When are Plurality Rule Voting Games[¤] Dominance-Solvable?

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First Version: March 1999
This version: December 1999

Abstract

[¤]We would like to thank Tilman Borgers, Martin Cripps, Jean-Francois Mertens, and Myrna Wooders for very valuable advice. We would also like to thank seminar participants at the ESRC Game Theory Workshop at UCL, The CORE-Franqui Summer School in Political Economy, the International Conference of Game Theory at SUNY, Stony Brook, Queen Mary College, University of London, The Indian Statistical Institute, and the Universities of Nottingham, Southampton, and York.

Abstract

This paper studies the dominance-solvability (by iterated deletion of weakly dominated strategies) of plurality rule voting games. For $K \geq 3$ alternatives and $n \geq 3$ voters, we find sufficient conditions for the game to be dominance-solvable (DS) and not to be DS. These conditions can be stated in terms of only one statistic of the game, the largest proportion of voters who agree on which alternative is worst in a sequence of subsets of the original set of alternatives. When n is large, "almost all" games can be classified as either DS or not DS. If the game is DS, a Condorcet Winner always exists when $n \geq 4$, and the outcome is always the Condorcet Winner when the electorate is sufficiently replicated.

1. Introduction

Plurality voting is the dominant electoral rule in most democracies. Nevertheless, its properties are still not well-understood. One literature has studied plurality rule under the simplifying assumption that voters vote sincerely i.e. for their most preferred alternative (Lepelley (1993), Gehrlein (1998), Gehrlein and Lepelley (1998)). Even with this simpli…cation, plurality rule is far from the most well-behaved voting rule. For example, in contrast to the case with majority rule between pairs of alternatives, it is possible that even when a Condorcet winner[1] (CW) exists, it is not selected by plurality rule. By imposing a stochastic structure on the voting problem (every voter has a preference ordering drawn at random from the possible orderings over a …nite set of alternatives, and all orderings are equally likely), the literature on sincere voting can make statements about which kind of voting rule is most likely to choose the CW, and avoid choosing the Condorcet loser. Plurality rule performs rather badly in this setting; for example, Gehrlein(1998) shows that with plurality voting, selection of a CW is less likely, and selection of a Condorcet loser is more likely, than with approval voting.

Of course, the assumption that voters vote sincerely is very strong, and has no game theoretic foundations. In many circumstances[2], if other voters are voting sincerely, it is rational for a voter not to vote for their most preferred candidate (i.e. sincere voting is not a Nash equilibrium). In practice, strategic (non-sincere) voting seems to be quite common where plurality rule voting is used. For example, in parliamentary elections in the UK and Germany evidence suggests that candidates who were perceived to be running third were deserted by their support-

[1] An alternative that beats (is beaten by) every other in pair-wise voting with majority rule is called the Condorcet Winner (Loser).

[2] A simple case is where voter 1 prefers x to y to z, voter 2 prefers y to x to z, and voter 3 prefers z to y to x: Assume (as we do below) that ties are broken by selecting each winning alternative with equal probability. Suppose now that voters 1 and 2 vote sincerely. If 3 votes sincerely, each of the three alternatives occurs with probability 1/3. If 3 votes for his second-ranked alternative y; the outcome is y with probability 1. If voter 3 prefers the second outcome, he will vote strategically.

ers (Cox(1997), Chapter 4). Moreover there is some experimental evidence that voters do vote strategically in three candidate elections (Forsythe et al. (1996)).

The main problem that arises when we move away from sincere voting is that multiple voting equilibria are pervasive with plurality rule. For example[3], consider the "canonical" plurality voting game where voters vote simultaneously, preferences are common knowledge, and ties are broken according to a "neutral" rule that treats alternatives symmetrically[4]. It is obvious that with at least three voters, any candidate may win in a Nash equilibrium: if all other voters vote for this candidate, then it is a (weak) best response for any voter to also vote for that candidate[5], as she cannot a¤ect the outcome, however she votes.

The reason this problem arises is that Nash equilibrium allows any possible beliefs on the part of voters, as long as they are consistent. For example, suppose that it is common knowledge that a candidate, z is worst for all voters. Nevertheless, there is a Nash equilibrium where every voter votes for z because he believes that all other voters will vote for z. The obvious response to this problem is to look for equilibrium re…nements, such as ruling out weakly dominated voting strategies (Besley and Coate(1997)). However, it turns out that standard re…nements have little bite in this canonical plurality rule game. De Sinopoli(1999) shows that with more than four voters, if an alternative is not a strict Condorcet loser, there is a

[3] The multiple equilibrium problem also arises when agents have incomplete information about some aspect of the structure of the game (Myatt(1999), Myerson and Weber(1993), Myerson(1998)). For example, Myerson(1998) studies "scoring" voting rules (of which plurality voting is a special case) in an environment where there are three alternatives, each voter is equally likely to have three possible preference orderings over these alternatives (in the base case), and the number of voters is a Poisson random variable. The equilibrium is de…ned for the limiting case as the expected number of voters becomes large, and allows voters to make small mistakes. Even in this setting, plurality rule generates multiple equilibria; there is an equilibrium where any one of the three candidates can win with probability one.

[4] The only such rule is: if there are K alternatives with the most number of votes, then each of these alternatives is selected as winner with probability $1=K$ (Myerson(1998)):

[5] This is in contrast to pairwise majority voting over a prede…ned binary agenda. If agents play weakly undominated actions at each stage, there is always a unique outcome. However, when a CW does not exist, the outcome is agenda-dependent.

perfect Nash equilibrium where that alternative is an outcome with probability at least 0.5. Moreover, there is by definition only one strict Condorcet loser in any set of alternatives. It follows from this result that imposing the weaker refinement of weakly undominated Nash equilibrium will rule out at most one alternative as a Nash outcome.

We take a different approach to this problem of multiplicity of Nash equilibria in this paper. First, we argue below that eliminating weakly dominated strategies is very reasonable in the plurality rule game; it simply amounts to no-one voting for her worst-ranked alternative[6]. But, there is nothing to stop voters going a step further and recalculating which strategies are weakly dominated for them given that other voters will not use weakly dominated strategies. In other words, if we iteratively eliminate weakly dominated strategies, it is possible that we could substantially narrow down the set of possible outcomes in the plurality voting game. It has long been recognized that iterated deletion may be a powerful tool for predicting outcomes in voting games. For example, Farquarson (1969) called this procedure "sophisticated voting", and he called a voting game "determinate" if sophisticated voting led to a unique outcome.

Our paper addresses the question of when the plurality rule voting game is determinate, or dominance solvable. The main contribution is to derive conditions that are sufficient for the game to be dominance-solvable and not to be dominance-solvable. Moreover, as the number of voters, n, becomes large, these conditions are asymptotically necessary and sufficient for dominance-solvability. The conditions are most easily stated in the case of three alternatives, when they involve just one summary statistic of the game, namely the largest fraction of players that agree on which alternative is worst, q. When this fraction is greater than 2/3, the game is always dominance-solvable; when this fraction is less than or equal to $2/3 - \pi_n$, for some $\pi_n > 0$, the game is never dominance-solvable. Moreover, there is at most only one possible value of q between $2/3$ and $2/3 - \pi_n$, and π_n goes to zero asymptotically.

[6] Lemma 1 below shows that with more than three voters, the only voting strategy that is weakly dominated is the one where the voter votes for her worst alternative.

The intuition for the sufficiency condition is straightforward. First, voting for one's worst alternative is weakly dominated, so if a sufficient fraction of the voters agree on which is worst, all voters can deduce that this alternative cannot win if voters do not vote for weakly dominated alternatives. But if this alternative cannot win, a vote for it is "wasted" i.e. weakly dominated wherever it appears in a voter's preference ordering, so the game is reduced to one of just two alternatives by iterated deletion, and two-alternative voting games are always dominance-solvable.

The intuition behind the sufficient condition for the game not to be dominance-solvable is more subtle. When there is sufficient disagreement on the worst alternative, the space of weakly undominated strategy profiles is rich enough to ensure that for any voter i, voting for her middle-ranked (or best) alternative is a unique best response (i.e. not weakly dominated) to some weakly undominated profile of voting strategies of the other players. This means that iterated deletion cannot proceed beyond deleting the strategy of voting for one's worst alternative.

In the completely general case with n voters and K alternatives, the conditions generalize from the three-alternative case. In a game with K alternatives, let q_K be the largest fraction of players who agree on which alternative (say z_K) is worst. When z_K is deleted from the feasible set; let q_{K-1} be the largest fraction of players who agree on which remaining alternative (say z_{K-1}) is worst, and so on. This procedure generates a sequence $q_K, q_{K-1}, \ldots q_3$. Our sufficient condition for dominance-solvability is that each element in the sequence be sufficiently large ($q_k > (k-1)/k$) i.e. there should be sufficient agreement on which alternative is worst across the players. Our sufficient condition for non-dominance-solvability is that there is an $3 \le l \le K$, such that for all $k > l$, q_k is sufficiently large, but q_l fails to be sufficiently large (i.e. $q_l \le (l-1)/l - \pi_n^l$, for some $\pi_n^l > 0$). Moreover, as $n \to \infty$, $\pi_n^k \to 0$. We are able to show that if we increase the number of voters without changing the distribution of preferences across alternatives (replicating the electorate), for a large enough electorate, we have necessary and sufficient conditions[7] for the game to be dominance-solvable. Finally, when the sufficient

[7] Subject to a weak regularity condition - see Theorem 4 below.

conditions for dominance-solvability hold, we show that the only strategies that survive iterated deletion involve every voter voting for one of two alternatives (a strong form of Duverger's Law, Cox (1997)), and moreover, every voter votes "sincerely" over this pair i.e. for her more-preferred alternative of the two.

