Quasi-Polynomial Time Algorithms for Free Quantum Games in Bounded Dimension

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Abstract

In a recent landmark result [Ji *et al.*, arXiv:2001.04383 (2020)], it was shown that approximating the value of a two-player game is undecidable when the players are allowed to share quantum states of unbounded dimension. In this paper, we study the computational complexity of two-player games when the dimension of the quantum systems is bounded by T. More specifically, we give a semidefinite program of size $\exp\big(\mathcal{O}\big(T^{12}(\log^2(AT)+\log(Q)\log(AT))/\epsilon^2\big)\big)$ to compute additive ϵ-approximations on the value of two-player free games with $T\times T$-dimensional quantum entanglement, where A and Q denote the number of answers and questions of the game, respectively. For fixed dimension T, this scales polynomially in Q and quasi-polynomially in A, thereby improving on previously known approximation algorithms for which worst-case run-time guarantees are at best exponential in Q and A. For the proof, we make a connection to the quantum separability problem and employ improved multipartite quantum de Finetti theorems with linear constraints that we derive via quantum entropy inequalities.

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1 Introduction

Thanks to the celebrated discovery by John Bell [4], it is well-known that quantum correlations can be used to overcome locality constraints, which was one of the earliest examples of advantages provided by quantum correlations over classical correlations. This led to the development of numerous quantum information processing tasks which make use of quantum correlations as a resource to outperform their classical analogues. In general, understanding the differences in the performance of distinct correlation sets for a given task is important both fundamentally and practically. A common way to measure the quantitative advantages

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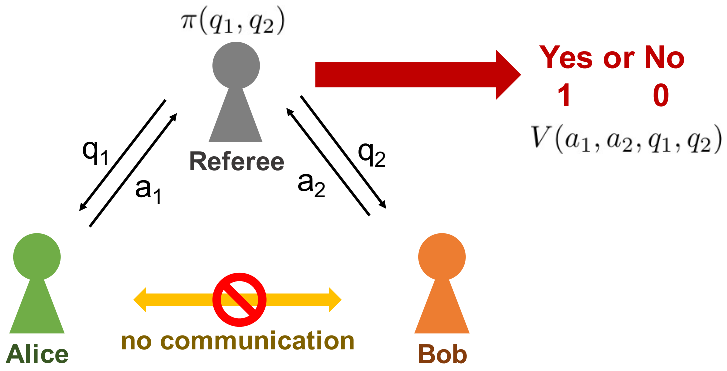

Figure 1 Two-player games. The referee gives Alice and Bob questions $q_1 \in Q_1$ and $q_2 \in Q_2$ according to the question probability distribution $\pi(q_1, q_2)$, and then Alice and Bob give answers $a_1 \in A_1$ and $a_2 \in A_2$ back to the referee depending on the questions they received. The referee decides whether Alice and Bob win or lose according to the rule function $V : A_1 \times A_2 \times Q_1 \times Q_2 \to \{0, 1\}$, where 0 denotes losing the game, and 1 denotes winning the game. Alice and Bob cannot communicate with each other during the game, but they can agree on a strategy beforehand. We are interested in determining the values of the game, i.e., the maximum achievable winning probabilities, for different classes of strategies. For simplicity, we assume that $|Q_1| = |Q_2| = Q$ and $|A_1| = |A_2| = A$.

of different sets of correlations is via a two-player game G (illustrated in Figure 1). In a two-player game, the performance of a given correlation set is quantified by the maximum achievable winning probability. For example, the classical value $\omega_C(G)$ is the maximum winning probability that can be achieved using shared randomness between the two players, while the quantum value $\omega_Q(G)$ is the maximum winning probability that can be achieved by sharing arbitrary quantum states between the players.

In general, it is hard to compute $\omega_C(G)$ and $\omega_Q(G)$ for the given description of a two-player game G. Approximating $\omega_C(G)$ within some constant multiplicative factor is NP-hard [2, 3], while approximating $\omega_Q(G)$ has recently been shown not to be possible for an algorithm running in finite time [20]. Despite these general hardness results, there are some special classes of two-player games for which $\omega_C(G)$ and $\omega_Q(G)$ can be approximated in polynomial time [10, 21, 1, 9]. In particular, for *free games*, i.e., games where the questions for the two players are chosen independently, there exists a quasi-polynomial time algorithm that can approximate $\omega_C(G)$ within any constant additive error [1, 9]. Also, in practice, the Navascués-Pironio-Acín (NPA) hierarchy [27, 29] provides semidefinite programming (SDP) upper bounds on $\omega_Q(G)$ which give approximately tight bounds for many games of interest.

1.1 Contributions

In this paper, we study the dimension-bounded quantum value $\omega_{Q(T)}(G)$ – the maximum winning probability that can be achieved by sharing quantum states of fixed dimension $T \times T$. It is easy to see that $\omega_{Q(1)}(G) = \omega_C(G)$ and $\omega_Q(G) = \sup_{T \geq 1} \omega_{Q(T)}(G)$. Computing $\omega_{Q(T)}(G)$ is of particular interest since it can be used as a *dimension witness* for an underlying system in semi-device-independent quantum information processing protocols, see for example [15]. SDP upper bounds have been derived for $\omega_{Q(T)}(G)$ in [25, 28, 26]. In [25], the authors exploit a connection to the quantum separability problem, and in [28, 26], the authors employ a moment matrix technique similar to the NPA hierarchy to derive SDP relaxations with better performance than the ones in [25]. However, the worst case runtime guarantees for these works is either not analytically quantified or is at best exponential in the number of questions Q and the number of answers A of the game G.

In our work, we provide approximation algorithms for $\omega_{Q(T)}(G)$ whose runtime has an improved dependence on both A and Q. More specifically, we construct a new hierarchy of SDP relaxations, providing a sequence of upper bounds for $\omega_{Q(T)}(G)$ for a given game G, and then derive analytical bounds on the convergence speed. This gives an upper bound on the computational complexity of calculating $\omega_{Q(T)}(G)$ in terms of the size of the game G. For the case of free games, a semidefinite program of size

$$\exp\left(\mathcal{O}\left(\frac{T^{12}}{\epsilon^2}\log(AT)\left(\log(Q)+\log(AT)\right)\right)\right) \tag{1}$$

is sufficient for computing additive ϵ-approximations of $\omega_{Q(T)}(G)$, where A and Q denote the number of answers and questions, respectively. The dependence is quasi-polynomial in A and polynomial in Q thus improving on the best previously known approximation algorithms [25, 28, 26], for which only exponential bounds in A and Q are known. In the classical limit ($T = 1$), our result recovers the quasi-polynomial time approximation scheme for computing $\omega_C(G)$ for two-player free games – which has a matching hardness result assuming the Exponential Time Hypothesis [1, 9]. Besides analysing free games, we give an algorithm for general games as well, leading to approximation algorithms that are still quasi-polynomial in A but exponential in Q.

We construct our SDP relaxations by drawing a connection to a variant of the quantum separability problem where the optimisation variables are additionally subject to some linear constraints. Similar variants of the quantum separability problem have been studied in [35, 34, 6]. The main tool we use to obtain the analytical convergence speed is improved multipartite quantum de Finetti theorems with linear constraints, which we derive in our work. One of the contributions towards this result, which we believe is of independent interest, is an improved version of the optimal loss in distinguishability relative to quantum side information.

1.2 Preliminaries on two-player games

A non-local game is a mathematical formulation for the correlations between distant parties. In this paper, we will consider *two-player games* where only two distant parties are involved. In this formulation, the correlation between two parties is considered to be a resource to win the games.

In a two-player game G, two spatially separated agents, Alice and Bob, need to provide correct answers $a_1 \in A_1$ and $a_2 \in A_2$ to the referee depending on the questions $q_1 \in Q_1$ and $q_2 \in Q_2$ they received (see Figure 1). The correct answers are determined by a given rule function of G

$$V : A_1 \times A_2 \times Q_1 \times Q_2 \to \{0, 1\}, \tag{2}$$

where 0 means the answer is incorrect, and 1 means the answer is correct. The questions q_1 and q_2 are chosen by the referee according to a given probability distribution $\pi(q_1, q_2)$ of G. A specific two-player game G can be represented by the pair of rule function $V(a_1, a_2, q_1, q_2)$ and question probability distribution $\pi(q_1, q_2)$, and hereafter we will denote a game G as (V, π). Alice and Bob cannot communicate with each other during the game, but they can agree on a strategy beforehand as well as make use of systems whose correlations lie within a given class. When only classical shared randomness is allowed, the correlations take the form

$$p(a_1, a_2|q_1, q_2) = e(a_1|q_1)d(a_2|q_2), \tag{3}$$

where $e(a_1|q_1)$ and $d(a_2|q_2)$ are conditional probability distributions for Alice and Bob respectively. That is, $\sum_{a_1} e(a_1|q_1) = 1\ \forall q_1 \in Q_1$, and $\sum_{a_2} d(a_2|q_2) = 1\ \forall q_2 \in Q_2$. When quantum resources are allowed, the correlations have a more general form

$$p(a_1, a_2|q_1, q_2) = \mathrm{tr}\left[\rho_{T\hat{T}}\left(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2)\right)\right], \tag{4}$$

where $\rho_{T\hat{T}}$ is a possibly entangled quantum state shared by Alice and Bob, and $\{E_T(a_1|q_1)\}_{a_1}$ and $\{D_{\hat{T}}(a_2|q_2)\}_{a_2}$ are positive-operator valued measurements (POVMs) performed by Alice and Bob respectively for given q_1 and q_2, i.e., $\sum_{a_1} E_T(a_1|q_1) = \mathbb{I}_T\ \forall q_1 \in Q_1$ and $\sum_{a_2} D_{\hat{T}}(a_2|q_2) = \mathbb{I}_{\hat{T}}\ \forall q_2 \in Q_2$.

