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On stability in ideal compressible hydrodynamics

by

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Abstract.

The stability of parallel and horizontal shear flow of an ideal, compressible and adiabatic fluid in a constant gravity field is investigated in 2 and 3 dimensions. The method applied is based on a study of systems of transport equations similar to those appearing in geometrical acoustics. The results obtained are in agreement with known results for compressible fluids, and furthermore some results known only for incompressible fluids are seen to be valid also for compressible fluids.

Introduction.

The literature in hydrodynamics is dominated by studies on incompressible fluids; only a minor part is concerned with compressible fluids. Mathematically speaking, the incompressible fluid is an extremely singular limit of a compressible fluid, in fact the equations describing the two different objects are mathematically so different that it probably is more correct to consider them as two completely independent mathematical models. Experience as well as physical intuition, however, indicates that in many cases the two models most likely will give essentially the same description of the phenomena studied, but to actually prove this mathematically in advance is probably very difficult. The simplest way of showing it is most likely to consider the same problems for the two models and then compare the obtained results. It seems almost certain that the problems of stability belong to this category, since we are there concerned with what happens as the time $t \to +\infty$; even small differences in the models may therefore add up to essential differences in these problems.

The reason for the lack of literature on compressible fluids, which is especially noticable for the stability problems (see [1], [7]), is probably that the incompressible case has been thought to be mathematically more tractable than the compressible one. In the last decades, however, the available theory for hyperbolic equations has grown substantially, it seems therefore now possible to study ideal compressible fluids much more satisfactorily than before. This author has in [3] developed a theory of stability for hyperbolic equations which in this work is shown to be directly

applicable to problems in ideal compressible hydrodynamics. Primarily, this work is meant as an illustration of the applicability of the methods in [3], but in addition to giving results which are in agreement with known results, it also gives some minor results which are new as far as this author knows. In later works the methods in [3] will be seen to give further new results.

In section 1 the linearized equations of perturbations to be studied are found, and in section 2 some general properties of these equations are established. In section 3 the acoustic waves are studied in general. It is shown that the equations describing the change in the amplitudes of these waves due to inhomogenities etc., can be transformed to an extremely simple form. In section 4 this is used to show that the acoustic waves do not give rise to any instabilities in shear flow. In section 4 and 5 the gravity waves are considered in different types of shear flows. The results obtained are in agreement with known results for compressible as well as incompressible ideal fluids.

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In section 1 the linearized equations of perturbations to be studied are found, and in section 2 some general properties of these equations are established. In section 3 the acoustic waves are studied in general. It is shown that the equations describing the change in the amplitudes of these waves due to inhomogenities etc., can be transformed to an extremely simple form. In section 4 this is used to show that the acoustic waves do not give rise to any instabilities in shear flow. In section 4 and 5 the gravity waves are considered in different types of shear flows. The results obtained are in agreement with known results for compressible as well as incompressible ideal fluids.

1. Formulation of the problem.

We shall study the stability properties of an ideal, compressible and adiabatic fluid. The governing equations are

$$\underline{v}_t + \underline{v}\cdot\nabla\underline{v} = -\frac{1}{\rho}\nabla p + \nabla\phi$$

$$\rho_t + \underline{v}\cdot\nabla\rho + \rho\nabla\cdot\underline{v} = 0 \qquad (1.1)$$

$$\frac{\partial}{\partial t}(p\rho^{-\gamma}) + \underline{v}\cdot\nabla(p\rho^{-\gamma}) = 0$$

Here $\underline{v}$ is the velocity, ρ the density, p the pressure, ϕ a given potential for the external forces acting on the fluid, and γ is a constant. We shall consider both 2- and 3-dimensional flows, i.e. the velocity $\underline{v}$ and the deloperator ∇ in (1.1) may be 2- or 3-dimensional. The fluid may also be bounded or unbounded, the boundary conditions will be specified later.

Let $\underline{U}$, Q, P denote the velocity, the density and the pressure respectively, of a known solution of (1.1). For the present $\underline{U}$, Q, P may be arbitrary functions of the spacevariables and the time t, satisfying (1.1); we shall later restrict ourselves to special types of solutions. In order to study the stability properties of the solution $\underline{U}$, Q, P we introduce perturbations of it by substituting the following expressions into (1.1)

$$\underline{v} = \underline{U} + \underline{u} \quad , \quad \rho = Q + q \quad , \quad p = P + \sigma \qquad (1.2)$$

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$$\underline{v} = \underline{U} + \underline{u} \quad , \quad \rho = Q + q \quad , \quad p = P + s \qquad (1.2)$$

The equations for the perturbations are

$$\underline{u}_t + \underline{U}\cdot\nabla\underline{u} + \underline{u}\cdot\nabla\underline{U} + \underline{u}\cdot\nabla\underline{u} = \{\frac{1}{Q} - \frac{1}{Q+q}\}\nabla P - \frac{1}{Q+q}\nabla\sigma$$

$$q_t + \underline{U}\cdot\nabla q + \underline{u}\cdot\nabla Q + \underline{u}\cdot\nabla q + Q\nabla\cdot\underline{u} + q\nabla\cdot\underline{U} + q\nabla\cdot\underline{u} = 0 \qquad (1.3)$$

$$\sigma_t + \underline{U}\cdot\nabla\sigma + \underline{u}\cdot\nabla P + \underline{u}\cdot\nabla\sigma + \gamma P\nabla\cdot\underline{u} + \gamma\sigma\nabla\cdot\underline{U} + \gamma\sigma\nabla\cdot\underline{u} = 0$$

Since the solution $\underline{U}$, Q, P of (1.1) corresponds to the trivial solution $\underline{u} = q = \sigma = 0$ of (1.3), we want to study the stability properties of this trivial solution. We shall confine our study to the linearized version of the equations (1.3), which can be written in the following way

$$\underline{u}_t + \underline{U}\cdot\nabla\underline{u} + \frac{1}{Q}\nabla\sigma - \frac{1}{Q^2}q\nabla P + \underline{u}\cdot\nabla\underline{U} = 0$$

$$q_t + \underline{U}\cdot\nabla q + Q\nabla\cdot\underline{u} + q\nabla\cdot\underline{U} + \underline{u}\cdot\nabla Q = 0 \qquad (1.4)$$

$$\sigma_t + \underline{U}\cdot\nabla\sigma + \gamma P\nabla\cdot\underline{u} + \gamma\sigma\nabla\cdot\underline{U} + \underline{u}\cdot\nabla P = 0$$

This is a hyperbolic system which can be transformed into symmetric form by the following transformation of the dependent variables

$$\underline{w} = F\underline{u} \quad , \quad s_1 = \frac{QE}{F}\{cq - \frac{1}{c}\sigma\} \quad , \quad s_2 = \frac{F}{Qc}\sigma \qquad (1.5)$$

Here F and E are scalar weightfunctions which are completely at our disposal, we only assume that $F \neq 0$, $E \neq 0$ everywhere. The quantity c is the local soundspeed, it is given by

$$c = \sqrt{\frac{\gamma P}{Q}} \qquad (1.6)$$

When we introduce the transformation (1.5) into (1.4), we get

$$\underline{w}_t + \underline{U}\cdot\nabla\underline{w} + c\nabla s_2 - \frac{F^2}{Q^3E}\frac{1}{c}\nabla P\, s_1$$

$$+ \{\frac{1}{Qc}(\frac{\gamma}{2} - 1)\nabla P + \frac{c}{2Q}\nabla Q - \frac{c}{F}\nabla F\}s_2 \tag{1.7}$$

$$+ \underline{w}\cdot\nabla\underline{U} - \frac{1}{F}(F_t + \underline{U}\cdot\nabla F)\underline{w} = 0$$

$$s_{1t} + \underline{U}\cdot\nabla s_1 + \frac{EQ}{F^2}\{c\nabla Q - \frac{1}{c}\nabla P\}\cdot\underline{w} \tag{1.8}$$

$$+ \{\frac{1}{F}(F_t + \underline{U}\cdot\nabla F) - \frac{1}{E}(E_t + \underline{U}\cdot\nabla E) + \frac{1}{2}(3 + \gamma)\nabla\cdot\underline{U}\}s_1 = 0$$

$$s_{2t} + \underline{U}\cdot\nabla s_2 + c\nabla\cdot\underline{w} + \{\frac{1}{Qc}\nabla P - \frac{c}{F}\nabla F\}\cdot\underline{w} \tag{1.9}$$

$$+ \{\frac{1}{2}(\gamma - 1)\nabla\cdot\underline{U} - \frac{1}{F}(F_t + \underline{U}\cdot\nabla F)\}s_2 = 0$$

As we shall see in the next section, this is a symmetric hyperbolic system with characteristics of constant multiplicity. We shall in this paper study the stability properties of the trivial solution $\underline{w} = s_1 = s_2 = 0$ of this system by the method developed in Eckhoff [3].

