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ON INCREASING OF RESOLUTION OF SATELLITE IMAGES VIA THEIR FUSION WITH IMAGERY AT HIGHER RESOLUTION

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Abstract. In this paper we propose a new statement of the spatial increasing resolution problem of MODIS-like multi-spectral images via their fusion with Lansat-like imagery at higher resolution. We give a precise definition of the solution to the indicated problem, postulate assumptions that we impose at the initial data, establish existence and uniqueness result, and derive the corresponding necessary optimality conditions. For illustration, we supply the proposed approach by results of numerical simulations with real-life satellite images.

Key words: Satellite data fusion, image interpolation, image processing, variational approach, objective functional with non-standard growth conditions..

2010 Mathematics Subject Classification: 49Q20, 49K10, 49J45, 26B30.

1. Introduction

Following in some aspects the paper [5], we propose a new variational approach to the spatial increasing resolution of multi spectral MODIS-like images via their fusion with Lansat-like imagery at higher resolution. Our approach is based on the variational model in Sobolev-Orlicz space with a non-standard growth condition of the objective functional and on the assumption that, to a large extent, the image topology in the each spectral channel is contained in the topographic map of its spectral energy. We discuss the well foundedness of the above approach, the consistency of the corresponding variational problem, and show that this problem admits a unique solution. We also derive some optimality conditions and supply our approach by results of numerical simulations with the real satellite images.

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2. Preliminaries

We begin with some notation. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with a Lipschitz boundary $\partial\Omega$. Let $I : \Omega \to \mathbb{R}^m$, with $m \geqslant 3$, be a multispectral image containing the usual R, G, B channels I_R, I_G, I_B, and arguably some others ones like the infrared channel I_{NIR}, i.e.,

$$I(x) = [I_R(x), I_G(x), I_B(x), \dots]^t \in \mathbb{R}^m, \quad \forall\, x \in \Omega.$$

We say that $Y_I : \Omega \to \mathbb{R}$ is the panchromatic component of the multispectral image $I : \Omega \to \mathbb{R}^m$ (or in other words Y_I is the spectral energy coming from the RGB-channels) if the following representation

$$Y_I(x) = \alpha_R I_R(x) + \alpha_G I_G(x) + \alpha_B I_B(x), \quad \forall\, x \in \Omega$$

holds for some weight coefficients $\alpha_R, \alpha_G, \alpha_B \geqslant 0$. In particular, if

$$\alpha_R = 0.299, \quad \alpha_G = 0.587, \quad \alpha_B = 0.114$$

then Y_I can be interpreted as the luma component of I and it represents the perceptual brightness of the multispectral image $I : \Omega \to \mathbb{R}^m$.

For each $\lambda \in \mathbb{R}$, we define the upper level set of the spectral energy Y_I as follows

$$X_\lambda = \{x \in \Omega : Y_I(x) \geqslant \lambda\}.$$

Then the spectral energy Y_I can be recovered from its level sets by the reconstruction formula

$$Y_I(x) = \sup\{\lambda \;:\; x \in X_\lambda\}, \quad \forall\, x \in \Omega.$$

Hereinafter, we will refer to the family of connected components of the upper level sets of Y_I as the topographic map of Y_I.

Let $S_H \subset \Omega$ and $S_L \subset \Omega$ be two sample grids on Ω such that

$$S_H = \left\{ (x_i, y_j) \;\middle|\; \begin{array}{l} x_1 = x_H,\ x_i = x_1 + \Delta_{H,x}(i-1),\ i = 1, \dots, N_x, \\ y_1 = y_H,\ y_j = y_1 + \Delta_{H,y}(j-1),\ j = 1, \dots, N_y, \end{array} \right\},$$

$$S_L = \left\{ (x_i, y_j) \;\middle|\; \begin{array}{l} x_1 = x_L,\ x_i = x_1 + \Delta_{L,x}(i-1),\ i = 1, \dots, M_x, \\ y_1 = y_L,\ y_j = y_1 + \Delta_{L,y}(j-1),\ j = 1, \dots, M_y, \end{array} \right\},$$

where $N_x >> M_x$ and $N_y >> M_y$.

Let $H : \Omega \to \mathbb{R}^3$ be a given multispectral (Landsat-like) image which is sampled at the grid of high resolution S_H. We suppose that, in practice, this image can be identified with an 3-D array

$$H = \left\{ \begin{bmatrix} H_R(x_i, y_j) \\ H_G(x_i, y_j) \\ H_B(x_i, y_j) \end{bmatrix},\ i = 1, \dots, N_x,\ j = 1, \dots, N_y \right\}.$$

Let $L : \Omega \to \mathbb{R}^4$ be a given multispectral (MODIS-like) image which is sampled at the grid of low resolution S_L, and it has 4 spectral channels R, G, B, and NIR. So, we can indentify this image with an 4-D array

$$L = \left\{ \begin{bmatrix} L_R(x_i, y_j) \\ L_G(x_i, y_j) \\ L_B(x_i, y_j) \\ L_{NIR}(x_i, y_j) \end{bmatrix}, \ i = 1, \dots, M_x, \ j = 1, \dots, M_y \right\}.$$

The problem, we are going to consider, can be formally stated as follows: Using only the data $H : S_H \to \mathbb{R}^3$ and $L : S_L \to \mathbb{R}^4$, we have to increase the resolution of the four-band image $L : S_L \to \mathbb{R}^4$ via its fusion with the three-band image $H : S_H \to \mathbb{R}^3$ at higher resolution such that the following properties for the retrieved high resolution multispectral image $I : S_H \to \mathbb{R}^4$ would be satisfied:

(i) The image $I : \Omega \to \mathbb{R}^4$ we are going to retrieve, should be of bounded variation, $I \in BV(\Omega; \mathbb{R}^4)$.

(ii) The topographic maps for each spectral channel at higher resolution must have a similar structure to the topographic map of the spectral energy Y_H coming from the RGB-channels of $H : S_H \to \mathbb{R}^3$.

(iii) The spectral energies Y_I and Y_H should be as close as possible with respect to the $L^2(\Omega)$-norm.

(iv) The sampled values of $I : \Omega \to \mathbb{R}^4$ on the grid of low resolution S_L should be as close as possible in L^2-metric to the multispectral imagery $L : S_L \to \mathbb{R}^4$.

(v) The NIR-channel I_{NIR} for the retrieved high resolution multispectral image $I : \Omega \to \mathbb{R}^4$ should be in the same regression relationship with I_R, I_G, I_B channels as L_{NIR} with L_R, L_G, L_B, that is, if

$$L_{NIR}(x_i, y_j) = \gamma_R L_R(x_i, y_j) + \gamma_G L_G(x_i, y_j) + \gamma_B L_B(x_i, y_j), \quad \forall\, (x_i, y_j) \in S_L$$

is a linear regression model which is fitted using the least squares approach, then

$$I_{NIR}(x_i, y_j) = \gamma_R I_R(x_i, y_j) + \gamma_G I_G(x_i, y_j) + \gamma_B I_B(x_i, y_j), \quad \forall\, (x_i, y_j) \in S_H$$

with the same regression coefficients $\gamma_R, \gamma_G, \gamma_B \in \mathbb{R}$.

3. Auxiliaries

3.1. BV-Space

By $BV(\Omega)$ we denote the space of all functions $u \in L^1(\Omega)$ for which their distributional derivatives are representable by finite Borel measures in Ω, i.e.

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi\, dD_i u, \quad \forall\, \phi \in C_0^\infty(\Omega),\ i = 1, 2$$

for some $\mathbb{R}^2$-valued measure $Du = (D_1 u, D_2 u) \in \mathcal{M}^2(\Omega)$. It can be shown that $BV(\Omega)$, endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space, where

$$\begin{aligned} |Du|(\Omega) &:= \int_\Omega d|Du| \\ &= \sup\left\{ \int_\Omega u \operatorname{div} \varphi\, dx \ :\ \varphi \in C_0^1(\Omega; \mathbb{R}^2),\ |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega \right\} \end{aligned} \tag{3.1}$$

stands for the total variation of u in Ω. It is clear that $|Du|(\Omega) = \int_\Omega |\nabla u|\, dx$ if u is continuously differentiable in Ω.

The following embedding results for BV-function plays a crucial role for qualitative analysis of variational problems that we study in this paper.

Proposition 3.1. [4, p.378] Let Ω be an open bounded Lipschitz subset of $\mathbb{R}^2$. Then the embedding $BV(\Omega; \mathbb{R}^M) \hookrightarrow L^2(\Omega; \mathbb{R}^M)$ is continuous and the embeddings $BV(\Omega; \mathbb{R}^M) \hookrightarrow L^p(\Omega; \mathbb{R}^M)$ are compact for all p such that $1 \leqslant p < 2$. Moreover, there exists a constant $C_{em} > 0$ which depends only on Ω and p such that for all u in $BV(\Omega; \mathbb{R}^M)$,

$$\left(\int_\Omega |u|^p\, dx \right)^{1/p} \leqslant C_{em} \|u\|_{BV(\Omega; \mathbb{R}^M)}, \quad \forall\, p \in [1, 2].$$

According to the Radon-Nikodym theorem, if $u \in BV(\Omega)$ then there exists $\nabla u \in L^1(\Omega; \mathbb{R}^2)$ and a measure $D_s u$ singular with respect to the 2-dimensional Lebesgue measure $\mathcal{L}^2 \llcorner \Omega$ restricted to Ω, such that $Du = \nabla u \mathcal{L}^2 \llcorner \Omega + D_s u$.

We recall that if $u \in BV(\Omega)$, then almost all its level sets $\{x \in \Omega : u(x) \geqslant \lambda\}$ are sets of finite perimeter. Hence, at almost all points of almost all level sets of $u \in BV(\Omega)$ we may define a normal vector $\theta(x)$. This vector field of normals $\theta(x)$ can be also defined as the Radon-Nikodym derivative of the measure Du with respect to $|Du|$, i.e., it formally satisfies the following relations

$$(\theta, Du) = |Du| \quad \text{and} \quad |\theta| \leqslant 1 \text{ a.e. in } \Omega.$$

In the sequel, we will refer to the vector field θ as the vector field of unit normals to the topographic map of a function u. Further information on BV-functions and their properties can be found in [1, 4].