A key question is the nature of the winning alternative(s) when the game is dominance-solvable. Here, the benchmark is majority voting with a binary agenda. Here, it is well-known that the subgame-perfect equilibrium outcome[8] is agenda-independent if and only if a CW exists, and in this case, the unique outcome is the CW. We show that the relationship between dominance-solvability and CWs is much less sharp. When the number of voters is at least four, dominance-solvability implies that a CW exists, but the reverse implication does not hold. Moreover, even if the game is dominance-solvable, and a CW exists, the outcome may not be the CW! On the other hand, when the electorate is su¢ciently replicated, then the outcome is always some CW whenever the game is dominance-solvable.

Related literature is as follows. The only work of which we are aware on re…nements of Nash equilibrium with plurality voting is De Sinopoli(1998), as described above. De Sinopoli and Turrini(1999) showed that iterated deletion of weakly dominated strategies may be applied to eliminate some of the Nash equilibria in the citizen-candidate model of Besley and Coate (1997). They show that in a four alternative, seven-person example, that iterated weak dominance eliminates all the Nash equilibria except for one. To our knowledge there are no general results on conditions for dominance solvability of strategic-form games. However, Borgers(1992), Borgers and Janssen(1992) have results on the dominance-solvability of Bertrand and Cournot games. For example, Borgers(1992) shows that in a model of Bertrand price competition, under some conditions, the set of prices that survive iterated deletion is close to the Walrasian price, and Borgers and Janssen(1992) have similar results for the Cournot case[9]. More recently, Mar-

[8] This outcome is always unique under the assumption that voters do not play weakly undominated strategies, and that (in the case of an even number of voters), a tie-breaking rule is speci…ed.

[9] They show that the Cournot outputs are the only outputs that survive iterated deletion if

iotti (1999) has provided a class of games (called maximum games) which are dominance solvable.

The layout of the paper is as follows. The model is outlined in Section 2. Examples and preliminary results are presented in Section 3. Our analysis of the three alternative case is in Section 4, and the more general case in Section 5. Section 6 discusses some extensions and concludes.

2. The Model

There is a set $N = \{1, ..., n\}$ of voters with $n \geq 3$ and a set $X = \{x_1, ..., x_K\}$ of alternatives. Voter $i \in N$ has utility function $u_i : X \to \Re$. The voting game is as follows. Each voter has one vote, which she can cast for any one of the K alternatives (i.e. no abstentions are allowed). The alternative with the largest number of votes wins (plurality rule). If two or more alternatives have the greatest number of votes, every alternative in this set is selected with equal probability. All voters vote simultaneously.

This game can be written more formally in strategic form as follows. Let $V_i = X$ be the strategy set of i; with generic element v_i. If $v_i = x_k$, voter i votes for alternative x_k. Let v be the strategy profile $v = (v_1, ..., v_n)$. Let $\omega_k(v)$ be the number of votes for alternative x_k if the strategy profile is v. Also, let the winset $W(v) \subset X$ be defined as

$$W(v) = \{x_k \in X \mid \omega_k(v) \geq \omega_l(v); \ \ x_l \in X\}$$

This is the set of alternatives that receive the most number of votes. Let $L(Y)$; $Y \subset X$ denote the lottery over Y where each alternative in Y is chosen with equal probability. Then, given a strategy profile v, depending on the context, we refer to either $W(v)$ or $L(W(v))$ as the outcome of the game[10].

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and only if the underlying Walrasian market is globally stable under cobweb dynamics.

[10] Also, if $W = \{x\}$, we write $L(W) = \{x\}$.

So, we can write down the payoff to i as a function of the strategy profile v as

$$u_i(v) = \frac{1}{\#W(v)} \sum_{x_k \in W(v)} u_i(x_k)$$

This completes the description of the plurality rule game in strategic form. We denote the game formally by $\Gamma = (u_i; V_i)_{i \in N}$ where of course $V_i = X$; so sometimes we write $\Gamma = (u_i; X)_{i \in N}$.

We will impose the following regularity condition which ensures that the order of deletion of weakly dominated strategies does not matter.

A1: For all v, v' s.t. $W(v) \neq W(v')$, $u_i(v) \neq u_i(v')$, $i \in N$

This says that no player is indifferent between any two different winsets[11]. It is satisfied generically in the space of utility functions on X.

Finally, the following notation will be useful. Let $\omega_{-i}(v_{-i})$ be a vector recording the total votes for each alternative in X given a strategy profile v_{-i} i.e. when individual i is not included. Also, let $\Omega_{-i} = \{\omega_{-i}(v_{-i}) | v_{-i} \in V_{-i}\}$. We suppress the dependence of ω_{-i} on v_{-i} except when needed and refer to ω_{-i} as a vote distribution. Clearly i's best response to ω_{-i} depends only on the information in ω_{-i}.

Two comments are in order at this point. First, we do not allow mixed strategies; as discussed in the Conclusions below, this is not a major restriction. Second, we do not allow voters to abstain; this is without loss of generality because abstention is always a weakly dominated strategy for any voter (Brams(1994)), and so will be deleted at the first round of the iterated deletion process.

A Nash equilibrium in the game is defined in the usual way as a strategy profile v^* where v_i^* is a best response to v_{-i}^*, $i \in N$. Say that $W^* = W(v^*)$ or $L^* = L(W^*)$ is the Nash equilibrium outcome. Let x_i^{max} be i's (unique) most preferred alternative in X, and say that i votes sincerely if $v_i = x_i^{max}$. A Nash equilibrium is sincere if all voters vote sincerely in that equilibrium. Note that any alternative in $x_k \in X$ is a Nash equilibrium outcome, for it is supported by

[11] A1 implies that no voter is indifferent between any pair of alternatives.

the Nash equilibrium strategies that $v_i = x_k$, all i. This observation motivates our analysis in this paper.

Finally, define an alternative $x \in X$ to be a Condorcet winner if $\#\{i \in N | x \succ_i y\} \geq \#\{i \in N | y \succ_i x\}$, all $y \neq x$; and say that x is a strict Condorcet loser if all the inequalities hold strictly, and weak otherwise. As we have assumed strict preferences, if the number of voters, n, is odd, the CW is strict i.e. unique, but if n is even, this is not necessarily the case (Moulin(1983), p 29). In the former case, denote the unique CW by x^{cw}, and in the latter case, denote the set of CWs by X^{cw}.

3. Characterizing Undominated Nash Equilibrium

In this section, we show that requiring strategies simply to be undominated in our voting game does not significantly narrow down the set of predicted outcomes. First, we establish the following useful Lemma which almost fully[12] characterizes weakly dominated strategies in the voting game. With three alternatives, say that $i \in N$ has dominated middle alternative (DMA) preferences if he prefers a lottery (with equal probabilities) over all three alternatives (i.e. $L(X)$) to his middle-ranked (i.e., second-ranked) alternative[13].

Lemma 1. In the plurality rule game, the strategy of voting for one's worst alternative is always weakly dominated, and the strategy of voting for one's best alternative is never weakly dominated (Brams (1994)). Moreover, if $n = K = 3$, voting for one's middle-ranked alternative is weakly dominated iff that player has DMA preferences. If $n > 3$, voting for one's worst alternative is the only weakly dominated strategy.

This result, along with all others, is proved in the Appendix[14]. This result

[12] Except for the case $n = 3, K > 3$.

[13] Note that we do not need to distinguish between strict and weak DMA preferences because from (A1), if a voter prefers $L(X)$ to his middle-ranked alternative, he always does so strictly.

[14] De Sinopoli, 1999, proves the sufficiency of DMA preferences for the middle-ranked alternative of a voter to be weakly dominated.

illustrates an important difference between three-player games and those with more than three players. In the latter case, the set of preference profiles is rich enough so that when i votes for any alternative in X except his worst-ranked, we can find a $v_{-i} \in V_{-i}$ such that this strategy for i is a unique best response to v_{-i}. By contrast, in the case of three voters and three alternatives, whether a voter's middle-ranked alternative is weakly dominated or not depends on cardinal preferences.

Define an alternative $x \in X$ to be a Condorcet loser if $\#\{i \in N \mid x \succ_i y\} \leq \#\{i \in N \mid y \succ_i x\}$, all $y \neq x$; and say that x is a strict Condorcet loser if all the inequalities hold strictly, and weak otherwise. There can be at most one strict Condorcet loser, but if n is even, there can be multiple weak Condorcet losers. We then have the following result, which mostly follows from De Sinopoli (1999):

Proposition 1. Assume $n > 3$. If $x \in X$ is not a strict Condorcet loser, then there is an undominated Nash equilibrium where x occurs with probability of at least 0.5. Furthermore, if x is not a weak Condorcet loser either, then there is an undominated Nash equilibrium where x occurs with probability 1. Assume $n = K = 3$. If $x \in X$ is not a strict Condorcet loser, and no voter has DMA preferences, then there is an undominated Nash equilibrium where x occurs with probability 1.

The above result suggests that when a Condorcet loser does not exist, then all alternatives must be possible unique outcomes of undominated Nash equilibria, and this is indeed the case, as the following example shows.

Example 1. The ordinal preferences of voters 1,...6 over alternatives x, y, z are given below:

$$\begin{aligned} 1, 2 &: x \succ y \succ z \\ 3, 4 &: y \succ z \succ x \\ 5, 6 &: z \succ x \succ y \end{aligned}$$

Note that there is neither a Condorcet winner nor a Condorcet loser in this example. By Lemma 1, in an undominated Nash equilibrium, a voter can vote for either his top-ranked or middle-ranked alternative. So, consider the following strategy profiles

$$
\begin{aligned}
v_1^*; v_2^*; v_5^*; v_6^* &= x;\; v_3^*; v_4^* = y \\
v_1^{**}; v_2^{**}; v_3^{**}; v_4^{**} &= y;\; v_5^{**}; v_6^{**} = z \\
v_3^{***}; v_4^{***}; v_5^{***}; v_6^{***} &= z;\; v_1^{***}; v_2^{***} = x
\end{aligned}
$$

It is clear that v^* is an undominated Nash equilibrium where x is the unique outcome, v^{**} is an undominated Nash equilibrium where y is the unique outcome, and finally, v^{***} is an undominated Nash equilibrium where z is the unique outcome. ¤

4. Results for Three Alternatives

The case of three alternatives is of course special, but in this case, our results can presented in a simple and intuitive way, which helps prepare for discussion of the general many-alternative case in the next Section. The general results can also be strengthened somewhat in this case. Moreover, comparative studies of voting systems tend to work with the three-alternative case as it is simplest case that serves to differentiate alternative systems (e.g. majority voting, plurality voting, approval voting); see for example, Myerson and Weber(1993), Myerson(1999), and it is also the simplest case where strategic voting may occur. In practice, some important political contests typically have three candidates or less e.g. presidential elections in the US (Levin and Nalebuff, 1995). We begin by defining iterated deletion of weakly dominated strategies: all definitions here also apply to the general case.

