Editing to Eulerian Graphs*

Konrad K. Dabrowski[1], Petr A. Golovach[2], Pim van 't Hof[2], and Daniël Paulusma[1]

1 School of Engineering and Computing Sciences, Durham University, Science Laboratories, South Road, Durham DH1 3LE, United Kingdom
{konrad.dabrowski,daniel.paulusma}@durham.ac.uk

2 Department of Informatics, University of Bergen, PB 7803, 5020 Bergen, Norway
{petr.golovach,pim.vanthof}@ii.uib.no

Abstract

We investigate the problem of modifying a graph into a connected graph in which the degree of each vertex satisfies a prescribed parity constraint. Let ea, ed and vd denote the operations edge addition, edge deletion and vertex deletion respectively. For any $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$, we define Connected Degree Parity Editing(S) (CDPE(S)) to be the problem that takes as input a graph G, an integer k and a function $\delta\colon V(G) \to \{0,1\}$, and asks whether G can be modified into a connected graph H with $d_H(v) \equiv \delta(v) \pmod 2$ for each $v \in V(H)$, using at most k operations from S. We prove that

- if $S = \{\mathsf{ea}\}$ or $S = \{\mathsf{ea}, \mathsf{ed}\}$, then CDPE($S$) can be solved in polynomial time;
- if $\{\mathsf{vd}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$, then CDPE($S$) is NP-complete and W[1]-hard when parameterized by k, even if $\delta \equiv 0$.

Together with known results by Cai and Yang and by Cygan, Marx, Pilipczuk, Pilipczuk and Schlotter, our results completely classify the classical and parameterized complexity of the CDPE(S) problem for all $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$. We obtain the same classification for a natural variant of the CDPE(S) problem on directed graphs, where the target is a weakly connected digraph in which the difference between the in- and out-degree of every vertex equals a prescribed value.

As an important implication of our results, we obtain polynomial-time algorithms for Eulerian Editing problem and its directed variant. To the best of our knowledge, the only other natural non-trivial graph class $\mathcal{H}$ for which the $\mathcal{H}$-Editing problem is known to be polynomial-time solvable is the class of split graphs.

1998 ACM Subject Classification G.2.2 Graph Theory, F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases Eulerian graphs, graph editing, polynomial algorithm

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2014.97

1 Introduction

Graph modification problems play a central role in algorithmic graph theory, partly due to the fact that they naturally arise in numerous practical applications. A graph modification problem takes as input a graph G and an integer k, and asks whether G can be modified

* The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013)/ERC Grant Agreement n. 267959 and from EPSRC Grant EP/K025090/1.

34th Int'l Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2014).
Editors: Venkatesh Raman and S. P. Suresh; pp. 97–108
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

into a graph belonging to a prescribed graph class $\mathcal{H}$, using at most k operations of a certain type. The most common operations that are considered in this context are edge additions ($\mathcal{H}$-Completion), edge deletions ($\mathcal{H}$-Edge Deletion), vertex deletions ($\mathcal{H}$-Vertex Deletion), and a combination of edge additions and edge deletions ($\mathcal{H}$-Editing). The intensive study of graph modification problems has produced a plethora of classical and parameterized complexity results (see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 15, 16, 17, 19, 20]).

An undirected graph is Eulerian if it is connected and every vertex has even degree, while a directed graph is Eulerian if it is strongly connected[1] and balanced, i.e. the in-degree of every vertex equals its out-degree. Eulerian graphs form a well-known graph class both within algorithmic and structural graph theory. Several groups of authors have investigated the problem of deciding whether a given graph can be made Eulerian using a small number of operations. Boesch et al. [1] presented a polynomial-time algorithm for Eulerian Completion, and Cai and Yang [4] showed that the problems Eulerian Vertex Deletion and Eulerian Edge Deletion are NP-complete [4]. When parameterized by k, it is known that Eulerian Vertex Deletion is W[1]-hard [4], while Eulerian Edge Deletion is fixed-parameter tractable [7]. Cygan et al. [7] showed that the classical and parameterized complexity results for Eulerian Vertex Deletion and Eulerian Edge Deletion also hold for the directed variants of these problems.

Our Contribution. We generalize, extend and complement known results on graph modification problems dealing with Eulerian graphs and digraphs. The main contribution of this paper consists of two non-trivial polynomial-time algorithms: one for solving the Eulerian Editing problem, and one for solving the directed variant of this problem. Given the aforementioned NP-completeness result for Eulerian Edge Deletion and the fact that $\mathcal{H}$-Editing is NP-complete for almost all natural graph classes $\mathcal{H}$ [2, 20], we find it particularly interesting that Eulerian Editing turns out to be polynomial-time solvable. To the best of our knowledge, the only other natural non-trivial graph class $\mathcal{H}$ for which $\mathcal{H}$-Editing is known to be polynomial-time solvable is the class of split graphs [13].

In fact, our polynomial-time algorithms are implications of two more general results. In order to formally state these results, we need to introduce some terminology. Let ea, ed and vd denote the operations edge addition, edge deletion and vertex deletion, respectively. For any set $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$ and non-negative integer k, we say that a graph G can be (S, k)-*modified* into a graph H if H can be obtained from G by using at most k operations from S. We define the following problem for every $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$:

CDPE(S):	Connected Degree Parity Editing(S)
Instance:	A graph G, an integer k and a function $\delta\colon V(G) \to \{0, 1\}$.
Question:	Can G be (S, k)-modified into a connected graph H with $d_H(v) \equiv \delta(v) \pmod 2$ for each $v \in V(H)$?

Inspired by the work of Cygan et al. [7] on directed Eulerian graphs, we also study a natural directed variant of the CDBE(S) problem. Denoting the in- and out-degree of a vertex v in a digraph G by $d_G^-(v)$ and $d_G^+(v)$, respectively, we define the following problem for every $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$:

[1] Replacing "strongly connected" by "weakly connected" yields an equivalent definition of Eulerian digraphs, as it is well-known that a balanced digraph is strongly connected if and only it is weakly connected (see e.g. [7]).

Table 1 A summary of the results for CDPE(S) and CDBE(S). All results are new except those for which a reference is given. The number of allowed operations k is the parameter in the parameterized results, and if a parameterized result is stated, then the corresponding problem is NP-complete.

