Homotopy method for seismic modeling in strongly scattering acoustic media with density variation

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SUMMARY

The wave equation for acoustic media with variable density and velocity can be transformed into an integral equation of the Lippmann-Schwinger type; but for a 4-dimensional state vector involving the gradient of the pressure field as well as the pressure field itself. The Lippmann-Schwinger equation can in principle be solved exactly via matrix inversion, but the computational cost of matrix inversion scales like N^3, where N is the number of grid blocks. The computational cost can be significantly reduced if one solves the Lippmann-Schwinger equation iteratively. However, the popular Born series is only guaranteed to converge if the contrasts and the size of the model (relative to the wavelength) are relatively small. In this study, we have used the so-called homotopy analysis method to derive an iterative method of the Lippmann-Schwinger equation which is guaranteed to converge independent of the contrasts and size of the model. The computational cost of our convergent scattering series scales as N^2 times the number of iterations. Our algorithm, which is based on the homotopy analysis method, involves a convergence control operator that we select using a randomized matrix factorization. We illustrate the performance of the new convergent scattering series by seismic wavefield modeling in a strongly scattering salt model with variable density and velocity.

INTRODUCTION

Seismic wavefield modeling is a simulation of seismic wave propagation in an assumed structure of the subsurface (Yilmaz, 2001). It is a valuable tool for seismic interpretation and an essential part of seismic inversion (Carcione et al., 2002). Therefore, many approaches to seismic modeling have already been developed. One class of them called integral equation methods consider the total wavefield in actual media as a superposition of wavefields in the reference media due to a real source and a virtual source which is caused by the contrast between actual media and reference media (Jakobsen and Ursin, 2015). After discretization of the relevant integral operators, the integral equation transformed from the wave equation can be solved very accurately through matrix inversion (Jakobsen and Ursin, 2015). However, matrix inversion with a computational cost that scales like the number of grid blocks cubed is very costly. In order to avoid matrix inversion, we can modify some highly developed methods in quantum physics, such as the well-known Born series, to solve the integral equation. However, the conventional Born series is only guaranteed to converge if the contrast and the size of the model (relative to the wavelength) is relatively small (Jakobsen and Ursin, 2015).

In order to find a convergent scattering series solution in the case of strongly scattering media, we can use the homotopy analysis method (HAM) developed by Liao (2003) to solve the integral equation. Huang and Greenhalgh (2018) first introduced the modern HAM into geophysical applications. Jakobsen et al. (2020a) have proposed a general scattering series solution of the Lippmann-Schwinger equation through HAM in the special case of a fixed density. Provided that the so-called convergence control operator is chosen in a specific manner, the novel scattering series is guaranteed to converge in strong scattering cases. Jakobsen et al. (2020b) have used different variants of the homotopy continuation method and discussed their relationship with renormalization group theory. In this paper, we extend the HAM scattering series to the case of variable velocity and density. The original HAM scattering series derived by Jakobsen et al. (2020a) is guaranteed to converge independent of the scattering potential. Therefore, it is attractive when more model properties are involved and high accuracy is required.

The paper is structured as follows. First we transform the wave equation for acoustic media into an integral equation of the Lippmann-Schwinger type; but for a 4-dimensional state vector including the pressure field and the gradient of the pressure field. Then we discuss the exact solution based on matrix inversion as well as the conventional Born series. Next we use HAM to derive homotopy scattering series and randomized matrix factorization to select the convergence control operator. Finally we use numerical examples to demonstrate the performance of the proposed method and compare it with conventional Born series.

THEORY

Integral equation and conventional Born series

The acoustic wave equation in the frequency domain can be written as (Cerveny, 2005)

$$\left(\nabla\cdot\frac{1}{\rho(\mathbf{x})}\nabla+\frac{\omega^2}{\kappa(\mathbf{x})}\right)P(\mathbf{x})=-S(\mathbf{x}_s), \tag{1}$$

where ∇ is the spatial gradient operator, ω is the angular frequency, $\kappa(\mathbf{x})=\rho(\mathbf{x})\cdot v(\mathbf{x})^2$ is elastic moduli related to density $\rho(\mathbf{x})$ and velocity $v(\mathbf{x})$, $P(\mathbf{x})$ is the pressure wavefield in acoustic media and $S(\mathbf{x}_s)$ represents the source located at $\mathbf{x}_s$ position. The wavefield $P(\mathbf{x})$, the source function $S(\mathbf{x}_s)$ and the following Green's functions are all dependent on the angular frequency ω, but we will in the following supress this dependency.