4.1. Iterated Deletion of Weakly Dominated Strategies

As argued below, due to assumption A1, in our model, the outcome of iterated deletion of weakly dominated strategies does not depend on the order in which strategies are deleted. However, for expositional convenience, for the most part, we will assume an order of deletion as in Moulin(1983). Let $NWD_i(S_i; S_{-i}) \subseteq V_i$ be the set of strategies for i which are not weakly dominated[15] by any $v_i' \in S_i$; given $S_{-i} \subseteq V_{-i}$. Let $V_i^0 = V_i$, and define recursively

$$V_i^m = NWD_i(V_i^{m-1}; V_{-i}^{m-1});\ i \in N;\ m = 1, 2, \ldots \tag{4.1}$$

Also, say that a v_i is weakly dominated relative to V^{m-1} if it is not in V_i^m. As X is finite, this algorithm converges after a finite number of steps to $V^\infty = V_1^\infty \times V_2^\infty \ldots \times V_n^\infty$: the set of weakly iteratively undominated strategy profiles. The set of iteratively weakly undominated outcomes is $U = \{W(v) | v \in V^\infty\}$.

The game is said to be dominance-solvable (DS) if either V^∞ only contains one strategy profile v^∞, or V^∞ contains several strategy profiles; which are payoff-equivalent for all players. If the game is dominance-solvable[16], by A1, every $v^\infty \in V^\infty$ generates the same outcome which we denote by W^∞ (so $U = \{W^\infty\}$). For if not, by A1, all players would not be indifferent between two profiles in V^∞, contradicting the assumed payoff-equivalence of elements of V^∞. So, we refer to W^∞, or equivalently $L^\infty = L(W^\infty)$ as the solution outcome.

Note that, following Moulin(1983), at each "round" of deletion we delete the weakly dominated strategies of all players, and then proceed to the next round. However, by assumption A1, this is without loss of generality in the following sense. Let $\hat{V}^\infty$ be the set of iteratively undominated strategies following some different procedure[17] for deletion. Then, the game is dominance-solvable by our procedure iff it is dominance-solvable by the other procedure[18]. Moreover, if the

[15] Formally,

$$D_i = \{v_i \in S_i | \exists v_i' \in S_i | u_i(v_i'; v_{-i}) \geq u_i(v_i; v_{-i})\ \forall v_{-i} \in S_{-i};\ u_i(v_i'; v_{-i}) > u_i(v_i; v_{-i}) \text{ some } v_{-i} \in S_{-i}\}$$

[16] If a Nash equilibrium v^* is in V^∞, we say that it is iteratively (weakly) undominated. If a game is dominance-solvable it is well-known that any $v^\infty \in V^\infty$ is a Nash equilibrium, so dominance-solvability implies that there exists a unique iteratively undominated Nash equilibrium outcome.

[17] For example, "player-by-player" deletion, where (say) player 1's weakly dominated strategies are deleted from his strategy set, then the weakly dominated strategies of player 2 are calculated, then player 2's weakly dominated strategies are deleted from his strategy set, and so on.

[18] This follows from the results of Marx and Swinkels(1997). First, by our assumption A1,

game is dominance solvable, the outcome is the same under either procedure i.e. $W(v^\infty) = W(\hat{v}^\infty)$; all $v^\infty \in V^\infty$; $\hat{v}^\infty \in \hat{V}^\infty$.

4.2. Sufficient Conditions for Dominance-Solvability and Non-Dominance Solvability

Let the set of alternatives $X = \{x; y; z\}$. Let $N_x; N_y; N_z$ be the sets of voters that rank $x; y$ or z respectively as worst, and let $n_x; n_y; n_z$ be the numbers of voters in each set. Also, define $q = \max_{a \in X} n_a/n$; this is the largest fraction of voters who agree on which alternative is worst, and let $b = \arg\max_{a \in X} n_a$ be the alternative that most rank worst. So, b is easily remembered as denoting a "bottom-ranked" alternative.

Now define a critical value of q as:

$$q_n = \begin{cases} 1 - \frac{1}{n} - \frac{1}{n}\left\lceil \frac{n+1}{3} \right\rceil; & n \text{ odd} \\ 1 - \frac{1}{n}\left\lceil \frac{n+2}{3} \right\rceil; & n \text{ even} \end{cases} \quad (4.2)$$

where $\lceil x \rceil$ denotes the smallest integer larger than x. Note that $q_n < 2/3$, and $\lim_{n \to \infty} q_n = 2/3$. Finally, say that in game $\Gamma = (u_i; X)_{i \in N}$, preferences are polarized over alternative $x \in X$ if there is an $M \subset N$ such that all $i \in M$ rank x highest, and $i \in N/M$ rank x lowest[19]. Preferences over alternative x are non-polarized otherwise.

We then have the following result, most of which follows directly from Theorems 1 and 2 below, setting $K = 3$;

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their TDI condition is satisfied in the game $\Gamma = (u_i; X)_{i \in N}$. So, by their Corollary 1, $V^\infty; \hat{V}^\infty$ only differ by the addition or removal of redundant strategies and a renaming of strategies. But a redundant strategy is payoff-equivalent for all players to some other strategy (Marx and Swinkels(1997), Definition 5), and so by A1, must give the same outcome as some other strategy. However, condition A1 guarantees that no player is indifferent between any two outcomes, and therefore under this regularity condition, the two notions of iterated dominance are equivalent.

[19] This is a general definition, and does not only apply to the case of $K = 3$. We will assume that b is unique i.e. that $q > 1/2$. This is without loss of generality, as we use the definition of b only in the case where $q \geq 2/3$.

Proposition 2. Assume that $K = 3$. If (i) $q > 2/3$, or (ii) $q = 2/3$, and preferences are not polarized over b, the game is dominance-solvable. If $q \leq q_n$, then the game is not dominance-solvable.

The intuition for this result is as described in the introduction. Note that Proposition 2 can classify the game as dominance-solvable or not in almost all cases, the exceptions being (i) when $q = 2/3$ and preferences over b are polarized, (ii) $q = q_n + 1/n < 2/3$; (iii) $q_n + 1/n < q < 2/3$. Finally, it is possible to show that (iii) never occurs, so we have;

Corollary 1. The only possible circumstances[20] in which the plurality voting game cannot be classified as dominance-solvable or not are (i) when $q = 2/3$ and preferences over b are polarized, (ii) $q = q_n + 1/n < 2/3$.

The following examples illustrate what may happen in these two exceptional cases. The first example shows that the non-polarization condition is needed for dominance-solvability when $q = 2/3$.

Example 2. The ordinal preferences of the six voters are as follows.

$$1,2,3,4 \;:\; x \succ y \succ z$$
$$5,6 \;:\; z \succ x \succ y$$

Also, voters 1-4 have non-DMA preferences. Note that $q = 4/6 = 2/3$, and preferences over z are polarized, so this game is not classified by Proposition 2. In fact, the game is not dominance-solvable. To show this, we prove that for any voter, it is a weakly undominated strategy relative to V^1 to vote for her second-ranked alternative. A similar argument (left to the reader) then shows that it is a weakly undominated strategy relative to V^1 to vote for her first-ranked alternative. These two statements together then imply that iterated deletion stops at the first round.

By Lemma 1, $V_i^1 = \{x, y\}$, $i = 1, \ldots, 4$; $V_i^1 = \{x, z\}$, $i = 5, 6$. Define $\Omega^1_{-i} = \{\omega_{-i}(v_{-i}) \mid v_{-i} \in V^1_{-i}\}$. We show that for every voter i, there exists $\hat{\omega}_{-i} \in \Omega^1_{-i}$ such

[20] Note that $q = q_n + 1/n < 2/3$ may not occur e.g. $q_4 = q_6 = 0.5$, but $q_4 + 1/4 = 3/4 > 2/3$, $q_6 + 1/6 = 2/3$.

that her second-ranked alternative is a unique best response to σ_{-i}. Specifically, for $i = 1, \ldots, 4$, y is a unique best response to $\sigma_{-i} = (1, 2, 2)$, as $1, \ldots, 4$ have DMA preferences. Also, for $i = 5, 6$, x is a best response to $\sigma_{-i} = (2, 3, 0)$. Finally, it is easily checked that $\sigma_{-i} \in \Sigma^1_{-i}$, all $i = 1, \ldots, 6$. $\square$

The next example shows that whether or not the game is DS when $q = q_n + \frac{1}{n} < 2/3$ depends on whether preferences over b are polarized.

Example 3. The ordinal preferences of the five voters are as follows.

$$
\begin{aligned}
1, 2, 3 &: x \succ y \succ z \\
4 &: x \succ z \succ y \\
5 &: z \succ x \succ y
\end{aligned}
$$

Note that $q_n = 2/5$, $q = 3/5$, so, $q = q_n + \frac{1}{n} < 2/3$. Also, preferences are not polarized over b so the game is not classified by Proposition 2. We will show that the game is dominance-solvable.