S	CDPE(S)	CDBE(S)
ea, ed	P	P
ea	P	P
ed	FPT [7]	FPT [7]
vd	W[1]-hard [4]	W[1]-hard [7]
ea, vd	W[1]-hard	W[1]-hard
ed, vd	W[1]-hard	W[1]-hard
ea, ed, vd	W[1]-hard	W[1]-hard

CDBE(S):	Connected Degree Balance Editing(S)
Instance:	A digraph G, an integer k and a function $\delta\colon V(G) \to \mathbb{Z}$.
Question:	Can G be (S,k)-modified into a weakly connected digraph H with $d_H^+(v) - d_H^-(v) = \delta(v)$ for each $v \in V(H)$?

In Section 3, we prove that CDPE(S) can be solved in polynomial time when $S = \{\mathsf{ea}\}$ and when $S = \{\mathsf{ea}, \mathsf{ed}\}$. The first of these two results extends the aforementioned polynomial-time result by Boesch et al. [1] on Eulerian Completion and the second yields the first polynomial-time algorithm for Eulerian Editing, as these problems are equivalent to CDPE($\{\mathsf{ea}\}$) and CDPE($\{\mathsf{ea}, \mathsf{ed}\}$), respectively, when we set $\delta \equiv 0$. The complexity of the problem drastically changes when vertex deletion is allowed: we prove that for every subset $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$ with $\mathsf{vd} \in S$, the CDPE(S) problem is NP-complete and W[1]-hard with parameter k, even when $\delta \equiv 0$. This complements results by Cai and Yang [4] stating that CDPE(S) is NP-complete and W[1]-hard with parameter k when $S = \{\mathsf{vd}\}$ and $\delta \equiv 0$ or $\delta \equiv 1$. Our results, together with the aforementioned results due to Cygan et al. [7][2] and Cai and Yang [4], yield a complete classification of both the classical and the parameterized complexity of CDPE(S) for all $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$; see the middle column of Table 1.

In Section 4, we use different and more involved arguments to classify the classical and parameterized complexity of the CDBE(S) problem for all $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$. Interestingly, the classification we obtain for CDBE(S) turns out to be identical to the one we obtained for CDPE(S). In particular, our proof of the fact that CDBE(S) is polynomial-time solvable when $S = \{\mathsf{ea}\}$ and $S = \{\mathsf{ea}, \mathsf{ed}\}$ implies that the directed variants of Eulerian Completion and Eulerian Editing are not significantly harder than their undirected counterparts. All results on CDBE(S) are summarized in the right column of Table 1.

We would like to emphasize that there are no obvious hardness reductions between the different problem variants. The parameter k in the problem definitions represents the budget for all operations in total; adding a new operation to S may completely change the problem, as there is no way of forbidding its use. Hence, our polynomial-time algorithms for CDPE($\{\mathsf{ea}, \mathsf{ed}\}$) and CDBE($\{\mathsf{ea}, \mathsf{ed}\}$) do not generalize the polynomial-time algorithms for CDPE($\{\mathsf{ea}\}$) and CDBE($\{\mathsf{ea}\}$), and as such require significantly different arguments. In

[2] The FPT-results by Cygan et al. [7] only cover CDPE($\{\mathsf{ed}\}$) and CDBE($\{\mathsf{ed}\}$) when $\delta \equiv 0$, but it can easily be seen that their results carry over to CDPE($\{\mathsf{ed}\}$) and CDBE($\{\mathsf{ed}\}$) for any function δ.

particular, our main result, stating that Eulerian Editing is polynomial-time solvable, is not a generalization of the fact that Eulerian Completion is polynomial-time solvable and stands in no relation to the FPT-result by Cygan et al. [7] for Eulerian Edge Deletion.

We end this section by mentioning two similar graph modification frameworks in the literature that formed a direct motivation for the framework defined in this paper. Mathieson and Szeider [17] considered the Degree Constraint Editing(S) problem, which is that of testing whether a graph G can be k-modified into a graph H in which the degree of every vertex belongs to some list associated with that vertex. They classified the parameterized complexity of this problem for all $S \subseteq \{\mathsf{ea}, \mathsf{ed}, \mathsf{vd}\}$. Golovach [11] performed a similar study where the resulting graph must in addition be connected.

2 Preliminaries

We consider finite graphs $G = (V, E)$ that may be undirected or directed; in the latter case we will always call them digraphs. All our undirected graphs will be without loops or multiple edges; in particular, this is the case for both the input and the output graph in every undirected problem we consider. Similarly, for every directed problem that we consider, we do not allow the input or output digraph to contain multiple arcs. In our proofs we will also make use of *directed multigraphs*, which are digraphs that are permitted to have multiple arcs.

We denote an edge between two vertices u and v in a graph by uv. We denote an arc between two vertices u and v by (u, v), where u is the *tail* of (u, v) and v is the *head*. The disjoint union of two graphs G_1 and G_2 is denoted $G_1 + G_2$. The complete graph on n vertices is denoted K_n and the complete bipartite graph with classes of size s and t is denoted $K_{s,t}$.

Let $G = (V, E)$ be a graph or a digraph. Throughout the paper we assume that $n = |V|$ and $m = |E|$. For $U \subseteq V$, we let $G[U]$ be the graph (digraph) with vertex set U and an edge (arc) between two vertices u and v if and only if this is the case in G; we say that $G[U]$ is *induced by* U. We write $G - U = G[V \setminus U]$. For $E' \subseteq E$, we let $G(E')$ be the graph (digraph) with edge (arc) set E' whose vertex set consists of the end-vertices of the edges in E'; we say that $G(E')$ is *edge-induced by* E'. Let S be a set of (ordered) pairs of vertices of G. We let $G - S$ be the graph (digraph) obtained by deleting all edges (arcs) of $S \cap E$ from G, and we let $G + S$ be the graph (digraph) obtained by adding all edges (arcs) of $S \setminus E$ to G. We may write $G - e$ or $G + e$ if $S = \{e\}$.

