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Regular matrix methods of summability and real interpolation

Abstract. We show that the Banach–Saks property with respect to a regular positive matrix method of summability is inherited by the real interpolation spaces from a space forming the interpolation family and possessing this property. The proof refers to the Galvin–Prikry theorem on Ramsey sets. The results apply to several matrix methods of summability, such as Cesàro, Nørlund or Hölder methods.

1. Introduction. One of the fundamental questions in interpolation theory is whether a property of Banach spaces forming the interpolation couple or family is inherited by interpolation spaces. A similar question can be posed for interpolated operators. The main results in the literature concern reflexivity, weak and strong compactness, or more general, properties related to closed operator ideals (see [10]).

The Banach–Saks property lies within reflexive spaces and, respectively, weakly compact operators. The interpolation theorem for the classical Banach–Saks property, that is, defined by arithmetic means, was obtained by Beauzamy in [1], where the main tool used in the proof is Rosenthal's result, closely related to spreading models.

In our paper we propose a different approach to this problem. Instead of spreading models, we refer more directly to the Galvin–Prikry theorem on

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Ramsey sets [4]. We apply a result concerning behavior of regular summation for direct sums of Banach spaces recently proved in [6]. We obtain a general result for all positive matrix methods of summability which include the lower triangular matrix of arithmetic means. We also extend results to the interpolation of finite families of Banach spaces developed by Sparr [11] and Yoshikawa [12].

2. Real interpolation. In interpolation of Banach spaces, passing from two to several spaces brings some difficulties. If $n > 1$, the interpolation spaces defined by J- and K-methods for Banach $(n+1)$-tuples are no longer equivalent (see [11, Remark 5.1]). Nevertheless, the inheritance of some properties is similar for couples and families of Banach spaces. We show that this is the case for regular summation and spaces defined by the K-method. We briefly recall basic facts on interpolation of finite families of Banach spaces (for details see [11], [12]).

Let $n \geq 1$ and $A_0, \ldots, A_n$ be Banach spaces, linearly and continuously embedded in a Hausdorff topological vector space V. Then $\vec{A} = (A_0, \ldots, A_n)$ is called a Banach $(n+1)$-tuple. The spaces

$$\Delta(\vec{A}) = A_0 \cap \ldots \cap A_n, \quad \Sigma(\vec{A}) = A_0 + \ldots + A_n$$

are Banach spaces with norms, respectively,

$$\|a\|_{\Delta(\vec{A})} = \max_{0 \leq k \leq n} \|a\|_{A_k}, \quad \|a\|_{\Sigma(\vec{A})} = \inf \sum_{k=0}^{n} \|a_k\|_{A_k},$$

where the infimum is taken over all decompositions

$$a = a_0 + \ldots + a_n$$

with $a_k \in A_k$ for $k = 0, \ldots, n$.

Let

$$\mathbb{R}^n_+ = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : t_k > 0,\ k = 1, \ldots, n\}$$

and

$$H^{n+1}_+ = \left\{ (\theta_0, \ldots, \theta_n) \in \mathbb{R}^{n+1} : \sum_{k=0}^{n} \theta_k = 1,\ \theta_k > 0,\ k = 0, \ldots, n \right\}.$$

For $\theta = (\theta_0, \ldots, \theta_n) \in H^{n+1}_+$ let $\xi_k = (\xi^k_1, \ldots, \xi^k_n)$ be given by the formula

$$\xi^k_i = \begin{cases} \theta_i - 1 & \text{if } k > 0 \text{ and } i = k \\ \theta_i & \text{otherwise} \end{cases},$$

where $i = 1, \ldots, n$ and $k = 0, \ldots, n$.

For $a \in \Sigma(\vec{A})$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$ and $1 \leq q < \infty$ we put

$$K_q(t, a) = \inf \left(\|a_0\|^q_{A_0} + \sum_{k=1}^{n} t^q_k \|a_k\|^q_{A_k} \right)^{1/q},$$

where the infimum is taken as for $\|a\|_{\Sigma(\vec{A})}$.