First, note that after the first round of deletion, by Lemma 1, $V_i^1 = \{x, y\}$, $i = 1, \ldots, 3$, $V_i^1 = \{x, z\}$, $i = 4, 5$. We show that for voter 4, $v_4 = z$ is weakly dominated by $v_4 = x$. Suppose to the contrary that there exists a $\omega_{-4} \in \Sigma^1_{-4}$ such that z is a unique best response to ω_{-4}. This requires that 4 must be able to affect the outcome by voting z, given some $\omega_{-4} \in \Sigma^1_{-4}$. The only such vote distributions are $\omega_{-4} = (2, 1, 1), (1, 2, 1)$. But it is clear that voter 4 does better voting for x in response to each of these, a contradiction. [For example, if he votes for x rather than z against $(2, 1, 1)$, the outcome is x rather than $L(x, z)$, which voter 4 obviously prefers.] So, at the end of the second round of deletion, z can get at most one vote and so cannot win, in which case $v_5 = z$ is weakly dominated for voter 5. The game is then reduced to one where each player can vote for (at most) one of two alternatives x, y and is thus dominance-solvable.

Now suppose that voter 4's preferences change so that $z \succ x \succ y$. Then, preferences are polarized over z. We can now show that with this change in preferences the game is not dominance-solvable. Following the argument in Example 2, we show that every voter i's second ranked alternative is a best response to some

$\sigma_{-i} \in \Sigma^1_{-i}$. First z is now a unique best response for 4 to $\sigma_{-4} = (2,1,1)$. Also, x is a unique best response by 5 to $\sigma_{-5} = (2,2,0)$; and y is a unique best response by 1,2,3 to $\sigma_{-i} = (0,2,2)$. Also, $\sigma_{-i} \in \Sigma^1_{-i}$, $i = 1, \ldots 6$. A similar argument shows that no top-ranked alternative is weakly dominated either. So, iterated deletion stops after the first round. $\square$

An important implication of Corollary 1 is that there are at most two values (out of n possible values) of q such that the game cannot be classified as dominance-solvable or not. So, in this sense, non-classification of the game becomes "unlikely", but not impossible[21], as the number of voters increases. If we hold the distribution of preferences across voters constant as n increases, we can prove a sharper result, namely we can provide necessary and sufficient conditions for games with more than a critical number of voters to be DS.

This can be formalized as follows, and this formalization also applies to the general case. Let $\Gamma_n = (u_i, X)_{i \in N}$ be the plurality voting game with a fixed number $n \geq 3$ players. Note that in any such game, there are $K!$ possible strict preference orderings over the K alternatives. Let ϕ^n_l, $l = 1, \ldots K!$ be the fractions of players in Γ_n who have the lth possible preference ordering. So a distribution of preferences on X across players is characterized by $\phi^n = \{\phi^n_l\}^{K!}_{l=1}$. Define the $m-$replica game $\Gamma_{n,m} = (u_i, X)^{nm}_{i=1}$, $m = 1, 2..$ to be a game with nm voters but with $\phi^{nm} = \phi^n$, all m; i.e. where the different "types" of voters in Γ_n are replicated by the factor m. The key feature of the $m-$replica game is that the distribution of preference profiles in the population of players does not change as n gets large.

Holding the preference distribution ϕ^n fixed, define q_n to be the largest fraction of voters who agree on the worst alternative in $X = \{x, y, z\}$. We make the following assumption about Γ_n;

A2. $q_n \neq \frac{2}{3}$.

This simply rules out the case where q_n is on the boundary. Then, we have the following result, which follows directly from Theorem 3 below.

[21] Inspection of the proof of Corollary 1 reveals that $q_n + 1/n < 2/3$ unless n is even and has a remainder when divided by three of less than 2.

Proposition 3. Consider any game Γ_n for which A2 holds. Then there is an m_0 such that for all $m > m_0$, $\Gamma_{n,m}$ is dominance-solvable iff $q_n > \frac{2}{3}$.

In other words, if the replicated electorate is large enough, and condition A2 holds, then the game can always be classified as DS or not DS.

Finally, the question arises as to how "likely" the sufficient conditions for dominance-solvability in Proposition 2 are to hold. A standard way of posing this question is to suppose that the preferences of each member of the electorate are determined by random draw from the $K!$ possible preference profiles, and every possible preference profile is equally likely to be drawn . These are the so-called "impartial culture" assumptions (Gehrlein(1988)). Under these assumptions, in the case of $K = 3$, it is easy to calculate that the sufficient conditions for dominance-solvability in Proposition 2 are satisfied with probabilities that depend on n as follows:

Table 1

# of voters	4	5	6	7	8	9	10	20	100
probability	0.333	0.136	0.273	0.136	0.059	0.123	0.059	0.003	0.000

As Table 1 indicates, the greater the size of the electorate, the less likely it is that the condition is satisfied. The intuition[22] is that the probability that the randomly chosen preferences of a voter rank any particular alternative (say z) worst is 1/3, so the number of voters ranking z worst, n_z, is binomially distributed with probability of success 1/3. As $n \to \infty$, n_z/n converges in probability to the degenerate distribution with unit probability mass at $n_z/n = 1/3$. However, note that there is no real reason to suppose that preferences are randomly generated like this; rather, it is plausible that the common social factors that determine preferences will imply that preferences are correlated, positively or negatively

[22] The values in the Table are calculated as follows. If n is not a multiple of 3, the value in the Table is 3 times $\Pr(y > 2/3)$, where $y \sim b(n, 1/3)$. If n is a multiple of 3, we add to this term $\Pr(y = 2/3).(1 - \left(\frac{1}{3}\right)^{n/3})$, where the second term is the probability that preferences over (say) z are non-polarized, given that exactly 2/3 of the voters rank z worst.

(Sen(1976), p 165). Either kind of correlation would tend to raise n_z/n, making dominance solvability more likely, and so the probabilities in the above Table may better be interpreted as lower bounds.

Finally it is worth noting that the sufficient condition for dominance solvability, $q > 2/3$, in Theorem 1 is logically unrelated to Sen's (1976) concept of value restriction (which guarantees existence of a Condorcet winner), even though it appears to bear a close resemblance. Our condition is weaker in the sense that it requires less than full agreement across voters, but stronger in that agreement has to be over the worst-ranked alternative.

4.3. Dominance-Solvability and Condorcet Winners

We now turn to a characterisation of the solution outcome in the event that the game is dominance-solvable, and in particular how this outcome relates to the CW, whenever the latter exists. The benchmark here is "pairwise" majority voting i.e. majority voting with a binary agenda[23]. Define the majority voting game to be the multi-stage game where at each stage, a set N of voters vote for either one of a pair of subsets of alternatives, as specified by a binary agenda. It is well-known that if voters do not use weakly dominated strategies, the majority voting game has a unique subgame-perfect equilibrium[24], but the equilibrium outcome in general depends on the agenda. In fact, (i) the outcome is agenda-independent if and only if there exists a CW in X, and (ii) in that case, the outcome is always the CW itself.

With plurality rule, the relationship between dominance-solvability and Condorcet Winners is much less sharp. First, there is the question of whether dominance-solvability implies existence of a Condorcet Winner, or vice-versa. Consider first the case of at least four voters. Here, from Proposition 2, the game is dominance-

[23] An agenda is binary if at each stage of the voting, a voter can vote for either of two subsets of alternatives.

[24] If the number of voters is even, we require a tie-breaking rule to ensure uniqueness. An obvious such rule is that if two (subsets of) alternatives get equal numbers of votes, each is chosen with probability 0.5.

solvable if $q > q_n$ i.e. $q \geq q_n + 1/n$. It is easy to check[25] that $q_n + 1/n > 0.5$. So, if the game is dominance-solvable, there is a Condorect loser. Consequently, as $K = 3$, by a well-known result (Gehrlein(1988)), there is a Condorcet winner. On the other hand, it is clear that the reverse implication is not true. Consider Example 3 in the second case when the game is not dominance-solvable. There is in fact a Condorcet Winner in this case, namely x.

Finally, if $n = 3$, it is not even the case that dominance-solvability requires the existence of a Condorcet winner, as the following example shows.

Example 4. Ordinal preferences over the three alternatives are as in the Condorcet paradox i.e.

$$
\begin{aligned}
1 &: x \succ z \succ y \\
2 &: y \succ x \succ z \\
3 &: z \succ y \succ x
\end{aligned}
$$

Then obviously a CW does not exist. Also, suppose that all voters have DMA preferences. Then, by Lemma 1, we can delete the bottom- and middle-ranked alternatives from each player's strategy set, so the game is DS, and the outcome is $L^1 = L(\{x, y, z\})$. ¤

We can summarise our discussion as follows:

Proposition 4. If $n > 3$, then if the game is dominance-solvable, a Condorcet Winner exists but the reverse implication does not hold. If $n = 3$, neither implication holds.

We can now turn to the second question, that of whether the solution outcome is a Condorcet winner in the event that the game is dominance-solvable. It is easy to show that if our sufficient conditions for dominance-solvability are satisfied, then this is the case:

---

[25] In the odd case, $q_n + 1/n > 0.5$ if $0.5n > [\frac{n+1}{3}]$. As $[\frac{n+1}{3}] < \frac{n+1}{3} + 1$, it is sufficient that $0.5n > \frac{n+1}{3} + 1$. This holds for $n \geq 9$. Finally, in the cases $n = 5, 7$, $q_n + 1/n = 3/5$, $4/7$ respectively. The proof in the even case is similar.

Proposition 5. Whenever the sufficient conditions stated in Proposition 2 for dominance-solvability hold, then (i) W^{∞} contains at most two alternatives, (ii) if n is odd, a unique CW x^{cw} exists, and $W^{\infty} = \{x^{cw}\}$; (iii) if n is even, at least one Condorcet winner exists (i.e. $X^{cw} \neq \emptyset$), and $W^{\infty} \subseteq X^{cw}$.

