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On combining independent tests in case of log-logistic distribution

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We take a look at the log-logistic distribution via Bahadur's stochastic comparison of asymptotic relative efficiency by combining infinitely independent tests of hypotheses. We discuss the six free-distribution combination producers namely; Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values for testing simple hypotheses against one-sided alternative. These methods are compared via the exact Bahadur slope (EBS). Moreover, several comparisons among the six procedures using the exact Bahadur's slopes were obtained. We further employ numerical study to investigate these comparisons behavior. These non-parametric procedures depend on the p-value of the individual tests combined.

keywords: Asymptotic relative efficiency, log-logistic distribution, combining independent tests, Bahadur efficiency, Bahadur slope.

1 Introduction

The combination of n independent tests of hypotheses is an important statistical practice. If H_0 is a simple hypotheses. Bahadur's stochastic comparison is one of the most common approach in asymptotic relative efficiency for two test procedures in which the probabilities of the two types of errors (I and II) changes with increasing sample size, and the manner of the alternatives are behave. In comparison of test procedures, let $H_0 : F \in \mathscr{F}_0$ is to be tested, where $\mathscr{F}_0$ is a family of distributions, for any test procedure T_n. The function $\gamma_n(T, F) = P_F(T_n \text{ rejects } H_0)$, for distribution functions F, represents the power function of T_n. Under H_0, $\gamma_n(T, F)$ represents the probability of a *Type I* error. The size of the test is $\alpha_n(T, \mathscr{F}_0) = \sup\limits_{F\in\mathscr{F}_0} \gamma_n(T, F)$. For $F \notin \mathscr{F}_0$, the probability of a *Type II* error is $\beta_n(T, F) = 1-\gamma_n(T, F)$. We are interesting in studying consistent tests, that is for fixed $F \notin \mathscr{F}_0$, $\beta_n(T, F) \to 0$ as $n \to \infty$, and unbiased tests that is $F \notin \mathscr{F}_0$, $\gamma_n(T, F) \geq \alpha_n(T, \mathscr{F}_0)$. To compare two test procedures through their power functions, we will use the asymptotic relative efficiency (ARE) for two test procedures T_A and T_B, with sample sizes n_1 and n_2 respectively, then

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the ratio n_1/n_2 goes to some limit. This limit is the ARE of T_B relative to T_A. In Bahadur approach, the following behaviors are satisfied: the *Type I* error is $\alpha_n \to 0$, the *Type II* error is $\beta_n \to 0$, and the alternatives is $F^n = F$ fixed. Asymptotic relative efficiency have been considered by many authors. Kallenberg (1981) showed that in testing problems in multivariate exponential families the LR test is deficient in the sense of Bahadur of order $O(log\ n)$. Abu-Dayyeh and El-Masri (1994) studied six free-distribution methods (sum of P-values, inverse normal, logistic, Fisher, minimum of P-values and maximum of P-values) of combining infinitely number of independent tests when the P-values are IID rv's distributed with uniform distribution under the null hypothesis versus triangular distribution with essential support $(0, 1)$ under the alternative hypothesis. They proved that the sum of P-values method is the best method. Abu-Dayyeh et al. (2003) they combined infinity number of independent tests for testing simple hypotheses against one-sided alternative for normal and logistic distributions, they used four methods of combining (Fisher, logistic, sum of P-values and inverse normal). Al-Masri (2010) studied six methods of combining independent tests. He showed under conditional shifted Exponential distribution that the inverse normal method is the best among six combination methods. Al-Talib et al. (2020) considered combining independent tests in case of conditional normal distribution with probability density function $X|\theta \sim N(\gamma\theta, 1)$, $\theta \in [a, \infty], a \geq 0$ when $\theta_1, \theta_2, ...$ have a distribution function (DF) F_θ. They concluded that the inverse normal procedure is the best procedure. AL-MASRI (2021) considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of Laplace distribution $\mathbb{L}(\gamma, 1)$. They showed that the sum of p-values procedure is better than all other procedures under the null hypothesis, and the inverse normal procedure is better than the other procedures under the alternative hypothesis.

