学位論文

Profile analysis and tests for mean vectors with two-step monotone missing data

（2-ステップ単調欠測データのもとでのプロフィール分析と平均ベクトルに対する検定）

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Chapter 1

Introduction

In this study, we consider the profile analysis and the tests for the mean vectors when the observations have two-step monotone missing data. We often encounter the problem of missing data in many practical situations. In this study, we assume that missing pattern is two-step monotone missing data. In Chapter 2, we discuss profile analysis for two sample and multi-sample problem with two-step monotone missing data. This is the content of Onozawa et al. [12]. Profile analysis is known as statistical method when we are interested in comparing the profiles of several groups. There are three hypotheses known as parallelism hypothesis, level hypothesis and flatness hypothesis. In normal population, the profile analysis for two sample problem have been discussed by using Hotelling's T^2-type statistic (see, e.g., Morrison [10]). And Srivastava [17] gave a profile analysis of several groups based on the likelihood ratio. For the assumption of nonnormality, Okamoto et al. [11] discussed profile analysis in elliptical populations. Further Maruyama [9] obtained asymptotic expansions of the null distributions of some test statistics for general distributions. On the other hand, when the missing observations are of the monotone-type, the test for the equality of means have been discussed by many authors. In one sample problem, Chang and Richards [3] considered the T^2-type test statistic for the mean vector with two-step monotone missing data. Further, Anderson and Olkin [2] obtained the maximum likelihood estimators (MLEs) of mean vector and covariance matrix for two-step monotone missing data, and Kanda and Fujikoshi [6] discussed the distribution of these MLEs and expanded for K-step monotone missing data. Further, Seko et al. [15] discussed the T^2-type test statistic and likelihood ratio test statistic using linear interpolation. In two sample problem, by the same way as Anderson and Olkin [2], the MLEs have been obtained (see, e.g., Shutoh et al. [16]). Linear interpolation approximation for the null distribution of the Hotelling's T^2-type test statistic and the likelihood ratio test statistic with two-step monotone missing data were reported by Seko et al. [14].

The organization of Chapter 2 is as follows. In Section 2.1, we consider a profile analysis for complete data. In Section 2.2, we derive the MLEs of $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ when the missing observations are of the two-step monotone-type. In Section 2.3, we give the T^2-type test statistics for profile analysis with two-step monotone missing data. In Section 2.4, we give the likelihood ratio test statistic for the parallelism hypothesis with two-step monotone missing data. In Section 2.5, we perform a Monte Carlo simulation to investigate the accuracy for the null distributions of these statistics. Finally, in Section 2.6, we conclude Chapter 2.

Chapter 3 is the content of Onozawa et al. [13]. In Chapter 3, we consider the tests for a single mean vector and two mean vectors with two-step monotone missing data. In one sample problem, Jinadasa and Tracy [5] obtained a closed form expression for the MLEs for the mean vector and the covariance matrix in the case of K-step monotone missing data. Krishnamoorthy and Pannala [7] considered the likelihood ratio test statistic with K-step monotone missing data. Meanwhile, it is difficult to obtain an exact covariance matrix for the MLE of the mean vector since the T^2-type test statistic is complicated with missing data. Therefore, Krishnamoorthy and Pannala [8] provided a simplified T^2-type test statistic and approximated the upper percentiles using the F distribution. They adjusted the freedom of F distribution using the expected value and the variance of the test statistic. They discussed K-step monotone missing data although they only provided MLEs with up to three-step monotone missing data. Yagi and Seo [18] provided the approximate upper percentiles of the simplified T^2-type test statistic using linear interpolation with the notation used by Jinadasa and Tracy [5]. Yagi et al. [19] provided an asymptotic expansion for the null distribution of the simplified T^2 test statistic and improved the χ^2 approximation for the statistic in the case of two and K-step monotone missing data. In a two sample problem, Yu et al. [21] reported a simplified T^2-type test statistic with K-step monotone missing data using Krishnamoorthy and Pannala [8]'s idea. Moreover, Yagi and Seo [18] also considered a two sample problem with the pivotal quantities similar to the Hotelling's T^2-type statistic in Yu, et al. [21]. Yagi et al. [20] extended the result of the one sample problem in their recent work [19]. In Chapter 3, we propose test statistics for one and two sample problems with two-step monotone missing data under consideration by Krishnamoorthy and Pannala [7] and Yu et al. [21]. In a one sample problem, the part of the likelihood ratio in Krishnamoorthy and Pannala [7] is used as the test statistics we have. The two sample problem is considered in the same manner as that of the one sample problem. Further using the asymptotic expansion of the distribution, we present the transformed test statistics based on the Bartlett adjustment. The detailed explanation of the Bartlett adjustment is found in e.g., Anderson [1]. The improved transformations for the general test statistic are discussed by Fujikoshi [4]. The organization of Chapter 3 is as follows. In Section 3.1, we first present the assumption and notations. In Section 3.2, we derive an asymptotic expansion of the null distribution of the new test statistics. In Section 3.3, we consider transformed test statistics for the proposed test statistics and approximated upper percentiles of the distribution. In Section 3.4, we describe the Monte Carlo simulation that was implemented to investigate the accuracy for the null distributions of these statistics. Finally, in Section 3.5, we conclude this study.

Chapter 2

Tests for Profile Analysis Based on Two-step Monotone Missing Data

2.1 Profile analysis for complete data

In this section, we consider the test statistics when the data have non-missing observations. Let the p-dimensional random vector $\boldsymbol{x}_j^{(\ell)}$ be independently distributed as $N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma})$ $(j = 1, \ldots, N_1^{(\ell)},\ \ell = 1, 2)$, where $\boldsymbol{\mu}^{(\ell)} = (\mu_1^{(\ell)}, \ldots, \mu_p^{(\ell)})'$. Let the ℓ-th sample mean vector, the ℓ-th sample covariance matrix, and the pooled sample covariance matrix be

$$\begin{aligned} \overline{\boldsymbol{x}}^{(\ell)} &= \frac{1}{N_1^{(\ell)}} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{x}_j^{(\ell)},\ \boldsymbol{S}_\ell = \frac{1}{N_1^{(\ell)} - 1} \sum_{j=1}^{N_1^{(\ell)}} (\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})(\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})', \\ \boldsymbol{S} &= \frac{(N_1^{(1)} - 1)\boldsymbol{S}_1 + (N_1^{(2)} - 1)\boldsymbol{S}_2}{N_1^{(1)} + N_1^{(2)} - 2}, \end{aligned}$$

respectively. When carrying out a profile analysis for two samples, we first consider the parallelism hypothesis that is expressed as

$$H_{P_2} : \boldsymbol{C}\boldsymbol{\mu}^{(1)} = \boldsymbol{C}\boldsymbol{\mu}^{(2)} \quad \text{vs.} \quad A_{P_2} : \text{not } H_{P_2},$$

where $\boldsymbol{C}$ is a $(p-1) \times p$ matrix of rank $p-1$ such that $\boldsymbol{C}\boldsymbol{1}_p = \boldsymbol{0}$ and $\boldsymbol{1}_p$ is a p-vector of ones. The test statistic for testing hypothesis H_{P_2} can be written as

$$T_{Pc}^2 = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})'\boldsymbol{C}' \left\{ \frac{N_1^{(1)} + N_1^{(2)}}{N_1^{(1)} N_1^{(2)}} (\boldsymbol{C}\boldsymbol{S}\boldsymbol{C}') \right\}^{-1} \boldsymbol{C}(\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)}).$$

In normal populations,

$$T^2_{Pc} \sim \frac{(N_1^{(1)} + N_1^{(2)} - 2)(p-1)}{N_1^{(1)} + N_1^{(2)} - p} F_{p-1, N_1^{(1)} + N_1^{(2)} - p}.$$

If the parallelism hypothesis is true, we test the level hypothesis or the flatness hypothesis. The level hypothesis is expressed as

$$H_{L_2} | H_{P_2} : \mathbf{1}_p' \boldsymbol{\mu}^{(1)} = \mathbf{1}_p' \boldsymbol{\mu}^{(2)} \ \text{ vs. } \ A_{L_2} : \text{not } H_{L_2} | H_{P_2}.$$

The test statistic for testing hypothesis H_{L_2} can be written as

$$T^2_{Lc} = (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})' \mathbf{1}_p \left\{ \frac{N_1^{(1)} + N_1^{(2)}}{N_1^{(1)} N_1^{(2)}} (\mathbf{1}_p' \boldsymbol{S} \mathbf{1}_p) \right\}^{-1} \mathbf{1}_p' (\overline{\boldsymbol{x}}^{(1)} - \overline{\boldsymbol{x}}^{(2)})$$

In normal populations,

$$T^2_{Lc} \sim F_{1, N_1^{(1)} + N_1^{(2)} - 2}.$$

Further, the flatness hypothesis is expressed as

$$H_{F_2} | H_{P_2} : \boldsymbol{C}(\boldsymbol{\mu}^{(1)} + \boldsymbol{\mu}^{(2)}) = \mathbf{0} \ \text{ vs. } \ A_{F_2} : \text{not } H_{F_2} | H_{P_2}$$

The test statistic for testing hypothesis H_{F_2} can be written as

$$T^2_{Fc} = \overline{\boldsymbol{x}}_{12}' \boldsymbol{C}' \left\{ \frac{1}{N_1^{(1)} + N_1^{(2)}} \boldsymbol{C} \boldsymbol{S} \boldsymbol{C}' \right\}^{-1} \boldsymbol{C} \overline{\boldsymbol{x}}_{12},$$

where

$$\overline{\boldsymbol{x}}_{12} = \frac{N_1^{(1)}}{N_1^{(1)} + N_1^{(2)}} \overline{\boldsymbol{x}}^{(1)} + \frac{N_1^{(2)}}{N_1^{(1)} + N_1^{(2)}} \overline{\boldsymbol{x}}^{(2)}.$$

In normal populations,

$$T^2_{Fc} \sim \frac{(N_1^{(1)} + N_1^{(2)} - 2)(p-1)}{N_1^{(1)} + N_1^{(2)} - p} F_{p-1, N_1^{(1)} + N_1^{(2)} - p}.$$

In addition, we consider a parallelism hypothesis of several groups when the data have non-missing observations. Let $\boldsymbol{x}_1^{(\ell)}, \ldots, \boldsymbol{x}_{N_1^{(\ell)}}^{(\ell)}$ be $N_1^{(\ell)}$ independent observations from $N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma})$ $(\ell = 1, \ldots, k)$. Then we consider the primarily testing the parallelism hypothesis as follows:

$$H_{P_k} : \boldsymbol{C}\boldsymbol{\mu}^{(1)} = \cdots = \boldsymbol{C}\boldsymbol{\mu}^{(k)} \ \text{ vs. } \ A_{P_k} : \text{not } H_{P_k}.$$

The MLEs of $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ under A_{P_k} are

$$\overline{\boldsymbol{x}}^{(\ell)} = \frac{1}{N_1^{(\ell)}} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{x}_j^{(\ell)}, \quad \widehat{\boldsymbol{\Sigma}}_c = \frac{1}{N_1} \sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} (\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})(\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}}^{(\ell)})',$$

respectively, where $N_1 = \sum_{\ell=1}^{k} N_1^{(\ell)}$. In contrast, the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ under H_{P_k} are

$$\overline{\boldsymbol{x}} = \frac{1}{N_1} \sum_{\ell=1}^{k} \sum_{j=1}^{N^{(\ell)}} \boldsymbol{x}_j^{(\ell)}, \quad \widetilde{\boldsymbol{\Sigma}}_c = \frac{1}{N_1} \sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} (\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}})(\boldsymbol{x}_j^{(\ell)} - \overline{\boldsymbol{x}})',$$

respectively. For complete data, using these MLEs, we can construct the following likelihood ratio:

$$\Lambda_c = \frac{|\boldsymbol{C}\widehat{\boldsymbol{\Sigma}}_c\boldsymbol{C}'|^{\frac{1}{2}N_1}}{|\boldsymbol{C}\widetilde{\boldsymbol{\Sigma}}_c\boldsymbol{C}'|^{\frac{1}{2}N_1}}.$$

The likelihood ratio test statistic, $-2\log\Lambda_c$, is asymptotically distributed as a χ^2 distribution with $(p-1)(k-1)$ degrees of freedom as $N_1^{(\ell)}$s tend to infinity (see Srivastava [17]). Hence, we reject H_{P_k} when $-2\log\Lambda_c > \chi^2_{(p-1)(k-1),\alpha}$, where $\chi^2_{(p-1)(k-1),\alpha}$ is the upper 100α percentile of a χ^2 distribution with $(p-1)(k-1)$ degrees of freedom. However, convergence to the asymptotic χ^2 distribution can be improved by considering an asymptotic expansion for the likelihood ratio statistic and deriving the modified likelihood ratio statistic as $-2\rho_{c_1}\log\Lambda_c$, where

$$\rho_{c_1} = 1 - \frac{1}{2N_1}(p+k+1).$$

2.2 MLEs

We consider the case when the missing observations are of the two-step monotone-type. Observations $\{x_{\ell j}^{(\ell)}\}$ can be written in the following form:

$$\begin{pmatrix} x_{11}^{(\ell)} & \cdots & x_{1p_1}^{(\ell)} & x_{1,p_1+1}^{(\ell)} & \cdots & x_{1p}^{(\ell)} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{N_1^{(\ell)}1}^{(\ell)} & \cdots & x_{N_1^{(\ell)}p_1}^{(\ell)} & x_{N_1^{(\ell)},p_1+1}^{(\ell)} & \cdots & x_{N_1^{(\ell)}p}^{(\ell)} \\ x_{N_1^{(\ell)}+1,1}^{(\ell)} & \cdots & x_{N_1^{(\ell)}+1,p_1}^{(\ell)} & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{N^{(\ell)}1}^{(\ell)} & \cdots & x_{N^{(\ell)}p_1}^{(\ell)} & * & \cdots & * \end{pmatrix},$$

where $*$ denotes missing component. Let $\boldsymbol{x}_j^{(\ell)} \equiv (\boldsymbol{x}_{1j}^{(\ell)'}, \boldsymbol{x}_{2j}^{(\ell)'})'$ $(j = 1, \ldots, N_1^{(\ell)}$, $\ell = 1, \ldots, k)$ be a p-dimensional observation vector from the ℓ-th group with complete data. Let $\boldsymbol{x}_{1j}^{(\ell)}$ $(j = N_1^{(\ell)}+1,\ldots,N^{(\ell)})$ be p_1-dimensional vectors based on $N_2^{(\ell)}$ $(= N^{(\ell)} - N_1^{(\ell)})$ observations. Now, we assume the distribution of observation vectors:

$$\begin{aligned}&\boldsymbol{x}_j^{(\ell)} \sim N_p(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma})\ (j = 1,\ldots,N_1^{(\ell)},\ \ell = 1,\ldots,k),\\ &\boldsymbol{x}_{1j}^{(\ell)} \sim N_{p_1}(\boldsymbol{\mu}_1^{(\ell)}, \boldsymbol{\Sigma}_{11})\ (j = N_1^{(\ell)}+1,\ldots,N^{(\ell)},\ \ell = 1,\ldots,k),\end{aligned}$$

respectively, where

$$\boldsymbol{\mu}^{(\ell)} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \end{pmatrix},\ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

and $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ are partitioned according to the blocks of the data set. Therefore, $\boldsymbol{\mu}_j^{(\ell)}$ $(j = 1, 2)$ is a p_j-dimensional vector and $\boldsymbol{\Sigma}_{jm}$ $(j, m = 1, 2)$ is a $p_j \times p_m$ matrix.

We give some notations for the sample mean vectors. Let $\overline{\boldsymbol{x}}_{1T}^{(\ell)}$ be the sample mean vector of $\boldsymbol{x}_{11}^{(\ell)},\ldots,\boldsymbol{x}_{1N^{(\ell)}}^{(\ell)}$. Let $(\overline{\boldsymbol{x}}_{1F}^{(\ell)\prime},\ \overline{\boldsymbol{x}}_{2F}^{(\ell)\prime})'$ be the sample mean vector of $\boldsymbol{x}_1^{(\ell)},\ldots,\boldsymbol{x}_{N_1^{(\ell)}}^{(\ell)}$, where $\overline{\boldsymbol{x}}_{jF}^{(\ell)\prime} : p_j \times 1$ $(j = 1, 2)$. That is,

$$\overline{\boldsymbol{x}}_{1T}^{(\ell)} = \frac{1}{N^{(\ell)}}\sum_{j=1}^{N^{(\ell)}} \boldsymbol{x}_{1j}^{(\ell)},\quad \overline{\boldsymbol{x}}_{1F}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{x}_{1j}^{(\ell)},\quad \overline{\boldsymbol{x}}_{2F}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{x}_{2j}^{(\ell)}.$$

Since the MLEs based on the complete data case cannot be used, we have to estimate $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ under two-step monotone missing data. Let $\widehat{\boldsymbol{\mu}}^{(\ell)}$ and $\widehat{\boldsymbol{\Sigma}}$ be the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. These have the same patterns of partition as $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$. The likelihood function is

$$\begin{aligned}&L(\boldsymbol{\mu}^{(\ell)}, \boldsymbol{\Sigma})\\ &= \prod_{\ell=1}^{k}\left[\prod_{j=1}^{N_1^{(\ell)}} \frac{1}{(2\pi)^{\frac{p}{2}} \mid \boldsymbol{\Sigma} \mid^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_j^{(\ell)} - \boldsymbol{\mu}^{(\ell)})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}_j^{(\ell)} - \boldsymbol{\mu}^{(\ell)})\right\}\right.\\ &\left.\times \prod_{j=N_1^{(\ell)}+1}^{N^{(\ell)}} \frac{1}{(2\pi)^{\frac{p_1}{2}} \mid \boldsymbol{\Sigma}_{11} \mid^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_{1j}^{(\ell)} - \boldsymbol{\mu}_1^{(\ell)})'\boldsymbol{\Sigma}_{11}^{-1}(\boldsymbol{x}_{1j}^{(\ell)} - \boldsymbol{\mu}_1^{(\ell)})\right\}\right].\end{aligned}$$

Let $\boldsymbol{A}$ be a $p \times p$ transformation matrix:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{O} \\ -\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{I}_{p_2} \end{pmatrix}.$$

Then we have

$$\boldsymbol{A}\boldsymbol{x}_j^{(\ell)} = \begin{pmatrix} \boldsymbol{x}_{1j}^{(\ell)} \\ \boldsymbol{x}_{2j}^{(\ell)} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{x}_{1j}^{(\ell)} \end{pmatrix} \sim N_p(\boldsymbol{A}\boldsymbol{\mu}^{(\ell)}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}'),$$

where the mean vector and the covariance matrix of transformed observation vectors are

$$\boldsymbol{A}\boldsymbol{\mu}^{(\ell)} = \boldsymbol{\eta}^{(\ell)} = \begin{pmatrix} \boldsymbol{\eta}_1^{(\ell)} \\ \boldsymbol{\eta}_2^{(\ell)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1^{(\ell)} \end{pmatrix},$$

$$\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Sigma}_{22\cdot 1} \end{pmatrix}$$

and $\boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$. It should be noted that $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ have one-to-one correspondence with $\boldsymbol{\eta}^{(\ell)}$ and $\boldsymbol{\Psi}$, where

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{\Sigma}_{22\cdot 1} \end{pmatrix}.$$

For parameters $\boldsymbol{\eta}^{(1)}, \ldots, \boldsymbol{\eta}^{(k)}$ and $\boldsymbol{\Psi}$, the likelihood function is

$$\begin{aligned} & L(\boldsymbol{\eta}^{(1)}, \ldots, \boldsymbol{\eta}^{(k)}, \Psi) \\ & = \text{Const.} \times \mid \boldsymbol{\Psi}_{11} \mid^{-\frac{1}{2}N} \mid \boldsymbol{\Psi}_{22} \mid^{-\frac{1}{2}N_1} \\ & \times \exp\left\{ -\frac{1}{2} \sum_{\ell=1}^{k} \sum_{j=1}^{N^{(\ell)}} (\boldsymbol{x}_{1j}^{(\ell)} - \boldsymbol{\eta}_1^{(\ell)})' \boldsymbol{\Psi}_{11}^{-1} (\boldsymbol{x}_{1j}^{(\ell)} - \boldsymbol{\eta}_1^{(\ell)}) \right\} \\ & \times \exp\left\{ -\frac{1}{2} \sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} (\boldsymbol{x}_{2j}^{(\ell)} - \boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}^{(\ell)} - \boldsymbol{\eta}_2^{(\ell)})' \boldsymbol{\Psi}_{22}^{-1} (\boldsymbol{x}_{2j}^{(\ell)} - \boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}^{(\ell)} - \boldsymbol{\eta}_2^{(\ell)}) \right\}, \end{aligned}$$

where $N = \sum_{\ell=1}^{k} N^{(\ell)}$. Differentiating the log likelihood function, we get that

$$\begin{aligned} \widehat{\boldsymbol{\eta}}_1^{(\ell)} &= \overline{\boldsymbol{x}}_{1T}^{(\ell)}, \\ \widehat{\boldsymbol{\eta}}_2^{(\ell)} &= \overline{\boldsymbol{x}}_{2F}^{(\ell)} - \widehat{\boldsymbol{\Psi}}_{21}\overline{\boldsymbol{x}}_{1F}^{(\ell)}, \end{aligned}$$

and that

$$\begin{aligned} \widehat{\boldsymbol{\Psi}}_{11} &= \frac{1}{N} \sum_{\ell=1}^{k} \sum_{j=1}^{N^{(\ell)}} (\boldsymbol{x}_{1j}^{(\ell)} - \overline{\boldsymbol{x}}_{1T}^{(\ell)})(\boldsymbol{x}_{1j}^{(\ell)} - \overline{\boldsymbol{x}}_{1T}^{(\ell)})', \\ \widehat{\boldsymbol{\Psi}}_{21} &= \left[\sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{z}_{2j}^{(\ell)} \boldsymbol{z}_{1j}^{\prime(\ell)} \right] \left[\sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{z}_{1j}^{(\ell)} \boldsymbol{z}_{1j}^{\prime(\ell)} \right]^{-1}, \\ \widehat{\boldsymbol{\Psi}}_{22} &= \frac{1}{N_1} \left\{ \sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{z}_{2j}^{(\ell)} \boldsymbol{z}_{2j}^{\prime(\ell)} \right. \\ & \left. - \left[\sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{z}_{2j}^{(\ell)} \boldsymbol{z}_{1j}^{\prime(\ell)} \right] \left[\sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{z}_{1j}^{(\ell)} \boldsymbol{z}_{1j}^{\prime(\ell)} \right]^{-1} \left[\sum_{\ell=1}^{k} \sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{z}_{1j}^{(\ell)} \boldsymbol{z}_{2j}^{\prime(\ell)} \right] \right\}, \\ \boldsymbol{z}_{1j}^{(\ell)} &= \boldsymbol{x}_{1j}^{(\ell)} - \overline{\boldsymbol{x}}_{1F}^{(\ell)}, \quad \boldsymbol{z}_{2j}^{(\ell)} = \boldsymbol{x}_{2j}^{(\ell)} - \overline{\boldsymbol{x}}_{2F}^{(\ell)}. \end{aligned}$$