Remark 3.1. In practice, at the discrete level, $\theta(x,y)$ can be defined by the rule $\theta(x_i,y_j) = \frac{Du(x_i,y_j)}{|Du(x_i,y_j)|}$ when $Du(x_i,y_j) \neq 0$, and $\theta = 0$ when $Du(x_i,y_j) = 0$. However, as was mentioned in [5], a better choice for $\theta(x,y)$ would be to compute it as $\xi(t) = \frac{DU(t,\cdot)}{|DU(t,\cdot)|}$ for some small value of $t > 0$, where $U(t,x,y)$ is a solution of the following initial-boundary value problem with $1D$-Laplace operator in the right hand side

$$\frac{\partial U}{\partial t} = \operatorname{div}\left(\frac{DU}{|DU|}\right), \quad t \in (0,+\infty),\ (x,y) \in \Omega, \tag{3.2}$$

$$U(0,x,y) = u(x,y), \quad (x,y) \in \Omega, \tag{3.3}$$

$$\frac{\partial U(0,x,y)}{\partial \nu} = 0, \quad t \in (0,+\infty),\ (x,y) \in \partial\Omega. \tag{3.4}$$

As a result, for any $t > 0$, there can be found a vector field

$$\xi \in L^\infty(\Omega;\mathbb{R}^2) \ \text{ with } \ \|\xi(t)\|_{L^\infty(\Omega;\mathbb{R}^2)} \leqslant 1$$

such that

$$(\xi(t), U(t,\cdot)) = |DU(t,\cdot)| \text{ in } \Omega, \quad \xi(t)\cdot\nu = 0 \text{ on } \partial\Omega, \tag{3.5}$$

and $U_t(t,x,y) = \operatorname{div}\xi(t,x,y)$ in the sense of distributions on Ω for a.a. $t > 0$.

We notice that following this procedure, for small value of $t > 0$, we do not distort the geometry of the function $u(x,y)$ in an essential way. Moreover, it can be shown that this regularization of the vector field $\theta(x,y) = \frac{DU(x,y)}{|DU(x,y)|}$ satisfies condition $\operatorname{div}\theta \in L^2(\Omega)$.

3.2. On Orlicz Spaces

Let $p(\cdot)$ be a measurable exponent function on Ω such that

$$1 \leqslant \alpha \leqslant p(x) \leqslant \beta < \infty \quad \text{a.e. in } \Omega, \tag{3.6}$$

where α and β are given constants. Let $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ be the corresponding conjugate exponent. It is clear that

$$1 \leqslant \underbrace{\frac{\beta}{\beta-1}}_{\beta'} \leqslant p'(x) \leqslant \underbrace{\frac{\alpha}{\alpha-1}}_{\alpha'} \ \text{a.e. in } \Omega, \tag{3.7}$$

where β' and α' stand for the conjugates of constant exponents. Denote by $L^{p(\cdot)}(\Omega)$ the set of all measurable functions $f(x)$ on Ω such that $\int_\Omega |f(x)|^{p(x)}\,dx < \infty$. Then $L^{p(\cdot)}(\Omega)$ is a reflexive separable Banach space with respect to the Luxemburg norm (see [7,8] for the details)

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \ : \ \rho_p(\lambda^{-1}f) \leqslant 1\right\}, \tag{3.8}$$

where $\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx$.

It is well-known that $L^{p(\cdot)}(\Omega)$ is reflexive provided $\alpha > 1$, and its dual is $L^{p'(\cdot)}(\Omega)$, that is, any continuous functional $F = F(f)$ on $L^{p(\cdot)}(\Omega)$ has the form (see [21, Lemma 13.2])

$$F(f) = \int_\Omega fg\,dx, \quad \text{with} \;\; g \in L^{p'(\cdot)}(\Omega).$$

As for the infimum in (3.8), we have the following result.

Proposition 3.2. The infimum in (3.8) is attained if $\rho_p(f) > 0$. Moreover,

$$\text{if} \;\; \lambda_* := \|f\|_{L^{p(\cdot)}(\Omega)} > 0, \;\; \text{then} \;\; \rho_p(\lambda_*^{-1} f) = 1. \tag{3.9}$$

Taking this result and condition $1 \leqslant \alpha \leqslant p(x) \leqslant \beta$ into account, we see that

$$\frac{1}{\lambda_*^\beta}\int_\Omega |f(x)|^{p(x)}\;dx \leqslant \int_\Omega \left|\frac{f(x)}{\lambda_*}\right|^{p(x)}\;dx \leqslant \frac{1}{\lambda_*^\alpha}\int_\Omega |f(x)|^{p(x)}\;dx,$$
$$\frac{1}{\lambda_*^\beta}\int_\Omega |f(x)|^{p(x)}\;dx \leqslant 1 \leqslant \frac{1}{\lambda_*^\alpha}\int_\Omega |f(x)|^{p(x)}\;dx.$$

provided $\lambda_* \geqslant 1$. And we arrive at the reverse inequality if $0 < \lambda_* < 1$. Hence, (see [7, 8, 20] for the details)

$$\begin{aligned} &\|f\|^\alpha_{L^{p(\cdot)}(\Omega)} \leqslant \int_\Omega |f(x)|^{p(x)}\,dx \leqslant \|f\|^\beta_{L^{p(\cdot)}(\Omega)}, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} > 1,\\ &\|f\|^\beta_{L^{p(\cdot)}(\Omega)} \leqslant \int_\Omega |f(x)|^{p(x)}\,dx \leqslant \|f\|^\alpha_{L^{p(\cdot)}(\Omega)}, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} < 1, \end{aligned} \tag{3.10}$$

and, therefore,

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)} - 1 \leqslant \int_\Omega |f(x)|^{p(x)}\,dx \leqslant \|f\|^\beta_{L^{p(\cdot)}(\Omega)} + 1, \quad \forall\, f \in L^{p(\cdot)}(\Omega), \tag{3.11}$$

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \int_\Omega |f(x)|^{p(x)}\,dx, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} = 1. \tag{3.12}$$

The following estimates are well-known (see, for instance, [7, 8, 20]): if $f \in L^{p(\cdot)}(\Omega)$ then

$$\|f\|_{L^\alpha(\Omega)} \leqslant (1 + |\Omega|)^{1/\alpha}\,\|f\|_{L^{p(\cdot)}(\Omega)}, \tag{3.13}$$

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leqslant (1 + |\Omega|)^{1/\beta'}\,\|f\|_{L^\beta(\Omega)}, \quad \beta' = \frac{\beta}{\beta - 1}, \quad \forall\, f \in L^\beta(\Omega). \tag{3.14}$$

Let $\left\{f_k \in L^{p(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ be a given sequence, where the exponent $p \in C(\overline{\Omega})$ satisfies property (3.6). We say that the sequence $\left\{f_k \in L^{p(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ is bounded if (see [15, Section 6.2])

$$\limsup_{k\to\infty} \int_\Omega |f_k(x)|^{p(x)}\,dx < +\infty. \tag{3.15}$$

Definition 3.1. A bounded sequence $\left\{f_k \in L^{p(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ is weakly convergent in the Orlicz space $L^{p(\cdot)}(\Omega)$ to a function $f \in L^{p(\cdot)}(\Omega)$, if

$$\lim_{k\to\infty}\int_\Omega f_k\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2). \tag{3.16}$$

For our further analysis, we make use of the following lower semicontinuity property of the $L^{p(\cdot)}$-norm with respect to the weak convergence in $L^{p(\cdot)}(\Omega)$ (we refer to [21, Lemma 13.3] for details).

Proposition 3.3. If a bounded sequence $\left\{f_k \in L^{p(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ converges weakly in $L^\alpha(\Omega)$ to f, where $\alpha > 1$ is defined in (3.6), then $f \in L^{p(\cdot)}(\Omega)$, $f_k \rightharpoonup f$ in $L^{p(\cdot)}(\Omega)$, and

$$\liminf_{k\to\infty}\int_\Omega |f_k(x)|^{p(x)}\,dx \geqslant \int_\Omega |f(x)|^{p(x)}\,dx. \tag{3.17}$$

Remark 3.2. Arguing in a similar manner as in [21, Lemma 13.3] and using the estimate

$$\liminf_{k\to\infty}\int_\Omega \frac{1}{p_k(x)}|f_k(x)|^{p_k(x)}\,dx \geqslant \int_\Omega f(x)\varphi(x)\,dx - \int_\Omega \frac{1}{p_k'(x)}|\varphi(x)|^{p'(x)}\,dx,$$

which is valid for any smooth function φ, it is easy to show that the lower semicontinuity property (3.17) can be generalized as follows

$$\liminf_{k\to\infty}\int_\Omega \frac{1}{p(x)}|f_k(x)|^{p(x)}\,dx \geqslant \int_\Omega \frac{1}{p(x)}|f(x)|^{p(x)}\,dx. \tag{3.18}$$

We need the following result that leads to the analog of the Hölder inequality in Lebesgue spaces with variable exponents (for the details we refer to [7, 8]).