2 Stochastic Comparison And A Measure Of ARE

To measure the strength of the observed sample as evidence against the null hypothesis is to compute the significance level of the observed value of the test statistic, which is another way to compare two test procedures. When the alternative is true yields strong evidence against the null hypothesis. Bahadur et al. (1960) Introduced a notion of "stochastic comparison" and corresponding measure of asymptotic relative efficiency. We consider iid observations $X_1, \ldots, X_n$ in a sample space having a distribution by parameter $\theta \in \Theta$. For testing the hypothesis $H_0 : \theta \in \Theta_0$ by a real-valued test statistic T_n, where H_0 becomes rejected for sufficiently large values of T_n. Let G_{θ_n} denoted the DF of T_n under the $\theta-$distribution of $X_1, \ldots, X_n$. The level attained is the indicator of the significance of the observed data against the null hypothesis is given by $L_n = L_n(X_1, \ldots, X_n) = sup_{\theta \in \Theta_0}[1 - G_{\theta_n}(T_n)]$. Where $sup_{\theta \in \Theta_0}[1 - G_{\theta_n}(t)]$ is the maximum probability, under any one of the null hypothesis models, that the experiment will lead to a test statistic exceeding t. Also, Bahadur et al. (1960) suggests stochastic comparison of two test sequences $T_A = T_{A_n}$ and $T_B = T_{B_n}$ in terms of their performances with respect to level attained, as follows. Under the nonnull $\theta-$distribution, the test T_{A_n} is more successful than the test T_{B_n} at the sample $X_1, \ldots, X_n$ if $L_{A_n}(X_1, \ldots, X_n) < L_{B_n}(X_1, \ldots, X_n)$. Equivalently, defining $K_n = -2 \log L_n$, T_{A_n} is more successful than T_{B_n} at the observed sample if $K_{A_n} > K_{B_n}$. In this case, for $\theta \in \Theta_0$, L_n converges in $\theta-$distribution to some nondegenerate random variable, and under an alternative $\theta \notin \Theta_0$, $L_n \to 0$ at an exponential rate of θ.

3 Log-Logistic Distribution

The Log-Logistic distribution is often used in decision making business, decision making with project management and audio dithering. Let X be a random variable following the Log-Logistic distribution. The distribution function (cdf) of X can take the following form

$$F(x;\xi) = \Psi_{Logistic}\left(\frac{\ln(x)-\xi}{\sigma}\right) I_{\mathbb{R}^+}(x) \tag{1}$$

The probability density function (pdf) of X is given by

$$f(x;\xi) = \frac{1}{\sigma x}\varphi_{Logistic}\left(\frac{\ln(x)-\xi}{\sigma}\right) I_{\mathbb{R}^+}(x), \xi \in \mathbb{R}, \sigma \in \mathbb{R}^+. \tag{2}$$

Where I_A is the indicator function and $\Psi_{Logistic}(h) = \left(1+e^{-h}\right)^{-1}$ is the cdf of the standard logistic distribution with $L(0,1)$. The mean of X is $E\left(X\right) = \frac{\sigma\pi}{\sin(\pi\sigma)}$; if $\sigma > 1$, else undefined. For more details see Al-Omari and Zamanzade (2018) and RRL et al. (2010).

4 Definitions And Preliminaries

This section lays out some basic tools to Bahadur's stochastic comparison theory that used in this article

Definition *(Bahadur efficiency and exact Bahadur slope (EBS))* Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with a probability density function $f(x,\theta)$, and we want to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \Theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \leq t_i\right)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that

$$C_i(\theta) = \lim_{\theta\to\infty} -2n^{-1}\ln\left(L_n^i\right)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B\left(T_1, T_2\right) = C_1(\theta)/C_2(\theta)$. Serfling (2009)

Theorem 1. (Large deviation theorem) *Let $X_1, X_2, \ldots, X_n$ be IID, with distribution F and put $S_n = \sum_{i=1}^n X_i$. Assume existence of the moment generating function (mgf) $M(z) = E_F\left(e^{zX}\right)$, z real, and put $m(t) = \inf_z e^{-z(X-t)} = \inf_z e^{-zt}M(z)$. The behavior of large deviation probabilities $P\left(S_n \geq t_n\right)$, where $t_n \to \infty$ at rates slower than $O(n)$. The case $t_n = tn$, if $-\infty < t \leq EY$, then $P\left(S_n \leq nt\right) \leq \left[m(t)\right]^n$, the*

$$-2n^{-1}\ln P_F\left(S_n \geq nt\right) \to -2\ln m(t) \quad a.s. \quad \left(F_\theta\right).$$

Serfling (2009)

Theorem 2. (Bahadur theorem) *Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

1. *Under $H_1 : \theta \in \Theta - \{\theta_0\}$:*

$$n^{-\frac{1}{2}}T_n \to b(\theta) \quad a.s. \quad \left(F_\theta\right),$$

where $b(\theta) \in \Re$.

2. *There exists an open interval I containing $\{b(\theta) : \theta \in \Theta - \{\theta_0\}\}$, and a function g continuous on I, such that*

$$\lim_n -2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = \lim_n -2n^{-1} \log \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = g(t), \qquad t \in I.$$

If $\{T_n\}$ satisfied (1)-(2), then for $\theta \in \Theta - \{\theta_0\}$

$$-2n^{-1} \log \sup_{\theta \in \Theta_0} [1 - F_{\theta_n}(T_n)] \to C(\theta) \qquad a.s. \qquad (F_\theta).$$

Bahadur et al. (1960)

Theorem 3. *Let $X_1, \ldots, X_n$ be i.i.d. with probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. For $j = 1, 2$, let $T_{n,j} = \sum_{i=1}^n f_i(x_i)/\sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let φ_j be the test based on $T_{n,j}$. Assume $\mathbb{E}_\theta(f_i(x)) > 0, \forall \theta \in \Theta$, $\mathbb{E}_0(f_i(x)) = 0$, $Var(f_i(x)) > 0$ for $j = 1, 2$. Then*
1. If the derivative $b'_j(0)$ is finite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\frac{b'_1(0)}{b'_2(0)}\right]^2,$$

where $b_i(\theta) = \mathbb{E}_\theta(f_j(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.
2. If the derivative $b'_j(0)$ is infinite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\lim_{\theta \to 0} \frac{b'_1(\theta)}{b'_2(\theta)}\right]^2.$$