We thus obtain the MLEs of $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ in general:

$$\widehat{\boldsymbol{\mu}}^{(\ell)} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1^{(\ell)} \\ \widehat{\boldsymbol{\mu}}_2^{(\ell)} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_{1T}^{(\ell)} \\ \overline{\boldsymbol{x}}_{2F}^{(\ell)} - \widehat{\boldsymbol{\Psi}}_{21}(\overline{\boldsymbol{x}}_{1F}^{(\ell)} - \overline{\boldsymbol{x}}_{1T}^{(\ell)}) \end{pmatrix},$$

$$\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_{11} & \widehat{\boldsymbol{\Sigma}}_{12} \\ \widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\boldsymbol{\Sigma}}_{22} \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{\Psi}}_{11} & \widehat{\boldsymbol{\Psi}}_{11}\widehat{\boldsymbol{\Psi}}_{12} \\ \widehat{\boldsymbol{\Psi}}_{21}\widehat{\boldsymbol{\Psi}}_{11} & \widehat{\boldsymbol{\Psi}}_{22} + \widehat{\boldsymbol{\Psi}}_{21}\widehat{\boldsymbol{\Psi}}_{11}\widehat{\boldsymbol{\Psi}}_{12} \end{pmatrix}.$$

2.3 Two sample profile analysis with two-step monotone missing data

By using the MLEs which are given in Section 2.2, we give T^2-type statistics. In this section, let $k = 2$. T^2-type statistic under H_{P_2} can be written as

$$T_{Pm}^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\boldsymbol{C}'\{\boldsymbol{C}\widehat{\boldsymbol{\Xi}}\boldsymbol{C}'\}^{-1}\boldsymbol{C}(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}),$$

where $\widehat{\boldsymbol{\Xi}}$ is the MLE of $\boldsymbol{\Xi} = \{\mathrm{Cov}[\widehat{\boldsymbol{\mu}}^{(1)}] + \mathrm{Cov}[\widehat{\boldsymbol{\mu}}^{(2)}]\}$,

$$\widehat{\boldsymbol{\Xi}} = \begin{pmatrix} \dfrac{N}{N^{(1)}N^{(2)}}\widehat{\boldsymbol{\Sigma}}_{11} & \dfrac{N}{N^{(1)}N^{(2)}}\widehat{\boldsymbol{\Sigma}}_{12} \\ \dfrac{N}{N^{(1)}N^{(2)}}\widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2^{(1)}] + \widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2^{(2)}] \end{pmatrix}$$

and

$$\widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2^{(1)}] + \widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2^{(2)}] = \sum_{\ell=1}^{2}\left\{\frac{1}{N_1^{(\ell)}}\left(\widehat{\boldsymbol{\Sigma}}_{22} - \frac{N_2^{(\ell)}}{N^{(\ell)}}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12}\right) + \frac{N_2^{(\ell)}p_1}{N^{(\ell)}N_1^{(\ell)}(N_1^{(\ell)} - p_1 - 2)}\widehat{\boldsymbol{\Sigma}}_{22\cdot 1}\right\}.$$

For details of the MLEs, see Kanda and Fujikoshi [6]. T_{Pm}^2 is asymptotically distributed as a χ^2 distribution with $p-1$ degrees of freedom when $N_1^{(\ell)}$s are large.

The T^2-type statistic under H_{L_2} can be written as

$$T_{Lm}^2 = (\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)})'\boldsymbol{1}_p\{\boldsymbol{1}_p'\widehat{\boldsymbol{\Xi}}\boldsymbol{1}_p\}^{-1}\boldsymbol{1}_p'(\widehat{\boldsymbol{\mu}}^{(1)} - \widehat{\boldsymbol{\mu}}^{(2)}).$$

T_{Lm}^2 is asymptotically distributed as a χ^2 distribution with 1 degree of freedom when $N_1^{(\ell)}$s are large.

When we consider the case under H_{F_2}, we can join the two samples and regard it as a one sample problem. The T^2-type statistic under H_{F_2} can be written as

$$T_{Fm}^2 = (\boldsymbol{C}\widehat{\boldsymbol{\mu}})'\{\boldsymbol{C}\widehat{\mathrm{Cov}}(\widehat{\boldsymbol{\mu}})\boldsymbol{C}'\}^{-1}(\boldsymbol{C}\widehat{\boldsymbol{\mu}}),$$

where

$$\widehat{\boldsymbol{\mu}} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1 \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_{1T} \\ \overline{\boldsymbol{x}}_{2F} - \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\overline{\boldsymbol{x}}_{1F} - \overline{\boldsymbol{x}}_{1T}) \end{pmatrix},$$

$$\widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}] = \begin{pmatrix} \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{11} & \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{12} \\ \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2] \end{pmatrix},$$

$$\widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2] = \frac{1}{N_1}\left(\widehat{\boldsymbol{\Sigma}}_{22} - \frac{N_2}{N}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12}\right) + \frac{N_2 p_1}{NN_1(N_1 - p_1 - 2)}\widehat{\boldsymbol{\Sigma}}_{22\cdot 1}$$

and

$$\overline{\boldsymbol{x}}_{1T} = \frac{1}{N}\sum_{\ell=1}^{2}\sum_{j=1}^{N^{(\ell)}}\boldsymbol{x}_{1j}^{(\ell)}, \quad \overline{\boldsymbol{x}}_{1F} = \frac{1}{N_1}\sum_{\ell=1}^{2}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{x}_{1j}^{(\ell)}, \quad \overline{\boldsymbol{x}}_{2F} = \frac{1}{N_1}\sum_{\ell=1}^{2}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{x}_{2j}^{(\ell)},$$

$$N_2 = \sum_{\ell=1}^{k} N_2^{(\ell)}.$$

These estimators are extended for the MLEs obtained by Kanda and Fujikoshi [6]. T_{Fm}^2 is asymptotically distributed as a χ^2 distribution with $p-1$ degrees of freedom when $N_1^{(\ell)}$s are large.

However, the upper percentiles of the χ^2 distribution are not a good approximation for the T^2-type statistic when the sample size is small, and it is difficult to obtain the exact upper percentiles of these statistics when the data have missing observations. Hence, we give the approximate upper percentiles based on the idea of Seko et al. [15] where it is assumed that the true upper percentiles exist between $T^2_{p-1,N_1-p,\alpha}$ and $T^2_{p-1,N-p,\alpha}$. This idea is based on linear interpolation. $F^*_{1,\alpha}$ can give the approximate upper percentiles of T_{Pm} and T_{Fm}.

$$F^*_{1,\alpha} = T^2_{p-1,N_1-p,\alpha} - \frac{Np - N_2p_2}{Np}\left(T^2_{p-1,N_1-p,\alpha} - T^2_{p-1,N-p,\alpha}\right),$$

where

$$T^2_{p-1,N-p,\alpha} = \frac{(N-2)(p-1)}{N-p}F_{p-1,N-p,\alpha},$$

$$T^2_{p-1,N_1-p,\alpha} = \frac{(N_1-2)(p-1)}{N_1-p}F_{p-1,N_1-p,\alpha}$$

and $F_{p,q,\alpha}$ is the upper 100α percentile of F distribution with p and q degrees of freedom. Further, $F^*_{2,\alpha}$ can give the approximate upper percentiles of T_{Lm}.

$$F^*_{2,\alpha} = T^2_{1,N_1-2,\alpha} - \frac{Np - N_2p_2}{Np}(T^2_{1,N_1-2,\alpha} - T^2_{1,N-2,\alpha}),$$

where

$$T^2_{1,N-2,\alpha} = F_{1,N-2,\alpha},$$
$$T^2_{1,N_1-2,\alpha} = F_{1,N_1-2,\alpha}.$$

2.4 Parallelism hypothesis for several groups with two-step monotone missing data

We have two-step monotone missing data when $k \geq 3$ as in Section 2.2. First, we transform the observation vectors using $\boldsymbol{C}$. Then we have

$$\boldsymbol{u}_j^{(\ell)} = \boldsymbol{C}\boldsymbol{x}_j^{(\ell)} \sim N_{p-1}(\boldsymbol{\theta}^{(\ell)}, \boldsymbol{\Gamma}),$$
$$\boldsymbol{u}_{1j}^{(\ell)} = \boldsymbol{C}_1\boldsymbol{x}_{1j}^{(\ell)} \sim N_{p_1-1}(\boldsymbol{\theta}_1^{(\ell)}, \boldsymbol{\Gamma}_{11}),$$

where $\boldsymbol{\theta}^{(\ell)} = \boldsymbol{C}\boldsymbol{\mu}^{(\ell)}$, $\boldsymbol{\Gamma} = \boldsymbol{C}\boldsymbol{\Sigma}\boldsymbol{C}'$ and $\boldsymbol{C}_1$ is a $(p_1-1)\times p_1$ matrix of rank (p_1-1) such that $\boldsymbol{C}_1\boldsymbol{1}_{p_1} = \boldsymbol{0}$ and $\boldsymbol{1}_{p_1}$ is a p_1-vector of ones.

$$\boldsymbol{\theta}^{(\ell)} = \begin{pmatrix} \boldsymbol{\theta}_1^{(\ell)} \\ \boldsymbol{\theta}_2^{(\ell)} \end{pmatrix}, \ \boldsymbol{\Gamma} = \begin{pmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} \\ \boldsymbol{\Gamma}_{21} & \boldsymbol{\Gamma}_{22} \end{pmatrix}.$$

$\boldsymbol{\theta}^{(\ell)}$ and $\boldsymbol{\Gamma}$ are partitioned according to blocks of the data set. It should be noted that $\boldsymbol{\theta}_1 : (p_1-1)\times 1$, $\boldsymbol{\theta}_2 : p_2 \times 1$, $\boldsymbol{\Gamma}_{11} : (p_1-1)\times(p_1-1)$, $\boldsymbol{\Gamma}_{12} = \boldsymbol{\Gamma}_{21}' : (p_1-1)\times p_2$, $\boldsymbol{\Gamma}_{22} : p_2 \times p_2$. To construct a likelihood ratio, we obtain the MLEs of $\boldsymbol{\theta}^{(\ell)}$ and $\boldsymbol{\Gamma}$ in general and under the hypothesis H_{P_k}. can be obtained in the same way as earlier:

$$\widehat{\boldsymbol{\theta}}^{(\ell)} = \begin{pmatrix} \widehat{\boldsymbol{\theta}}_1^{(\ell)} \\ \widehat{\boldsymbol{\theta}}_2^{(\ell)} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{u}}_{1T}^{(\ell)} \\ \overline{\boldsymbol{u}}_{2F}^{(\ell)} - \widehat{\boldsymbol{\Phi}}_{21}(\overline{\boldsymbol{u}}_{1F}^{(\ell)} - \overline{\boldsymbol{u}}_{1T}^{(\ell)}) \end{pmatrix},$$
$$\widehat{\boldsymbol{\Gamma}} = \begin{pmatrix} \widehat{\boldsymbol{\Gamma}}_{11} & \widehat{\boldsymbol{\Gamma}}_{12} \\ \widehat{\boldsymbol{\Gamma}}_{21} & \widehat{\boldsymbol{\Gamma}}_{22} \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{\Phi}}_{11} & \widehat{\boldsymbol{\Phi}}_{11}\widehat{\boldsymbol{\Phi}}_{12} \\ \widehat{\boldsymbol{\Phi}}_{21}\widehat{\boldsymbol{\Phi}}_{11} & \widehat{\boldsymbol{\Phi}}_{22} + \widehat{\boldsymbol{\Phi}}_{21}\widehat{\boldsymbol{\Phi}}_{11}\widehat{\boldsymbol{\Phi}}_{12} \end{pmatrix},$$

where

$$\overline{\boldsymbol{u}}_{1T}^{(\ell)} = \frac{1}{N^{(\ell)}}\sum_{j=1}^{N^{(\ell)}} \boldsymbol{u}_{1j}^{(\ell)}, \ \overline{\boldsymbol{u}}_{1F}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{u}_{1j}^{(\ell)}, \ \overline{\boldsymbol{u}}_{2F}^{(\ell)} = \frac{1}{N_1^{(\ell)}}\sum_{j=1}^{N_1^{(\ell)}} \boldsymbol{u}_{2j}^{(\ell)}$$

and

$$\widehat{\boldsymbol{\Phi}}_{11} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N^{(\ell)}}(\boldsymbol{u}_{1j}^{(\ell)} - \overline{\boldsymbol{u}}_{1T}^{(\ell)})(\boldsymbol{u}_{1j}^{(\ell)} - \overline{\boldsymbol{u}}_{1T}^{(\ell)})',$$

$$\widehat{\boldsymbol{\Phi}}_{21}= \left[\sum_{\ell=1}^{k}\sum_{J=1}^{N_1^{(\ell)}}\boldsymbol{y}_{2j}^{(\ell)}\boldsymbol{y}_{1j}'^{(\ell)}\right]\left[\sum_{\ell=1}^{k}\sum_{J=1}^{N_1^{(\ell)}}\boldsymbol{y}_{1j}^{(\ell)}\boldsymbol{y}_{1j}'^{(\ell)}\right]^{-1},$$

$$\widehat{\boldsymbol{\Phi}}_{22}=\frac{1}{N_1}\left\{\sum_{\ell=1}^{k}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{y}_{2j}^{(\ell)}\boldsymbol{y}_{2j}'^{(\ell)}\right.$$

$$\left. - \left[\sum_{\ell=1}^{k}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{y}_{2j}^{(\ell)}\boldsymbol{y}_{1j}'^{(\ell)}\right]\left[\sum_{\ell=1}^{k}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{y}_{1j}^{(\ell)}\boldsymbol{y}_{1j}'^{(\ell)}\right]^{-1}\left[\sum_{\ell=1}^{k}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{y}_{1j}^{(\ell)}\boldsymbol{y}_{2j}'^{(\ell)}\right]\right\},$$

$$\boldsymbol{y}_{1j}^{(\ell)} = \boldsymbol{u}_{1j}^{(\ell)} - \overline{\boldsymbol{u}}_{1F}^{(\ell)},\ \boldsymbol{y}_{2j}^{(\ell)} = \boldsymbol{x}_{2j}^{(\ell)} - \overline{\boldsymbol{u}}_{2F}^{(\ell)}.$$

Similarly, the MLEs of $\boldsymbol{\theta}^{(\ell)}$ and $\boldsymbol{\Gamma}$ under H_{P_k} are as follows:

$$\widetilde{\boldsymbol{\theta}}^{(\ell)} = \begin{pmatrix} \widetilde{\boldsymbol{\theta}}_1^{(\ell)} \\ \widetilde{\boldsymbol{\theta}}_2^{(\ell)} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{u}}_{1F}^{(\ell)} \\ \overline{\boldsymbol{u}}_{2F}^{(\ell)} - \widetilde{\boldsymbol{\Phi}}_{21}(\overline{\boldsymbol{u}}_{1F}^{(\ell)} - \overline{\boldsymbol{u}}_{1T}^{(\ell)}) \end{pmatrix},$$

$$\widetilde{\boldsymbol{\Gamma}} = \begin{pmatrix} \widetilde{\boldsymbol{\Gamma}}_{11} & \widetilde{\boldsymbol{\Gamma}}_{12} \\ \widetilde{\boldsymbol{\Gamma}}_{21} & \widetilde{\boldsymbol{\Gamma}}_{22} \end{pmatrix} = \begin{pmatrix} \widetilde{\boldsymbol{\Phi}}_{11} & \widetilde{\boldsymbol{\Phi}}_{11}\widetilde{\boldsymbol{\Phi}}_{12} \\ \widetilde{\boldsymbol{\Phi}}_{21}\widetilde{\boldsymbol{\Phi}}_{11} & \widetilde{\boldsymbol{\Phi}}_{22} + \widetilde{\boldsymbol{\Phi}}_{21}\widetilde{\boldsymbol{\Phi}}_{11}\widetilde{\boldsymbol{\Phi}}_{12} \end{pmatrix},$$

where

$$\overline{\boldsymbol{u}}_{1T} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N^{(\ell)}}\boldsymbol{u}_{1j}^{(\ell)},\ \overline{\boldsymbol{u}}_{1F} = \frac{1}{N_1}\sum_{\ell=1}^{k}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{u}_{1j}^{(\ell)},\ \overline{\boldsymbol{u}}_{2F} = \frac{1}{N_1}\sum_{\ell=1}^{k}\sum_{j=1}^{N_1^{(\ell)}}\boldsymbol{u}_{2j}^{(\ell)}$$

and

$$\widetilde{\boldsymbol{\Phi}}_{11} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N^{(\ell)}}(\boldsymbol{u}_{1j}^{(\ell)} - \overline{\boldsymbol{u}}_{1T})(\boldsymbol{u}_{1j}^{(\ell)} - \overline{\boldsymbol{u}}_{1T})',$$

$$\widetilde{\boldsymbol{\Phi}}_{21} = \left[\sum_{\ell=1}^{k}\sum_{j=1}^{N^{(\ell)}}\boldsymbol{w}_{2j}^{(\ell)}\boldsymbol{w}_{1j}'^{(\ell)}\right]\left[\sum_{\ell=1}^{k}\sum_{j=1}^{N^{(\ell)}}\boldsymbol{w}_{1j}^{(\ell)}\boldsymbol{w}_{1j}'^{(\ell)}\right]^{-1},$$

$$\widetilde{\boldsymbol{\Phi}}_{22} = \frac{1}{N_1}\sum_{\ell=1}^{k}\left\{\sum_{j=1}^{N^{(\ell)}}\boldsymbol{w}_{2j}^{(\ell)}\boldsymbol{w}_{2j}'^{(\ell)}\right.$$

$$\left. - \left[\sum_{j=1}^{N^{(\ell)}}\boldsymbol{w}_{2j}^{(\ell)}\boldsymbol{w}_{1j}'^{(\ell)}\right]\left[\sum_{j=1}^{N^{(\ell)}}\boldsymbol{w}_{1j}^{(\ell)}\boldsymbol{w}_{1j}'^{(\ell)}\right]^{-1}\left[\sum_{j=1}^{N^{(\ell)}}\boldsymbol{w}_{1j}^{(\ell)}\boldsymbol{w}_{2j}'^{(\ell)}\right]\right\},$$

$$\boldsymbol{w}_{1j}^{(\ell)} = \boldsymbol{u}_{1j}^{(\ell)} - \overline{\boldsymbol{u}}_{1F},\ \boldsymbol{w}_{2j}^{(\ell)} = \boldsymbol{u}_{2j}^{(\ell)} - \overline{\boldsymbol{u}}_{2F}.$$

We have a likelihood ratio for the parallelism hypothesis as follows:

$$\Lambda_m = \prod_{\ell=1}^{k} \frac{L(\widetilde{\boldsymbol{\theta}}_1^{(\ell)}, \widetilde{\boldsymbol{\theta}}_2^{(\ell)}, \widetilde{\boldsymbol{\Gamma}})}{L(\widehat{\boldsymbol{\theta}}_1^{(\ell)}, \widehat{\boldsymbol{\theta}}_2^{(\ell)}, \widehat{\boldsymbol{\Gamma}})} = \frac{|\widehat{\boldsymbol{\Gamma}}^*|^{\frac{1}{2}N_1}}{|\widetilde{\boldsymbol{\Gamma}}^*|^{\frac{1}{2}N_1}} \times \frac{|\widehat{\boldsymbol{\Gamma}}_{11}|^{\frac{1}{2}N_2}}{|\widetilde{\boldsymbol{\Gamma}}_{11}|^{\frac{1}{2}N_2}},$$

where

$$\widehat{\boldsymbol{\Gamma}}^* = \begin{pmatrix} \widehat{\boldsymbol{\Gamma}}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \widehat{\boldsymbol{\Gamma}}_{22} - \widehat{\boldsymbol{\Gamma}}_{21}\widehat{\boldsymbol{\Gamma}}_{11}^{-1}\widehat{\boldsymbol{\Gamma}}_{12} \end{pmatrix}, \quad \widetilde{\boldsymbol{\Gamma}}^* = \begin{pmatrix} \widetilde{\boldsymbol{\Gamma}}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \widetilde{\boldsymbol{\Gamma}}_{22} - \widetilde{\boldsymbol{\Gamma}}_{21}\widetilde{\boldsymbol{\Gamma}}_{11}^{-1}\widetilde{\boldsymbol{\Gamma}}_{12} \end{pmatrix}.$$

Then the likelihood ratio statistic $-2\log\Lambda_m$ is asymptotically distributed as a χ^2 distribution with $(p-1)(k-1)$ degrees of freedom as $N_1^{(\ell)}$s tend to infinity. Hence, we reject H_{P_k} when $-2\log\Lambda_m > \chi^2_{(p-1)(k-1),\alpha}$. However, it is difficult to obtain the modified likelihood ratio statistic directly when the data have missing observations. As such, much like in the two-sample case, we use ρ_m that improves convergence to a χ^2 distribution, and put it into the test statistic:

$$\rho_m = \left\{ \frac{1}{\rho_{c_1}} - \frac{Np - N_2p_2}{Np}\left(\frac{1}{\rho_{c_1}} - \frac{1}{\rho_{c_2}}\right) \right\}^{-1},$$

where

$$\rho_{c_1} = 1 - \frac{1}{2N_1}(p+k+1),$$
$$\rho_{c_2} = 1 - \frac{1}{2N}(p+k+1)$$

and $\rho_{c_1}, \rho_{c_2} \neq 0$. Then we reject H_{P_k} when $-2\rho_m \log\Lambda_m > \chi^2_{(p-1)(k-1),\alpha}$.