Proposition 3.4. If $f \in L^{p(\cdot)}(\Omega)^N$ and $g \in L^{p'(\cdot)}(\Omega)^N$, then $(f, g) \in L^1(\Omega)$ and

$$\int_\Omega (f, g)\;dx \leqslant 2\|f\|_{L^{p(\cdot)}(\Omega)^N}\|g\|_{L^{p'(\cdot)}(\Omega)^N}. \tag{3.19}$$

3.3. Sobolev Spaces with Variable Exponent

We recall here the well-known facts concerning the Sobolev spaces with variable exponent. Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 < \alpha \leqslant p(x) \leqslant \beta < \infty$ a.e. in Ω, where α and β are given constants. We associate with it the so-called Sobolev-Orlicz space

$$W^{1,p(\cdot)}(\Omega) := \left\{ y \in W^{1,1}(\Omega)\ :\ \int_\Omega \left[|y(x)|^{p(x)} + |\nabla y(x)|^{p(x)}\right]\,dx < +\infty \right\} \tag{3.20}$$

and equip it with the norm $\|y\|_{W_0^{1,p(\cdot)}(\Omega)} = \|y\|_{L^{p(\cdot)}(\Omega)} + \|\nabla y\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}$.

It is well-known that, in general, unlike classical Sobolev spaces, smooth functions are not necessarily dense in $W = W_0^{1,p(\cdot)}(\Omega)$. Hence, with the given variable exponent $p = p(x)$ $(1 < \alpha \leqslant p \leqslant \beta)$ there can be associated another Sobolev space with variable exponent,

$$H = H^{1,p(\cdot)}(\Omega) \text{ as the closure of the set } C^\infty(\overline{\Omega}) \text{ in } W^{1,p(\cdot)}(\Omega)\text{-norm.}$$

Since the identity $W = H$ is not always valid, it makes sense to say that an exponent $p(x)$ is regular if $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.

The following result reveals an important property ensuring the regularity of exponent $p(x)$.

Proposition B.1. *Assume that there exists $\delta \in (0,1]$ such that $p \in C^{0,\delta}(\overline{\Omega})$. Then the set $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, and, therefore, $W = H$.*

Proof. Let $p \in C^{0,\delta}(\overline{\Omega})$ be a given exponent. Since

$$\lim_{t\to 0} |t|^\delta \log(|t|) = 0 \quad \text{with an arbitrary } \delta \in (0,1], \tag{3.21}$$

it follows from the Hölder continuity of $p(\cdot)$ that

$$|p(x) - p(y)| \leqslant C|x-y|^\delta \leqslant \left[\sup_{x,y\in\Omega} \||x-y|^\delta \log(|x-y|^{-1})\right] \omega(|x-y|),\ \forall\, x,y \in \Omega, \tag{3.22}$$

where $\omega(t) = C/\log(|t|^{-1})$, and $C > 0$ is some positive constant.

Then property (3.21) implies that $p(\cdot)$ is a log-Hölder continuous function. So, to deduce the density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$ it is enough to refer to Theorem 13.10 in [21].

4. Main Assumptions

Let $H : S_H \to \mathbb{R}^3$ and $L : S_L \to \mathbb{R}^4$ be given digital images. Hereinafter, we assume that their continuous counterparts $H : \Omega \to \mathbb{R}^3$ and $L : \Omega \to \mathbb{R}^4$ are such that

$$Y_H \in BV(\Omega) \quad \text{and} \quad Y_L \in L^2(\Omega), \tag{4.1}$$

where Y_H and Y_L stand for the spectral energies of the H and L images, respectively.

Before proceed further, we associate with the spectral energy Y_H the so-called texturity characteristic $p : \Omega \to \mathbb{R}$ following the rule

$$p(x) := \mathfrak{F}(Y_H(x)) = 1 + g\left(\left|\left(\nabla G_\sigma * Y_H\right)(x)\right|\right), \quad \forall\, x \in \Omega, \tag{4.2}$$

where g:$[0,\infty) \to (0,\infty)$ is the edge-stopping function which we take it in the form of the Cauchy law $g(t) = \frac{1}{1+(t/a)^2}$ with an appropriate $a > 0$, $\left(\nabla G_\sigma * Y_H\right)(x)$ determines the convolution of function Y_H with the two-dimensional Gaussian filter kernel G_σ,

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad x \in \mathbb{R}^2, \tag{4.3}$$

$$\left(\nabla G_\sigma * Y_H\right)(x) := \int_\Omega \nabla G_\sigma(x-y) Y_H(y)\, dy, \quad \forall\, x \in \Omega. \tag{4.4}$$

Here, the parameter $\sigma>0$ determines the spatial size of the image details which are removed by this $2D$ filter.

Since the magnitude $g\left(\left|\left(\nabla G_\sigma * Y_H\right)(x)\right|\right)$ is close to one at those points, where the spectral energy Y_H is slowly varying, and this value is close to zero at the edges of Y_H, it follows that the function $p(x)$ can be interpreted as a texturity characteristic of the panchromatic image Y_H.

The following result plays a crucial role in the sequel.

Lemma 4.1. *Let $Y_H \in L^1(\Omega)$ be a given spectral energy. Let*

$$p = 1 + g\left(\left|\left(\nabla G_\sigma * Y_H\right)\right|\right)$$

be the corresponding texturity characteristic. Then

$$p \in C^{0,1}(\overline{\Omega}), \tag{4.5}$$

$$\alpha := 1 + \delta \leqslant p(x) \leqslant \beta := 2, \quad \forall\, x \in \Omega, \tag{4.6}$$

where $\delta = \dfrac{a^2}{a^2 + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}\|Y_H\|^2_{L^1(\Omega)}}$.

Proof. By smoothness of the Gaussian filter kernel G_σ, we have

$$\left|\left(\nabla G_\sigma * Y_H\right)(x)\right| \leqslant \int_\Omega \left|\nabla G_\sigma(x-y)\right| Y_H(y)\, dy \leqslant \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\|Y_H\|_{L^1(\Omega)},$$

$$\begin{aligned} p(x) &= 1 + \frac{a^2}{a^2 + \left(\left|\left(\nabla G_\sigma * Y_H\right)(x)\right|\right)^2} \\ &\geqslant 1 + \frac{a^2}{a^2 + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}\|Y_H\|^2_{L^1(\Omega)}}, \quad \forall\, x \in \Omega. \end{aligned}$$

From this the existence of a positive value $\delta \in (0,1)$ such that estimate (4.6) holds true follows. Combining this fact with $\max_{x\in\overline{\Omega}} |p(x)| \leqslant \beta := 2$, we arrive at the announced estimate (4.6).

As for the property (4.5), we make use of the estimate

$$\begin{aligned} |p(x) - p(y)| &\leqslant a^2 \left| \frac{\left|\left(\nabla G_\sigma * Y_H\right)(x)\right|^2 - \left|\left(\nabla G_\sigma * Y_H\right)(y)\right|^2}{\left(a^2 + \left|\left(\nabla G_\sigma * Y_H\right)(x)\right|^2\right)\left(a^2 + \left|\left(\nabla G_\sigma * Y_H\right)(y)\right|^2\right)} \right| \\ &\leqslant \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\|Y_H\|_{L^1(\Omega)}}{a^2} \left|\left|\left(\nabla G_\sigma * Y_H\right)(x)\right| - \left|\left(\nabla G_\sigma * Y_H\right)(y)\right|\right| \\ &\leqslant \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1^2|\Omega|}{a^2} \int_\Omega \left|\nabla G_\sigma(x-z) - \nabla G_\sigma(y-z)\right| dz, \ \forall\, x,y \in \Omega, \end{aligned} \tag{4.7}$$

with $\gamma_1 = \max_{x\in\Omega} |Y_H(x)|$. Then, by smoothness of the function $\nabla G_\sigma(\cdot)$, we deduce: there exists a positive constant $C_G > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1^2|\Omega| C_G}{a^2}|x-y|, \ \forall\, x,y \in \Omega.$$

Setting $C := \dfrac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1^2|\Omega|C_G}{a^2}$, we finally see that

$$p(\cdot) \in \mathfrak{S} = \left\{ h \in C^{0,1}(\Omega) \;\middle|\; \begin{array}{c} |h(x) - h(y)| \leqslant C|x-y|, \ \forall\, x, y, \in \Omega, \\ 1 < \alpha \leqslant h(\cdot) \leqslant \beta \text{ in } \overline{\Omega}. \end{array} \right\} \tag{4.8}$$

□

The algorithm that we propose to realize for the spatial interpolation of MODIS-like multi-band color images to the Landsat-like imagery at high resolution, is essentially grounded on the following key assumptions.

Assumption 1. The MODIS image $L : S_L \to \mathbb{R}^4$ and the Landsat image $H : S_H \to \mathbb{R}^3$ are rigidly co-registered.

This means that there exists an affine transformation $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$\mathcal{F}(x) = Bx + a, \quad \forall\, x \in \mathbb{R}^2, \tag{4.9}$$

where

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

such that the MODIS-like image after the affine transformation $L\left(\mathcal{F}^{-1}(\cdot)\right) : S_L \to \mathbb{R}^4$ and Landsat-like image $H : S_H \to \mathbb{R}^3$, after the bilinear resampling to the grid of low resolution S_L, could be successfully matched.

In practice, the co-registration procedure can be realized using, for instance, the open-source LSReg v2.0.2 software [16, 19] that has been used in a number of recent studies [9, 17], or the rigid co-registration approach that was recently developed by the EOS Company [11, 12]. However, in both cases, in order to find an appropriate affine transformation, we propose to apply the above mentioned procedure not to the original images, but rather to their spectral energies

$$Y_L(x_i, y_j) = \alpha_R L_R(x_i, y_j) + \alpha_G L_G(x_i, y_j) + \alpha_B L_B(x_i, y_j), \quad \forall\, (x_i, y_j) \in S_L$$

and

$$Y_H(x_i, y_j) = \alpha_R H_R(x_i, y_j) + \alpha_G H_G(x_i, y_j) + \alpha_B H_B(x_i, y_j), \quad \forall\, (x_i, y_j) \in S_H, \tag{4.10}$$

where the last one should be previously resampled to the grid of low resolution S_L.

Assumption 2. The low resolution pixels in the image $L : S_L \to \mathbb{R}^4$ are formed from the high resolution pixels of $I : S_H \to \mathbb{R}^4$ by a low pass filtering (the se-called subsampling procedure).