Al-Masri (2010)

Theorem 4. *If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0, respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any α, then if φ_j is the test based on $T_n^{(j)}$, $j = 1, 2$, then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta).$$

Serfling (2009)

Corollary 1. *If T_n is the uniformly most powerful test for all α, then it is the best via EBS. Serfling (2009)*

Theorem 5. *Let $U_1, U_2, \ldots$ be i.i.d. like U with probability density function f and suppose that we want to test $H_0 : U_i \sim U(0, 1)$ vs. $H_1 : U_i \sim f$ on $(0, 1)$ but not $U(0, 1)$. Then $C_{max}(f) = -2 \ln(ess.sup_f(u))$*
where $ess.sup_f(u) = sup\{u : f(u) > 0\}$ w.p.1 under f. Abu-Dayyeh and El-Masri (1994)

Theorem 6. *If $\pi(\ln \pi)^2 f(\pi) \to 0$ as $\pi \to 0$, then $C_T(f) = 0$. Abu-Dayyeh and El-Masri (1994)*

Theorem 7.

$$2t \leq m_S(t) \leq et, \quad \forall : 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z > 0} e^{-zt} \frac{e^z - 1}{z}.$$

Al-Masri (2010)

Theorem 8. *1.* $m_L(t) \geq 2te^{-t}, \quad \forall t \geq 0,$

2. $m_L(t) \leq te^{1-t}, \quad \forall t \geq 0.852,$

3. $m_L(t) \leq t\left(\frac{t^2}{1+t^2}\right)^3 e^{1-t}, \quad \forall t \geq 4,$
where $m_L(t) = \inf_{z \in (0,1)} e^{-zt}\pi z\ csc(\pi z)$ *and csc is an abbreviation for cosecant function. Al-Masri (2010)*

Theorem 9. *For* $x > 0$*,*

$$\phi(x)\left[\frac{1}{x} - \frac{1}{x^3}\right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ *is the pdf of standard normal distribution. Al-Masri (2010)*

Theorem 10. *For* $x > 0$*,*

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

Al-Masri (2010)

Lemma 1. *1.* $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$

2. $m_L(t) \leq \dfrac{e^{-t^2/(t+1)}\left(\frac{\pi t}{t+1}\right)}{sin\left(\frac{\pi t}{t+1}\right)}$

3. $\begin{cases} m_s(t) = \inf_{z>0} \frac{e^{-zt}(1-e^{-z})}{z} \leq \inf_{z>0} \frac{e^{-zt}}{z} \leq -et, & t < 0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$
Al-Masri (2010)

5 The Basic Problem

Consider testing the hypothesis

$$H_0^{(i)} : \eta_i = \eta_0^i,\ vs\ , H_1^{(i)} : \eta_i \in \Omega_i - \{\eta_0^i\} \tag{3}$$

such that $H_0^{(i)}$ becomes rejected for large values of some real valued continuous random variable $T^{(i)}$, $i = 1, 2, \ldots, n$. The n hypotheses are combined into one as

$$H_0^{(i)} : (\eta_1, ..., \eta_n) = (\eta_0^1, ..., \eta_0^n),\ vs\ , H_1^{(i)} : (\eta_1, ..., \eta_n) \in \left\{\prod_{i=1}^{n} \Omega_i - \{(\eta_0^1, ..., \eta_0^n)\}\right\} \tag{4}$$

For $i = 1, 2, \ldots, n$ the p-value of the i-th test is given by

$$P_i(t) = P_{H_0^{(i)}}\left(T^{(i)} > t\right) = 1 - F_{H_0^{(i)}}(t) \tag{5}$$

where $F_{H_0^{(i)}}(t)$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0,1)$ under $H_0^{(i)}$.
If considering the special case where $\eta_i = \theta$ and $\eta_0^i = \theta_0$ for $i = 1, \ldots, n$, and also assume that $T^{(1)}, \ldots, T^{(n)}$ are independent, then (4) reduces to

$$H_0 : \theta = \theta_0,\ vs\ , H_1 : \theta \in \Omega - \{\theta_0\} \tag{6}$$