2.5 Simulation studies

In this section, we examine the accuracy of the approximations of the proposed test statistics. The Monte Carlo simulation for the upper percentiles of T^2-type statistics and the likelihood ratio test statistic is implemented for selected values of parameters. The settings of the parameters α, p $(= p_1 + p_2)$ and M $(= M_1 + M_2)$ for the simulation are as follows:

$$\begin{aligned} k &= 2, 3, 6, \\ \alpha &= 0.05, \\ (p_1, p_2) &= (2,2),\ (3,1),\ (6,2),\ (2,6), \\ (M_1, M_2) &= (10,10),\ (20,10),\ (50,10),\ (100,10), \\ &\quad (10,100),\ (20,100),\ (50,100),\ (100,100), \end{aligned}$$

where $M_j = N_j^{(\ell)}$ $(j = 1, 2)$. Further, we compare their type I error rates. As a numerical experiment, we carry out 1,000,000 replications. It should be noted that our results may be applicable to the case where the sample size differs for each population. However, for simplicity, we show the results under the same sample size. Tables 1-3 list the percentiles of T^2-type statistics and the values of F_1^* and F_2^*. They also list the results for the comparison of the type I error rates under the T^2-type statistics when the null hypothesis is rejected, using F_1^*, F_2^* and a χ^2 distribution. The T^2-type statistics are closer to the χ^2 distribution when the sample size is large. Comparing the type I error rates, we have that $F_{1,\alpha}^*$ and $F_{2,\alpha}^*$ seem to be closer to 0.05 than the percentiles of the χ^2 distribution especially when the sample size is small. The value tends to be closer to 0.05 under the level hypothesis than under the parallelism hypothesis and the flatness hypothesis.

Tables 4 and 5 which are compare $-2\log\Lambda_m$ and $-2\rho_m\log\Lambda_m$, list the percentiles and type I error rates using a χ^2 distribution. $-2\log\Lambda_m$ and $-2\rho_m\log\Lambda_m$ are close to the χ^2 distribution when the sample size is large. Furthermore, $-2\rho_m\log\Lambda_m$ is closer to the χ^2 distribution than $-2\log\Lambda_m$.

Table 1 : Upper percentiles and type I error rates of T^2_{Pm} and F^*_1 values.

						percentile		type I error rate	
p	p_1	p_2	M	M_1	M_2	T^2_{Pm}	F^*_1	F^*_1	χ^2
4	2	2	20	10	10	9.671	9.540	0.052	0.089
$\chi^2_{3,0.05} = 7.815$			30	20	10	8.750	8.684	0.051	0.071
			60	50	10	8.212	8.194	0.050	0.059
			110	100	10	8.001	8.014	0.050	0.054
			110	10	100	9.176	9.339	0.047	0.078
			120	20	100	7.996	8.446	0.050	0.064
			150	50	100	8.075	8.061	0.050	0.056
			200	100	100	7.974	7.950	0.051	0.054
	3	1	20	10	10	9.198	9.308	0.048	0.080
			30	20	10	8.664	8.644	0.050	0.069
			60	50	10	8.182	8.191	0.050	0.058
			110	100	10	8.020	8.013	0.050	0.055
			110	10	100	8.261	8.676	0.042	0.060
			120	20	100	8.137	8.221	0.048	0.057
			150	50	100	7.987	8.010	0.050	0.054
			200	100	100	7.953	7.936	0.050	0.053
8	6	2	20	10	10	18.184	20.645	0.030	0.120
$\chi^2_{7,0.05} = 14.067$			30	20	10	17.200	17.288	0.049	0.108
			60	50	10	15.465	15.444	0.050	0.076
			110	100	10	14.787	14.779	0.050	0.063
			110	10	100	14.195	18.371	0.014	0.052
			120	20	100	15.011	15.655	0.041	0.067
			150	50	100	14.774	14.685	0.049	0.061
			200	100	100	14.498	14.499	0.050	0.058
	2	6	20	10	10	26.607	23.487	0.073	0.251
			30	20	10	18.428	17.640	0.060	0.131
			60	50	10	15.624	15.470	0.052	0.078
			110	100	10	14.811	14.783	0.050	0.063
			110	10	100	25.615	25.559	0.050	0.234
			120	20	100	17.715	17.534	0.052	0.117
			150	50	100	15.306	15.160	0.052	0.072
			200	100	100	14.695	14.600	0.052	0.061

Table 2 : Upper percentiles and type I error rates of T^2_{Lm} and F^*_2 values.

						percentile		type I error rate	
p	p_1	p_2	M	M_1	M_2	T^2_{Lm}	F^*_2	F^*_2	χ^2
4	2	2	20	10	10	4.171	4.177	0.050	0.059
$\chi^2_{1,0.05} = 3.841$			30	20	10	4.037	4.022	0.050	0.056
			60	50	10	3.924	3.923	0.050	0.052
			110	100	10	3.880	3.885	0.050	0.051
			110	10	100	4.208	4.125	0.052	0.060
			120	20	100	4.026	3.971	0.052	0.055
			150	50	100	3.911	3.895	0.051	0.052
			200	100	100	3.879	3.871	0.050	0.051
	3	1	20	10	10	4.050	4.138	0.048	0.056
			30	20	10	4.003	4.014	0.050	0.055
			60	50	10	3.928	3.922	0.050	0.052
			110	100	10	3.885	3.885	0.050	0.051
			110	10	100	3.985	4.005	0.049	0.054
			120	20	100	3.939	3.926	0.050	0.053
			150	50	100	3.884	3.884	0.050	0.051
			200	100	100	3.871	3.868	0.050	0.051
8	6	2	20	10	10	3.415	4.138	0.032	0.039
$\chi^2_{1,0.05} = 3.841$			30	20	10	3.979	4.014	0.049	0.054
			60	50	10	3.918	3.922	0.050	0.052
			110	100	10	3.887	3.885	0.050	0.051
			110	10	100	2.845	4.005	0.023	0.026
			120	20	100	3.850	3.926	0.048	0.050
			150	50	100	3.877	3.884	0.050	0.051
			200	100	100	3.870	3.868	0.050	0.051
	2	6	20	10	10	4.233	4.217	0.050	0.061
			30	20	10	4.067	4.030	0.051	0.057
			60	50	10	3.923	3.924	0.050	0.052
			110	100	10	3.890	3.885	0.050	0.051
			110	10	100	4.261	4.245	0.050	0.061
			120	20	100	4.061	4.017	0.051	0.056
			150	50	100	3.919	3.905	0.050	0.052
			200	100	100	3.894	3.874	0.051	0.052

Table 3 : Upper percentiles and type I error rates of T^2_{Fm} and F^*_1 values.

						percentile		type I error rate	
p	p_1	p_2	M	M_1	M_2	T^2_{Fm}	F^*_1	F^*_1	χ^2
4	2	2	20	10	10	10.699	9.540	0.069	0.112
$\chi^2_{3,0.05} = 7.815$			30	20	10	9.072	8.684	0.057	0.078
			60	50	10	8.301	8.194	0.052	0.061
			110	100	10	8.065	8.014	0.051	0.056
			110	10	100	10.672	9.339	0.072	0.112
			120	20	100	8.898	8.446	0.059	0.074
			150	50	100	8.212	8.061	0.053	0.059
			200	100	100	8.017	7.950	0.051	0.055
	3	1	20	10	10	10.071	9.308	0.100	0.100
			30	20	10	8.913	8.644	0.055	0.075
			60	50	10	8.294	8.191	0.052	0.061
			110	100	10	8.060	8.013	0.051	0.055
			110	10	100	9.414	8.676	0.085	0.085
			120	20	100	8.479	8.221	0.055	0.065
			150	50	100	8.106	8.010	0.052	0.057
			200	100	100	7.980	7.936	0.051	0.054
8	6	2	20	10	10	24.303	20.645	0.084	0.230
$\chi^2_{7,0.05} = 14.067$			30	20	10	18.111	17.288	0.061	0.127
			60	50	10	15.663	15.444	0.053	0.080
			110	100	10	14.838	14.779	0.051	0.064
			110	10	100	21.011	18.371	0.077	0.168
			120	20	100	16.274	15.655	0.059	0.091
			150	50	100	14.982	14.774	0.053	0.067
			200	100	100	14.606	14.499	0.052	0.060
	2	6	20	10	10	30.222	23.487	0.103	0.314
			30	20	10	30.222	19.236	0.070	0.148
			60	50	10	15.834	15.470	0.055	0.083
			110	100	10	14.904	14.783	0.052	0.065
			110	10	100	30.757	25.559	0.086	0.324
			120	20	100	18.966	17.534	0.068	0.144
			150	50	100	15.630	15.160	0.057	0.079
			200	100	100	14.842	14.600	0.054	0.064

Table 4 : Upper percentiles and type I error rates using $-2\log\Lambda_m$ and $-2\rho_m\log\Lambda_m$ values for $k=3$.

						percentile		type I error rate	
p	p_1	p_2	M	M_1	M_2	LRT	modified LRT	LRT	modified LRT
4	2	2	20	10	10	14.314	13.108	0.086	0.060
$\chi^2_{6,0.05}=12.592$			30	20	10	13.437	12.789	0.066	0.054
			60	50	10	12.923	12.631	0.056	0.051
			110	100	10	12.771	12.615	0.053	0.050
			110	10	100	14.132	13.126	0.082	0.060
			120	20	100	13.306	12.840	0.064	0.055
			150	50	100	12.894	12.702	0.056	0.052
			200	100	100	12.726	12.620	0.053	0.051
	3	1	20	10	10	13.961	12.906	0.078	0.056
			30	20	10	13.287	12.671	0.064	0.051
			60	50	10	12.893	12.604	0.056	0.050
			110	100	10	12.757	12.602	0.053	0.050
			110	10	100	13.544	12.967	0.069	0.057
			120	20	100	13.051	12.747	0.059	0.053
			150	50	100	12.782	12.630	0.054	0.051
			200	100	100	12.718	12.623	0.052	0.051
8	6	2	20	10	10	28.011	24.822	0.128	0.067
$\chi^2_{14,0.05}=23.685$			30	20	10	25.836	24.039	0.084	0.055
			60	50	10	24.586	23.760	0.064	0.051
			110	100	10	24.166	23.726	0.057	0.051
			110	10	100	26.788	25.009	0.102	0.070
			120	20	100	25.089	24.204	0.071	0.057
			150	50	100	24.285	23.851	0.059	0.052
			200	100	100	24.033	23.762	0.055	0.051
	2	6	20	10	10	29.686	25.521	0.168	0.079
			30	20	10	26.312	24.333	0.094	0.059
			60	50	10	24.669	23.826	0.064	0.052
			110	100	10	24.201	23.758	0.057	0.051
			110	10	100	29.538	25.110	0.164	0.072
			120	20	100	26.219	24.371	0.092	0.060
			150	50	100	24.613	23.952	0.064	0.054
			200	100	100	24.165	23.832	0.057	0.052

Table 5 : Upper percentiles and type I error rates
using $-2\log\Lambda_m$ and $-2\rho_m \log\Lambda_m$values for $k = 6$.

						percentile		type I error rate	
p	p_1	p_2	M	M_1	M_2	LRT	modified LRT	LRT	modified LRT
4	2	2	20	10	10	27.213	25.642	0.085	0.059
$\chi^2_{15,0.05} = 24.996$			30	20	10	26.079	25.215	0.066	0.053
			60	50	10	25.462	25.066	0.057	0.051
			110	100	10	25.243	25.031	0.053	0.050
			110	10	100	27.213	25.642	0.085	0.059
			120	20	100	25.918	25.298	0.063	0.054
			150	50	100	25.395	25.135	0.055	0.052
			200	100	100	25.189	25.044	0.053	0.051
	3	1	20	10	10	26.759	25.372	0.077	0.055
			30	20	10	25.951	25.125	0.064	0.052
			60	50	10	25.407	25.016	0.056	0.050
			110	100	10	25.211	25.000	0.053	0.050
			110	10	100	26.158	25.410	0.067	0.056
			120	20	100	25.580	25.175	0.058	0.052
			150	50	100	25.301	25.094	0.054	0.051
			200	100	100	25.157	25.028	0.052	0.050
8	6	2	20	10	10	54.759	50.882	0.114	0.061
$\chi^2_{35,0.05} = 49.802$			30	20	10	52.432	50.154	0.080	0.053
			60	50	10	50.959	49.888	0.062	0.051
			110	100	10	50.382	49.808	0.056	0.050
			110	10	100	53.064	50.957	0.088	0.062
			120	20	100	51.423	50.305	0.068	0.055
			150	50	100	50.581	49.802	0.058	0.052
			200	100	100	50.248	49.895	0.054	0.051
	2	6	20	10	10	56.757	51.821	0.148	0.072
			30	20	10	53.077	50.585	0.088	0.058
			60	50	10	51.083	49.992	0.064	0.052
			110	100	10	50.431	49.854	0.056	0.051
			110	10	100	56.585	51.391	0.145	0.067
			120	20	100	52.916	50.608	0.086	0.058
			150	50	100	51.004	50.150	0.062	0.053
			200	100	100	50.427	49.993	0.056	0.052

2.6 Concluding Remarks

We discussed profile analysis when the observations have two-step monotone missing data. In Section 2.2, we first derived the MLEs of several groups. In Section 2.3, we constructed the T^2-type statistics under the three hypotheses for a two sample problem using the MLEs given in Section 2.2. We gave the likelihood ratio test statistic under the parallelism hypothesis for several groups in Section 2.4. Finally, we performed a Monte Carlo simulation for the type I error rates in Section 2.5. As a result, we confirmed that $F^*_{1,\alpha}$ and $F^*_{2,\alpha}$ are better approximations than the upper percentiles of a χ^2 distribution. We confirmed that both $-2\log\Lambda_m$ and $-2\rho_m\log\Lambda_m$ are closer to the χ^2 distribution as the sample size becomes large. We can also see that $-2\rho_m\log\Lambda_m$ is always closer to the χ^2 distribution than $-2\log\Lambda_m$ for any sample size. Therefore, we confirmed that convergence to the asymptotic χ^2 distribution is improved by inputting ρ_m into the likelihood ratio statistic $-2\log\Lambda_m$

Chapter 3

New test statistics for one and two mean vectors with two-step monotone missing data

3.1 Assumption and notations

Let us consider the ℓ-th data set $\boldsymbol{X}^{(\ell)}$ that has the same monotone pattern as that of the two-step monotone missing data ($*$: missing part $\ell = 1, 2$) :

$$\boldsymbol{X}^{(\ell)} = \begin{pmatrix} \boldsymbol{X}_{11}^{(\ell)} & \boldsymbol{X}_{12}^{(\ell)} \\ \boldsymbol{X}_{21}^{(\ell)} & * \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_{1(12)}^{(\ell)} \\ \boldsymbol{X}_{21}^{(\ell)} \quad * \end{pmatrix},$$

where $\boldsymbol{X}_{11}^{(\ell)}$ is a $n_1^{(\ell)} \times p_1$ block matrix, $\boldsymbol{X}_{12}^{(\ell)}$ is a $n_1^{(\ell)} \times p_2$ block matrix, $\boldsymbol{X}_{21}^{(\ell)}$ is a $n_2^{(\ell)} \times p_1$ block matrix, and $\boldsymbol{X}^{(\ell)}$ is independent and distributed as a multivariate normal distribution with a common covariance matrix. Next, we assume the distribution of the observation vectors in the following manner:

$$\begin{aligned} \mathrm{vec}(\boldsymbol{X}_{1(12)}^{(\ell)\prime}) &\sim N_{n_1^{(\ell)}p}(\mathrm{vec}(\boldsymbol{\mu}^{(\ell)}\mathbf{1}_{n_1^{(\ell)}}^{\prime}), \boldsymbol{I}_{n_1^{(\ell)}} \otimes \boldsymbol{\Sigma}), \\ \mathrm{vec}(\boldsymbol{X}_{21}^{(\ell)\prime}) &\sim N_{n_2^{(\ell)}p_1}(\mathrm{vec}(\boldsymbol{\mu}_1^{(\ell)}\mathbf{1}_{n_2^{(\ell)}}^{\prime}), \boldsymbol{I}_{n_2^{(\ell)}} \otimes \boldsymbol{\Sigma}_{11}), \end{aligned}$$

respectively, where

$$\boldsymbol{\mu}^{(\ell)} = \begin{pmatrix} \boldsymbol{\mu}_1^{(\ell)} \\ \boldsymbol{\mu}_2^{(\ell)} \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^{\prime} & \boldsymbol{\Sigma}_{22} \end{pmatrix},$$

and $\boldsymbol{\mu}^{(\ell)}$ and $\boldsymbol{\Sigma}$ are partitioned according to the blocks of the data set. Therefore, $\boldsymbol{\mu}_j^{(\ell)}$ $(j = 1, 2)$ is a p_j-dimensional vector and $\boldsymbol{\Sigma}_{jm}$ $(j, m = 1, 2)$ is a $p_j \times p_m$ matrix. Further, let $\boldsymbol{\mu}^{(\ell)}$ be the p dimensional mean vector of $\boldsymbol{X}^{(\ell)}$, where

$p = p_1 + p_2$. Let $\overline{\boldsymbol{x}}^{(\ell)}_{(12)1}$ be the sample mean vector, $\boldsymbol{S}^{(\ell)}_{(12)1}$ be the unbiased sample covariance matrix of $\boldsymbol{X}^{(\ell)}_{(12)1}$. Let $\overline{\boldsymbol{x}}^{(\ell)}_{1(12)}(= (\overline{\boldsymbol{x}}^{(\ell)'}_{11}, \overline{\boldsymbol{x}}^{(\ell)'}_{12})')$ be the sample mean vector, and $\boldsymbol{S}^{(\ell)}_{1(12)}$ be the unbiased sample covariance matrix of $\boldsymbol{X}^{(\ell)}_{1(12)}$, where

$$\begin{pmatrix} \boldsymbol{X}^{(\ell)}_{11} \\ \boldsymbol{X}^{(\ell)}_{21} \end{pmatrix} = \boldsymbol{X}^{(\ell)}_{(12)1}, \quad \boldsymbol{S}^{(\ell)}_{1(12)} = \begin{pmatrix} \boldsymbol{S}^{(\ell)}_{1(12),11} & \boldsymbol{S}^{(\ell)}_{1(12),12} \\ \boldsymbol{S}^{(\ell)'}_{1(12),12} & \boldsymbol{S}^{(\ell)}_{1(12),22} \end{pmatrix}.$$

3.2 Asymptotic expansion for the distribution of the test statistic

3.2.1 One Sample Problem

We first consider the one sample problem. Further, in this case, we define the notation of the dataset by omitting "(ℓ)" from superscript of the notation defined in the previous section, for example, $\boldsymbol{X}^{(\ell)} = \boldsymbol{X}$. Then, the simplified T^2-type statistic for the hypothesis is the following:

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0 \tag{3.1}$$

(without loss of generality, we can assume in (3.1) that $\boldsymbol{\mu}_0 = \boldsymbol{0}$) is given by

$$Q = Q_1 + Q_2, \tag{3.2}$$

where

$$Q_i = N_{3-i}\widehat{\boldsymbol{\eta}}_i'\widehat{\Delta}_{ii}^{-1}\widehat{\boldsymbol{\eta}}_i, \quad N_i = \sum_{j=1}^{i} n_j, \quad i = 1, 2,$$
$$\widehat{\boldsymbol{\eta}}_1 = \overline{\boldsymbol{x}}_{(12)1}, \quad \widehat{\boldsymbol{\eta}}_2 = \overline{\boldsymbol{x}}_{12} - \boldsymbol{S}_{1(12),21}\boldsymbol{S}^{-1}_{1(12),11}\overline{\boldsymbol{x}}_{11},$$
$$\widehat{\boldsymbol{\Delta}}_{11} = \frac{N_2 - 1}{N_2}\boldsymbol{S}_{(12)1}, \quad \widehat{\boldsymbol{\Delta}}_{22} = \frac{N_1 - 1}{N_1}\left[\boldsymbol{S}_{1(12),22} - \boldsymbol{S}_{1(12),21}\boldsymbol{S}^{-1}_{1(12),11}\boldsymbol{S}_{1(12),12}\right].$$

We note that this statistic was originally obtained by Krishnamoorthy and Pannala [8]. We also note that Q_1 and Q_2 are not exactly independent. We propose a test using R_2, which is independent of Q_1, instead of Q_2. We suggest

$$Q_M = Q_1 + R_2 \tag{3.3}$$

as a new test statistic, where

$$R_2 = \frac{Q_2}{1 + Q_{2d}}, \quad Q_{2d} = \frac{N_1}{N_1 - 1}\overline{\boldsymbol{x}}_{11}'\boldsymbol{S}^{-1}_{1(12),11}\overline{\boldsymbol{x}}_{11}.$$

The likelihood ratio statistic with two-step monotone missing data using R_2 is discussed by Krishnamoorthy and Pannala [7]. We can derive an asymptotic expansion of the test statistic more accurately because Q_1 and R_2 are exactly independent. Without loss of generality, we may assume that $\boldsymbol{\Sigma} = \boldsymbol{I} = \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{p_2} \end{pmatrix}$.