By using the source representation integral and partial integration, we obtain the following integral equation (Cerveny, 2005; Jakobsen and Ursin, 2015):

$$\begin{aligned}P(\mathbf{x})=&P^{(0)}(\mathbf{x})+\omega^2\int_D d\mathbf{x}'G^{(0)}(\mathbf{x},\mathbf{x}')\,\chi_\kappa(\mathbf{x}')\,P(\mathbf{x}')\\&-\int_D d\mathbf{x}'\nabla_{\mathbf{x}'}G^{(0)}(\mathbf{x},\mathbf{x}')\cdot\chi_\rho(\mathbf{x}')\,\nabla_{\mathbf{x}'}P(\mathbf{x}'),\end{aligned} \tag{2}$$

where $P^{(0)}(\mathbf{x})$ and $G^{(0)}(\mathbf{x},\mathbf{x}')$ are the wavefield and Green's function for the arbitrary reference medium, $\chi_\rho(\mathbf{x}) \equiv 1/\rho(\mathbf{x}) - 1/\rho^{(0)}(\mathbf{x})$ is the contrast in density and $\chi_\kappa(\mathbf{x}) \equiv 1/\kappa(\mathbf{x}) - 1/\kappa^{(0)}(\mathbf{x})$ is the contrast in bulk moduli, D is the scattering domain where the contrasts are non-zero. Explicit analytical formulae for 2D and 3D Green's functions in the case of homogeneous acoustic media are given by Cerveny (2005).

As we can see, equation (2) includes the spatial derivative of the pressure field and the pressure field itself which makes the forward modeling more complicated. In order to solve equation (2) using the known methods of solving Lippmann-Schwinger equations without spatial derivatives, we need the gradient of the pressure field. From equation (2), we obtain

$$\begin{aligned}\nabla_{\mathbf{x}} P(\mathbf{x}) =& \nabla_{\mathbf{x}} P^{(0)}(\mathbf{x}) + \omega^2 \int_D d\mathbf{x}' \nabla_{\mathbf{x}} G^{(0)}(\mathbf{x},\mathbf{x}')\chi_\kappa(\mathbf{x}') P(\mathbf{x}') \\ &- \int_D d\mathbf{x}' \nabla_{\mathbf{x}} \nabla_{\mathbf{x}'} G^{(0)}(\mathbf{x},\mathbf{x}') \cdot \chi_\rho(\mathbf{x}') \nabla_{\mathbf{x}'} P(\mathbf{x}').\end{aligned} \tag{3}$$

Integral equations (2-3) can be combined into a single integral equation of the Lippmann-Schwinger type (Jakobsen and Ursin, 2015):

$$\Psi(\mathbf{x}) = \Psi^{(0)}(\mathbf{x}) + \int_D d\mathbf{x}' \mathscr{G}^{(0)}(\mathbf{x},\mathbf{x}') \mathscr{V}(\mathbf{x}') \Psi(\mathbf{x}'), \tag{4}$$

where $\Psi(\mathbf{x}) = (P(\mathbf{x}), \nabla_{\mathbf{x}} P(\mathbf{x}))^{\mathsf{T}}$ and $\Psi^{(0)}(\mathbf{x}) = \left(P^{(0)}(\mathbf{x}), \nabla_{\mathbf{x}} P^{(0)}(\mathbf{x})\right)^{\mathsf{T}}$ are 4×1 vectors of the wavefields and the spatial derivative of wavefields in the actual and reference media;