It follows that the p-values $P_1, \ldots, P_n$ are also iid rv's that have a $U(0,1)$ distribution under H_0, and under H_1 have a distribution whose support is a subset of the interval $(0,1)$ and is not a $U(0,1)$ distribution. Therefore, if f is the probability density function (pdf) of P, then (6) is equivalent to

$$H_0 : P \sim U(0,1), \;\; vs \;, H_1 : P \nsim U(0,1) \tag{7}$$

where P has a pdf f with support subset of the interval $(0,1)$.
In this paper we will study the case when

$$f(x;\xi) = \frac{1}{x}\varphi_{Logistic}\left(\ln(x) - \xi\right) I_{\mathbb{R}^+}(x), \xi \in \mathbb{R},$$

where $\sigma = 1$.
By sufficiency we may assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \ldots, n$. Then we consider the sequence $\left\{T^{(n)}\right\}$ of independent test statistics, thus is we will take a random sample $X_1, \ldots, X_n$ of size n and let $n \to \infty$ and compare the six non-parametric methods via exact Bahadur slope (EBS).
The producers that we will used in this paper are Fisher, logistic, sum of P-values, inverse normal, Tippett's method and maximum of p-values. These producers are based on p-values of the individual statistics T_i, and reject H_0 if

$$\Psi_{Fisher} = -2\sum_{i=1}^{n} \ln(P_i) > \chi^2_{2n,\alpha} \tag{8}$$

$$\Psi_{logistic} = -\sum_{i=1}^{n} \ln\left(\frac{P_i}{1-P_i}\right) > b_\alpha \tag{9}$$

$$\Psi_{Normal} = -\sum_{i=1}^{n} \Phi^{-1}(P_i) > \sqrt{n}\Phi^{-1}(1-\alpha) \tag{10}$$

$$\Psi_{Sum} = -\sum_{i=1}^{n} P_i > C_\alpha \tag{11}$$

$$\Psi_{Max} = -max\; P_i < \alpha^{\frac{1}{n}} \tag{12}$$

$$\Psi_T = -min\; P_i < 1 - (1-\alpha)^{\frac{1}{n}}\,. \tag{13}$$

where Φ is the DF of standard normal distribution.

6 Derivation of the EBS

In this section we will study testing problem (7). We will compare the six methods Fisher, logistic, sum of P-values,the inverse normal, Tippett's method and maximum of p-values using EBS.
Let $X_1, \ldots, X_n$ be IID with probability density function (2) and we want to test (7). The i-th P-value is given by

$$P_i(t_i) = P_{H_0}(P_i \geq t_i) = 1 - F_{H_0}(t_i) = 1 - \Psi_{Logistic}\left(\ln(t_i)\right),\; \forall i = 1, \ldots n. \tag{14}$$

The next sections give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), and sum of p-values (C_S), maximum of p-values (C_{max}) and Tippett's method (C_T) methods.

6.1 The EBS For Fisher Combination Method

Theorem 11. *The exact Bahadur's slope (EBS's) of the Fisher's procedure is*

$$C_F(\xi) = b_F(\xi) - 2\ln(b_F(\xi)) + 2\ln(2) - 2$$

$$= -2 + \frac{2\xi e^{\xi}}{e^{\xi}-1} - 2\ln\left(\frac{2\xi e^{\xi}}{e^{\xi}-1}\right) + 2\ln 2$$

Proof. By Equation (8) and by Bahadur's Theorem (2) (1) it follows that

$$b_F(\xi) = -2\,\mathbb{E}^{H_1}\ln\left[P_n(X_n)\right].$$

Now by Equations (14) and (2) it follows that

$$b_F(\xi) = -2\int_0^{\infty}\frac{1}{x}\ln\left[1-\Psi_{Logistic}\left(\ln(x)\right)\right]\varphi_{Logistic}\left(\ln(x)-\xi\right)\,dx = \frac{2\xi e^{\xi}}{e^{\xi}-1}.$$

Now under H_0, and by using Large deviation Theorem (1), it follows that $M_F(z) = \mathbb{E}_F\left(e^{-2\ln(x)z}\right) = \int_0^1 e^{-2\ln(x)z}\,dx$. Set $t = -\ln(x)$ implies $dt = -e^t dx$. It then follows that $M_F(z) = \int_0^1 e^{-x(1-2t)}\,dx = (1-2z)^{-1}$, $Z < 1/2$. Then, $m_F(t) = \inf_{z>0} e^{-zt}(1-2z)^{-1} = \frac{t}{2}e^{1-t/2}$, now by Bahadur's Theorem (2) (2), we complete the proof, that is $C_F(\xi) = -2\ln(m_F(b_F(\xi))) = -2\ln\left(\frac{b_F(\xi)}{2}e^{1-\frac{b_F(\xi)}{2}}\right) = b_F(\xi) - 2\ln(b_F(\xi)) + 2\ln(2) - 2$. □

6.2 The EBS For Logistic Combination Method

Theorem 12. *The exact Bahadur's slope (EBS's) of the logistic procedure is* $C_L(\xi) = -2\ln(m(b_L(\xi)))$, *where* $m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\ csc(\pi z)$ *and* $b_L(\xi) = \xi$.

Proof. Similar to the proof of Theorem (11). □

6.3 The EBS For Sum Of p-values Combination Method

Theorem 13. *The exact Bahadur's slope (EBS's) of the Sum of p-values procedure is* $C_S(\xi) = -2\ln(m(b_S(\xi)))$, *where* $m_S(t) = \inf_{z>0} e^{-zt}\frac{1-e^{-z}}{z}$ *and* $b_S(\xi) = \frac{e^{\xi}-\xi e^{\xi}-1}{\left(e^{\xi}-1\right)^2}$.