The quantitative advantage of each set of correlations can be captured by the maximum winning probabilities achievable using the given correlation set. For a given two-player game $G = (V, \pi)$, the classical value is defined as

$$\omega_C(V, \pi) := \max_{(e,d)} \sum_{a1,q1,a2,q2} \pi(q_1, q_2)V(a_1, a_2, q_1, q_2)e(a_1|q_1)d(a_2|q_2), \tag{5}$$

and the quantum value is given by

$$\omega_Q(V, \pi) := \sup_{\substack{(E\otimes D,\rho)\\ \text{on } \mathcal{H}_{T\hat{T}}}} \sum_{\substack{a1,q1\\ a2,q2}} \pi(q_1, q_2)V(a_1, a_2, q_1, q_2)\, \mathrm{tr}\left[\rho_{T\hat{T}}\left(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2)\right)\right]. \tag{6}$$

Here, the optimisation is taken over not only states and measurements but also the Hilbert space $\mathcal{H}_{T\hat{T}}$. We can define the dimension-bounded quantum value as

$$\omega_{Q(T)}(V, \pi) := \max_{\substack{(E\otimes D,\, \rho)\\ \text{on}\\ \mathbb{C}^T \otimes \mathbb{C}^T}} \sum_{\substack{a1,q1\\ a2,q2}} \pi(q_1, q_2)V(a_1, a_2, q_1, q_2)\, \mathrm{tr}\left[\rho_{T\hat{T}}\left(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2)\right)\right], \tag{7}$$

which is the central object of investigation in this paper.

If not stated otherwise, we assume that the choice of questions for Alice and Bob are independent, i.e., $\pi(q_1, q_2) = \pi_1(q_1)\pi_2(q_2)$, which corresponds to *free games*. We denote $\mathcal{H}_A^{\otimes n}$ as A^n, and $\dim(\mathcal{H}_A)$ as $|A|$. For simplicity, we assume that $|Q_1| = |Q_2| = Q$ and $|A_1| = |A_2| = A$.

2 Derivation of semidefinite programming relaxations

2.1 Connection with quantum separability

Quantum separability problems are a special type of optimisation problems, where the optimisation is taken over the set of separable quantum states. We show that computing $\omega_{Q(T)}(V, \pi)$ for a given two-player game (V, π) can be rephrased as an instance of the tripartite quantum separability problem subject to additional linear constraints.

▶ **Lemma 1.** *For a two-player free game with $V(a_1,a_2,q_1,q_2)$, $\pi(q_1,q_2)=\pi_1(q_1)\pi_2(q_2)$, and $|T|^2$-dimensional quantum correlation, we have*

$$\omega_{Q(T)}(V,\pi) = |T|^2 \cdot \max_{(E,D,\rho)} \operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right)\left(E_{A_1Q_1T}\otimes D_{A_2Q_2\hat{T}}\otimes\rho_{S\hat{S}}\right)\right]$$
$$\begin{aligned} s.t. \quad & \rho_{S\hat{S}} \geq 0\,, \qquad \operatorname{tr}\left[\rho_{S\hat{S}}\right]=1 \\ & E_{A_1Q_1T} = \sum_{a_1,q_1} \pi_1(q_1)\,|a_1q_1\rangle\langle a_1q_1|_{A_1Q_1}\otimes\frac{E_T(a_1|q_1)}{|T|} \geq 0 \\ & D_{A_2Q_2\hat{T}} = \sum_{a_2,q_2} \pi_2(q_2)\,|a_2q_2\rangle\langle a_2q_2|_{A_2Q_2}\otimes\frac{D_{\hat{T}}(a_2|q_2)}{|T|} \geq 0 \\ & \operatorname{tr}_{A_1}\left[E_{A_1Q_1T}\right] = \sum_{q_1}\pi_1(q_1)\,|q_1\rangle\langle q_1|_{Q_1}\otimes\frac{\mathbb{I}_T}{|T|} \\ & \operatorname{tr}_{A_2}\left[D_{A_2Q_2\hat{T}}\right] = \sum_{q_2}\pi_2(q_2)\,|q_2\rangle\langle q_2|_{Q_2}\otimes\frac{\mathbb{I}_{\hat{T}}}{|T|}, \end{aligned} \tag{8}$$

where $\Phi_{T\hat{T}|S\hat{S}} = |\Phi\rangle\langle\Phi|_{T\hat{T}|S\hat{S}}$ is the (non-normalised) maximally-entangled state, $|\Phi\rangle_{T\hat{T}|S\hat{S}} = \sum_i |i\rangle_{T\hat{T}}|i\rangle_{S\hat{S}}$, and $V_{A_1A_2Q_1Q_2}$ is a diagonal matrix whose entries are given by the rule function $V(a_1,a_2,q_1,q_2)$.

To prove Lemma 1, we need a slightly modified version of the swap trick.

▶ **Lemma 2.** *Let M_{AB} be a linear operator on $\mathcal{H}_A\otimes\mathcal{H}_B$, and N_A be a linear operator on $\mathcal{H}_A$. Then, it holds that*

$$\operatorname{tr}\left[(N_A\otimes\mathbb{I}_B)M_{AB}\right] = \operatorname{tr}\left[\left(F_{\hat{A}|A}\otimes\mathbb{I}_B\right)\left(N_{\hat{A}}\otimes M_{AB}\right)\right], \tag{9}$$

where $F_{\hat{A}|A}$ denotes the swap operator between $\hat{A}$ and A.

Proof. By inspection, we have that

$$\begin{aligned} & \operatorname{tr}\left[\left(F_{\hat{A}|A}\otimes\mathbb{I}_B\right)\left(N_{\hat{A}}\otimes M_{AB}\right)\right] \\ &= \operatorname{tr}\left[\left(F_{\hat{A}|A}\otimes\mathbb{I}_B\right)\left(\sum_{i,j} n_{ij}\,|i\rangle\langle j|_{\hat{A}}\otimes\sum_{k,\ell,s,t} m_{(k\ell)(st)}\,|k\rangle\langle\ell|_A\otimes|s\rangle\langle t|_B\right)\right] \\ &= \operatorname{tr}\left[\sum_{i,j,k,\ell,s,t} n_{ij}\,m_{(k\ell)(st)}\,|k\rangle\langle j|_{\hat{A}}\otimes|i\rangle\langle\ell|_A\otimes|s\rangle\langle t|_B\right] \\ &= \sum_{i,j,s,t} n_{ij}\,m_{(ji)(st)} = \operatorname{tr}\left[(N_A\otimes\mathbb{I}_B)M_{AB}\right], \end{aligned} \tag{10}$$

where we used $N_{\hat{A}} = \sum_{i,j} n_{ij}\,|i\rangle\langle j|_{\hat{A}}$ and $M_{AB} = \sum_{k,\ell,s,t} m_{(k\ell)(st)}\,|k\rangle\langle\ell|_A\otimes|s\rangle\langle t|_B$. ◀

Proof of Lemma 1. Let us start from the expression for $\omega_{Q(T)}$ in Eq. (7). For free games, i.e. $\pi(q_1, q_2) = \pi_1(q_1)\pi_2(q_2)$, we can write

$$\omega_{Q(T)}(V,\pi) = |T|^2 \max_{E,D,\rho} \operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2} \otimes \rho_{T\hat{T}}\right)\left(E_{A_1Q_1T} \otimes D_{A_2Q_2\hat{T}}\right)\right] \tag{11}$$

$$\begin{aligned}
\text{s.t.}\quad & \rho_{T\hat{T}} \geq 0\,, \qquad \operatorname{tr}\left[\rho_{T\hat{T}}\right] = 1 \\
& E_{A_1Q_1T} = \sum_{a_1,q_1} \pi_1(q_1)\, |a_1q_1\rangle\langle a_1q_1|_{A_1Q_1} \otimes \frac{E_T(a_1|q_1)}{|T|} \geq 0 \\
& D_{A_2Q_2\hat{T}} = \sum_{a_2,q_2} \pi_2(q_2)\, |a_2q_2\rangle\langle a_2q_2|_{A_2Q_2} \otimes \frac{D_{\hat{T}}(a_2|q_2)}{|T|} \geq 0 \\
& \operatorname{tr}_{A_1}\left[E_{A_1Q_1T}\right] = \sum_{q_1} \pi_1(q_1)\, |q_1\rangle\langle q_1|_{Q_1} \otimes \frac{\mathbb{I}_T}{|T|} \\
& \operatorname{tr}_{A_2}\left[D_{A_2Q_2\hat{T}}\right] = \sum_{q_2} \pi_2(q_2)\, |q_2\rangle\langle q_2|_{Q_2} \otimes \frac{\mathbb{I}_{\hat{T}}}{|T|},
\end{aligned}$$

where we define $V_{A_1A_2Q_1Q_2} := \sum_{a_1,a_2,q_1,q_2} V(a_1,a_2,q_1,q_2)\, |a_1,a_2,q_1,q_2\rangle\langle a_1,a_2,q_1,q_2|$. Then, using Lemma 2 we can rewrite the objective function in Eq. (11) as

$$\begin{aligned}
&\operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2} \otimes \rho_{T\hat{T}}\right)\left(E_{A_1Q_1T} \otimes D_{A_2Q_2\hat{T}}\right)\right] \\
&= \operatorname{tr}\left[\left(\mathbb{I}_{A_1A_2Q_1Q_2} \otimes \rho_{T\hat{T}}\right)\left(\left(V_{A_1A_2Q_1Q_2} \otimes \mathbb{I}_{T\hat{T}}\right)\left(E_{A_1Q_1T} \otimes D_{A_2Q_2\hat{T}}\right)\right)\right] \\
&= \operatorname{tr}\left[\left(\mathbb{I}_{A_1A_2Q_1Q_2} \otimes F_{T\hat{T}|S\hat{S}}\right)\left(\left(\left(V_{A_1A_2Q_1Q_2} \otimes \mathbb{I}_{T\hat{T}}\right)\left(E_{A_1Q_1T} \otimes D_{A_2Q_2\hat{T}}\right)\right) \otimes \rho_{S\hat{S}}\right)\right] \\
&\hspace{25em} \text{(by Lemma 2)} \\
&= \operatorname{tr}\left[\left(\left(V_{A_1A_2Q_1Q_2} \otimes F_{T\hat{T}|S\hat{S}}\right)\left(E_{A_1Q_1T} \otimes D_{A_2Q_2\hat{T}} \otimes \rho_{S\hat{S}}\right)\right)\right],
\end{aligned} \tag{12}$$

which has a similar form to the objective function in Lemma 1 with the exception that $F_{T\hat{T}|S\hat{S}}$ replaces $\Phi_{T\hat{T}|S\hat{S}}$. To complete the proof, we write the swap operator $F_{A|\hat{A}}$ in terms of the (non-normalised) maximally-entangled state $\Phi_{A|\hat{A}} = |\Phi\rangle\langle\Phi|_{A|\hat{A}}$, where $|\Phi\rangle_{A|\hat{A}} = \sum_{i=1}^{d_A} |i\rangle_A |i\rangle_{\hat{A}}$. Namely, we have $F_{A|\hat{A}} = \Phi_{A|\hat{A}}^{T_A}$, where T_A denotes the transposition over the A subsystem. Redefining the variable ρ as ρ^T, we then immediately obtain Eq. (8) as this last step leaves the constraints invariant. ◀

In Lemma 1, the optimisation is now taken over all product states with respect to the tripartition $A_1Q_1T|A_2Q_2\hat{T}|S\hat{S}$ satisfying the stated linear constraints. Since product states are extreme points in the set of separable states, we can equivalently think of the above as an optimisation over the convex hull of the feasible states, where the feasible states are all product states satisfying the linear constraints. This gives the claimed connection to the quantum separability problem.