Let us now fix $1 \le p < \infty$. The Sparr real interpolation space $A_{\theta p;K}$ with respect to $\vec{A}$ consists of all $a \in \Sigma(\vec{A})$ for which the norm

$$(1)\qquad \|a\|_{\theta p;K} = \left(\int_{\mathbb{R}^n_+} \left(t_1^{-\theta_1} \dots t_n^{-\theta_n} K_1(t,a) \right)^p \frac{dt_1}{t_1} \dots \frac{dt_n}{t_n} \right)^{1/p}$$

is finite. The space $A_{\theta p;K}$ is a Banach space such that the inclusions

$$\Delta(\vec{A}) \subset A_{\theta p;K} \subset \Sigma(\vec{A})$$

are continuous. This construction of interpolation spaces is called the K-method.

The Sparr space can also be defined by a discrete norm. Given $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$, we write $2^v = (2^{v_1}, \dots, 2^{v_n})$. For vectors $\xi \in \mathbb{R}^n$ and $v \in \mathbb{Z}^n$ let $\xi \cdot v$ denote its standard scalar product. Then the norm

$$\|a\|_{\theta p;K_q} = \left(\sum_{v \in \mathbb{Z}^n} \left(2^{-\xi_0 \cdot v} K_q(2^v, a) \right)^p \right)^{1/p}$$

is equivalent to that of (1) if we put $q = 1$ (see [11, Remark 4.5]). Moreover, the norms $\|\cdot\|_{\theta p;K_q}$ and $\|\cdot\|_{\theta p;K_r}$ are equivalent for all $1 \le q, r < \infty$.

Our results in this paper depend on the behavior of regular summation for vector-valued l_p-spaces. Therefore we will use an interpolation method of Yoshikawa [12], where such spaces are directly involved.

For $1 \le p < \infty$, $\xi \in \mathbb{R}^n$ and a Banach space X let $l_p(\xi, X)$ be the Banach space of all families $w = (w(v))_{v \in \mathbb{Z}^n}$ such that $w(v) \in X$ for every $v \in \mathbb{Z}^n$ and

$$\|w\|_{l_p(\xi,X)} = \left(\sum_{v \in \mathbb{Z}^n} \left(2^{-\xi \cdot v} \|w(v)\|_X \right)^p \right)^{1/p}$$

is finite. Let us notice that

$$(2)\qquad \|a\|_{\theta p;K_p} = \inf \left(\sum_{k=0}^n \|a_k\|^p_{l_p(\xi_k, A_k)} \right)^{1/p},$$

where the infimum is taken over all decompositions $a = \sum_{k=0}^n a_k(v)$ for every $v \in \mathbb{Z}^n$ with $a_k = (a_k(v))_{v \in \mathbb{Z}^n} \in l_p(\xi_k, A_k)$ and $k = 0, \dots, n$.

Particularly useful in our considerations will be another equivalent norm introduced by Yoshikawa (see [12, Définition 1.6]). For $a \in \Sigma(\vec{A})$, $\theta \in H^{n+1}_+$ and $1 \le p < \infty$ let

$$(3)\qquad \|a\|_{\theta p} = \inf \left\{ \max_{0 \le k \le n} \|a_k\|_{l_p(\xi_k, A_k)} \right\},$$

where the infimum is taken as for $\|a\|_{\theta p;K_p}$. Since norms (2) and (3) are equivalent, the space

$$A_{\theta p} = \left\{ a \in \Sigma(\vec{A}) : \|a\|_{\theta p} < \infty \right\}$$

is equal to the Sparr space $A_{\theta p;K}$ up to an equivalent norm (see [11, Remark 4.6]).