Proof. By Equation (11) and by Bahadur's Theorem (2) (1) it follows that $b_S(\xi) = -\,\mathbb{E}^{H_1}\left[P_n(X_n)\right]$. Now by Equation (14), clearly, $\frac{T_S}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_S(\xi) = -\,\mathbb{E}^{H_1}(X)$. So, by Equation (2) it follows that $b_S(\xi) = -\int_0^{\infty}\frac{1}{x}\left[1-\Psi_{Logistic}\left(\ln(x)\right)\right]\varphi_{Logistic}\left(\ln(x)-\xi\right)\,dx = \frac{e^{\xi}-\xi e^{\xi}-1}{\left(e^{\xi}-1\right)^2}$. Now, by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt}M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -x \sim U(-1,0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem (2) we complete the proof, we conclude that $C_S(\xi) = -2\ln(m_S(b_S(\xi)))$. □

6.4 The EBS For Inverse Normal Combination Method

Theorem 14. *The exact Bahadur's slope (EBS's) of the inverse normal procedure is* $C_N(\xi) = -2\ln(m(b_N(\xi))) = b_N^2(\xi) = e^{2\xi}\,\mathbb{E}^2_{N(0,1)}\left\{\frac{v}{M_{Bin(2,\Phi(v))}(\xi)}\right\}$.

Proof. By Equation (10) and by Bahadur's Theorem (2) (1) it follows that $b_N(\xi) = -\,\mathbb{E}^{H_1}\left[P_n(X_n)\right]$. Now by Equation (14), clearly, $\frac{T_N}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_N(\xi) = -\,\mathbb{E}^{H_1}\,\Phi^{-1}(X)$. So, by Equation (2) it follows that

$$b_N(\xi) = -\int_0^\infty \frac{1}{x}\Phi^{-1}(1-\Psi_{Logistic}\,(\ln(x)))\varphi_{Logistic}\,(\ln(x)-\xi)\;dx = -\int_0^\infty \Phi^{-1}\left(\frac{1}{1+x}\right)\frac{e^\xi}{(x+e^\xi)^2}\,dx.$$

On substituting $v = \Phi^{-1}\left(\frac{1}{1+x}\right)$, implies $\frac{1}{1+x} = \Phi(v)$, $\frac{dx}{dv} = \frac{\phi(v)}{\Phi^2(v)}$. It follows that $b_N(\xi) = \int_{\Re} \frac{-e^\xi v\phi(v)}{(1-\Phi(v)+\Phi(v)e^\xi)^2}\,dv = -e^\xi\,\mathbb{E}_{N(0,1)}\left\{\frac{v}{M_{Binomial(2,\Phi(v))}(\xi)}\right\}$. Now under H_0, and by Large deviation Theorem (1), it follows that $M_N(z) = \mathbb{E}_N\left(e^{-z\Phi^{-1}(X)}\right) = \int_0^1 e^{-z\Phi^{-1}(X)}\,dx$. Set $w = -\Phi^{-1}(x)$ implies $x = 1-\Phi(w)$, then $dx = -\phi(w)dw$. It then follows that $M_N(z) = \int_{\Re} e^{wz}\phi(w)\,dw = M_{N(0,1)}(z) = e^{z^2/2}$. Then, $m_N(t) = \inf_{z>0} e^{-zt}e^{z^2/2} = e^{-t^2/2}$, now by Bahadur's Theorem (2) (2), we complete the proof, that is

$$C_N(\xi) = -2\ln(m_N(b_N(\xi))) = -2\ln(e^{-b_N^2(\xi)/2}) = b_N^2(\xi) = e^{2\xi}\,\mathbb{E}^2_{N(0,1)}\left\{\frac{v}{M_{Bin(2,\Phi(v))}(\xi)}\right\}.$$

□

6.5 The EBS For Maximum of p-values Method

Theorem 15. *The exact Bahadur's slope (EBS's) of the maximum of p-values is* $C_{max}(\xi) = \infty$.

Proof. By Theorem (5), $C_{max}(\xi) = -2\ln\left(ess.sup_\xi(u)\right)$ where $ess.sup_\xi(u) = Sup\left(u : f(u) > 0\right)$ w.p.1 under ξ, then $ess.sup_f(u) = 0$. It follows that $C_{max}(\xi) = \infty$. □

6.6 The EBS For Tippett's Method

Theorem 16. *The exact Bahadur's slope (EBS's) of the Tippett's method is* $C_T(\xi) = 0$.

Proof. By Theorem (6), Equation (2), it follows that $\lim_{t\to 0} t(\ln t)^2 \frac{\varphi_{Logistic}\,(\ln(t)-\xi)}{t}$ $= \lim_{t\to 0} t(\ln t)^2 \frac{e^\xi}{(t+e^\xi)^2}$. Clearly, by using L'Hoptial rule twice, $\lim_{t\to 0} t(\ln t)^2 = 0$ and $\lim_{t\to 0} \frac{e^\xi}{(t+e^\xi)^2}$ $= e^{-\xi}$ which implies $C_T(\xi) = 0$. □

7 Comparison of the EBSs

In this section, we will compare the EBSs that obtained in Section (6). We will find the limit of the ratio of the EBSs of any two methods under study when $\xi \to 0$ and $\xi \to \infty$.