2.2 Hierarchy of semidefinite programming relaxations

In the previous section, we showed that $\omega_{Q(T)}(V,\pi)$ can be rephrased as a variant of the quantum separability problem which is subject to additional linear constraints. However, solving quantum separability problems is known to be NP-hard [16, 17], and our mapping does not necessarily make the problem more approachable. Fortunately, there are well-known relaxations for the quantum separability condition; the Doherty-Parrilo-Spedalieri (DPS) hierarchy [12] based on extendibility, which is strongly related to the notion of monogamy of entanglement [33].

▶ **Definition 3** (Extendibility). *A bipartite quantum state ρ_{AB} is n-extendible if there exists a multipartite quantum state ρ_{AB^n} such that*

$$\mathrm{tr}_{B^{n-1}}\left[\rho_{AB^n}\right] = \rho_{AB}, \quad \left(\mathcal{I}_A \otimes \mathcal{U}^{\pi}_{B^n}\right)\left(\rho_{AB^n}\right) = \rho_{AB^n} \quad \forall \pi \in \mathcal{S}(B^n), \tag{13}$$

where $\mathcal{S}(B^n)$ is the symmetric group over B^n, $\mathcal{U}^{\pi}_{B^n}(\cdot) = U^{\pi}_{B^n}(\cdot)(U^{\pi}_{B^n})^{\dagger}$ is the adjoint representation of the group, and $U^{\pi}_{B^n}$ is a unitary permutation operator acting on B^n.

Extendible states have two main advantages. Firstly, deciding if a state is n-extendible can be done efficiently via SDPs [11, 12]; for fixed n, the computation resources scale polynomially in the system dimension. Secondly, it is shown that a quantum state is n-extendible for all $n \geq 2$ if and only if the state is separable [14, 30]. Thus, the set of n-extendible states is a good outer approximation for the separable set and converges to the separable set when $n \to \infty$. The same idea can be generalised to the tripartite case as well; (n_1, n_2)-extendible states ρ_{ABC} with the two-fold extension $\rho_{AB^{n_1}C^{n_2}}$. As in the bipartite case, the set of (n_1, n_2)-extendible states converges to the set of tripartite separable states when $n_1 \to \infty$ and $n_2 \to \infty$ [13].

To derive SDP relaxations for $\omega_{Q(T)}(V, \pi)$ in Eq. (8), we can simply replace the optimisation variables with (n, n)-extendible states with respect to the appropriate tripartition.

$$\mathrm{sdp}_n(V, \pi, T) := |T|^2 \max_{\rho} \mathrm{tr}\left[\left(V_{A_1A_2Q_1Q_2} \otimes \Phi_{T\hat{T}|S\hat{S}}\right)\rho_{(A_1Q_1T)(A_2Q_2\hat{T})(S\hat{S})}\right] \tag{14}$$

$$\text{s.t.} \quad \rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n} \geq 0\,, \quad \mathrm{tr}\left[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\right] = 1 \tag{15}$$

$$\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n} \text{ perm. inv. on } (A_2Q_2\hat{T})^n \text{ wrt } (A_1Q_1T)(S\hat{S})^n \tag{16}$$

$$\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n} \text{ perm. inv. on } (S\hat{S})^n \text{ wrt } (A_1Q_1T)(A_2Q_2\hat{T})^n \tag{17}$$

$$\mathrm{tr}_{A_1}[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}] = \left(\sigma_{Q_1} \otimes \frac{\mathbb{I}_T}{|T|}\right) \otimes \rho_{(A_2Q_2\hat{T})^n(S\hat{S})^n} \tag{18}$$

$$\mathrm{tr}_{A_2}[\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}] = \left(\sigma_{Q_2} \otimes \frac{\mathbb{I}_{\hat{T}}}{|T|}\right) \otimes \rho_{(A_1Q_1T)(A_2Q_2\hat{T})^{(n-1)}(S\hat{S})^n} \tag{19}$$

$$\rho^{T_{A_1Q_1T}}_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n} \geq 0\,, \quad \rho^{T_{(A_2Q_2\hat{T})^n}}_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n} \geq 0, \ \ldots\,, \tag{20}$$

where $\sigma_{Q_i} = \sum_{q_i} \pi_i(q_i)\,|q_i\rangle\!\langle q_i|_{Q_i}$ for $i = 1, 2$, and the last line Eq. (20) contains all positive partial transpose (PPT) conditions with respect to all the cuts

$$A_1Q_1T : A_2^1Q_2^1\hat{T}^1 : \cdots : A_2^nQ_2^n\hat{T}^n : S^1\hat{S}^1 : \cdots : S^n\hat{S}^n. \tag{21}$$

Note that in addition to the n-extendibility conditions Eq. (16)–(17) enforced by the DPS hierarchy, we arrive at the additional linear constraints, Eq. (18)–(19), originating from the constraints in Eq. (8). These additional constraints are crucial in order to obtain the improved complexity bounds. Furthermore, we are able to combine our SDPs with the NPA constraints [27], so that our new hierarchy is guaranteed to produce at least as good outputs as the ones produced by the NPA hierarchy (see the full version [19, Section 5]),

$$\mathrm{sdp}_n^{\mathrm{NPA}}(V, \pi, T) := \mathrm{sdp}_n(V, \pi, T) \text{ with } \Gamma_n\big(\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}\big) \geq 0, \tag{22}$$

where $\Gamma_n(\rho)$ denotes the n-th level NPA matrix.

It is worth noting that $\mathrm{sdp}_n(V, \pi, T)$ in Eq. (14) is naturally upper bounded by 1.

▶ **Proposition 4.** *Let $sdp_n(V,\pi,T)$ be the n-th level SDP relaxation for the two-player free game with rule matrix V, probability distribution $\pi(q_1,q_2)=\pi_1(q_1)\pi_2(q_2)$, and $|T|^2$-dimensional quantum correlation. Then, we have that*

$$0 \leq sdp_n(V,\pi,T) \leq 1. \tag{23}$$

The proof can be found in the full version [19, Proposition 5].

3 Convergence of the hierarchy

3.1 Tripartite quantum de Finetti theorem with additional linear constraints

Quantum de Finetti theorems provide a quantitative bound on how close n-extendible states are to the set of separable states in trace distance as a function of both n and the system's dimensions. This information can be converted to the upper bound on the accuracy of our SDP relaxations. However, since the quantum separability problem for $\omega_{Q(T)}(V,\pi)$ in Eq. (8) is subject to the additional linear constraints, we cannot directly exploit the standard quantum de Finetti theorem and need an adapted version (we refer to [6, Example 3.7] for a discussion of counterexamples). What we need is an upper bound on how close n-extendible states satisfying the linear constraints are to the separable states satisfying the same linear constraints.

In this paper, we derive improved multipartite quantum de Finetti theorems with additional linear constraints employing the information-theoretic proof technique based on quantum entropy inequalities [8, 9]. Using this adapted quantum de Finetti theorems is crucial to obtain the improved complexity bounds on approximating $\omega_{Q(T)}(V,\pi)$ in the next section. Here, we state the tripartite version of the theorem.

▶ **Theorem 5.** *Let $\rho_{AB^{n_1}C^{n_2}}$ be a quantum state which is invariant under permutations on B^{n_1} with respect to AC^{n_2} and on C^{n_2} with respect to AB^{n_1}, satisfying for linear maps $\mathcal{E}_{A\to\tilde{A}}$, $\Lambda_{B\to\tilde{B}}$, and $\Gamma_{C\to\tilde{C}}$ and operators $\mathbf{X}_{\tilde{A}}$, $\mathbf{Y}_{\tilde{B}}$, and $\mathbf{Z}_{\tilde{C}}$ that*

$$\left(\mathcal{E}_{A\to\tilde{A}}\otimes\mathcal{I}_{B^{n_1}C^{n_2}}\right)\left(\rho_{AB^{n_1}C^{n_2}}\right)=\mathbf{X}_{\tilde{A}}\otimes\rho_{B^{n_1}C^{n_2}} \qquad \textit{linear constraint on } A \tag{24}$$

$$\left(\Lambda_{B\to\tilde{B}}\otimes\mathcal{I}_{B^{n_1-1}C^{n_2}}\right)\left(\rho_{B^{n_1}C^{n_2}}\right)=\mathbf{Y}_{\tilde{B}}\otimes\rho_{B^{n_1-1}C^{n_2}} \qquad \textit{linear constraint on } B \tag{25}$$

$$\left(\mathcal{I}_{B^{n_1}C^{n_2-1}}\otimes\Gamma_{C\to\tilde{C}}\right)\left(\rho_{B^{n_1}C^{n_2}}\right)=\mathbf{Z}_{\tilde{C}}\otimes\rho_{B^{n_1}C^{n_2-1}} \qquad \textit{linear constraint on } C. \tag{26}$$

Then, there exist a probability distribution $\{p_i\}_{i\in I}$ and sets of quantum states $\{\sigma_A^i\}_{i\in I}$, $\{\omega_B^i\}_{i\in I}$ and $\{\tau_C^i\}_{i\in I}$ such that we have that

$$\left\|\rho_{ABC}-\sum_{i\in I}p_i\sigma_A^i\otimes\omega_B^i\otimes\tau_C^i\right\|_1 \leq \min\left\{18^{3/2}\sqrt{|ABC|},4|BC|\right\}\times\sqrt{2\ln 2}\left(\sqrt{\frac{\log|A|+8\log|B|}{n_2}+\frac{\log|A|}{n_1}}\right) \tag{27}$$

$$\mathcal{E}_{A\to\tilde{A}}\left(\sigma_A^i\right)=\mathbf{X}_{\tilde{A}},\quad \Lambda_{B\to\tilde{B}}\left(\omega_B^i\right)=\mathbf{Y}_{\tilde{B}},\quad \Gamma_{C\to\tilde{C}}\left(\tau_C^i\right)=\mathbf{Z}_{\tilde{C}} \qquad \forall i\in I. \tag{28}$$

Like any other de Finetti theorem, Theorem 5 can be understood as a statement on the monogamy of entanglement; a multipartite system, described by an extendible state, cannot possess much entanglement between any tripartition. Instead of directly working with the trace distance, we prove the above theorem via quantum entropy inequalities and chain rules. This approach allows us to carefully quantify how correlations are divided between different partitions of the extendible states.