Initially, we consider a statistic expansion of Q_1. Let

$$\overline{\boldsymbol{x}}_{1(12)} = \frac{1}{\sqrt{N_2}}\boldsymbol{z}, \quad \boldsymbol{S}_{(12)1} = \boldsymbol{I}_{p_1} + \frac{1}{\sqrt{N_2 - 1}}\boldsymbol{V}.$$

Yagi et al. [19] have provided the expansions of Q_1 and Q_2. According to their, Q_1 is expanded as

$$Q_1 = \boldsymbol{z}'\boldsymbol{z} - \frac{1}{\sqrt{N_2}}\boldsymbol{z}'\boldsymbol{V}\boldsymbol{z} + \frac{1}{N_2}(\boldsymbol{z}'\boldsymbol{z} + \boldsymbol{z}'\boldsymbol{V}^2\boldsymbol{z}) + O_p(N_2^{-\frac{3}{2}}),$$

and the characteristic function of Q_1 is

$$\phi_{Q_1}(t) = E[\exp(itQ_1)] = u^{-\frac{1}{2}p_1} + \frac{1}{N_1}\sum_{j=0}^{2}\beta_{j,1}u^{-\frac{1}{2}p_1 - j} + O(N_1^{-2}),$$

where

$$\beta_{0,1} = -\frac{1}{4(1+r_2)}p_1(p_1+2),\ \beta_{1,1} = 0,\ \beta_{2,1} = -\beta_{0,1},\ \ r_2 = \frac{n_2}{n_1},\ u = 1 - 2it.$$

Similarly, let

$$\begin{aligned} &\overline{\boldsymbol{x}}_{11} = \frac{1}{\sqrt{N_1}}\boldsymbol{z}_1,\ \overline{\boldsymbol{x}}_{12} = \frac{1}{\sqrt{N_1}}\boldsymbol{z}_2, \\ &\boldsymbol{S}_{1(12)} = \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{p_2} \end{pmatrix} + \frac{1}{\sqrt{N_1 - 1}}\begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix}. \end{aligned}$$

Then, Q_2 can be expanded as

$$Q_2 = \boldsymbol{z}_2'\boldsymbol{z}_2 + \frac{1}{\sqrt{N_1}}A_1 + \frac{1}{N_1}A_2 + O_p(N_1^{-\frac{3}{2}}),$$

where

$$\begin{aligned} A_1 =& -2\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{z}_1 - \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{z}_2, \\ A_2 =& 2(\boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{11}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{V}_{22}\boldsymbol{V}_{21}\boldsymbol{z}_1) \\ &+ \boldsymbol{z}_1'\boldsymbol{V}_{12}\boldsymbol{V}_{21}\boldsymbol{z}_1 + \boldsymbol{z}_2'\boldsymbol{z}_2 + \boldsymbol{z}_2'\boldsymbol{V}_{21}\boldsymbol{V}_{12}\boldsymbol{z}_2 + \boldsymbol{z}_2'\boldsymbol{V}_{22}^2\boldsymbol{z}_2. \end{aligned}$$

Therefore, the characteristic function of Q_2 is

$$\phi_{Q_2}(t) = E[\exp(itQ_2)] = u^{-\frac{1}{2}p_2} + \frac{1}{N_1}\sum_{j=0}^{2}\beta_{j,2}u^{-\frac{1}{2}p_2-j} + O(N_1^{-2}),$$

where

$$\beta_{0,2} = -\frac{1}{4}p_2(4p_1 + p_2 + 2), \quad \beta_{1,2} = p_1p_2, \quad \beta_{2,2} = \frac{1}{4}p_2(p_2 + 2).$$

Further, we derive the characteristic function of R_2. The denominator of R_2 can be expanded as follows:

$$(1 + Q_{2d})^{-1} = 1 - \frac{1}{N_1}\boldsymbol{z}_1'\boldsymbol{z}_1 + O_p(N_1^{-\frac{3}{2}}).$$

Then, R_2 is expanded as

$$R_2 = \boldsymbol{z}_2'\boldsymbol{z}_2 + \frac{1}{\sqrt{N_1}}A_1 + \frac{1}{N_1}(A_2 - \boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2) + O_p(N_1^{-\frac{3}{2}}).$$

This means that an extra term $-\frac{1}{N_1}(\boldsymbol{z}_1'\boldsymbol{z}_1\boldsymbol{z}_2'\boldsymbol{z}_2)$ has been added to the expansion of Q_2. Therefore, the characteristic function of R_2 can be expressed as

$$\begin{aligned}\phi_{R_2}(t) =& E[\exp(itR_2)] \\ =& u^{-\frac{1}{2}p_2} + \frac{1}{N_1}\Big[(\beta_{0,2} + \frac{1}{2}p_1p_2)u^{-\frac{1}{2}p_2} + (\beta_{1,2} - \frac{1}{2}p_1p_2)u^{-\frac{1}{2}p_2-1} \\ & +\beta_{2,2}u^{-\frac{1}{2}p_2-2}\Big] + O(N_1^{-\frac{3}{2}}).\end{aligned}$$

Then, the characteristic function of Q_M is

$$\phi_{Q_M}(t) = \phi_{Q_1}(t)\phi_{R_2}(t) = u^{-\frac{1}{2}p_2} + \frac{1}{N_1}\sum_{j=0}^{2}\beta_j^*u^{-\frac{p}{2}-j} + O(N_1^{-\frac{3}{2}}),$$

where

$$\beta_0^* = -\frac{1}{4}\left\{\frac{1}{1+r_2}p_1(p_1+2) + p_2(2p_1 + p_2 + 2)\right\}, \quad \beta_1^* = \frac{1}{2}p_1p_2,$$
$$\beta_2^* = \frac{1}{4}\left\{\frac{1}{1+r_2}p_1(p_1+2) + p_2(p_2+2)\right\}.$$

Therefore,

$$Pr(Q_M \le x) = G_p(x) + \frac{1}{N_1}\sum_{j=0}^{2}\beta_j^*G_{p+2j}(x) + O(N_1^{-2}),$$

where $G_f(x)$ is the distribution function of χ^2-variate with f degrees of freedom. An approximation to the 100α percentile of Q_M is given by

$$q_{MAE}(\alpha) = \chi_p^2(\alpha) - \frac{1}{N_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\beta_0^* - \frac{\beta_2^*}{p+2}\chi_p^2(\alpha)\right\}\right], \tag{3.4}$$

where $\chi_p^2(\alpha)$ is the upper 100α percentile ofχ^2 distribution with p degrees of freedom.

3.2.2 Two Sample Problem

In this section, we consider the two sample case. Let us consider the following hypothesis:

$$H_0 : \boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} \quad \text{vs.} \quad H_1 : \boldsymbol{\mu}^{(1)} \neq \boldsymbol{\mu}^{(2)}$$

Yu et al. [21] provided the simplified T^2 statistic for the two sample case. We express the simplified T^2 statistic as follows:

$$Q = Q_1 + Q_2, \tag{3.5}$$

where

$$\begin{aligned}
Q_i &= \frac{N_{3-i}^{(1)}N_{3-i}^{(2)}}{N_{3-i}^{(1)} + N_{3-i}^{(2)}}(\widehat{\boldsymbol{\eta}}_i^{(1)} - \widehat{\boldsymbol{\eta}}_i^{(2)})'\widehat{\Delta}_{ii_{p\ell}}^{-1}(\widehat{\boldsymbol{\eta}}_i^{(1)} - \widehat{\boldsymbol{\eta}}_i^{(2)}),\ i = 1, 2, \quad N_i^{(\ell)} = \sum_{j=1}^{i} n_j^{(\ell)}, \\
\widehat{\boldsymbol{\eta}}_1^{(\ell)} &= \overline{\boldsymbol{x}}_{(12)1}^{(\ell)}, \\
\widehat{\boldsymbol{\eta}}_2^{(\ell)} &= \overline{\boldsymbol{x}}_{12}^{(\ell)} - \sum_{\ell=1}^{2}\left\{(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(12),21}^{(\ell)}\right\}\sum_{\ell=1}^{2}\left\{(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(12),11}^{(\ell)}\right\}^{-1}\overline{\boldsymbol{x}}_{11}^{(\ell)}, \\
\widehat{\boldsymbol{\Delta}}_{11_{p\ell}} &= \frac{1}{N_2^{(1)} + N_2^{(2)}}\sum_{\ell=1}^{2}(N_2^{(\ell)} - 1)\boldsymbol{S}_{(12)1}^{(\ell)}, \\
\widehat{\boldsymbol{\Delta}}_{22_{p\ell}} &= \frac{1}{N_1^{(1)} + N_1^{(2)}}\left[\sum_{\ell=1}^{2}\left\{(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(12),22}^{(\ell)}\right\} - \sum_{\ell=1}^{2}\left\{(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(12),21}^{(\ell)}\right\}\right. \\
&\qquad \left.\sum_{\ell=1}^{2}\left\{(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(12),11}^{(\ell)}\right\}^{-1}\sum_{\ell=1}^{2}\left\{(N_1^{(\ell)} - 1)\boldsymbol{S}_{1(12),12}^{(\ell)}\right\}\right].
\end{aligned}$$

Then, we suggest

$$Q_M = Q_1 + R_2 \tag{3.6}$$

as a new test statistic, where

$$R_2 = \frac{Q_2}{1+Q_{2d}},$$
$$Q_{2d} = \frac{N_1^{(1)}N_1^{(2)}}{N_1^{(1)}+N_1^{(2)}}(\overline{\boldsymbol{x}}_{11}^{(1)} - \overline{\boldsymbol{x}}_{11}^{(2)})' \left\{ \sum_{\ell=1}^{2}(N_1^{(\ell)}-1)\boldsymbol{S}_{1(12),11}^{(\ell)} \right\}^{-1} (\overline{\boldsymbol{x}}_{11}^{(1)} - \overline{\boldsymbol{x}}_{11}^{(2)}).$$

Without loss of generality, we may assume that $\boldsymbol{\Sigma} = \boldsymbol{I} = \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{p_2} \end{pmatrix}$.

In a derivation that is similar to the one sample case, at the beginning we consider a statistic expansion of Q_1. Let $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)} = \boldsymbol{\mu} = (\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')'$, where $\boldsymbol{\mu}_i$ is a $p_i \times 1$ vector. Then, let

$$\overline{\boldsymbol{x}}_{(12)1}^{(\ell)} = \boldsymbol{\mu}_1 + \frac{1}{\sqrt{N_2^{(\ell)}}}\boldsymbol{u}^{(\ell)}, \quad \boldsymbol{S}_{(12)1}^{(\ell)} = \boldsymbol{I}_{p_1} + \frac{1}{\sqrt{N_2^{(\ell)}-1}}\boldsymbol{W}^{(\ell)}, \quad \ell = 1, 2.$$

Yagi et al. [20] have provided the expansions of Q_1 and Q_2. According to their work, Q_1 is expanded as

$$Q_1 = \boldsymbol{u}'\boldsymbol{u} - \frac{1}{\sqrt{N_2}}\boldsymbol{u}'\boldsymbol{W}\boldsymbol{u} + \frac{1}{N_2}\left\{ \frac{2}{1+t}\boldsymbol{u}'\boldsymbol{u} + \boldsymbol{u}'\boldsymbol{W}^2\boldsymbol{u} \right\} + O_p(N_2^{-\frac{3}{2}}),$$

where

$$\boldsymbol{u} = \left(\frac{t}{1+t}\right)^{\frac{1}{2}} \left(\boldsymbol{u}^{(1)} - \frac{1}{\sqrt{t}}\boldsymbol{u}^{(2)}\right), \ \boldsymbol{W} = \frac{1}{1+t}\boldsymbol{W}^{(1)} + \frac{\sqrt{t}}{1+t}\boldsymbol{W}^{(2)}, \ t = \frac{N_2^{(2)}}{N_2^{(1)}}.$$

Then, the characteristic function of Q_1 can be written as

$$\phi_{Q_1}(t) = u^{-\frac{1}{2}p_1} + \frac{1}{N_2^{(1)}}\sum_{j=0}^{2}\gamma_{j,1}u^{-\frac{1}{2}p_1-j} + O(N_2^{(1)^{-2}}),$$

where

$$\gamma_{0,1} = -\frac{1}{4(1+t)}p_1(p_1+4), \ \gamma_{1,1} = \frac{1}{2(1+t)}p_1, \ \gamma_{2,1} = \frac{1}{4(1+t)}p_1(p_1+2).$$

Then, let

$$\overline{\boldsymbol{x}}_{11}^{(\ell)} = \boldsymbol{\mu}_1 + \frac{1}{\sqrt{N_1^{(\ell)}}}\boldsymbol{u}_1^{(\ell)}, \quad \overline{\boldsymbol{x}}_{12}^{(\ell)} = \boldsymbol{\mu}_2 + \frac{1}{\sqrt{N_1^{(\ell)}}}\boldsymbol{u}_2^{(\ell)},$$
$$\boldsymbol{S}_{1(12)}^{(\ell)} = \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{p_2} \end{pmatrix} + \frac{1}{\sqrt{N_1^{(\ell)}-1}}\begin{pmatrix} \boldsymbol{W}_{11}^{(\ell)} & \boldsymbol{W}_{12}^{(\ell)} \\ \boldsymbol{W}_{21}^{(\ell)} & \boldsymbol{W}_{22}^{(\ell)} \end{pmatrix}, \quad \ell = 1, 2.$$

Then, Q_2 is expanded as

$$Q_2 = \boldsymbol{u}_2'\boldsymbol{u}_2 + \frac{1}{\sqrt{N_1^{(1)}}}A_3 + \frac{1}{N_1^{(1)}}A_4 + O_p(N_1^{(1)^{-\frac{3}{2}}}),$$

where

$$\begin{aligned}
A_3 &= -2\boldsymbol{u}_2'\boldsymbol{W}_{21}\boldsymbol{u}_1 - \boldsymbol{u}_2'\boldsymbol{W}_{22}\boldsymbol{u}_2, \\
A_4 &= 2(\boldsymbol{u}_2'\boldsymbol{W}_{21}\boldsymbol{W}_{11}\boldsymbol{u}_1 + \boldsymbol{u}_2'\boldsymbol{W}_{21}^2\boldsymbol{u}_1) + \boldsymbol{u}_1'\boldsymbol{W}_{12}\boldsymbol{W}_{21}\boldsymbol{u}_1 + \boldsymbol{u}_2'\boldsymbol{W}_{21}\boldsymbol{W}_{12}\boldsymbol{u}_2 \\
&\quad + \frac{2}{1+q}\boldsymbol{u}_2'\boldsymbol{u}_2 + \boldsymbol{u}_2'\boldsymbol{W}_{22}^2\boldsymbol{u}_2,
\end{aligned}$$

$$\boldsymbol{u}_1 = \left(\frac{q}{1+q}\right)^{\frac{1}{2}}\left(\boldsymbol{u}_1^{(1)} - \frac{1}{\sqrt{q}}\boldsymbol{u}_1^{(2)}\right), \quad \boldsymbol{u}_2 = \left(\frac{q}{1+q}\right)^{\frac{1}{2}}\left(\boldsymbol{u}_2^{(1)} - \frac{1}{\sqrt{q}}\boldsymbol{u}_2^{(2)}\right),$$

$$\begin{pmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{pmatrix} = \frac{1}{1+q}\begin{pmatrix} \boldsymbol{W}_{11}^{(1)} & \boldsymbol{W}_{12}^{(1)} \\ \boldsymbol{W}_{21}^{(1)} & \boldsymbol{W}_{22}^{(1)} \end{pmatrix} + \frac{\sqrt{q}}{1+q}\begin{pmatrix} \boldsymbol{W}_{11}^{(2)} & \boldsymbol{W}_{12}^{(2)} \\ \boldsymbol{W}_{21}^{(2)} & \boldsymbol{W}_{22}^{(2)} \end{pmatrix},$$

$$q = \frac{N_1^{(2)}}{N_1^{(1)}}.$$

Therefore, the characteristic function of Q_2 is

$$\phi_{Q_2}(t) = u^{-\frac{1}{2}p_2} + \frac{1}{N_1^{(1)}}\sum_{j=0}^{2}\gamma_{j,2}u^{-\frac{1}{2}p_1-j} + O(N_1^{(1)^{-2}}),$$

where

$$\gamma_{0,2}=\frac{1}{4(1+q)}p_2(4p_1+p_2+4),\ \gamma_{1,2}=\frac{1}{2(1+q)}p_2(2p_1+1),\ \gamma_{2,2}=\frac{1}{4(1+q)}p_2(p_2+2).$$

Further, we derive the characteristic function of R_2. The denominator of R_2 can be expanded as follows:

$$(1+Q_{2d})^{-1}=1 - \frac{1}{N_1^{(1)}} \times \frac{q}{(1+q)^2}\left(\boldsymbol{u}_1^{(1)} - \frac{1}{\sqrt{q}}\boldsymbol{u}_1^{(2)}\right)'\left(\boldsymbol{u}_1^{(1)} - \frac{1}{\sqrt{q}}\boldsymbol{u}_1^{(2)}\right) + O_p(N_1^{(1)^{-\frac{3}{2}}}).$$

Therefore,

$$R_2 = \boldsymbol{u}_2'\boldsymbol{u}_2 + \frac{1}{\sqrt{N_1^{(1)}}}A_3 + \frac{1}{N_1^{(1)}}\left(A_4 - \frac{1}{1+q}\boldsymbol{u}_2'\boldsymbol{u}_2\boldsymbol{u}_1'\boldsymbol{u}_1\right) + O_p(N_1^{(1)^{-\frac{3}{2}}}).$$

The characteristic function of R_2 is

$$\begin{aligned}\phi_{R_2}(t) &= E[\exp(itR_2)]\\ &= u^{-\frac{1}{2}p_2} + \frac{1}{N_1}\sum_{j=0}^{2}\gamma_{j,3}u^{-\frac{1}{2}p_2-j} + O(N_1^{(1)^{-2}}),\end{aligned}$$

where,

$$\gamma_{0,3}=-\frac{1}{4(1+q)}p_2(2p_1+p_2+4),\quad \gamma_{1,3}=\frac{1}{2(1+q)}p_2(p_1+1),\quad \gamma_{2,3}=\frac{1}{4(1+q)}p_2(p_2+2).$$

Conforming the order of Q_1 and R_2 to ν_1,

$$\begin{aligned}\phi_{Q_1}(t) &= u^{-\frac{1}{2}p_1} + \frac{1}{\nu_1}\Bigg\{-\frac{1}{4(1+s)}p_1(p_1+4)u^{-\frac{1}{2}p_1}\\ &\quad +\frac{1}{2(1+s)}p_1u^{-\frac{1}{2}p_1-1} + \frac{1}{4(1+s)}p_1(p_1+2)u^{-\frac{1}{2}p_1-2}\Bigg\} + O(\nu_1^{-2}),\\ \phi_{R_2}(t) &= u^{-\frac{1}{2}p_2} + \frac{1}{\nu_1}\Bigg\{-\frac{1}{4}p_2(p_2+2p_1+4)u^{-\frac{1}{2}p_2}\\ &\quad +\frac{1}{2}p_2(p_1+1)u^{-\frac{1}{2}p_2-1} + \frac{1}{4}p_2(p_2+2)u^{-\frac{1}{2}p_2-2}\Bigg\} + O(\nu_1^{-2}).\end{aligned}$$

Since $Q_M = Q_1 + R_2$, the characteristic function of Q_M is

$$\begin{aligned}\phi_{Q_M}(t) &= \phi_{Q_1}(t)\phi_{R_2}(t)\\ &= u^{-\frac{1}{2}p} + \frac{1}{\nu_1}\sum_{j=0}^{2}\gamma_j^* u^{-\frac{1}{2}p-j} + O(\nu_1^{-2}),\end{aligned}$$

where

$$\gamma_0^* = -\frac{1}{4}p_2(2p_1+p_2+4) - \frac{1}{4(1+s)}p_1(p_1+4),\quad \gamma_1^* = \frac{1}{2}p_2(p_1+1) + \frac{1}{2(1+s)}p_1,$$
$$\gamma_2^* = \frac{1}{4}p_2(p_2+2) + \frac{1}{4(1+s)}p_1(p_1+2),\quad \nu_i = n_i^{(1)} + n_i^{(2)}\ (i=1,2),\quad s=\frac{\nu_2}{\nu_1}.$$