7.1 The Limiting ratio of the EBS for different tests when $\xi \to 0$

Lemma 2. $\lim_{\xi\to 0} \frac{C_{\mathfrak{D}}(\xi)}{C_{max}(\xi)} = 0$, *where* $C_{\mathfrak{D}} \in \{C_F, C_L, C_S, C_N, C_T\}$.

Proof. It follows by Theorem (11) and Theorem (15) that $\lim_{\xi\to 0} \frac{2\xi e^{\xi}}{e^{\xi}-1} = 2$, it then follows that $\lim_{\xi\to 0} C_F(\xi) = -2 + \lim_{\xi\to 0} \frac{2\xi e^{\xi}}{e^{\xi}-1} - 2\lim_{\xi\to 0} \ln\left(\frac{2\xi e^{\xi}}{e^{\xi}-1}\right) + 2\ln 2 = 0$. Thus $e_B(T_F, T_{max}) \to 0$. For C_L, we use Theorem (12) then $\lim_{\xi\to 0} b_L(\xi) = \lim_{\xi\to 0} \xi = 0$, hence with Lemma (1)(1) we have that $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$ implies $C_L(\xi) \leq 2b_L(\xi)$. Clearly, $\lim_{\xi\to 0} C_L(\xi) \leq 2\lim_{\xi\to 0} b_L(\xi) = 0$. So $\lim_{\xi\to 0} C_L(\xi) = 0$. Thus $e_B(T_L, T_{max}) \to 0$. In the same way for $C_S(\xi)$, we use Theorem (13) to show that $b_S(\xi) \to -\frac{1}{2}$, hence with Lemma (1) (3) we have that $m_S(t) \geq -2t$ implies $C_S(\xi) \leq -2\ln(-2b_S(\xi))$. Clearly, $\lim_{\xi\to 0} C_S(\xi) \leq -2\ln(2) - 2\lim_{\xi\to 0} \ln(-b_S(\xi)) = -2\ln(2) - 2\ln\left(\frac{1}{2}\right) = 0$. So $\lim_{\xi\to 0} C_S(\xi) = 0$. Thus $e_B(T_S, T_{max}) \to 0$. Finally, by Theorem (14) it is easy to show that $\lim_{\xi\to 0} b_N(\xi) = \lim_{\xi\to 0} \int_{\Re} \frac{-e^{\xi} v\phi(v)}{\left(1-\Phi(v)+\Phi(v)e^{\xi}\right)^2} dv \to -\int_{\Re} v\phi(v) = \mathbb{E}_{N(0,1)} V = 0$. Clearly, $C_N(\xi)$ is converges to 0 as $\xi \to 0$. Thus $e_B(T_N, T_{max}) = e_B(T_T, T_N) \to 0$. □

Lemma 3. $\lim_{\xi\to 0} \frac{C_T(\xi)}{C_{\mathfrak{D}}(\xi)} = 0$, *where* $T_{\mathfrak{D}} \in \{T_F, T_L, T_S, T_N\}$.

Proof. Similar to the proof of the previous lemma. □

Lemma 4. $\lim_{\xi\to 0} \frac{C_S(\xi)}{C_F(\xi)} > 1$.

Proof. By Theorems (11), (13) and (3)(1) it follows that $b'_F(\xi) = \frac{2e^{\xi}(e^{\xi}-\xi-1)}{(e^{\xi}-1)^2}$ It follows that $b'_F(0) = 1$. Clearly that $b'_F(0)$ is finite. Also, in the same way $b'_S(\xi) = \frac{e^{\xi}(2+e^{\xi}(\xi-2)+\xi)}{(e^{\xi}-1)^3}$. It follows that $b'_S(0) = \frac{1}{6}$. Also $b'_S(0)$ is finite. Now under $H_0 : h_F(x) = -2\ln\left[1-\Psi_{Logistic}(\ln(x))\right] \sim \chi^2_2$ and $h_S(x) = -\left[1-\Psi_{Logistic}(\ln(x))\right] \sim U(-1,0)$, so $Var_{\xi=0}(h_F(x)) = 4$ and $Var_{\xi=0}(h_S(x)) = \frac{1}{12}$, also, $\frac{b'_S(0)}{b'_F(0)} = \frac{1}{6}$. By applying Theorem (3) we get $\lim_{\xi\to 0} \frac{C_S(\xi)}{C_F(\xi)} = e_B(T_S, T_F) = \frac{4}{3} > 1$. □

Lemma 5. $\lim_{\xi\to 0} \frac{C_L(\xi)}{C_F(\xi)} > 1$.

Proof. Similar to the proof of the previous lemma. □

Lemma 6. $\lim_{\xi\to 0} \frac{C_N(\xi)}{C_F(\xi)} > 1$.