For k given quantum systems $A_1, ..., A_k$ and a classical system R described by the global state $\rho_{A_1 A_2 \cdots A_k R}$, the conditional multipartite quantum mutual information is defined as

$$I(A_1 : A_2 : \ldots : A_k|R) := \sum_{i=1}^{k} S(A_i R) - S(A_1 A_2 \ldots A_k R) - S(R), \tag{29}$$

where $S(A_i) = -\operatorname{tr}[\rho_{A_i} \log \rho_{A_i}]$ is the von Neumann entropy [23] of the marginal state ρ_{A_i}. This quantity has a few useful mathematical properties. One is its relation to the bipartite ones [9, Lemma 3]

$$I(A_1 : \ldots : A_k|R) = I(A_1 : A_2|R) + I(A_1 A_2 : A_3|R) + \ldots + I(A_1 \ldots A_{k-1} : A_k|R), \tag{30}$$

and another one is the chain rule

$$I(AB : C|D) = I(B : C|D) + I(A : C|BD). \tag{31}$$

The conditional multipartite quantum mutual information is mathematically equivalent to the relative entropy distance between the state and the tensor product of its conditional marginals

$$I(A_1 : A_2 : \ldots : A_k|R) = D(\rho_{A_1 \cdots A_k|R} \| \rho_{A_1|R} \otimes \ldots \otimes \rho_{A_k|R}), \tag{32}$$

where $\rho_{A_i|R}$ is the marginal state of the conditional $\rho_{A_1 \cdots A_k|R} = \rho_R^{-1/2} \rho_{A_1 \cdots A_k} \rho_R^{-1/2}$, and $D(\rho||\sigma) = \operatorname{tr}[\rho(\log \rho - \log \sigma)]$ is the relative entropy between ρ and σ whenever $\operatorname{supp}(\rho) \subset \operatorname{supp}(\sigma)$. The relative entropy can be further related to the trace distance via Pinsker's inequality. As the tensor product of marginal states is a separable state, if we can find an upper bound on the conditional multipartite quantum mutual information of an extendible state $\rho_{AB^{n_1}C^{n_2}}$, we can show Eq. (27) in Theorem 5.

For the first ingredient, we derive a general upper bound on the conditional multipartite quantum mutual information of a state with classical subsystems.

▶ **Lemma 6.** *Consider a quantum state $\rho_{AZ^{n_1}W^{n_2}}$ classical on the Z- and W-systems. Then, there exist $0 \le \bar{m} < n_1$ and $0 \le \bar{l} < n_2$ such that*

$$I(A : Z_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}} W^{\bar{l}}) \le \frac{\log|A|}{n_1} + \frac{\log|A| + \log|Z|}{n_2}. \tag{33}$$

Moreover, by Pinsker's inequality, this implies that

$$\mathbb{E}_{z^{\bar{m}} w^{\bar{l}}} \left\{ \left\| \rho_{AZ_{\bar{m}+1}W_{\bar{l}+1}|z^{\bar{m}}w^{\bar{l}}} - \rho_{A|z^{\bar{m}}w^{\bar{l}}} \otimes \rho_{Z_{\bar{m}+1}|z^{\bar{m}}w^{\bar{l}}} \otimes \rho_{W_{\bar{l}+1}|z^{\bar{m}}w^{\bar{l}}} \right\|_1^2 \right\} \le 2\ln 2 \left(\frac{\log|A|}{n_1} + \frac{\log|A| + \log|Z|}{n_2} \right). \tag{34}$$

Here, we use the notation $\rho_{A|z}$ for the conditional state after measurement on classical system Z when the measurement outcome is z, i.e.,

$$\rho_{A|z} := \frac{\operatorname{tr}_Z[\rho_{AZ}(\mathbb{I}_A \otimes |z\rangle\langle z|_Z)]}{\operatorname{tr}[\rho_{AZ}(\mathbb{I}_A \otimes |z\rangle\langle z|_Z)]}. \tag{35}$$

The proof of Lemma 6 is as follows.

Proof of Lemma 6. The multipartite quantum mutual information $I(A : Z_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}}W^{\bar{l}})$ can be expressed in terms of bipartite ones using Eq. (30):

$$I(A : Z_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}}W^{\bar{l}}) = I(A : Z_{\bar{m}+1}|Z^{\bar{m}}W^{\bar{l}}) + I(AZ_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}}W^{\bar{l}}). \tag{36}$$

The two terms in the right hand side (RHS) are the bipartite mutual information between quantum and classical systems, and this allows us to find an upper bound for each term using the chain rule in Eq. (31). Additionally, we also make use of a general upper bound

$$I(A : Z|X) \leq \log |A| \tag{37}$$

for a classical-quantum state ρ_{AZX} with classical Z and X systems [19, Lemma 13].

First term: For any l, it holds that

$$I(A : Z^{n_1}|W^l) = \sum_{m=0}^{n_1-1} I(A : Z_{m+1}|Z^m W^l) \leq \log |A|, \tag{38}$$

where the first equality is the chain rule in Eq. (31) and the second inequality is found by applying Eq. (37) to $I(A : Z^{n_1}|W^l)$. Then, summing over all l gives us

$$\sum_{m=0}^{n_1-1} \sum_{l=0}^{n_2-1} I(A : Z_{m+1}|Z^m W^l) \leq n_2 \log |A|. \tag{39}$$

Second term: Using the same argument, for any m, it holds that

$$I(AZ_{m+1} : W^{n_2}|Z^m) = \sum_{l=0}^{n_2-1} I(AZ_{m+1} : W_{l+1}|Z^m W^l) \leq \log |AZ_{m+1}|, \tag{40}$$

and summing over m gives us

$$\sum_{m=0}^{n_1-1} \sum_{l=0}^{n_2-1} I(AZ_{m+1} : W_{l+1}|Z^m W^l) \leq n_1 (\log |A| + \log |Z|). \tag{41}$$

Combining Eq. (39) and Eq. (41) gives

$$\begin{aligned} & n_2 \log |A| + n_1 (\log |A| + \log |Z|) \\ & \geq \sum_{m=0}^{n_1-1} \sum_{l=0}^{n_2-1} \left[I(A : Z_{m+1}|Z^m W^l) + I(AZ_{m+1} : W_{l+1}|Z^m W^l) \right] \\ & \geq n_1 n_2 \left[I(A : Z_{\bar{m}+1}|Z^{\bar{m}}W^{\bar{l}}) + I(AZ_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}}W^{\bar{l}}) \right], \end{aligned} \tag{42}$$

where $\bar{m}$ and $\bar{l}$ are the indices of the smallest element in the sum. Dividing both sides by $n_1 n_2$ gives us the desired relation,

$$\begin{aligned} I(A : Z_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}}W^{\bar{l}}) &= I(A : Z_{\bar{m}+1}|Z^{\bar{m}}W^{\bar{l}}) + I(AZ_{\bar{m}+1} : W_{\bar{l}+1}|Z^{\bar{m}}W^{\bar{l}}) \\ &\leq \frac{\log |A|}{n_1} + \frac{\log |A| + \log |Z|}{n_2}. \end{aligned} \tag{43}$$

This ends the proof of Eq. (33). Then, using Eq. (32) and Pinsker's inequality we can obtain Eq. (34). ◀

As another ingredient, we derive two different types of informationally complete measurements that achieve the optimal loss in distinguishability.

▶ **Lemma 7.**

1. ([8, Lemma 14]) *There exist fixed measurements $\mathcal{M}_A$, $\mathcal{M}_B$, and $\mathcal{M}_C$ with at most $|A|^8$, $|B|^8$, and $|C|^8$ outcomes, respectively, such that for every traceless Hermitian operator γ_{ABC} on $\mathcal{H}_{ABC}$ we have*

$$\|\gamma_{ABC}\|_1 \leq 18^{3/2}\sqrt{|ABC|} \cdot \|(\mathcal{M}_A \otimes \mathcal{M}_B \otimes \mathcal{M}_C)(\gamma_{ABC})\|_1. \tag{44}$$

2. *There exists a fixed measurement $\mathcal{M}_B$ with at most $|B|^6$ outcomes such that for every traceless Hermitian operator γ_{AB} on $\mathcal{H}_{AB}$ we have*

$$\|\gamma_{AB}\|_1 \leq 2|B| \cdot \|(\mathcal{I}_A \otimes \mathcal{M}_B)(\gamma_{AB})\|_1. \tag{45}$$

The first part is straightforward from [8, Lemma 14]. We remark that when a traceless Hermitian operator already has a classical subsystem, i.e., γ_{ABCZ} with classical Z-system, the dimension factor only includes the dimension of the quantum systems

$$\|\gamma_{ABCZ}\|_1 \leq 18^{3/2}\sqrt{|ABC|} \cdot \|(\mathcal{M}_A \otimes \mathcal{M}_B \otimes \mathcal{M}_C \otimes \mathcal{I}_Z)(\gamma_{ABCZ})\|_1. \tag{46}$$

This follows easily as $\|\sum_z \rho_A^z \otimes |z\rangle\langle z|\|_1 = \sum_z \|\rho_A^z\|_1$ for classical-quantum states ρ_{AZ}.