2. The characteristic equations.

We shall in this section calculate the characteristic roots and eigenvectors associated with the hyperbolic system (1.7, 8 & 9) in cartesian spacecoordinates. We consider first the 3-dimensional case, i.e. the case with three independent space variables x^1, x^2, x^3, and $\underline{w} = \{w_1, w_2, w_3\}$ 3-dimensional. If we introduce the 5-dimensional vector

$$\tilde{u} = \{w_1, w_2, w_3, s_1, s_2\} \tag{2.1}$$

the system (1.7, 8 & 9) can be written

$$\tilde{u}_t + A^1\tilde{u}_{x^1} + A^2\tilde{u}_{x^2} + A^3\tilde{u}_{x^3} + B\tilde{u} = 0 \tag{2.2}$$

Here we have treated $\tilde{u}$ as a columnvector, and from (1.7, 8 & 9) the 5×5 matrices A^1, A^2, A^3, B are seen to be

$$A^1 = \begin{Bmatrix} U_1 & 0 & 0 & 0 & c \\ 0 & U_1 & 0 & 0 & 0 \\ 0 & 0 & U_1 & 0 & 0 \\ 0 & 0 & 0 & U_1 & 0 \\ c & 0 & 0 & 0 & U_1 \end{Bmatrix}, \quad A^2 = \begin{Bmatrix} U_2 & 0 & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 & c \\ 0 & 0 & U_2 & 0 & 0 \\ 0 & 0 & 0 & U_2 & 0 \\ 0 & c & 0 & 0 & U_2 \end{Bmatrix}$$

$$A^3 = \begin{Bmatrix} U_3 & 0 & 0 & 0 & 0 \\ 0 & U_3 & 0 & 0 & 0 \\ 0 & 0 & U_3 & 0 & c \\ 0 & 0 & 0 & U_3 & 0 \\ 0 & 0 & c & 0 & U_3 \end{Bmatrix} \tag{2.3}$$

$$B = \{b_1\ b_2\ b_3\ b_4\ b_5\} \tag{2.4}$$

where b_1, b_2, b_3, b_4, b_5 are the following 5-dimensional column vectors

$$b_1 = \begin{Bmatrix} U_{1x^1} - \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) \\ U_{2x^1} \\ U_{3x^1} \\ \frac{EQ}{F^2}(cQ_{x^1} - \frac{1}{c}P_{x^1}) \\ \frac{1}{Qc}P_{x^1} - \frac{c}{F}F_{x^1} \end{Bmatrix}, \quad b_2 = \begin{Bmatrix} U_{1x^2} \\ U_{2x^2} - \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) \\ U_{3x^2} \\ \frac{EQ}{F^2}(cQ_{x^2} - \frac{1}{c}P_{x^2}) \\ \frac{1}{Qc}P_{x^2} - \frac{c}{F}F_{x^2} \end{Bmatrix}$$

$$b_3 = \begin{Bmatrix} U_{1x^3} \\ U_{2x^3} \\ U_{3x^3} - \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) \\ \frac{EQ}{F^2}(cQ_{x^3} - \frac{1}{c}P_{x^3}) \\ \frac{1}{Qc}P_{x^3} - \frac{c}{F}F_{x^3} \end{Bmatrix}$$

$$b_4 = \begin{Bmatrix} -\frac{F^2}{Q^3E}\frac{1}{c}P_{x^1} \\ -\frac{F^2}{Q^3E}\frac{1}{c}P_{x^2} \\ -\frac{F^2}{Q^3E}\frac{1}{c}P_{x^3} \\ \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) - \frac{1}{E}(E_t + \underline{U}\cdot\nabla E) + \frac{1}{2}(3 + \gamma)\nabla\cdot\underline{U} \\ 0 \end{Bmatrix}$$

$$b_5 = \begin{bmatrix} \frac{1}{Qc}(\frac{\gamma}{2} - 1)P_{x^1} + \frac{c}{2Q}Q_{x^1} - \frac{c}{F}F_{x^1} \\ \frac{1}{Qc}(\frac{\gamma}{2} - 1)P_{x^2} + \frac{c}{2Q}Q_{x^2} - \frac{c}{F}F_{x^2} \\ \frac{1}{Qc}(\frac{\gamma}{2} - 1)P_{x^3} + \frac{c}{2Q}Q_{x^3} - \frac{c}{F}F_{x^3} \\ 0 \\ \frac{1}{2}(\gamma - 1)\nabla\cdot\underline{U} - \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) \end{bmatrix}$$

From (2.3) we see that (2.2) is a symmetric hyperbolic system. The characteristic equation associated with (2.2) is

$$\det\{-\lambda I + \xi^1 A^1 + \xi^2 A^2 + \xi^3 A^3\}$$
$$\equiv (\underline{\xi}\cdot\underline{U} - \lambda)^3\{(\underline{\xi}\cdot\underline{U} - \lambda)^2 - c^2|\xi|^2\} = 0 \tag{2.5}$$

where
$$\underline{\xi} = \{\xi^1, \xi^2, \xi^3\}$$
$$|\xi| = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2} \tag{2.6}$$

The characteristic roots are seen from (2.5) to be

$$\Omega^1 = \underline{\xi}\cdot\underline{U} \qquad (q^1 = 3)$$
$$\Omega^2 = \underline{\xi}\cdot\underline{U} + c|\xi| \qquad (q^2 = 1) \tag{2.7}$$
$$\Omega^3 = \underline{\xi}\cdot\underline{U} - c|\xi| \qquad (q^3 = 1)$$

From (2.7) we see that (2.2) has characteristics of constant

multiplicity. The roots Ω^2 and Ω^3 are seen to be nonlinear with respect to $\underline{\xi}$, they correspond to the acoustic waves in the fluid. The triple root Ω^1 is seen to be linear with respect to $\underline{\xi}$, it corresponds to the gravity waves (mass waves, entropy waves).

The eigenvectors associated with the characteristic roots (2.7) can be chosen to be

$$
\begin{aligned}
r^{11} &= \ell^{11} = \frac{1}{|\xi|}\{\xi^2, -\xi^1, 0, \xi^3, 0\} \\
r^{12} &= \ell^{12} = \frac{1}{|\xi|}\{0, \xi^3, -\xi^2, \xi^1, 0\} \\
r^{13} &= \ell^{13} = \frac{1}{|\xi|}\{-\xi^3, 0, \xi^1, \xi^2, 0\} \qquad (2.8) \\
r^{21} &= \ell^{21} = \frac{1}{\sqrt{2}|\xi|}\{\xi^1, \xi^2, \xi^3, 0, |\xi|\} \\
r^{31} &= \ell^{31} = \frac{1}{\sqrt{2}|\xi|}\{\xi^1, \xi^2, \xi^3, 0, -|\xi|\}
\end{aligned}
$$

Let us now consider the 2-dimensional case, i.e. the case with two space variables x^1, x^2, and $\underline{w} = \{w_1, w_2\}$ 2-dimensional. If we introduce the 4-dimensional vector

$$u^* = \{w_1, w_2, s_1, s_2\} \qquad (2.9)$$

the system (1.7, 8 & 9) can be written

$$u^*_t + A^{*1}u^*_{x^1} + A^{*2}u^*_{x^2} + B^*u^* = 0 \qquad (2.10)$$

The 4×4 matrices A^{*1} and A^{*2} are obtained from (2.3) by deleting the third row and the third column in A^1 and A^2 respectively. The 4×4 matrix B^* is obtained from (2.4) by deleting the third row and the third column. The characteristic equation associated with the symmetric hyperbolic system (2.10) is

$$\det \{-\lambda I + \xi^1 A^{*1} + \xi^2 A^{*2}\}$$

$$\equiv (\underline{\xi} \cdot \underline{U} - \lambda)^2 \{(\underline{\xi} \cdot \underline{U} - \lambda)^2 - c^2 |\xi|^2\} = 0 \qquad (2.11)$$

where $\underline{\xi}$ and $|\xi|$ are the 2-dimensional analogues of (2.6). The characteristic roots are

$$\Omega^1 = \underline{\xi} \cdot \underline{U} \qquad (q^1 = 2)$$

$$\Omega^2 = \underline{\xi} \cdot \underline{U} + c|\xi| \qquad (q^2 = 1) \qquad (2.12)$$

$$\Omega^3 = \underline{\xi} \cdot \underline{U} - c|\xi| \qquad (q^3 = 1)$$

The eigenvectors associated with these roots can be chosen to be

$$R^{11} = \{0, 0, 1, 0\}$$

$$R^{12} = \frac{1}{|\xi|}\{\xi^2, -\xi^1, 0, 0\}$$

$$(2.13)$$

$$R^{21} = \frac{1}{\sqrt{2}|\xi|}\{\xi^1, \xi^2, 0, |\xi|\}$$

$$R^{31} = \frac{1}{\sqrt{2}|\xi|}\{\xi^1, \xi^2, 0, -|\xi|\}$$

From the expressions we have found in this section, it is a straightforward matter to calculate the transport equations as well as the stability equations (see [3]). For the acoustic waves we shall do this in the next section. For the gravity waves the expressions become rather complicated in general, we shall therefore calculate the transport equations only in special cases for these waves in section 4 and 5.

