ is homotopy equivalent to X.*

Proof. Since $A \subseteq X$ is s-dense, we have by definition that $\{B_X^{d_L}(a,s)\}_{a\in A}$ is a cover of X.

Since $s \leq \rho$, the definition of convexity radius implies that each ball $B_X^{d_L}(a,s)$ is geodesically convex, hence by Lemma 3.15, $\{B_X^{d_L}(a,s)\}_{a\in A}$ is a good cover of X.

Identifying $\mathscr{C}_X^{d_L}(A,s)$ with the nerve complex of $\{B_X^{d_L}(a,s)\}_{a\in A}$ (as done in Example 1.24) and applying the Nerve lemma 1.20, we get that

$$|\mathscr{C}_X^{d_L}(A,s)| \simeq X.$$

□

3.3 Persistences: homotopy and homology groups.

In this section we define persistence modules and persistence (abelian) groups, and interleavings between them. We show interleavings between homology and homotopy groups induced by Čech complexes constructed from a geodesic subspace and a dense enough sample, and we understand such groups as persistence groups (Proposition 3.25). We conclude giving an isomorphism between fundamental groups (Theorem 3.31) and an analogous isomorphism for the first homology groups with coefficients in an abelian group (Corollary 3.34).

Definition 3.19. *A **persistence module** M is a set of vector spaces $\{M_t\}_{t\in\mathbb{R}}$ so that for each $s \leq t \in \mathbb{R}$ there is a linear map $\varphi_M(s,t) : M_s \to M_t$ such that for all $r \leq s \leq t \in \mathbb{R}$, $\varphi_M(t,t) = id_{M_t}$ and $\varphi_M(r,t) = \varphi_M(s,t) \circ \varphi_M(r,s)$.*

Definition 3.20. *A* ***morphism of persistence modules*** $f : M \to N$ *is a collection of morphisms* $\{f_t : M_t \to N_t\}_{t\in\mathbb{R}}$ *such that for every* $s \leq t \in \mathbb{R}$*, the following diagram commutes*

$$\begin{array}{ccc} M_s & \xrightarrow{\varphi_M(s,t)} & M_t \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f_t} \\ N_s & \xrightarrow{\varphi_N(s,t)} & N_t. \end{array}$$

Example 3.21. *The homology groups with coefficients in a field* $\mathbb{F}$*, denoted by* $H_n(\_, \mathbb{F})$ *for* $n = 0, 1, 2, \ldots$*, are vector spaces, and therefore persistence modules (we discuss this in Appendix A2, in particular in Remark A.27).*

We can also define **persistence abelian groups** by taking abelian groups instead of vector spaces, and by taking homomorphisms $\varphi(s, t)$ as the morphisms verifying the same conditions as for the case of persistence modules. We can consider as persistence abelian groups the simplicial homology groups $H_n(|K|)$ for an abstract simplicial complex K, the singular homology groups $H_n(X)$ for a topological space X, the homology groups with coefficients in an abelian group G, $H_n(\_, G)$, and the homotopy groups for higher dimension $\pi_n(\_, \bullet)$, since for $n \geq 2$ these are always abelian ([9], page 340). We present homotopy groups and simplicial and singular homology in Appendix A2.

Moreover, we can define **persistence groups** by taking groups (without having to be abelian) as the persistence objects, and homomorphisms $\varphi(s, t)$ as the morphisms like for the case of persistence modules. We can consider the fundamental group $\pi_1(\_, \bullet)$ as a persistence group. The fundamental group is also defined in Appendix A2.

However, in the definition of persistence modules we work over all real numbers, but for the homology or homotopy groups of Čech complexes we only take values greater than zero. Hence, we establish that if $0 \geq t \in \mathbb{R}$, then $M_t = 0$.

We use the term **persistences** when we are including persistence modules and persistence (abelian) groups.

Definition 3.22. *For* $\delta, \epsilon \geq 0$*, a* (δ, ϵ)***-interleaving*** *between persistences* $\{A_t\}_{t\in\mathbb{R}}$ *and* $\{B_t\}_{t\in\mathbb{R}}$ *consists of a collection of morphisms* $f_t : A_t \longrightarrow B_{t+\delta}$ *and* $g_s : B_s \longrightarrow A_{s+\epsilon}$ *such that the following diagrams commute:*

$$\begin{array}{ccc} B_s & \xrightarrow{g_s} & A_{s+\epsilon} \\ & \searrow{\scriptstyle \varphi_B(s,s+\epsilon+\delta)} & \downarrow{\scriptstyle f_{s+\epsilon}} \\ & & B_{s+\epsilon+\delta} \end{array} \qquad \begin{array}{ccc} A_t & \xrightarrow{f_t} & B_{t+\delta} \\ & \searrow{\scriptstyle \varphi_A(t,t+\delta+\epsilon)} & \downarrow{\scriptstyle g_{t+\delta}} \\ & & A_{t+\delta+\epsilon} \end{array}$$

and in case of having $s < t$*, the following two diagrams also commute:*

$$\begin{array}{ccccccc}
B_s & \xrightarrow{g_s} & A_{s+\epsilon} & \qquad & A_s & \xrightarrow{f_s} & B_{s+\delta} \\
\Big\downarrow \varphi_B(s,t) & & \Big\downarrow \varphi_A(s+\epsilon,t+\epsilon) & & \Big\downarrow \varphi_A(s,t) & & \Big\downarrow \varphi_B(s+\delta,t+\delta) \\
B_t & \xrightarrow{g_t} & A_{t+\epsilon} & & A_t & \xrightarrow{f_t} & B_{t+\delta}.
\end{array}$$

Also, we can denote $A_s(\epsilon) := A_{s+\epsilon}$ and $B_t(\delta) := B_{t+\delta}$.

For X a geodesic subspace, $A \subseteq X$ an s-dense subset and $B \subseteq X$, we present in the following Proposition 3.25, a $(0, s)$-interleaving between the persistence groups $\{\pi_n(|\mathscr{C}_B^{d_L}(A,p)|, \bullet)\}_{p>0}$ and $\{\pi_n(|\mathscr{C}_B^{d_L}(X,p)|, \bullet)\}_{p>0}$, and a $(0, s)$-interleaving between $\{H_n(|\mathscr{C}_B^{d_L}(A,p)|)\}_{p>0}$ and $\{H_n(|\mathscr{C}_B^{d_L}(X,p)|)\}_{p>0}$, for any $n \geq 1$.

But it order to prove such result, we need to introduce contiguous simplicial maps together with the well-known result Lemma 3.24 (whose classical proof can be found in [19], Chapter 3.5, Lemma 2):

Definition 3.23. *Two simplicial maps $F, G : K_1 \to K_2$ are* ***contiguous*** *if for every simplex $\sigma \in K_1$, $F(\sigma) \bigcup G(\sigma)$ is a simplex in K_2.*

Lemma 3.24. *If $F, G : K_1 \to K_2$ are contiguous simplicial maps, then $|F|, |G| : |K_1| \to |K_2|$ are homotopic.*

The next result is from [23] (Proposition 3.3), in a new version for Čech complexes, with a different proof working at the level of simplicial complexes and for homology and homotopy in higher dimension.

Proposition 3.25. *Let $X \subseteq \mathbb{R}^d$ be a geodesic subspace, $A \subseteq X$ an s-dense subset with a base point $\bullet \in A$, $B \subseteq X$ (although B would typically be X) and $p > 0$. Then, we get the following commutative diagrams for any $n \in \mathbb{N}$:*

$$\begin{array}{ccc}
\pi_n(|\mathscr{C}_B^{d_L}(A,p)|, \bullet) & \xrightarrow{\varphi_A^{\pi_n}(p,p+s)} & \pi_n(|\mathscr{C}_B^{d_L}(A,p+s)|, \bullet) \\
\Big\downarrow i_p^{\pi_n,A} & \nearrow \nu_p^{\pi_n,A} & \Big\downarrow i_{p+s}^{\pi_n,A} \\
\pi_n(|\mathscr{C}_B^{d_L}(X,p)|, \bullet) & \xrightarrow[\varphi_X^{\pi_n}(p,p+s)]{} & \pi_n(|\mathscr{C}_B^{d_L}(X,p+s)|, \bullet)
\end{array}$$

and

$$\begin{array}{ccc}
H_n(|\mathscr{C}_B^{d_L}(A,p)|) & \xrightarrow{\varphi_A^{H_n}(p,p+s)} & H_n(|\mathscr{C}_B^{d_L}(A,p+s)|) \\
\Big\downarrow i_p^{H_n,A} & \nearrow \nu_p^{H_n,A} & \Big\downarrow i_{p+s}^{H_n,A} \\
H_n(|\mathscr{C}_B^{d_L}(X,p)|) & \xrightarrow[\varphi_X^{H_n}(p,p+s)]{} & H_n(|\mathscr{C}_B^{d_L}(X,p+s)|)
\end{array}$$

Proof. The idea is to prove that both the upper and lower triangles of the two diagrams commute, and so the two whole diagrams are commutative.
We first define a projection map from the vertex set of $\mathscr{C}_B^{d_L}(X,p)$ to the vertex set of $\mathscr{C}_B^{d_L}(A,p+s)$, by the definition of A being an s-dense subset of X:

$$\pi : X \to A;\ x \mapsto \pi(x) \quad s.t. \quad d_L(x,\pi(x)) < s \text{ and } \pi|_A = id_A.$$

Now we define the following simplicial map:

$$\nu_p^A : \mathscr{C}_B^{d_L}(X,p) \to \mathscr{C}_B^{d_L}(A,p+s)$$

such that for a simplex $\sigma \in \mathscr{C}_B^{d_L}(X,p)$ we have that $\sigma \subseteq X$ so that there exists an $b \in B$ with $d_L(b,x) < p$ for all $x \in \sigma$. We define the image of σ to be $\nu_p^A(\sigma) :=$ $= \{\pi(x) \quad | \quad x \in \sigma\}$, so that there exists an $b \in B$ with $d_L(b,\pi(x)) \leq d_L(b,x) +$ $+ d_L(x,\pi(x)) < p+s$, for all $\pi(x) \in \nu_p^A(\sigma)$. Hence $\nu_p^A(\sigma) \in \mathscr{C}_B^{d_L}(A,p+s)$.

For the upper triangle, we have the following diagram at the level of simplicial complexes:

$$\begin{array}{ccc} \mathscr{C}_B^{d_L}(A,p) & \xhookrightarrow{\varphi_A(p,p+s)} & \mathscr{C}_B^{d_L}(A,p+s) \\ \big\downarrow i_p^A & \nearrow \nu_p^A & \\ \mathscr{C}_B^{d_L}(X,p) & & \end{array}$$

where $\varphi_A(p,p+s)$ is the inclusion from the filtration of $\{\mathscr{C}_B^{d_L}(A,p)\}_{p>0}$ and i_p^A is the first inclusion of Lemma 3.2 for $\alpha = p$.