Therefore,

$$Pr(Q_M \le x) = G_p(x) + \frac{1}{\nu_1}\sum_{j=0}^{2}\gamma_j^* G_{p+2j}(x) + O(\nu_1^{-2}),$$

where $G_f(x)$ is the distribution function of χ^2-variate with f degrees of freedom. An approximation to the 100α percentile of Q_M is given by

$$q_M(\alpha) = \chi_p^2(\alpha) - \frac{1}{\nu_1}\left[\frac{2\chi_p^2(\alpha)}{p}\left\{\gamma_0^* - \frac{\gamma_2^*}{p+2}\chi_p^2(\alpha)\right\}\right]. \tag{3.7}$$

3.3 Transformed test statistics

3.3.1 One Sample case

The statistics Q_1 and Q_2 that are transformed by the Bartlett collection are given by

$$Q_1^* = \left(1 - \frac{1}{N_2}(p_1 + 2)\right) Q_1, \quad R_2^* = \left(1 - \frac{1}{N_1}(p_1 + p_2 + 2)\right) R_2.$$

Therefore, we suggest the transformed test statistic of Q_M as

$$Q_M^* = Q_1^* + R_2^*. \tag{3.8}$$

Moreover, Fujikoshi [4] suggested the Bartlett-type correction for general test statistics. Using this method, the Bartlett-type corrections for Q_1 and R_2 are

$$Y_1 = \left\{N_2 - \frac{1}{2}(p_1 + 2)\right\} \log\left(1 + \frac{1}{N_2}Q_1\right), \quad \text{for} \ \ N_2 - \frac{1}{2}(p_1 + 2) > 0$$

and

$$Y_{2M} = \left\{N_1 - \frac{1}{2}(2p_1 + p_2 + 2)\right\} \log\left(1 + \frac{1}{N_1}R_2\right), \ \text{for} \ N_1 - \frac{1}{2}(2p_1 + p_2 + 2) > 0,$$

respectively. Then, we suggest the transformed statistic of Q_M by the Bartlett-type correction as

$$Y_M = Y_1 + Y_{2M}. \tag{3.9}$$

We also have the Barllett correction of Q_M as

$$Q_M^\dagger = \left(1 + \frac{1}{N_1}c_1\right) Q_M, \tag{3.10}$$

where

$$c_1 = \frac{1}{p}\left\{\frac{1}{1+t}p_1(p_1 + 2) + p_2(p + 2)\right\}.$$

Using the result of Fujikoshi [4], we can obtain the Bartlett-type correction as

$$Y_M^\dagger = (N_1 a + b)\log\left(1 + \frac{1}{N_1 a}Q_M\right), \quad \text{for} \ \ a > 0 \ \ \text{and} \ \ N_1 a + b > 0, \tag{3.11}$$

where

$$a = p(p+2)\left\{\frac{1}{1+t}p_1(p_1+2) + p_2p_1 + 2\right\}^{-1},$$
$$b = -\frac{a}{2p}\left\{\frac{1}{1+t}p_1(p_1+2) + p_2(2p_1+p_2+2)\right\}.$$

Krishnarmoorthy and Pannala [8] proposed approximated upper percentiles of Q distributions using F distribution. Applying its approximation to Q_M, we propose $q_{MKP}(\alpha)$ as

$$q_{MKP}(\alpha) = d_1 F_{p,\nu_{\rm KP}}(\alpha), \qquad \text{for} \quad N_1 > p+4, \tag{3.12}$$

where

$$\begin{aligned}
d_1 &= M_1\frac{\nu_{\rm KP}-2}{\nu_{\rm KP}},\ \nu_{\rm KP} = \frac{4pM_2 - 2(p+2)M_1^2}{pM_2 - (p+2)M_1^2},\\
M_1 &= E(Q_M) = E(Q_1) + E(R_2)\\
&= \frac{N_2p_1}{N_2-p_1-2} + \frac{N_1p_2}{N_1-p-2},\\
M_2 &= E(Q_M^2)\\
&= E(Q_1^2) + 2E(Q_1)E(R_2) + E(R_2^2),\\
E(Q_1^2) &= \frac{N_2^2p_1(p_1+2)}{(N_2-p_1-2)(N_2-p_1-4)}, \quad E(R_2^2) = \frac{N_1^2p_2(p_2+2)}{(N_1-p-2)(N_1-p-4)}.
\end{aligned}$$

This approximation $q_{MKP}(\alpha)$ is closer to the truth than the approximation for Q that Krishnarmoorthy and Pannala [8] originally proposed because Q_1 and R_2 are independent, although Q_1 and Q_2 are not independent.

3.3.2 Two Sample case

We suggest the transformed test statistics similar to that of the one sample case. The statistic Q_1 and Q_2 that are transformed by the Bartlett collection are given by

$$Q_1^* = \left(1 - \frac{1}{\nu_1+\nu_2}(p_1+3)\right)Q_1, \quad R_2^* = \left(1 - \frac{1}{\nu_1}(p_1+p_2+3)\right)R_2.$$

Therefore, we suggest the transformed test statistic of Q_M as

$$Q_M^* = Q_1^* + R_2^*. \tag{3.13}$$

Moreover, by applying the Bartlett-type correction of Q_1 and R_2, we obtain the following:

$$Y_1 = \{(\nu_1+\nu_2)a_{Y_1} + b_{Y_1}\}\log\left(1 + \frac{1}{(\nu_1+\nu_2)a_{Y_1}}Q_1\right),\ \text{for}\ (\nu_1+\nu_2)a_{Y_1} + b_{Y_1} > 0,$$

where

$$a_{Y_1} = 1 + s, \quad b_{Y_1} = -\frac{1}{2}(p_1 + 4)$$

and

$$Y_{2M} = \left\{\nu_1 - \frac{1}{2}(2p_1 + p_2 + 4)\right\} \log\left(1 + \frac{1}{\nu_1} R_2\right), \text{ for } \nu_1 - \frac{1}{2}(2p_1 + p_2 + 4) > 0,$$

respectively. Then, we suggest the transformed statistic of Q_M by the Bartlett-type correction as

$$Y_M = Y_1 + Y_{2M}. \tag{3.14}$$

We have the Bartlett correction of Q_M as

$$Q_M^{\dagger} = \left(1 + \frac{1}{\nu_1} c_1\right) Q_M, \tag{3.15}$$

where

$$c_1 = \frac{1}{p}\left\{\frac{1}{1+s} p_1(p_1 + 3) + p_2(p_1 + p_2 + 3)\right\}.$$

In addition, using the result of Fujikoshi [4], we can obtain the Bartlett-type correction as

$$Y_M^{\dagger} = (\nu_1 a + b) \log\left(1 + \frac{1}{\nu_1 a} Q_M\right), \quad \text{for } \ a > 0, \ N_1 a + b > 0, \tag{3.16}$$

where

$$a = p(p+2)\left\{\frac{1}{1+s} p_1(p_1 + 2) + p_2(p_2 + 2)\right\}^{-1},$$
$$b = -\frac{a}{2p}\left\{-\frac{1}{1+s} p_1(p_1 + 4) + p_2(-2p + p_2 - 4)\right\}.$$

Applying the approximation proposed by Yu et al. [21] to Q_M, we propose $q_{MYKP}(\alpha)$ as

$$q_{MYKP}(\alpha) = d_2 F_{p,\nu_{\mathrm{YKP}}}(\alpha), \quad \text{for } N_1^{(1)} + N_1^{(2)} > p + 5, \tag{3.17}$$

where

$$
\begin{aligned}
d_2 &= G_1 \frac{\nu_{\mathrm{YKP}} - 2}{\nu_{\mathrm{YKP}}}, \quad \nu_{\mathrm{YKP}} = \frac{4pG_2 - 2(p+2)G_1^2}{pG_2 - (p+2)G_1^2}, \\
G_1 &= E(Q_M) = E(Q_1) + E(R_2) \\
&= \frac{(N_2^{(1)} + N_2^{(2)})p_1}{N_2^{(1)} + N_2^{(2)} - p_1 - 3} + \frac{(n_1^{(1)} + n_1^{(2)})p_2}{n_1^{(1)} + n_1^{(2)} - p - 3}, \\
G_2 &= E(Q_M^2) \\
&= E(Q_1^2) + 2E(Q_1)E(R_2) + E(R_2^2), \\
E(Q_1^2) &= \frac{(N_2^{(1)} + N_2^{(2)})^2 p_1(p_1+2)}{(N_2^{(1)} + N_2^{(2)} - p_1 - 3)(N_2^{(1)} + N_2^{(2)} - p_1 - 5)}, \\
E(R_2^2) &= \frac{(N_1^{(1)} + N_1^{(2)})^2 p_2(p_2+2)}{(N_1^{(1)} + N_1^{(2)} - p - 3)(N_1^{(1)} + N_1^{(2)} - p - 5)}.
\end{aligned}
$$

3.4 Simulation studies

In this section, we describe the accuracy of the approximations and the asymptotic behavior of approximate upper percentiles of the test statistic Q_M for the one sample and two sample problems. We compare the approximate upper percentiles of Q_M, some of the transformed test statistics proposed in Section 4, and the test using Krishnamoorthy and Pannala [8] and Yu et al. [21]'s approximation. Further, we compare the test Q_M we proposed with test Q. Their upper 100α percentiles and Type I errors are defined as the following:

1. q_M and $\alpha_{\chi^2_M} = 100\mathrm{Pr}(Q_M > \chi^2_p(\alpha))$: for test Q_M in (3.3) and (3.6)

2. q_{MAE} and $\alpha_{MAE} = 100\mathrm{Pr}(Q_M > q_{MAE}(\alpha))$: for asymptotic expansion approximation (MAE) test in (3.4) and (3.7)

3. (a) q_{MKP} and $\alpha_{MKP} = 100\mathrm{Pr}(Q_M > q_{MKP}(\alpha))$

   : for applying Krishnamoorthy and Pannala's approximation to Q_M in (3.12) (one sample)

   (b) q_{MYKP} and $\alpha_{MYKP} = 100\mathrm{Pr}(Q_M > q_{MYKP}(\alpha))$

   : for applying Yu, Krishnamoorthy and Pannala's approximation to Q_M in (3.17) (two sample)

4. $q_{Q^*_M}$ and $\alpha_{Q^*_M} = 100\mathrm{Pr}(Q^*_M > \chi^2_p(\alpha))$: for test Q^*_M with Bartlett correction in (3.8) and (3.13)

5. $q_{Q^\dagger_M}$ and $\alpha_{Q^\dagger_M} = 100\mathrm{Pr}(Q^\dagger_M > \chi^2_p(\alpha))$: for test $Q^\dagger_M$ with Bartlett correction in (3.10) and (3.15)

6. q_{Y_M} and $\alpha_{Y_M} = 100\Pr(Y_M > \chi^2_p(\alpha))$: for test Y_M based on Bartlett-type correction in (3.9) and (3.14)

7. $q_{Y_M^\dagger}$ and $\alpha_{Y_M^\dagger} = 100\Pr(Y_M^\dagger > \chi^2_p(\alpha))$: for test $Y_M^\dagger$ with Bartlett-type correction in (3.11) and (3.16)

8. q and $\alpha_{\chi^2} = 100\Pr(Q > \chi^2_p(\alpha))$: for test Q

9. q_{AE} and $\alpha_{AE} = 100\Pr(Q > q_{AE}(\alpha))$: for asymptotic expansion approximation (AE) test

10. (a) q_{KP} and $\alpha_{KP} = 100\Pr(Q > q_{KP}(\alpha))$

    : for applying Krishnamoorthy and Pannala's approximation to Q (one sample)

    (b) q_{YKP} and $\alpha_{YKP} = 100\Pr(Q > q_{YKP}(\alpha))$

    : for applying Yu, Krishnamoorthy and Pannala's approximation to Q (two sample)

11. q_{Q^*} and $\alpha_{Q^*} = 100\Pr(Q^* > \chi^2_p(\alpha))$: for test Q^* with Bartlett correction

12. $q_{Q^\dagger}$ and $\alpha_{Q^\dagger} = 100\Pr(Q^\dagger > \chi^2_p(\alpha))$: for test $Q^\dagger$ with Bartlett correction

13. q_Y and $\alpha_Y = 100\Pr(Y > \chi^2_p(\alpha))$: for test Y_M based on Bartlett-type correction

14. $q_{Y^\dagger}$ and $\alpha_{Y^\dagger} = 100\Pr(Y^\dagger > \chi^2_p(\alpha))$: for test $Y^\dagger$ with Bartlett-type correction

We note that 1 – 7 are about test Q_M, and 8 – 14 are about test Q. A Monte Carlo simulation (10^6) was conducted, considering a significance level $\alpha = 0.05$. The settings of the parameters $p(= p_1 + p_2)$, and sample size for the simulation are as follows:

Case $1 (p_1, p_2) = (2, 2),\ (4, 4)$

$$\text{(I) } (n_1, n_2) = (m, m),\ (m, \frac{m}{2}),\ (m, 2m),\ m = 10, 20, 30, 40, 50, 100, 200, 400$$

$$\text{(II) } (n_1, n_2) = (m, 10),\ (m, 20),\ (m, 50),\ (m, 100),\ m = 10, 20, 30, 50, 100$$

Case $2\ p_1 = 2, 4, 6, 8, 10, p_2 = 2 : fix,\ n_1 = n_2 = 30,$

where, in the two sample case, we set $n_1 = n_1^{(1)} = n_1^{(2)},\ n_2 = n_2^{(1)} = n_2^{(2)}$. Tables 1–4 list the simulation results for (I). Tables 5–8 list the for (II), which are obtained by changing the sample size of the complete data (n_1) to a larger value and fixing the sample size of missing data (n_2). Tables 9 and 10 list the simulation results for *Case* 2, which are obtained increasing the dimension of p_1 and fixing the missing data of p_2. Tables 3,4,7,8, and 10 are for the two sample case. The upper percentiles of the test statistics are closer to the upper percentiles of χ^2 distribution when n_1 is large. Comparing q_{MAE} and q_{AE}, q_{MAE} is a better approximation to the true value q_M. In particular, the transformed test statistics were observed to have a faster convergence. Comparing the type I error, α_{MKP} and α_{MYKP} values are approximately 5.00 when the sample size is large. α_{Y_M} and $\alpha_{Y_M^\dagger}$ are always stable (almost 5.00 or more) for any sample size although we find that the type I error using q_{Y_M} gose away from 5.00 for large p, as shown in Tables 9 and 10. Judging in this light, the result using q_{MKP} and q_{MYKP} are good approximation even when p is large. Additionally, $q_{Y_M^\dagger}$ was observed to be more conservative than q_{MKP}.

Table 1. The upper percentiles of test statistics and empirical Type I errors for the one sample case, $(p_1, p_2) = (2, 2)$.

1mm n_1	n_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{MKP} (α_{MKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{KP} (α_{KP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
10	10	20.77	14.11	22.03	10.68	12.46	9.47	10.68	26.95	15.33	27.65	9.20	13.48	8.79	10.8
		(23.94)	(11.7)	(4.35)	(6.85)	(9.24)	(4.97)	(7.27)	(30.03)	(14.64)	(4.75)	(4.61)	(10.00)	(3.70)	(7.55)
20	20	12.77	11.8	12.86	9.99	10.21	9.48	9.69	13.96	12.41	13.92	9.96	10.47	9.46	9.74
		(11.88)	(6.42)	(4.88)	(5.93)	(6.32)	(4.98)	(5.42)	(14.17)	(7.11)	(5.05)	(5.85)	(6.74)	(4.95)	(5.52)
30	30	11.41	11.03	11.43	9.80	9.89	9.49	9.57	12.03	11.44	11.96	9.84	10.02	9.52	9.59
		(9.07)	(5.64)	(4.96)	(5.62)	(5.78)	(5.01)	(5.18)	(10.36)	(5.94)	(5.10)	(5.67)	(6.03)	(5.07)	(5.22)
40	40	10.82	10.64	10.85	9.70	9.74	9.47	9.52	11.24	10.95	11.2	9.74	9.84	9.51	9.53
		(7.85)	(5.32)	(4.95)	(5.42)	(5.50)	(4.96)	(5.06)	(8.73)	(5.47)	(5.07)	(5.49)	(5.67)	(5.04)	(5.09)
50	50	10.54	10.41	10.54	9.67	9.69	9.49	9.52	10.85	10.66	10.8	9.71	9.77	9.52	9.53
		(7.23)	(5.22)	(4.99)	(5.36)	(5.40)	(5.00)	(5.06)	(7.92)	(5.33)	(5.09)	(5.43)	(5.54)	(5.07)	(5.08)
100	100	9.99	9.95	9.98	9.58	9.59	9.49	9.5	10.12	10.07	10.09	9.61	9.62	9.52	9.50
		(6.05)	(5.07)	(5.01)	(5.19)	(5.20)	(5.01)	(5.03)	(6.34)	(5.10)	(5.07)	(5.24)	(5.27)	(5.07)	(5.04)
200	200	9.72	9.72	9.73	9.53	9.53	9.49	9.49	9.79	9.78	9.78	9.54	9.54	9.50	9.49
		(5.49)	(5.01)	(5.00)	(5.09)	(5.09)	(5.00)	(5.00)	(5.63)	(5.02)	(5.02)	(5.11)	(5.12)	(5.02)	(5.00)
400	400	9.60	9.60	9.61	9.51	9.51	9.49	9.49	9.64	9.63	9.63	9.52	9.52	9.49	9.49
		(5.23)	(5.00)	(5.00)	(5.04)	(5.04)	(5.00)	(5.00)	(5.30)	(5.00)	(5.01)	(5.06)	(5.06)	(5.01)	(5.00)
10	5	21.49	14.52	22.63	10.85	12.17	9.47	10.37	27.79	15.83	28.26	9.41	12.97	8.87	10.36
		(25.33)	(11.88)	(4.41)	(7.05)	(8.83)	(4.97)	(6.73)	(31.11)	(14.63)	(4.82)	(4.90)	(9.33)	(3.88)	(6.74)
20	10	13.01	12.00	13.10	10.03	10.19	9.47	9.62	14.28	12.66	14.15	10.06	10.47	9.50	9.65
		(12.41)	(6.45)	(4.88)	(5.99)	(6.27)	(4.97)	(5.28)	(14.70)	(7.11)	(5.15)	(6.00)	(6.69)	(5.03)	(5.33)
30	15	11.54	11.16	11.59	9.81	9.87	9.47	9.53	12.20	11.60	12.11	9.89	10.03	9.53	9.54
		(9.39)	(5.61)	(4.93)	(5.63)	(5.74)	(4.97)	(5.08)	(10.71)	(5.90)	(5.13)	(5.74)	(6.01)	(5.08)	(5.11)
40	20	10.96	10.75	10.97	9.75	9.77	9.49	9.52	11.40	11.07	11.31	9.81	9.88	9.56	9.52
		(8.13)	(5.36)	(4.98)	(5.50)	(5.56)	(5.01)	(5.06)	(9.04)	(5.54)	(5.14)	(5.63)	(5.76)	(5.14)	(5.08)
50	25	10.62	10.49	10.63	9.68	9.70	9.48	9.50	10.95	10.76	10.88	9.74	9.79	9.54	9.51
		(7.38)	(5.22)	(4.98)	(5.39)	(5.42)	(4.99)	(5.03)	(8.08)	(5.34)	(5.12)	(5.50)	(5.59)	(5.11)	(5.05)
100	50	10.01	9.99	10.02	9.58	9.58	9.48	9.49	10.16	10.12	10.13	9.61	9.62	9.51	9.49
		(6.08)	(5.04)	(4.99)	(5.18)	(5.19)	(4.99)	(5.00)	(6.39)	(5.07)	(5.05)	(5.24)	(5.26)	(5.06)	(5.00)
200	100	9.74	9.74	9.75	9.53	9.53	9.48	9.49	9.81	9.81	9.80	9.55	9.55	9.50	9.48
		(5.53)	(5.01)	(4.99)	(5.09)	(5.09)	(4.99)	(4.99)	(5.69)	(5.01)	(5.03)	(5.12)	(5.13)	(5.03)	(4.99)
400	200	9.59	9.61	9.62	9.49	9.49	9.47	9.47	9.63	9.65	9.64	9.50	9.50	9.48	9.47
		(5.22)	(4.96)	(4.96)	(5.01)	(5.01)	(4.96)	(4.96)	(5.29)	(4.97)	(4.98)	(5.02)	(5.02)	(4.98)	(4.96)
10	20	20.22	13.70	21.54	10.54	12.81	9.47	11.06	26.29	14.83	27.13	8.99	14.02	8.70	11.34
		(22.79)	(11.68)	(4.32)	(6.64)	(9.74)	(4.97)	(7.91)	(29.09)	(14.77)	(4.69)	(4.29)	(10.73)	(3.54)	(8.43)
20	40	12.54	11.59	12.64	9.93	10.24	9.49	9.77	13.69	12.16	13.71	9.88	10.50	9.43	9.86
		(11.37)	(6.42)	(4.86)	(5.85)	(6.38)	(5.01)	(5.55)	(13.60)	(7.12)	(4.98)	(5.71)	(6.77)	(4.88)	(5.74)
30	60	11.25	10.89	11.29	9.76	9.87	9.48	9.59	11.84	11.27	11.82	9.78	10.00	9.49	9.64
		(8.70)	(5.58)	(4.93)	(5.54)	(5.73)	(4.99)	(5.22)	(9.95)	(5.89)	(5.03)	(5.55)	(5.96)	(5.00)	(5.29)
40	80	10.75	10.54	10.75	9.70	9.76	9.50	9.56	11.14	10.82	11.10	9.72	9.84	9.51	9.58
		(7.67)	(5.37)	(5.00)	(5.43)	(5.54)	(5.02)	(5.16)	(8.53)	(5.54)	(5.08)	(5.46)	(5.69)	(5.05)	(5.20)
50	100	10.46	10.33	10.46	9.66	9.69	9.50	9.54	10.76	10.56	10.71	9.68	9.75	9.52	9.55
		(7.06)	(5.24)	(5.01)	(5.34)	(5.42)	(5.02)	(5.10)	(7.68)	(5.36)	(5.07)	(5.39)	(5.53)	(5.06)	(5.13)
100	200	9.94	9.91	9.94	9.56	9.57	9.49	9.50	10.07	10.02	10.05	9.58	9.60	9.50	9.50
		(5.95)	(5.06)	(5.00)	(5.16)	(5.18)	(5.00)	(5.02)	(6.22)	(5.09)	(5.03)	(5.18)	(5.22)	(5.03)	(5.03)
200	400	9.72	9.70	9.71	9.54	9.54	9.50	9.50	9.78	9.76	9.76	9.55	9.55	9.51	9.50
		(5.49)	(5.04)	(5.03)	(5.11)	(5.11)	(5.03)	(5.03)	(5.61)	(5.05)	(5.05)	(5.12)	(5.13)	(5.05)	(5.03)
400	800	9.58	9.59	9.59	9.49	9.49	9.47	9.47	9.61	9.62	9.62	9.50	9.50	9.48	9.47
		(5.19)	(4.98)	(4.97)	(5.01)	(5.01)	(4.97)	(4.97)	(5.25)	(4.98)	(4.98)	(5.02)	(5.02)	(4.98)	(4.98)