Proposition 5 follows directly from Theorem 1 below. However, what if the game is dominance-solvable, but the sufficient conditions for dominance-solvability do not hold? This is a possibility, as Example 3 shows. In this case, at least with three voters, it is possible that the solution outcome is not a Condorcet Winner, as the following example shows.

Example 5. Ordinal preferences over the three alternatives are as follows.

$$\begin{aligned} 1 &: x \succ z \succ y \\ 2 &: z \succ x \succ y \\ 3 &: y \succ z \succ x \end{aligned}$$

The unique CW is z. Assume moreover that only voters 1 and 3 have DMA preferences. Then, By Lemma 1, $V_1^1 = \{x\}, V_3^1 = \{y\}$. So as voter 2 has non-DMA preferences, his unique best response to $v_{-2} = (x, y)$ is $v_2 = x$. So the game is DS, and $W^1 = \{x\} \neq \{z\}$. $\square$

However, the case of three voters is not well-behaved in that whether or not the middle-ranked alternative is weakly dominated or not depends on cardinal preferences over the alternatives (cf. Lemma 1). We conjecture, but have not been able to prove, that with at least four voters, every solution outcome must be a Condorcet winner. We certainly have an asymptotic result of this kind, i.e. Corollary 2 below. This says that if the electorate is replicated sufficiently often, and if the game is dominance-solvable, the outcome is a Condorcet winner.

5. General Results

We now consider the case of an arbitrary number $K \geq 3$ of alternatives. In this Section, we assume throughout that $n \geq 4$; as we have seen above, the game with three voters has some special properties. For any $Y \subset X$, define $\Gamma = (u_i, Y)_{i \in N}$ to be the plurality game defined in Section 1 above, with a fixed set of n players, but a set $Y \subset X$ of alternatives. The preferences of players are the restriction of the preferences over the set X, to the subset Y. For any such game, let $Q(Y)$ be the largest set of voters who agree on a worst alternative in Y, and define

$$q(Y) = \frac{\#Q(Y)}{n} \tag{5.1}$$

This fraction plays a crucial role in what follows. Denote the worst alternative in Y for voters in $Q(Y)$ by $b(Y)$. Without loss of generality, we will restrict our attention to games where $b(Y)$ is a Condorcet loser, i.e. $q(Y) > 0.5$, so $Q(Y)$ is unique.

Let $X \equiv X_K$; and define the following sets recursively;

$$X_{l-1} = X_l / \{b(X_l)\}; \; l = K, \ldots 2 \tag{5.2}$$

Each set is obtained from the previous one by deleting the alternative in the previous set that is worst-ranked by the most players, and the initial set is just X. These sets are uniquely defined for any sequence of games where for each game, at least a simple majority agree on the worst alternative (there exists a Condorcet loser). Note that $\#X_l = l$. We now have our general sufficient conditions for dominance-solvability.

Theorem 1. Assume that for all $l = 3, \ldots K$, either (i) $q(X_l) > \frac{l-1}{l}$; or (ii) $q(X_l) = \frac{l-1}{l}$ and preferences are not polarized over $b(X_l)$ in game $\Gamma = (u_i, X_l)_{i \in N}$. Then the game $\Gamma = (u_i, X)_{i \in N}$ is dominance solvable. Moreover, the solution outcome W^{∞} is that alternative in X_2 which is preferred to the other by a strict majority of voters, or $L(X_2)$ if equal numbers of voters prefer each alternative in X_2. Also, whenever (i) or (ii) hold, then (a) if n is odd, a unique CW x^{cw} exists, and $W^{\infty} = \{x^{cw}\}$; (b) if n is even, at least one Condorcet winner exists (i.e. $X^{cw} \neq \emptyset$), $W^{\infty} \subset X^{cw}$.

The conditions require that for the sequence of sets of alternatives $(X_K, X_{K-1}, \ldots X_3)$, there is sufficient agreement amongst the voters about which alternative is worst. Moreover, the solution outcome is generated by sincere voting over the set of the two-element alternatives, X_2, that remains when the alternatives ranked worst by the most voters have been sequentially deleted.

Two further remarks are appropriate at this point. First, if the game is DS, then the only iteratively undominated strategies involve voting for one of two alternatives in X_2. This is consistent with Duverger's Law, which asserts that "plurality rule tends to produce a two-party system" (Cox(1997)). Second, our sufficient conditions for DS are quite strong in that they imply the existence of a Condorcet winner, but they have the attractive feature that any alternative in the solution outcome W^1 is always a CW.

The sufficiency conditions in Theorem 1 can be quickly and easily checked, and the solution outcome computed, as the following example shows.

Example 6. Ordinal preferences for five voters over four alternatives in X are given below. Note that W is the Condorcet winner.

$$\begin{aligned} 1 &: W \succ Y \succ X \succ Z \\ 2 &: X \succ W \succ Y \succ Z \\ 3 &: W \succ X \succ Y \succ Z \\ 4 &: X \succ W \succ Y \succ Z \\ 5 &: Z \succ W \succ X \succ Y \end{aligned}$$

First, four voters agree that Z is worst in X, so $q(X_4) = 4/5$. So, we can delete Z from X to obtain $X_3 = \{W; X; Y\}$. Again, four voters agree that Y is worst in X_3, so $q(X_3) = 4/5$. To check that the conditions of the Theorem are satisfied, note that

$$q(X_4) = \frac{4}{5} > \frac{4-1}{4} = \frac{3}{4};\ q(X_3) = \frac{4}{5} > \frac{3-1}{4} = \frac{2}{3}$$

Finally, $X_2 = \{W; X\}$. As voters 1,3,5 prefer W to X, $W^1 = \{W\}$. $\square$

We now present sufficient conditions for the game $\Gamma(u_i; X)_{i \in N}$ not to be DS, and a characterization of U, the set of iteratively weakly undominated outcomes in this case. Consider the sequence of sets (5.2) above, and the associated sequence of fractions $\{q(X_l)\}_{l=3}^{K}$. Also, for any game with l alternatives, define the critical fractions:

$$q_n^l = \begin{cases} q_n \text{ in (4.2)}, & l = 3 \\ 1 - \frac{1}{n} - \frac{1}{n}\left\lceil \frac{n+3l-8}{l} \right\rceil; & l > 3;\ n \text{ odd} \\ 1 - \frac{1}{n}\left\lceil \frac{n+3l-7}{l} \right\rceil; & l > 3;\ n \text{ even} \end{cases} \quad (5.3)$$

where [x] denotes the smallest integer larger than x. Note that $q_n^l < (l-1)/l$, and $\lim_{n\to\infty} q_n^l = (l-1)/l$. Obviously, q_n^3 in (5.3) is equal to q_n in (4.2). Then we have:

Theorem 2. *If there exists an $l \in \{3; 4; ...; K\}$ such that (i) $q(X_k) > \frac{k-1}{k}$, all $k > l$; (ii) $q(X_l) < q_n^l$; or $q(X_l) = q_n^l + \frac{1}{n}$ and preferences over $b(X_l)$ are* polarized, then the game $\Gamma = (u_i; X)_{i \in N}$ is not DS. In this case, the set of iteratively undominated outcomes is $U = \{W(v) | v_i \in X_l / b_i\}$; where b_i is voter i's bottom-ranked alternative in X_l.

Note from Theorem 2 that we are also able to characterise the set of iteratively weakly undominated outcomes even if the game is not dominance-solvable.

Theorems 1 and 2 together provide conditions under which a game is classifiable as dominance-solvable or not, and these conditions leave few games unclassified. Indeed, we can generalize Corollary 1 as follows. It can be shown that any $k \in \{3, 4, ..., K\}$; there is at most one possible value of $q(X_k)$; namely $q_n^k + 1/n$; for which $q_n^k < q(X_k) < \frac{k-1}{k}$. Consequently, from inspection of Theorems 1 and 2, there is at most two possible values of each $q(X_l)$ for which the game cannot always be classified as DS or not, $q_n^k + 1/n$ and $\frac{k-1}{k}$. That is to say, if $q(X_k) \neq q_n^k + 1/n; \frac{k-1}{k}$; $k \in \{3, 4, ..., K\}$, then the game can always be classified.

If we hold the distribution of preferences across voters constant as n increases, we can prove a sharper result, namely we can provide necessary and sufficient conditions for games with more than a critical number of voters to be DS. Consider the $m-$replica game as defined in the previous Section. For any preference distribution ϕ, and set of alternatives $Y \subset X$, define $q(\phi; Y)$ as in (5.1) to be the largest fraction of voters who agree on the worst alternative in Y. Also, recall the definition of the sequence of subsets of alternatives $X_K; X_{K-1}; ... X_3$ defined above. We make the following assumption about Γ_n;

A3. $q(\phi^n; X_l) \neq \frac{l-1}{l}$, $l = 3, ..., K$.

Then, we have the following result.

Theorem 3. Consider any game Γ_n for which A3 holds. Then there is an m_0 such that for all $m > m_0$, $\Gamma_{n,m}$ is dominance-solvable iff $q(\phi^n; X_l) > \frac{l-1}{l}$; $l = 3, ..., K$.

In other words, if the replicated electorate is large enough, and condition A3 holds, then the game can always be classified as DS or not DS. An obvious corollary of Theorems 1 and 3 is the following:

Corollary 2. Consider any game Γ_n for which A2 holds. Then there is an m_0

such that if $m > m_0$, and $\Gamma_{n,m}$ is dominance-solvable, at least one Condorect winner exists ($X^{cw} \neq \emptyset$) and any solution outcome is a Condorcet winner i.e. $W^{\infty} \subset X^{cw}$.

This is the most general statement of the relationship between dominance-solvability and Condorcet winners.