3. The acoustic waves.

The characteristic roots Ω^2 and Ω^3 which correspond to the acoustic waves in the fluid, are nonlinear with respect to $\underline{\xi}$. From [3] we know that instead of the transport equations, we should consider the stability equations in a stability research. The compression terms are seen to be

$$K^2\sigma_1^2 = \frac{1}{2}\sum_\mu \frac{\partial^2\Omega^2}{\partial x^\mu \partial\xi^\mu}\sigma_1^2 = \frac{1}{2}\{\nabla\cdot\underline{U} + \frac{1}{|\xi|}\underline{\xi}\cdot\nabla c\}\sigma_1^2$$

$$K^3\sigma_1^3 = \frac{1}{2}\sum_\mu \frac{\partial^2\Omega^3}{\partial x^\mu \partial\xi^\mu}\sigma_1^3 = \frac{1}{2}\{\nabla\cdot\underline{U} - \frac{1}{|\xi|}\underline{\xi}\cdot\nabla c\}\sigma_1^3 \qquad (3.1)$$

The stability equations for the characteristic root Ω^2 are in the 3-dimensional case

$$\frac{dx^i}{dt} = U_i + \frac{c}{|\xi|}\xi^i$$

$$i = 1,2,3 \qquad (3.2)$$

$$\frac{d\xi^i}{dt} = -\underline{\xi}\cdot\underline{U}_{x^i} - c_{x^i}|\xi|$$

$$\frac{d\sigma_1^2}{dt} = \{-r^{21}Br^{21} + K^2\}\sigma_1^2$$

$$= \{-\frac{1}{2|\xi|^2}\underline{\xi}\cdot\nabla\underline{U}\cdot\underline{\xi} + \frac{1}{F}(F_t + \underline{U}\cdot\nabla F)$$

$$- \frac{1}{2|\xi|}\underline{\xi}\cdot(\frac{c}{2P}\nabla P + \frac{c}{2Q}\nabla Q - \frac{2c}{F}\nabla F)$$

$$- \frac{1}{4}(\gamma-1)\nabla\cdot\underline{U} + \frac{1}{2}(\nabla\cdot\underline{U} + \frac{1}{|\xi|}\underline{\xi}\cdot\nabla c)\}\sigma_1^2$$

$$\Rightarrow \frac{d\sigma_1^2}{dt} = \{\frac{1}{2|\xi|^2}\xi\cdot\frac{d}{dt}\xi + \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) + \frac{1}{4}(3-\gamma)\nabla\cdot\underline{U}$$

$$+ \frac{c}{|\xi|}\xi\cdot(\frac{1}{F}\nabla F + \frac{1}{c}\nabla c - \frac{1}{4P}\nabla P - \frac{1}{4Q}\nabla Q)\}\sigma_1^2 \tag{3.3}$$

We have that

$$\frac{1}{F}\nabla F + \frac{1}{c}\nabla c - \frac{1}{4P}\nabla P - \frac{1}{4Q}\nabla Q = \frac{\sqrt[4]{PQ}}{Fc}\nabla\left(\frac{Fc}{\sqrt[4]{PQ}}\right) \tag{3.4}$$

The expression (3.4) is seen to vanish if we choose

$$F = \frac{\sqrt{\gamma}\sqrt[4]{PQ}}{c} = \sqrt{\frac{Q}{c}} \tag{3.5}$$

For this choice of F, we get from (1.1) and (1.6) that

$$\frac{1}{F}(F_t + \underline{U}\cdot\nabla F) = -\frac{1}{4}(3-\gamma)\nabla\cdot\underline{U} \tag{3.6}$$

Thus for F given by (3.5) the equation (3.3) reduces to

$$\frac{d\sigma_1^2}{dt} = \frac{1}{4}\{\frac{1}{|\xi|^2}\frac{d}{dt}|\xi|^2\}\sigma_1^2 \Rightarrow \sigma_1^2 = \sigma_{10}^2\sqrt{\frac{|\xi|}{|\xi_0|}} \tag{3.7}$$

Here $|\xi|$ is determined by the bicharacteristic equations (3.2) (which are the equations for the sound "rays"), and the subindex 0 refers to the values at $t = 0$ on the ray considered.

If instead of (3.5) we let F be given by

$$F = \sqrt{Q} \tag{3.8}$$

it follows in the same way as above that

$$\sigma_1^2 = \sigma_{10}^2 \sqrt{\frac{c|\xi|}{c_0|\xi_0|}} \tag{3.9}$$

This seemingly more complicated expression will be seen below to be more natural than (3.7), since $(\sigma_1^2)^2$ in this case corresponds in some sense to the local energy in the acoustic wave.

The simple relations (3.7) and (3.9), which hold for <u>any</u> given solution $\underline{U}$, Q, P of (1.1), indicates that the compression terms and therefore the stability equations have a profound physical significance. For any other choice of the compression terms, it will not in general be possible to choose the weightfunction F such that (3.7) or (3.9) are satisfied. From the derivation in [3], we know that the solutions of the stability equations in some sense represents a mean value of the amplitude of the acoustic waves.

When $\underline{U}$, Q, P are independent of t, Ω^2 is an integral of the bicharacteristic equations (3.2), i.e.

$$\frac{d}{dt}\{\underline{\xi}\cdot\underline{U} + c|\xi|\} = 0 \tag{3.10}$$

The expression (3.9) can therefore in this case be written in the following way

$$\sigma_1^2 = \sigma_{10}^2\{1 + \frac{1}{c_0|\xi_0|}(\underline{\xi}_0\cdot\underline{U}_0 - \underline{\xi}\cdot\underline{U})\}^{\frac{1}{2}} \tag{3.11}$$

In section 4 and 5 we shall see that in many cases we are able to conclude from (3.11) that $\sigma_1^2 = 0$ is stable.

For the special cases where $\nabla\underline{U} \equiv 0$, i.e. the cases <u>without shear</u>, the right hand side of (3.3) is seen to vanish if F is

given by (3.8). For this choice therefore, σ_1^2 is a constant along the bicharacteristics. The physical interpretation of this and the expression (3.11) is simple: When F is given by (3.8), $(\sigma_1^2)^2$ corresponds in some sense to a mean value of the local energy density in the acoustic wave. Thus when $\nabla U \equiv 0$ the energy carried by the acoustic wave is conserved, while there will in general be an exchange of energy between the acoustic wave and the basic flow $\underline{U}$, Q, P when $\nabla \underline{U} \not\equiv 0$.

The discussion of the other family of acoustic waves, corresponding to the charcteristic root Ω^3, is completely analogous. The stability equations are

$$\frac{dx^i}{dt} = U_i - \frac{c}{|\xi|}\xi^i \qquad i = 1,2,3 \tag{3.12}$$

$$\frac{d\xi^i}{dt} = -\underline{\xi}\cdot\underline{U}_{x^i} + c_{x^i}|\xi|$$

$$\begin{aligned}\frac{d\sigma_1^3}{dt} &= \{-r^{31}Br^{31} + K^3\}\sigma_1^3 \\ &= \{\frac{1}{2|\xi|^2}\,\underline{\xi}\cdot\frac{d}{dt}\,\underline{\xi} + \frac{1}{F}(F_t + \underline{U}\cdot\nabla F) \\ &\quad - \frac{c}{|\xi|}\,\underline{\xi}\cdot(\frac{1}{F}\nabla F + \frac{1}{c}\nabla c - \frac{1}{4P}\nabla P - \frac{1}{4Q}\nabla Q) \\ &\quad + \frac{1}{4}(3 - \gamma)\nabla\cdot\underline{U}\}\sigma_1^3\end{aligned} \tag{3.13}$$

If F is given by (3.5), we get

$$\frac{d\sigma_1^3}{dt} = \frac{1}{4}\{\frac{1}{|\xi|^2}\,\frac{d}{dt}|\xi|^2\}\sigma_1^3 \Rightarrow \sigma_1^3 = \sigma_{10}^3\sqrt{\frac{|\xi|}{|\xi_0|}} \tag{3.14}$$

If F is given by (3.8), we get

$$\sigma_1^3 = \sigma_{10}^3 \sqrt{\frac{c|\xi|}{c_0|\xi_0|}} \tag{3.15}$$

When $\underline{U}$, Q, P are independent of t, Ω^3 is an integral of the bicharacteristic equations (3.12). In this case, therefore, (3.15) can be written

$$\sigma_1^3 = \sigma_{10}^3 \{1 + \frac{1}{c_0|\xi_0|}(\underline{\xi}\cdot\underline{U} - \underline{\xi}_0\cdot\underline{U}_0)\}^{\frac{1}{2}} \tag{3.16}$$

For the cases without shear, the right hand side in (3.13) is seen to vanish if we let F be given by (3.8). Thus σ_1^3 is a constant along the bicharacteristic in this case.
The physical interpretation of these results is identical as in the corresponding cases discussed above for the root Ω^2, it is therefore omitted here.