As a consequence of this Assumption, we can suppose that there exists an impulse response K such that

$$L(x_i, y_j) = [\mathcal{K} * I](x_i, y_j), \quad \forall\, i = 1, \dots, M_x,\ \forall\, j = 1, \dots, M_y. \tag{4.11}$$

where $[\mathcal{K} * I]$ stands for the convolution operator. In particular, if $\mathcal{K} = [k_{p,q}]_{p,q=1,\dots,K}$ is a squared matrix, then

$$[\mathcal{K} * I](x_i, y_j) = \sum_{p=1}^{K}\sum_{q=1}^{K} k_{p,q} I(x_{i-p+1}, y_{j-q+1})$$

provided $I(x, y) = 0$ if $(x, y) \notin \Omega$. For practical implementation, we usually set

$$k_{p,q} = \frac{1}{K^2}, \quad \forall\, p, q = 1, \dots, K$$

with an appropriate choice of $K \in \mathbb{N}$.

Assumption 3. The spectral energy Y_I of the retrieved high resolution multispectral image $I : \Omega \to \mathbb{R}^4$ is an element of the Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$, where $p(\cdot)$ is defined in (4.2), and

$$Y_I(x) = \alpha_R I_R(x) + \alpha_G I_G(x) + \alpha_B I_B(x), \quad \forall\, x \in \Omega$$

with $\alpha_R = 0.299$, $\alpha_G = 0.587$, and $\alpha_B = 0.114$.

Assumption 4. The topographic maps for each spectral channels I_R, I_G, I_B, and I_{NIR} of the retrieved image $I : \Omega \to \mathbb{R}^4$ have a similar structure to the topographic map of the spectral energy Y_H of the Landsat image $H : \Omega \to \mathbb{R}^3$.

As follows from this Assumption, all spectral channels of the retrieved image should share the geometry of the panchromatic image Y_H in Ω. It means that, due to the property $Y_H \in BV(\Omega)$, for almost all points of almost all level sets of Y_H we can define a normal vector $\theta(x)$, i.e., it formally satisfies $(\theta, Y_H) = |\nabla Y_H|$ and $|\theta| \leqslant 1$ a.e. in Ω (see Remark 3.1 for the details). So, if $\theta \in L^\infty(\Omega, \mathbb{R}^2)$ is a vector field with indicated properties, it follows that $\theta(x)$ has the direction of the normal to the level lines of Y_H. Therefore, the counterclockwise rotation of angle $\pi/2$, denoted by $\theta^\perp$, represents the tangent vector to the level lines of Y_H. In this case, if the spectral channels of $I : \Omega \to \mathbb{R}^4$ share the geometry of the panchromatic image Y_H, we have

$$\left(\theta^\perp, \nabla I_i\right)_{\mathbb{R}^2} = 0, \quad i \in \{R, G, B, NIR\} \text{ in } \Omega.$$

5. Statement of the Spatial Interpolation Problem

The problem of spatial interpolation of the MODIS-like image $L : S_L \to \mathbb{R}^4$ to the resolution of three-band Landsat-like image $H : S_H \to \mathbb{R}^3$ consists in the restoration of the four-band image $I:\Omega\to\mathbb{R}^4$ such that properties (i)–(v) would be satisfied. To do so, we propose at the first stage to compute the high resolution images $I_R, I_G, I_B:\Omega\to\mathbb{R}$ as a solution of the following constrained minimization problem

$$\inf_{(I_R,I_G,I_B)\in\Xi}\mathcal{J}\left(I_R,I_G,I_B\right), \tag{5.1}$$

where Ξ denotes the set of admissible images, and $\mathcal{J}\left(I_R,I_G,I_B\right)$ stands for the energy functional. Here, we define the set Ξ as follows: $(I_R,I_G,I_B)\in\Xi$ if and only if

(A) $(I_R,I_G,I_B)\in W^{1,p(\cdot)}(\Omega;\mathbb{R}^3)$, where $p(\cdot)$ stands for the texturity characteristic of the spectral energy $Y_H\in BV(\Omega)$;

(B) the following pointwise inequalities

$$0\leqslant I_R(x,y)\leqslant\max_{(x_i,y_j)\in S_L}L_R(x_i,y_j)\ \text{a.e. in}\ \ \Omega, \tag{5.2}$$

$$0\leqslant I_G(x,y)\leqslant\max_{(x_i,y_j)\in S_L}L_G(x_i,y_j)\ \text{a.e. in}\ \ \Omega, \tag{5.3}$$

$$0\leqslant I_B(x,y)\leqslant\max_{(x_i,y_j)\in S_L}L_B(x_i,y_j)\ \text{a.e. in}\ \ \Omega. \tag{5.4}$$

hold true.

As for the energy functional $\mathcal{J}:\Xi\to\mathbb{R}$, we construct it in the form

$$\mathcal{J}=\mathcal{J}_0+\gamma\mathcal{J}_1+\lambda\mathcal{J}_2+\mu\mathcal{J}_3, \tag{5.5}$$

where

$$\begin{aligned}\mathcal{J}_0\left(I_R,I_G,I_B\right)=&\int_\Omega\frac{1}{p(x)}|\nabla I_R(x)|^{p(x)}\,dx+\int_\Omega\frac{1}{p(x)}|\nabla I_G(x)|^{p(x)}\,dx\\&+\int_\Omega\frac{1}{p(x)}|\nabla I_B(x)|^{p(x)}\,dx\end{aligned} \tag{5.6}$$

$$\begin{aligned}\mathcal{J}_1\left(I_R,I_G,I_B\right)=&\int_\Omega\left|\left(\theta^\perp,\nabla I_R\right)\right|^\alpha dx+\int_\Omega\left|\left(\theta^\perp,\nabla I_G\right)\right|^\alpha dx\\&+\int_\Omega\left|\left(\theta^\perp,\nabla I_B\right)\right|^\alpha dx,\end{aligned} \tag{5.7}$$

$$\mathcal{J}_2\left(I_R,I_G,I_B\right)=\int_\Omega\left[\alpha_R I_R+\alpha_G I_G+\alpha_B I_B-Y_H\right]^2\,dx, \tag{5.8}$$

$$\begin{aligned}\mathcal{J}_3\left(I_R,I_G,I_B\right)=&\sum_{i=1}^{M_x}\sum_{j=1}^{M_y}\left(\left[\mathcal{K}*I_R\right](x_i,y_j)-L_R(x_i,y_j)\right)^2\\&+\sum_{i=1}^{M_x}\sum_{j=1}^{M_y}\left(\left[\mathcal{K}*I_G\right](x_i,y_j)-L_G(x_i,y_j)\right)^2\\&+\sum_{i=1}^{M_x}\sum_{j=1}^{M_y}\left(\left[\mathcal{K}*I_B\right](x_i,y_j)-L_B(x_i,y_j)\right)^2,\end{aligned} \tag{5.9}$$

and

$$\theta(x,y) = \frac{DU(t,x,y)}{|DU(t,x,y)|} \text{ for small values of } t > 0. \tag{5.10}$$

Here, the exponent $\alpha > 0$ is defined by (4.6), and $U(t,x,y)$ is the solution of the parabolic problem (3.2)–(3.4) with the initial condition

$$U(0,x,y) = Y_H(x,y) = \alpha_R H_R(x,y) + \alpha_G H_G(x,y) + \alpha_B H_B(x,y), \quad \forall\,(x,y) \in \Omega.$$

The main motivation for such choice of the energy functional is rather clear. As follows from (5.9), each term in $\mathcal{J}_3\,(I_R, I_G, I_B)$ represents an L^2-distortion between a particular spectral channel in the MODIS image $L : S_L \to \mathbb{R}^4$ and the corresponding channel of the retrieved image $I : S_H \to \mathbb{R}^4$ which is resampled to the grid of low resolution S_L. So, the $\mathcal{J}_3$-term should be minimal and it is mainly motivated by Assumption 2.

As for the term $\mathcal{J}_2\,(I_R, I_G, I_B)$, it reflects the fact that the spectral energy $Y_I = \alpha_R I_R + \alpha_G I_G + \alpha_B I_B$ of the retrieved image should be as close as possible to the spectral energy of the Landsat image $H : S_H \to \mathbb{R}^3$. We interpret this closedness in the sense of L^2-norm.

Now about the term $\mathcal{J}_1\,(I_R, I_G, I_B)$. As was mentioned before, the main goal, we are going to follows in the spatial interpolation problem, is to preserve the following property: the geometry of each spectral channels in the retrieved image should be as close as possible to the geometry of the spectral energy of the Landsat image $H : S_H \to \mathbb{R}^3$. Formally, it means that the following relations have to be satisfied

$$\left(\theta^\perp, \nabla I_R\right)_{\mathbb{R}^2} = 0, \quad \left(\theta^\perp, \nabla I_G\right)_{\mathbb{R}^2} = 0, \quad \left(\theta^\perp, \nabla I_B\right)_{\mathbb{R}^2} = 0 \quad \text{a.e. in } \Omega.$$

Hence, the magnitude

$$\int_\Omega \left[\left| \left(\theta^\perp, \nabla I_R\right) \right|^\alpha + \left| \left(\theta^\perp, \nabla I_G\right) \right|^\alpha + \left| \left(\theta^\perp, \nabla I_B\right) \right|^\alpha \right] dx$$

must be small enough, where $\theta = \theta(x,y)$ stands for the vector field of unit normals to the topographic map of the spectral energy $Y_H = \alpha_R H_R + \alpha_G H_G + \alpha_B H_B$.

The first term $\mathcal{J}_0\,(I_R, I_G, I_B)$ is the regularization. Since $p(x) \approx 1$ in places in Ω where edges or discontinuities are present in the spectral energy Y_H of the image H, and $p(x) \approx 2$ in places where $Y_H(x)$ is smooth or contains homogeneous features, the main benefit of the model (5.1) is the manner in which it accommodates the local image information. The places where the gradient is sufficiently large (i.e. likely edges), we deal with the so-called TV-based diffusion [18], whereas the places where the gradient is close to zero (i.e. homogeneous regions), the model becomes isotropic. Specifically, the type of anisotropy at these ambiguous regions varies according to the strength of the gradient. This enables the model to have a much lower dependence on the approximation schemes for the variable exponent $p(x)$ and other thresholds.