3. Results. Let $[c_{ij}]$ be an infinite matrix of scalars. A sequence (x_j) in a Banach space X is called summable with respect to $[c_{ij}]$ if $\sum_{j=0}^{\infty} c_{ij}x_j$ is convergent for each $i \in \mathbb{N}$ and the sequence (y_i) with $y_i = \sum_{j=0}^{\infty} c_{ij}x_j$ converges in X. The matrix $[c_{ij}]$ is called a regular method of summability if every sequence (x_j) converging to $x \in X$ is summable to x with respect to $[c_{ij}]$.

We say that X has the Banach–Saks property with respect to the regular method of summability $[c_{ij}]$ (in short, the BS property with respect to $[c_{ij}]$), if every bounded sequence in X has a subsequence summable with respect to $[c_{ij}]$.

If $c_{ij} \geq 0$ for all i, j, a regular method of summability $[c_{ij}]$ is called positive. This class of summability methods includes many classical methods, such as the Hölder method (H, k) of order $k \geq 1$, the regular Nørlund method (N, p_n) and the Cesàro method (C, k) with $k \geq 1$ (see [2], [5] for details).

The Silverman–Toeplitz theorem [3] states that a matrix $[c_{ij}]$ is a regular method of summability if and only if:

(i) there is $M > 0$ such that $\sum_{j=0}^{\infty} |c_{ij}| < M$ for every row i;
(ii) $\lim_{i\to\infty} c_{ij} = 0$ for every column j;
(iii) $\lim_{i\to\infty} \sum_{j=0}^{\infty} c_{ij} = 1$.

The key result which we use in the proof of our theorem depends on the following lemma proved in [6] with the use of the Galvin–Prikry theorem [4].

Lemma 3.1. *Let X be a Banach space with the Banach–Saks property with respect to the regular method of summability $[c_{ij}]$. Let (x_j) be a bounded sequence in X. For every $\varepsilon > 0$ there exist $N \geq 0$ and a subsequence (x'_j) of (x_j) such that for every subsequence (x''_j) of (x'_j) and all $m, n \geq N$,*

$$\left\| \sum_{j=0}^{\infty} c_{mj}x''_j - \sum_{j=0}^{\infty} c_{nj}x''_j \right\| < \varepsilon.$$

Let Y be a Banach space with a basis (e_v) such that if $|\alpha_v| \leq |\beta_v|$ for all $v \in \mathbb{N}$, then

$$\left\| \sum_{v=0}^{\infty} \alpha_v e_v \right\|_Y \leq \left\| \sum_{v=0}^{\infty} \beta_v e_v \right\|_Y .$$

In particular, every unconditional basis of a real Banach space with unconditional constant 1 has this property (see [8]).

Given a sequence (X_v) of Banach spaces, by a direct sum $Y(X_v)$ we mean the space of all sequences $x = (x(v))$ such that $x(v) \in X_v$ for every $v \in \mathbb{N}$ and $\sum_{v=0}^{\infty} \|x(v)\|_{X_v} e_v \in Y$. Then $Y(X_v)$ is a Banach space with the norm

$$\|x\|_{Y(X_v)} = \left\| \sum_{v=0}^{\infty} \|x(v)\|_{X_v} e_v \right\|_Y .$$

The Cauchy criterion provided by Lemma 3.1 leads to the following general result proved in [6]. In particular, this result can be applied to spaces $l_p(\xi, X)$ used in the discrete interpolation methods.

Theorem 3.2. *A direct sum $Y(X_v)$ has the Banach–Saks property with respect to the regular positive method of summability $[c_{ij}]$ if and only if Y and all X_v have this property.*

Our aim is to prove that the BS property with respect to $[c_{ij}]$ is carried over to interpolation spaces by even one space forming the interpolation family and possessing this property. The direct use of norm (3) is not enough to obtain such result. On the other hand, as noticed in [7], the conclusion of Proposition 2.6 in [12] (that norm (3) can be expressed by a logarithmically convex equality) is not justified. Nevertheless, using standard methods applied by Lions and Peetre in [9] for Banach couples, we can obtain for Banach families a logarithmically convex estimate up to a constant factor. There is a constant $c_\theta > 0$ such that for every $a \in A_{\theta p}$,

$$\|a\|_{\theta p} \leq c_\theta \prod_{k=0}^{n} \|a_k\|_{l_p(\xi_k, A_k)}^{\theta_k},$$

where $a = \sum_{k=0}^{n} a_k$ as in the definition of $\|a\|_{\theta p}$ (see [7]).