Let $G = (V, E)$ be a graph. A component of G is a maximal connected subgraph of G. The complement of G is the graph $\overline{G} = (V, \overline{E})$ with vertex set V and an edge between two distinct vertices u and v if and only if $uv \notin E$. A *matching* M in G is a set of edges, in which no edge has a common end-vertex with some other edge. For a vertex $v \in V$, we let $N_G(v) = \{u \mid uv \in E\}$ denote its *(open) neighbourhood*. The *degree* of v is denoted $d_G(v) = |N_G(v)|$. The graph G is *even* if all its vertices have even degree, and it is *Eulerian* if it is even and connected. We say that a set $D \subseteq E$ is an *edge cut* in G if G is connected but $G - D$ is not. An edge cut of size 1 is called a *bridge* in G.

Let $G = (V, E)$ be a digraph. If (u, v) is an arc, then (v, u) is the *reverse* of this arc. For a subset $F \subseteq E$, we let $F^R = \{(u, v) | (v, u) \in F\}$ denote the set of arcs whose reverse is in F. The *underlying* graph of G is the undirected graph with vertex set V where two vertices $u, v \in V$ are adjacent if and only if (u, v) or (v, u) is an arc in G. We say that G is *(weakly) connected* if its underlying graph is connected. A *component* of G is a connected component of its underlying graph. An arc $a \in E$ is a *bridge* in G if it is a bridge in the underlying graph of G. A vertex u is an *in-neighbour* or *out-neighbour* of a vertex v if $(u, v) \in E$ or

$(v,u) \in E$, respectively. Let $N_G^-(v) = \{u \mid (u,v) \in E\}$ and $N_G^+(v) = \{u \mid (v,u) \in E\}$, where we call $d_G^-(v) = |N_G^-(v)|$ and $d_G^+(v) = |N_G^+(v)|$ the *in-degree* and *out-degree* of v, respectively. A vertex $v \in V$ is *balanced* if $d_G^+(v) = d_G^-(v)$. Recall that G is *Eulerian* if it is connected and *balanced*, that is, the out-degree of every vertex is equal to its in-degree.

Let $G = (V,E)$ be a graph and let $T \subseteq V$. A subset $J \subseteq E$ is a *T-join* if the set of odd-degree vertices in $G(J)$ is precisely T. If G is connected and $|T|$ is even then G has at least one T-join. In Section 3 we need to find a *minimum* T-join, that is, one of minimum size. We use the following result of Edmonds and Johnson [9] to do so.

▶ **Lemma 1** ([9]). *Let $G = (V,E)$ be a graph, and let $T \subseteq V$. Then a minimum T-join (if one exists) can be found in $O(n^3)$ time.*

Lemma 1 was used by Cygan et al. [7] to solve $\mathcal{H}$-Edge Deletion in polynomial time when $\mathcal{H}$ is the class of even graphs. It would immediately yield a polynomial-time algorithm for CDPE($\{\mathsf{ed}\}$) if we dropped the connectivity condition.

We need a variant of Lemma 1 for digraphs in Section 4. Let $G = (V,E)$ be a directed multigraph and let $f : T \to \mathbb{Z}$ be a function for some $T \subseteq V$. A multiset $E' \subseteq E$ with $T \subseteq V(G(E'))$ is a *directed f-join* in G if the following two conditions hold: $d^+_{G(E')}(v) - d^-_{G(E')}(v) = f(v)$ for every $v \in T$ and $d^+_{G(E')}(v) - d^-_{G(E')}(v) = 0$ for every $v \in V(G(E')) \setminus T$. A directed f-join is *minimum* if it has minimum size. The next lemma was used by Cygan et al. [7] to solve $\mathcal{H}$-Edge Deletion in polynomial time when $\mathcal{H}$ is the class of balanced digraphs; it would also yield a polynomial-time algorithm for CDBE($\{\mathsf{ed}\}$) if we dropped the connectivity condition.

▶ **Lemma 2** ([7]). *Let $G = (V,E)$ be a directed multigraph and $f : T \to \mathbb{Z}$ be a function for some $T \subseteq V$. A minimum directed f-join F (if one exists) can be found in $O(nm \log n \log\log m)$ time. Moreover, F consists of mutually arc-disjoint directed paths from vertices u with $f(u) > 0$ to vertices v with $f(v) < 0$.*

3 Connected Degree Parity Editing

We will show that CDPE(S) is polynomial-time solvable if $S = \{\mathsf{ea}\}$ or $S = \{\mathsf{ea}, \mathsf{ed}\}$ and that it is NP-complete and W[1]-hard with parameter k if $\mathsf{vd} \in S$.

First, let $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$. Let (G, δ, k) be an instance of CDPE(S) with $G = (V,E)$. Let A be a set of edges not in G, and let D be a set of edges in G, with $D = \emptyset$ if $S = \{\mathsf{ea}\}$. We say that (A, D) is a *solution* for (G, δ, k) if its *size* $|A| + |D| \leq k$, the congruence $d_H(u) \equiv \delta(u) \pmod 2$ holds for every vertex u and the graph $H = G + A - D$ is connected; if H is not connected then (A, D) is a *semi-solution* for (G, δ, k). If $S = \{\mathsf{ea}\}$ we may denote the solution by A rather than (A, D) (since $D = \emptyset$). We consider the optimization version for CDPE(S). The input is a pair (G, δ), and we aim to find the minimum k such that (G, δ, k) has a solution (if one exists). We call such a solution *optimal* and denote its size by $opt_S(G, \delta)$. We say that a (semi)-solution for (G, δ, k) is also a (semi)-solution for (G, δ). If (G, δ, k) has no solution for any value of k, then (G, δ) is a *no-instance* of CDPE(S) and $opt_S(G, \delta) = +\infty$.

Let $T = \{v \in V \mid d_G(v) \not\equiv \delta(v) \pmod 2\}$. Define $G_S = K_n$ if $S = \{\mathsf{ea}, \mathsf{ed}\}$ and $G_S = \overline{G}$ if $S = \{\mathsf{ea}\}$. Note that if $S = \{\mathsf{ea}\}$ then G_S contains no edges of G, so in this case any T-join in G_S can only contain edges in $E(\overline{G})$. The following key lemma is an easy observation.