We can compose i_p^A with ν_p^A, getting $\nu_p^A \circ i_p^A : \mathscr{C}_B^{d_L}(A,p) \to \mathscr{C}_B^{d_L}(A,p+s)$, where $\sigma \in \mathscr{C}_B^{d_L}(A,p)$ goes to $(\nu_p^A \circ i_p^A)(\sigma) = \sigma$ (since $\pi|_A = id_A$).

Hence, $\nu_p^A \circ i_p^A = \varphi_A(p,p+s)$, so this upper triangle is commutative, with $(\nu_p^A \circ i_p^A)$ and $\varphi_A(p,p+s)$ inducing equal maps on the homotopy and homology groups.

So for any n, we can denote such homomorphisms by $\nu_p^{\pi_n,A} \circ i_p^{\pi_n,A} = \varphi_A^{\pi_n}(p,p+s)$ for the homotopy groups and $\nu_p^{H_n,A} \circ i_p^{H_n,A} = \varphi_A^{H_n}(p,p+s)$ for the homology groups.

For the lower triangle, we have the following diagram at the level of simplicial complexes:

$$\begin{array}{ccc} & & \mathscr{C}_B^{d_L}(A,p+s) \\ & \nearrow \nu_p^A & \big\downarrow i_{p+s}^A \\ \mathscr{C}_B^{d_L}(X,p) & \xhookrightarrow[\varphi_X(p,p+s)]{} & \mathscr{C}_B^{d_L}(X,p+s) \end{array}$$

where $\varphi_X(p, p+s)$ is the inclusion from the filtration of $\{\mathscr{C}_B^{d_L}(X,p)\}_{p>0}$ and i_{p+s}^A is the first inclusion of Lemma 3.2 for $\alpha = p+s$.

We compose ν_p^A with i_{p+s}^A, getting $i_{p+s}^A \circ \nu_p^A : \mathscr{C}_B^{d_L}(X,p) \to \mathscr{C}_B^{d_L}(X,p+s)$ where $\sigma \in \mathscr{C}_B^{d_L}(X,p)$ goes to $(i_{p+s}^A \circ \nu_p^A)(\sigma) = \nu_p^A(\sigma) \in \mathscr{C}_B^{d_L}(X,p+s)$.

We want to show that $(i_{p+s}^A \circ \nu_p^A)$ and $\varphi_X(p,p+s)$ are contiguous. In order to do so, we have to show that $\nu_p^A(\sigma) \cup \sigma$ is a simplex in $\mathscr{C}_B^{d_L}(X,p+s)$, for every simplex $\sigma \in \mathscr{C}_B^{d_L}(X,p)$.

We procede taking a point $b \in B$ such that for any $x \in \nu_p^A(\sigma) \cup \sigma$, then we have either $x \in \nu_p^A(\sigma)$, which implies $d_L(b,x) < p+s$; or $x \in \sigma$, which implies $d_L(b,x) < p < p+s$. Therefore, $\nu_p^A(\sigma) \cup \sigma \in \mathscr{C}_B^{d_L}(X,p+s)$ for any $\sigma \in \mathscr{C}_B^{d_L}(X,p)$, so $(i_{p+s}^A \circ \nu_p^A)$ and $\varphi_X(p,p+s)$ are contiguous.

By Lemma 3.24, $|i_{p+s}^A \circ \nu_p^A| \simeq |\varphi_X(p,p+s)|$, and by Propositions A.18, such homotopic maps induce equal maps on the homotopy and homology groups . So for any n, we can denote the homomorphisms between the homotopy groups by $i_{p+s}^{\pi_n,A} \circ \nu_p^{\pi_n,A} = \varphi_X^{\pi_n}(p,p+s)$, and $i_{p+s}^{H_n,A} \circ \nu_p^{H_n,A} = \varphi_X^{H_n}(p,p+s)$ for the homomorphisms between the homology groups.

□

We proceed to define piecewise linear paths in abstract simplicial complexes and to discuss how they are determined by a sequence of vertices.

Definition 3.26. *Let K be an abstract simplicial complex with vertex set V. Let $f : I \to |K|$ be a path such that there exists a sequence of vertices $v_0, \ldots, v_k \in V$, and a subdivision $0 = a_0 < \ldots < a_k = 1$ of I, so that for every $i = 0, \ldots, k$ we have:*

1. *$f(a_i) = v_i$.*

   There, $f(a_i)$ is a map $f(a_i) : V \to I$ such that

$$f(a_i)(v) := \begin{cases} 1 & \text{if } v = v_i \\ 0 & \text{if } v \neq v_i \end{cases}$$

   and $\alpha \in |K|$ is a vertex if there exists $v_i \in V$ such that $\alpha(v_i) = 1$, where for any other $v \in V$ with $v \neq v_i$, $\alpha(v) = 0$. So we can write $\alpha = v_i$.

2. $f((1-t)a_i + ta_{i+1}) = (1-t)v_i + tv_{i+1}$ *for all* $t \in I$, *where*

$$((1-t)v_i + tv_{i+1})(v) = \begin{cases} (1-t) & \text{if } v = v_i \\ t & \text{if } v = v_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Such f *is called a* ***piecewise linear path****.*

Piecewise linear loops are defined in an analogous way as for the case of usual loops. Moreover, two paths are path homotopic if they are homotopic relative to the subspace $\{0,1\}$, as defined in Appendix A2.

Proposition 3.27. *If two piecewise linear paths go through the exact same vertices, following the same order, then they are path homotopic.*

Therefore, we can say that the sequence of vertices uniquely determines the piecewise linear path, up to path homotopy.

Proof. We take two piecewise linear paths $f, g : I \to |K|$ through the vertices $v_0, \dots, v_k$, and two respective subdivisions of I given by $0 = a_0 < \dots < a_k = 1$ and $0 = b_0 < \dots < b_k = 1$, so that $f(a_i) = g(b_i) = v_i$.

Then, the map $h : I \to I$ defined by $h((1-t)a_i + ta_{i+1}) = (1-t)b_i + tb_{i+1}$, is a reparametrization such that $f = g \circ h$, and since both paths have the same endpoints, we obtain $f \simeq_{\{0,1\}} g$. □

Definition 3.28. *Let* $X \subseteq \mathbb{R}^d$ *be a geodesic subspace and* $A \subseteq X$*.*

An ***l-loop*** *is a piecewise linear loop* $L : I \to |\mathscr{C}_X^{d_L}(A,p)|$*, for* $p \geq l > 0$*, determined by a sequence of vertices* $a_0, \dots, a_k = a_0 \in A$ *with* $d_L(a_i, a_{i+1}) < l$*, for all* $i = 0, \dots, k$*.*

A ***filling*** *of an l-loop* $L : I \to |\mathscr{C}_X^{d_L}(A,p)|$ *is a loop* $\gamma : I \to X$ *obtained by connecting* a_i *to* a_{i+1} *by a geodesic, for every* $i = 0, \dots, k$*.*

An ***r-sample*** *of a loop* $\gamma : I \to X$ *is a choice* $0 \leq t_0 < \dots < t_m \leq 1$ *with* $\gamma(t_i) \in A$ *for all* i*,* $\gamma([t_{i-1}, t_{i+1}]) \subseteq B_X^{d_L}(\gamma(t_i), r)$*,* $\gamma([0,t_0] \cup [t_{m-1}, 1]) \subseteq B_X^{d_L}(\gamma(t_m), r)$ *and* $\gamma([0,t_1] \cup [t_m, 1]) \subseteq B_X^{d_L}(\gamma(t_0), r)$*.*

An r-sample of a loop $\gamma : I \to X$ *induces an r-loop determined by the vertices* $\gamma(t_0), \dots, \gamma(t_m) = \gamma(t_0)$ *in* $\mathscr{C}_X^{d_L}(A,p)$*.*

The next result is from [25] (Proposition 3.2 (4)).

Proposition 3.29. *Let* X *be geodesic,* $A \subseteq X$ *and* $p > 0$*.*

Given a loop $\gamma : I \to X$*, then two r-loops induced by any two r-samples of* γ *are path homotopic in* $|\mathscr{C}_X^{d_L}(A,p)|$*.*

Proof. Consider two r-samples $0 \leq t_0 < \ldots < t_m \leq 1$ and $0 \leq t'_0 < \ldots < t'_{m'} \leq 1$. We claim that each of the r-loops induced by them are path homotopic to the r-loop obtained by the r-sample $\{t_i\}_{i=0}^m \cup \{t'_i\}_{i=0}^{m'}$, and since being homotopic is a transitive property, both r-loops are path homotopic between them.

Using induction, we can add one element at a time, so, it suffices to prove the following claim: given an r-sample α determined by $0 \leq s_0 <, \ldots, < s_k \leq 1$ and $y \in [0,1] \backslash \{s_i\}_{i=0}^k$, the r-sample α' obtained by adding y to the collection $\{s_i\}_{i=0}^k$, induces an r-loop, denoted as well by α', which is path homotopic to the r-loop induced by α, and which we denote as well by α.

In order to prove the claim, we consider two cases:

- if $y \in (s_i, s_{i+1})$ for some i, then both r-loops are identical except for the interval $[s_i, s_{i+1}]$. Hence, the path homotopy required by the claim is induced by the triangle consisting of the vertices $\{\gamma(s_i), \gamma(y), \gamma(s_{i+1})\}$ in $\mathscr{C}_X^{d_L}(A, p)$ (since $\{\gamma(s_i), \gamma(y), \gamma(s_{i+1})\} \subseteq A$ and all of them are in the ball $B_X^{d_L}(\gamma(y), p)$)).

  We denote such triangle by T, and since T is contractible, every path in T is homotopic to a point. Hence, the paths $\alpha|_{[s_i,s_{i+1}]}, \alpha'|_{[s_i,s_{i+1}]} : [s_i, s_{i+1}] \to T$ are path-homotopic.

  That is because $id_T \simeq c_{x_0}$ for $c_{x_0} : T \to \{x_0\} \subseteq T$ the constant map, and if we have a path $\gamma : I \to T$, then $\widetilde{c}_{x_0} = c_{x_0} \circ \gamma : I \to \{x_0\}$ is the constant path. Clearly, $id_T \circ \gamma = \gamma$, so

$$\widetilde{c}_{x_0} = c_{x_0} \circ \gamma \simeq id_T \circ \gamma = \gamma.$$

- If $y \notin (s_i, s_{i+1})$, then the triangle $\{\gamma(s_0), \gamma(y), \gamma(s_k)\}$ does the job.