$$\mathscr{V}(\mathbf{x}') = \begin{pmatrix} \chi_\kappa(\mathbf{x}') & \mathbf{0} \\ \mathbf{0} & \chi_\rho(\mathbf{x}') \end{pmatrix}$$

is a 4×4 scattering potential operator, the $\chi_\rho(\mathbf{x}')$ is a 3×3 diagonal matrix;

$$\mathscr{G}^{(0)}(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \omega^2 G^{(0)}(\mathbf{x},\mathbf{x}') & -\nabla_{\mathbf{x}'} G^{(0)}(\mathbf{x},\mathbf{x}') \\ \omega^2 \nabla_{\mathbf{x}} G^{(0)}(\mathbf{x},\mathbf{x}') & -\nabla_{\mathbf{x}} \nabla_{\mathbf{x}'} G^{(0)}(\mathbf{x},\mathbf{x}') \end{pmatrix}$$

is a 4×4 operator of Green's function and its first- and second-order spatial derivatives. The dimensions of the above vectors and operators will be reduced in the case of an 2D application.

After discretization of the Lippmann-Schwinger equation (4), we can arrange $\Psi(\mathbf{x})$ and $\Psi^{(0)}(\mathbf{x})$ at the discretized scatterering volume positions into vectors $\psi = (\Psi_1, ..., \Psi_N)^{\mathsf{T}}$ and $\psi^{(0)} = (\Psi_1^{(0)}, ..., \Psi_N^{(0)})^{\mathsf{T}}$, where N is the number of discretized grid blocks (Jakobsen and Ursin, 2015). The Lippmann-Schwinger equation (4) can be rewritten exactly in matrix notation as

$$\psi = \psi^{(0)} + g^{(0)} V \psi, \tag{5}$$

where ψ and $\psi^{(0)}$ are both $4N \times 1$ vectors, $g^{(0)}$ and V of size $4N \times 4N$ are the matrix elements of operators $\mathscr{G}^{(0)}$ and $\mathscr{V}$. From equation (5), we can get the following exact reference solution:

$$\psi = \left(I - g^{(0)} V\right)^{-1} \psi^{(0)}, \tag{6}$$

where I is $4N \times 4N$ identity matrix. Equation (6) for calculation of the reference solution by matrix inversion is applicable in any case, independent of the strength of the scattering potential V. However, the computational cost of inversion of a huge full matrix $\left(I - g^{(0)} V\right)$ scales like $(4N)^3$, which is very costly due to the large number of grid blocks in practical applications. To avoid huge matrix inversion and reduce computational cost, an iterative method based on the well-known Born series (Morse and Feshbach, 1954)

$$\psi = \left[\sum_{m=0}^{\infty} \left(g^{(0)} V\right)^m\right] \psi^{(0)} \tag{7}$$

should be a better choice for the solution of equation (5) in practical use. The Born series is very simple and easy to use. However, it is only guarantted to converge when the spectral radius, $\rho(g^{(0)}V)$, is smaller than one, which is often not the case for realistic seismic models.

Homotopy method for the Lippmann-Schwinger equation

In order to make the iterative method convergent, we use the the homotopy analysis method (HAM) to solve the Lippmann-Schwinger equation. The HAM can be used to solve nonlinear problems in this form (Liao, 2003) :

$$\mathscr{N}[\psi] = 0, \tag{8}$$

where $\mathscr{N}$ is the nonlinear operator and ψ is the target function. Then we can define the homotopy operator $\mathscr{H}$ (Liao, 2003) as

$$\mathscr{H}[\Phi, \lambda] \equiv (1-\lambda)\mathscr{L}[\Phi(\lambda) - \psi_0] + \lambda H \mathscr{N}[\Phi(\lambda)], \tag{9}$$

where $\lambda \in [0,1]$ is the embedding parameter, H is the convergence control operator, ψ_0 is the initial guess and $\mathscr{L}$ is an linear operator that must be chosen to satisfy $\mathscr{L}[0] = 0$.