The proof of the second part is given in Section 5. The main idea is to identify the one-way quantum teleportation protocol as a candidate for the optimal measurement and is largely inspired by [22, Theorem 16]. Our result improves on the factor $\sqrt{18}B^{3/2}$ given in [9, Eq.(68)]. Moreover, as there exist quantum states ρ_{AB} and σ_{AB} such that [24]

$$\|\rho_{AB} - \sigma_{AB}\|_1 = 2 \quad \text{and} \quad \sup_{\mathcal{M}_B} \|(\mathcal{I}_A \otimes \mathcal{M}_B)(\rho_{AB} - \sigma_{AB})\|_1 = \frac{2}{|B|+1}, \tag{47}$$

our result establishes that the dimension dependence for the optimal loss in distinguishability relative to quantum side information is $\Theta(|B|)$. This answers a question left open in [6].

Then, for the extendible state $\rho_{AB^{n_1}C^{n_2}}$ in Theorem 5, applying the optimal measurement $\mathcal{M}$ as specified in Lemma 7 to the state (to make it partially classical), and applying Lemma 6 to the resulting classical-quantum state allows us to derive Theorem 5.

Proof of Theorem 5. Let $\mathcal{M}_{B\to Y}$ be a quantum-to-classical measurement from B to the classical system Y, and $\mathcal{M}_{C\to Z}$ be a quantum-to-classical measurement from C to the classical system Z. We apply these measurements to the quantum state $\rho_{AB^{n_1}C^{n_2}}$ and will denote the outcome classical-quantum state as $\rho_{AY^{n_1}Z^{n_2}}$. Then, according to Lemma 6, there exist $m \in \{0, \cdots, n_1 - 1\}$ and $\ell \in \{0, \cdots, n_2 - 1\}$ such that

$$\mathbb{E}_{y^m z^\ell}\left\{\left\|\rho_{AY_{m+1}Z_{\ell+1}|y^m z^\ell} - \rho_{A|y^m z^\ell} \otimes \rho_{Y_{m+1}|y^m z^\ell} \otimes \rho_{Z_{\ell+1}|y^m z^\ell}\right\|_1^2\right\} \leq 2\ln 2\left(\frac{\log|A|}{n_1} + \frac{\log|A| + \log|Y|}{n_2}\right). \tag{48}$$

As $\rho_{AB^{n_1}C^{n_2}}$ is invariant under permutations of the systems B^{n_1} and C^{n_2}, we can always find m and l satisfying Eq. (48).

Now, let us define

$$\gamma_{ABC} \equiv \rho_{AB_{m+1}C_{\ell+1}|y^m z^\ell} - \rho_{A|y^m z^\ell} \otimes \rho_{B_{m+1}|y^m z^\ell} \otimes \rho_{C_{\ell+1}|y^m z^\ell}. \tag{49}$$

Note that

$$\mathcal{I}_A \otimes \mathcal{M}_{B\to Y} \otimes \mathcal{M}_{C\to Z}(\gamma_{ABC}) = \rho_{AY_{m+1}Z_{\ell+1}|y^m z^\ell} - \rho_{A|y^m z^\ell} \otimes \rho_{Y_{m+1}|y^m z^\ell} \otimes \rho_{Z_{\ell+1}|y^m z^\ell}. \tag{50}$$

Using the second part of Lemma 7 iteratively, we can obtain

$$\begin{aligned}\|\gamma_{ABC}\|_1 &\leq 2|C|\|(\mathcal{I}_{AB}\otimes\mathcal{M}_{C\to Z})(\gamma_{ABC})\|_1\\ &\leq 2|B|\times 2|C|\|(\mathcal{I}_{AC}\otimes\mathcal{M}_{B\to Y})(\mathcal{I}_{AB}\otimes\mathcal{M}_{C\to Z})(\gamma_{ABC})\|_1\\ &= 4|BC|\|(\mathcal{I}_{A}\otimes\mathcal{M}_{B\to Y}\otimes\mathcal{M}_{C\to Z})(\gamma_{ABC})\|_1, \end{aligned} \tag{51}$$

with $|Y|\leq|B|^6$. We can also exploit the first part of Lemma 7 to obtain

$$\begin{aligned}\|\gamma_{ABC}\|_1 &\leq \sqrt{18^3|ABC|}\,\|(\mathcal{M}_{A}\otimes\mathcal{M}_{B\to Y}\otimes\mathcal{M}_{C\to Z})(\gamma_{ABC})\|_1\\ &\leq \sqrt{18^3|ABC|}\,\|(\mathcal{I}_{A}\otimes\mathcal{M}_{B\to Y}\otimes\mathcal{M}_{C\to Z})(\gamma_{ABC})\|_1\end{aligned} \tag{52}$$

with $|Y|\leq|B|^8$, where the second inequality follows from the monotonicity of the trace norm under completely positive and trace preserving (CPTP) maps. Depending on the dimensions, we can freely choose the tighter bound between the two cases. Combining Eq. (48) with the above two results we obtain

$$\begin{aligned}&\mathbb{E}_{y^m z^\ell}\left\{\left\|\rho_{AB_{m+1}C_{\ell+1}|y^m z^\ell}-\rho_{A|y^m z^\ell}\otimes\rho_{B_{m+1}|y^m z^\ell}\otimes\rho_{C_{\ell+1}|y^m z^\ell}\right\|_1^2\right\}\\ &\qquad\leq \min\left\{\sqrt{18^3|ABC|},4|BC|\right\}^2\times 2\ln 2\left(\frac{\log|A|}{n_1}+\frac{\log|A|+8\log|B|}{n_2}\right).\end{aligned} \tag{53}$$

Then, we have

$$\begin{aligned}&\left\|\rho_{AB_{m+1}C_{\ell+1}}-\mathbb{E}_{y^m z^\ell}\left\{\rho_{A|y^m z^\ell}\otimes\rho_{B_{m+1}|y^m z^\ell}\otimes\rho_{C_{\ell+1}|y^m z^\ell}\right\}\right\|_1\\ &\leq\ \mathbb{E}_{y^m z^\ell}\left\{\left\|\rho_{AB_{m+1}C_{\ell+1}|y^m z^\ell}-\rho_{A|y^m z^\ell}\otimes\rho_{B_{m+1}|y^m z^\ell}\otimes\rho_{C_{\ell+1}|y^m z^\ell}\right\|_1\right\}\\ &\leq\ \sqrt{\mathbb{E}_{y^m z^\ell}\left\{\left\|\rho_{AB_{m+1}C_{\ell+1}|y^m z^\ell}-\rho_{A|y^m z^\ell}\otimes\rho_{B_{m+1}|y^m z^\ell}\otimes\rho_{C_{\ell+1}|y^m z^\ell}\right\|_1^2\right\}}\\ &\leq\ \min\left\{\sqrt{18^3|ABC|},4|BC|\right\}\times\sqrt{2\ln 2}\left(\sqrt{\frac{\log|A|}{n_1}+\frac{\log|A|+8\log|B|}{n_2}}\right),\end{aligned} \tag{54}$$

where we used the triangular inequality for Schatten p-norms in the second line and the concavity of the square function in the third line. As $\mathbb{E}_{y^m z^\ell}\left\{\rho_{A|y^m z^\ell}\otimes\rho_{B_{m+1}|y^m z^\ell}\otimes\rho_{C_{\ell+1}|y^m z^\ell}\right\}$ is a separable state with respect to the tripartition $A|B|C$, this proves the first half of the theorem.

The remaining part is to check whether $\rho_{A|y^m z^\ell}$, $\rho_{B_{m+1}|y^m z^\ell}$ and $\rho_{C_{\ell+1}|y^m z^\ell}$ satisfy the desired linear constraints. Let us denote $M_{B_i}^{y_i}$ and $M_{C_i}^{z_i}$ as the POVM elements of the measurements $\mathcal{M}_{B_i\to Y_i}$ and $\mathcal{M}_{C_i\to Z_i}$ corresponding to the measurement outcomes y_i and z_i, respectively. Then, we find

$$\begin{aligned}&\mathcal{E}_{A\to\tilde{A}}\left(\sigma_A^i\right)=\mathcal{E}_{A\to\tilde{A}}\left(\rho_{A|y^m z^\ell}\right)\\ &=\frac{\mathrm{Tr}_{B^m C^\ell}\left[(\mathbb{I}_A\otimes M_{B_1}^{y_1}\otimes\cdots\otimes M_{B_m}^{y_m}\otimes M_{C_1}^{z_1}\otimes\cdots\otimes M_{C_\ell}^{z_\ell})\mathcal{E}_{A\to\tilde{A}}\left(\rho_{AB^m C^\ell}\right)\right]}{\mathrm{Tr}\left[(\mathbb{I}_A\otimes M_{B_1}^{y_1}\otimes\cdots\otimes M_{B_m}^{y_m}\otimes M_{C_1}^{z_1}\otimes\cdots\otimes M_{C_\ell}^{z_\ell})\rho_{AB^m C^\ell}\right]}\\ &=\frac{\mathrm{Tr}_{B^m C^\ell}\left[(\mathbb{I}_A\otimes M_{B_1}^{y_1}\otimes\cdots\otimes M_{B_m}^{y_m}\otimes M_{C_1}^{z_1}\otimes\cdots\otimes M_{C_\ell}^{z_\ell})\left(\mathcal{X}_{\tilde{A}}\otimes\rho_{B^m C^\ell}\right)\right]}{\mathrm{Tr}\left[(\mathbb{I}_A\otimes M_{B_1}^{y_1}\otimes\cdots\otimes M_{B_m}^{y_m}\otimes M_{C_1}^{z_1}\otimes\cdots\otimes M_{C_\ell}^{z_\ell})\rho_{AB^m C^\ell}\right]}\\ &=\mathcal{X}_{\tilde{A}}.\end{aligned} \tag{55}$$