□

Definition 3.30. *Given a persistence $\{A_t\}_{t \in \mathbb{R}}$, we say that $c > 0$ is a:*

- ***left critical value**, if for all small enough $\varepsilon > 0$ the morphism $\varphi_A(c - \varepsilon, c) : A_{c-\varepsilon} \to A_c$ is not an isomorphism;*

- ***right critical value**, if for all small enough $\varepsilon > 0$ the morphism $\varphi_A(c, c + \varepsilon) : A_c \to A_{c+\varepsilon}$ is not an isomorphism.*

*We call $c > 0$ a **critical value** if it is either of the above.*

For X a geodesic subspace and $r > 0$, let c be the largest critical value of the persistence $\{\pi_1(|\mathscr{C}_X^{d_L}(X, p)|, \bullet)\}_{p>0}$, that is smaller than r (if no critical value is smaller than r, we set $c \in (0, r)$) such that there are no critical values in $(c, r]$.

Therefore, for all $\varepsilon > 0$ such that $c + \varepsilon < r$, $c + \varepsilon$ is a non-critical value, so we can pick an $\varepsilon' > 0$ such that $\varphi_X^{\pi_1}(c + \varepsilon - \varepsilon', r) : \pi_1(|\mathscr{C}_X^{d_L}(X, c + \varepsilon - \varepsilon')|, \bullet) \longrightarrow$ $\longrightarrow \pi_1(|\mathscr{C}_X^{d_L}(X, r)|, \bullet)$ is an isomorphism.

Hence, we can pick $c' > 0$ very close to c, making $\varphi_X^{\pi_1}(c', r)$ an isomorphism.

The following theorem is based on Theorem 4.2 and Proposition 4.1 in [23], but for the case of Čech complexes.

Theorem 3.31. *Under the conditions above, let $A \subseteq X$ be $(r - c')$-dense, with $\bullet \in A$. Then,*

$$i_r^{\pi_1, A} : \pi_1(|\mathscr{C}_X^{d_L}(A, r)|, \bullet) \longrightarrow \pi_1(|\mathscr{C}_X^{d_L}(X, r)|, \bullet)$$

is an isomorphism.

Proof. First we notice that every loop in $|K|$, for K an abstract simplicial complex, is path homotopic to a piecewise linear loop ([27], Theorem 3.6).

By Proposition 3.25 for $B = X$ and $n = 1$, we have the following commutative diagram:

$$\begin{array}{ccc} & & \pi_1(|\mathscr{C}_X^{d_L}(A, r)|, \bullet) \\ & \overset{\nu_{c'}^{\pi_1, A}}{\nearrow} & \downarrow i_r^{\pi_1, A} \\ \pi_1(|\mathscr{C}_X^{d_L}(X, c')|, \bullet) & \xrightarrow[\varphi_X^{\pi_1}(c', r)]{} & \pi_1(|\mathscr{C}_X^{d_L}(X, r)|, \bullet) \end{array}$$

and $\varphi_X^{\pi_1}(c', r)$ being an isomorphism implies that $i_r^{\pi_1, A}$ is surjective.

To show that $i_r^{\pi_1, A}$ is injective, we need to prove that $\nu_{c'}^{\pi_1, A}$ is surjective, because for such commutative diagram, if $\nu_{c'}^{\pi_1, A}$ and $i_r^{\pi_1, A}$ are surjective and $\varphi_X^{\pi_1}(c', r)$ is injective, then $i_r^{\pi_1, A}$ is injective.

We take an r-loop $L_A : I \longrightarrow |\mathscr{C}_X^{d_L}(A, r)|$ representing an equivalence class on $\pi_1(|\mathscr{C}_X^{d_L}(A, r)|, \bullet)$, and determined by the vertices $\bullet = a_0, \ldots, a_k = \bullet \in A$.

Let $\gamma : I \longrightarrow X$ be a filling of L_A, so that we take a c'-sample of γ such that no vertex of γ lies in the midpoint of γ_i, where γ_i is the geodesic segment of γ between a_i and a_{i+1}. Hence, we can consider the c'-loop induced by such c'-sample, and we denote it by L.

Now, $L : I \longrightarrow |\mathscr{C}_X^{d_L}(X, c')|$ is a c'-loop such that $\nu_{c'}^{\pi_1, A}$ maps each vertex of L on γ_i to the closer endpoint of γ_i, which is either a_i or a_{i+1}. This implies that L is mapped to L_A with repetitions of points, so the image of L under $\nu_{c'}^{\pi_1, A}$ is, for example, given by $\bullet = a_0, a_0, a_1, a_2, a_2, \ldots, a_k = \bullet \in A$, which is the composition of paths (as defined in Appendix A2) $[a_0, a_0] * [a_0, a_1] * [a_1, a_2] * \ldots$, where $[a_i, a_j]$

denotes the path from a_i to a_j. Moreover, if $i = j$, then $[a_i, a_i]$ is the constant path $\tilde{c}_{a_i}$, and $\tilde{c}_{a_i} * [a_i, a_j]$ is path homotopic to $[a_i, a_j]$. We can also write L_A as the composition of paths $[a_0, a_1] * [a_1, a_2] * \ldots * [a_{k-1}, a_k]$. Therefore, the image of L under $\nu_{c'}^{\pi_1, A}$ is path homotopic to L_A in $|\mathscr{C}_X^{d_L}(A, r)|$, so $\nu_{c'}^{\pi_1, A}$ is surjective.

We proceed showing the injectivity of $i_r^{\pi_1, A}$. Since $i_r^{\pi_1, A}$ is surjective, for all $[L'] \in \pi_1(|\mathscr{C}_X^{d_L}(X, r)|, \bullet)$ there exists $[L_A] \in \pi_1(|\mathscr{C}_X^{d_L}(A, r)|, \bullet)$ such that $i_r^{\pi_1, A}([L_A]) = [L']$, and since $\nu_{c'}^{\pi_1, A}$ is surjective, let

$$[L_{A_1}] = \nu_{c'}^{\pi_1, A}([L_1]) \quad \text{and} \quad [L_{A_2}] = \nu_{c'}^{\pi_1, A}([L_2]) \quad \text{in} \quad \pi_1(|\mathscr{C}_X^{d_L}(A, r)|, \bullet) \tag{3.2}$$

with $[L_1], [L_2] \in \pi_1(|\mathscr{C}_X^{d_L}(X, c')|, \bullet)$.

If $i_r^{\pi_1, A}([L_{A_1}]) = i_r^{\pi_1, A}([L_{A_2}]) \in \pi_1(|\mathscr{C}_X^{d_L}(X, r)|, \bullet)$, then by (3.2), $i_r^{\pi_1, A}(\nu_{c'}^{\pi_1, A}([L_1])) =$

$= i_r^{\pi_1, A}(\nu_{c'}^{\pi_1, A}([L_2]))$, which by the commutativity of the diagram is $\varphi_X^{\pi_1}(c', r)([L_1]) =$

$= \varphi_X^{\pi_1}(c', r)([L_2])$, and since $\varphi_X^{\pi_1}(c', r)$ is in particular injective, we have that $[L_1] = [L_2]$. Then, since $\nu_{c'}^{\pi_1, A}$ is well-defined, $\nu_{c'}^{\pi_1, A}([L_1]) = \nu_{c'}^{\pi_1, A}([L_2])$, which by (3.2) is precisely $[L_{A_1}] = [L_{A_2}]$. □

Definition 3.32. *For a group H, a subgroup N of H is a* ***normal subgroup*** *if $hnh^{-1} \in N$, for all $h \in H$ and $n \in N$.*

The ***commutator subgroup*** *of H is the smallest normal subgroup such that the quotient group of H by this subgroup is abelian. It is denoted by $[H, H]$, and it is generated by all commutators, i.e, elements of the form $[g, h] := g^{-1}h^{-1}gh$, for $g, h \in H$.*

The abelianization of the fundamental group of a topological space can be identified with the first homology group of such space. This is shown in Hurewicz Theorem ([9], Theorem 4.32).

Theorem 3.33 (Hurewicz Theorem)**.** *If X is a path-connected space and $n \geq 1$, then there exists a group homomorphism $h_* : \pi_n(X) \longrightarrow H_n(X)$.*

Moreover, for $n = 1$, such homomorphism induces an isomorphism

$$\bar{h}_* : Ab(\pi_1(X)) := \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \longrightarrow H_1(X)$$

between the abelianization of the fundamental group and the first homology group.

Corollary 3.34. *Let X be a geodesic subspace and $A \subseteq X$ be $(r-c')$-dense with $(r-c') < r/2$ for the same conditions as in Theorem 3.31, and let G be an abelian group. Then,*

$$i_r^{H_1,A} : H_1(|\mathscr{C}_X^{d_L}(A,r)|;G) \longrightarrow H_1(|\mathscr{C}_X^{d_L}(X,r)|;G)$$

is an isomorphism.

Proof. By Theorem 3.31 we have that $i_r^{\pi_1,A} : \pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet) \longrightarrow \pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet)$ is an isomorphism.

Then, we define an isomorphism between their commutator subgroups as follows:

for any $[g,h] \in [\pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet), \pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet)]$, then $i_r^{\pi_1,A}([g,h]) = = [i_r^{\pi_1,A}(g), i_r^{\pi_1,A}(h)] \in [\pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet), \pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet)]$.

And for any $[g,h] \in [\pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet), \pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet)]$, then $[(i_r^{\pi_1,A})^{-1}(g), (i_r^{\pi_1,A})^{-1}(h)] \in [\pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet), \pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet)]$.

So, the quotients of such fundamental groups by their respective commutator subgroups, are isomorphic between them, and we denote them by $Ab(\pi_1(\_,\bullet))$:

$$Ab(\pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet)) \cong Ab(\pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet)) \tag{3.3}$$

The subspace X is geodesic, which implies being path-connected. Hence, $|\mathscr{C}_X^{d_L}(X,r)|$ is also path connected.

Let a_1 and a_2 be two arbitrary points in $|\mathscr{C}_X^{d_L}(A,r)|$, and since $A \subseteq X$ and $|\mathscr{C}_X^{d_L}(X,r)|$ is path-connected, there exists a piecewise linear path $L_X : I \longrightarrow \longrightarrow |\mathscr{C}_X^{d_L}(X,r)|$ connecting a_1 to a_2. There exists a vertex x^1 in L_X such that $d_L(a_1,x^1) = r-c'$, and since $A \subseteq X$ is $(r-c')$-dense, there must exist an element $a^1 \in A$ such that $d_L(x^1,a^1) < r-c'$ and hence $d_L(a_1,a^1) < 2(r-c')$. We can take such a^1 as the next vertex in a piecewise linear path in $|\mathscr{C}_X^{d_L}(A,r)|$ connecting a_1 with a_2. We continue on the path L_X until we find a point x^2 such that $d_L(x^2,a^1) = r-c'$, so there must exists $a^2 \in A$ such that $d_L(x^2,a^2) < r-c'$ and $d_L(a^1,a^2) < 2(r-c')$. We repeat the process until we find x^k in L_X such that $d_L(x^k,a_2) < r-c'$.