If we set $\mathscr{H}[\Phi, \lambda] = 0$, then we can obtain the zero-order deformation equation (Liao, 2003) :

$$(1-\lambda)\mathscr{L}[\Phi(\lambda) - \psi_0] = -\lambda H \mathscr{N}[\Phi(\lambda)]. \tag{10}$$

When we let $\lambda = 0$, we can find $\mathscr{L}[\Phi(0) - \psi_0] = 0$, which means that $\Phi(0) = \psi_0$ which is the initial guess of our nonlinear problem. When we set $\lambda = 1$ then $\mathscr{N}[\Phi(1)] = 0$, which means that $\Phi(1) = \psi$, and ψ is the solution of equation (8). According to the analysis above, if we modify λ from 0 to 1 gradually and then we can get the exact solution ψ of our nonlinear problem from the intial guess ψ_0.

Then we differentiate both sides of the equation (10) m times with respect to λ, divide the result by $m!$ and set $\lambda = 0$. Finally we obtain the mth-order deformation (Liao, 2003) :

$$\mathscr{L}[\psi_m - \chi_m \psi_{m-1}] = -H R_m, \tag{11}$$

where

$$\psi_m \equiv \frac{1}{m!} \frac{\partial^m \Phi(\lambda)}{\partial \lambda^m}\bigg|_{\lambda=0} \qquad \chi_m = \begin{cases} 0 & \text{if } m \le 1 \\ 1 & \text{if } m \ge 2 \end{cases}$$

and

$$R_m = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial \lambda^{m-1}} \mathscr{N}[\Phi(\lambda)]\right)\bigg|_{\lambda=0}.$$

After summing up all the $\psi_m \lambda^m$, we can find that

$$\sum_{m=0}^{\infty} \psi_m \lambda^m = \sum_{m=0}^{\infty} \frac{\partial^m \Phi(\lambda)}{\partial \lambda^m}\bigg|_{\lambda=0} \lambda^m = \Phi(\lambda) \tag{12}$$

is the Maclaurin series of $\Phi(\lambda)$. If we set $\lambda = 1$ in equation (12), then $\Phi(1) = \psi = \sum_{m=0}^{\infty} \psi_m$, which means we can find a new series solution from equation (11).

In order to use the above HAM to solve the Lippmann-Schwinger equation (5), first we define the linear operator $\mathscr{L}$ and nonlinear operator $\mathscr{N}$ as (Jakobsen et al., 2020a)

$$\mathscr{L}[\psi] = \psi, \quad \mathscr{N}[\psi] = \psi - \psi^{(0)} - g^{(0)} V \psi. \tag{13}$$

If we insert equation (13) into equation (11), then we can obtain the HAM scattering series of the Lippmann-Schwinger equation (Jakobsen et al., 2020a):

$$\psi = \left[\sum_{m=0}^{\infty} M^m \right] H \psi^{(0)}, \tag{14}$$

where

$$M \equiv I - H + H g^{(0)} V. \tag{15}$$

Equation (7), (14) and (15) show that the HAM scattering series of the Lippmann-Schwinger equation is more flexible than the conventional Born series, because it introduces a convergence control operator H. It is possible to find a suitable H to make sure $\rho(M) < 1$ and the HAM scattering series of the Lippmann-Schwinger equation converges, even if $\rho(g^{(0)}V) > 1$ and the Born series diverges.

For choosing a suitable H, Eikrem et al. (2020) have proposed a method based on randomized matrix factorization. From the above discussion, we must make $\rho(M) < 1$ to get a converging HAM scattering series. According to the equation (15), if H approximates $\left(I - g^{(0)} V\right)^{-1}$, then $\rho(M)$ approaches 0. In order to find a good approximation of $\left(I - g^{(0)} V\right)^{-1}$, first we approximate $g^{(0)}V$ by a product of two low rank matrices

$$g^{(0)} V \approx U W^{\mathsf{T}}, \tag{16}$$

where U and W are $4N \times k$ matrices, N is the number of total grid blocks and k is a relatively small number and T is the complex conjugate transpose. Then $\left(I - g^{(0)} V\right)^{-1}$ can be approximated as

$$\left(I - g^{(0)} V\right)^{-1} \approx \left(I - U W^{\mathsf{T}}\right)^{-1}. \tag{17}$$