Proof. By Theorems (11), (14), (3)(1) and Lemma (4) it follows that $b'_F(0) = 1$. Also, $b'_N(\xi) = \int_{\Re} \frac{e^{\xi} v\phi(v)\left(\left(1+e^{\xi}\right)\Phi(v)-1\right)}{\left(1-\Phi(v)+\Phi(v)e^{\xi}\right)^3} dv$. Now $b'_N(0) = \int_{\Re} v\phi(v)\left(2\Phi(v)-1\right) dv. = \int_{\Re} -v\phi(v)\, dv + \int_{\Re} 2v\phi(v)\Phi(v)\, dv = 0 + \int_{\Re} \phi^2(v)\, dv = \frac{1}{\sqrt{\pi}} < \infty$. Now under $H_0 : h_F(x) = -2\ln\left(1-\Psi_{Logistic}(\ln(x))\right) \sim$

χ^2_2 and $h_N(x) = -\Phi^{-1}\left(1 - \Psi_{Logistic}\left(\ln(x)\right)\right) \sim N(0,1)$, so $Var_{\xi=0}(h_F(x)) = 4$ and $Var_{\xi=0}(h_N(x)) = 1$, also, $\frac{b'_N(0)}{b'_F(0)} = \frac{1}{\sqrt{\pi}}$. By applying Theorem (3) we get numerically that $\lim\limits_{\xi\to 0} \frac{C_N(\xi)}{C_F(\xi)} = e_B\left(T_N, T_F\right) = \frac{4}{\pi} > 1$. □

Lemma 7. $\lim\limits_{\xi\to 0} \frac{C_N(\xi)}{C_L(\xi)} > 1$.

Proof. By lemmas (5) and (6) it follows that $b'_N(0) = \frac{1}{\sqrt{\pi}}$, and $b'_L(0) = 1$. Now under H_0 : $h_L(x) \sim Logitic(0,1)$, so $Var_{\xi=0}(h_L(x)) = \frac{\pi^2}{3}$, also, $\frac{b'_N(0)}{b'_L(0)} = \frac{1}{\sqrt{\pi}}$. By applying Theorem (3) we get numerically that $\lim\limits_{\xi\to 0} \frac{C_N(\xi)}{C_L(\xi)} = e_B\left(T_N, T_L\right) = \frac{\pi}{3} > 1$. □

7.2 The Limiting ratio of the EBS for different tests when $\xi \to \infty$

Lemma 8. $\lim\limits_{\xi\to\infty} \frac{C_L(\xi)}{C_F(\xi)} = 1$.

Proof. It follows by Theorem (11), Theorem (12) and lemma (1)(1) that $C_L(\xi) \le 2b_L(\xi)$. So $\lim\limits_{\xi\to\infty} \frac{C_L(\xi)}{C_F(\xi)} \le \lim\limits_{\xi\to\infty} \frac{2b_L(\xi)}{b_F(\xi) - 2\ln(b_F(\xi)) + 2\ln(2) - 2}$. Clearly, it is sufficient to obtain $\lim\limits_{\xi\to\infty} \frac{2b_L(\xi)}{b_F(\xi)}$. Now, $\lim\limits_{\xi\to\infty} \frac{2b_L(\xi)}{b_F(\xi)} \le \lim\limits_{\xi\to\infty} \frac{2\xi}{\frac{2\xi e^{\xi}}{e^{\xi}-1}} = \lim\limits_{\xi\to\infty} e^{-\xi}\left(e^{\xi} - 1\right) = 1$. Therefore, $\lim\limits_{\xi\to\infty} \frac{C_L(\xi)}{C_F(\xi)} \le 1$. In the same way, by Theorem (8)(2) we have $C_L(\xi) \ge -2 + 2b_L(\xi) - 2\ln\left(b_L(\xi)\right)$. So $\lim\limits_{\xi\to\infty} \frac{C_L(\xi)}{C_F(\xi)} \ge \lim\limits_{\xi\to\infty} \frac{-2 + 2b_L(\xi) - 2\ln\left(b_L(\xi)\right)}{b_F(\xi) - 2\ln(b_F(\xi)) + 2\ln(2) - 2}$. Clearly, it is sufficient to obtain $\lim\limits_{\xi\to\infty} \frac{2b_L(\xi)}{b_F(\xi)}$. Now, $\lim\limits_{\xi\to\infty} \frac{2b_L(\xi)}{b_F(\xi)} \ge \lim\limits_{\xi\to\infty} \frac{2\xi}{\frac{2\xi e^{\xi}}{e^{\xi}-1}} = \lim\limits_{\xi\to\infty} e^{-\xi}\left(e^{\xi} - 1\right) = 1$. Therefore, $\lim\limits_{\xi\to\infty} \frac{C_L(\xi)}{C_F(\xi)} \ge 1$. By pinching theorem, we have $\lim\limits_{\xi\to\infty} \frac{C_L(\xi)}{C_F(\xi)} = 1$. □

Lemma 9. $\lim\limits_{\xi\to\infty} \frac{C_S(\xi)}{C_F(\xi)} = 1$.