Then, we have obtained a piecewise linear path $L_A : I \to |\mathscr{C}_X^{d_L}(A,r)|$ given by the vertices $a_1, a^1, a^2, \ldots, a_2$ such that the distance between consecutive vertices is less than $2(r-c') < r$, by hyptothesis. So $|\mathscr{C}_X^{d_L}(A,r)|$ is path-connected.

By Hurewicz Theorem 3.33,

$$Ab(\pi_1(|\mathscr{C}_X^{d_L}(A,r)|,\bullet)) \cong H_1(|\mathscr{C}_X^{d_L}(A,r)|)$$

and

$$Ab(\pi_1(|\mathscr{C}_X^{d_L}(X,r)|,\bullet)) \cong H_1(|\mathscr{C}_X^{d_L}(X,r)|)$$

so from (3.3), $H_1(|\mathscr{C}_X^{d_L}(A,r)|) \cong H_1(|\mathscr{C}_X^{d_L}(X,r)|)$.

And finally Corollary A.26 in Appendix A2 gives us that

$$H_1(|\mathscr{C}_X^{d_L}(A,r)|;G) \cong H_1(|\mathscr{C}_X^{d_L}(A,r)|) \otimes G$$

and

$$H_1(|\mathscr{C}_X^{d_L}(X,r)|;G) \cong H_1(|\mathscr{C}_X^{d_L}(X,r)|) \otimes G$$

so that we obtain the desired isomorphism between the first homology groups with coefficients in G.

$\square$

Chapter 4

Example: the square.

We want to study a particular example of a Euclidean subspace whose reach is 0, where our results from Chapter 2 cannot be applied (as in most cases we are not able to find radii in order to construct subspace balls and get a good cover). This space is going to be the square.

However, we find that for a simple case where the balls have centers inside the square and these centers are selected in such a way that they are at a particular distance from their consecutive one, and the radius is the same for every ball, then we can reconstruct the homotopy type of the square. We also present different counterexamples, and we finish with a different approach where the square is considered as a geodesic subspace, with other sampling parameters rather than the reach (like the convexity radius), and so we can find a good cover that reconstructs its homotopy type. We also work with the thickening of the square in the cases where the centers of the balls are outside the square.

Let X be the square, that is, the boundary of $[0,1]\times[0,1]\subseteq\mathbb{R}^2$, with the restriction of the Euclidean metric, and let O be the geometric point situated in its center. The two diagonals form the set Y (Definition 1.3) of all points $y\in\mathbb{R}^d$ such that there exist $x_1,x_2\in X$ with $d(x_1,y)=d(x_2,y)=d(X,y)$. Hence, its closure $\overline{Y}$ is the two diagonals with their four end points as the vertices of the square. So the reach $\tau=d(X,\overline{Y})$ is zero.

Case 1: Centers of the balls in the square and equal radii.

Let $A=\{a_1,\dots,a_N\}\subset X$ where $\|a_i-a_{i+1}\|=l>0$ for every $a_i\in A$, such that $\{B_{\mathbb{R}^2}(a_i,r)\cap X\}_{a_i\in A}$ is a good cover of X (and therefore r must be bigger than $\frac{l}{2}$ so that X is covered).

Then, by the Nerve Lemma 1.20,

$$|N(\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A})| \simeq X. \quad (4.1)$$

We now take the collection of balls $\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A}$ with A as before. Since they are convex and finite intersections are again convex, by Proposition 1.21, we have a good cover of the union of such balls, and by the convex version of the Nerve Lemma 1.22, we have

$$|N(\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A})| \simeq \bigcup_i B_{\mathbb{R}^2}(a_i, r). \quad (4.2)$$

It is known that arbitrarily small perturbations in the location of points can have serious effects on the topology of the simplicial complexes, that is why the nerve of a cover is not always homotopy equivalent to the nerve of another similar cover.

We want to study when the nerve of $\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A}$ is homotopy equivalent to the nerve of $\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A}$ by determining the radius r.

By Figure 4.1, $sin 45 = \frac{r}{l}$, hence $r = \frac{l}{\sqrt{2}}$.

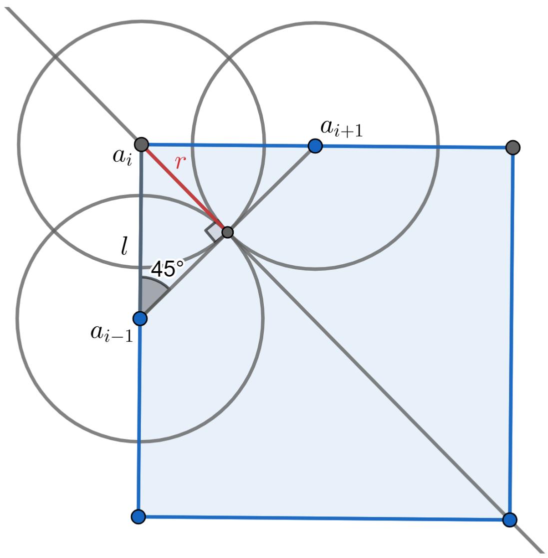


Figure 4.1: Case 1

So if $r \in \left(0, \frac{l}{\sqrt{2}}\right]$, then the two nerves are equal, because each simplex in $N(\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A})$ with vertices two lines in the square $L_i := B_{\mathbb{R}^2}(a_i, r) \cap X$ that have non-empty intersection, corresponds to a simplex in $N(\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A})$ with vertices the correspondent open balls $B_{\mathbb{R}^2}(a_i, r)$ which do also have non-empty intersection. By renaming in the nerve complexes the line L_i in the same way as the open ball $B_{\mathbb{R}^2}(a_i, r)$ for each i, we get the same vertex sets and the same simplices (as explained more generally in Example 1.24).

Therefore, $N(\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A})$ is equal to $N(\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A})$ when $0 < r \leq \dfrac{l}{\sqrt{2}}$, for $l = \|a_i - a_{i+1}\|$ for all $a_i \in A$.

Following the notation of Figure 4.1, if $r > \dfrac{l}{\sqrt{2}}$ but small enough so that $\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A}$ does not cover completely the interior area of X, then we see in Figure 4.2 that the subcomplex with vertices the lines L_i, L_{i+1} and L_{i-1} in the nerve complex of $\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A}$ is contractible, and the simplex with vertices $B_{\mathbb{R}^2}(a_i, r)$, $B_{\mathbb{R}^2}(a_{i+1}, r)$ and $B_{\mathbb{R}^2}(a_{i-1}, r)$ in the nerve of $\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A}$, is a filled triangle, and also contractible.

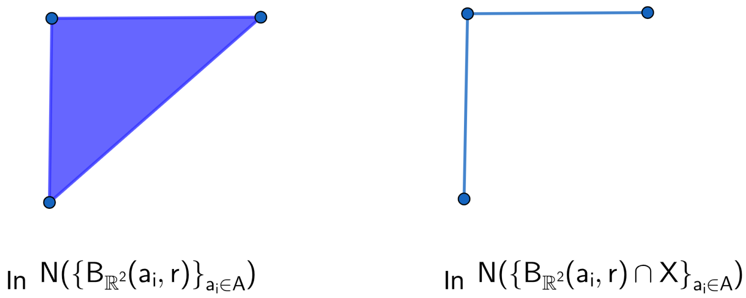


Figure 4.2: Nerves case 1

Therefore, they are homotopy equivalent, and $|N(\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A})| \simeq |N(\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A})|$.

So, by (4.1) and (4.2),

$$X \simeq \bigcup_i B_{\mathbb{R}^2}(a_i, r)$$

or equivalently by identifying $N(\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A})$ with $\mathscr{C}_{\mathbb{R}^2}(A, r)$, $X \simeq |\mathscr{C}_{\mathbb{R}^2}(A, r)|$.

A counterexample that we can present when the radius is too big and $\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A}$ covers completely the interior area of X, is Figure 4.3, where we take $A \subset X$ as the set of the four vertices of the square, so that $l = 1$ and $r > \frac{1}{\sqrt{2}}$.

Here we have that $N(\{B_{\mathbb{R}^2}(a_i, r)\}_{a_i \in A})$ is the tetrahedron with its interior, which is contractible and hence homotopy equivalent to a point; while $N(\{B_{\mathbb{R}^2}(a_i, r) \cap X\}_{a_i \in A})$ is just the square, and therefore is not contractible.

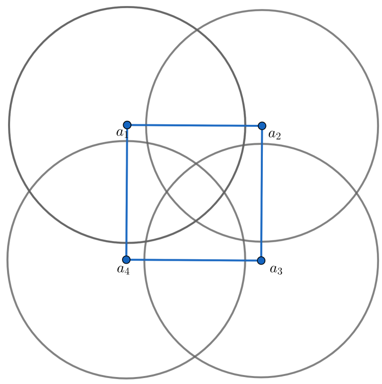


Figure 4.3: Counterexample case 1

Case 2: Centers of the balls in the square and different radii.

Following the reasoning of Case 1, we find the following counterexample where $A \subset X$ consist of the four vertices of the square.