6. Extensions and Conclusions

6.1. Mixed Strategies

Following many studies of voting behaviour (e.g. Moulin(1983), Besley-Coate(1997)), we have assumed pure voting strategies. To what extent do our results extend to the case where voters can randomize? Note that both the sufficient conditions for dominance-solvability and non-dominance-solvability are derived by finding conditions under which we can or cannot find a pure strategy that dominates another. Clearly, the sufficient conditions for dominance-solvability are unaffected by the inclusion of mixed strategies, in the sense that they remain sufficient even when mixtures are allowed. Our sufficient conditions for non-dominance-solvability imply that the remaining strategies for player i are a unique best response to some strategy profile for the other players v_{-i} that has not yet been deleted. Thus, such strategies cannot be dominated by a mixed strategy either, and our sufficient conditions remain sufficient, even allowing for mixed strategies. The only open question is whether we can use mixed strategies to classify games in our "indeterminate" region of values of q_n^l.

6.2. Indifference

Our assumption A1 does not allow voters to be indifferent between alternatives. So, with indifference, A1 cannot hold, and so in general, the order of deletion of dominated strategies matters. There are two alternatives here. One is to make assumptions sufficient to ensure that the Marx-Swinkels Transference of Decision Maker Indifference (TDI) condition is satisfied. This condition says that whenever a voter is indifferent between two profiles where only his strategy changes, then all players are indifferent between these two profiles. An assumption[26] which implies TDI in our model is that if some $i \in N$ is indifferent between two equi-probability lotteries $L(Y), L(Z)$, $Y, Z \subset X$, then so are all $j \in N$. With this assumption, all voters are indifferent between the same subset of alternatives.

The second is to accept that the order matters, and focus on the outcome with some "plausible" order of deletion. The order of iteration we used to prove Theorems 2 and 3 is of some interest. Iterated deletion is applied to the game $\Gamma = (u_i; X)_{i \in N}$ until the alternative ranked worst by the highest number of voters (say b) and only that alternative, is deleted from all strategy sets, so the game is reduced to $\Gamma = (u_i; X/b)_{i \in N}$, and so on. This procedure is known as the Coombs social choice function (Moulin (1983), p24). If we want to apply this order of deletion with indifference, the problem is that z may not be uniquely defined. But, given some tie-breaking rule, we may be able to proceed as before.

6.3. Justifying Iterated Deletion of Weakly Dominated Strategies

It is well-known that if the structure of the game and rationality of the players are common knowledge, then players must play only those strategies that survive iterated deletion of strictly dominated strategies. If in addition, the beliefs of each player over the strategies that other players might play are common knowledge, then players must only play strategies corresponding to some Nash equilibrium (Brandenberger and Dekel(1989)).

A common knowledge justification for iterated weak dominance[27] has recently been presented by Rajan(1998). His results are established in a Bayesian framework where each player has a infinite hierarchy of prior beliefs. The first element

[26] This assumption is satisfied, for example, in a citizen-candidate voting game where voters are of $K \leq n$ types, and every voter has strict preferences over different types, satisfying A1, and is indifferent between two candidates of a given type. Then any voter is only indifferent between $L(Y), L(Z)$ if Z can be obtained from Y by deleting candidates of some type and replacing them by others of the same type, in which case all voters are indifferent.

[27] See also Stahl(1993).

of this hierarchy is a probability measure on the strategies that other players will play; the second is a probability measure on the prior beliefs of other players over strategies, and so on. Rajan's result is that if it is common knowledge that agents are "hierarchically Bayesian rational" then each agent plays a strategy that survives iterated weak dominance. Hierarchical Bayesian rationality requires that player i believes that for each player $j \neq i$, the probability that j will play any non-best response strategy is infinitesimal relative to the probability that j plays a best response strategy, given j's own beliefs.

6.4. Conclusions

This paper has presented conditions sufficient for a plurality voting game to be dominance-solvable, and sufficient for it not to be dominance-solvable. These conditions can be stated in terms of only one statistic of the game, the largest proportion of voters who agree on which alternative is worst in a sequence of subsets of the original set of alternatives, where each subset is derived from the previous one by deleting the alternative that most voters rank as worst in the previous subset. When the number of voters is large, "almost all" games can be classified as either DS or not DS. If the game is DS, the outcome is usually but not always the Condorcet Winner, whenever it exists.

Dominance solvability has attractive features: it imposes weak requirements on player's rationality and moreover in voting games it usually leads to the choice of the CW whenever it exists. However, it is not the only approach to dealing with the multiplicity of Nash equilibrium. De Sinopoli(1999) studies (trembling hand) perfection and properness in the plurality voting game above. He shows that with more than four voters, if an alternative is not a (weak) Condorcet loser, there is a perfect Nash equilibrium where that alternative is an outcome. [With an odd number of voters, and no indifference, there is only one Condorcet loser].

Finally, a natural question that arises from this analysis is whether our approach to characterizing sufficient conditions for dominance-solvability and non-dominance-solvability remain the same for voting games defined with other scoring rules, like Borda and Approval Voting. This is a topic for future work.

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A. Appendix.

Proof of Lemma 1. (i) The first two statements are well-known (Brams (1994)). To prove the third, let the alternatives be $X = \{x, y, z\}$. W.l.o.g., let the preferences of some $i \in \{1, 2, 3\}$ be $x \succ_i y \succ_i z$. Note that the vote of i only affects the outcome if he faces a vote distribution in the set $\tilde{\Omega}_{-i} = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$. Then, i strictly prefers to play $v_i = x$ rather than $v_i = y$ against $(1, 1, 0)$ and $(1, 0, 1)$, and i weakly prefers to play $v_i = x$ rather than $v_i = y$ against $(0, 1, 1)$ iff his preferences are (weakly) DMA.

(ii) To prove the fourth statement, we proceed as follows. Suppose w.l.o.g. that voter i's preferences are : $x_1 \succ_i x_2 \succ_i \dots \succ_i x_j \succ_i x_{j+1} \dots \succ_i x_K$. So, it is sufficient to show that for any $j < K$, there exists some $\omega^j_{-i} \in \Omega_{-i}$ such that $v_i = x_j$ is a unique best response to ω^j_{-i}; for then, $v_i = x_j$ cannot be weakly dominated relative to V.

Let $\omega^j_{-i} = (\omega^j_1, \omega^j_2, \dots, \omega^j_K)$ where ω^j_l is the number of votes (excluding i's) for alternative x_l. If n is odd, construct ω^j_{-i} so that $\omega^j_j = \omega^j_{j+1} = (n-1)/2$; $\omega^j_l = 0, \forall l \neq j, j+1$. As $n > 3$, note that $\omega^j_j, \omega^j_{j+1} > \omega^j_l + 1, \forall l \neq j, j+1$. So, if i plays x_j against ω^j_{-i}, the outcome is x_j; if i plays x_{j+1} against ω^j_{-i}, the outcome is x_{j+1}, and finally if i plays x_l, $l \neq j, j+1$ against ω^j_{-i}, the outcome is x_j or x_{j+1} with equal probability. As i strictly prefers the first outcome to the second or third, x_j is a unique best response to ω^j_{-i}, as claimed.

If n is even, construct ω^j_{-i} so that $\omega^j_j = n/2 - 1$, $\omega^j_{j+1} = n/2$; $\omega^j_l = 0, \forall l \neq j, j+1$. As $n > 3$, note that $\omega^j_j, \omega^j_{j+1} > \omega^j_l, \forall l \neq j, j+1$. So, if i plays x_j against ω^j_{-i}, the outcome is x_j or x_{j+1} with equal probability. If i plays x_{j+1} or x_l, $l \neq j, j+1$ against ω^j_{-i}, the outcome is x_{j+1}. As i strictly prefers the first outcome to the second, x_j is a unique best response to ω^j_{-i}, as claimed. $\square$

Proof of Proposition 1. If $n > 3$, this follows from De Sinopoli (1999), Proposition 8, and the fact that if an equilibrium is perfect, it is undominated. If $n = 3$, then if x is not a strict Condorcet loser, it is not a weak Condorcet loser, either, so at least two voters out of three will prefer it to some other alternative, y. Moreover, these two voters prefer x to an equal-probability lottery over the three alternatives. So, the strategy profile where each voter votes for her most preferred alternative in $\{x, y\}$ is an undominated Nash equilibrium with outcome x. $\square$

Proof of Proposition 2. If $n \geq 4$, the Proposition follows directly from Theorems 1 and 2 below, setting $K = 3$. If $n = 3$, and $q > 2/3$, then $q = 1$, so all voters agree that some alternative (say z) is worst. So, after one round of deletion, all voters have deleted z from their strategy sets, and so the voting game is reduced to one of (at most) two alternatives for each player, and so is dominance solvable. If $q = 2/3$, let $b = z$. Then the only possible kind of preference profile is one where two voters (say 1,2) rank z worst, and voter 3 ranks z middle. W.l.o.g, suppose that $y \succ_3 z \succ_3 x$. Then at the first round of deletion, $V_1^1, V_2^1 \subset \{x, y\}$, $V_3^1 \subset \{y, z\}$. We can now show that $v_3 = z$ is a weakly dominated strategy relative to V^1_{-3}. Then, z will be deleted at the second round, and the game will be reduced to one of (at most) two alternatives for each player, and thus be dominance-solvable.

Note that $\Omega^1_{-3} = \{\omega_{-3}(v_{-3}) \mid v_{-3} \in V^1_{-3}\} \subset \{(2,0,0),(1,1,0),(0,2,0)\}$. Note that $v_3 = y$ is a weakly better response to each of these vote distributions than $v_3 = z$, and a strictly better response to $(1,1,0)$. So, z is weakly dominated relative to V^1, as required.

Finally, note that $q_3 = 0$, but $q > 0$ by definition, so $q \leq q_n$ is an empty condition when $n = 3$. $\square$

Proof of Corollary 1. It suffices to show that there exists at most one possible value of q in $(q_n, 2/3)$, namely $q_n + \frac{1}{n}$. Let $m = \left\lceil \frac{2n}{3} \right\rceil - 1$. Clearly, $\frac{m}{n}$ is the largest fraction less than $2/3$, as $\left\lceil \frac{2n}{3} \right\rceil \geq \frac{2n}{3}$.