In the cases without shear we found that both σ_1^2 and σ_1^3 are constants along the bicharacteristics, when F is given by (3.8). Thus $\sigma_1^2 = \sigma_1^3 = 0$ is obviously stable in this case if we do not consider boundary conditions which may act as a source of energy. It is natural to ask whether the stability properties of the acoustic waves actually depend on the choice of F, or whether the choice of F is just a matter of convenience in handling the equations. To settle this, we consider an unbounded "atmosphere" in static equilibrium in a constant gravity field. Thus, let $\phi = - gx^3$ and consider the following solution of (1.1)

$$\underline{U} \equiv 0 \ , \quad Q = D\exp\{-\mu x^3\} \ , \quad P = D\frac{g}{\mu}\exp\{-\mu x^3\} \tag{3.17}$$

If F is given by (3.8), we get

$$\sigma_1^3 = \sigma_{10}^3 \sqrt{\frac{c|\underline{\xi}|}{c_0|\underline{\xi}_0|}} \qquad (3.15)$$

When $\underline{U}$, Q, P are independent of t, $\underline{\xi}^2$ is an integral of the bicharacteristic equations (3.12). In this case, therefore, (3.15) can be written

$$\sigma_1^3 = \sigma_{10}^3 (1 + \frac{1}{c_0|\underline{\xi}_0|}(\underline{\xi}\cdot\underline{U} - \underline{\xi}_0\cdot\underline{U}_0))^{\frac{1}{2}} \qquad (3.16)$$

For the cases without shear, the right hand side in (3.13) is seen to vanish if we let F be given by (3.8). Thus σ_1^3 is a constant along the bicharacteristic in this case.
The physical interpretation of these results is identical as in the corresponding cases discussed above for the root τ^2, it is therefore omitted here.

In the cases without shear we found that both σ_1^2 and σ_1^3 are constants along the bicharacteristics, when F is given by (3.8). Thus $\sigma_1^2 = \sigma_1^3 = 0$ is obviously stable in this case if we do not consider boundary conditions which may act as a source of energy. It is natural to ask whether the stability properties of the acoustic waves actually depend on the choice of F, or whether the choice of F is just a matter of convenience in handling the equations. To settle this, we consider an unbounded "atmosphere" in static equilibrium in a constant gravity field. Thus, let $\phi = - gx^3$ and consider the following solution of (1.1)

$$\underline{U} = 0 \ , \quad Q = D \exp(- \mu x^3) \ , \quad P = D \frac{g}{\mu} \exp(- \mu x^3) \qquad (3.17)$$

where g, D, μ are positive constants. From the equations (3.2) and (3.12) it follows that $\underline{\xi}$ is constant along the bicharacteristics, and that the bicharacteristics are straight lines. The equations (3.3) and (3.13) become

$$\frac{d\sigma_1^2}{dt} = \{\frac{1}{F}(F_t + \frac{c}{|\xi|}\underline{\xi}\cdot\nabla F) + \frac{c\mu\xi^3}{2|\xi|}\}\sigma_1^2$$
$$\frac{d\sigma_1^3}{dt} = \{\frac{1}{F}(F_t - \frac{c}{|\xi|}\underline{\xi}\cdot\nabla F) - \frac{c\mu\xi^3}{2|\xi|}\}\sigma_1^3 \qquad (3.18)$$

From these equations we see that $\sigma_1^2 = 0$ and $\sigma_1^3 = 0$ cannot be stable simultaneously unless either F depends on t explicitly, or the right hand sides of (3.18) essentially vanish. Since we consider the stability of a static solution (3.17) of (1.1), it is unnatural to let F depend on t. Thus in order to assure stability of the acoustic waves, F is essentially determined within a constant factor by (3.8). The physical interpretation of this is straight forward in view of our considerations about the energy of the acoustic waves.

In the above calculations we have seen that weightfunctions may be very essential in a study of stability problems. We have only introduced two scalar weightfunctions F and E in the hyperbolic system (1.7, 8 & 9), while the most general weightfunctionmatrix will consist of 25 functions in the 3-dimensional case and 16 functions in the 2-dimensional case. It is therefore natural to ask whether there is anything to gain by introducing more weightfunctions in the equations. If we compare (3.7) and (3.14) with the general expressions obtained in section 10 in [3] for the effect of the weightfunctions, it seems that the expressions we have obtained are the simplest possible in the general case.

Thus as far as the acoustic waves are concerned, it suffices to consider only one weightfunction F, since the choice of E is seen to have no influence at all. As a matter of fact, we shall see that E does not have any influence on the stability properties of the gravity waves either. The choice of a weightfunction is therefore not necessarily essential in a stability research.

From the theory of stability developed in [3], we know that in general the stability equations will only give necessary conditions for stability. In order to obtain sufficient conditions, the effect of the distortion coefficients in the W.K.B-expansion have to be examined. So far the available theory for handling this problem is rather limited, but it seems possible to treat some special cases. Such cases are for instance those where plane waves remain plane waves. The acoustic waves are seen to have this property if

$$\frac{\partial^2 U}{\partial x^i \partial x^j} = \frac{\partial^2 c}{\partial x^i \partial x^j} \equiv 0 \quad ; \quad i,j = 1,2,3 \tag{3.19}$$

Even though these conditions are satisfied only in very special cases, it seems worth while studying more closely cases where (3.19) are satisfied, in order to gain intuition about the distortion coefficients. So far we have not considered this in detail, but we do not expect the distortion coefficients to give rise to any instabilities for the acoustic waves.

In order to examine the effect on the stability properties of the distortion coefficients, the compression effect has to be taken into account. This can most easily be done when the compression effect can be compensated for in the weightfunctions.

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$$\frac{\partial^2 U}{\partial x^i \partial x^j} = \frac{\partial^2 G}{\partial x^i \partial x^j} = 0 \;\; ; \;\; i,j = 1,2,3 \qquad (3.19)$$

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In order to examine the effect on the stability properties of the distortion coefficients, the compression effect has to be taken into account. This can most easily be done when the compression effect can be compensated for in the weightfunctions.

It is easily found that a sufficient condition for this is that

$$\nabla \cdot \underline{U} \equiv 0 \qquad (3.20)$$

In fact, if (3.20) is satisfied and we put $K^2 = K^3 = 0$ into (3.3) and (3.13), then these equations can be transformed into (3.9) and (3.15) respectively if instead of (3.8) we let F be given by

$$F = \sqrt{Qc} \qquad (3.21)$$

In the above discussion we have seen that in order to assure the stability of the acoustic waves, we are in a constructive way led to specific choices of the weightfunction F. These choices (3.5, 8 & 21) are analogous to transformations considered earlier by various authors Yih [7], Eckart [2].

4. Two-dimensional parallel flow.

In this section we consider the two-dimensional version of (1.1) with

$$\phi = - gx^2 \tag{4.1}$$

where $g > 0$ is the acceleration of gravity which is assumed to be a constant. We want to study the stability properties of parallel flows, i.e. solutions of (1.1) of the form

$$U_1 = U_1(x^2),\ U_2 = 0,\ Q = Q(x^2),\ P = P(x^2) \tag{4.2}$$

It is readily seen that (4.2) is a solution of (1.1) for arbitrary functions U_1 and Q if and only if the function P satisfies

$$P_{x^2} = - gQ \tag{4.3}$$

For given Q, (4.3) determines P within an additive constant. For physical reasons we only consider non-negative functions P, Q.

The fluid considered is obviously assumed to be unbounded along the x^1-axis. Along the x^2-axis the fluid may be bounded or unbounded. If the fluid is unbounded upwards in the positive x^2-direction, the function Q must decrease sufficiently fast when $x^2 \to +\infty$, in order that P shall be non-negative everywhere. If the fluid is bounded in the positive x^2-direction, we shall assume that it is bounded either by a rigid wall or that the boundary is free. For the value of x^2 at the boundary, the boundary conditions are respectively

$$w_2 = 0 \tag{4.4}$$

$$s_2 = 0 \tag{4.5}$$

If the fluid is bounded in the negative x^2-direction, we assume the rigid wall boundary condition (4.4) there.