We are now in a position to define what we mean by the solution of spatial interpolation problem that was stated in Section 2. Taking into account the properties (i)–(v) that we imposed and the structure of the energy functional $\mathcal{J} : \Xi \to \mathbb{R}$, we say that a four-band image $I^0 = \left[I^0_R, I^0_G, I^0_B, I^0_{NIR}\right]^t : S_H \to \mathbb{R}^4$ is the result of fusion of MODIS-like multi spectral image $L : S_L \to \mathbb{R}^4$ with the Landsat-like color image $H : S_H \to \mathbb{R}^3$ at higher resolution if:

- the triplet $\left(I^0_R, I^0_G, I^0_B\right)$ is a solution of constrained minimization problem (5.1), i.e.,

$$\left(I^0_R, I^0_G, I^0_B\right) \in \Xi \quad \text{and} \quad \mathcal{J}\left(I^0_R, I^0_G, I^0_B\right) = \inf_{(I_R, I_G, I_B)\in\Xi} \mathcal{J}\left(I_R, I_G, I_B\right)$$

- The spectral channel $I^0_{NIR} : \Omega \to \mathbb{R}$ is defined as follows

$$I^0_{NIR}(x,y) = \gamma_R I^0_R(x,y) + \gamma_G I^0_G(x,y) + \gamma_B I^0_B((x,y), \quad \forall\, (x,y) \in \Omega,$$

where

$$\begin{bmatrix} \gamma_R \\ \gamma_G \\ \gamma_B \end{bmatrix} = \left[\int_\Omega \begin{pmatrix} L_R^2 & L_R L_G & L_R L_B \\ L_R L_G & L_R^2 & L_G L_B \\ L_R L_B & L_G L_B & L_B^2 \end{pmatrix} dx\right]^{-1} \int_\Omega \begin{bmatrix} L_{NIR} L_R \\ L_{NIR} L_G \\ L_{NIR} L_B \end{bmatrix} dx. \tag{5.11}$$

Here, the last equality is a formal representation of the solution to the following linear regression problem

$$\int_\Omega \left[\gamma_R L_R + \gamma_G L_G + \gamma_B L_B - L_{NIR}\right]^2 dx \overset{\gamma_R, \gamma_G, \gamma_B}{\longrightarrow} \inf.$$

Remark 5.1. As an alternative way to define the NIR spectral channel I^0_{NIR}, we can propose the following one: define I^0_{NIR} as a solution of the constrained minimization problem

$$\mathcal{Z}\left(I^0_{NIR}\right) = \inf_{I \in \Xi_{NIR}} \mathcal{Z}\left(I\right),$$

where

$$\begin{aligned} \mathcal{Z}\left(I\right) = \int_\Omega \frac{1}{p(x)} |\nabla I(x)|^{p(x)}\, dx + \gamma \int_\Omega \left| \left(\theta^\perp, \nabla I\right) \right|^\alpha dx \\ + \lambda \int_\Omega \left[\gamma_R I^0_R + \gamma_G I^0_G + \gamma_B I^0_B - I\right]^2 dx, \end{aligned}$$

$$\Xi_{NIR} = \left\{ I \in W^{1,p(\cdot)}(\Omega) \ : \ 0 \leqslant I(x,y) \leqslant \max_{(x_i,y_j)\in S_L} L_{NIR}(x_i, y_j) \text{ a.e. in } \Omega \right\},$$

and the weight coefficients $\gamma_R, \gamma_G, \gamma_B$ are defined by (5.11).

Remark 5.2. Another variant for the setting of the spatial interpolation problem is to consider, instead of the energy term $\mathcal{J}_1$ in (5.5), the following functional (this approach was firstly proposed in [5])

$$\mathcal{J}_1(I_R, I_G, I_B) = \left(\int_\Omega |\nabla I_R|\, dx + \int_\Omega I_R \operatorname{div}\theta\, dx\right) + \left(\int_\Omega |\nabla I_G|\, dx + \int_\Omega I_G \operatorname{div}\theta\, dx\right) + \left(\int_\Omega |\nabla I_B|\, dx + \int_\Omega I_B \operatorname{div}\theta\, dx\right).$$

The main motivation for such choice of the functional $\mathcal{J}_1$ is rather clear. Since the geometry of each spectral channel in the retrieved image should be as close as possible to the geometry of the spectral energy of the Landsat image $H : S_H \to \mathbb{R}^3$, it means that the following relations have to be satisfied

$$|\nabla I_R| = (\theta, \nabla I_R),\ |\nabla I_G| = (\theta, \nabla I_G),\ |\nabla I_B| = (\theta, \nabla I_B),\ \text{a.e. in}\ \Omega.$$

Hence, the magnitude

$$\int_\Omega [(|\nabla I_R| - (\theta, \nabla I_R)) + (|\nabla I_G| - (\theta, \nabla I_G)) + (|\nabla I_B| - (\theta, \nabla I_B))]\, dx$$

must be small enough, where $\theta = \theta(x, y)$ stands for the vector field of unit normals to the topographic map of the spectral energy $Y_H = \alpha_R H_R + \alpha_G H_G + \alpha_B H_B$. By default, we assume that this field is zero along the boundary $\partial\Omega$. Then, making use of the Green's formula, we deduce:

$$\begin{aligned}
\int_\Omega (|\nabla I_R| - \theta \cdot \nabla I_R)\, dx &= \int_\Omega |\nabla I_R|\, dx + \int_\Omega I_R \operatorname{div}\theta\, dx,\\
\int_\Omega (|\nabla I_G| - \theta \cdot \nabla I_G)\, dx &= \int_\Omega |\nabla I_B|\, dx + \int_\Omega I_B \operatorname{div}\theta\, dx,\\
\int_\Omega (|\nabla I_B| - \theta \cdot \nabla I_B)\, dx &= \int_\Omega |\nabla I_B|\, dx + \int_\Omega I_B \operatorname{div}\theta\, dx.
\end{aligned}$$

6. Existence Result and Optimality Conditions

6.1. On Existence and Uniqueness of Retrieved Image at High Resolution

We begin this section with the following existence result.

Theorem 6.1. *Let $H : S_H \to \mathbb{R}^3$ and $L : S_L \to \mathbb{R}^4$ be given images such that their spectral energies satisfy conditions* (4.1)*. Then, under Assumptions 1–4, there exists a unique triplet $(I_R, I_G, I_B) \in \Xi \subset BV(\Omega; \mathbb{R}^3)$ such that (I_R, I_G, I_B) is a minimizer to constrained minimization problem* (5.1)*.*

Proof. First of all, we notice that minimization problem (5.1) is consistent, moreover, $\mathcal{J}(I_R, I_G, I_B) < +\infty$ for each feasible triplet $(I_R, I_G, I_B) \in \Xi$. Indeed, in view of the pointwise estimates (5.2)–(5.4), we have $I_R, IG, I_B \in L^2(\Omega)$. Hence,

$$\mathcal{J}_2\left(I_R, I_G, I_B\right) < +\infty.$$

It remains to notice that the inclusion $I \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^4)$ and the estimates

$$\begin{aligned}\int_\Omega \left|\left(\theta^\perp, \nabla I_A\right)\right|^\alpha dx &\leqslant \|\theta\|^\alpha_{L^\infty(\Omega;\mathbb{R}^2)}\|\nabla I_A\|^\alpha_{L^\alpha(\Omega;\mathbb{R}^2)}\\ &\overset{\text{by (3.13)–(3.15)}}{\leqslant} (1+|\Omega|)\ \|\theta\|^\alpha_{L^\infty(\Omega;\mathbb{R}^2)}\|\nabla I_A\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}\\ &\leqslant (1+|\Omega|)\ \|\theta\|^\alpha_{L^\infty(\Omega;\mathbb{R}^2)}\|I_A\|_{W^{1,p(\cdot)}(\Omega;\mathbb{R}^2)}, \quad A \in \{R, G, B\}\end{aligned}$$

imply $\mathcal{J}_1\left(I_R, I_G, I_B\right) < +\infty$. As a result, consistency of the problem (5.1) immediately follows from (5.5)–(5.9) and definition of the set Ξ.

