A note on Severi varieties of nodal curves on Enriques surfaces

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Abstract Let $|L|$ be a linear system on a smooth complex Enriques surface S whose general member is a smooth and irreducible curve of genus p, with $L^2 > 0$, and let $V_{|L|,\delta}(S)$ be the Severi variety of irreducible δ-nodal curves in $|L|$. We denote by $\pi : X \longrightarrow S$ the universal covering of S. In this note we compute the dimensions of the irreducible components V of $V_{|L|,\delta}(S)$. In particular we prove that, if C is the curve corresponding to a general element $[C]$ of V, then the codimension of V in $|L|$ is δ if $\pi^{-1}(C)$ is irreducible in X and it is $\delta - 1$ if $\pi^{-1}(C)$ consists of two irreducible components.

1 Introduction

Let S be a smooth complex projective surface and L a line bundle on S such that the complete linear system $|L|$ contains smooth, irreducible curves (such a line bundle, or linear system, is often called a *Bertini system*). Let

$$p := p_a(L) = \frac{1}{2} L \cdot (L + K_S) + 1,$$

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be the arithmetic genus of any curve in $|L|$.

For any integer $0 \leq \delta \leq p$, consider the locally closed, functorially defined subscheme of $|L|$

$$V_{|L|,\delta}(S) \text{ or simply } V_{|L|,\delta}$$

parameterizing irreducible curves in $|L|$ having only δ nodes as singularities; this is called the *Severi variety* of δ-nodal curves in $|L|$. We will let $g := p - \delta$, the geometric genus of the curves in $V_{|L|,\delta}$.

It is well-known that, if $V_{|L|,\delta}$ is non-empty, then all of its irreducible components V have dimension $\dim(V) \geq \dim|L| - \delta$. More precisely, the Zariski tangent space to $V_{|L|,\delta}$ at the point corresponding to C is

$$T_{[C]}V_{|L|,\delta} \simeq H^0(L \otimes \mathcal{J}_N)/< C >, \tag{1}$$

where $\mathcal{J}_N = \mathcal{J}_{N|S}$ is the ideal sheaf of subscheme N of S consisting of the δ nodes of C (see, e.g., [4, §1]). Thus, $V_{|L|,\delta}$ is *smooth of dimension* $\dim|L| - \delta$ at $[C]$ if and only if the set of nodes N imposes independent conditions on $|L|$. In this case, $V_{|L|,\delta}$ is said to be *regular* at $[C]$. An irreducible component V of $V_{|L|,\delta}$ will be said to be *regular* if the condition of regularity is satisfied at any of its points, equivalently, if it is smooth of dimension $\dim|L| - \delta$.

The *existence and regularity problems of* $V_{|L|,\delta}(S)$ have been studied in many cases and are the most basic problems one may ask on Severi varieties. We only mention some of known results. In the case $S \simeq \mathbb{P}^2$, Severi proved the existence and regularity of $V_{|L|,\delta}(S)$ in [14]. The description of the tangent space is due to Severi and later to Zariski [15]. The existence and regularity of $V_{|L|,\delta}(S)$ when S is of general type has been studied in [4] and [3]. Further regularity results are provided in [10]. More recently Severi varieties on K3 surfaces have received a lot of attention for many reasons. In this case Severi varieties are known to be regular (cf. [13]) and are nonempty on general K3 surfaces by Mumford and Chen (cf. [12], [2]).

As far as we know, Severi varieties on Enriques surfaces have not been studied yet, apart from [8, Thm. 4.12] which limits the singularities of a general member of the Severi variety $V^g_{|L|}$ of irreducible genus g curves in $|L|$, and gives a sufficient condition for the density of the latter in the Severi variety $V_{|L|,p-g}$ of $(p-g)$-nodal curves. In particular, the existence problem is mainly open and we intend to treat it in a forthcoming article. The result of this paper is Proposition 1, which answers the regularity question for Severi varieties of nodal curves on Enriques surfaces.

2 Regularity of Severi varieties on Enriques surfaces

Let S be a smooth Enriques surface, i.e. a smooth complex surface with nontrivial canonical bundle $\omega_S \ncong \mathcal{O}_S$, such that $\omega_S^{\otimes 2} \simeq {}_S$ and $H^1(\mathcal{O}_S) = 0$. We denote linear (resp. numerical) equivalence by $\sim$ (resp. $\equiv$).

Let L be a line bundle on S such that $L^2 > 0$. It is well-known that $|L|$ contains smooth, irreducible curves if and only if it contains irreducible curves (see [5, Thm.