Let us first consider the acoustic waves. From (3.2 & 11), (3.12 & 16) and (4.2) it follows that the amplitudes of the acoustic waves at the time t and at the point x^2 are given by

$$\sigma_1^2 = \sigma_{10}^2[1 + \frac{\xi_0^1}{c(x_0^2)|\xi_0|}\{U_1(x_0^2) - U_1(x^2)\}]^{\frac{1}{2}} \tag{4.6}$$

$$\sigma_1^3 = \sigma_{10}^3[1 + \frac{\xi_0^1}{c(x_0^2)|\xi_0|}\{U_1(x^2) - U_1(x_0^2)\}]^{\frac{1}{2}} \tag{4.7}$$

where the subindex 0 refers to the values at $t = 0$ on the rays considered. The expressions (4.6) and (4.7) are valid as long as the rays do not hit the boundaries. When a ray of one of the acoustic families (superindex 2 or 3 for σ) hits the boundary, it is easy to show that it is reflected, i.e. the "information" carried by the ray up to the boundary is carried into the fluid again on another ray. If the rigid wall boundary condition (4.4) is assumed, the reflected ray belongs to the same family (i.e. the same superindex for σ), and the initialvalues of x^1, x^2, ξ^1, ξ^2 and σ_1^2 or σ_1^3 for the reflected ray are the values of these quantities on the ray hitting the boundary at the point of reflection, with the exception that ξ^2 changes sign. If the free boundary condition (4.5) is assumed, the reflected ray belongs to the other family, i.e. a ray with superindex 2 is reflected

into a ray with superindex 3 and vice versa. In this case the initialvalues of x^1, x^2, ξ^1, ξ^2 and σ_1^2 or σ_1^3 for the reflected ray are the values of these quantities on the ray hitting the boundary at the point of reflection, with the exception that the superindex for σ changes.

With this background we can conclude from (4.6 & 7) that the change in the amplitude along a ray essentially only depends on the change in x^2, regardless of whether the ray is reflected or not when the fluid is bounded. Furthermore we see from (4.6 & 7) that $\sigma_1^2 = \sigma_1^3 = 0$ is stable if there exist a constant M such that

$$|U_1| \leq M \tag{4.8}$$

everywhere in the fluid. Obviously, (4.8) is always satisfied if the fluid is bounded in both x^2-directions. When the fluid is unbounded in one or both x^2-directions, (4.8) represents a restriction. However, a flow which does not satisfy (4.8), is certainly not realistic physically. Thus we conclude that as far as our study goes, the acoustic waves do not give rise to any instabilities for physically realistic flows (4.2).

We shall now consider the gravity waves. In our calculations we shall keep the weightfunctions F and E unspecified, but we shall assume them to depend on x^2 only, in order to be consistent with (4.2). If we consider a fluid which is unbounded in the x^2-direction, the weightfunction F cannot be chosen arbitrarily as we saw in section 3, since we insist that the acoustic waves shall be stable. However, we shall see that the stability properties of the gravity waves are independent of both F and E, the restric-

tions on F will therefore not cause any difficulty.

In the case we are considering, the matrix B* in (2.10) becomes

$$B^* = \begin{pmatrix} 0 & U_{1x^2} & 0 & 0 \\ 0 & 0 & \frac{F^2 g}{Q^2 Ec} & H \\ 0 & \frac{EQ}{F^2}(cQ_{x^2} + \frac{gQ}{c}) & 0 & 0 \\ 0 & -\frac{g}{c} - \frac{c}{F}F_{x^2} & 0 & 0 \end{pmatrix} \qquad (4.9)$$

where $H = (1-\frac{\gamma}{2})\frac{g}{c} + \frac{c}{2Q}Q_{x^2} - \frac{c}{F}F_{x^2}$

The compression terms for the gravity waves are in general seen to contain a factor $\nabla\cdot\underline{U}$. For the solution (4.2) therefore, the compression terms vanish identically. Thus the stability equations and the transport equations are identical for the gravity waves in the case we are considering, they are seen to be

$$\frac{dx^1}{dt} = U_1$$

$$\frac{dx^2}{dt} = 0$$

$$\frac{d\xi^1}{dt} = 0 \qquad (4.10)$$

$$\frac{d\xi^2}{dt} = -\xi^1 U_{1x^2}$$

$$\frac{d\sigma_1^1}{dt} = -R^{11}B^*R^{12}\sigma_2^1$$

$$\frac{d\sigma_2^1}{dt} = -R^{12}B^*R^{11}\sigma_1^1 - R^{12}B^*R^{12}\sigma_2^1 \qquad (4.11)$$

All the other terms in the general form of the transport equations obtained in [3] are seen to vanish. From (2.13) and (4.9) we get

$$\begin{aligned} R^{11}B^*R^{12} &= -\frac{\xi^1}{|\xi|}\frac{EQ}{F^2}(cQ_{x^2}+\frac{gQ}{c}) \\ R^{12}B^*R^{11} &= -\frac{\xi^1}{|\xi|}\frac{F^2 g}{Q^2 Ec} \\ R^{12}B^*R^{12} &= -\frac{\xi^1\xi^2}{|\xi|^2}U_{1x^2} \end{aligned} \tag{4.12}$$

The bicharacteristic equations (4.10) can be directly integrated, the solutions are

$$\begin{aligned} &x^1 = x_0^1 + U_1(x_0^2)t \quad , \quad x^2 = x_0^2 \\ &\xi^1 = \xi_0^1 \quad , \quad \xi^2 = \xi_0^2 - \xi_0^1 U_{1x^2}(x_0^2)t \end{aligned} \tag{4.13}$$

where the subindex 0 refers to the initialvalues at $t = 0$. From (4.13) it is clear that the bicharcteristics associated with the gravity waves never reach the boundaries of the fluid, if they start from within the fluid. Thus it is clear that within the fluid the boundary conditions will have no effect on the amplitude of the gravity waves. It is also clear that the boundary conditions can only play a secondary role in the distortion coefficients for the gravity waves, and therefore in the stability problem altogether in view of the results we have earlier obtained for the acoustic waves.

According to the theory of stability developed in [3], we now have to substitute the expressions (4.13) into the amplitude

equations (4.11) and then study the stability properties of the trivial solution $\sigma_1^1 = \sigma_2^1 = 0$ for all values of x_0^1, x_0^2, ξ_0^1, ξ_0^2. If $\xi_0^1 = 0$, the quantities (4.12) are seen to vanish and consequently in this case $\sigma_1^1 = \sigma_{10}^1$ and $\sigma_2^1 = \sigma_{20}^1$. Thus no instabilities can be detected unless $\xi_0^1 \neq 0$, we therefore assume this in the following. With this assumption the quantities (4.13) are independent of t if and only if

$$U_{1x^2}(x_0^2) = 0 \qquad (4.14)$$

For those points x_0^2 where (4.14) is satisfied, if any, (4.11) is an autonomous linear system. The stability properties of the solution $\sigma_1^1 = \sigma_2^1 = 0$ are therefore determined by the eigenvalues of the coefficientmatrix in (4.11), which are found to be

$$\lambda_{\pm} = \pm i \frac{\xi_0^1}{|\xi_0|} N \qquad (4.15)$$

Here $i = \sqrt{-1}$, and N is the local Väisälä-Brunt frequency which is given by (see [2], [7])

$$- N^2 = g\frac{Q_{x^2}}{Q} + (\frac{g}{c})^2 \qquad (4.16)$$

From this we can conclude that a necessary condition for the trivial solution of (4.11) to be stable for all values of x_0^1, x_0^2, ξ_0^1, ξ_0^2, is that N is a real quantity at those points x_0^2 where (4.14) is satisfied. The case $N = 0$ has to be given special attention since $\lambda_+ = \lambda_- = 0$ is a multiple eigenvalue. However, it is easily seen directly from (4.11) that in this case

$$\sigma_1^1 = \sigma_{10}^1 , \quad \sigma_2^1 = \sigma_{20}^1 + \sigma_{10}^1 \frac{\xi_0^1}{|\xi_0|} \frac{F^2 g}{Q^2 Ec} t \tag{4.17}$$

$\sigma_1^1 = \sigma_1^2 = 0$ is therefore unstable when $N = 0$.