$$
\begin{aligned}
&\Lambda_{B\to\tilde{B}}\left(\omega_B^i\right) = \Lambda_{B\to\tilde{B}}\left(\rho_{B_{m+1}|y^m z^\ell}\right) && (56)\\
&= \frac{\mathrm{Tr}_{B^m C^\ell}\left[(\mathbb{I}_{\tilde{B}} \otimes M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell})\Lambda_{B\to\tilde{B}}\left(\rho_{B^{m+1}C^\ell}\right)\right]}{\mathrm{Tr}\left[(M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes \mathbb{I}_{B_{m+1}} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell})\rho_{B^{m+1}C^\ell}\right]}\\
&= \frac{\mathrm{Tr}_{B^m C^\ell}\left[(\mathbb{I}_{\tilde{B}} \otimes M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell})\left(\mathcal{Y}_{\tilde{B}} \otimes \rho_{B^m C^\ell}\right)\right]}{\mathrm{Tr}\left[(M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes \mathbb{I}_{B_{m+1}} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell})\rho_{B^{m+1}C^\ell}\right]}\\
&= \mathcal{Y}_{\tilde{B}}.
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma_{C\to\tilde{C}}\left(\tau_C^i\right) = \Gamma_{C\to\tilde{C}}\left(\rho_{C_{\ell+1}|y^m z^\ell}\right) && (57)\\
&= \frac{\mathrm{Tr}_{B^m C^\ell}\left[(\mathbb{I}_{\tilde{C}} \otimes M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell})\Gamma_{C\to\tilde{C}}\left(\rho_{B^m C^{\ell+1}}\right)\right]}{\mathrm{Tr}\left[(M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell} \otimes \mathbb{I}_{C_{\ell+1}})\left(\rho_{B^m C^{\ell+1}}\right)\right]}\\
&= \frac{\mathrm{Tr}_{B^m C^\ell}\left[(\mathbb{I}_{\tilde{C}} \otimes M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell})\left(\mathcal{Z}_{\tilde{C}} \otimes \rho_{B^m C^\ell}\right)\right]}{\mathrm{Tr}\left[(M_{B_1}^{y_1} \otimes \cdots \otimes M_{B_m}^{y_m} \otimes M_{C_1}^{z_1} \otimes \cdots \otimes M_{C_\ell}^{z_\ell} \otimes \mathbb{I}_{C_{\ell+1}})\left(\rho_{B^m C^{\ell+1}}\right)\right]}\\
&= \mathcal{Z}_{\tilde{C}}.
\end{aligned}
$$

◀

Theorem 5 describes a general setting; both the extendible state and the linear constraints do not have any refined structures. However, in our case, we have more information about the state and the constraints. The extendible state $\rho_{(A_1Q_1T)(A_2Q_2\hat{T})^n(S\hat{S})^n}$ in $\mathrm{sdp}_n(V,\pi,T)$ to which we apply the de Finetti theorem already has some classical subsystems, and the linear constraints are partial trace constraints. We can exploit this information to obtain a better bound in the quantum de Finetti theorem. We state this special case as a lemma.

▶ **Lemma 8.** *Let $\rho_{(AX\tilde{X})B^{n_1}(CZ\tilde{Z})^{n_2}}$ be a quantum state with classical $X\tilde{X}$- and $Z\tilde{Z}$-systems invariant under permutation on B^{n_1} and $(CZ\tilde{Z})^{n_2}$ with respect to the other systems, satisfying*

$$\mathrm{tr}_X\left[\rho_{(AX\tilde{X})B^{n_1}(CZ\tilde{Z})^{n_2}}\right] = \mathcal{X}_{A\tilde{X}} \otimes \rho_{B^{n_1}(CZ\tilde{Z})^{n_2}} \tag{58}$$

$$\mathrm{tr}_Z\left[\rho_{(AX\tilde{X})B^{n_1}(CZ\tilde{Z})^{n_2}}\right] = \mathcal{Z}_{C\tilde{Z}} \otimes \rho_{(AX\tilde{X})B^{n_1}(CZ\tilde{Z})^{n_2-1}} \tag{59}$$

for some operators $\mathcal{X}_{A\tilde{X}}$, and $\mathcal{Z}_{C\tilde{Z}}$. Then, there exist a probability distribution $\{p_i\}_{i\in I}$ and sets of quantum states $\{\sigma^i_{AX\tilde{X}}\}_{i\in I}$, $\{\omega^i_B\}_{i\in I}$ and $\{\tau^i_{CZ\tilde{Z}}\}_{i\in I}$ such that

$$
\begin{aligned}
&\left\|\rho_{(AX\tilde{X})B(CZ\tilde{Z})} - \sum_{i\in I} p_i\, \sigma^i_{AX\tilde{X}} \otimes \omega^i_B \otimes \tau^i_{CZ\tilde{Z}}\right\|_1\\
&\leq \min\left\{18^{3/2}\sqrt{|ABC|}, 4|BC|\right\} \times \sqrt{4\ln 2}\left(\sqrt{\frac{\log|X| + 8\log|B|}{n_2}} + \frac{\log|X|}{n_1}\right)
\end{aligned}
\tag{60}
$$

with $\mathrm{tr}_X\left[\sigma^i_{AX\tilde{X}}\right] = \mathcal{X}_{A\tilde{X}}$ and $\mathrm{tr}_Z\left[\tau^i_{CZ\tilde{Z}}\right] = \mathcal{Z}_{C\tilde{Z}}$ for all $i \in I$.

The proof of Lemma 8 is similar to the one of Theorem 5 apart from the following two ingredients – leading to the tighter bound in Eq. (60) in comparison to Eq. (27):

- The partial trace constraints allow us to use a stronger bound on the conditional quantum mutual information in the proof of Lemma 6 (instead of Eq. (37)). Namely, for a quantum state ρ_{ABCD} satisfying $\mathrm{tr}_A[\rho_{ABCD}] = \rho_B \otimes \rho_{CD}$, we have that

  $$I(AB:C|D)_\rho = I(B:C|D) + I(A:C|DB) \leq 2\log|A|. \tag{61}$$

  Using this results in a better bound with $|X|$ instead of $|AX\tilde{X}|$ in the square root part of Eq. (60). Please see Section 4.2, especially Lemma 6 and Lemma 7, in the full version [19] for a more detailed discussion.

- As we remarked in Eq. (46) after Lemma 7, the dimension factor only comes from the measurements on the quantum systems. This is why there is no $|X\tilde{X}Z\tilde{Z}|$ contribution in the first part of Eq. (60).

3.2 Convergence of the hierarchy

Lemma 8 allows us to find an upper bound on the accuracy of the SDP relaxations in Eq. (14). We derive analytical bounds on the convergence speed of our SDP hierarchy in terms of the dimension $|T|$ and the size of the game.

▶ **Theorem 9.** *Let $sdp_n(V,\pi,T)$ be the n-th level SDP relaxation for the two-player free game with rule matrix V, probability distribution $\pi(q_1,q_2)=\pi_1(q_1)\pi_2(q_2)$, and quantum correlation of dimension $|T|^2$. Then, we have*

$$0 \leq sdp_n(V,\pi,T) - \omega_{Q(T)}(V,\pi) \leq O\left(|T|^6\sqrt{\frac{\log|T||A|}{n}}\right). \tag{62}$$

Hence, we have $\omega_{Q(T)}(V,\pi) = \lim_{n\to\infty} sdp_n(V,\pi,T)$.

Proof. Let $\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}}$ be the optimal state of the n-th level relaxation $\mathrm{sdp}_n(V,\pi,T)$. The state should be (n,n)-extendible since all feasible states must be (n,n)-extendible states satisfying the linear constraints. Then, we have

$$\begin{aligned}
&\mathrm{sdp}_n(V,\pi,T) = |T|^2 \operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right)\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}}\right]\\
&= |T|^2 \operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right)\left(\sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}\right)\right]\\
&\quad + |T|^2 \operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right)\left(\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}} - \sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}\right)\right]\\
&\leq \omega_{Q(T)}(V,\pi)\\
&\quad + |T|^2 \operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right)\left(\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}} - \sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}\right)\right],
\end{aligned} \tag{63}$$

where $\sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}$ is one of the close separable states to $\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}}$ specified by Lemma 8. As $\mathrm{sdp}_n(V,\pi,T)$ is an upper bound for $\omega_{Q(T)}(V,\pi)$ we obtain

$$\begin{aligned}
&\left|\mathrm{sdp}_n(V,\pi,T) - \omega_{Q(T)}(V,\pi)\right|\\
&\leq |T|^2 \left|\operatorname{tr}\left[\left(V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right)\left(\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}} - \sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}\right)\right]\right|\\
&\leq |T|^2 \left\|V_{A_1A_2Q_1Q_2}\otimes\Phi_{T\hat{T}|S\hat{S}}\right\|_\infty \left\|\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}} - \sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}\right\|_1\\
&\qquad\qquad\qquad \text{(by Hölder's inequality)}\\
&= |T|^2 \|V_{A_1A_2Q_1Q_2}\|_\infty \left\|\Phi_{T\hat{T}|S\hat{S}}\right\|_\infty \left\|\rho_{A_1Q_1TA_2Q_2\hat{T}S\hat{S}} - \sum_i p_i\,\sigma^i_{A_1Q_1T}\otimes\omega^i_{A_2Q_2\hat{T}}\otimes\tau^i_{S\hat{S}}\right\|_1
\end{aligned}$$

$$= |T|^4 \left\| \rho_{A_1 Q_1 T A_2 Q_2 \hat{T} S \hat{S}} - \sum_i p_i \, \sigma^i_{A_1 Q_1 T} \otimes \omega^i_{A_2 Q_2 \hat{T}} \otimes \tau^i_{S\hat{S}} \right\|_1$$
$$\left(\text{by } \|V_{A_1 A_2 Q_1 Q_2}\|_\infty = 1, \ \left\|\Phi_{T\hat{T}|S\hat{S}}\right\|_\infty = |T|^2\right)$$
$$\leq |T|^4 \left[18^{3/2} |T|^2 \left(\sqrt{2 \ln 2}\right) \left(\sqrt{\frac{\log |A_1| + 8 \log |S\hat{S}|}{n}} + \frac{\log |A_1|}{n} \right) \right] \qquad \text{(by Lemma 8)}$$
$$= 18^{3/2} |T|^6 \left(\sqrt{2 \ln 2}\right) \left(\sqrt{\frac{\log |A| + 16 \log |T|}{n}} + \frac{\log |A|}{n} \right). \tag{64}$$

Here, we set $A = T$, $X = A_1$, $\tilde{X} = Q_1$ $B = S\hat{S}$, $C = \hat{T}$, $Z = A_2$, and $\tilde{Z} = Q_2$ when we applied Lemma 8. ◀

It is worth noting that neither the PPT nor NPA constraints are used to derive this convergence speed.