▶ **Lemma 3.** *Let $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$. Let (G, δ) be an instance of* CDPE(S) *and $A \subseteq E(\overline{G})$, $D \subseteq E(G)$. Then (A, D) is a semi-solution of* CDPE(S) *if and only if $A \cup D$ is a T-join in G_S.*

We extend the result of Boesch et al. [1] for $\delta \equiv 0$ to arbitrary δ. Our proof is based around similar ideas but we also had to do some further analysis. The main difference in the two proofs is the following. If $\delta \equiv 0$ then none of the added edges in a solution will be a bridge in the modified graph (as the number of vertices of odd degree in a graph is always even). However this is no longer true for arbitrary δ and extra arguments are needed. We omit the proof of our result.

▶ **Theorem 4.** *Let $S = \{\mathsf{ea}\}$. Then* CDPE(S) *can be solved in $O(n^3)$ time.*

We are now ready to present the main result of this section. Recall that proving this result requires significantly different arguments than the ones used in the proof of Theorem 4. Let $S = \{\mathsf{ea}, \mathsf{ed}\}$ and let (G, δ) be an instance of CDPE(S). If F is a T-join in $G_S = K_n$, let $D = F \cap E(G)$ and $A = F \setminus D$. Then by Lemma 3, (A, D) is a semi-solution. Note that if F is a minimum T-join in G_S then it is a matching in which every vertex of T is incident to precisely one edge of F, so $|F| = \frac{1}{2}|T|$. We will show how this allows us to calculate $opt_S(G, \delta)$ directly from the structure of G, without having to find a T-join. We will also show that there are only trivial no-instances for this problem.

▶ **Theorem 5.** *Let $S = \{\mathsf{ea}, \mathsf{ed}\}$. Then* CDPE($S$) *can be solved in $O(n + m)$ time and an optimal solution (if one exists) can be found in $O(n^3)$ time.*

Proof. Let $S = \{\mathsf{ea}, \mathsf{ed}\}$ and let (G, δ) be an instance of CDPE(S). By Lemma 3, we may assume that $|T|$ is even, otherwise (G, δ) is a no-instance. If $G = K_2$ and $T = V(G)$, or $G = K_1 + K_1$ and $T = \emptyset$, then (G, δ) is a no-instance. If $G = K_2$ and $T = \emptyset$ then, trivially, $opt_S(G, \delta) = 0$, and if $G = K_1 + K_1$ and $T = V(G)$ then $opt_S(G, \delta) = 1$. To avoid these trivial instances, we therefore assume that G contains at least three vertices. Under these assumptions we will show that $opt_S(G, \delta)$ is always finite and give exact formulas for the value of $opt_S(G, \delta)$. Let p be the number of components of G that do not contain any vertex of T and let q be the number of components of G that contain at least one vertex of T. We prove the following series of statements:

- $opt_S(G, \delta) = 0$ if $p = 1, q = 0$,
- $opt_S(G, \delta) = \max\{3, p\}$ if $p \geq 2, q = 0$,
- $opt_S(G, \delta) = \frac{1}{2}|T| + 1$ if $p = 0, q = 1$, $G[T] = K_{1,r}$, for some $r \geq 1$, and each edge of $G[T]$ is a bridge of G,
- $opt_S(G, \delta) = \max\{p + q - 1, p + \frac{1}{2}|T|\}$ in all other cases.

Note that if $p = 1, q = 0$, then the first statement applies and the trivial solution $(A, D) = (\emptyset, \emptyset)$ is optimal. We now consider the remaining three cases separately.

Case 1: *$p \geq 2$ and $q = 0$.*
Then $T = \emptyset$, so by Lemma 3 for any semi-solution (A, D), every vertex in $G_S(A \cup D)$ must have even degree in $G_S(A \cup D)$. In other words, every vertex of G must be incident to an even number of edges in $A \cup D$. Since $p \geq 2$, the graph G is disconnected, so any solution (A, D) is non-empty. This means that $G_S(A \cup D)$ must contain a cycle, so $opt_S(G, \delta) \geq 3$ if a solution exits. Suppose $p = 2$. As G has at least three vertices, it contains a component containing an edge xy. Let z be a vertex in its other component. We set $A = \{xz, yz\}$ and $D = \{xy\}$ to obtain a solution for (G, δ). Since $|A| + |D| = 3$, this solution is optimal. Suppose $p \geq 3$. Since $G + A - D$ must be connected for any solution (A, D), every component in G must contain at least one vertex incident to an edge of A. By Lemma 3, this vertex must be incident to an even number of edges of $A \cup D$, meaning that it must be incident to at least two such edges. Therefore $opt_S(G, \delta) \geq p$. Indeed, if we choose vertices $v_1, \ldots, v_p$,

one from each component of G, then setting $A = \{v_1v_2, v_2v_3, \dots, v_{p-1}v_p, v_pv_1\}$ and $D = \emptyset$ gives a solution of size p, which is therefore optimal. This concludes Case 1.

Case 2: $p = 0, q = 1$, $G[T] = K_{1,r}$ *for some* $r \geq 1$ *and each edge of* $G[T]$ *is a bridge of* G.
Then G is connected. Let v_0 be the central vertex of the star and let $v_1, \dots, v_r$ be the leaves. By Lemma 3, in any semi-solution (A, D), every vertex of T must be incident to an odd number of edges in $A \cup D$, so $opt_S(G, \delta) \geq \frac{1}{2}|T|$. Suppose (A, D) is a semi-solution of size $|A| + |D| = \frac{1}{2}|T|$. Then $A \cup D$ must be a matching with each edge joining a pair of vertices of T. However, then $v_0v_i \in A \cup D$ for some i. Since $v_0v_i \in E(G)$, we must have $v_0v_i \in D$. However, since v_0v_i is a bridge of G, v_0 and v_i must then be in different components of $G + A - D$, so $G + A - D$ is not connected and (A, D) is not a solution. Therefore $opt_S(G, \delta) \geq \frac{1}{2}|T| + 1$.