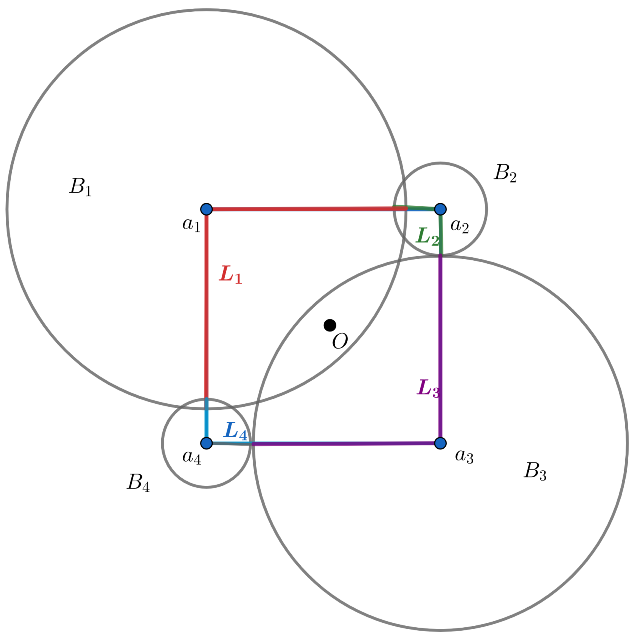


Figure 4.4: Counterexample case 2

According to the notation of Figure 4.4, we have the two following nerve complexes

$N(\{B_{\mathbb{R}^2}(a_i, r_i)\}_{a_i \in A}) = \{\{B_1, B_2\}, \{B_1, B_4\}, \{B_4, B_3\}, \{B_3, B_2\}, \{B_1, B_3\}, \{B_1\}, \{B_2\}, \{B_3\}, \{B_4\}\}$

$N(\{B_{\mathbb{R}^2}(a_i, r_i) \cap X\}_{a_i \in A}) = \{\{L_1, L_2\}, \{L_1, L_4\}, \{L_4, L_3\}, \{L_3, L_2\}, \{L_1\}, \{L_2\}, \{L_3\}, \{L_4\}\}$

which can be represented by Figure 4.5

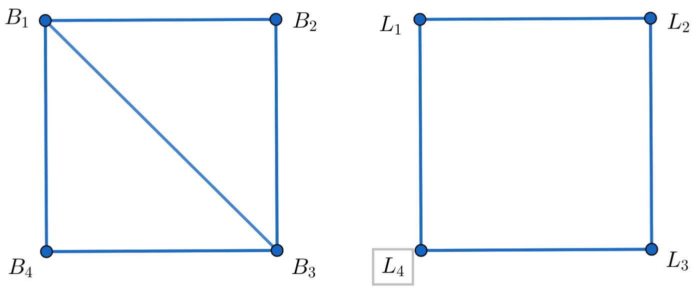


Figure 4.5: Nerves case 2.1

The first one are two triangles, homotopy equivalent to two circles touching at one point, and therefore its fundamental group is a free group (defined in Appendix A2) with two generators. The second nerve is the square, which is homeomorphic to the circle, and therefore its fundamental group is a free group with one generator. So, the fundamental groups are not isomorphic between them, and the geometric realizations of the nerve complexes cannot be homotopy equivalent (by Proposition A.18).

Such counterexample can also happen at a more local level, when we have a hole in the union of Euclidean balls. We take $A \subseteq X$, with no further conditions on the distance between its consecutive points.

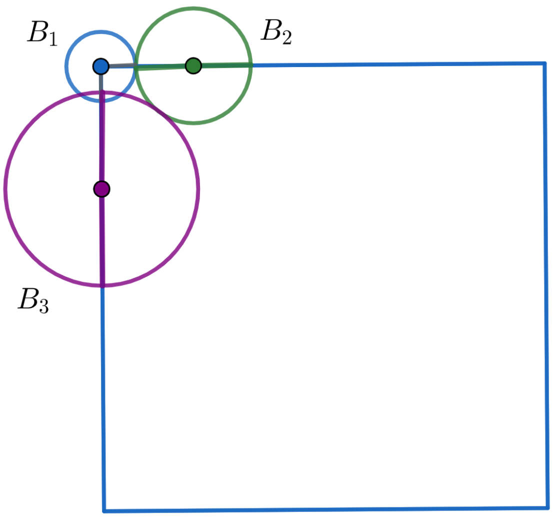


Figure 4.6: Counterexample case 2.2

As seen in Figure 4.7, the subcomplex in $N(\{B_{\mathbb{R}^2}(a_i, r_i) \cap X\}_{a_i \in A})$ with vertices the lines L_1, L_2, L_3 is contractible, whereas the simplex in $N(\{B_{\mathbb{R}^2}(a_i, r_i)\}_{a_i \in A})$ with vertices B_1, B_2, B_3 is a non-filled triangle, and therefore is not contractible. So these two nerves are not homotopy equivalent.

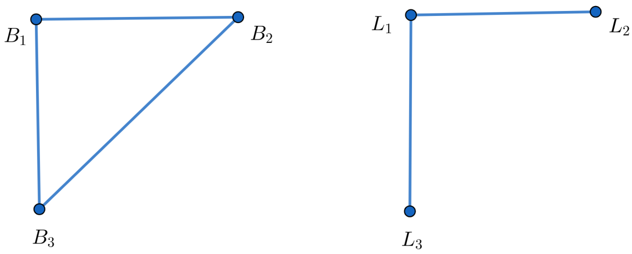


Figure 4.7: Nerves case 2.2

Case 3: Centers of the balls in the square or on its proximity, and equal radii.

In this case we want to show that the thickening of the square is homotopy equivalent to the union of some Euclidean balls, and that the square is a deformation retract of such union of balls, therefore, the square and its thickening are homotopy equivalent. However, that fails, because we find collections of balls whose nerve is not homotopy equivalent to the nerve of such balls intersected with the thickening.

Let X be the square and let

$$X^r := \{p \in \mathbb{R}^2 \mid d(p, X) < r\} \tag{4.3}$$

be the thickening of X, where $0 < r \leq \frac{1}{4}$.

We take $A \subseteq X^r$ and $r' \in \left(\overrightarrow{d}_H(X^r, A), \frac{1}{4}\right)$ such that $\{B_{\mathbb{R}^2}(a_i, r') \cap X^r\}_{a_i \in A}$ is a good cover of X^r.

We also take the collection of balls $\{B_{\mathbb{R}^2}(a_i, r')\}_{a_i \in A}$ forming a good cover of $\bigcup_{a_i \in A} B_{\mathbb{R}^2}(a_i, r')$.

The union of such Euclidean balls can present holes, hence we have the following situation:

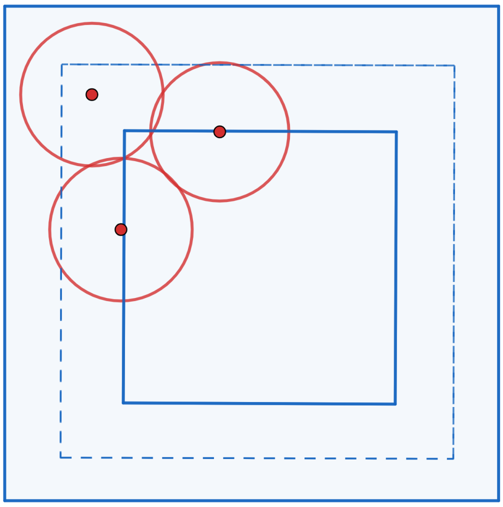


Figure 4.8: Counterexample case 3

which is the same as in the previous counterexample 4.6.

Nevertheless, for this case of equal radii, we have seen in [1] a corollary (Corollary 6.6) which applied to a long exact sequence of homology groups constructed from [9] (page 117), can give a $(0, \epsilon)$-interleaving between the persistence groups $H_n(|\mathscr{C}_{X^r}(A, r')|)$ and $H_n(|\mathscr{C}_{\mathbb{R}^2}(A, r')|)$, for $n \in \mathbb{N}$, where ϵ seems to take the value $log2$. So, if we have more time to work on it and formalize it better, we could conclude that even though the nerve complexes $N(\{B_{\mathbb{R}^2}(a_i, r') \cap X^r\}_{a_i \in A})$ and $N(\{B_{\mathbb{R}^2}(a_i, r')\}_{a_i \in A})$ are not homotopy equivalent, they are very close in terms of their homology, understanding their homology groups as persistence abelian groups.

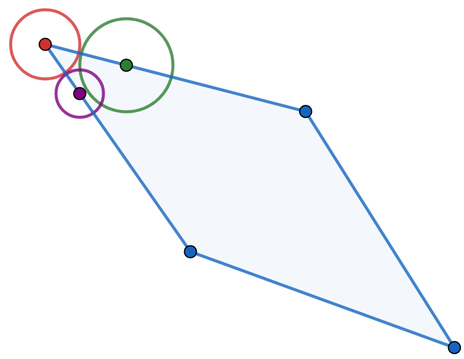
Case 4: Centers of the balls in the square or on its proximity, and different radii.

This case leads to the same counterexample as in Case 3, just setting the radii of $\{B_{\mathbb{R}^2}(a_i, r_i)\}_{a_i \in A}$ as $\overrightarrow{d}_H(X^r, A) < r_i < d_E(O, a_i)$, for each i.

Case 5: Generalization.

We show that for other figures with reach 0, we can also find a counterexample, and therefore this reconstruction approach is difficult to generalize.

We take the rhomboid, whose reach is 0 by the same argument as for the square. The following picture shows that we can find balls that cover the space, but whose union contains holes, and therefore the nerve of such collection of balls would not be homotopy equivalent to the nerve of the balls itersected with the rhomboid, as seen in detail in the counterexample of Figure 4.6.



Case 6: Square as a geodesic subspace.

Now we consider the square $X = \partial([0,1] \times [0,1]) \subseteq \mathbb{R}^2$ as a geodesic subspace. In particular, it is an embedded metric planar graph, that is, a subset of $\mathbb{R}^2$ that is homeomorphic to a 1-dimensional simplicial complex, where the length metric d_L is the shortest path distance on X.

The square X has a finite number of vertices, so since $b = 4$ is the length of its shortest simple cycle (that it, a loop that passes only one time through each vertex), then, in [5] it appears that the convexity radius is given for this case by the formula $\rho = \frac{b}{4} = 1$.

We take an s-dense subset $A \subseteq X$, whose points are going to be the centers of the balls that form a cover of X, and such balls have all equal radii s. We choose s to be $0 < s \leq \rho = 1$. By definition of s-dense subset, the collection of open intervals $\{B_X^{d_L}(a_i, s)\}_{a_i \in A}$ is a cover of X, and by Lemma 3.18, $|\mathscr{C}_X^{d_L}(A, s)| \simeq X$, or equivalently, $|N(\{B_X^{d_L}(a_i, s)\}_{a_i \in A})| \simeq X$.

Hence, we have successfully achieved a geometric reconstruction of the square X, since we have found a simplicial complex $\mathscr{C}_X^{d_L}(A, s)$ whose geometric realization has the same homotopy type as X.