Let $\left\{(I^k_R, I^k_G, I^k_B)\right\}_{k\in\mathbb{N}} \subset \Xi$ be a minimizing sequence to the problem (5.1). i.e.,

$$\lim_{k\to\infty} \mathcal{J}(I^k_R, I^k_G, I^k_B) = \inf_{(I_R,I_G,I_B)\in\Xi} \mathcal{J}\left(I_R, I_G, I_B\right).$$

Then there exists a constant $\widehat{C} > 0$ such that

$$\sup_{k\in\mathbb{N}} \mathcal{J}(I^k_R, I^k_G, I^k_B) \leqslant \widehat{C}.$$

From this and (5.5), we deduce that

$$\int_\Omega \frac{1}{p(x)}\left[|\nabla I_R(x)|^{p(x)} + |\nabla I_G(x)|^{p(x)} + |\nabla I_B(x)|^{p(x)}\right] dx \leqslant \widehat{C}, \tag{6.1}$$

$$\int_\Omega \left[\alpha_R I^k_R + \alpha_G I^k_G + \alpha_B I^k_B - Y_H\right]^2 dx \leqslant \widehat{C}. \tag{6.2}$$

Since $\alpha := 1+\delta \leqslant p(x) \leqslant \beta := 2$ for all $x \in \Omega$, it follows from (6.1) and (3.11) that

$$\sup_{k\in\mathbb{N}}\left[\|\nabla I_R\|^\alpha_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)} + \|\nabla I_G\|^\alpha_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)} + \|\nabla I_B\|^\alpha_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}\right] \leqslant \alpha\left(\widehat{C}+3\right). \tag{6.3}$$

On th other hand, utilizing estimate (6.2) and non-negativity of I^k_R, I^k_G, I^k_B, and Y_H, we obtain

$$\int_\Omega \left[\alpha_R I^k_R + \alpha_G I^k_G + \alpha_B I^k_B\right]^2 dx \leqslant 2\widehat{C} + 2\int_\Omega Y^2_H\, dx.$$

Hence,

$$\begin{aligned}(1+|\Omega|)^{-1}\sup_{k\in\mathbb{N}}\|I^k_R\|^2_{L^{p(\cdot)}(\Omega)} &\overset{\text{by (3.14)}}{\leqslant} \sup_{k\in\mathbb{N}}\|I^k_R\|^2_{L^2(\Omega)} \leqslant 2\alpha_R^{-2}\left[\widehat{C} + \|Y_H\|^2_{L^2(\Omega)}\right],\\ (1+|\Omega|)^{-1}\sup_{k\in\mathbb{N}}\|I^k_G\|^2_{L^{p(\cdot)}(\Omega)} &\overset{\text{by (3.14)}}{\leqslant} \sup_{k\in\mathbb{N}}\|I^k_G\|^2_{L^2(\Omega)} \leqslant 2\alpha_G^{-2}\left[\widehat{C} + \|Y_H\|^2_{L^2(\Omega)}\right],\\ (1+|\Omega|)^{-1}\sup_{k\in\mathbb{N}}\|I^k_B\|^2_{L^{p(\cdot)}(\Omega)} &\overset{\text{by (3.14)}}{\leqslant} \sup_{k\in\mathbb{N}}\|I^k_B\|^2_{L^2(\Omega)} \leqslant 2\alpha_B^{-2}\left[\widehat{C} + \|Y_H\|^2_{L^2(\Omega)}\right].\end{aligned} \tag{6.4}$$

As a result, it follows from (6.3) and (6.4) that

$$\sup_{k\in\mathbb{N}}\|I_R^k\|_{W^{1,p(\cdot)}(\Omega)}=\sup_{k\in\mathbb{N}}\left[\|I_R^k\|_{L^{p(\cdot)}(\Omega)}+\|\nabla I_R^k\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}\right]\leqslant\mathcal{Q}_R,$$
$$\sup_{k\in\mathbb{N}}\|I_G^k\|_{W^{1,p(\cdot)}(\Omega)}=\sup_{k\in\mathbb{N}}\left[\|I_G^k\|_{L^{p(\cdot)}(\Omega)}+\|\nabla I_G^k\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}\right]\leqslant\mathcal{Q}_G,$$
$$\sup_{k\in\mathbb{N}}\|I_B^k\|_{W^{1,p(\cdot)}(\Omega)}=\sup_{k\in\mathbb{N}}\left[\|I_B^k\|_{L^{p(\cdot)}(\Omega)}+\|\nabla I_B^k\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}\right]\leqslant\mathcal{Q}_B$$

with $\mathcal{Q}_A=\left[2\left(1+|\Omega|\right)\alpha_A^{-2}\left(\widehat{C}+\|Y_H\|^2_{L^2(\Omega)}\right)\right]^{1/2}+\left[\alpha\left(\widehat{C}+3\right)\right]^{1/\alpha}$, $A\in\{R,G,B\}$.

Thus, the sequence $\left\{(I_R^k,I_G^k,I_B^k)\right\}_{k\in\mathbb{N}}\subset\Xi$ is bounded in $W^{1,p(\cdot)}(\Omega;\mathbb{R}^3)$. Therefore, in view of Proposition 3.3, there exists a subsequence of

$$\left\{(I_R^k,I_G^k,I_B^k)\right\}_{k\in\mathbb{N}}\subset\Xi,$$

still denoted by the same index, and functions $(I_R^0,I_G^0,I_B^0)\in W^{1,p(\cdot)}(\Omega;\mathbb{R}^3)$ such that

$$(I_R^k,I_G^k,I_B^k)\rightharpoonup(I_R^0,I_G^0,I_B^0)\text{ weakly in }W^{1,p(\cdot)}(\Omega;\mathbb{R}^3),\tag{6.5}$$
$$(I_R^k,I_G^k,I_B^k)\rightharpoonup(I_R^0,I_G^0,I_B^0)\text{ weakly in }W^{1,\alpha}(\Omega;\mathbb{R}^3),\tag{6.6}$$
$$(I_R^k,I_G^k,I_B^k)\to(I_R^0,I_G^0,I_B^0)\text{ strongly in }L^{\alpha}(\Omega;\mathbb{R}^3),\tag{6.7}$$

and

$$\int_\Omega\frac{1}{p(x)}|\nabla I_R^0|^{p(x)}\,dx\leqslant\liminf_{k\to\infty}\int_\Omega\frac{1}{p(x)}|\nabla I_R^k|^{p(x)}\,dx,\tag{6.8}$$
$$\int_\Omega\frac{1}{p(x)}|\nabla I_G^0|^{p(x)}\,dx\leqslant\liminf_{k\to\infty}\int_\Omega\frac{1}{p(x)}|\nabla I_G^k|^{p(x)}\,dx,\tag{6.9}$$
$$\int_\Omega\frac{1}{p(x)}|\nabla I_B^0|^{p(x)}\,dx\leqslant\liminf_{k\to\infty}\int_\Omega\frac{1}{p(x)}|\nabla I_B^k|^{p(x)}\,dx.\tag{6.10}$$

Moreover, passing to a subsequence if necessary and taking into account the inequalities (5.2)–(5.4) and (3.13), we have:

$$(I_R^k(x,y),I_G^k(x),I_B^k(x))\to(I_R^0(x),I_G^0(x),I_B^0(x))\quad\text{for a.e. }x\in\Omega,\tag{6.11}$$
$$(I_R^k,I_G^k,I_B^k)\rightharpoonup(I_R^0,I_G^0,I_B^0)\quad\text{weakly in }L^2(\Omega;\mathbb{R}^3),\tag{6.12}$$
$$\left(\theta^\perp,\nabla I_A^k\right)\rightharpoonup\left(\theta^\perp,\nabla I_A^0\right)\quad\text{weakly in }L^\alpha(\Omega)\text{ for }A\in\{R,G,B\}.\tag{6.13}$$

Hence, without loss of generality, we can suppose that the limit triplet (I_R^0,I_G^0,I_B^0) satisfies the pointwise restrictions (5.2)–(5.4), and, as a consequence, we deduce: $(I_R^0,I_G^0,I_B^0)\in\Xi$ is a feasible solution to the problem (5.1).

Let us show that $(I_R^0, I_G^0, I_B^0) \in \Xi$ is a minimizer to the problem (5.1). Indeed, utilizing convergence properties (6.5)–(6.12), we get

$$\liminf_{k\to\infty} \mathcal{J}_0(I_R^k, I_G^k, I_B^k) \overset{\text{by (6.8)–(6.10)}}{\geqslant} \mathcal{J}_0(I_R^0, I_G^0, I_B^0),$$

$$\liminf_{k\to\infty} \mathcal{J}_1(I_R^k, I_G^k, I_B^k) \overset{\text{by (6.13)}}{\geqslant} \mathcal{J}_1(I_R^0, I_G^0, I_B^0),$$

$$\liminf_{k\to\infty} \mathcal{J}_2(I_R^k, I_G^k, I_B^k) \overset{\text{by (6.12)}}{\geqslant} \mathcal{J}_2(I_R^0, I_G^0, I_B^0),$$

$$\liminf_{k\to\infty} \mathcal{J}_3(I_R^k, I_G^k, I_B^k) \overset{\text{by (6.11),(6.12)}}{\geqslant} \mathcal{J}_3(I_R^0, I_G^0, I_B^0).$$

Hence, $\liminf_{k\to\infty} \mathcal{J}(I_R^k, I_G^k, I_B^k) \geqslant \mathcal{J}(I_R^0, I_G^0, I_B^0)$, and, therefore,

$$\begin{aligned} \inf_{(I_R,I_G,I_B)\in\Xi} \mathcal{J}\left(I_R, I_G, I_B\right) &= \lim_{k\to\infty} \mathcal{J}(I_R^k, I_G^k, I_B^k) = \liminf_{k\to\infty} \mathcal{J}(I_R^k, I_G^k, I_B^k) \\ &\geqslant \mathcal{J}(I_R^0, I_G^0, I_B^0) \geqslant \inf_{(I_R,I_G,I_B)\in\Xi} \mathcal{J}\left(I_R, I_G, I_B\right). \end{aligned}$$

Thus, (I_R^0, I_G^0, I_B^0) is a minimiser to the problem (5.1).

It remains to show that (I_R^0, I_G^0, I_B^0) is a unique minimizer for this problem. Indeed, let us assume the converse. Let $(I_R^0, I_G^0, I_B^0) \in \Xi$ and $(I_R^*, I_G^*, I_B^*) \in \Xi$ be two minimizers for the problem (5.1). Then by the strict convexity of norm $\|\cdot\|_{L^2(\Omega)}$ and convexity of the set of feasible solutions Ξ, we have

$$\begin{aligned} \mathcal{J}\left(\frac{I_R^0 + I_R^*}{2}, \frac{I_G^0 + I_G^*}{2}, \frac{I_B^0 + I_B^*}{2}\right) &< \frac{1}{2}\mathcal{J}(I_R^0, I_G^0, I_B^0) + \frac{1}{2}\mathcal{J}(I_R^*, I_G^*, I_B^*) \\ &= \inf_{(I_R,I_G,I_B)\in\Xi} \mathcal{J}\left(I_R, I_G, I_B\right) \end{aligned}$$

which brings us into a conflict with the initial assumptions. Thus, (I_R^0, I_G^0, I_B^0) is a unique minimizer to the problem (5.1). The proof is complete. □

For further convenience, we rewrite the energy functional $\mathcal{J} : \Xi \to \mathbb{R}$ in the integral form. To this end, we set: let $\delta_{(i,j)}$ be the Dirac's delta on the point $(i.j) \in \Omega$. Let $\Pi_L = \sum_{(i,j)\in S_L} \delta_{(i,j)}$ be the Dirac's comb on Ω defined by the grid S_L. Then we may write $\mathcal{J}_3$ in integral terms

$$\mathcal{J}_3\left(I_R, I_G, I_B\right) = \sum_{A\in\{R,G.B\}} \int_\Omega \Pi_L \left(\left[\mathcal{K} * I_A\right](x) - L_A(x)\right)^2 dx, \tag{6.14}$$

where L_A, $A \in \{R, G.B\}$ denotes an arbitrary extension of $L_A(i, j)$ as continuous functions from S_L to Ω. Since the terms above are multiplied by Π_L, the integral terms in (6.14) do not depend on the particular extensions of L_A, $A \in \{R, G.B\}$.