4.1 and Prop. 8.2]); in other words, *on Enriques surfaces the Bertini linear systems are the linear systems that contain irreducible curves*. Moreover, by [6, Prop. 2.4], this is equivalent to L being nef and not of the form $L \sim P + R$, with $|P|$ an elliptic pencil and R a smooth rational curve such that $P \cdot R = 2$ (in which case $p = 2$). If $|L|$ is a Bertini linear system, the adjunction formula, the Riemann–Roch theorem, and Mumford vanishing yield that

$$L^2 = 2(p-1) \quad \text{and} \quad \dim |L| = p - 1$$

(see, e.g., [5, 7]).

Let K_S be the canonical divisor. It defines an étale double cover

$$\pi : X \longrightarrow S \tag{2}$$

where X is a smooth, projective $K3$ surface (that is, $\omega_X \simeq O_X$ and $H^1(O_X) = 0$), endowed with a fixed-point-free involution ι, which is the universal covering of S. Conversely, the quotient of any $K3$ surface by a fixed-point-free involution is an Enriques surface.

Let $C \subset S$ be a reduced and irreducible curve of genus $g \geq 2$. We will henceforth denote by $\nu_C : \widetilde{C} \to C$ the normalization of C and define $\eta_C := O_C(K_S) = O_C(-K_S)$, a nontrivial 2-torsion element in $\operatorname{Pic}^0 C$, and $\eta_{\widetilde{C}} := \nu_C^* \eta_C$. The fact that η_C is nontrivial follows from the cohomology of the restriction sequence

$$0 \longrightarrow O_S(K_S - C) \longrightarrow O_S(K_S) \longrightarrow \eta_C \longrightarrow 0,$$

which yields $h^0(\eta_C) = h^1(K_S - C) = h^1(C) = 0$, the latter vanishing as C is big and nef. One has the fiber product

$$\begin{tikzcd}
(\pi^{-1}C) \times_C \widetilde{C} \arrow[r] \arrow[d] & \widetilde{C} \arrow[d, "\nu_C"] \\
(\pi^{-1}C) \arrow[r, "\pi|_{\pi^{-1}(C)}"'] & C,
\end{tikzcd}$$

where $\pi|_{\pi^{-1}(C)}$ and the upper horizontal map are the double coverings induced respectively by η_C and $\eta_{\widetilde{C}}$. By standard results on coverings of complex manifolds (cf. [1, Sect. I.17]), two cases may happen:

- $\eta_{\widetilde{C}} \not\simeq O_{\widetilde{C}}$ and $\pi^{-1}C$ is irreducible, as in Fig. 1;
- $\eta_{\widetilde{C}} \simeq O_{\widetilde{C}}$ and $\pi^{-1}C$ consists of two irreducible components conjugated by the involution ι. These two components are *not* isomorphic to C, as η_C is nontrivial, as in Fig. 2 (each component of $\tilde{C}$ is a partial normalization of C).

As mentioned in the Introduction, it is well-known that *any* irreducible component of a Severi variety on a $K3$ surface is regular when nonempty (see, e.g., [4, Ex. 1.3]; see also [8, §4.2]). The corresponding result on Enriques surfaces is the following.

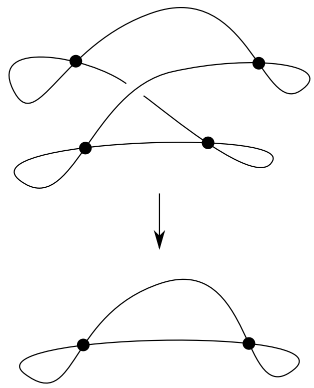

Fig. 1 $\eta_{\tilde{C}} = \nu_C^*(\eta_C) \neq 0$

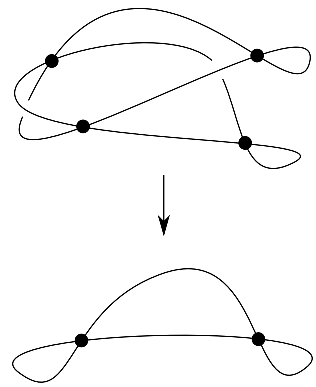

Fig. 2 $\eta_{\tilde{C}} = \nu_C^*(\eta_C) = 0$

First note that, in the above notation, the dimension of the Severi variety of genus $g = p_g(C)$ curves in $|L| = |C|$ at the point $[C]$ satisfies the inequality

$$\dim_{[C]}\big(V^g_{|L|}\big) \geq h^0(\omega_{\tilde{C}} \otimes \eta_{\tilde{C}}) = \begin{cases} g-1 & \text{if } \eta_{\tilde{C}} \not\simeq \mathcal{O}_{\tilde{C}} \\ g & \text{if } \eta_{\tilde{C}} \simeq \mathcal{O}_{\tilde{C}} \end{cases} \tag{3}$$

(see [8, Proofs of Thm. 4.12 and Cor. 2.7]). Our result implies that the latter is in fact an equality when C is nodal, and gives a concrete geometric description of the situation in both cases.