Next we show how to find a solution of size $\frac{1}{2}|T| + 1$. Since $|T|$ is even, r must be odd. First suppose that $r = 1$. Since G is connected and v_0v_1 is a bridge, $G \setminus \{v_0v_1\}$ has exactly two components. Since G contains at least three vertices, one of these components contains another vertex x. Without loss of generality assume $xv_0 \in E(G)$, in which case $xv_1 \notin E(G)$. Then setting $A = \{xv_1\}$ and $D = \{xv_0\}$ gives a solution of size $|A|+|D| = 2 = \frac{1}{2}|T|+1$, so this solution is optimal. Now suppose $r \geq 3$. Let $A = \{v_1v_2, v_2v_3\} \cup \{v_{2i}v_{2i+1} \mid 2 \leq i \leq \frac{1}{2}(r-1)\}$ and $D = \{v_0v_2\}$. Then (A, D) is a semi-solution and since $v_0, \dots, v_r$ are all in the same component of $G+A-D$, we find that (A, D) is a solution. Since $|A|+|D| = 2+\frac{1}{2}(r-1)-1+1 = \frac{1}{2}|T| + 1$, this solution is optimal. This concludes Case 2.

Case 3: $q \geq 1$ *and Case 2 does not hold.*
Then $T \neq \emptyset$. Let $G_1, \dots, G_p$ be the components of G without vertices of T and let $G' = G - V(G_1) \cup \dots \cup V(G_p)$. Note that $G' = G$ if $p = 0$ and that G' is not the empty graph, as $q > 0$. Choose $v_i \in V(G_i)$ for $i \in \{1, \dots, p\}$.

We first show that $opt_S(G, \delta) \geq \max\{p + q - 1, p + \frac{1}{2}|T|\}$. Since G has $p + q$ components, any solution (A, D) must contain at least $p + q - 1$ edges in A to ensure that $G + A - D$ is connected, so $opt_S(G, \delta) \geq p + q - 1$. If (A, D) is a solution then every component G_i must contain a vertex incident to some edge in A. By Lemma 3, this vertex must be incident to an even number of edges of $A \cup D$, meaning that it must be incident to at least two such edges. By Lemma 3, every vertex of T must be incident to some edge in $A \cup D$. Therefore $A \cup D$ must contain at least $p + \frac{1}{2}|T|$ edges, so $opt_S(G, \delta) \geq p + \frac{1}{2}|T|$.

We now show how to find a solution of size $\max\{p + q - 1, p + \frac{1}{2}|T|\}$. We start by finding a maximum matching M in $\overline{G[T]}$. Let U be the set of vertices in T that are not incident to any edge in M. We divide the argument into two cases, depending on the size of U.

Case 3a: $U = \emptyset$.
In this case, by Lemma 3, setting $A = M$ and $D = \emptyset$ gives a semi-solution. Now suppose that $uv, u'v' \in M$, such that uv is not a bridge in $G + M$ and the vertices u and u' are in different components of $G + M$. Let $M' = M \setminus \{uv, u'v'\} \cup \{u'v, uv'\}$. Then M' is also a maximum matching in $\overline{G[T]}$. However, $G + M'$ has one component less than $G + M$. Indeed, since uv is not a bridge in $G + M$, the vertices u, u', v, v' must all be in the same component of $G + M'$. Therefore, if such edges $uv, u'v' \in M$ exist, we replace M by M'. We do this exhaustively until no further such pairs of edges exist. At this point either every edge in M is a bridge in $G + M$ or every edge in M is in the same component of $G + M$. We consider these possibilities separately.

First suppose that every edge in M is a bridge in $G + M$. Choose $uv \in M$ and let $Q_1, \dots, Q_k$ be the components of $G + M$, with $u, v \in V(Q_1)$. Note that since every edge in M is a bridge, $k = p + q - |M|$. Now let $x_i \in V(Q_i)$ for $i \in \{2, \dots, k\}$. Let $D = \emptyset$

and let $A = M$ if $k = 1$ and $A = M \setminus \{uv\} \cup \{ux_2, x_2x_3, \ldots, x_{k-1}x_k, x_kv\}$ otherwise. Now every vertex in $G + A - D$ has the same degree parity as in $G + M$, so (A, D) is a semi-solution by Lemma 3. The graph $G + A - D$ is connected, so (A, D) is a solution. As $|A| + |D| = |M| - 1 + p + q - |M| + 0 = p + q - 1$, we find that (A, D) is an optimal solution.

Now suppose that every edge in M is in the same component of $G + M$. Note that $G_1, \ldots, G_p$ are the remaining components of $G + M$. Choose $uv \in M$. Let $D = \emptyset$ and let $A = M$ if $p = 0$ and $A = M \setminus \{uv\} \cup \{uv_1, v_1v_2, \ldots, v_{p-1}v_p, v_pv\}$ otherwise. Then every vertex in $G + A - D$ has the same parity as in $G + M$ and $G + A - D$ is connected, so by Lemma 3 (A, D) is a solution. Since $|A| + |D| = \frac{1}{2}|T| - 1 + p + 1 = p + \frac{1}{2}|T|$, this solution is optimal. This concludes Case 3a.

Case 3b: $U \neq \emptyset$.
Note that $z = |U|$ must be even since $|T|$ is even. Every pair of vertices in U must be non-adjacent in $\overline{G}$, as otherwise M would not be maximum. Therefore $G[U]$ is a clique. Let $U = \{u_1, \ldots, u_z\}$.

We claim that $Q = G' + M$ is connected. Clearly every vertex of the clique U must be in the same component of $Q = G' + M$. Suppose for contradiction that Q_1 is a component of Q that does not contain U. Then Q_1 must contain some edge $w_1w_2 \in M$. However, in this case $M' = M \setminus \{w_1w_2\} \cup \{u_1w_1, u_2w_2\}$ is a larger matching in $\overline{G[T]}$ than M, which contradicts the maximality of M. Therefore Q is connected.

Let $M' = \{u_1u_2, u_3u_4, \ldots, u_{z-1}u_z\}$. If $z \geq 4$ then since U is a clique, $G' + M - M'$ is connected. If $p = 0$ set $A = M$ and $D = M'$. If $p > 0$ set $A = M \cup \{u_1v_1, v_1v_2, \ldots, v_{p-1}v_p, v_pu_2\}$ and $D = M' \setminus \{u_1u_2\}$. Then $G + A - D$ is connected, so (A, D) is a solution by Lemma 3. This solution has size $|A| + |D| = p + \frac{1}{2}|T|$, so it is optimal.