6.2. Optimality Conditions

In order to derive some optimality conditions to the problem (5.1) and characterize its solution (I_R^0, I_G^0, I_B^0), we check that the functional $\mathcal{J} : \Xi \to \mathbb{R}$ is Gâteaux differentiable. To this end, we note that

$$\frac{|\nabla I_A^0(x) + t\nabla h_A(x)|^{p(x)} - |\nabla I_A^0(x)|^{p(x)}}{p(x)t} \to \left(|\nabla I_A^0(x)|^{p(x)-2}\nabla I_A^0(x), \nabla h_A(x)\right) \quad \text{as} \quad t \to 0$$

almost everywhere in Ω for all $A \in \{R, G.B\}$ and $h = (h_R, h_G, h_B) \in W^{1,p(\cdot)}(\Omega)^3$. Indeed, by convexity, we have $|\xi|^p - |\eta|^p \leqslant 2p\left(|\xi|^{p-1} + |\eta|^{p-1}\right)|\xi - \eta|$. Then

$$\begin{aligned}\left|\frac{|\nabla I_A^0(x) + t\nabla h_A(x)|^{p(x)} - |\nabla I_A^0(x)|^{p(x)}}{p(x)t}\right| &\leqslant 2\left(|\nabla I_A^0(x) + t\nabla h_A(x)|^{p(x)-1} + |\nabla I_A^0(x)|^{p(x)-1}\right)|\nabla h_A(x)| \\ &\leqslant \text{const}\left(|\nabla I_A^0(x)|^{p(x)-1} + |\nabla h_A(x)|^{p(x)-1}\right)|\nabla h_A(x)|. \quad (6.15)\end{aligned}$$

Taking into account that

$$\begin{aligned}\||\nabla I_A^0(x)|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(\Omega;\mathbb{R}^2)} &\overset{\text{by (3.11) and (3.7)}}{\leqslant} \left(\int_\Omega |\nabla I_A^0(x)|^{p(x)}\,dx + 1\right)^{\frac{1}{2}} \\ &\overset{\text{by (3.11)}}{\leqslant} \left(\|\nabla I_A^0|^2_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)} + 2\right)^{\frac{1}{2}}, \\ \int_\Omega |\nabla I_A^0(x)|^{p(x)-1}|\nabla h_A(x)|\,dx &\overset{\text{by (3.19)}}{\leqslant} 2\||\nabla I_A^0(x)|^{p(x)-1}\|_{L^{p'(\cdot)}(\Omega)}\|h_A(x)|\|_{L^{p(\cdot)}(\Omega)},\end{aligned}$$

and $\int_\Omega |\nabla h_A(x)|^{p(x)}\,dx \overset{\text{by (3.11)}}{\leqslant} \|\nabla h_A\|^2_{L^{p(\cdot)}(\Omega)} + 1$, we see that the right hand side of inequality (6.15) is an $L^1(\Omega)$ function. Therefore,

$$\int_\Omega \frac{|\nabla I_A^0(x) + t\nabla h_A(x)|^{p(x)} - |\nabla I_A^0(x)|^{p(x)}}{p(x)t}\,dx \to \int_\Omega \left(|\nabla I_A^0(x)|^{p(x)-2}\nabla I_A^0(x), \nabla h_A(x)\right)dx \quad \text{as} \quad t \to 0$$

by the Lebesgue dominated convergence theorem for each $h = (h_R, h_G, h_B) \in$

$W^{1,p(\cdot)}(\Omega;\mathbb{R}^3)$ and $A \in \{R, G.B\}$. Thus,

$$\lim_{t\to 0} \frac{\mathcal{J}_0\left(I_R^0 + th_R, I_G^0 + th_G, I_B^0 + th_B\right) - \mathcal{J}_0\left(I_R^0, I_G^0, I_B^0\right)}{t}$$
$$= \int_\Omega \Big(|\nabla I_R^0(x)|^{p(x)-2} \nabla I_R^0(x), \nabla h_R(x) \Big)\, dx$$
$$+ \int_\Omega \Big(|\nabla I_G^0(x)|^{p(x)-2} \nabla I_G^0(x), \nabla h_G(x) \Big)\, dx$$
$$+ \int_\Omega \Big(|\nabla I_B^0(x)|^{p(x)-2} \nabla I_B^0(x), \nabla h_B(x) \Big)\, dx. \quad (6.16)$$

Arguing in a similar manner, it can be shown that (see [2, Section A 14] for the details)

$$\lim_{t\to 0} \frac{\mathcal{J}_1\left(I_R^0 + th_R, I_G^0 + th_G, I_B^0 + th_B\right) - \mathcal{J}_1\left(I_R^0, I_G^0, I_B^0\right)}{t}$$
$$= \alpha \int_\Omega \left| \left(\theta^\perp, \nabla I_R^0\right) \right|^{\alpha-2} \left(\theta^\perp, \nabla I_R^0\right) \left(\theta^\perp, \nabla h_R\right)\, dx$$
$$+ \alpha \int_\Omega \left| \left(\theta^\perp, \nabla I_G^0\right) \right|^{\alpha-2} \left(\theta^\perp, \nabla I_G^0\right) \left(\theta^\perp, \nabla h_G\right)\, dx$$
$$+ \alpha \int_\Omega \left| \left(\theta^\perp, \nabla I_B^0\right) \right|^{\alpha-2} \left(\theta^\perp, \nabla I_B^0\right) \left(\theta^\perp, \nabla h_B\right)\, dx. \quad (6.17)$$

Setting

$$\Lambda = \theta^\perp \left(\theta^\perp\right)^t = \begin{bmatrix} \theta_1^\perp \theta_1^\perp & \theta_1^\perp \theta_2^\perp \\ \theta_2^\perp \theta_1^\perp & \theta_2^\perp \theta_2^\perp \end{bmatrix},$$

the Gâteaux differential of $\mathcal{J}_1$ can be rewritten as follows

$$\lim_{t\to 0} \frac{\mathcal{J}_1\left(I_R^0 + th_R, I_G^0 + th_G, I_B^0 + th_B\right) - \mathcal{J}_1\left(I_R^0, I_G^0, I_B^0\right)}{t}$$
$$= \alpha \int_\Omega \left| \left(\theta^\perp, \nabla I_R^0\right) \right|^{\alpha-2} \left(\Lambda \nabla I_R^0, \nabla h_R\right)\, dx$$
$$+ \alpha \int_\Omega \left| \left(\theta^\perp, \nabla I_G^0\right) \right|^{\alpha-2} \left(\Lambda \nabla I_G^0, \nabla h_G\right)\, dx$$
$$+ \alpha \int_\Omega \left| \left(\theta^\perp, \nabla I_B^0\right) \right|^{\alpha-2} \left(\Lambda \nabla I_B^0, \nabla h_B\right)\, dx. \quad (6.18)$$

As for the rest terms in the cost functional $\mathcal{J} : \Xi \to \mathbb{R}$, we have the following representation for their Gâteaux derivatives (for the proof and its substantiation we refer to [13, Section 3]).

Proposition 6.1. Let $H : S_H \to \mathbb{R}^3$ and $L : S_L \to \mathbb{R}^4$ be given images such that their spectral energies satisfy conditions (4.1) Then the functionals $\mathcal{J}_2, \mathcal{J}_3$:

$L^2(\Omega;\mathbb{R}^3) \to \mathbb{R}$ are convex and Gâteaux differentiable in $L^2(\Omega;\mathbb{R}^3)$ with

$$\mathcal{J}_2'(I^0)[h] = 2 \sum_{A\in\{R,G.B\}} \int_\Omega \alpha_A \left(\alpha_R I_R^0 + \alpha_G I_G^0 + \alpha_B I_B^0 - Y_H\right) h_A\, dx, \tag{6.19}$$

$$\begin{aligned}\mathcal{J}_3'(I^0)[h] &= 2 \sum_{A\in\{R,G.B\}} \int_\Omega \Pi_L \left(\left[\mathcal{K} * I_A^0\right] - L_A\right) \left[\mathcal{K} * h_A\right] dx \\ &= 2 \sum_{A\in\{R,G.B\}} \int_\Omega \Pi_L \left[\mathcal{K}^* * \left(\left[\mathcal{K} * I_A^0\right] - L_A\right)\right] h_A\, dx, \end{aligned}\tag{6.20}$$

for all $h = (h_R, h_G, h_B) \in L^2(\Omega;\mathbb{R}^3)$.

We are now in a position to derive an optimality system for a unique minimizer $(I_R^0, I_G^0, I_B^0) \in \Xi \subset BV(\Omega;\mathbb{R}^3)$ to constrained minimization problem (5.1). Following the standard technique which is based on the use of the Lagrange principle [10] and utilizing Proposition 4.4, we arrive at the following result.