Proof. It follows by Theorem (11), Theorem (13) and lemma (1)(3) that $C_S(\xi) \le -2\ln(2) - 2\ln(-b_S(\xi))$. So $\lim\limits_{\xi\to\infty} \frac{C_S(\xi)}{C_F(\xi)} \le \lim\limits_{\xi\to\infty} \frac{-2\ln(2) - 2\ln(-b_S(\xi))}{b_F(\xi) - 2\ln(b_F(\xi)) + 2\ln(2) - 2}$. Clearly, it is sufficient to obtain $\lim\limits_{\xi\to\infty} \frac{-2\ln(-b_S(\xi))}{b_F(\xi)}$. Now, by L'Hopitals rule, we have $\lim\limits_{\xi\to\infty} \frac{-2\ln(-b_S(\xi))}{b_F(\xi)}$ $= \lim\limits_{\xi\to\infty} \frac{e^{-\xi}\left(1 - e^{-\xi}\right)\left(-2\ln\left(e^{\xi} - 1\right) + \ln\left(1 + e^{\xi}(\xi - 1)\right)\right)}{\xi} \le 1$. Therefore, $\lim\limits_{\xi\to\infty} \frac{C_S(\xi)}{C_F(\xi)} \le 1$. In the same way, lemma (1)(3) that $C_S(\xi) \ge -2 - 2\ln(-b_S(\xi))$. Therefore, $\lim\limits_{\xi\to\infty} \frac{C_S(\xi)}{C_F(\xi)} \ge 1$. By pinching theorem, we have $\lim\limits_{\xi\to\infty} \frac{C_S(\xi)}{C_F(\xi)} = 1$. □

Lemma 10. $\lim_{\xi\to\infty} \frac{C_F(\xi)}{C_N(\xi)} = 1.$

Proof. By Theorems (11), (14), (9), (10) and L'Hopitals rule, we have $\lim_{\xi\to\infty} \frac{C_F(\xi)}{C_N(\xi)} = \lim_{\xi\to\infty} \frac{b_F(\xi) - 2\ln(b_F(\xi)) + 2\ln(2) - 2}{e^{2\xi}\,\mathbb{E}^2_{N(0,1)}\left\{\frac{v}{M_{Bin(2,\Phi(v))}(\xi)}\right\}}$. Clearly, it is sufficient to obtain $\lim_{\xi\to\infty} \frac{b_F(\xi)}{e^{2\xi}\,\mathbb{E}^2_{N(0,1)}\left\{\frac{v}{M_{Bin(2,\Phi(v))}(\xi)}\right\}} = 1$. Which implies that $\lim_{\xi\to\infty} \frac{C_F(\xi)}{C_N(\xi)} = 1$. Thus $\lim_{\xi\to\infty} \frac{C_S(\xi)}{C_N(\xi)} = 1$. □

Lemma 11. $\lim_{\xi\to\infty} \frac{C_S(\xi)}{C_N(\xi)} = 1.$

Proof. By using the limiting properties and Lemma (9) and Lemma (10). □

Lemma 12. $\lim_{\xi\to\infty} \frac{C_L(\xi)}{C_N(\xi)} = 1.$

Proof. By using the limiting properties and Lemma (8) and Lemma (10). □

8 Comparison of the EBS for the six methods

By the results of Section (7.1) the best procedure, which has a higher EBS, is the maximum of p-values since it has the highest limit as $\xi \to 0$, then the sum of p-values, the inverse normal, logistic, Fisher and Tippett's procedure, respectively. Whereas, from the results of Section (7.2) for large values of ξ the sum of p-values, the inverse normal, logistic and Fisher methods remain the same, since they have the same limit as ξ approaches to infinity. For the other values of ξ, the EBS's will be compared numerically. These results are summarized in Table 1.

Table 1: EBS for the log-logistic distribution

ξ	$C_F(\xi)$	$C_L(\xi)$	$C_S(\xi)$	$C_N(\xi)$
0.01	0.00002500	0.00003039	0.00003333	0.00003183
0.05	0.00062498	0.00075972	0.00083326	0.00079574
0.1	0.00249965	0.00303824	0.00333222	0.00318254
0.25	0.01561146	0.01896148	0.02079004	0.01987249
0.5	0.06228418	0.07542218	0.08264598	0.07922904
1	0.24660312	0.29526990	0.32266708	0.31283918
1.5	0.54575562	0.64284946	0.69863471	0.68933628
2	0.94894929	1.09683459	1.18167736	1.19177037
2.5	1.44324502	1.63627862	1.74306019	1.80030286
3	2.01501124	2.24326519	2.36054060	2.49468754
3.5	2.65111225	2.90354503	3.02076352	3.25616867
4	3.33969927	3.60616108	3.71666039	4.06860537
4.5	4.07060702	4.34278934	4.44345147	4.91888808
5	4.83543923	5.10710076	5.19700214	5.79684761
5.5	5.62745248	5.89424348	5.97364911	6.69487783
6	6.44133634	6.70044401	6.77020560	7.60744953

9 Numerical values of the EBS for the log-logistic distribution using SRS

Through Table 1, we have the EBS using the SRS for the log-logistic distribution under study. We observe that, when ξ is small, the best method is the sum of p-values followed by the inverse normal, logistic and Fisher methods, respectively, and when ξ becomes big the inverse normal method becomes the best than the sum of p-values followed by logistic and Fisher methods, respectively.

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