Now suppose that $z \leq 3$. Then $z = 2$. If $p > 0$, let $A = M \cup \{u_1v_1, v_1v_2, \ldots, v_{p-1}v_p, v_pu_2\}$ and $D = \emptyset$. Then $G + A - D$ is connected, so (A, D) is a solution by Lemma 3. This solution has size $|A| + |D| = p + \frac{1}{2}|T|$, so it is optimal. Assume that $p = 0$, so $G + M$ contains only one component. If u_1u_2 is not a bridge in $G + M$, let $A = M$ and $D = \{u_1u_2\}$. Then $G + M$ is connected, so (A, D) is a solution. This solution has size $|A| + |D| = p + \frac{1}{2}|T|$, so it is optimal.

Now assume that u_1u_2 is a bridge in $Q = G + M$. Let Q_1 and Q_2 denote the components of $Q - \{u_1u_2\}$ with $u_1 \in V(Q_1)$ and $u_2 \in V(Q_2)$. Note that u_1u_2 is also a bridge in G. We claim that the edges of M are either all in Q_1 or all in Q_2. Suppose for contradiction that $y_1z_1 \in E(Q_1) \cap M$ and $y_2z_2 \in E(Q_2) \cap M$. Then $M' = M \setminus \{y_1z_1, y_2z_2\} \cup \{u_1y_2, u_2y_1, z_1z_2\}$ would be a larger matching in $\overline{G[T]}$ than M, contradicting the maximality of M. Without loss of generality, we may therefore assume that all edges of M are in Q_1.

Let $M = \{x_1y_1, \ldots, x_ry_r\}$, where $r = \frac{1}{2}|T| - 1$. We claim that u_1 must be adjacent in G to all vertices of $T \setminus \{u_1\}$. Suppose for contradiction that u_1 is non-adjacent in G to some vertex of $T \setminus \{u_1\}$. Since $u_1u_2 \in E(G)$, this vertex would have to be incident to some edge in M. Without loss of generality, assume $u_1x_1 \notin E(G)$. Then $M' = M \setminus \{x_1y_1\} \cup \{u_1x_1, u_2y_1\}$ would be a larger matching in $\overline{G[T]}$ than M, contradicting the maximality of M. Therefore u_1 is adjacent in G to every vertex of $T \setminus \{u_1\}$. In particular, since $p = 0$, it follows that $q = 1$ and G is connected.

Suppose that every edge between u_1 and $T \setminus \{u_1\}$ is a bridge in G. Then no two vertices of $T \setminus \{u_1\}$ can be adjacent, and $G[T] = K_{1,r}$. However, then Case 2 applies, which we assumed was not the case. Without loss of generality, we may therefore assume that u_1x_1 is not a bridge in G. Let $A = M \setminus \{x_1y_1\} \cup \{y_1u_2\}$ and $D = \{u_1x_1\}$. Then $G + A - D$ is connected, so (A, D) is a solution. Since $|A| + |D| = \frac{1}{2}|T| - 1 - 1 + 1 + 1 = p + \frac{1}{2}|T|$, this solution is optimal. This concludes Case 3b and therefore also concludes Case 3.

It is clear that $opt_S(G,\delta)$ can be computed in $O(n+m)$ time. We also observe that the above proof is constructive, that is, we not only solve the decision variant of CDPE(ea, ed) but we can also find an optimal solution. To do so, we must find a maximum matching in $\overline{G[T]}$. This takes $O(n^{5/2})$ time [18]. However, the bottleneck is in Case 3a, where we are glueing components by replacing two matching edges by two other matching edges, which takes $O(n^2)$ time. As the total number of times we may need to do this is $O(n)$, this procedure may take $O(n^3)$ time in total. Hence, we can obtain an optimal solution in $O(n^3)$ time. ◀

The proof of the next result has been omitted.

▶ **Theorem 6.** *Let $\{vd\} \subseteq S \subseteq \{vd, ed, ea\}$. Then* CDPE($S$) *is NP-complete and W[1]-hard when parameterized by k, even if $\delta \equiv 0$.*

4 Connected Degree Balance Editing

We will show that CDBE(S) is polynomial-time solvable if $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$ and that it is NP-complete and W[1]-hard with parameter k if $\mathsf{vd} \in S$.

Let $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$. Let (G, δ, k) be an instance of CDBE(S) with $G = (V, E)$. Let A be a set of arcs not in G, and let D be a set of arcs in G, with $D = \emptyset$ if $S = \{\mathsf{ea}\}$. We say that (A, D) is a *solution* for (G, δ, k) if its *size* $|A| + |D| \leq k$, the equation $d^+_H(u) - d^-_H(u) = \delta(u)$ holds for every vertex u and the graph $H = G + A - D$ is connected; if H is not connected then (A, D) is a *semi-solution* for (G, δ, k). Just as in Section 3 we consider the optimization version for CDBE(S) and we use the same terminology.

Let (G, δ) be an instance of (the optimization version) of CDBE(S) where $G = (V, E)$. Let $T = T_{(G,\delta)}$ be the set of vertices v such that $d^+_G(v) - d^-_G(v) \neq \delta(v)$. Define a function $f_{(G,\delta)} : T \to \mathbb{Z}$ by $f(v) = f_{(G,\delta)}(v) = \delta(v) - d^+_G(v) + d^-_G(v)$ for every $v \in T$.

We construct a directed multigraph G_S with vertex set V and arc set determined as follows. If $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$, for each pair of distinct vertices u and v in G, if $(u, v) \notin E$, add the arc (u, v) to G_S (these arcs are precisely those that can be added to G). If $S = \{\mathsf{ea}, \mathsf{ed}\}$, for each pair of distinct vertices u and v, if $(u, v) \in E$, add the arc (v, u) to G_S (these arcs are precisely those whose reverse can be deleted from G). Note that adding a (missing) arc has the same effect on the degree balance of the vertices in a digraph as deleting the reverse of the arc (if it exists). Also observe that G_S becomes a directed multigraph rather than a digraph only if $S = \{\mathsf{ea}, \mathsf{ed}\}$ and there are distinct vertices u and v such that $(u, v) \in E$ and $(v, u) \notin E$ applies. Moreover, G_S contains at most two copies of any arc, and if there are two copies of (u, v) then (v, u) is not in G_S.