Proposition 1. *Let L be a Bertini linear system, with $L^2 > 0$, on a smooth Enriques surface S. Then the Severi variety $V_{|L|,\delta}(S)$ is smooth and every irreducible component $V \subseteq V_{|L|,\delta}(S)$ has either dimension $g-1$ or g; in the former case the component is regular. Furthermore, with the notation introduced above,*

1. *for any curve C in a $(g-1)$-dimensional irreducible component V, $\pi^{-1}C$ is irreducible (whence an element in $V_{|\pi^*L|,2\delta}(X)$);*
2. *for any g-dimensional component V, there is a line bundle L' on X with $(L')^2 = 2(p-d)-2$ and $L' \cdot \iota^*L' = 2d$ for some integer d satisfying*

$$\frac{p-1}{2} \leq d \leq \delta,$$

such that $\pi^*L \simeq L' \otimes \iota^*L'$, and the curves parametrized by $V \subseteq V_{|L|,\delta}(S)$ are the birational images by π of the curves in $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by ι transversely (in $2d$ points). In other words, for any $[C] \in V$, we have $\pi^{-1}C = Y + \iota(Y)$, with $[Y] \in V_{|L'|,\delta-d}(X)$ and $[\iota(Y)] \in V_{|\iota^*L'|,\delta-d}(X)$ intersecting transversely.
Furthermore, if $L' \simeq \iota^*L'$, which is the case if S is general in moduli, then $d = \frac{p-1}{2}$ and $L \sim 2M$, for some $M \in \operatorname{Pic} S$ such that $M^2 = d$.

We will henceforth refer to components of dimension $g-1$ as *regular* and the ones of dimension g as *nonregular*. Note however that from a parametric perspective the Severi variety has the expected dimension and is smooth in both cases, as the fact that (3) is an equality indicates; we do not dwell on this here, and refer to [8] for a discussion of the differences between the parametric and Cartesian points of view (the latter is the one we adopted in this text).

Note that Proposition 1 does not assert that the Severi variety $V_{|L|,\delta}$ is necessarily non-empty: in such a situation, $V_{|L|,\delta}$ does not have any irreducible component and the statement is empty.

Proof. Pick any curve C in an irreducible component V of $V_{|L|,\delta}(S)$. Let $f : \widetilde{S} \to S$ be the blow-up of S at N, the scheme of the δ nodes of C, denote by $\mathfrak{e}$ the (total) exceptional divisor and by $\widetilde{C}$ the strict transform of C. Thus $f_{|\widetilde{C}} = \nu_C$ and we have

$$K_{\widetilde{S}} \sim f^*K_S + \mathfrak{e} \quad \text{and} \quad \widetilde{C} \sim f^*C - 2\mathfrak{e}.$$

From the restriction sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{S}}(\mathfrak{e}) \longrightarrow \mathcal{O}_{\widetilde{S}}(\widetilde{C} + \mathfrak{e}) \longrightarrow \omega_{\widetilde{C}}(\eta_{\widetilde{C}}) \longrightarrow 0$$

we find

$$\begin{aligned} \dim T_{[C]}V_{|L|,\delta}(S) = \dim |L \otimes \mathcal{J}_N| &= h^0(L \otimes \mathcal{J}_N) - 1 = h^0(f^*L - \mathfrak{e}) - 1 \\ &= h^0(\mathcal{O}_{\widetilde{S}}(\widetilde{C} + \mathfrak{e})) - 1 = h^0(\omega_{\widetilde{C}}(\eta_{\widetilde{C}})) \\ &= \begin{cases} g-1, & \text{if } \eta_{\widetilde{C}} \not\simeq \mathcal{O}_{\widetilde{C}}, \\ g, & \text{if } \eta_{\widetilde{C}} \simeq \mathcal{O}_{\widetilde{C}}. \end{cases} \end{aligned} \tag{4}$$

In the upper case, by (1), we have that $V_{|L|,\delta}$ is smooth at $[C]$ of dimension $g-1 = p-\delta-1 = \dim |L \otimes \mathcal{J}_N|$.