In order to draw conclusions about the stability properties of the flow (4.2) from the above results, care has to be taken to avoid the critical cases discussed at the end of §6 in [3]. In fact, such critical cases may exist if for instance N is a real quantity everywhere and $N = 0$ for a set of values of x^2 of measure zero. However, we are able to draw the following conclusion: A necessary condition for stability of the flow (4.2) is that the inequality

$$\frac{Q_{x^2}}{Q} < - \frac{g}{c^2} \tag{4.18}$$

holds almost everywhere in the set of points where (4.14) is satisfied.

The physical interpretation of the inequality (4.18) is simple and well known (see [2], [7]): Since for each fluid element $p\rho^{-\gamma}$ is conserved by (1.1), it is easy to show that the total force acting on the fluid element is pointing in the direction that opposes the fluid element from moving out of its position of static equilibrium if and only if (4.18) is satisfied. Even though the condition (4.18) and its interpretation is well known, this author has not found any results in the literature which actually prove that the unstable motions are always dynamically possible when (4.18) is violated in some region. Thus in addition to serving as an illustration of the applicability of the theory of stability developed in [3], the result established

above may also fill a minor gap in the literature.

We now consider the points with shear, i.e. the points x_0^2 where (4.14) is not satisfied. For these points (4.11) becomes a nonautonomous linear system where all the coefficients (4.12) tend to zero when $t \to +\infty$. In the appendix it is shown that under certain conditions, which clearly are satisfied here, the stability problem can be solved by introducing $\tau = \ln t$ as a new independent variable. If we do this, (4.11) can be written

$$\frac{d\sigma_1^1}{d\tau} = \frac{\xi_0^1 \frac{EQ}{F^2}(cQ_{x^2} + \frac{gQ}{c})\sigma_2^1}{\{(\xi_0^1 U_{1x^2})^2 - 2\xi_0^1\xi_0^2 U_{1x^2}e^{-\tau} + |\xi_0|^2 e^{-2\tau}\}^{\frac{1}{2}}}$$

$$\frac{d\sigma_2^1}{d\tau} = \frac{\xi_0^1 \frac{F^2 g}{Q^2 Ec}\sigma_1^1}{\{(\xi_0^1 U_{1x^2})^2 - 2\xi_0^1\xi_0^2 U_{1x^2}e^{-\tau} + |\xi_0|^2 e^{-2\tau}\}^{\frac{1}{2}}} - \frac{\{(\xi_0^1 U_{1x^2})^2 - \xi_0^1\xi_0^2 U_{1x^2}e^{-\tau}\}\sigma_2^1}{(\xi_0^1 U_{1x^2})^2 - 2\xi_0^1\xi_0^2 U_{1x^2}e^{-\tau} + |\xi_0|^2 e^{-2\tau}} \tag{4.19}$$

Asymptotically as $\tau \to +\infty$, this system becomes

$$\frac{d\sigma_1^1}{d\tau} = \frac{\xi_0^1}{|\xi_0^1 U_{1x^2}|}\frac{EQ}{F^2}(cQ_{x^2} + \frac{gQ}{c})\sigma_2^1$$

$$\frac{d\sigma_2^1}{d\tau} = \frac{\xi_0^1}{|\xi_0^1 U_{1x^2}|}\frac{F^2 g}{Q^2 Ec}\sigma_1^1 - \sigma_2^1 \tag{4.20}$$

The coefficient matrix in this system corresponds to the matrix A in the appendix, its eigenvalues are found to be

$$\lambda_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - R} \tag{4.21}$$

Here R is the local Richardson number (see [1], [4], [5], [7] for the incompressible analogue of R), which here is given by

$$R = \left(\frac{N}{U_{1x^2}}\right)^2 \tag{4.22}$$

In view of the results shown in the appendix, it now follows from (4.21) that the trivial solution of (4.11) is asymptotically stable if $R > 0$, which is equivalent to (4.18), it is stable if $R = 0$, and it is unstable if $R < 0$. From this and the theory in [3] it is now a straight forward matter to draw conclusions about the stability of the flow (4.2). We can summarize the results found in the following

<u>Theorem.</u> A necessary condition for stability of the flow (4.2) is that in every point

$$\frac{Q_{x^2}}{Q} \leq -\frac{g}{c^2} \tag{4.23}$$

In the points without shear it is necessary that the strict inequality is satisfied almost everywhere.

In the marginal case where the equality sign in (4.23) holds, the discussion above seems to indicate that a shear in the flow have a stabilizing effect. However, if we allow 3-dimensional perturbations, this seemingly stabilizing effect of the shear disappears as we shall see in the next section.

At the points without shear, i.e. the points where (4.14) is satisfied, we see from (4.15) that the amplitude of the gravity waves oscillates with the Väisälä-Brunt frequency N if $\xi_0^2 = 0$,

while it oscillates with a frequency less than N if $\xi_0^2 \neq 0$. Thus N is in this case seen to be the characteristic frequency of the gravity waves as we would expect in view of other theories [2], [7].

At the points in the fluid where (4.14) is not satisfied, we see from (4.21) that when $R \neq \frac{1}{4}$ the amplitudes of the gravity waves are asymptotically as $t \to +\infty$ given by

$$\sigma_\nu^1 = c_{1\nu} t^{-\frac{1}{2}+\sqrt{\frac{1}{4}-R}} + c_{2\nu} t^{-\frac{1}{2}-\sqrt{\frac{1}{4}-R}} \; ; \; \nu = 1,2 \qquad (4.24)$$

where $c_{1\nu}$, $c_{2\nu}$ are constants along the bicharacteristics. Thus we see that in this case the gravity waves have an oscillatory character only when $R > \frac{1}{4}$, while there is pure damping when $0 < R < \frac{1}{4}$. When $R = \frac{1}{4}$ we find that asymptotically as $t \to +\infty$

$$\sigma_\nu^1 = c_{1\nu} t^{-\frac{1}{2}} + c_{2\nu} t^{-\frac{1}{2}} \ln t \; ; \; \nu = 1,2 \qquad (4.25)$$

The expressions (4.24 & 25) are completely analogous to the results obtained for the continuous spectrum in incompressible fluids by Engevik [4], [5].

In the results obtained above, we have not detected any instabilities similar to those found in the incompressible case when $0 < R < \frac{1}{4}$ (see [1], [5], [7]). However, from the theory of stability developed in [3], we know that in general the results obtained above will only give necessary conditions for stability. In order to obtain sufficient conditions, the effect of the distortion coefficients in the W.K.B-expansion has to be examined. Since Ω^1 is linear with respect to $\underline{\xi}$, it seems possible to handle this problem, but so far we have not studied it any further.

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At the points in the fluid where (4.14) is not satisfied, we see from (4.21) that when $R \neq \frac{1}{4}$ the amplitudes of the gravity waves are asymptotically as $t \to +\infty$ given by

$$a_\nu^1 = c_{1\nu} t^{-\frac{1}{2}+\sqrt{\frac{1}{4}-R}} + c_{2\nu} t^{-\frac{1}{2}-\sqrt{\frac{1}{4}-R}} \; ; \; \nu = 1,2 \qquad (4.24)$$

where $c_{1\nu}, c_{2\nu}$ are constants along the bicharacteristic. Thus we see that in this case the gravity waves have an oscillatory character only when $R > \frac{1}{4}$, while there is pure damping when $0 < R < \frac{1}{4}$. When $R = \frac{1}{4}$ we find that asymptotically as $t \to +\infty$

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We expect that the distortion coefficients may give rise to instabilities in some cases when $0 < R < \frac{1}{4}$, since the amplitudes (4.26) are non-oscillating and only weakly damped for R in this region.

5. Three-dimensional horizontal flow.

In this section we consider the three-dimensional version of (1.1) with

$$\phi = -gx^3 \tag{5.1}$$

where $g > 0$ is still the constant acceleration of gravity. We want to study the stability properties for solutions of (1.1) of the form

$$\begin{gathered} U_1 = U_1(x^3)\ ,\ U_2 = U_2(x^3)\ ,\ U_3 = 0 \\ Q = Q(x^3)\ ,\ P = P(x^3) \end{gathered} \tag{5.2}$$

It is readily seen that (5.2) is a solution of (1.1) for arbitrary functions U_1, U_2 and Q if and only if the function P satisfies

$$P_{x^3} = -gQ \tag{5.3}$$

For given Q, this equation determines P.

We see that the solution (5.2) represents a three-dimensional horizontal flow where the direction as well as the magnitude of the velocity depend on the height x^3. Obviously the fluid is assumed to be unbounded in the horizontal directions, i.e. the x^1 and x^2-directions, while the fluid may be bounded or unbounded in the vertical direction, i.e. the x^3-direction. If the fluid is bounded in the x^3-direction, the effect of the boundaries is

completely analogous in this case as in the case discussed in the previous section. Also the discussion of the amplitudes of the acoustic waves becomes completely analogous; these matters are therefore left to the reader. Thus we shall limit our discussion to the amplitudes of the gravity waves.