Note: $\chi^2_4(0.05) = 9.49$

Table 2. The upper percentiles of test statistics and empirical Type I errors for one the sample case, $(p_1, p_2) = (4, 4)$.

n_1	n_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{MKP} (α_{MKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{KP} (α_{KP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
20	20	27.90	21.51	28.35	17.01	18.83	14.94	17.05	34.84	23.49	34.73	15.49	20.03	14.66	17.18
		(26.84)	(11.66)	(4.72)	(7.11)	(9.54)	(4.10)	(7.49)	(35.65)	(15.18)	(5.05)	(4.99)	(10.60)	(3.77)	(7.73)
30	30	21.58	19.51	21.66	16.36	16.91	15.13	16.05	24.38	20.83	24.12	16.13	17.48	15.36	16.15
		(16.58)	(7.56)	(4.93)	(6.31)	(7.13)	(4.37)	(5.91)	(21.59)	(8.99)	(5.21)	(5.89)	(7.88)	(4.76)	(6.07)
40	40	19.51	18.51	19.56	16.08	16.34	15.21	15.76	21.20	19.50	21.01	16.07	16.70	15.50	15.83
		(12.59)	(6.33)	(4.94)	(5.91)	(6.31)	(4.51)	(5.42)	(15.84)	(7.09)	(5.20)	(5.87)	(6.82)	(4.99)	(5.53)
50	50	18.53	17.91	18.53	15.97	16.12	15.29	15.68	19.75	18.70	19.54	16.02	16.39	15.57	15.73
		(10.64)	(5.86)	(5.00)	(5.75)	(5.99)	(4.63)	(5.29)	(12.97)	(6.35)	(5.23)	(5.80)	(6.37)	(5.09)	(5.37)
100	100	16.83	16.71	16.84	15.70	15.74	15.38	15.54	17.32	17.10	17.24	15.77	15.85	15.56	15.55
		(7.34)	(5.19)	(4.99)	(5.32)	(5.38)	(4.78)	(5.05)	(8.23)	(5.31)	(5.12)	(5.41)	(5.54)	(5.08)	(5.07)
200	200	16.14	16.11	16.14	15.61	15.62	15.45	15.52	16.36	16.31	16.31	15.65	15.66	15.55	15.52
		(6.10)	(5.05)	(5.00)	(5.17)	(5.18)	(4.90)	(5.02)	(6.48)	(5.09)	(5.07)	(5.23)	(5.26)	(5.06)	(5.02)
400	400	15.82	15.81	15.82	15.56	15.56	15.48	15.52	15.93	15.91	15.90	15.58	15.59	15.54	15.52
		(5.53)	(5.02)	(5.01)	(5.09)	(5.10)	(4.96)	(5.01)	(5.70)	(5.03)	(5.05)	(5.12)	(5.13)	(5.04)	(5.02)
20	10	28.51	22.00	28.96	17.15	18.53	14.92	16.68	35.62	24.12	35.29	15.76	19.59	14.76	16.66
		(28.26)	(11.73)	(4.73)	(7.27)	(9.14)	(4.06)	(6.92)	(36.79)	(15.03)	(5.15)	(5.30)	(10.00)	(3.93)	(6.91)
30	15	21.98	19.84	22.05	16.43	16.85	15.12	15.92	24.92	21.25	24.49	16.32	17.44	15.44	15.96
		(17.38)	(7.60)	(4.94)	(6.42)	(7.05)	(4.36)	(5.69)	(22.33)	(8.95)	(5.35)	(6.13)	(7.75)	(4.89)	(5.75)
40	20	19.82	18.75	19.84	16.15	16.35	15.20	15.71	21.60	19.81	21.28	16.21	16.74	15.56	15.74
		(13.17)	(6.36)	(4.97)	(6.00)	(6.30)	(4.49)	(5.34)	(16.43)	(7.10)	(5.33)	(6.04)	(6.84)	(5.09)	(5.39)
50	25	18.74	18.11	18.75	16.00	16.11	15.28	15.63	19.99	18.95	19.76	16.11	16.39	15.61	15.64
		(11.06)	(5.88)	(4.98)	(5.80)	(5.98)	(4.62)	(5.20)	(13.45)	(6.34)	(5.29)	(5.92)	(6.39)	(5.17)	(5.23)
100	50	16.95	16.81	16.95	15.73	15.76	15.38	15.53	17.47	17.23	17.34	15.83	15.89	15.59	15.54
		(7.58)	(5.21)	(5.00)	(5.37)	(5.41)	(4.80)	(5.05)	(8.53)	(5.33)	(5.18)	(5.52)	(5.62)	(5.14)	(5.06)
200	100	16.18	16.16	16.19	15.62	15.62	15.45	15.51	16.42	16.37	16.36	15.67	15.68	15.55	15.51
		(6.18)	(5.04)	(4.99)	(5.18)	(5.19)	(4.90)	(5.00)	(6.59)	(5.07)	(5.07)	(5.25)	(5.28)	(5.08)	(5.00)
400	200	15.84	15.83	15.84	15.56	15.57	15.48	15.51	15.95	15.94	15.92	15.59	15.59	15.54	15.51
		(5.58)	(5.02)	(5.01)	(5.10)	(5.10)	(4.96)	(5.01)	(5.77)	(5.03)	(5.05)	(5.14)	(5.14)	(5.05)	(5.01)
20	40	27.30	21.01	27.83	16.85	19.11	14.93	17.43	34.04	22.85	34.24	15.19	20.42	14.52	17.76
		(25.52)	(11.68)	(4.67)	(6.91)	(9.97)	(4.05)	(8.09)	(34.63)	(15.39)	(4.91)	(4.59)	(11.20)	(3.56)	(8.61)
30	60	21.19	19.18	21.31	16.26	16.95	15.11	16.18	23.88	20.40	23.80	15.94	17.51	15.25	16.36
		(15.64)	(7.50)	(4.88)	(6.15)	(7.20)	(4.36)	(6.09)	(20.58)	(8.99)	(5.06)	(5.62)	(7.91)	(4.60)	(6.42)
40	80	19.28	18.26	19.30	16.04	16.39	15.21	15.86	20.90	19.18	20.76	15.96	16.72	15.43	15.98
		(12.04)	(6.36)	(4.97)	(5.84)	(6.39)	(4.52)	(5.59)	(15.21)	(7.14)	(5.15)	(5.68)	(6.84)	(4.88)	(5.78)
50	100	18.30	17.71	18.33	15.90	16.10	15.27	15.71	19.44	18.44	19.34	15.89	16.33	15.49	15.78
		(10.15)	(5.82)	(4.97)	(5.64)	(5.95)	(4.62)	(5.34)	(12.39)	(6.30)	(5.12)	(5.59)	(6.28)	(4.97)	(5.45)
100	200	16.72	16.61	16.74	15.68	15.72	15.37	15.54	17.17	16.98	17.14	15.71	15.80	15.52	15.56
		(7.13)	(5.16)	(4.96)	(5.27)	(5.34)	(4.78)	(5.05)	(7.99)	(5.29)	(5.06)	(5.33)	(5.47)	(5.03)	(5.08)
200	400	16.08	16.06	16.09	15.58	15.60	15.43	15.51	16.28	16.24	16.26	15.61	15.63	15.52	15.51
		(5.99)	(5.03)	(4.98)	(5.13)	(5.15)	(4.88)	(5.00)	(6.36)	(5.06)	(5.03)	(5.17)	(5.21)	(5.01)	(5.01)
400	800	15.77	15.78	15.79	15.53	15.53	15.45	15.49	15.86	15.87	15.87	15.54	15.54	15.50	15.48
		(5.44)	(4.97)	(4.96)	(5.03)	(5.04)	(4.91)	(4.97)	(5.61)	(4.98)	(4.98)	(5.06)	(5.06)	(4.99)	(4.96)

Note: $\chi^2_8(0.05) = 15.51$

Table 3. The upper percentiles of test statistics and empirical Type I errors for two sample $(p_1, p_2) = (2, 2)$.

n_1	n_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{YMKP} (α_{YMKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{YKP} (α_{YKP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
10	10	13.54	12.15	13.66	10.01	10.33	9.89	9.77	14.96	12.63	14.90	9.98	10.66	9.87	10.01
		(13.62)	(6.98)	(4.86)	(5.96)	(6.51)	(5.84)	(5.57)	(16.20)	(8.10)	(5.06)	(5.88)	(7.00)	(5.77)	(6.00)
20	20	11.08	10.82	11.11	9.71	9.77	9.68	9.54	11.53	11.06	11.48	9.76	9.87	9.72	9.63
		(8.48)	(5.46)	(4.96)	(5.45)	(5.56)	(5.41)	(5.11)	(9.44)	(5.81)	(5.09)	(5.54)	(5.76)	(5.49)	(5.29)
30	30	10.49	10.38	10.50	9.64	9.66	9.62	9.51	10.76	10.53	10.71	9.68	9.72	9.65	9.57
		(7.16)	(5.21)	(5.00)	(5.31)	(5.35)	(5.27)	(5.06)	(7.73)	(5.39)	(5.08)	(5.38)	(5.47)	(5.33)	(5.17)
40	40	10.24	10.15	10.22	9.62	9.64	9.60	9.53	10.43	10.27	10.37	9.65	9.68	9.63	9.56
		(6.59)	(5.17)	(5.04)	(5.26)	(5.30)	(5.23)	(5.08)	(6.99)	(5.28)	(5.11)	(5.32)	(5.38)	(5.28)	(5.16)
50	50	10.07	10.02	10.06	9.58	9.59	9.57	9.50	10.21	10.12	10.17	9.61	9.62	9.59	9.53
		(6.22)	(5.09)	(5.01)	(5.19)	(5.21)	(5.17)	(5.03)	(6.53)	(5.18)	(5.06)	(5.24)	(5.27)	(5.22)	(5.09)
100	100	9.77	9.75	9.76	9.53	9.54	9.53	9.49	9.83	9.80	9.82	9.55	9.55	9.54	9.51
		(5.59)	(5.02)	(5.01)	(5.09)	(5.09)	(5.08)	(5.01)	(5.73)	(5.06)	(5.03)	(5.12)	(5.13)	(5.10)	(5.04)
200	200	9.64	9.62	9.62	9.52	9.52	9.52	9.50	9.67	9.64	9.65	9.53	9.53	9.53	9.51
		(5.32)	(5.03)	(5.03)	(5.07)	(5.07)	(5.07)	(5.03)	(5.38)	(5.05)	(5.04)	(5.09)	(5.09)	(5.08)	(5.04)
400	400	9.56	9.55	9.55	9.50	9.50	9.49	9.49	9.57	9.57	9.57	9.50	9.50	9.50	9.49
		(5.14)	(5.00)	(5.00)	(5.02)	(5.02)	(5.01)	(5.00)	(5.17)	(5.01)	(5.01)	(5.03)	(5.03)	(5.02)	(5.01)
10	5	13.89	12.40	13.97	10.09	10.30	9.86	9.69	15.33	12.87	15.19	10.10	10.60	9.89	9.90
		(14.37)	(7.11)	(4.91)	(6.11)	(6.47)	(5.77)	(5.41)	(16.92)	(8.24)	(5.15)	(6.05)	(6.93)	(5.79)	(5.81)
20	10	11.26	10.94	11.25	9.77	9.81	9.67	9.55	11.73	11.18	11.61	9.83	9.92	9.74	9.64
		(8.85)	(5.54)	(5.02)	(5.54)	(5.62)	(5.39)	(5.12)	(9.84)	(5.90)	(5.18)	(5.66)	(5.84)	(5.51)	(5.32)
30	15	10.58	10.46	10.58	9.65	9.67	9.60	9.51	10.86	10.62	10.79	9.71	9.74	9.65	9.57
		(7.32)	(5.21)	(5.00)	(5.32)	(5.35)	(5.23)	(5.03)	(7.91)	(5.42)	(5.11)	(5.43)	(5.49)	(5.33)	(5.17)
40	20	10.27	10.22	10.28	9.60	9.61	9.56	9.49	10.47	10.33	10.43	9.64	9.66	9.61	9.54
		(6.67)	(5.11)	(4.98)	(5.23)	(5.24)	(5.16)	(5.00)	(7.08)	(5.25)	(5.08)	(5.31)	(5.36)	(5.24)	(5.11)
50	25	10.12	10.07	10.11	9.59	9.59	9.56	9.50	10.27	10.16	10.22	9.62	9.63	9.59	9.54
		(6.33)	(5.09)	(5.01)	(5.21)	(5.21)	(5.15)	(5.02)	(6.65)	(5.19)	(5.08)	(5.27)	(5.30)	(5.22)	(5.10)
100	50	9.80	9.78	9.79	9.55	9.55	9.53	9.50	9.87	9.83	9.84	9.56	9.57	9.55	9.52
		(5.64)	(5.04)	(5.02)	(5.11)	(5.12)	(5.09)	(5.02)	(5.79)	(5.09)	(5.06)	(5.15)	(5.16)	(5.13)	(5.06)
200	100	9.62	9.63	9.64	9.50	9.50	9.49	9.47	9.66	9.66	9.66	9.51	9.51	9.50	9.48
		(5.28)	(4.98)	(4.97)	(5.02)	(5.02)	(5.00)	(4.97)	(5.35)	(5.00)	(4.99)	(5.04)	(5.04)	(5.02)	(4.99)
400	200	9.56	9.56	9.56	9.50	9.50	9.50	9.49	9.58	9.57	9.57	9.50	9.51	9.50	9.49
		(5.15)	(5.00)	(5.00)	(5.03)	(5.03)	(5.02)	(5.00)	(5.19)	(5.01)	(5.01)	(5.04)	(5.04)	(5.03)	(5.01)
10	20	13.28	11.91	13.39	9.99	10.41	9.87	9.89	14.61	12.39	14.64	9.91	10.72	9.80	10.12
		(13.02)	(7.02)	(4.87)	(5.95)	(6.67)	(5.81)	(5.81)	(15.57)	(8.09)	(4.97)	(5.76)	(7.17)	(5.63)	(6.25)
20	40	10.95	10.70	10.98	9.68	9.76	9.65	9.56	11.37	10.94	11.35	9.70	9.85	9.67	9.63
		(8.15)	(5.44)	(4.95)	(5.40)	(5.55)	(5.35)	(5.15)	(9.07)	(5.74)	(5.03)	(5.43)	(5.72)	(5.37)	(5.31)
30	60	10.37	10.30	10.41	9.59	9.62	9.57	9.49	10.62	10.45	10.62	9.61	9.67	9.59	9.54
		(6.88)	(5.14)	(4.94)	(5.20)	(5.27)	(5.17)	(5.01)	(7.41)	(5.29)	(5.00)	(5.25)	(5.37)	(5.21)	(5.11)
40	80	10.14	10.09	10.16	9.58	9.59	9.56	9.50	10.32	10.21	10.30	9.59	9.63	9.58	9.53
		(6.39)	(5.09)	(4.97)	(5.18)	(5.22)	(5.15)	(5.02)	(6.76)	(5.19)	(5.02)	(5.22)	(5.28)	(5.19)	(5.09)
50	100	10.01	9.97	10.01	9.56	9.57	9.55	9.50	10.14	10.07	10.12	9.58	9.60	9.57	9.52
		(6.09)	(5.06)	(4.99)	(5.15)	(5.16)	(5.14)	(5.02)	(6.38)	(5.13)	(5.02)	(5.18)	(5.22)	(5.17)	(5.06)
100	200	9.72	9.73	9.74	9.51	9.51	9.51	9.47	9.78	9.78	9.79	9.52	9.52	9.52	9.48
		(5.48)	(4.98)	(4.97)	(5.05)	(5.05)	(5.04)	(4.97)	(5.61)	(5.01)	(4.98)	(5.07)	(5.07)	(5.06)	(4.99)
200	400	9.59	9.61	9.61	9.49	9.49	9.49	9.47	9.62	9.63	9.64	9.50	9.50	9.49	9.48
		(5.22)	(4.97)	(4.97)	(5.01)	(5.01)	(5.00)	(4.97)	(5.29)	(4.98)	(4.98)	(5.02)	(5.02)	(5.01)	(4.98)
400	800	9.54	9.55	9.55	9.49	9.49	9.48	9.48	9.55	9.56	9.56	9.49	9.49	9.49	9.48
		(5.11)	(4.98)	(4.97)	(4.99)	(4.99)	(4.99)	(4.98)	(5.13)	(4.98)	(4.98)	(5.00)	(5.00)	(5.00)	(4.98)

Note: $\chi^2_8(0.05) = 9.49$

Table 4. The upper percentiles of test statistics and empirical Type I errors for the two sample case, $(p_1, p_2) = (4, 4)$.