(i) First assume n odd. We now show that $m = nq_n + 1$. For then, m/n will be the only feasible value of q in $(q_n, 2/3]$ other than $2/3$. Then we require $\left\lceil \frac{2n}{3} \right\rceil = n + 1 - \left\lceil \frac{n+1}{3} \right\rceil$. Now, $n = 3k + r$, where $r \in \{0, 1, 2\}$. So,

$$\left\lceil \frac{2n}{3} \right\rceil = \left\lceil 2k + \frac{2r}{3} \right\rceil = 2k + r \tag{A.1}$$

Also,

$$n+1-\left\lceil\frac{n+1}{3}\right\rceil = 3k+r+1-\left\lceil k+\frac{r+1}{3}\right\rceil \tag{A.2}$$
$$= 3k+r+1-k-1$$
$$= 2k+r$$

So, comparing (A.1) and (A.2), we have the desired result.

(ii) Assume n even. We now show that $m \leq nq_n + 1$. Then we require $\left\lceil\frac{2n}{3}\right\rceil \leq n+2-\left\lceil\frac{n+2}{3}\right\rceil$. But $n = 3k + r;\ r \in \{0, 1, 2\}$; so

$$n+2-\left\lceil\frac{n+2}{3}\right\rceil = 3k+r+2-\left\lceil k+\frac{r+2}{3}\right\rceil \tag{A.3}$$
$$= 2k+r+2-\max\{1,r\}$$
$$\geq 2k+r$$

Comparing (A.1) and (A.3), we have the desired result. So, if m/n is in the interval $(q_n, 2/3)$, it is the only such fraction of n in that interval. ∎

Proof of Theorem 1. Define

$$\bar{V}_i/b(Y) = \begin{cases} Y/b(Y); & i \in Q(Y) \\ Y & i \notin Q(Y) \end{cases}$$

and also $\bar{V}/b(Y) = (\bar{V}_1/b(Y), \ldots, \bar{V}_n/b(Y))$. Let $y = \#Y$. The following intermediate result is obvious but useful:

Lemma A0. In any game, $\Gamma = (u_i, Y)_{i \in N}$, if all players have the same strategy set i.e. $V_i = V_j\ \forall i \neq j$; and each player has at least two strategies, then any player's top ranked strategy in Y is a unique best response to at least one strategy profile of the other players.

We can now state and prove three additional Lemmas.

Lemma A1. In the game $\Gamma = (u_i, Y)_{i \in N}$, if $q(Y) > \frac{y-1}{y}$, then $b(Y) \notin W(v)$ for all $v \in \bar{V}/b(Y)$.

Proof. Suppose to the contrary that $b(Y) \in W(v)$ for some $v \in \bar{V}\setminus b(Y)$. Then $b(Y)$ must get at least as many votes as all other alternatives in Y. But $b(Y)$ can get at most $n(1 - q(Y))$ votes, as $q(Y)$ players have deleted $b(Y)$ from their strategy sets. So, the total number of votes cast is at most $T = yn(1 - q(Y))$. Now, if $q(Y) > \frac{y-1}{y}$, $yn(1 - q(Y)) < n$, so $T < n$; a contradiction since we do not allow abstentions. $\square$

Lemma A2. In the game $\Gamma = (u_i; Y)_{i \in N}$, if $q(Y) = \frac{y-1}{y}$, and preferences are non-polarized, then $v_i = b(Y)$ is weakly dominated for i relative to $\bar{V}\setminus b(Y)$.

Proof. First we prove that if $b(Y) \in W(v)$ for some $v \in \bar{V}\setminus b(Y)$; then, all alternatives must be in $W(v)$. For suppose not; let $x \notin W(v)$. Then, as $b(Y)$ can get at most $n(1 - q(Y))$ votes, so all $y - 1$ alternatives in $W(v)$ get $n(1 - q(Y))$ votes. Thus, the total number of votes cast is at most $T = (y - 1)n(1 - q(Y)) + n(1 - q(Y)) - 1$ where $n(1 - q(Y)) - 1$ is the most votes that x can get and not be in the winset. Now, as $q(Y) = \frac{y-1}{y}$, $T = n - 1$; a contradiction since we do not allow abstentions.

As the game is non-polarized, $b(Y)$ is not ranked best in Y by some $i \notin Q(Y)$. From the above, at any profile $v_{-i} \in \bar{V}\setminus b(Y)$, there are two possibilities for this voter i. Either when $v_i = b(Y)$, $b(Y) \notin W(v)$; in which case he does weakly better by voting for his most preferred alternative. Or when $v_i = b(Y)$, $b(Y) \in W(v)$; in which case all alternatives have equal numbers of votes, in which case, he could do strictly better by voting for his most preferred alternative. $\square$

Lemma A3. If (i) $q(Y) > \frac{y-1}{y}$; or (ii) $q(Y) = \frac{y-1}{y}$ and the game $\Gamma = (u_i; Y)_{i \in N}$ is non-polarized, then this game can be reduced to the game $\Gamma = (u_i; Y\setminus b(Y))_{i \in N}$ by iteratively deleting weakly dominated strategies.

Proof (i) First, from Lemma 1, $b(Y)$ is weakly dominated relative to $V = Y^n$ for all players in $Q(Y)$, and so can be deleted from their strategy sets to get $\bar{V}\setminus b(Y) = (\bar{V}_1\setminus b(Y); \ldots; \bar{V}_n\setminus b(Y))$.

(ii) If $q(Y) > \frac{y-1}{y}$; Lemma A1 implies that $b(Y) \notin W(v)$; for any $v \in \bar{V}\setminus b(Y)$. Thus, from Lemma A0, $v_i = b(Y)$ is weakly dominated by the top ranked strategy in $Y\setminus b(Y)$; for all players $i \in N\setminus Q(Y)$ relative to $\bar{V}\setminus b(Y)$, and so can be deleted.

(iii) Lemma A2 implies that $b(Y)$ is weakly dominated relative to $\bar{V}\setminus b(Y)$ for some $i \in N \backslash Q(Y)$. So, we can delete $b(Y)$ from the strategy set of $i \in N/Q(Y)$ to give a set of strategy profiles W. But then by the argument of Lemma A1, $b(Y) \notin W(v), v \in W$, as it can get at most $n(1 - q(Y)) - 1$ votes. So, $b(Y)$ is weakly dominated for all players relative to W and thus can be deleted from all the remaining players' strategy sets. $\square$

We can now return to the proof of Theorem 1. Under (i) or (ii) in the Theorem, by Lemma A3, $\Gamma = (u_i; X_l)_{i \in N}$ can be reduced to $\Gamma = (u_i; X_{l-1})_{i \in N}$ by iterated deletion of weakly dominated strategies. So, iterating, $\Gamma = (u_i; X_l)_{i \in N}$ can be reduced to $\Gamma = (u_i; X_2)_{i \in N}$ where each player has only two strategies. In the game $\Gamma = (u_i; X_2)_{i \in N}$, the only undominated strategy is to vote sincerely, and so the game is dominance-solvable, with outcome as described in the Theorem.

To prove the last part, let $X_2 = \{x, y\}$. First suppose that n is odd. Then w.l.o.g, suppose x beats y in a majority vote, so $W^{\infty} = \{x\}$. Then, x must be a CW. For suppose not. Then, x must be ranked worse than some $w \notin X_2$ by a majority of voters, and thus must be in X/X_2, contrary to assumption.

Now suppose that n is even. Then, either the above argument applies (i.e. $W^{\infty} = \{x\}$, where x is a CW), or equal numbers of voters prefer x to y and vice versa, in which case $W^{\infty} = \{x, y\}$. Again, x, y must both be CWs. For suppose not. Then, one or both of x, y must be ranked worse than some $w \notin X_2$ by a majority of voters, and thus must be in X/X_2, contrary to assumption. $\square$

Proof of Theorem 2. Let l be the first $k \in \{3, 4, ..., K\}$ for which $q(X_k) \leq q_n^k$. Then, by the proof of Theorem 1, the game $\Gamma = (u_i; X)_{i \in N}$ can be reduced to $\Gamma = (u_i; X_l)_{i \in N}$ by iterated deletion of weakly dominated strategies. Let b_i be i's bottom-ranked alternative in X_l. Then, by Lemma 1, $v_i = b_i$ is weakly dominated in $\Gamma = (u_i; X_l)_{i \in N}$, so we can delete b_i from player i's strategy set to get

$$T_i = X_l / b_i, \; i \in N$$

We now show that no $v_i \in T_i$ is weakly dominated relative to T_{-i}; this implies that $T = \times_i T_i$ is a full reduction of V by weak dominance (Marx and Swinkels(1997), Definition 3) and hence by Corollary 1 of Marx and Swinkels, the set of iteratively undominated outcomes is $U = \{W(v) \mid v_i \in X_l / b_i, \; i \in N\}$.

W.l.o.g., let $X_l = \{x_1, \ldots x_l\}$, and let $N(x_m) = \{i \in N | b_i = x_m\}$, $1 \leq m \leq l$. So,

$$T_i = X_l / x_m, \; i \in N(x_m) \qquad \text{(A.4)}$$

It now suffices to show that for every $i \in N$, there exists $v_{-i} \in T_{-i}$ such that it is a unique best response for i to vote for any alternative $x \in T_i$. For then, no alternative in T_i can be weakly dominated for i.