Combining equation (17) with the Sherman-Morrison-Woodbury formula

$$(A - BC)^{-1} = A^{-1} + A^{-1} B \left(I - C A^{-1} B\right)^{-1} C A^{-1}, \tag{18}$$

we get

$$\left(I - g^{(0)} V\right)^{-1} \approx \left(I - U W^{\mathsf{T}}\right)^{-1} = I + U \left(I_k - W^{\mathsf{T}} U\right)^{-1} W^{\mathsf{T}}, \tag{19}$$

where I_k is $k \times k$ identity matrix. It can be seen from equation (19) that we can use a small matrix inversion to obtain an approximation of a huge matrix inversion. Finally the H can be selected as (Eikrem et al., 2020)

$$H = I + U \left(I_k - W^{\mathsf{T}} U\right)^{-1} W^{\mathsf{T}}. \tag{20}$$

This simple method of choosing H works well for small models and for low frequencies, but for larger models a more efficient method based on hierarchical matrices is also presented in Eikrem et al. (2020).

NUMERICAL EXAMPLES

We use a resampled SEG/EAGE salt model (Figure 1, upper) to test the validity of our method (Aminzadeh et al., 1997). The bulk density of our model (Figure 1, lower) apart from the salt dome portion is obtained from velocity by Gardner's relation: $\rho = 0.31 V_p^{0.25}$ (Gardner et al., 1974). The density of the salt dome portion equal to the density of halite salt, which is 2163 kg/m^3 (Mavko et al., 2020). The grid size of our model is 1390 m wide and 290 m deep. The discrete grid size of the real-space in horizontal and vertical direction are both 10 m. The number of grid blocks is 4031 ($N = 139 \times 29 = 4031$). We use Ricker wavelet to simulate a source term located exactly in the middle of the upper row of the model.

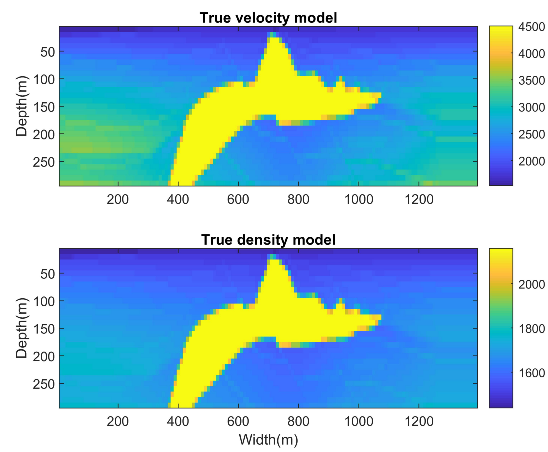


Figure 1: Modified version of the SEG/EAGE salt model, where the density variations (bottom) are related with the velocity variations (top) via Gardners relation.

To calculate H, we use a small matrix inversion with $k = 800$. All the forward modeling results are shown in the frequency domain at 20 Hz. The wavefield in acoustic media is pressure. Figure 2 shows the reference wavefield obtained via matrix inversion. Figure 3 shows wavefields computed via conventional Born series and HAM series. To quantify the convergence performance of different iterative methods, we compute the normalized overall difference which is defined as $\delta_i = \left\| \psi_i - \psi^{(r)} \right\| / \left\| \psi^{(r)} \right\|$, where $\psi^{(r)}$ is the reference wavefield and ψ_i is the iterative wavefield after ith iteration. Figure 4 represents a comparison of the convergence performance of the HAM series and the Born series. In Figure 3 and Figure 4, we can see that the HAM wavefield is visually equal to the reference wavefield and the normalized overall difference close to 0 after about 70 iterations while the Born series diverges.