n_1	n_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{YMKP} (α_{YMKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{YKP} (α_{YKP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
10	10	30.56	22.09	31.22	17.09	19.48	16.53	17.44	38.98	23.64	38.99	15.16	20.95	15.50	18.19
		(31.08)	(13.56)	(4.65)	(7.18)	(10.30)	(6.79)	(8.12)	(40.59)	(18.61)	(4.99)	(4.59)	(11.56)	(4.99)	(9.13)
20	20	20.02	18.80	20.10	16.07	16.39	15.94	15.80	21.83	19.57	21.65	16.05	16.78	15.92	16.11
		(13.69)	(6.59)	(4.92)	(5.90)	(6.38)	(5.75)	(5.48)	(17.16)	(7.80)	(5.18)	(5.82)	(6.94)	(5.69)	(5.98)
30	30	18.19	17.70	18.20	15.88	15.99	15.80	15.63	19.15	18.22	19.01	15.93	16.19	15.85	15.80
		(9.98)	(5.68)	(4.98)	(5.59)	(5.77)	(5.49)	(5.20)	(11.90)	(6.27)	(5.18)	(5.67)	(6.09)	(5.57)	(5.49)
40	40	17.40	17.15	17.42	15.77	15.82	15.73	15.56	18.06	17.54	17.95	15.83	15.97	15.78	15.69
		(8.49)	(5.36)	(4.98)	(5.43)	(5.52)	(5.37)	(5.09)	(9.77)	(5.73)	(5.13)	(5.52)	(5.74)	(5.47)	(5.30)
50	50	16.98	16.82	16.99	15.72	15.75	15.69	15.55	17.48	17.13	17.39	15.78	15.86	15.75	15.65
		(7.66)	(5.23)	(5.00)	(5.35)	(5.40)	(5.30)	(5.07)	(8.60)	(5.50)	(5.12)	(5.44)	(5.57)	(5.40)	(5.23)
100	100	16.20	16.17	16.20	15.61	15.61	15.59	15.52	16.42	16.32	16.38	15.64	15.66	15.63	15.56
		(6.21)	(5.05)	(5.00)	(5.16)	(5.18)	(5.15)	(5.01)	(6.60)	(5.16)	(5.06)	(5.23)	(5.26)	(5.21)	(5.09)
200	200	15.86	15.84	15.85	15.57	15.57	15.56	15.52	15.96	15.91	15.93	15.59	15.59	15.58	15.54
		(5.60)	(5.04)	(5.02)	(5.10)	(5.10)	(5.09)	(5.03)	(5.78)	(5.08)	(5.06)	(5.13)	(5.14)	(5.12)	(5.06)
400	400	15.67	15.67	15.67	15.53	15.53	15.52	15.50	15.72	15.71	15.71	15.54	15.54	15.53	15.51
		(5.27)	(4.99)	(4.99)	(5.03)	(5.03)	(5.03)	(4.99)	(5.35)	(5.01)	(5.01)	(5.05)	(5.05)	(5.04)	(5.01)
10	5	31.41	22.65	31.95	17.29	19.11	16.45	17.00	40.04	24.20	39.68	15.49	20.35	15.54	17.57
		(32.85)	(13.76)	(4.71)	(7.41)	(9.79)	(6.62)	(7.42)	(41.88)	(18.74)	(5.14)	(4.98)	(10.77)	(5.05)	(8.19)
20	10	20.38	19.08	20.42	16.17	16.39	15.92	15.74	22.29	19.85	21.96	16.23	16.81	15.98	16.06
		(14.44)	(6.69)	(4.96)	(6.04)	(6.39)	(5.70)	(5.39)	(17.97)	(7.96)	(5.32)	(6.06)	(6.96)	(5.77)	(5.91)
30	15	18.40	17.89	18.41	15.91	16.00	15.77	15.60	19.41	18.40	19.21	16.03	16.23	15.87	15.80
		(10.45)	(5.71)	(4.98)	(5.65)	(5.78)	(5.45)	(5.15)	(12.43)	(6.36)	(5.26)	(5.80)	(6.15)	(5.60)	(5.48)
40	20	17.60	17.29	17.57	15.84	15.88	15.74	15.58	18.31	17.68	18.10	15.94	16.06	15.84	15.74
		(8.83)	(5.44)	(5.04)	(5.53)	(5.60)	(5.39)	(5.13)	(10.14)	(5.88)	(5.27)	(5.69)	(5.87)	(5.54)	(5.39)
50	25	17.11	16.94	17.11	15.75	15.77	15.67	15.54	17.64	17.25	17.51	15.83	15.90	15.76	15.66
		(7.88)	(5.27)	(5.01)	(5.39)	(5.44)	(5.27)	(5.06)	(8.86)	(5.58)	(5.19)	(5.53)	(5.65)	(5.41)	(5.26)
100	50	16.24	16.22	16.26	15.60	15.61	15.57	15.50	16.48	16.38	16.44	15.65	15.67	15.62	15.56
		(6.28)	(5.03)	(4.97)	(5.16)	(5.16)	(5.10)	(4.98)	(6.70)	(5.16)	(5.06)	(5.24)	(5.26)	(5.18)	(5.08)
200	100	15.88	15.86	15.87	15.57	15.57	15.56	15.52	15.99	15.94	15.96	15.60	15.60	15.58	15.54
		(5.64)	(5.03)	(5.01)	(5.11)	(5.11)	(5.08)	(5.02)	(5.83)	(5.08)	(5.06)	(5.15)	(5.15)	(5.12)	(5.06)
400	200	15.70	15.69	15.69	15.54	15.54	15.53	15.52	15.75	15.72	15.73	15.56	15.56	15.55	15.53
		(5.32)	(5.02)	(5.01)	(5.06)	(5.06)	(5.05)	(5.01)	(5.41)	(5.04)	(5.04)	(5.08)	(5.08)	(5.07)	(5.04)
10	20	29.90	21.53	30.62	16.94	19.94	16.40	17.97	38.13	23.08	38.42	14.89	21.61	15.24	18.89
		(29.53)	(13.60)	(4.62)	(7.00)	(10.97)	(6.55)	(8.94)	(39.31)	(18.65)	(4.89)	(4.25)	(12.47)	(4.59)	(10.17)
20	40	19.74	18.52	19.80	16.04	16.45	15.90	15.91	21.46	19.29	21.37	15.94	16.81	15.81	16.20
		(13.00)	(6.60)	(4.94)	(5.82)	(6.47)	(5.67)	(5.66)	(16.40)	(7.74)	(5.10)	(5.66)	(6.99)	(5.50)	(6.12)
30	60	17.97	17.52	18.01	15.82	15.98	15.76	15.66	18.90	18.03	18.82	15.84	16.17	15.77	15.82
		(9.60)	(5.66)	(4.95)	(5.51)	(5.77)	(5.43)	(5.24)	(11.42)	(6.19)	(5.10)	(5.53)	(6.05)	(5.43)	(5.51)
40	80	17.24	17.01	17.27	15.72	15.80	15.68	15.57	17.86	17.40	17.81	15.76	15.92	15.71	15.68
		(8.17)	(5.32)	(4.96)	(5.35)	(5.47)	(5.30)	(5.10)	(9.37)	(5.65)	(5.06)	(5.39)	(5.66)	(5.33)	(5.28)
50	100	16.86	16.71	16.87	15.69	15.74	15.65	15.56	17.33	17.02	17.27	15.72	15.83	15.69	15.64
		(7.41)	(5.22)	(4.99)	(5.29)	(5.37)	(5.24)	(5.08)	(8.27)	(5.44)	(5.07)	(5.34)	(5.51)	(5.29)	(5.21)
100	200	16.14	16.11	16.15	15.59	15.61	15.57	15.52	16.35	16.26	16.32	15.61	15.64	15.60	15.55
		(6.09)	(5.05)	(4.99)	(5.13)	(5.16)	(5.11)	(5.02)	(6.47)	(5.13)	(5.04)	(5.17)	(5.21)	(5.14)	(5.07)
200	400	15.82	15.81	15.82	15.55	15.55	15.55	15.51	15.91	15.89	15.90	15.56	15.57	15.56	15.53
		(5.53)	(5.01)	(5.00)	(5.08)	(5.08)	(5.07)	(5.00)	(5.70)	(5.05)	(5.03)	(5.10)	(5.10)	(5.09)	(5.03)
400	800	15.64	15.66	15.66	15.51	15.51	15.51	15.49	15.69	15.70	15.70	15.52	15.52	15.52	15.50
		(5.23)	(4.97)	(4.97)	(5.01)	(5.01)	(5.01)	(4.97)	(5.32)	(4.99)	(4.98)	(5.02)	(5.02)	(5.01)	(4.98)

Note: $\chi^2_8(0.05) = 15.51$

Table 5. The upper percentiles of test statistics and empirical Type I errors for the one sample case, $(p_1, p_2) = (2, 2), n_2 : fix$.

n_1	n_2	q_M	q_{MAE}	q_{MKP}	$q_{Q^*_M}$	$q_{Q^\dagger_M}$	q_{Y_M}	$q_{Y^\dagger_M}$	q	q_{AE}	q_{KP}	q_{Q^*}	$q^\dagger$	q_Y	$q^\dagger_Y$
		$(\alpha_{M\chi^2})$	(α_{MAE})	(α_{MKP})	$(\alpha_{Q^*_M})$	$(\alpha_{Q^\dagger_M})$	(α_{Y_M})	$(\alpha_{Y^\dagger_M})$	(α_{χ^2})	(α_{AE})	(α_{KP})	(α_{Q^*})	$(\alpha_{Q^\dagger})$	(α_Y)	$(\alpha_{Y^\dagger})$
10	10	20.77	14.11	22.03	10.68	12.46	9.47	10.68	26.95	15.33	27.65	9.20	13.48	8.79	10.80
		(23.94)	(11.70)	(4.35)	(6.85)	(9.24)	(4.97)	(7.27)	(30.03)	(14.64)	(4.75)	(4.61)	(10.00)	(3.70)	(7.55)
20	10	13.01	12.00	13.10	10.03	10.19	9.47	9.62	14.28	12.66	14.15	10.06	10.47	9.50	9.65
		(12.41)	(6.45)	(4.88)	(5.99)	(6.27)	(4.97)	(5.28)	(14.70)	(7.11)	(5.15)	(6.00)	(6.69)	(5.03)	(5.33)
30	10	11.65	11.23	11.67	9.85	9.90	9.49	9.54	12.32	11.69	12.19	9.93	10.06	9.56	9.54
		(9.59)	(5.66)	(4.97)	(5.70)	(5.78)	(5.00)	(5.10)	(10.95)	(5.96)	(5.19)	(5.84)	(6.07)	(5.15)	(5.11)
50	10	10.70	10.58	10.72	9.69	9.70	9.47	9.48	11.06	10.86	10.97	9.77	9.80	9.55	9.49
		(7.57)	(5.22)	(4.96)	(5.40)	(5.42)	(4.96)	(4.99)	(8.29)	(5.33)	(5.14)	(5.54)	(5.60)	(5.12)	(5.00)
100	10	10.08	10.05	10.09	9.59	9.59	9.48	9.49	10.24	10.20	10.19	9.64	9.65	9.53	9.49
		(6.25)	(5.06)	(4.99)	(5.22)	(5.22)	(4.99)	(5.00)	(6.57)	(5.09)	(5.09)	(5.31)	(5.32)	(5.08)	(5.00)
10	20	20.22	13.70	21.54	10.54	12.81	9.47	11.06	26.29	14.83	27.13	8.99	14.02	8.70	11.34
		(22.79)	(11.68)	(4.32)	(6.64)	(9.74)	(4.97)	(7.91)	(29.09)	(14.77)	(4.69)	(4.29)	(10.73)	(3.54)	(8.43)
20	20	12.77	11.80	12.86	9.99	10.21	9.48	9.69	13.96	12.41	13.92	9.96	10.47	9.46	9.74
		(11.88)	(6.42)	(4.88)	(5.93)	(6.32)	(4.98)	(5.42)	(14.17)	(7.11)	(5.05)	(5.85)	(6.74)	(4.95)	(5.52)
30	20	11.47	11.11	11.52	9.80	9.86	9.47	9.53	12.12	11.54	12.05	9.86	10.02	9.53	9.55
		(9.26)	(5.60)	(4.92)	(5.62)	(5.74)	(4.97)	(5.09)	(10.57)	(5.90)	(5.10)	(5.71)	(6.00)	(5.08)	(5.13)
50	20	10.67	10.52	10.66	9.71	9.73	9.50	9.52	11.01	10.79	10.91	9.78	9.82	9.57	9.52
		(7.52)	(5.27)	(5.03)	(5.43)	(5.46)	(5.03)	(5.07)	(8.23)	(5.39)	(5.17)	(5.54)	(5.64)	(5.16)	(5.08)
100	20	10.06	10.03	10.07	9.59	9.59	9.49	9.49	10.22	10.17	10.17	9.64	9.64	9.53	9.49
		(6.21)	(5.06)	(4.99)	(5.21)	(5.22)	(5.00)	(5.00)	(6.54)	(5.09)	(5.09)	(5.30)	(5.31)	(5.09)	(5.00)
10	50	19.89	13.29	21.12	10.43	13.26	9.50	11.54	25.91	14.33	26.69	8.82	14.68	8.64	12.07
		(21.79)	(11.79)	(4.36)	(6.51)	(10.27)	(5.02)	(8.67)	(28.24)	(15.10)	(4.70)	(4.02)	(11.53)	(3.43)	(9.54)
20	50	12.47	11.54	12.58	9.93	10.25	9.49	9.79	13.63	12.09	13.66	9.84	10.51	9.42	9.90
		(11.23)	(6.42)	(4.86)	(5.82)	(6.39)	(5.00)	(5.61)	(13.49)	(7.15)	(4.97)	(5.67)	(6.80)	(4.86)	(5.82)
30	50	11.30	10.93	11.32	9.79	9.89	9.49	9.60	11.90	11.31	11.86	9.81	10.02	9.51	9.63
		(8.83)	(5.63)	(4.96)	(5.58)	(5.78)	(5.01)	(5.22)	(10.09)	(5.93)	(5.07)	(5.60)	(6.00)	(5.05)	(5.30)
50	50	10.46	10.33	10.46	9.66	9.69	9.50	9.54	10.76	10.56	10.71	9.68	9.75	9.52	9.55
		(7.06)	(5.24)	(5.01)	(5.34)	(5.42)	(5.02)	(5.10)	(7.68)	(5.36)	(5.07)	(5.39)	(5.53)	(5.06)	(5.13)
100	50	10.02	9.99	10.02	9.58	9.59	9.49	9.49	10.17	10.12	10.13	9.62	9.63	9.52	9.49
		(6.13)	(5.06)	(5.00)	(5.19)	(5.20)	(5.00)	(5.01)	(6.44)	(5.09)	(5.07)	(5.26)	(5.28)	(5.07)	(5.02)
10	100	19.66	13.11	20.95	10.34	13.41	9.48	11.73	25.61	14.11	26.51	8.72	14.90	8.60	12.39
		(21.25)	(11.81)	(4.33)	(6.39)	(10.49)	(4.99)	(8.95)	(27.76)	(15.17)	(4.66)	(3.87)	(11.80)	(3.36)	(9.96)
20	100	12.27	11.39	12.45	9.86	10.23	9.46	9.81	13.37	11.91	13.53	9.76	10.47	9.37	9.95
		(10.80)	(6.37)	(4.77)	(5.69)	(6.38)	(4.95)	(5.67)	(13.02)	(7.12)	(4.82)	(5.49)	(6.78)	(4.76)	(5.92)
30	100	11.17	10.81	11.21	9.74	9.88	9.48	9.62	11.74	11.17	11.74	9.74	10.00	9.48	9.67
		(8.57)	(5.61)	(4.94)	(5.51)	(5.76)	(4.99)	(5.26)	(9.80)	(5.92)	(5.00)	(5.49)	(5.96)	(4.98)	(5.38)
50	100	10.46	10.33	10.46	9.66	9.69	9.50	9.54	10.76	10.56	10.71	9.68	9.75	9.52	9.55
		(7.06)	(5.24)	(5.01)	(5.34)	(5.42)	(5.02)	(5.10)	(7.68)	(5.36)	(5.07)	(5.39)	(5.53)	(5.06)	(5.13)
100	100	9.99	9.95	9.98	9.58	9.59	9.49	9.50	10.12	10.07	10.09	9.61	9.62	9.52	9.50
		(6.05)	(5.07)	(5.01)	(5.19)	(5.20)	(5.01)	(5.03)	(6.34)	(5.10)	(5.07)	(5.24)	(5.27)	(5.07)	(5.04)

Note: $\chi^2_4(0.05) = 9.49$

Table 6. The upper percentiles of test statistics and empirical Type I errors for the one sample case, $(p_1, p_2) = (4, 4), n_2 : fix$.

n_1	n_2	q_M	q_{MAE}	q_{MKP}	$q_{Q^*_M}$	$q_{Q^\dagger_M}$	q_{Y_M}	$q_{Y^\dagger_M}$	q	q_{AE}	q_{KP}	q_{Q^*}	$q^\dagger$	q_Y	$q^\dagger_Y$
		$(\alpha_{M\chi^2})$	(α_{MAE})	(α_{MKP})	$(\alpha_{Q^*_M})$	$(\alpha_{Q^\dagger_M})$	(α_{Y_M})	$(\alpha_{Y^\dagger_M})$	(α_{χ^2})	(α_{AE})	(α_{KP})	(α_{Q^*})	$(\alpha_{Q^\dagger})$	(α_Y)	$(\alpha_{Y^\dagger})$
10	10	20.77	14.11	22.03	10.68	12.46	9.47	10.68	26.95	15.33	27.65	9.20	13.48	8.79	10.80
		(23.94)	(11.70)	(4.35)	(6.85)	(9.24)	(4.97)	(7.27)	(30.03)	(14.64)	(4.75)	(4.61)	(10.00)	(3.70)	(7.55)
20	10	13.01	12.00	13.10	10.03	10.19	9.47	9.62	14.28	12.66	14.15	10.06	10.47	9.50	9.65
		(12.41)	(6.45)	(4.88)	(5.99)	(6.27)	(4.97)	(5.28)	(14.70)	(7.11)	(5.15)	(6.00)	(6.69)	(5.03)	(5.33)
30	10	22.20	20.00	22.26	16.50	16.84	15.13	15.87	25.20	21.46	24.68	16.41	17.43	15.51	15.88
		(17.84)	(7.63)	(4.95)	(6.50)	(7.00)	(4.38)	(5.60)	(22.80)	(8.96)	(5.42)	(6.28)	(7.71)	(5.00)	(5.62)
50	10	18.96	18.30	18.99	16.05	16.12	15.27	15.58	20.31	19.21	19.98	16.21	16.45	15.66	15.59
		(11.53)	(5.89)	(4.97)	(5.85)	(5.98)	(4.60)	(5.13)	(14.01)	(6.35)	(5.37)	(6.07)	(6.44)	(5.24)	(5.14)
100	10	17.09	16.95	17.11	15.76	15.77	15.38	15.51	17.66	17.42	17.49	15.90	15.94	15.63	15.51
		(7.85)	(5.21)	(4.98)	(5.41)	(5.43)	(4.79)	(5.01)	(8.85)	(5.33)	(5.22)	(5.62)	(5.69)	(5.20)	(5.01)
10	20	20.22	13.70	21.54	10.54	12.81	9.47	11.06	26.29	14.83	27.13	8.99	14.02	8.70	11.34
		(22.79)	(11.68)	(4.32)	(6.64)	(9.74)	(4.97)	(7.91)	(29.09)	(14.77)	(4.69)	(4.29)	(10.73)	(3.54)	(8.43)
20	20	12.77	11.80	12.86	9.99	10.21	9.48	9.69	13.96	12.41	13.92	9.96	10.47	9.46	9.74
		(11.88)	(6.42)	(4.88)	(5.93)	(6.32)	(4.98)	(5.42)	(14.17)	(7.11)	(5.05)	(5.85)	(6.74)	(4.95)	(5.52)
30	20	21.82	19.71	21.89	16.42	16.87	15.13	15.97	24.71	21.08	24.34	16.26	17.46	15.41	16.04
		(17.05)	(7.58)	(4.93)	(6.38)	(7.08)	(4.38)	(5.78)	(22.00)	(8.96)	(5.31)	(6.06)	(7.79)	(4.85)	(5.88)
50	20	18.78	18.16	18.82	15.99	16.10	15.24	15.60	20.07	19.03	19.82	16.11	16.40	15.60	15.61
		(11.15)	(5.83)	(4.95)	(5.77)	(5.93)	(4.57)	(5.16)	(13.57)	(6.30)	(5.29)	(5.92)	(6.37)	(5.14)	(5.18)
100	20	17.05	16.91	17.06	15.76	15.78	15.39	15.53	17.60	17.36	17.44	15.88	15.93	15.62	15.53
		(7.78)	(5.22)	(5.00)	(5.40)	(5.43)	(4.81)	(5.03)	(8.78)	(5.34)	(5.22)	(5.58)	(5.66)	(5.19)	(5.04)
20	50	27.21	20.87	27.70	16.84	19.24	14.94	17.58	33.90	22.67	34.12	15.15	20.58	14.51	17.98
		(25.22)	(11.69)	(4.69)	(6.90)	(10.11)	(4.10)	(8.30)	(34.30)	(15.47)	(4.90)	(4.55)	(11.40)	(3.52)	(8.93)
30	50	21.30	19.26	21.40	16.29	16.95	15.11	16.15	24.02	20.51	23.88	15.99	17.51	15.28	16.32
		(15.86)	(7.53)	(4.91)	(6.20)	(7.19)	(4.35)	(6.06)	(20.85)	(8.96)	(5.12)	(5.70)	(7.88)	(4.64)	(6.33)
50	50	18.53	17.91	18.53	15.97	16.12	15.29	15.68	19.75	18.70	19.54	16.02	16.39	15.57	15.73
		(10.64)	(5.86)	(5.00)	(5.75)	(5.99)	(4.63)	(5.29)	(12.97)	(6.35)	(5.23)	(5.80)	(6.37)	(5.09)	(5.37)
100	50	16.92	16.81	16.95	15.71	15.73	15.37	15.51	17.44	17.23	17.34	15.81	15.87	15.57	15.52
		(7.52)	(5.17)	(4.96)	(5.34)	(5.37)	(4.77)	(5.00)	(8.48)	(5.29)	(5.13)	(5.48)	(5.57)	(5.11)	(5.02)
20	100	26.94	20.52	27.39	16.74	19.53	14.93	17.93	33.51	22.21	33.83	14.95	20.94	14.41	18.53
		(24.42)	(11.89)	(4.71)	(6.78)	(10.53)	(4.08)	(8.83)	(33.61)	(15.81)	(4.85)	(4.30)	(11.94)	(3.40)	(9.76)
30	100	21.02	18.98	21.12	16.23	17.03	15.12	16.29	23.63	20.14	23.62	15.84	17.57	15.20	16.54
		(15.23)	(7.57)	(4.90)	(6.10)	(7.31)	(4.37)	(6.28)	(20.13)	(9.09)	(5.01)	(5.48)	(8.04)	(4.52)	(6.67)
50	100	16.92	16.81	16.95	15.71	15.73	15.37	15.51	17.44	17.23	17.34	15.81	15.87	15.57	15.52
		(7.52)	(5.17)	(4.96)	(5.34)	(5.37)	(4.77)	(5.00)	(8.48)	(5.29)	(5.13)	(5.48)	(5.57)	(5.11)	(5.02)
100	100	16.83	16.71	16.84	15.70	15.74	15.38	15.54	17.32	17.10	17.24	15.77	15.85	15.56	15.55
		(7.34)	(5.19)	(4.99)	(5.32)	(5.38)	(4.78)	(5.05)	(8.23)	(5.31)	(5.12)	(5.41)	(5.54)	(5.08)	(5.07)

Note: $\chi^2_8(0.05) = 15.51$

Table 7. The upper percentiles of test statistics and empirical Type I errors for the two sample case, $(p_1, p_2) = (2, 2), n_2 : fix$.