Let now X^r be the thickening of X, as defined in (4.3), with $0 \leq r \leq \dfrac{1}{4}$ and convexity radius ρ. We understand X^r as a geodesic subspace, so that we can take an s-dense subspace $A \subseteq X^r$, with $0 < s \leq \rho$, such that $\{B_{X^r}^{d_L}(a_i, s)\}_{a_i \in A}$ is a good cover of X^r. By Lemma 3.18 and by definition of good cover, we get:

$$|\mathscr{C}_{X^r}^{d_L}(A, s)| \simeq X^r, \qquad \bigcup_{a_i \in A} B_{X^r}^{d_L}(a_i, s) = X^r. \tag{4.4}$$

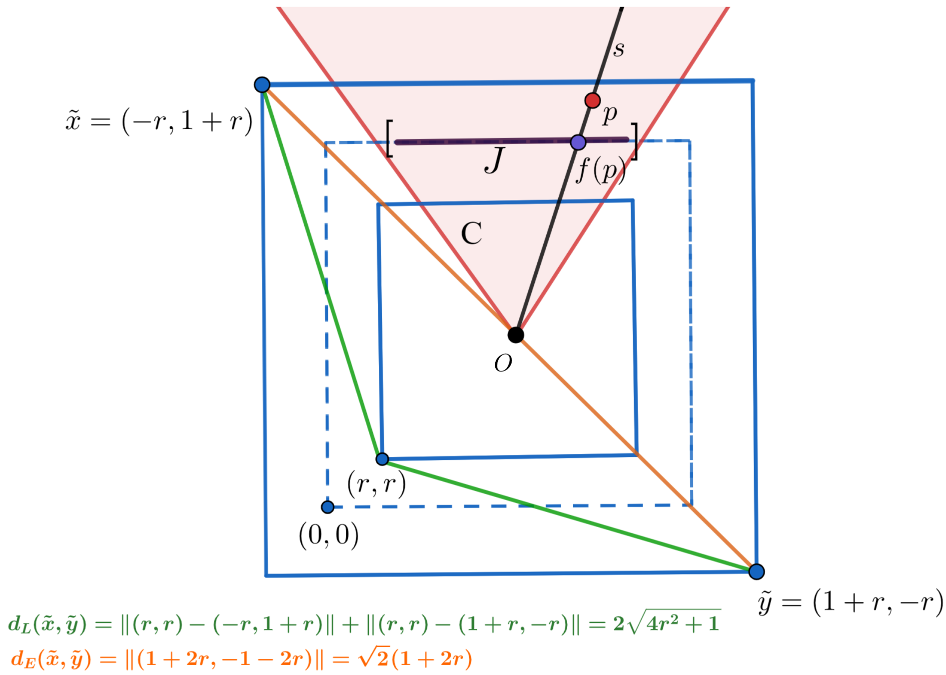


Figure 4.9: Case 6

Now we define the map $f : X^r \to X$ such that $f(p) := s \bigcap X$, where s is the half-line with the geometric point O placed in the center of the square as its initial point, and that passes through $p \in X^r$. It is clear that f restricted to X is the identity in X, so we just have to show that f is continuous in order to be a retract. Hence, for any closed interval $J \subseteq X$, we have to check that $f^{-1}(J) \subseteq X^r$ is closed.

By Figure 4.9, the preimage of J is $R := C \bigcap X^r$, where C is the cone from O including J, and $C \subseteq \mathbb{R}^2$ is closed. Therefore, R is closed in X^r (since $X^r \subseteq \mathbb{R}^2$ has the subspace topology, so if C is closed in $\mathbb{R}^2$, then $C \bigcap X^r$ is closed in X^r).

We can also consider $X^r \subseteq \mathbb{R}^d$ with the restriction of the Euclidean metric d_E, and following the notation of Figure 4.9, the distortion of X^r is

$$\delta := sup_{x \neq y} \frac{d_L(x,y)}{d_E(x,y)} = \frac{d_L(\tilde{x},\tilde{y})}{d_E(\tilde{x},\tilde{y})} = \frac{2\sqrt{4r^2+1}}{\sqrt{2}(1+2r)} = \frac{\sqrt{2(4r^2+1)}}{1+2r} < \infty$$

and from the inequalities $d_E(x,y) \leq d_L(x,y) \leq \delta d_E(x,y)$ presented in (3.1), we deduce that since the distortion is finite, then the metric topology in X^r induced by d_L is the same as the metric topology in X^r induced by the restriction of the Euclidean distance. So, the geodesic $\gamma(t)$ between two close enough points $x, y \in X^r$ is precisely the straight line segment $tx + (1-t)y$. Hence, the square X is a deformation retract (defined in Appendix A1) of its thickening X^r by the following homotopy relative to X:

$$H : X^r \times I \longrightarrow X^r;\ (p,t) \mapsto \gamma(t) = tf(p) + (1-t)p.$$

So, by Proposition A.7, X is homotopy equivalent to X^r. Finally, by (4.4), we obtain that

$$\bigcup_{a_i \in A} B^{d_L}_{X^r}(a_i, s) \simeq X \quad \text{and} \quad |\mathscr{C}^{d_L}_{X^r}(A,s)| \simeq X.$$

Moreover, the square and the circle as geodesic subspaces are isometric, i.e, there exists a bijective map between them that preserves distances, but the thickening of the square is not isometric to the thickening of the circle, because for any two points in the square, the goedesic in its thickening connecting them has not the same length as the geodesic of the thickening of the circle connecting their corresponding points lying in the circle. This shows that we can follow this last approach for other geodesic subspaces that are isometric to the square, but not if their thickenings are involved.

Appendix A

Appendix.

The goal of this appendix is to explain concepts and results that have been used throughout the thesis. In Section A1 we present basic notions of algebraic topology. In Section A2 we begin defining the fundamental group and higher homotopy groups, continuing with Δ-complexes and their relation with abstract simplicial complexes, and afterwards we define simplicial and singular homology, together with singular homology with coefficients in an abelian group. We also state some results showing properties of induced maps on homotopy or homology groups from maps between topological spaces, and we conclude with the Universal Coefficient Theorem for Homology, which determines when homology groups (defined with integer coefficients) tensored by an abelian group are isomorphic to homology groups with coefficients in that abelian group. In Section A3 we give the definitions of convergence, continuity and gradient in a metric space. Finally, in Section A4 we briefly introduce Rips complexes.

A.1 Basic notions in algebraic topology.

This section is guided by Hatcher's Algebraic Topology [9] and [18].

Let X, Y be topological spaces and $A \subseteq X$ a topological subspace.

Definition A.1. *A* ***homotopy*** *is a continuous map* $H : X \times [0,1] \to Y$.

It is understood as a family of continuous maps $\{h_t\}$ *for each* $t \in [0,1]$, *with* $h_t : X \to Y$ *such that* $H(x,t) = h_t(x)$.

Two continuous maps $f, g : X \to Y$ *are* ***homotopic*** *if there exists a homotopy* H *connecting them, which means that* $H(x,0) = f(x)$ *and* $H(x,1) = g(x)$ *for all* $x \in X$. *We denote it by* $f \simeq g$.

Definition A.2. *A* ***retraction*** *of* X *onto* A *is a continuous map which fixes* A, *that is,* $r : X \to A$ *continuous, with* $r(X) = A$, *such that* $r|_A = id_A$. *Equivalently,*

$r \circ i = id_A$ where $i : A \hookrightarrow X$ is the inclusion map. We can also say that A is a retract of X.

Definition A.3. *For two continuous maps $f, g : X \to Y$, **a homotopy relative to the set A** is a homotopy $H : X \times [0,1] \to Y$ such that $H(a,t) = H(a,0)$ for all $a \in A$ and $t \in [0,1]$. We say that f and g are homotopic relative to A, and we denote it by $f \simeq_A g$.*

It is clear that in order to have the possibility to define a homotopy relative to $A \subseteq X$ between $f, g : X \to Y$ continuous, f and g must agree in A.

Definition A.4. *A **deformation retraction** of X onto A is a retract taken as $r : X \to X$ (since $r(X) = A$) such that $r \simeq_A id_X$, where such homotopy $H : X \times [0,1] \to X$ relative to A verifies that $H(x,0) = id_X$ and $H(x,1) = r(x)$, for every $x \in X$; and $H(a,t) = id_A$, for every $a \in A$ and $t \in [0,1]$.*

We say that A is a deformation retract of X.

Remark A.5. *A deformation retract of X onto $A \subset X$ can also be defined as a retract $r : X \to A$ such that $i \circ r \simeq id_X$, where $i : A \hookrightarrow X$ is the inclusion map.*

Definition A.6. *A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists $g : Y \to X$ continuous such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.*

*The map g is said to be an homotopy inverse of f, and the spaces X and Y to be **homotopy equivalent** or **to have the same homotopy type**. We denote it by $X \simeq Y$.*

Proposition A.7. *If A is a deformation retract of X, then X and A have the same homotopy type.*

Proof. There exists a retraction $r : X \to A$ such that $i \circ r \simeq id_X$, with $i : A \hookrightarrow X$ continuous, by Remark A.5. We also have that $r \circ i = id_A$, hence we get the definition of $r : X \to A$ homotopy equivalence, with i as its inverse. □

Remark A.8. *Being homotopic, homotopy equivalent or homotopic relative to A are equivalence relations.*

Definition A.9. *The space X is **contractible** if the identity map on X is homotopic to a constant map $c_{x_0} : X \to \{x_0\} \subset X$, for $x_0 \in X$.*

A.2 Homotopy groups, simplicial and singular homology.

The next definitions are guided by [9] (page 21 for the fundamental group, page 340 for higher homotopy groups, and page 130 for Δ-complexes).

We recall that paths were introduced in Definition 3.4.

Definition A.10. *Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. The* ***composition of paths*** *f and g, denoted by $f * g$, is again a path defined by*

$$(f * g)(t) := \begin{cases} f(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

The paths $f, g : I \to X$ are ***path homotopic*** *if $f \simeq_{\{0,1\}} g$, and such relative homotopy is called a* ***homotopy of paths****.*

The ***fundamental group****, or first homotopy group, of X with base point $x_0 \in X$, denoted by $\pi_1(X, x_0)$, is the set of path homotopic classes of loops based on x_0, together with the operation $[f] * [g] := [f * g]$ for any $[f], [g] \in \pi_1(X, x_0)$.*

Definition A.11. *Let I^n be the n-dimensional unit cube. Let $f, g : I^n \to X$ be continuous maps, so that we can define the following operation:*

$$(f + g)(t_1, \dots, t_n) := \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{if } 0 \le t_1 \le \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

where $f + g$ is also a continuous map from I^n to X.

Two maps $f, g : I^n \to X$ are said to be homotopic if $f \simeq_{\partial I^n} g$, where ∂I denotes the boundary of I.

The equivalence class of homotopic maps $f : I^n \to X$ is denoted by $[f]$.

For $x_0 \in X$, the ***higher homotopy groups*** *for $n \ge 2$ are defined by*

$$\pi_n(X, x_0) := \{[f] \quad | \quad f : I^n \to X \text{ s.t. } f(t_1 \dots, t_n) = x_0 \ \forall (t_1, \dots, t_n) \in \partial I^n\}$$

together with the operation $[f_1] + [f_2] = [f_1 + f_2]$.

Definition A.12. *Let*

$$\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, \ t_i \ge 0 \ \forall i\}$$

with the subspace topology from the Euclidean space $\mathbb{R}^{n+1}$, be the ***standard n-simplex****, where t_i are the barycentric coordinates.*

An n-simplex is the smallest convex set in an Euclidean space containing $n+1$ points $v_0, \dots, v_n$ which are affinely independent. The points v_i are the vertices of the simplex $[v_0, \dots, v_n]$.

If we delete one of the $n + 1$ vertices of an n-simplex $[v_0, \dots, v_n]$, then the remaining n vertices span an $(n-1)$ simplex, called a **face** of $[v_0, \dots, v_n]$.