Let F be a minimum directed f-join in G_S (if one exists). Note that F may contains two copies of the same arc if G_S is a directed multigraph. Also note that for any pair of vertices u, v, either $(u, v) \notin F$ or $(v, u) \notin F$, otherwise $F' = F \setminus \{(u, v), (v, u)\}$ would be a smaller f-join in G_S, contradicting the minimality of F. We define two sets A_F and D_F which, as we will show, correspond to a semi-solution (A_F, D_F) of (G, δ). Initially set $A_F = D_F = \emptyset$. Consider the arcs in F. If F contains (u, v) exactly once then add (u, v) to A_F if $(u, v) \notin E$ and add (v, u) to D_F if $(u, v) \in E$ (in this case $(v, u) \in E$ holds). If F contains two copies of (u, v) then add (u, v) to A_F and (v, u) to D_F; note that by definition of F and G_S, in this case $S = \{\mathsf{ea}, \mathsf{ed}\}, (u, v) \notin E$ and $(v, u) \in E$. Observe that the sets A_F and D_F are not multisets.

If X and Y are sets, then $X \uplus Y$ is the multiset that consists of one copy of each element that occurs in exactly one of X and Y and two copies of each element that occurs in both. The next lemma provides the starting point for our algorithm. Its proof has been omitted.

▶ **Lemma 7.** *Let* $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$*. Let* (G, δ) *be an instance of* CDBE(S) *where* $G = (V, E)$*. The following holds:*

(i) *If* F *is a minimum directed* f*-join in* G_S*, then* (A_F, D_F) *is a semi-solution for* (G, δ) *of size* $|F|$*.*

(ii) *If* (A, D) *is a semi-solution for* (G, δ)*, then* $A \uplus D^R$ *is a directed* f*-join in* G_S *of size* $|A| + |D|$*.*

Let (G, δ) be an instance of CDBE(S). Let $p = p_{(G,\delta)}$ be the number of components of G that contain no vertex of T. Let $q = q_{(G,\delta)}$ be the number of components of G that contain at least one vertex of T. Let $t = t_{(G,\delta)} = \sum_{u \in T} |f(u)|$.

We now state the following lemma. Its proof (based on Lemmas 2 and 7) has been omitted.

▶ **Lemma 8.** *Let* $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$*. Let* (G, δ) *be an instance of* CDBE(S) *with* $q \geq 1$*. If* F *is a (given) minimum directed* f*-join in* G_S*, then* (G, δ) *has a solution that has size at most* $\max\{|F|, p + q - 1, p + \frac{1}{2}t\}$*, which can be found in* $O(nm)$ *time.*

The next result is our first main result of this section. We prove it by showing that the upper bound in Lemma 8 is also a lower bound for (almost) any instance of CDBE(S) with $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$ that has a semi-solution.

▶ **Theorem 9.** *For* $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$*,* CDBE($S$) *can be solved in time* $O(n^3 \log n \log \log n)$*.*

Proof. Let $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$, and let (G, δ) be an instance of CDBE(S). We first use Lemma 2 to check whether G_S has a directed f-join. Because G_S has at most $2n^2$ arcs, this takes $O(n^3 \log n \log \log n)$ time. If G_S has no directed f-join then (G, δ) has no semi-solution by Lemma 7, and thus no solution either. Assume that G_S has a directed f-join, and let F be a minimum directed f-join that can be found in time $O(n^3 \log n \log \log n)$ by Lemma 2. As before, p denotes the number of components of G that do not contain any vertex of T, while q is the number of components of G that contain at least one vertex of T, and $t = \sum_{u \in T} |f(u)|$.

We will prove the following series of statements:

- $opt_S(G, \delta) = 0$ if $p \leq 1$, $q = 0$,
- $opt_S(G, \delta) = p$ if $p \geq 2$, $q = 0$,
- $opt_S(G, \delta) = \max(|F|, p + q - 1, p + \frac{1}{2}t)$ if $q > 0$.

If $p \leq 1$ and $q = 0$ then $A = D = \emptyset$ is an optimal solution. If $p \geq 2$ and $q = 0$, to ensure connectivity and preserve degree balance, for every component of G there must be at least one arc whose head is in this component and at least one arc whose tail is in this component, thus any solution must contain at least p arcs. Let $G_1, \ldots, G_p$ be the components of G and arbitrarily choose vertices $v_i \in V(G_i)$ for $i \in \{1, \ldots, p\}$. Let $A = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{p-1}, v_p), (v_p, v_1)\}$ and $D = \emptyset$. Then (A, D) is a solution which has size p and is therefore optimal.

Suppose $q \geq 1$. By Lemma 8 we find a solution (A, D) for (G, δ) of size at most $\max\{|F|, p + q - 1, p + \frac{1}{2}t\}$ in $O(nm)$ time. Hence, the total running time is $O(n^3 \log n \log \log n)$, and it remains to show that any solution has size at least $\max(|F|, p + q - 1, p + \frac{1}{2}t)$.

Let (A, D) be an arbitrary solution. Then (A, D) is also semi-solution. Every semi-solution has size at least $|F|$ by Lemma 7 2. Therefore (A, D) has size at least $|F|$.

Since there are $p + q$ components in G, we must add at least $p + q - 1$ arcs to ensure $G + A - D$ is connected. Therefore (A, D) has size at least $p + q - 1$.

Finally, for every vertex u with $f(u) > 0$ (resp. $f(u) < 0$) we find that (A, D) must be such that at least $|f(u)|$ arcs are either in A and have u as a tail (resp. head) or else are in D and have u as a head (resp. tail). For every component containing only vertices v with $f(v) = 0$, there must be at least one arc in A whose head is in this component and at least one arc in A whose tail is in this component (to ensure connectivity and to ensure that the degree balance is not changed for any vertex in this component). Therefore we have that (A, D) has size at least $p + \frac{1}{2}t$. This completes the proof of Theorem 9. ◀

The proof of our second main result of this section has been omitted.

▶ **Theorem 10.** *Let* $\{\mathsf{vd}\} \subseteq S \subseteq \{\mathsf{vd}, \mathsf{ed}, \mathsf{ea}\}$. *Then* CDBE($S$) *is* NP*-complete and* W[1]*-hard when parameterized by* k*, even if* $\delta \equiv 0$.