Theorem 3.3. *Let $[c_{ij}]$ be a regular positive method of summability, $\theta \in H_+^{n+1}$ and $1 < p < \infty$. If l_p and at least one of the spaces $A_0, \ldots, A_n$ have the Banach–Saks property with respect to $[c_{ij}]$, then so have all real interpolation spaces $A_{\theta p}$ with respect to the Banach $(n+1)$-tuple $\vec{A} = (A_0, \ldots, A_n)$.*

Proof. We show that if A_0 has the BS property with respect to $[c_{ij}]$, then each bounded sequence $(a_j) \subset A_{\theta p}$ contains a subsequence (a'_j) such that $(\sum_{j=0}^{\infty} c_{ij} a'_j)$ is a Cauchy sequence. The proof for A_k with $1 \leq k \leq n$ runs similarly.

We restrict our attention to sequences (a_j) in the open unit ball of $A_{\theta p}$. Then, for every $j = 0, 1, 2, \ldots$, in the unit balls of

$$l_p(\xi_0, A_0), \ldots, l_p(\xi_n, A_n)$$

there exist, respectively,

$$x_{0j} = (a_{0j}(v))_{v \in \mathbb{Z}^n}, \ldots, x_{nj} = (a_{nj}(v))_{v \in \mathbb{Z}^n}$$

such that

$$a_{0j}(v) + \ldots + a_{nj}(v) = a_j$$

for all $v \in \mathbb{Z}^n$.

Fix $\varepsilon > 0$ and let $\theta = (\theta_0, \ldots, \theta_n) \in H_+^{n+1}$. By the Silverman–Toeplitz theorem, there is $M > 0$ such that $\sum_{j=0}^{\infty} |c_{ij}| < M$ for every $i \in \mathbb{N}$. Take $\varepsilon_0 > 0$ such that

$$c_\theta \varepsilon_0^{\theta_0} (2M)^{1-\theta_0} < \varepsilon.$$

Since l_p and A_0 have the BS property with respect to $[c_{ij}]$, by Theorem 3.2, $l_p(\xi_0, A_0)$ also has this property. Thus, for such chosen ε_0, there exist $N \geq 0$ and a subsequence (x'_{0j}) of (x_{0j}) such that for all $m, n \geq N$,

$$\left\| \sum_{j=0}^{\infty} c_{mj} x'_{0j} - \sum_{j=0}^{\infty} c_{nj} x'_{0j} \right\|_{l_p(\xi_0, A_0)} < \varepsilon_0.$$

For each $k = 1, \ldots, n$ let the subsequence (x'_{kj}) of (x_{kj}) correspond to the subsequence (x'_{0j}) of (x_{0j}), that is $x'_{kj} = (a'_{kj}(v))_{v \in \mathbb{Z}^n}$ with