In order to be consistent with (5.2), the weightfunctions F and E are assumed to depend on x^3 only. The matrix B given by (2.4) is then seen to take the form

$$B = \begin{bmatrix} 0 & 0 & U_{1x^3} & 0 & 0 \\ 0 & 0 & U_{2x^3} & 0 & 0 \\ 0 & 0 & 0 & \frac{F^2 g}{Q^2 Ec} & H \\ 0 & 0 & \frac{EQ}{F^2}(cQ_{x^3} + \frac{gQ}{c}) & 0 & 0 \\ 0 & 0 & -\frac{g}{c} - \frac{c}{F}F_{x^3} & 0 & 0 \end{bmatrix} \qquad (5.4)$$

where $H = (1 - \frac{\gamma}{2})\frac{g}{c} + \frac{c}{2Q}Q_{x^3} - \frac{c}{F}F_{x^3}$

Since $\nabla \cdot \underline{U} = 0$ for the solution (5.2), the stability equations are identical with the transport equations for the gravity waves in the case we are considering. The transport equations are found to be

$$\frac{dx^1}{dt} = U_1$$

$$\frac{dx^2}{dt} = U_2$$

$$\frac{dx^3}{dt} = 0 \qquad (5.5)$$

completely analogous in this case as in the case discussed in the previous section. Also the discussion of the amplitudes of the acoustic waves becomes completely analogous; these matters are therefore left to the reader. Thus we shall limit our discussion to the amplitudes of the gravity waves.

In order to be consistent with (5.2), the weightfunctions F and E are assumed to depend on x^3 only. The matrix B given by (2.4) is then seen to take the form

$$B = \begin{bmatrix} 0 & 0 & U_{1x^3} & 0 & 0 \\ 0 & 0 & U_{2x^3} & 0 & 0 \\ 0 & 0 & 0 & \frac{T^2 g}{Q^2 Ec} & H \\ 0 & 0 & \frac{EQ}{T^2}(cQ_{x^3} + \frac{gQ}{c}) & 0 & 0 \\ 0 & 0 & -\frac{g}{c} - \frac{c}{T}T_{x^3} & 0 & 0 \end{bmatrix} \qquad (5.4)$$

where $H = (1 - \frac{\gamma}{2})\frac{g}{c} + \frac{c}{2Q}Q_{x^3} - \frac{c}{T}T_{x^3}$.

Since $\nabla \cdot \underline{U} = 0$ for the solution (5.2), the stability equations are identical with the transport equations for the gravity waves in the case we are considering. The transport equations are found to be

$$\frac{dx^1}{dt} = U_1$$

$$\frac{dx^2}{dt} = U_2$$

$$\frac{dx^3}{dt} = 0 \qquad (5.5)$$

$$\frac{d\xi^1}{dt} = 0$$

$$\frac{d\xi^2}{dt} = 0 \qquad (5.5)$$

$$\frac{d\xi^3}{dt} = -\xi^1 U_{1x^3} - \xi^2 U_{2x^3}$$

$$\frac{d\sigma_1^1}{dt} = -r^{11}\cdot B\cdot r^{12}\sigma_2^1 - r^{11}\cdot B\cdot r^{13}\sigma_3^1$$

$$+ \frac{\partial\Omega^1}{\partial x^3} r^{11}\cdot\frac{\partial r^{12}}{\partial\xi^3}\sigma_2^1 + \frac{\partial\Omega^1}{\partial x^3} r^{11}\cdot\frac{\partial r^{13}}{\partial\xi^3}\sigma_3^1$$

$$= \{\frac{(\xi^2)^2 - (\xi^1)^2}{|\xi|^2}U_{1x^3} - \frac{2\xi^1\xi^2}{|\xi|^2}U_{2x^3}$$

$$+ \frac{\xi^2\xi^3}{|\xi|^2}\frac{EQ}{F^2}(cQ_{x^3} + \frac{gQ}{c})\}\sigma_2^1 \qquad (5.6)$$

$$+ \{\frac{(\xi^1)^2 - (\xi^2)^2}{|\xi|^2}U_{2x^3} - \frac{2\xi^1\xi^2}{|\xi|^2}U_{1x^3}$$

$$- \frac{\xi^1\xi^3}{|\xi|^2}\frac{EQ}{F^2}(cQ_{x^3} + \frac{gQ}{c})\}\sigma_3^1$$

$$\frac{d\sigma_2^1}{dt} = -r^{12}\cdot B\cdot r^{11}\sigma_1^1 - r^{12}\cdot B\cdot r^{12}\sigma_2^1$$

$$- r^{12}\cdot B\cdot r^{13}\sigma_3^1 + \frac{\partial\Omega^1}{\partial x^3} r^{12}\cdot\frac{\partial r^{11}}{\partial\xi^3}\sigma_1^1$$

$$= \{\frac{(\xi^1)^2}{|\xi|^2}U_{1x^3} + \frac{\xi^1\xi^2}{|\xi|^2}U_{2x^3} + \frac{\xi^2\xi^3}{|\xi|^2}\frac{F^2 g}{Q^2 Ec}\}\sigma_1^1$$

$$+ \{\frac{\xi^2\xi^3}{|\xi|^2}U_{2x^3} + \frac{\xi^1\xi^2}{|\xi|^2}(\frac{F^2 g}{Q^2 Ec} + \frac{EQc}{F^2}Q_{x^3} + \frac{EQ^2 g}{F^2 c})\}\sigma_2^1 \qquad (5.7)$$

$$+ \{-\frac{\xi^1\xi^3}{|\xi|^2}U_{2x^3} + \frac{(\xi^2)^2}{|\xi|^2}\frac{F^2 g}{Q^2 Ec}$$

$$- \frac{(\xi^1)^2}{|\xi|^2}\frac{EQ}{F^2}(cQ_{x^3} + \frac{gQ}{c})\}\sigma_3^1$$

$$\frac{d\sigma_3^1}{dt} = - r^{13}\cdot B\cdot r^{11}\sigma_1^1 - r^{13}\cdot B\cdot r^{12}\sigma_2^1$$

$$- r^{13}\cdot B\cdot r^{13}\sigma_3^1 + \frac{\partial\Omega^1}{\partial x^3} r^{13}\cdot\frac{\partial r^{11}}{\partial \xi^3}\sigma_1^1$$

$$= \{\frac{\xi^1\xi^2}{|\xi|^2}U_{1x^3} + \frac{(\xi^2)^2}{|\xi|^2}U_{2x^3} - \frac{\xi^1\xi^3}{|\xi|^2}\frac{F^2 g}{Q^2 Ec}\}\sigma_1^1$$

$$+ \{- \frac{\xi^2\xi^3}{|\xi|^2}U_{1x^3} - \frac{(\xi^1)^2}{|\xi|^2}\frac{F^2 g}{Q^2 Ec} \qquad (5.8)$$

$$+ \frac{(\xi^2)^2}{|\xi|^2}\frac{EQ}{F^2}(cQ_{x^3} + \frac{gQ}{c})\}\sigma_2^1$$

$$+ \{\frac{\xi^1\xi^3}{|\xi|^2}U_{1x^3} - \frac{\xi^1\xi^2}{|\xi|^2}(\frac{F^2 g}{Q^2 Ec} + \frac{EQc}{F^2}Q_{x^3} + \frac{EQ^2 g}{F^2 c})\}\sigma_3^1$$

The bicharacteristic equations (5.5) can be directly integrated, the solutions are

$$x^1 = x_0^1 + U_1(x_0^3)t \ , \ x^2 = x_0^2 + U_2(x_0^3)t$$

$$x^3 = x_0^3 \ , \ \xi^1 = \xi_0^1 \ , \ \xi^2 = \xi_0^2 \qquad (5.9)$$

$$\xi^3 = \xi_0^3 - \{\xi_0^1 U_{1x^3}(x_0^3) + \xi_0^2 U_{2x^3}(x_0^3)\}t$$

where the subindex 0 refers to the initialvalues at $t = 0$.