Assume next that we are in the lower case. Then, by the discussion prior to the proposition, we have $\pi^{-1}C = Y + \iota(Y)$ for an irreducible curve Y on X, such that π maps both Y and $\iota(Y)$ birationally, but not isomorphically, to C. In particular, Y and $\iota(Y)$ have geometric genus $p_g(Y) = p_g(\iota(Y)) = p_g(C) = p - \delta = g$. Set $L' := \mathcal{O}_X(Y)$ and $2d := Y \cdot \iota(Y)$. Note that d is an integer because, if $y = \iota(x) \in Y \cap \iota(Y)$, then $\iota(y) = x \in Y \cap \iota(Y)$. Since $Y \simeq \iota(Y)$ and π is étale, both Y and $\iota(Y)$ are nodal with $\delta - d$ nodes and they intersect transversely at $2d$ points, which are pairwise conjugate by ι, and therefore map to d nodes of C. Hence $d \leq \delta$. We have

$$p_a(Y) = p_a(\iota(Y)) = g + \delta - d = p - \delta + \delta - d = p - d. \tag{5}$$

whence

$$(L')^2 = 2(p-1-d).$$

By the Hodge index theorem, we have

$$4(p-1-d)^2 = \left((L')^2\right)^2 = (L')^2(\iota^*L')^2 \leq (L' \cdot \iota^*L')^2 = 4d^2,$$

whence $p-1 \leq 2d$.

By the regularity of Severi varieties on $K3$ surfaces, any irreducible component of $V_{|L'|,\delta-d}(X)$ has dimension $\dim|L'|-(\delta-d) = p_g(Y) = g$. Hence, V is g-dimensional; more precisely, the curves parameterized by V are the (birational) images by π of the curves in an irreducible component of $V_{|L'|,\delta-d}(X)$ intersecting their conjugates by ι transversely (in $2d$ points). By (4), it also follows that $\dim V = \dim T_{[C]}V_{|L|,\delta}(S)$, so that $[C]$ is a smooth point of $V_{|L|,\delta}(S)$.

To prove the final assertion of the proposition, observe that, by the regularity of Severi varieties on $K3$ surfaces, we may deform Y and $\iota(Y)$ on X to irreducible curves Y' and $\iota(Y')$ with any number of nodes $\leq \delta-d$ and intersecting transversally in $2d$ points; in particular, we may deform Y and $\iota(Y)$ to *smooth* curves Y' and $\iota(Y')$. Thus, $C' := \pi(Y')$ is a member of $V_{|L|,d}$, whence of geometric genus $p-d$. Since $\dim|Y'| = p_a(Y') = p_g(C') = p_a(C')-d = p-d$, the component of $V_{|L|,d}$ containing $[C']$ has dimension $\dim|L|-d+1 = p-d$. We thus have $\dim|L\otimes\mathcal{J}_{N'}| = \dim|L|-d+1$, where N' is the set of d nodes of C', hence N' does not impose independent conditions on $|L|$.

Assume now that $L' \simeq \iota^*L'$, which — as is well-known (see, e.g., [9, §11]) — is the case occurring for generic S, as then $\operatorname{Pic} X$ is precisely the invariant part under ι of $H_2(X,\mathbb{Z})$. Then $2d = L'\cdot\iota^*L' = (L')^2 = 2(p-1-d)$, so that $p-1 = 2d$. Since $L^2 = 2(p-1) = 4d$ and N' does not impose independent conditions on $|L|$, by [11, Prop. 3.7] there is an effective divisor $D \subset S$ containing N' satisfying $L-2D \geq 0$ and

$$L\cdot D - d \leq D^2 \overset{(i)}{\leq} \frac{1}{2}L\cdot D \overset{(ii)}{\leq} d, \tag{6}$$

with equality in (i) or (ii) only if $L \equiv 2D$; moreover, since $L-2D \geq 0$, the numerical equivalence $L \equiv 2D$ implies the linear equivalence $L \sim 2D$. Now since $N' \subset D$, we must have $L\cdot D = C'\cdot D \geq 2d$, hence the inequalities in (6) are all equalities, and thus $D^2 = d$ and $L \sim 2D$.

The following corollary is a straightforward consequence of Prop. 1 and the fact that the nodes on curves in a regular component in a Severi variety (on any surface and in particular on a K3 surface) can be independently smoothened.

Corollary 1. *If a Severi variety $V_{|L|,\delta}$ on an Enriques surface has a regular (resp., nonregular) component, then for any $0 \leq \delta' \leq \delta$ (resp., $d \leq \delta' \leq \delta$, with d as in Prop. 1), also $V_{|L|,\delta'}$ contains a regular (resp., nonregular) component.*

Acknowledgements The first and third author want to thank the organisers of BGMS Indam Workshop for the opportunity to give a talk. The first author acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. The first and second authors were members of project FOSICAV, which has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement n. 652782. The third author has been partially supported by GNSAGA of INDAM. The last author has been partially supported by grant n. 261756 of the Research Council of Norway and by Bergen Research Foundation.

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