W.l.o.g, let $i \in N(x_l)$, so $T_i = \{x_1, x_2 \ldots x_{l-1}\}$, and assume that $x_1 \succ_i x_2 \succ_i \ldots x_j \succ_i x_{j+1} \ldots \succ_i x_{l-1}$. Also, let $\omega_{-i} = (\omega_1, \omega_2, .. \omega_l)$, be a vote distribution over X_l, where ω_j is the number of votes for alternative x_j, and $\Omega_{-i}(T) = \{\omega_{-i}(v_{-i}) | v_{-i} \in T_{-i}\}$. So, we need to show that there exists $\omega_{-i} \in \Omega_{-i}(T)$ such that it is a unique best response for i to vote for any alternative $x_1, \ldots x_{l-1}$. Note two properties of a vote distribution ω_{-i} that must hold if it is to belong to $\Omega_{-i}(T)$. First, the total number of votes must add up to $n-1$;

$$\sum_{k=1} \omega_k = n - 1 \qquad \text{(A.5)}$$

Second, ω_k must be no greater than the number of voters (excluding i) who do not rank x_k worst in X_l i.e. for whom $x_k \in T_i$. Inspection of (A.4) implies that this requires

$$\omega_l \leq \sum_{k \neq l} n_k, \; \omega_j \leq \sum_{k \neq j} n_k - 1, \; j \neq l \qquad \text{(A.6)}$$

where $n_k = \# N(x_k)$.

Case I: n odd.

We know from the proof of Lemma 1 that $v_i = x_j$, $j < l$, is a unique best response in T_i to $\omega_{-i} = (\omega_1, \omega_2, \ldots; \omega_l)$ if

$$\omega_j = \omega_{j+1} > \omega_k + 1, \; k \neq j, j+1 \qquad \text{(A.7)}$$

So, it suffices to show that we can find $\hat{\omega}_{-i} \in \Omega_{-i}(T)$ where (A.5), (A.6),(A.7) are satisfied. We construct $\hat{\omega}_{-i}$ as follows. First, set $\omega_j = \omega_{j+1} = \omega$, where:

$$\omega = \begin{cases} \left[\frac{n+1}{3}\right] & \text{if } l = 3 \\ \left\lfloor \frac{n+3l-8}{l} \right\rfloor & \text{if } l > 3 \end{cases} \qquad \text{(A.8)}$$

where $[x]$ is the smallest integer greater than or equal to x: Let $t = (n - 1) - 2\omega$ be the number of remaining votes[28]: Note that we can always write $t = s(l - 2) + r;\ s \geq 0;\ 0 \leq r < l - 2$; where $s; r$ are integers. Now, distribute the remaining t votes over the remaining $l - 2$ alternatives as evenly as possible. That is, if $r = 0; s \geq 0$; give every remaining alternative s votes; if $r > 0$ and give every remaining alternative s votes and an additional vote to r of the $l - 2$ remaining alternatives. Clearly, π_{-i} satisfies (A.5) by construction.

Also, π_{-i} satisfies (A.7). To see this, note first that the maximum number of votes for any of the remaining alternatives $\omega_k;\ k \neq j; j+1$ is s if $r = 0$; and $s + 1$ if $r > 0$: Also, $s = (t - r)/(l - 2)$: But then, noting that if $l = 2;\ r \equiv 0$; (A.7) requires simply that

$$[\frac{n+1}{3}] > \frac{t}{l-2} + 1 \text{ if } l = 3 \tag{A.9}$$

$$\left\lfloor \frac{n+3l-8}{l} \right\rfloor > \frac{t-r}{l-2} + 2 \text{ if } l > 3;\ l-2 > r \geq 1 \tag{A.10}$$

Using the definition of t; the inequality (A.9) requires

$$[\frac{n+1}{3}] > \frac{n-1-2[\frac{n+1}{3}]}{l-2} + 1$$

which, using $[x] \geq x$; certainly holds. Also, the inequality (A.10) requires

$$\left\lfloor \frac{n+3l-8}{l} \right\rfloor > \frac{n-2-2\left\lfloor \frac{n+3l-8}{l} \right\rfloor}{l-2} + 2$$

which, again, using $[x] \geq x$; certainly holds for $l > 2$:

It remains to check that π_{-i} satisfies (A.6). From (A.7), a sufficient condition for (A.6) to be satisfied is that

$$\omega \leq \sum_{k \neq j} n_k - 1 \tag{A.11}$$

[28] It is easy to show that $t \geq 0$. For this, we require $n - 1 \geq 2[\frac{n+3l-8}{l}] = 6 + 2[\frac{n-8}{l}]$: Now the right-hand side of this inequality is largest when $l = 3$, so we only need $n \geq 6 + 2\left\lfloor \frac{n-8}{3} \right\rfloor$: It can easily be checked that this holds for $n \geq 6$; the case of $n = 5$ can be checked separately. Similarly, when $l = 3$; we can show that $n - 1 \geq 2[\frac{n+1}{3}]$ for $n \geq 9$; the cases $n = 5; 7$; can be checked separately.

Now let $n_k \leq \theta n, \forall k$. Note that as $\sum_{k=1}^{l} n_k = n$, $n_k \leq \theta n$ implies $\sum_{k \neq j} n_k \geq (1-\theta)n$, for any j. Then (A.11) is certainly satisfied if the following holds:

$$\omega \leq (1-\theta)n - 1 \tag{A.12}$$

Now let q_n^l be the largest value of θ such that (A.12) holds. So, q_n^l satisfies (A.12) with equality, i.e. $q_n^l = 1 - \frac{1}{n} - \frac{\omega}{n}$. Substituting out ω from (A.8), we get the expression (4.2) for $l = 3$, and the expression (5.3) for $l > 3$, for the case of n odd.
Case II: n even.

Here, the argument is the same, except we now choose $\tau_{-i} \in T_{-i}$ such that (A.13) below, rather than (A.7) is satisfied;

$$\omega_j = \omega_{j+1} - 1 > \omega_k, \; k \neq j, j+1 \tag{A.13}$$

For then, if (A.13) holds, by Lemma 1, $v_i = x_j$, $j < l$, is a unique best response by i to $\tau_{-i} \in T_{-i}$. The required vote distribution τ_{-i} is constructed as follows. First, we set $\omega_{j+1} = \omega$, $\omega_j = \omega - 1$, where

$$\omega = \begin{cases} [\frac{n+2}{3}] & \text{if } l = 3 \\ \lfloor \frac{n+3l-7}{l} \rfloor & \text{if } l > 3 \end{cases} \tag{A.14}$$

Also, distribute the remaining $t = n - 2\omega$ votes over the remaining $l - 2$ alternatives as evenly as possible[29], as before.

Clearly, τ_{-i} satisfies (A.5) by construction. Also, τ_{-i} satisfies (A.7). To see this, note by the argument in the odd case, that the maximum number of votes for any of the remaining alternatives ω_k, $k \neq j, j+1$ is s if $r = 0$, and $s + 1$ if $r > 0$, where $s = (t - r)/(l - 2)$. But then, noting that if $l = 2$, $r \equiv 0$, (A.7) requires simply that

$$[\frac{n+2}{3}] - 1 > \frac{t}{l-2} \quad \text{if } l = 3 \tag{A.15}$$

$$\left\lceil \frac{n+3l-7}{l} \right\rceil - 1 > \frac{t-r}{l-2} + 1 \quad \text{if } l > 3;\; l-2 > r \geq 1 \tag{A.16}$$

[29] It can be checked, as for the odd case that $t \geq 0$, since this is true iff $n - 2 \geq 2\omega$, i.e. $n - 2 \geq 4 + 2[\frac{n-7}{3}]$. The RHS is maximised when $l = 3$, thus it is sufficient to show that $n - 2 \geq 4 + 2[\frac{n-7}{3}]$. The latter holds for $n \geq 6$, and the case $n = 4$ can be checked separately.

where $t = s(l - 2) + r$, as before. Using the definition of t; (A.15) requires

$$[\frac{n+2}{3}] - 1 > n - 2[\frac{n+2}{3}] + 1$$

which, using $[x] \geq x$; certainly holds. Noting that the RHS of (A.16) is maximized when $r = 1$; the inequality (A.16) requires;

$$\frac{\lfloor n + 3l - 7 \rfloor}{l} > \frac{n - 2\lfloor\frac{n+3l-7}{l}\rfloor}{l - 2} + 2$$

which, again using $[x] \geq x$; certainly holds for $l > 3$: Finally, (A.6) is certainly satisfied by τ_{--} if

$$\omega_{j+1} \leq \sum_{k \neq j} n_k$$

which, from (A.14), reduces to

$$\omega \leq (1 - \theta)n \tag{A.17}$$

where $n_k \leq \theta n; \forall k$: Now let q_n^l be the largest value of θ such that (A.17) holds. So, q_n^l satisfies (A.17) with equality. Solving this expression for q_n^l;and using (A.14), we get the expression in (5.3) for the case of n even, and the expression in (4.2) for n even and $l = 3$. $\square$

Proof of Theorem 3 .(Sufficiency). If $q(\phi^n; X_l) > \frac{l-1}{l}$; $l = 3; \ldots K$ obviously $q(\phi^{nm}; X_l) > \frac{l-1}{l}$; $l = 3; \ldots K$; $m \geq 1$; so $\Gamma_{n;m}$ is DS for all $m \geq 1$ from Theorem 1.

(Necessity). Assume that n is odd. The proof for the even case is similar. If it is not the case that $q(\phi^n; X_l) > \frac{l-1}{l}$; $l = 3; \ldots K$; then, by A3, then there is some $l \in \{3; \ldots K\}$ such that

$$q(\phi^n; X_l) < \frac{l-1}{l};\ q(\phi^n; X_k) > \frac{k-1}{k};\ k > l \tag{A.18}$$

But then as $q_n^l < (l - 1)/l$, and $\lim_{n\to\infty} q_n^l = (l - 1)/l$, there exists a m_0 such that

$$q(\phi^{nm}; X_l) = q(\phi^n; X_l) \leq q_{nm}^l;\ m \geq m_0 \tag{A.19}$$

So, we conclude from (A.18), (A.19) that

$$q(\phi^{nm}; X_l) \leq q_{nm}^l;\ q(\phi^{nm}; X_k) \geq \frac{k-1}{k};\ k > l$$

for all $m \geq m_0$. So, by Theorem 2, $\Gamma_{n;m}$ is not DS for all $m \geq m_0$. $\square$