Theorem 6.2. *Let $(I_R^0, I_G^0, I_B^0) \in \Xi$ be a minimizer of constrained minimization problem* (5.1). *Then the following relations hold true*

$$\begin{aligned} - \operatorname{div}\left(|\nabla I_A^0(x)|^{p(x)-2}\nabla I_A^0(x)\right) - \gamma\alpha \operatorname{div}\left(\left|\left(\theta^\perp, \nabla I_A^0\right)\right|^{\alpha-2} \Lambda \nabla I_A^0\right) \\ +2\lambda\alpha_A \left(\alpha_R I_R^0 + \alpha_G I_G^0 + \alpha_B I_B^0 - Y_H\right) \\ +2\mu\Pi_L \left[\mathcal{K}^* * \left(\left[\mathcal{K} * I_A^0\right] - L_A\right)\right] = 0 \quad a.e.\ in\ \ \Omega, \end{aligned}\tag{6.21}$$

$$0 \leqslant I_A^0 \leqslant \max_{(x_i,y_j)\in S_L} L_A(x_i,y_j) \quad a.e.\ in\ \ \Omega, \tag{6.22}$$

$$\left(\nabla I_A^0, \nu\right) = 0 \quad on\ \partial\Omega, \tag{6.23}$$

for $A \in \{R, G, B\}$.

Remark 6.1. In practical implementation, it is reasonable to define an optimal triplet (I_R^0, I_G^0, I_B^0) using a 'gradient descent' strategy. Following the standard procedure, we can start from some initial RGB-components (I_R^*, I_G^*, I_B^*) and then to solve the following initial value problem for the system of quasi-linear parabolic equations with $2D$ elliptic operators in their principle part and Nuemann boundary conditions

$$\begin{aligned} \frac{\partial I_R^0}{\partial t} - \operatorname{div}\left(|\nabla I_R^0(x)|^{p(x)-2}\nabla I_R^0(x)\right) &= \gamma\alpha \operatorname{div}\left(\left|\left(\theta^\perp, \nabla I_R^0\right)\right|^{\alpha-2} \Lambda \nabla I_R^0\right) \\ &- 2\lambda\alpha_R \left(\alpha_R I_R^0 + \alpha_G I_G^0 + \alpha_B I_B^0 - Y_H\right) - 2\mu\Pi_L \left[\mathcal{K}^* * \left(\left[\mathcal{K} * I_R^0\right] - L_R\right)\right], \\ \frac{\partial I_G^0}{\partial t} - \operatorname{div}\left(|\nabla I_G^0(x)|^{p(x)-2}\nabla I_G^0(x)\right) &= \gamma\alpha \operatorname{div}\left(\left|\left(\theta^\perp, \nabla I_G^0\right)\right|^{\alpha-2} \Lambda \nabla I_G^0\right) \\ &- 2\lambda\alpha_G \left(\alpha_R I_R^0 + \alpha_G I_G^0 + \alpha_B I_B^0 - Y_H\right) - 2\mu\Pi_L \left[\mathcal{K}^* * \left(\left[\mathcal{K} * I_G^0\right] - L_G\right)\right], \\ \frac{\partial I_B^0}{\partial t} - \operatorname{div}\left(|\nabla I_B^0(x)|^{p(x)-2}\nabla I_B^0(x)\right) &= \gamma\alpha \operatorname{div}\left(\left|\left(\theta^\perp, \nabla I_B^0\right)\right|^{\alpha-2} \Lambda \nabla I_B^0\right) \\ &- 2\lambda\alpha_B \left(\alpha_R I_R^0 + \alpha_G I_G^0 + \alpha_B I_B^0 - Y_H\right) - 2\mu\Pi_L \left[\mathcal{K}^* * \left(\left[\mathcal{K} * I_B^0\right] - L_B\right)\right], \end{aligned}$$

$$\left(\nabla I_R^0, \nu\right) = 0,\ \left(\nabla I_G^0, \nu\right) = 0,\ \left(\nabla I_B^0, \nu\right) = 0 \quad \text{on } \partial\Omega,$$
$$0 \leqslant I_A^0 \leqslant \max_{(x_i, y_j) \in S_L} L_A(x_i, y_j) \quad \text{a.e. in } \Omega\ \forall A \in \{R, G, B\},$$
$$I_R^0(0, x) = I_R^*,\ I_G^0(0, x) = I_G^*,\ I_B^0(0, x) = I_B^*, \quad \forall x \in \Omega,$$

where we propose to take the triplet (I_R^*, I_G^*, I_B^*) as a result bicubic interpolation of the MODIS-like image $L : S_L \to \mathbb{R}^4$ onto the entire domain Ω.

7. Numerical Experiments

Fig. 7.1. The MODIS image with resolution $350m/pixel$

In order to illustrate the proposed algorithm for the spatial increasing resolution problem of MODIS-like multi-spectral images via their fusion with Lansat-like imagery at higher resolution. As input data we have used a MODIS image of some region with resolution $350m/pixel$ (see Fig. 7.1). This region represents a typical agricultural area with medium sides fields of various shapes.

We also have the image of the same territory with resolution $25m/pixel$ that was made by Lansat satellite at higher resolution. Figure 7.2 shows the spectral channels of this image.

Figure 7.3 displays the reconstructed images corresponding to the data given by Figures 7.1 and 7.2. In order to validate the obtained result, we have provided the following calculations.

- Closednees of the means $\rho_2 = |\mathrm{Mean}\, I - \mathrm{Mean}\, L| = 0$;
- Closedness of the variances $\rho_3 = 100 \frac{|\mathrm{Var}\, I - \mathrm{Var}\, L|}{\mathrm{Var}\, L} \approx 6\%$;
- ERGAS metric

$$ERGAS = 100 \frac{h}{l} \sqrt{\frac{1}{3} \sum_{k=1}^{3} \left(\frac{\mathrm{RMSE(k)}}{\mu_0(k)} \right)^2} = 2.24,$$

where h/l is the ratio between the size of the high spatial resolution image pixel and the size of the pixel in the MODIS-like image.

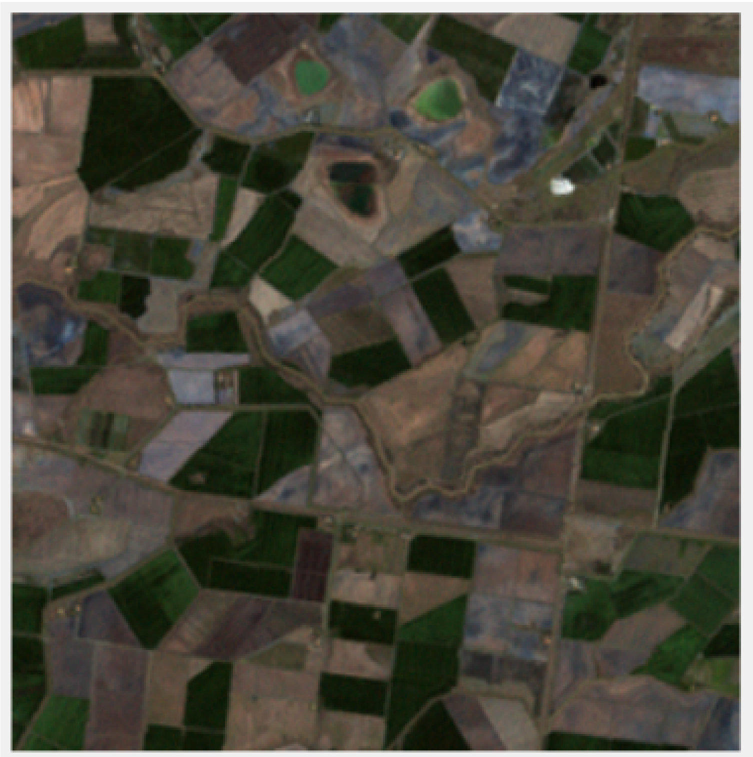

Fig. 7.2. The Lansat image with resolution $25m/pixel$

It is worth to notice that in view of the suggestions of Prof. L. Wald, if the ERGAS value is less than 3, the spectral quality of an image is satisfactory.

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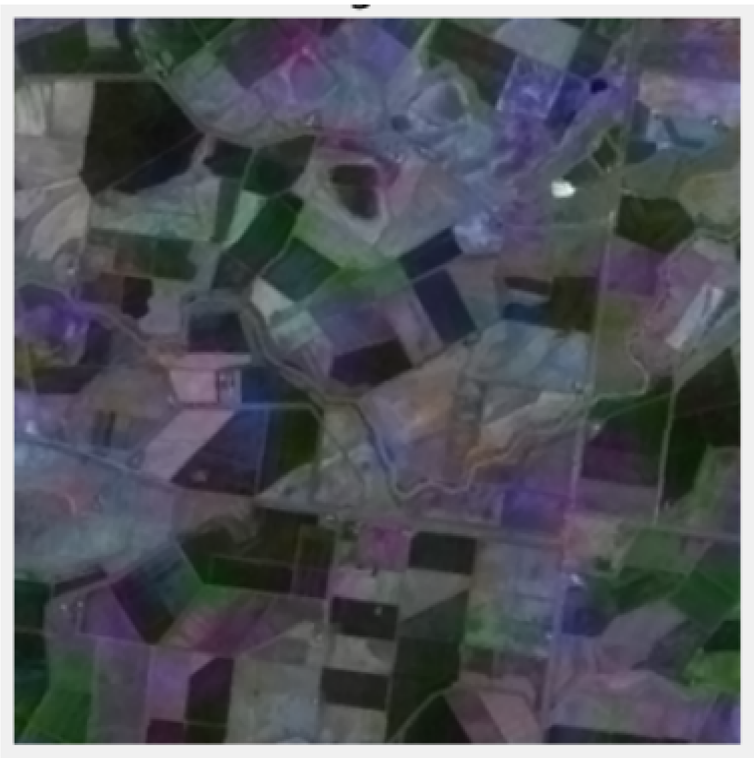

Fig. 7.3. The retrieved image at high resolution $25m/pixel$ following the proposed approach

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