Theorem 9 allows us to provide an upper bound on the computational complexity of calculating $\omega_{Q(T)}(V, \pi)$ for two-player free games. To achieve a constant error ϵ, it is sufficient to go up to the following level of the hierarchy:

$$\mathcal{O}\left(|T|^6 \sqrt{\frac{\log |TA|}{n}} \right) \leq \epsilon \qquad \Longleftrightarrow \qquad n \geq \mathcal{O}\left(|T|^{12} \frac{\log |TA|}{\epsilon^2} \right). \tag{65}$$

The resulting size of the program is stated in Eq. (1), where the dependence is quasi-polynomial in A and polynomial in Q. Our result is the quantum extension of the quasi-polynomial time approximation scheme for computing classical values $\omega_C(V, \pi)$ of two-player free games developed in [1, 9].

3.2.1 General games

We hitherto assume that the choice of questions for Alice and Bob is independent, i.e., $\pi(q_1, q_2) = \pi_1(q_1)\pi_2(q_2)$, which corresponds to free games. We can use the same protocol that we used for free games to derive upper bounds on the computational complexity of calculating $\omega_{Q(T)}(V, \pi)$ of general games, when $\pi(q_1, q_2) \neq \pi_1(q_1)\pi_2(q_2)$. The key difference is that for general games we absorb $\pi(q_1, q_2)$ into the rule matrix $V(a_1, a_2, q_1, q_2)$ instead of $E_{A_1 Q_1 T}$ and $D_{A_2 Q_2 \hat{T}}$ when we connect $\omega_{Q(T)}(V, \pi)$ to the quantum separability problem in Lemma 1. This leaves some additional factor $|Q_1||Q_2|$ in the objective function, which leads to a worse upper bound on the computational complexity. For a general two-player game with $|T|^2$-dimensional quantum correlation, we can compute additive ϵ-approximations of $\omega_{Q(T)}(V, \pi)$ with a semidefinite program of size

$$\exp\left(\mathcal{O}\left(\frac{|T|^{12} |Q|^4 \left(\log^2 |A||T| + \log |A||T| \log |Q|\right)}{\epsilon^2} \right) \right), \tag{66}$$

where $|A|$ and $|Q|$ are the number of possible answers and questions, respectively. The dependence is still quasi-polynomial in $|A|$, but exponential in $|Q|$ in contrast to the case of free games in Eq. (1). The detailed derivation can be found in Appendix C of the full version [19, Appendix C].

4 Conclusions

In this paper, we study the characterisation of quantum correlations of fixed dimension and, more specifically, provide a converging hierarchy of SDP relaxations with improved analytical convergence speed for the set of fixed-dimensional quantum correlations. This is done by employing a variant of the quantum separability problem and multipartite quantum de Finetti theorems with additional linear constraints. Our result leads to an upper bound on the computational complexity of additive ϵ-approximation for $\omega_{Q(T)}(V,\pi)$ of two-player free games with $T \times T$-dimensional quantum correlation.

We conclude with a few remarks on possible future studies. Firstly, for a given level n, $\mathrm{sdp}_n(V,\pi,T)$ has a relatively large-sized optimisation variable. One possible way to improve this aspect is to exploit the symmetry embedded in the program to reduce the size of the optimisation variable. We could employ some existing symmetry-finding programs such as [31] to achieve this. Secondly, it is still not certain whether the T-dependence in Eq. (1) is optimal. In the classical limit ($T = 1$), our result matches the best-known classical result for free games in terms of A and Q – which also has a matching hardness result [1]. This implies that the dependence on A and Q in Eq. (1) is optimal, but there could be more efficient approximation algorithm in terms of T-dependence. For example, one could explore ϵ-net based methods as in [7, 32].

5 Proof of Lemma 7

In this section, we prove the second part of Lemma 7 which states that for a traceless Hermitian operator γ_{AB} on $\mathcal{H}_{AB}$, there exists a measurement $\mathcal{M}_B$ on $\mathcal{H}_B$ with at most $|B|^6$ outcomes such that $\|(\mathcal{I}_A \otimes \mathcal{M}_B)(\gamma_{AB})\|_1 \geq \frac{1}{2|B|}\|\gamma_{AB}\|_1$. The proof is inspired by [22, Theorem 16].

Proof of the second part of Lemma 7. Let us start with the maximally entangled state

$$\Phi_{A'|B'} = |\Phi\rangle\langle\Phi|_{A'|B'} \text{ where } |\Phi\rangle_{A'|B'} = \frac{1}{|A'||B'|}\sum_i |i\rangle_{A'}|i\rangle_{B'}, \text{ and } |A'| = |B'|. \tag{67}$$

We can create a separable state $\omega_{A'B'}$ by mixing $\Phi_{A'|B'}$ with another separable state $\sigma_{A'B'} = \frac{\mathbb{I}_{A'B'} - \Phi_{A'|B'}}{|B'|^2-1}$ as

$$\omega_{A'B'} = \frac{1}{|B'|}\Phi_{A'B'} + \frac{|B'|-1}{|B'|}\sigma_{A'B'} \in \mathrm{SEP(A'\!:B')}, \tag{68}$$

where SEP(A': B') denotes the set of separable states with respect to the bipartition $A'|B'$. Hence, we can write $\omega_{A'B'} = \sum_i p_i \omega^i_{A'} \otimes \omega^i_{B'}$ for some probability distribution $\{p_i\}_i$ and states $\{\omega^i_{A'}\}_i$ and $\{\omega^i_{B'}\}_i$ with at most $|A'B'|^2$ elements [18]. Next, we define a measurement $\mathcal{M}_B$ with operators $\{\tilde{M}_B(i,k)\}_{i,k}$, as well as a set of measurements $\{\mathcal{M}^{i,k}_A\}_{i,k}$ with operators $\{\tilde{M}^{i,k}_A(j)\}_j$ as

$$\tilde{M}_B(i,k) = \mathrm{tr}_{B'}\left[p_i U^\dagger_B(k)\sqrt{\omega^i_{B'}}\Phi_{BB'}\sqrt{\omega^i_{B'}}U_B(k)\right] \quad \text{and} \tag{69}$$

$$\tilde{M}^{i,k}_A(j) = \mathrm{tr}_{A'}\left[\sqrt{\omega^i_{A'}}U^\dagger_{A'}(k)N_{AA'}(j)U_{A'}(k)\sqrt{\omega^i_{A'}}\right], \tag{70}$$

where $U(k)$ denote generalised Pauli operators, $\omega^i_{A'}$ and $\omega^i_{B'}$ are the elements of the decomposition of $\omega_{A'B'}$, and $\{N_{AA'}(j)\}_j$ are measurement operators defined later. We can check that both definitions indeed correspond to valid measurements:

$$\sum_{i,k} \tilde{M}_B(i,k) = \mathbb{I}_B, \sum_j \tilde{M}_A^{i,k}(j) = \mathbb{I}_A, \text{ and } \tilde{M}_B(i,k), \tilde{M}_A^{i,k}(j) \geq 0 \;\; \forall i,k,j. \tag{71}$$

The goal is to show that $\mathcal{M}_B$ defined in Eq. (69) gives rise to Eq. (45). Before showing that, however, it is helpful to understand where these measurements came from. They are related to the quantum teleportation protocol [5]. Without loss of generality, let us assume that $|A| \geq |B| = |A'| = |B'|$. Then, the quantum teleportation protocol from B to A is a quantum channel defined as [5]

$$\tau_{ABA'B' \to AA'}(\cdot) = \sum_{k=1}^{|B|^2} U_{A'}(k) \operatorname{tr}_{BB'}\left[(\cdot)\left(\mathbb{I}_{AA'} \otimes U_B(k)\Phi_{BB'}U_B^\dagger(k)\right)\right] U_{A'}^\dagger(k). \tag{72}$$

For a traceless Hermitian operator γ_{AB}, we then consider

$$\|\tau_{ABA'B'\to AA'}(\gamma_{AB} \otimes \omega_{A'B'})\|_1 = \sum_j |\operatorname{tr}[N_{AA'}(j)(\tau_{ABA'B'\to AA'}(\gamma_{AB} \otimes \omega_{A'B'}))]|, \tag{73}$$

where we used the expression $\|X_A\|_1 = \max_{\{M_A(i)\}_i} \sum_i |\operatorname{tr}[M_A(i)X_A]|$ for the trace norm with corresponding arg max $\{N_{AA'}(j)\}_j$ to be used in Eq. (70). Then, we have

$$\begin{aligned}
&\|\tau_{ABA'B'\to AA'}(\gamma_{AB} \otimes \omega_{A'B'})\|_1 \\
&= \sum_j \left| \sum_k \operatorname{tr}\left[N_{AA'}(j)\left(U_{A'}(k)\operatorname{tr}_{BB'}\left[(\gamma_{AB}\otimes\omega_{A'B'})\left(\mathbb{I}_{AA'}\otimes U_B(k)\Phi_{BB'}U_B^\dagger(k)\right)\right]U_{A'}^\dagger(k)\right)\right]\right| \\
&= \sum_j \left| \sum_k \operatorname{tr}\left[\left(U_{A'}^\dagger(k)N_{AA'}(j)U_{A'}(k)\otimes\mathbb{I}_{BB'}\right)\left((\gamma_{AB}\otimes\omega_{A'B'})\left(\mathbb{I}_{AA'}\otimes U_B(k)\Phi_{BB'}U_B^\dagger(k)\right)\right)\right]\right| \\
&= \sum_j \left| \sum_k \operatorname{tr}\left[\left(U_{A'}^\dagger(k)N_{AA'}(j)U_{A'}(k)\otimes U_B^\dagger(k)\Phi_{BB'}U_B(k)\right)\left(\gamma_{AB}\otimes\left(\sum_i p_i\omega_{A'}^i\otimes\omega_{B'}^i\right)\right)\right]\right| \\
&= \sum_j \left| \sum_{i,k} \operatorname{tr}\left[\left(\left(\sqrt{\omega_{A'}^i}U_{A'}^\dagger(k)N_{AA'}(j)U_{A'}(k)\sqrt{\omega_{A'}^i}\right)\right.\right.\right. \\
&\qquad\left.\left.\left.\otimes\left(p_iU_B^\dagger(k)\sqrt{\omega_{B'}^i}\Phi_{BB'}\sqrt{\omega_{B'}^i}U_B(k)\right)\right)(\gamma_{AB}\otimes\mathbb{I}_{A'B'})\right]\right| \\
&= \sum_j \left| \sum_{i,k} \operatorname{tr}\left[\gamma_{AB}\left(\tilde{M}_A^{i,k}(j)\otimes\tilde{M}_B(i,k)\right)\right]\right|.
\end{aligned} \tag{74}$$