When we substitute (5.9) into (5.6, 7 & 8) we obtain a closed linear system for the amplitudes σ_1^1, σ_2^1, σ_3^1. We want to study the stability properties for the trivial solution $\sigma_1^1 = \sigma_2^1 = \sigma_3^1 = 0$ of this system for all possible values of x_0^1, x_0^2, x_0^3, ξ_0^1, ξ_0^2, ξ_0^3. It is easily seen that the system (5.6, 7 & 8) is autonomous if and only if

$$e \overset{\text{def}}{=} \xi_0^1 U_{1x^3}(x_0^3) + \xi_0^2 U_{2x^3}(x_0^3) = 0 \tag{5.10}$$

For all possible values of x_0^3, ξ_0^1, ξ_0^2 which satisfy (5.10), the stability properties for the solution $\sigma_1^1 = \sigma_2^1 = \sigma_3^1 = 0$ are therefore determined by the eigenvalues of the coefficient-matrix in (5.6, 7 & 8). These eigenvalues are found to be

$$\lambda_1 = 0 \ , \ \lambda_2 = i \frac{\sqrt{(\xi_0^1)^2 + (\xi_0^2)^2}}{|\xi_0|} N \ , \ \lambda_3 = - \lambda_2 \tag{5.11}$$

The local Väisälä-Brunt frequency N (which is given by (4.16) with x^2 replaced by x^3) is seen to appear here in essentially the same way as in the 2-dimensional case (4.15). The case $\sqrt{(\xi_0^1)^2 + (\xi_0^2)^2} N = 0$ has to be given special attention since $\lambda = 0$ is then a multiple eigenvalue. If $\xi_0^1 = \xi_0^2 = 0$, the right hand sides in (5.6, 7 & 8) all vanish and consequently $\sigma_1^1 = \sigma_{10}^1$, $\sigma_2^1 = \sigma_{20}^1$, $\sigma_3^1 = \sigma_{30}^1$. Thus no instabilities can be detected unless $(\xi_0^1)^2 + (\xi_0^2)^2 \neq 0$, we therefore assume this in the following. With this assumption, the coefficient matrix in (5.6, 7 & 8) can never vanish when $e = N = 0$. Hence the eigenvalue $\lambda = 0$ has an index ≥ 2 (see [6]), we therefore conclude that the trivial solution of (5.6, 7 & 8) is unstable in this case.

Since for all possible values of x_0^3 we can obviously find values of ξ_0^1 and ξ_0^2 such that (5.10) is satisfied and $(\xi_0^1)^2 + (\xi_0^2)^2 \neq 0$, we can summarize the results found so far in the following

<u>Theorem.</u> A necessary condition for stability of the flow (5.2) is that

$$\frac{Q_{x^3}}{Q} < -\frac{g}{c^2} \tag{5.12}$$

holds almost everywhere in the fluid.

We observe here that the seemingly stabilizing effect of the shear found in the preceding section for the marginal cases where equality holds in (5.12), has now disappeared completely. Thus the results we have obtained here by studying the special cases where (5.10) is satisfied, are stronger than those we obtained in the previous section altogether. The above analysis which holds regardless of whether there is shear or not, is based on the fact that we here allow 3-dimensional perturbations. In fact, (5.10) means that the perturbation essentially acts orthogonal to the basic flow.

We now consider values of x_0^3, ξ_0^1, ξ_0^2 such that (5.10) is not satisfied. The system (5.7, 8 & 9) then becomes a nonautonomous linear system where all coefficients tend to zero as $t \to +\infty$. As in the preceding section we therefore introduce $\tau = \ln t$ as a new independent variable in (5.7, 8 & 9). Asymptotically as $\tau \to +\infty$ this system then becomes

$$\frac{d\sigma_1^1}{d\tau} = -\frac{\xi_0^2 a}{e}\sigma_2^1 + \frac{\xi_0^1 a}{e}\sigma_3^1$$

$$\frac{d\sigma_2^1}{d\tau} = -\frac{\xi_0^2 b}{e}\sigma_1^1 - \frac{\xi_0^2 U_{2x^3}}{e}\sigma_2^1 + \frac{\xi_0^1 U_{2x^3}}{e}\sigma_3^1 \tag{5.13}$$

$$\frac{d\sigma_3^1}{d\tau} = \frac{\xi_0^1 b}{e}\sigma_1^1 + \frac{\xi_0^2 U_{1x^3}}{e}\sigma_2^1 - \frac{\xi_0^1 U_{1x^3}}{e}\sigma_3^1$$

where
$$a = \frac{EQ}{F^2}(cQ_{x^3} + \frac{gQ}{c})$$
$$b = \frac{F^2 g}{Q^2 Ec} \qquad (5.14)$$

The eigenvalues of the coefficient matrix in (5.13) are found to be

$$\lambda_1 = 0 \ , \ \lambda_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} - R^*} \ , \ \lambda_3 = -\frac{1}{2} - \sqrt{\frac{1}{4} - R^*} \qquad (5.15)$$

where

$$R^* = [(\xi_0^1)^2 + (\xi_0^2)^2]\ \frac{ab}{e^2}.$$

When (5.12) is satisfied, we find that for any ξ_0^1, ξ_0^2

$$R^* \geqq R = \frac{N^2}{(U_{1x^3})^2 + (U_{2x^3})^2} > 0 \qquad (5.16)$$

The trivial solution $\sigma_1^1 = \sigma_2^1 = \sigma_3^1 = 0$ of (5.13) is therefore seen from (5.15) to be stable in this case. Thus it follows from the results shown in the appendix that the trivial solution of (5.6, 7 & 8) must be stable when (5.12) is satisfied, consequently we conclude that we are not led to any additional conditions for stability when we consider values of x_0^3, ξ_0^1, ξ_0^2 such that $e \neq 0$.

The quantity R in (5.16) is completely analogous to the Richardson number (4.22), and we see that $R^* = R$ for those ξ_0^1, ξ_0^2 which satisfy

$$\xi_0^1 = \alpha U_{1x^3}(x_0^3) \ , \ \xi_0^2 = \alpha U_{2x^3}(x_0^3) \qquad (5.17)$$

for some $\alpha \neq 0$. Thus the discussion of the behaviour of the amplitudes of the gravity waves will be completely analogous to the discussion in the preceding section, it is therefore omitted.

Appendix.

We want to study the stability properties for the trivial solution $\underline{w} = 0$ of the following linear system of ordinary differential equations

$$\frac{d\underline{w}}{dt} = D(t)\cdot\underline{w} \tag{1}$$

where $\underline{w} = \{w_1, \ldots, w_n\}$ and $D(t)$ is an $n\times n$-matrix which is continuous for $t \geq 0$ and such that

$$\lim_{t\to+\infty} D(t) = 0 \tag{2}$$

We shall assume that we can find a constant matrix A and a scalar constant $\alpha > 1$ such that the matrix

$$B(t) \stackrel{\text{def}}{=} t^{\alpha}D(t) - t^{\alpha-1}A \tag{3}$$

is bounded for $t \in [1,+\infty)$. From Taylors theorem and the continuity of $D(t)$ it is easily seen that sufficient conditions for the existence of A and α is that $\frac{d}{ds}D(\frac{1}{s})$, $\frac{d^2}{ds^2}D(\frac{1}{s})$ exist and are continuous for $s \in [0,\varepsilon]$ for some $\varepsilon > 0$. In this case we may choose $\alpha = 2$ and

$$A = \lim_{t\to+\infty} \{tD(t)\} \tag{4}$$

In fact, A must in any case be given by (4), thus a necessary condition for the existence of A and α is that the limit (4) exists.

With the assumptions made above we see that (1) can be written

$$\frac{d\underline{w}}{dt} = \{\frac{1}{t}A + \frac{1}{t^{\alpha}}B(t)\}\cdot\underline{w} \quad , \quad t \in [1,+\infty) \tag{5}$$

where $\alpha > 1$ is a constant, A a constant $n\times n$-matrix, and $B(t)$ an $n\times n$-matrix which is such that

$$\| B(t) \| \leqq c \quad , \quad t \in [1,+\infty) \tag{6}$$

where c is a constant. In order to study the stabilityproperties of the trivial solution of (5), we introduce $\tau = \ln t$ as a new independent variable. The system (5) then becomes

$$\frac{d\underline{w}}{d\tau} = \{A + e^{(1-\alpha)\tau}B(e^{\tau})\}\cdot\underline{w} \tag{7}$$

Since $t \to +\infty$ is equivalent to $\tau \to +\infty$, the trivial solutions of (5) and (7) respectively, have identical stabilityproperties. From (6) it follows since $\alpha > 1$ that

$$\int_0^{+\infty} \| e^{(1-\alpha)\tau}B(e^{\tau}) \| \, d\tau \leqq \frac{c}{\alpha-1} \tag{8}$$

Thus we see that the conditions in theorem 3.1 and 3.3 in Roseau [6] are satisfied for the system (7), we have therefore established

Theorem. With the assumptions made, the trivial solution of (1) is stable if the trivial solution of

$$\frac{d\underline{v}}{d\tau} = A\cdot\underline{v} \tag{9}$$

is stable, where A is given by (4).

Furthermore, the trivial solution of (1) is

a) asymptotically stable if all the eigenvalues of A have negative real parts.

b) unstable if A has an eigenvalue with positive real part.

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