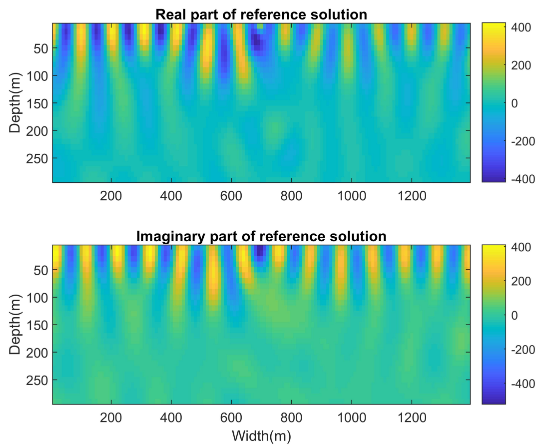
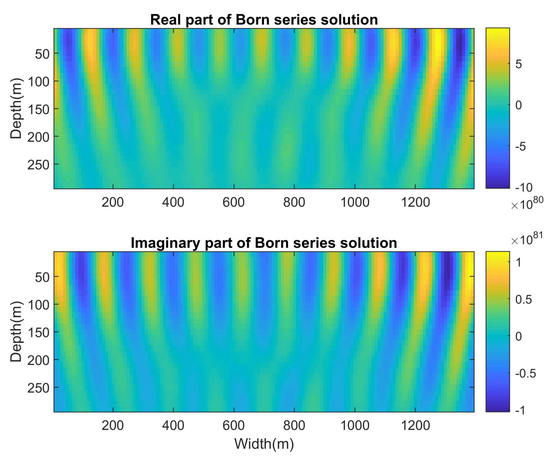
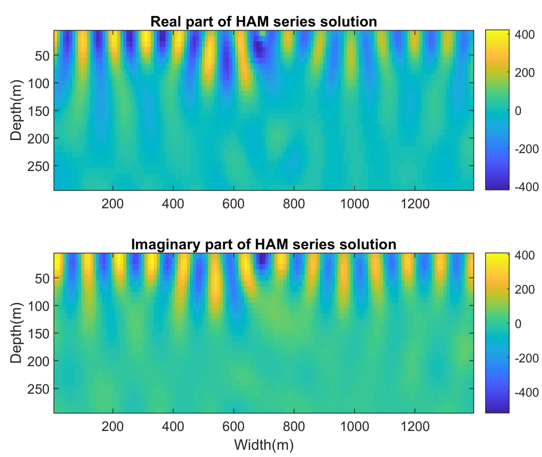


Figure 2: Reference wavefield within the model in Figure 1.



(a) Born series wavefield



(b) HAM series wavefield

Figure 3: Wavefields obtained via different scattering series.

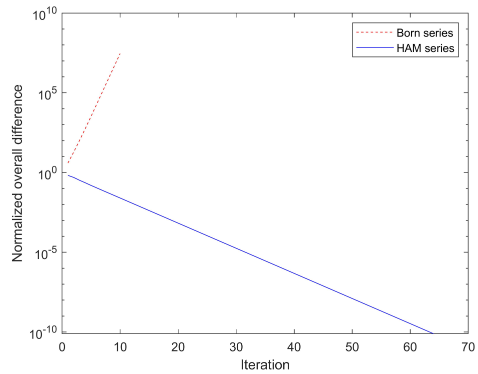


Figure 4: Normalized overall differences vs iteration for different scattering series.

CONCLUIONS AND DISCUSSIONS

We have introduced a numerical method for seismic wavefield modeling in strongly scattering acoustic media with variable velocity and density. Numerical examples show that the 4-dimensional integral equation of the Lippmann-Schwinger type can be used to do seismic wavefield modeling via either marix inversion or iterative methods. Compared to matrix inversion, the convergent iterative method can reduce the computational cost from $(4N)^3$ to $n \cdot (4N)^2$, where n is the number of iterations. Compared with the conventional Born series, the homotopy analysis scattering series can ensure convergence in strongly scattering case through introducing a suitable convergence control operator H.

Having developed a convergent scattering series solution of the wave equation for acoustic media with variable density and velocity, the next step may be to derive a corresponding convergent inverse scattering series (Weglein et al., 2003). However, convergence of the (direct) scattering series does not necessarily imply convergence of the corresponding inverse scattering series (Jakobsen et al., 2020a,b).

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