The measurement $\mathcal{M}_B$ defined in Eq. (69) now gives rise to

$$\|(\mathcal{I}_A \otimes \mathcal{M}_B)(\gamma_{AB})\|_1 \tag{75}$$

$$= \sum_{i,k} \left\|\operatorname{tr}_B\left[\left(\mathbb{I}_A \otimes \tilde{M}_B(i,k)\right)\gamma_{AB}\right]\right\|_1 \tag{76}$$

$$= \sum_{i,k} \max_{\{M_A^{i,k}(j)\}_j} \sum_j \left|\operatorname{tr}\left[\left(M_A^{i,k}(j)\otimes\tilde{M}_B(i,k)\right)\gamma_{AB}\right]\right| \tag{77}$$

$$\geq \sum_{i,k}\sum_j \left|\operatorname{tr}\left[\left(\tilde{M}_A^{i,k}(j)\otimes\tilde{M}_B(i,k)\right)\gamma_{AB}\right]\right| \tag{78}$$

$$\geq \sum_j \left|\sum_{i,k} \operatorname{tr}\left[\left(\tilde{M}_A^{i,k}(j)\otimes\tilde{M}_B(i,k)\right)\gamma_{AB}\right]\right| \tag{79}$$

$$= \left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\omega_{A'B'}\right)\right\|_1 \qquad \text{(by Eq. (74))} \tag{80}$$

$$= \left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\left(\frac{1}{|B|}\Phi_{A'B'}+\frac{|B|-1}{|B|}\sigma_{A'B'}\right)\right)\right\|_1 \tag{81}$$

$$= \left\|\frac{1}{|B|}\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\Phi_{A'B'}\right)+\frac{|B|-1}{|B|}\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\sigma_{A'B'}\right)\right\|_1 \tag{82}$$

$$\geq \frac{1}{|B|}\left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\Phi_{A'B'}\right)\right\|_1-\left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\left(\frac{|B|-1}{|B|}\sigma_{A'B'}\right)\right)\right\|_1, \tag{83}$$

where in the third line we substituted the measurement operators $\{\tilde{M}_A^{i,k}(j)\}_j$ instead of the maximisation, and in the last line we used the reverse triangular inequality. Note that the first term in the last line is equivalent to $\|\gamma_{AB}\|_1$ since $\Phi_{A'B'}$ is the maximally entangled state. Let us investigate the second term more closely. We have a chain of elementary implications

$$\begin{aligned}
&\frac{|B|-1}{|B|}\sigma_{A'B'}\leq\frac{|B|-1}{|B|}\sigma_{A'B'}+\frac{1}{|B|}\Phi_{A'B'}=\omega_{A'B'}\\
&\gamma_{AB}\otimes\frac{|B|-1}{|B|}\sigma_{A'B'}\leq\gamma_{AB}\otimes\omega_{A'B'}\\
&\left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\left(\frac{|B|-1}{|B|}\sigma_{A'B'}\right)\right)\right\|_1\leq\left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\omega_{A'B'}\right)\right\|_1\\
&\left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\left(\frac{|B|-1}{|B|}\sigma_{A'B'}\right)\right)\right\|_1\leq\sum_j\left|\sum_{i,k}\operatorname{tr}\left[\gamma_{AB}\left(\tilde{M}_A^{i,k}(j)\otimes\tilde{M}_B(i,k)\right)\right]\right| \qquad \text{(by Eq. (74))}\\
&\left\|\tau_{ABA'B'\to AA'}\left(\gamma_{AB}\otimes\left(\frac{|B|-1}{|B|}\sigma_{A'B'}\right)\right)\right\|_1\leq\left\|\left(\mathcal{I}_A\otimes\mathcal{M}_B\right)\left(\gamma_{AB}\right)\right\|_1 \qquad \text{(by Eq. (79))}
\end{aligned} \tag{84}$$

and substituting this into Eq. (83) yields the claim

$$\left\|\left(\mathcal{I}_A\otimes\mathcal{M}_B\right)\left(\gamma_{AB}\right)\right\|_1\geq\frac{1}{|B|}\|\gamma_{AB}\|_1-\left\|\left(\mathcal{I}_A\otimes\mathcal{M}_B\right)\left(\gamma_{AB}\right)\right\|_1. \tag{85}$$

It remains to quantify the number of measurement outcomes of $\mathcal{M}_B$ with measurement operators $\{\tilde{M}_B(i,k)\}_{i,k}$ defined in Eq. (70). The index i came from the number of elements in the separable state $\omega_{A'B'}$, which is at most $|A'B'|^2=|B|^4$, and the index k came from the number of generalised Pauli operators, which is $|B|^2$. Therefore, the number of outcomes is at most $|B|^6$. ◀

References

1 Scott Aaronson, Russell Impagliazzo, and Dana Moshkovitz. AM with multiple Merlins. In *IEEE 29th Conference on Computational Complexity*, pages 44–55, 2014.

2 Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM*, 45(3):501–555, 1998.

3 Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the ACM*, 45(1):70–122, 1998.

4 John S Bell. On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.

5 Charles H Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 70(13):1895, 1993.

6 Mario Berta, Francesco Borderi, Omar Fawzi, and Volkher B Scholz. Semidefinite programming hierarchies for constrained bilinear optimization. *Mathematical Programming*, pages 1–49, 2021.

7 Fernando GSL Brandão and Aram W Harrow. Estimating operator norms using covering nets. *arXiv:1509.05065*, 2015. arXiv:1509.05065.

8 Fernando GSL Brandão and Aram W Harrow. Product-state approximations to quantum states. *Communications in Mathematical Physics*, 342(1):47–80, 2016.

9 Fernando GSL Brandão and Aram W Harrow. Quantum de Finetti theorems under local measurements with applications. *Communications in Mathematical Physics*, 353(2):469–506, 2017.

10 Richard Cleve, Peter Høyer, Benjamin Toner, and John Watrous. Consequences and limits of nonlocal strategies. In *Proceedings 19th IEEE Annual Conference on Computational Complexity*, pages 236–249, 2004.

11 Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri. Distinguishing separable and entangled states. *Physical Review Letters*, 88(18):187904, 2002.

12 Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri. Complete family of separability criteria. *Physical Review A*, 69(2):022308, 2004.

13 Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri. Detecting multipartite entanglement. *Physical Review A*, 71(3):032333, 2005.

14 Mark Fannes, John T Lewis, and André Verbeure. Symmetric states of composite systems. *Letters in Mathematical Physics*, 15(3):255–260, 1988.

15 Rodrigo Gallego, Nicolas Brunner, Christopher Hadley, and Antonio Acín. Device-independent tests of classical and quantum dimensions. *Physical Review Letters*, 105(23):230501, 2010.

16 Sevag Gharibian. Strong NP-hardness of the Quantum Separability Problem. *Quantum Information and Computation*, 10(3&4):343–360, 2010.

17 Leonid Gurvits. Classical deterministic complexity of Edmonds' problem and quantum entanglement. In *Proceedings of the thirty-fifth annual ACM symposium on theory of computing*, pages 10–19, 2003.

18 Pawel Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. *Physics Letters A*, 232:333, 1997.

19 Hyejung H Jee, Carlo Sparaciari, Omar Fawzi, and Mario Berta. Characterising quantum correlations of fixed dimension. *arXiv preprint*, 2020. arXiv:2005.08883.

20 Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. MIP*= RE. *arXiv:2001.04383*, 2020. arXiv:2001.04383.

21 Julia Kempe, Oded Regev, and Ben Toner. Unique games with entangled provers are easy. *SIAM Journal on Computing*, 39(7):3207–3229, 2010.

22 Ludovico Lami, Carlos Palazuelos, and Andreas Winter. Ultimate data hiding in quantum mechanics and beyond. *Communications in Mathematical Physics*, 361(2):661–708, 2018.

23 Göran Lindblad. Entropy, information and quantum measurements. *Communications in Mathematical Physics*, 33(4):305–322, 1973.

24 William Matthews, Stephanie Wehner, and Andreas Winter. Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding. *Communications in Mathematical Physics*, 291(3):813–843, 2009.

25 Miguel Navascués, Gonzalo de la Torre, and Tamás Vértesi. Characterization of quantum correlations with local dimension constraints and its device-independent applications. *Physical Review X*, 4(1):011011, 2014.

26 Miguel Navascués, Adrien Feix, Mateus Araújo, and Tamás Vértesi. Characterizing finite-dimensional quantum behavior. *Physical Review A*, 92(4):042117, 2015.

27 Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, 2008.

28 Miguel Navascués and Tamás Vértesi. Bounding the set of finite dimensional quantum correlations. *Physical Review Letters*, 115(2):020501, 2015.

29 Stefano Pironio, Miguel Navascués, and Antonio Acín. Convergent relaxations of polynomial optimization problems with noncommuting variables. *SIAM Journal on Optimization*, 20(5):2157–2180, 2010.

30 GA Raggio and Reinhard F Werner. Quantum statistical mechanics of general mean field systems. *Helvetica Acta Physica*, 62:980, 1989.

31 Denis Rosset. SymDPoly: symmetry-adapted moment relaxations for noncommutative polynomial optimization. *arXiv:1808.09598*, 2018. `arXiv:1808.09598`.

32 Yaoyun Shi and Xiaodi Wu. Epsilon-net method for optimizations over separable states. *Theoretical Computer Science*, 598:51–63, 2015.

33 Barbara M Terhal. Is entanglement monogamous? *IBM Journal of Research and Development*, 48(1):71–78, 2004.

34 Xiao-Dong Yu, Timo Simnacher, H Chau Nguyen, and Otfried Gühne. Quantum-inspired hierarchy for rank-constrained optimization. *arXiv preprint*, 2020. `arXiv:2012.00554`.

35 Xiao-Dong Yu, Timo Simnacher, Nikolai Wyderka, H. Chau Nguyen, and Otfried Gühne. A complete hierarchy for the pure state marginal problem in quantum mechanics. *Nature Communications*, 12(1):1012, 2021.