n_1	n_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{YMKP} (α_{YMKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{YKP} (α_{YKP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
10	10	13.54	12.15	13.66	10.01	10.33	9.89	9.77	14.96	12.63	14.90	9.98	10.66	9.87	10.01
		(13.62)	(6.98)	(4.86)	(5.96)	(6.51)	(5.84)	(5.57)	(16.20)	(8.10)	(5.06)	(5.88)	(7.00)	(5.77)	(6.00)
20	10	11.26	10.94	11.25	9.77	9.81	9.67	9.55	11.73	11.18	11.61	9.83	9.92	9.74	9.64
		(8.85)	(5.54)	(5.02)	(5.54)	(5.62)	(5.39)	(5.12)	(9.84)	(5.90)	(5.18)	(5.66)	(5.84)	(5.51)	(5.32)
30	10	10.61	10.50	10.63	9.65	9.66	9.57	9.49	10.90	10.66	10.84	9.71	9.74	9.63	9.56
		(7.41)	(5.21)	(4.97)	(5.33)	(5.35)	(5.17)	(5.01)	(8.02)	(5.42)	(5.11)	(5.44)	(5.50)	(5.29)	(5.15)
50	10	10.17	10.12	10.17	9.59	9.60	9.53	9.49	10.33	10.21	10.28	9.63	9.65	9.57	9.54
		(6.43)	(5.09)	(5.00)	(5.21)	(5.22)	(5.09)	(5.01)	(6.76)	(5.20)	(5.09)	(5.29)	(5.31)	(5.16)	(5.10)
100	10	9.83	9.81	9.83	9.54	9.55	9.50	9.49	9.91	9.86	9.88	9.57	9.57	9.53	9.52
		(5.72)	(5.03)	(5.01)	(5.12)	(5.12)	(5.03)	(5.01)	(5.88)	(5.09)	(5.06)	(5.18)	(5.17)	(5.09)	(5.07)
10	20	13.28	11.91	13.39	9.99	10.41	9.87	9.89	14.61	12.39	14.64	9.91	10.72	9.80	10.12
		(13.02)	(7.02)	(4.87)	(5.95)	(6.67)	(5.81)	(5.81)	(15.57)	(8.09)	(4.97)	(5.76)	(7.17)	(5.63)	(6.25)
20	20	11.08	10.82	11.11	9.71	9.77	9.68	9.54	11.53	11.06	11.48	9.76	9.87	9.72	9.63
		(8.48)	(5.46)	(4.96)	(5.45)	(5.56)	(5.41)	(5.11)	(9.44)	(5.81)	(5.09)	(5.54)	(5.76)	(5.49)	(5.29)
30	20	10.56	10.43	10.55	9.66	9.68	9.62	9.52	10.83	10.58	10.76	9.71	9.75	9.67	9.58
		(7.27)	(5.24)	(5.02)	(5.35)	(5.39)	(5.28)	(5.07)	(7.83)	(5.44)	(5.12)	(5.43)	(5.52)	(5.37)	(5.19)
50	20	10.14	10.08	10.13	9.60	9.60	9.56	9.50	10.29	10.18	10.24	9.63	9.64	9.59	9.54
		(6.37)	(5.10)	(5.02)	(5.21)	(5.23)	(5.14)	(5.03)	(6.69)	(5.20)	(5.09)	(5.28)	(5.30)	(5.21)	(5.10)
100	20	9.83	9.80	9.81	9.55	9.55	9.52	9.50	9.90	9.85	9.87	9.57	9.57	9.54	9.52
		(5.70)	(5.04)	(5.02)	(5.13)	(5.13)	(5.06)	(5.02)	(5.86)	(5.09)	(5.06)	(5.17)	(5.17)	(5.11)	(5.07)
10	50	12.98	11.67	13.16	9.90	10.44	9.71	9.98	14.25	12.14	14.42	9.77	10.74	9.59	10.22
		(12.27)	(6.97)	(4.78)	(5.79)	(6.74)	(5.44)	(5.98)	(14.79)	(8.00)	(4.82)	(5.53)	(7.24)	(5.19)	(6.43)
20	50	10.93	10.66	10.94	9.69	9.78	9.65	9.58	11.34	10.90	11.32	9.70	9.86	9.66	9.65
		(8.09)	(5.47)	(4.97)	(5.41)	(5.58)	(5.33)	(5.18)	(8.99)	(5.75)	(5.04)	(5.42)	(5.73)	(5.34)	(5.33)
30	50	10.41	10.32	10.43	9.61	9.64	9.58	9.50	10.66	10.47	10.64	9.63	9.69	9.61	9.55
		(6.97)	(5.17)	(4.96)	(5.23)	(5.30)	(5.20)	(5.03)	(7.50)	(5.32)	(5.02)	(5.28)	(5.40)	(5.25)	(5.12)
50	50	10.07	10.02	10.06	9.58	9.59	9.57	9.50	10.21	10.12	10.17	9.61	9.62	9.59	9.53
		(6.22)	(5.09)	(5.01)	(5.19)	(5.21)	(5.17)	(5.03)	(6.53)	(5.18)	(5.06)	(5.24)	(5.27)	(5.22)	(5.09)
100	50	9.80	9.78	9.79	9.55	9.55	9.53	9.50	9.87	9.83	9.84	9.56	9.57	9.55	9.52
		(5.64)	(5.04)	(5.02)	(5.11)	(5.12)	(5.09)	(5.02)	(5.79)	(5.09)	(5.06)	(5.15)	(5.16)	(5.13)	(5.06)
10	100	12.87	11.56	13.06	9.89	10.48	9.61	10.04	14.14	12.03	14.32	9.75	10.80	9.48	10.30
		(12.05)	(6.99)	(4.77)	(5.75)	(6.80)	(5.27)	(6.09)	(14.57)	(8.01)	(4.81)	(5.47)	(7.31)	(4.99)	(6.55)
20	100	10.84	10.58	10.86	9.68	9.77	9.60	9.59	11.23	10.81	11.23	9.68	9.85	9.59	9.66
		(7.91)	(5.46)	(4.96)	(5.38)	(5.57)	(5.24)	(5.22)	(8.78)	(5.71)	(5.00)	(5.37)	(5.71)	(5.22)	(5.35)
30	100	10.35	10.25	10.36	9.60	9.64	9.57	9.52	10.58	10.40	10.57	9.61	9.68	9.57	9.56
		(6.83)	(5.19)	(4.97)	(5.24)	(5.31)	(5.17)	(5.07)	(7.34)	(5.32)	(5.01)	(5.25)	(5.39)	(5.19)	(5.14)
50	100	10.01	9.97	10.01	9.56	9.57	9.55	9.50	10.14	10.07	10.12	9.58	9.60	9.57	9.52
		(6.09)	(5.06)	(4.99)	(5.15)	(5.16)	(5.14)	(5.02)	(6.38)	(5.13)	(5.02)	(5.18)	(5.22)	(5.17)	(5.06)
100	100	9.77	9.75	9.76	9.53	9.54	9.53	9.49	9.83	9.80	9.82	9.55	9.55	9.54	9.51
		(5.59)	(5.02)	(5.01)	(5.09)	(5.09)	(5.08)	(5.01)	(5.73)	(5.06)	(5.03)	(5.12)	(5.13)	(5.10)	(5.04)

Note: $\chi^2_4(0.05) = 9.49$

Table 8. The upper percentiles of test statistics and empirical Type I errors for the two sample case, $(p_1, p_2) = (4, 4), n_2 : fix$.

n_1	n_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{YMKP} (α_{YMKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{YKP} (α_{YKP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
10	10	30.56	22.09	31.22	17.09	19.48	16.53	17.44	38.98	23.64	38.99	15.16	20.95	15.50	18.19
		(31.08)	(13.56)	(4.65)	(7.18)	(10.30)	(6.79)	(8.12)	(40.59)	(18.61)	(4.99)	(4.59)	(11.56)	(4.99)	(9.13)
20	10	20.38	19.08	20.42	16.17	16.39	15.92	15.74	22.29	19.85	21.96	16.23	16.81	15.98	16.06
		(14.44)	(6.69)	(4.96)	(6.04)	(6.39)	(5.70)	(5.39)	(17.97)	(7.96)	(5.32)	(6.06)	(6.96)	(5.77)	(5.91)
30	10	18.50	17.98	18.52	15.93	15.99	15.72	15.58	19.56	18.50	19.32	16.06	16.26	15.84	15.80
		(10.62)	(5.72)	(4.97)	(5.66)	(5.79)	(5.36)	(5.12)	(12.63)	(6.40)	(5.29)	(5.84)	(6.17)	(5.54)	(5.47)
50	10	17.25	17.05	17.23	15.78	15.80	15.63	15.55	17.81	17.36	17.63	15.91	15.96	15.75	15.69
		(8.18)	(5.30)	(5.02)	(5.45)	(5.48)	(5.20)	(5.06)	(9.20)	(5.65)	(5.25)	(5.64)	(5.72)	(5.40)	(5.30)
100	10	16.33	16.30	16.35	15.62	15.62	15.52	15.50	16.59	16.46	16.52	15.70	15.71	15.60	15.58
		(6.43)	(5.05)	(4.98)	(5.19)	(5.19)	(5.03)	(4.98)	(6.89)	(5.20)	(5.10)	(5.30)	(5.31)	(5.15)	(5.11)
10	20	29.90	21.53	30.62	16.94	19.94	16.40	17.97	38.13	23.08	38.42	14.89	21.61	15.24	18.89
		(29.53)	(13.60)	(4.62)	(7.00)	(10.97)	(6.55)	(8.94)	(39.31)	(18.65)	(4.89)	(4.25)	(12.47)	(4.59)	(10.17)
20	20	20.02	18.80	20.10	16.07	16.39	15.94	15.80	21.83	19.57	21.65	16.05	16.78	15.92	16.11
		(13.69)	(6.59)	(4.92)	(5.90)	(6.38)	(5.75)	(5.48)	(17.16)	(7.80)	(5.18)	(5.82)	(6.94)	(5.69)	(5.98)
30	20	18.32	17.81	18.33	15.90	16.00	15.79	15.61	19.33	18.33	19.13	15.99	16.23	15.88	15.81
		(10.30)	(5.71)	(4.99)	(5.64)	(5.79)	(5.49)	(5.18)	(12.23)	(6.34)	(5.24)	(5.77)	(6.14)	(5.61)	(5.51)
50	20	17.14	16.97	17.14	15.74	15.77	15.65	15.53	17.67	17.28	17.54	15.84	15.90	15.74	15.65
		(7.92)	(5.25)	(5.00)	(5.38)	(5.42)	(5.23)	(5.04)	(8.92)	(5.56)	(5.18)	(5.53)	(5.63)	(5.38)	(5.25)
100	20	16.30	16.28	16.32	15.62	15.62	15.54	15.50	16.56	16.43	16.50	15.68	15.69	15.60	15.57
		(6.39)	(5.04)	(4.97)	(5.17)	(5.17)	(5.05)	(4.98)	(6.84)	(5.18)	(5.09)	(5.28)	(5.30)	(5.16)	(5.10)
10	50	29.50	20.97	30.11	16.83	20.53	16.08	18.63	37.51	22.52	37.92	14.64	22.35	14.77	19.71
		(28.19)	(13.84)	(4.67)	(6.88)	(11.74)	(5.97)	(9.91)	(38.25)	(18.90)	(4.84)	(3.94)	(13.52)	(3.90)	(11.42)
20	50	19.64	18.44	19.72	16.01	16.45	15.88	15.93	21.34	19.21	21.29	15.90	16.80	15.76	16.22
		(12.88)	(6.61)	(4.92)	(5.81)	(6.50)	(5.62)	(5.70)	(16.25)	(7.73)	(5.05)	(5.59)	(7.02)	(5.40)	(6.17)
30	50	18.05	17.56	18.06	15.86	16.00	15.79	15.67	18.97	18.08	18.87	15.88	16.19	15.81	15.83
		(9.67)	(5.70)	(4.99)	(5.56)	(5.80)	(5.48)	(5.27)	(11.50)	(6.24)	(5.14)	(5.59)	(6.09)	(5.50)	(5.53)
50	50	16.98	16.82	16.99	15.72	15.75	15.69	15.55	17.48	17.13	17.39	15.78	15.86	15.75	15.65
		(7.66)	(5.23)	(5.00)	(5.35)	(5.40)	(5.30)	(5.07)	(8.60)	(5.50)	(5.12)	(5.44)	(5.57)	(5.40)	(5.23)
100	50	16.24	16.22	16.26	15.60	15.61	15.57	15.50	16.48	16.38	16.44	15.65	15.67	15.62	15.56
		(6.28)	(5.03)	(4.97)	(5.16)	(5.16)	(5.10)	(4.98)	(6.70)	(5.16)	(5.06)	(5.24)	(5.26)	(5.18)	(5.08)
10	100	29.28	20.72	29.90	16.75	20.77	15.87	18.92	37.28	22.27	37.72	14.53	22.71	14.49	20.11
		(27.57)	(13.97)	(4.67)	(6.79)	(12.06)	(5.61)	(10.32)	(37.73)	(19.01)	(4.83)	(3.79)	(13.96)	(3.49)	(11.97)
20	100	19.45	18.24	19.53	15.98	16.50	15.75	16.02	21.09	19.01	21.11	15.81	16.83	15.58	16.29
		(12.38)	(6.61)	(4.92)	(5.73)	(6.54)	(5.41)	(5.83)	(15.68)	(7.67)	(4.98)	(5.45)	(7.03)	(5.13)	(6.26)
30	100	17.91	17.40	17.89	15.84	16.02	15.74	15.72	18.79	17.92	18.71	15.82	16.18	15.71	15.86
		(9.35)	(5.71)	(5.02)	(5.53)	(5.82)	(5.39)	(5.35)	(11.09)	(6.20)	(5.10)	(5.49)	(6.08)	(5.34)	(5.59)
50	100	16.86	16.71	16.87	15.69	15.74	15.65	15.56	17.33	17.02	17.27	15.72	15.83	15.69	15.64
		(7.41)	(5.22)	(4.99)	(5.29)	(5.37)	(5.24)	(5.08)	(8.27)	(5.44)	(5.07)	(5.34)	(5.51)	(5.29)	(5.21)
100	100	16.20	16.17	16.20	15.61	15.61	15.59	15.52	16.42	16.32	16.38	15.64	15.66	15.63	15.56
		(6.21)	(5.05)	(5.00)	(5.16)	(5.18)	(5.15)	(5.01)	(6.60)	(5.16)	(5.06)	(5.23)	(5.26)	(5.21)	(5.09)

Note: $\chi^2_8(0.05) = 15.51$

Table 9. The upper percentiles of test statistics and empirical Type I errors for the one sample case, $(n_1, n_2) = (30, 30), p_2 : fix$.

p_1	p_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{MKP} (α_{MKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{KP} (α_{KP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
2	2	10.49	10.38	10.50	9.64	9.66	9.62	9.51	10.76	10.53	10.71	9.68	9.72	9.65	9.57
		(7.16)	(5.21)	(5.00)	(5.31)	(5.35)	(5.27)	(5.06)	(7.73)	(5.39)	(5.08)	(5.38)	(5.47)	(5.33)	(5.17)
4	2	15.69	14.95	15.70	13.08	13.25	12.18	12.78	17.04	15.67	16.88	13.06	13.63	12.58	12.92
		(11.14)	(6.07)	(4.99)	(5.84)	(6.13)	(4.28)	(5.33)	(13.72)	(6.83)	(5.19)	(5.77)	(6.73)	(4.98)	(5.61)
6	2	20.18	18.92	20.21	16.21	16.48	14.86	15.78	22.51	19.91	22.28	16.01	17.26	15.35	16.18
		(13.93)	(6.61)	(4.96)	(6.07)	(6.51)	(3.96)	(5.46)	(18.05)	(8.07)	(5.22)	(5.73)	(7.61)	(4.76)	(6.11)
8	2	25.04	22.98	25.09	19.25	19.70	17.49	18.73	28.77	24.22	28.46	18.70	21.10	17.86	19.52
		(17.54)	(7.44)	(4.96)	(6.37)	(7.03)	(3.81)	(5.66)	(23.58)	(9.84)	(5.23)	(5.51)	(8.81)	(4.36)	(6.88)
10	2	30.41	27.16	30.45	22.29	22.98	20.10	21.66	36.14	28.62	35.84	21.11	25.30	20.14	23.04
		(22.16)	(8.48)	(4.96)	(6.70)	(7.62)	(3.77)	(5.93)	(30.39)	(12.29)	(5.18)	(5.10)	(10.36)	(3.86)	(7.95)

Note: $\chi^2_4(0.05) = 9.49, \chi^2_6(0.05) = 12.59, \chi^2_8(0.05) = 15.51, \chi^2_{10}(0.05) = 18.31, \chi^2_{12}(0.05) = 18.31$

Table 10. The upper percentiles of test statistics and empirical Type I errors for the two sample case, $(n_1, n_2) = (30, 30), p_2 : fix$.

p_1	p_2	q_M ($\alpha_{M\chi^2}$)	q_{MAE} (α_{MAE})	q_{YMKP} (α_{YMKP})	$q_{Q^*_M}$ ($\alpha_{Q^*_M}$)	$q_{Q^\dagger_M}$ ($\alpha_{Q^\dagger_M}$)	q_{Y_M} (α_{Y_M})	$q_{Y^\dagger_M}$ ($\alpha_{Y^\dagger_M}$)	q (α_{χ^2})	q_{AE} (α_{AE})	q_{YKP} (α_{YKP})	q_{Q^*} (α_{Q^*})	$q^\dagger$ ($\alpha_{Q^\dagger}$)	q_Y (α_Y)	$q^\dagger_Y$ ($\alpha_{Y^\dagger}$)
2	2	10.49	10.38	10.50	9.64	9.66	9.62	9.51	10.76	10.53	10.71	9.68	9.72	9.65	9.57
		(7.16)	(5.21)	(5.00)	(5.31)	(5.35)	(5.27)	(5.06)	(7.73)	(5.39)	(5.08)	(5.38)	(5.47)	(5.33)	(5.17)
4	2	14.11	13.91	14.11	12.81	12.85	12.91	12.64	14.61	14.19	14.52	12.87	12.98	12.96	12.75
		(7.94)	(5.32)	(5.00)	(5.39)	(5.47)	(5.58)	(5.09)	(8.93)	(5.65)	(5.13)	(5.47)	(5.69)	(5.65)	(5.30)
6	2	17.69	17.38	17.69	15.83	15.89	16.09	15.58	18.46	17.76	18.32	15.87	16.12	16.14	15.78
		(9.03)	(5.46)	(5.01)	(5.53)	(5.62)	(6.00)	(5.11)	(10.48)	(5.97)	(5.18)	(5.58)	(5.96)	(6.06)	(5.45)
8	2	21.31	20.83	21.31	18.75	18.83	19.21	18.40	22.37	21.32	22.18	18.76	19.17	19.23	18.70
		(10.37)	(5.65)	(5.01)	(5.66)	(5.80)	(6.48)	(5.15)	(12.31)	(6.35)	(5.22)	(5.68)	(6.28)	(6.47)	(5.60)
10	2	24.99	24.30	25.00	21.58	21.70	22.30	21.14	26.37	24.88	26.16	21.55	22.16	22.27	21.54
		(11.82)	(5.84)	(5.00)	(5.79)	(5.94)	(6.97)	(5.17)	(14.36)	(6.77)	(5.22)	(5.71)	(6.57)	(6.88)	(5.75)

Note: $\chi^2_4(0.05) = 9.49, \chi^2_6(0.05) = 12.59, \chi^2_8(0.05) = 15.51, \chi^2_{10}(0.05) = 18.31, \chi^2_{12}(0.05) = 18.31$

3.5 Concluding Remarks

We considered testing for equality of one and two mean vectors when the observations have two-step monotone missing data. In this case, T^2-type statistic can be decomposed into sum forms that can be provided asymptotic expansion has been discussed in previous study. However, these are not independent. In this paper, we proposed new test statistics that can be expressed as sum of statistics that are independent. We also considered transformed test statistics that convergence quickly. Further, we performed a Monte Carlo simulation and confirmed that test statistics we proposed showed better χ^2 approximation than original test statistics. Finally, we compared some procedures and confirmed that $q_{MKP}(\alpha)$ in (3.12) and $q_{MYKP}(\alpha)$ in (3.17) are good approximations when the sample size of the complete data is large. Additionally, $Y^\dagger_M$ is a good approximation when considering the approximation accuracy and conservativeness. We recommend using the test statistics properly according to sample size and dimension. We are currently working on this problem with K-step ($K \geq 3$) monotone missing data.

Chapter 4

Conclusion

We considered some tests when the observations have two-step monotone missing data.

In Chapter 2, we discussed profile analysis. First, we derived the MLEs of several groups. Second, we considered two sample problem and gave the T^2-type test statistics under the three hypotheses using the MLEs given in Section 2.2. We also gave the likelihood ratio test statistic under the parallelism hypothesis for several groups. Then we proposed upper percentiles of the test statistics. Finally, we examine the accuracy of the approximations of the proposed test statistics by a Monte Carlo simulation. As a result, we confirmed that $F^*_{1,\alpha}$ and $F^*_{2,\alpha}$ are better approximations than the upper percentiles of a χ^2 distribution. We also confirmed that convergence to the asymptotic χ^2 distribution is improved by inputting ρ_m into the likelihood ratio statistic $-2\log\Lambda_m$.

In Chapter 3, we discussed testing for equality of mean vectors for one sample and two sample problem. We proposed new test statistics that can be expressed as sum of statistics that are independent. Then we derived an asymptotic expansion of the null distribution of the test statistics. We also considered transformed test statistics for the proposed test statistics and approximated upper percentiles of the distribution. Finally, we compared some procedures by a Monte Carlo simulation. Further, we confirmed that the test statistics we proposed showed better χ^2 approximation than the original test statistics.

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