$$a'_{0j}(v) + \ldots + a'_{nj}(v) = a'_j$$

for all $v \in \mathbb{Z}^n$. It follows that for all $m, n \geq N$,

$$\begin{aligned}
&\left\| \sum_{j=0}^{\infty} c_{mj} a'_j - \sum_{j=0}^{\infty} c_{nj} a'_j \right\|_{\theta p} \\
&\leq c_\theta \prod_{k=0}^{n} \left\| \sum_{j=0}^{\infty} c_{mj} x'_{kj} - \sum_{j=0}^{\infty} c_{nj} x'_{kj} \right\|_{l_p(\xi_k, A_k)}^{\theta_k} \\
&= c_\theta \left\| \sum_{j=0}^{\infty} c_{mj} x'_{0j} - \sum_{j=0}^{\infty} c_{nj} x'_{0j} \right\|_{l_p(\xi_0, A_0)}^{\theta_0} \prod_{k=1}^{n} \left\| \sum_{j=0}^{\infty} c_{mj} x'_{kj} - \sum_{j=0}^{\infty} c_{nj} x'_{kj} \right\|_{l_p(\xi_k, A_k)}^{\theta_k} \\
&\leq c_\theta \varepsilon_0^{\theta_0} \prod_{k=1}^{n} \left(\sum_{j=0}^{\infty} |c_{mj} - c_{nj}| \, \|x'_{kj}\|_{l_p(\xi_k, A_k)} \right)^{\theta_k} \\
&\leq c_\theta \varepsilon_0^{\theta_0} \prod_{k=1}^{n} \left[\left(\sum_{j=0}^{\infty} |c_{mj}| + \sum_{j=0}^{\infty} |c_{nj}| \right) \max_{j \geq 0} \|x'_{kj}\|_{l_p(\xi_k, A_k)} \right]^{\theta_k} \\
&\leq c_\theta \varepsilon_0^{\theta_0} \prod_{k=1}^{n} (2M)^{\theta_k} \\
&= c_\theta \varepsilon_0^{\theta_0} (2M)^{1-\theta_0} < \varepsilon.
\end{aligned}$$

Thus $(\sum_{j=1}^{\infty} c_{ij} a'_j)$ is a Cauchy sequence and $A_{\theta p}$ has the BS property with respect to $[c_{ij}]$. □

The K-method of Sparr [11] is a generalization of the Lions–Peetre [9] real interpolation method defined for Banach couples. For $n = 1$ the vector $\theta = (\theta_0, \theta_1) \in H^2_+$ is replaced by a parameter $\theta \in (0, 1)$ which corresponds to $(1 - \theta, \theta)$ in Sparr's notation. The following corollary extends Beauzamy's result [1] on the interpolation of the classical Banach–Saks property to all positive regular methods of summability.

Corollary 3.4. *If l_p and at least one of the spaces A_0, A_1 have the Banach–Saks property with respect to $[c_{ij}]$, then so have all Lions–Peetre's real interpolation spaces $A_{\theta p}$ with respect to the interpolation couple $\vec{A} = (A_0, A_1)$.*

References

[1] Beauzamy, B., *Propriété de Banach–Saks*, Studia Math. **66** (1979/80), 227–235.

[2] Boos, J., *Classical and Modern Methods in Summability. Assisted by Peter Cass*, Oxford University Press, Oxford, 2000.

[3] DeVito, C. L., *Functional Analysis*, Academic Press, Inc., New York–London, 1978.

[4] Galvin, F., Prikry, K., *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198.

[5] Hardy, G. H., *Divergent Series*, the Clarendon Press, Oxford, 1949.

[6] Kryczka, A., Kurlej, K., *Regular methods of summability for direct sums of Banach spaces*, J. Math. Anal. Appl. **494** no. 1, (2021), 124636.

[7] Kutzarova, D., Nikolova, L. I., Prus, S., *Infinite-dimensional geometric properties of real interpolation spaces*, Math. Nachr. **191** (1998), 215–228.

[8] Lindenstrauss, J., Tzafriri, L., *Classical Banach Spaces I*, Springer-Verlag, New York, 1977.

[9] Lions, J.-L., Peetre, J., *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. **19** (1964), 5–68.

[10] Pietsch, A., *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007.

[11] Spar, G., *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. **99** (1974), 247–316.

[12] Yoshikawa, A., *Sur la théorie d'espaces d'interpolation—les espaces de moyenne de plusieurs espaces de Banach*, J. Fac. Sci. Univ. Tokyo **16** (1970), 407–468.

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