The union of all the faces of Δ^n is the **boundary** of Δ^n, denoted by $\partial\Delta^n$. The **open simplex** of Δ^n is its interior, defined by $\mathring{\Delta}^n := \Delta^n \backslash \partial\Delta^n$.

Definition A.13. *A Δ-**complex** structure on a topological space X is a collection of continuous maps $\sigma_\alpha : \Delta^n \to X$, with n depending on the index α, such that:*

1. *The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$.*
2. *Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.*
3. *A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.*

The idea of Δ-complexes is to decompose a space into simplices allowing different faces of a simplex to coincide, and dropping the requirement of simplicial complexes where simplices are uniquely determined by their vertices. Therefore, it is said that Δ-complexes are a mild generalization of simplicial complexes.

We now explain such relationship between abstract simplicial complexes and Δ-complexes, so that we can justify defining simplicial homology for simplicial complexes in this thesis, as simplicial homology is in principle defined for Δ-complexes.

Claim A.14. *An abstract simplicial complex K induces a Δ-complex structure on its geometric realization $|K|$.*

Proof. From a simplicial complex K with vertex set V, we obtain the topological space $|K|$. For each q-simplex $\sigma \in K$, by the expression (1.3) in the construction of geometric realization in Chapter 1, $|\sigma| = \{\alpha : V \to I \mid \alpha(v) \neq 0 \Rightarrow v \in \sigma\}$ is in one-to-one correspondence with Δ^q, and since $|\sigma| \hookrightarrow |K| =: X$, we get a continuous map $\Delta^q \hookrightarrow X$. □

We proceed giving the definitions of simplicial and singular homology, from [9] (pages 106 and 108). We define free abelian group later in Definition A.19.

Definition A.15. *We define simplicial homology for a Δ-complex.*

To do that, it is necessary to give an order to the simplicies. Such order is the induced one by the ordering of the vertices of Δ^n.

Let C_n be the free abelian gorup generated by the maps $\sigma_\alpha : \Delta^n \to X$ of the Δ-complex. The elements of C_n are called simplicial n-chains, where a simplicial n-chain is a finite formal sum

$$\sum_\alpha c_\alpha \sigma_\alpha$$

with $c_\alpha \in \mathbb{Z}$ as the coefficients.

*We obtain a **simplicial chain complex** C, that is, the following sequence of homomorphisms of abelian groups:*

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots\dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where ∂_n is the ***boundary homomorphism*** *defined by*

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

such that $\partial_n \circ \partial_{n+1} = 0$, for each n. Such composition is equivalent to saying $im(\partial_{n+1}) \subseteq ker(\partial_n)$.

We define the n^{th}-cycle group by $Z_n := ker(\partial_n)$, and the n^{th}-boundary group by $B_n := im(\partial_{n+1})$.

Therefore, the ***n^{th}-simplicial homology group*** *is defined as the quotient group $H_n := \dfrac{Z_n}{B_n}$.*

The elements of H_n are called homology classes, and two cycles representing the same homology class are said to be homologous, which means that their difference is a boundary.

Remark A.16. *By Claim A.14, abstract simplicial complexes K induce Δ-complex structures on their geometric realizations $|K|$, so we can define their simplicial homology groups in the same way as for Δ-complexes, and we denote them by $H_n(|K|)$.*

By [9] (Theorem 2.27), these simplicial homology groups are isomorphic to singular homology groups for the topological space $|K|$ (with coefficients in $\mathbb{Z}$), which we proceed to define.

Definition A.17. *A* ***singular n-simplex*** *in a topological space X is a continuous map*

$$\sigma : \Delta^n \to X$$

where $\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1,\ t_i \geq 0\ \forall i\}$ is the standard n-simplex.

Let $C_n(X)$ be the free abelian group generated by singular n-simplices in X. The elements of $C_n(X)$, called singular n-chains, are finite formal sums

$$\sum_i c_i \sigma_i$$

for $c_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$.

The boundary homomorphism ∂_n of the ***singular chain complex*** *$C(X)$*

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots\dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

is defined by $\partial_n(\sigma_i) = \sum_{k=0}^{n} (-1)^k \sigma_i|_{[v_0,\dots,\hat{v}_k,\dots,v_n]}$, *for each* n.

We have that $\partial_n \circ \partial_{n+1} = 0$, *so as for the simplicial homology case, the* n^{th}***-singular homology group*** *of* X *is* $H_n(X) := \dfrac{ker\partial_n}{im\partial_{n+1}}$.

We can also generalize homology groups using chains of the form $\sum_i g_i\sigma_i$ where σ_i is a singular n-simplex in X as in the previous definition of singular homology, and now $g_i \in G$ for an abelian group G, rather than $\mathbb{Z}$.

Such n-chains form an abelian group $C_n(X; G) = C_n(X) \otimes G$, for $C_n(X)$ as in the previous definition. Then, following the same construction, we get $H_n(X; G)$ called the n^{th}**-homology group with coefficients in an abelian group** G.

The following discussion and proposition contain elementary results in homotopy theory ([9], page 111, page 340 and page 34 with Proposition 1.18 for the case of the fundamental group) and in homology theory ([22], Theorem 1.7 and Proposition 1.11).

A map $\varphi : X \to Y$ between topological spaces induces a map in the homotopy groups $\varphi_* : \pi_n(X, x_0) \to \pi_n(X, \varphi(x_0))$, for a base point $x_0 \in X$; defined by $\varphi_*([f]) := [\varphi \circ f]$, where $f : I^n \to X$ continuous.

It also induces a map in the homology groups, by first inducing a chain map in the singular chain complexes $\varphi_{C_*} : C(X) \to C(Y)$, defined by $\varphi_{C_n}(\sum c_i\sigma_i) :=$ $= \sum c_i(\varphi \circ \sigma_i)$ for each n, where $c_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$, or we can also write it by $\varphi_{C_n}(\sigma_i) := \varphi \circ \sigma_i$ in terms of the basis of $C_n(X)$. Then, we denote by $\partial(X)$ and $\partial(Y)$ the boundary homomorphisms in the singular chain complexes of X and Y, respectively. It holds that φ_{C_*} commutes with such boundary homomorphisms, that is, $\partial_n(Y) \circ \varphi_{C_n} = \varphi_{C_{n-1}} \circ \partial_n(X)$. Hence, φ_{C_*} sends cycles to cycles and boundaries to boundaries, inducing a map on homology $\varphi_{H_*} : H_n(X) \to H_n(Y)$.

Proposition A.18. *Let* X *and* Y *be two topological spaces, then for every* n*:*

1. *if* $\varphi : X \to Y$ *is a continuous map, then the induced maps* $\varphi_* : \pi_n(X, x_0) \to$ $\to \pi_n(X, \varphi(x_0))$ *and* $\varphi_{H_*} : H_n(X) \to H_n(Y)$ *are well-defined homomorphisms.*

2. *If* $\varphi, \psi : X \to Y$ *are homotopic, then* $\varphi_* = \psi_* : \pi_n(X, x_0) \to \pi_n(X, \varphi(x_0))$ *and* $\varphi_{H_*} = \psi_{H_*} : H_n(X) \to H_n(Y)$.

3. *If* $\varphi : X \to Y$ *is a homotopy equivalence, then* $\varphi_* : \pi_n(X, x_0) \to \pi_n(X, \varphi(x_0))$ *and* $\varphi_{H_*} : H_n(X) \to H_n(Y)$ *are isomorphisms.*

Now we discuss how the homology groups defined with coefficients in $\mathbb{Z}$ can determine the homology groups with coefficients in an abelian group G. We present the following definitions and results from [8].

Definition A.19. *An abelian group G is **free** if it has a basis, i.e, a set of linearly independent elements that generate G.*

Example A.20. *The group of integers $\mathbb{Z}$ under addition and with 0 as its identity element, is a free abelian group with the basis $\{1\}$.*

Definition A.21. *For G an abelian group, a **free presentation** of G, denoted by F, is a short exact sequence of abelian groups*

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0$$

such that F_0 and F_1 are free.

For B an abelian group, let $H_1(F \otimes B)$ denote the first homology group $ker(F_1 \otimes \otimes B \to F_0 \otimes B)$. More generally, $H_n(F \otimes B) \cong \dfrac{ker(f_n \otimes id_B)}{im(f_{n+1} \otimes id_B)}$, for $f_n : F_n \to F_{n-1}$ ([9] page 263).

Definition A.22. *For A, B abelian groups, the **torsion product** $Tor(A, B)$ is the abelain group defined by:*

$$Tor(A, B) := H_1(F \otimes B) \cong ker(F_1 \otimes B \to F_0 \otimes B)$$

for some choice of a free presentation F of A.

Remark A.23. *If A is a free abelian group, then $0 \to 0 \to A \xrightarrow{id_A} A \to 0$ is a free presentation of A, so $Tor(A, B) \cong 0$ for any abelian group B.*

The next lemma is from [9] (page 147).

Lemma A.24 (Splitting Lemma)**.** *For a short exact sequence of abelian groups $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$, the following statements are equivalent:*

1. *there exists a homomorphism $p : B \to A$ such that $p \circ i = id_A$.*
2. *There exists a homomorphism $s : C \to B$ such that $j \circ s = id_C$.*
3. *There exists an isomorphism $B \cong A \oplus C$.*

If these conditions are satisfied, the exact sequence is said to split.

Now we state the following result from [8] (Corollary 14) (also found it [9] Corollary 3A.4).

Corollary A.25 (Universal Coefficient Theorem for Homology)**.** *For a topological space X and an abelian group G, there is a natural short exact sequence*

$$0 \longrightarrow H_n(X) \otimes G \longrightarrow H_n(X;G) \longrightarrow Tor(H_{n-1}(X), G) \longrightarrow 0 \quad \text{for all } n > 0.$$

Moreover, this short exact sequence splits (but not naturally).

Corollary A.26. *For an abstract simplicial complex K, if its geometric realization $|K|$ is path-connected and G is an abelian group, then*

$$H_1(|K|;G) \cong H_1(|K|) \otimes G.$$

Proof. From Lemma A.24, the existence of a splitting in the short exact sequence of the Universal Coefficient Theorem for Homology A.25 implies that, for $n = 1$,

$$H_1(|K|;G) \cong (H_1(|K|) \otimes G) \oplus Tor(H_0(|K|), G).$$

Since $|K|$ is path-connected, $H_0(|K|) \cong \mathbb{Z}$ (by [22] Proposition 1.4), so by Example A.20 and Remark A.23, $Tor(H_0(|K|), G) \cong 0$. Hence,

$$H_1(|K|;G) \cong H_1(|K|) \otimes G.$$

□

Remark A.27. *A field is a particular case of abelian group, hence if we have a path-connected space X or $|K|$, the homology groups with integer coefficients also determine homology groups with coefficients in a field, and these last homology groups are vector spaces. We recall that they can be considered persistence modules, as explained in Section 3.3.*

For example, if $m \in \mathbb{Z}$ is prime, then (m) is a maximal ideal, and so $\mathbb{Z}/(m)$ is a field. Then,

$$H_n(\_, \mathbb{Z}/(m)) \cong H_n(\_) \otimes \mathbb{Z}/(m)$$

is a vector space.