5 Conclusions

By extending previous work [1, 4, 7] we completely classified both the classical and parameterized complexity of CDPE(S) and CDBE(S), as summarized in Table 1. Our work followed the framework used [11, 17] for (Connected) Degree Constraint Editing(S). Our study was motivated by Eulerian graphs. As such, the variants DPE(S) and DBE(S) of CDPE(S) and CDBE(S), respectively, in which the graph H is no longer required to be connected, were beyond the scope of this paper. It follows from results of Cai and Yang [4] and Cygan [7], respectively, that for $S = \{\mathsf{vd}\}$, DPE(S) and DBE(S) are NP-complete and, when parameterized by k, W[1]-hard, whereas they are polynomial-time solvable for $S = \{\mathsf{ed}\}$ as a result of Lemmas 1 and 2, respectively. The problems DPE(S) and DBE(S) are also polynomial-time solvable if $\{\mathsf{ea}\} \subseteq S \subseteq \{\mathsf{ea}, \mathsf{ed}\}$; this is in fact proven by combining Lemmas 1 and 3 for the undirected case, and Lemmas 2 and 7 for the directed case. We expect the remaining (hardness) results of Table 1 to carry over as well.

Let ℓ be an integer. Here is a natural generalization of CDPE(S).

ℓ-CDME(S):	Connected Degree Modulo-ℓ-Editing(S)
Instance:	A graph G, integer k and a function $\delta\colon V(G) \to \{0, \dots, \ell-1\}$.
Question:	Can G be (S, k)-modified into a connected graph H with $d_H(v) \equiv \delta(v) \pmod{\ell}$ for each $v \in V(H)$?

Note that 2-CDME(S) is CDPE(S). The following theorem shows that the complexity of 3-CDME(S) may differ from 2-CDME(S).

▶ **Theorem 11.** 3-CDME($\{\mathsf{ea}, \mathsf{ed}\}$) *is* NP*-complete even if* $\delta \equiv 2$.

Proof. Reduce from the Hamiltonicity problem, which is NP-complete for connected cubic graphs [10]. Let G be a connected cubic graph. Let $\delta(v) = 2$ for every $v \in V(G)$, and take $k = |E(G)| - |V(G)|$. Then G has a Hamiltonian cycle if and only if G can be (S, k)-modified into a connected graph H with $d_H(v) = 2 \pmod 3$ for all $v \in V(H)$. ◀

It is natural to ask whether 3-CDME($\{\mathsf{ea}, \mathsf{ed}\}$) is fixed-parameter tractable with parameter k.

Finally, another direction for future research is to investigate how the complexity of CDPE(S) and CDBE(S) changes if we permit other graph operations, such as edge contraction, to be in the set S.

References

1 Francis T. Boesch, Charles L. Suffel, and Ralph Tindell. The spanning subgraphs of Eulerian graphs. *Journal of Graph Theory*, 1(1):79–84, 1977.

2 Pablo Burzyn, Flavia Bonomo, and Guillermo Durán. NP-completeness results for edge modification problems. *Discrete Applied Mathematics*, 154(13):1824–1844, 2006.

3 Leizhen Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Information Processing Letters*, 58(4):171–176, 1996.

4 Leizhen Cai and Boting Yang. Parameterized complexity of even/odd subgraph problems. *Journal of Discrete Algorithms*, 9(3):231–240, 2011.

5 Katarína Cechlárová and Ildikó Schlotter. Computing the deficiency of housing markets with duplicate houses. In *IPEC 2010*, volume 6478 of *Lecture Notes in Computer Science*, pages 72–83. Springer, 2010.

6 Robert Crowston, Gregory Gutin, Mark Jones, and Anders Yeo. Parameterized Eulerian strong component arc deletion problem on tournaments. *Information Processing Letters*, 112(6):249–251, 2012.

7 Marek Cygan, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Ildikó Schlotter. Parameterized complexity of Eulerian deletion problems. *Algorithmica*, 68(1):41–61, 2014.

8 Frederic Dorn, Hannes Moser, Rolf Niedermeier, and Mathias Weller. Efficient algorithms for Eulerian extension and rural postman. *SIAM Journal on Discrete Mathematics*, 27:75–94, 2013.

9 Jack Edmonds and Ellis L. Johnson. Matching, Euler tours and the Chinese postman. *Mathematical Programming*, 5:88–124, 1973.

10 Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

11 Petr A. Golovach. Editing to a connected graph of given degrees. In *MFCS Part II*, volume 8635 of *Lecture Notes in Computer Science*, pages 324–335. Springer, 2014.

12 Petr A. Golovach. Editing to a graph of given degrees. In *IPEC 2014*, *Lecture Notes in Computer Science*. Springer, to appear.

13 Peter L. Hammer and Bruno Simeone. The splittance of a graph. *Combinatorica*, 1(3):275–284, 1981.

14 Wiebke Höhn, Tobias Jacobs, and Nicole Megow. On Eulerian extensions and their application to no-wait flowshop scheduling. *Journal of Scheduling*, 15(3):295–309, 2012.

15 Linda Lesniak and Ortrud R. Oellermann. An Eulerian exposition. *Journal of Graph Theory*, 10(3):277–297, 1986.

16 John M. Lewis and Mihalis Yannakakis. The node-deletion problem for hereditary properties is NP-complete. *Journal of Computer and System Sciences*, 20(2):219–230, 1980.

17 Luke Mathieson and Stefan Szeider. Editing graphs to satisfy degree constraints: A parameterized approach. *Journal of Computer and System Sciences*, 78(1):179–191, 2012.

18 Silvio Micali and Vijay V. Vazirani. An $O(\sqrt{|V|}|E|)$ algorithm for finding maximum matching in general graphs. In *FOCS 1980*, pages 17–27. IEEE Computer Society, 1980.

19 Hannes Moser and Dimitrios M. Thilikos. Parameterized complexity of finding regular induced subgraphs. *Journal of Discrete Algorithms*, 7(2):181–190, 2009.

20 Assaf Natanzon, Ron Shamir, and Roded Sharan. Complexity classification of some edge modification problems. *Discrete Applied Mathematics*, 113(1):109–128, 2001.