A.3 Definitions in metric spaces.

Definition A.28. *Let (M, d) be a metric space. A sequence of points $\{x_n\}_{n \in \mathbb{N}}$ in M is a* ***convergent sequence*** *to a point $x \in M$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, for every $n \geq N$. We denote it by $\{x_n\}_{n \in \mathbb{N}} \to x$ when $n \to \infty$.*

Definition A.29. *We present three definitions of* ***continuous map****:*

1. *Let X, Y be topological spaces. A map $f : X \to Y$ is continuous if for every open (or closed) subspace $V \subseteq Y$, $f^{-1}(V)$ is open (or closed) in X.*

2. *Let (X, d_1) and (Y, d_2) be metric spaces. A map $f : X \to Y$ is continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_1(x, x_0) < \delta$, then $d_2(f(x), f(x_0)) < \varepsilon$, for every $x \in X$ (assuming X is the domain of f). We say that f is continuous if it is continuous at every $x \in X$.*

3. *Let (X, d_1) and (Y, d_2) be metric spaces. A map $f : X \to Y$ is continuous if for every convergent sequence $\{x_n\}_{n\in\mathbb{N}} \to x$ in X, then $\{f(x_n)\}_{n\in\mathbb{N}} \to f(x)$ in Y.*

These three definitions are all equivalent between them if X and Y are metric spaces. In [16] (Theorem 21.1) it is proven that (1) is equivalent to (2), and also in [16] (Theorem 21.3), it is proven that (1) is equivalent to (3).

Definition A.30. *The **gradient** of a multivariable function $f : \mathbb{R}^d \to \mathbb{R}$ at a point $p \in \mathbb{R}^d$ is the map $gradf : \mathbb{R}^d \to \mathbb{R}^d$, that evaluated at p consists of the vector whose components are the partial derivatives of f at p, i.e,*

$$gradf(p) = \left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_d}(p) \right).$$

A.4 Rips complexes.

In this part of the appendix we present another type of simplicial complexes, the Rips complexes, together with some results relating them to Čech complexes from [24] and [25]. We also present Hausmann's Theorem A.32, which works as an equivalent result to the Nerve lemma but for the case of Rips complexes.

The motivation is mainly to help the reader to follow the bibliography of this thesis, since most of our results are new versions for Čech complexes of original ones for Rips complexes.

We have chosen the Čech complex as our tool to reconstruct a Euclidean subspace, rather than the Rips complex, because it is exactly the nerve of a cover, and therefore it can be more interesting from a geometric or theoretical point of view. However, Rips complexes are used more frequently because they are easier to compute, and from a more applied point of view, they can be preferable.

Definition A.31. *For X a metric space with metric d and $r > 0$, the **Rips complex** is an abstract simplicial complex defined by*

$$Rips(X, r) := \{\sigma = [x_0, \ldots, x_k] \subseteq X \mid Diam(\sigma) < r\}$$

where $Diam(\sigma) := sup_{x_i, x_j \in \sigma}\, d(x_i, x_j)$.

We now present Hausman's Theorem (Theorem 4.9 in [24]) which works as a reconstruction result in an analogous way as the Nerve Lemma does for Čech complexes.

Theorem A.32 (Hausman's Theorem.)**.** *Suppose X is a geodesic space with convexity radius $\rho > 0$ (for example, a compact Riemannian manifold). Then $X \simeq |Rips(X, q)|$, for $0 < q \leq \rho/2$.*

By the definitions, we always have for a metric space X and $r > 0$ the following inclusions:

$$\mathscr{C}^d_X(X, r/2) \hookrightarrow Rips(X, r) \hookrightarrow \mathscr{C}^d_X(X, r).$$

The following result ([24], Theorem 5.6) shows that these Rips and Čech complexes coincide locally.

Theorem A.33. *Let X be a geodesic space with convexity radius $\rho > 0$ and $0 \leq r \leq \frac{\rho}{2}$. Then, the inclusions*

$$\mathscr{C}^{d_L}_X(X, r/2) \hookrightarrow Rips(X, r) \hookrightarrow \mathscr{C}^{d_L}_X(X, r)$$

induce homotopy equivalences.

The following result gives an isomorphism between the persistence groups of Čech complexes and persistence groups of Rips complexes ([25], Theorem 10.6).

Theorem A.34. *For X a geodesic space and G an Abelian group, there are isomorphisms of persistences*

$$\{\pi_1(|\mathscr{C}^{d_L}_X(X, 3r)|, \bullet)\}_{r>0} \cong \{\pi_1(|Rips(X, 4r)|, \bullet)\}_{r>0}$$

and

$$\{H_1(|\mathscr{C}^{d_L}_X(X, 3r)|; G)\}_{r>0} \cong \{H_1(|Rips(X, 4r)|; G)\}_{r>0}.$$

Bibliography

[1] Ulrich Bauer, Michael Lesnick, *Induced Matchings and the Algebraic Stability of Persistence Barcodes.* arXiv:1311.3681v4 [math.AT] 30 Apr 2016. URL https://arxiv.org/pdf/1311.3681.pdf.

[2] Samir Chowdhury, Facundo Mémoli, *A Functorial Dowker Theorem and Persistent Homology of Asymmetric Networks.* arXiv: 1608.05432v3 [math.AT], 6 Jan 2018.

[3] Herbert Eldersbrunner, John Harer, *Computattional Topology: An Introduction.* Department of Computer Science and Mathematics Duke University, DOI: 10.1007/978-3-540-33259-6-7. Source: DBLP.

[4] Herbert Eldersbrunner, John Harer, *Persisten Homology - a Survey.* 1991 Mathematics Subject Classification. Primary 55N99,68W01; Secondary 57M99,55T05,52-02.

[5] B.T.Fasy,R.Komendarczyk, S.Majhi, C. Wenk, *On the Reconstruction of Geodesic Subspaces of* $\mathbb{R}^N$. arXiv:1810.10144v3 [math.AT], 21 Feb 2020. URL https://arxiv.org/pdf/1810.10144.pdf.

[6] Herbert Federer, *Curvature Measures.* Transactions of the American Mathematical Society, Vol. 93, No. 3 (Dec., 1959), pp. 418-491.

[7] Herbert Federer (auth.), Prof. Dr. B. Eckmann, Prof. Dr. B. L. van der Waerden (eds.), *Geometric Measure Theory.* Springer-Verlag Berlin Heidelberg, 1996. ISBN 13: 978-3-642-62010-2.

[8] Moritz Groth, *Lecture Notes on Algebraic Topology II, Section 02: Universal Coefficient Theorem in Singular Homology.*

[9] Allen Hatcher, *Algebraic Topology.* Cambridge University Press, 02 Jul 2002. ISBN 10 0521795400.

[10] I.M.James, *History of Topology.* Oxford University, UK. North Holland; 1 edition, March 13, 2006. ISBN 0-444-82375-1.

[11] M.Katetov, P.Simon, *The Mathematical Legacy of Eduard Čech.* Birkhäuser; 1993 edition, December 12, 2012. DOI 10.1007/978-3-0348-7524-0, ISBN 978-3-0348-7526-4.

[12] Jisu Kim, Jaehyeok Shin, Alessandro Rinaldo, Larry Wasserman, *Nerve Theorem on a Positive Reach set.* arXiv:1903.06955v1 [math.AT], 16 Mar 2019.

[13] Jisu Kim, Jaehyeok Shin, Frédéric Chazal, Alessandro Rinaldo, Larry Wasserman, *Homotopy Reconstruction via the Cech Complex and the Vietoris-Rips Complex.* arXiv:1903.06955v3 [math.AT] 12 May 2020. URL https://arxiv.org/abs/1903.06955.

[14] Joshua R. Mirth, *Master Thesis: Metric Thickenings of Euclidean Submanifolds.* Colorado State University, Fall 2017.

[15] Lars Moberg Salbu, *Master Thesis: Dowker's Theorem by Simplicial Sets and a Category of 0-Interleavings.* University of Bergen, 21 Jun 2019. URL http://hdl.handle.net/1956/20325.

[16] James Munkres, *Topology (Second Edition).* Springer, Graduate Texts in Mathematics, Pearson Education Limited, 2014. ISBN 13: 978-1-292-02362-5.

[17] P.Niyogi, S. Smale, S. Weinberg, *Finding the Homology of Submanifolds with High Confidence from Random Samples.* Discrete Comput. Geom. 39:419-441, Mar 2008. DOI 10.1007/s00454-008-9053-2.

[18] Jesús M. Ruiz, *Notas de Topología Algebraica muy elemental.* Departamento de Geometría y Topología, UCM.

[19] Edwin H. Spanier, *Algebraic Topology.* Springer-Verlag New-York, 1994.

[20] Michael Spivak, *Calculus (Fourth Edition).* Publish or Perish, Inc. PMB 377, 1302 Waugh Drive, Houston, Texas 77019. ISBN 978-0-914098-91-1.

[21] Wilson A.Sutherland, *Introduction to Metric and Topological Spaces.* Oxford University Press, 2009. ISBN 019956308X.

[22] James W.Vick, *Homology Theory. An Introduction to Algebraic Topology (Second Edition).* Graduate Texts in Mathematics, Springer-Verlag New York, 1994. DOI: 10.1007/978-1-4612-0881-5.

[23] Žiga Virk, *Approximations of 1-Dimensional Intrinsic Persistence of Geodesic Spaces and Their Stability.* arXiv:1711.05111v4 [math.GT], May 25, 2018. URL https://arxiv.org/pdf/1711.05111v4.pdf.

[24] Žiga Virk, *Rips complexes as nerves and a Functorial Dowker-Nerve Diagram.* arXiv:1906.04028 [math.MG], October 30, 2019. URL https://arxiv.org/pdf/1906.04028v2.pdf.

[25] Žiga Virk, *1-Dimensional Intrinsic Persistence of Geodesic Spaces.* arXiv:1709.05164 [math.GT], March 15, 2018. URL https://arxiv.org/pdf/1709.05164.pdf.

[26] Raymond O. Wells, Jr, *Differential and Complex Geometry: Origins, Abstractions and Embeddings.* Cham : Springer, 2017.

[27] Erland Wheeler, *Fundamental Groups of Simplicial Complexes.* arXiv:1605.08343 [math.AT], 15 Jan, 2017. URL https://arxiv.org/pdf/1605.08343.pdf.