Exploring Matrix Multiplication with Manipulatives

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Abstract: *The study of matrix operations and applications is required for students planning to take advanced courses, such as calculus, advanced statistics, or discrete mathematics. Furthermore, matrices appear on the ACT test and both SAT Mathematics Level 1 and Mathematics Level 2 subject tests. The student-centered activity presented here utilizes hands-on manipulatives, visual cues, group collaborative work, and whole-class reflective discourse. Finally, the activity engages students in several of the Standards for Mathematical Practice and helps students to better understand the process of matrix multiplication.*

Keywords: *Matrices, mathematical modeling, manipulatives*

Introduction

Matrices in the Mathematics Curriculum

Matrices remain a requisite topic for high school students planning to take advanced mathematics courses, such as calculus, advanced statistics, and discrete mathematics (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010, p. 57). Although not part of a typical traditional or integrated three-year mathematics course pathway, the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010) include content standards involving operations on matrices and the use of matrices in applications. In addition, student knowledge of and skills with matrices can be assessed on college entrance exams, such as the ACT test and both SAT Mathematics Level 1 and Mathematics Level 2 subject tests (ACT, 2020; The College Board, 2021).

As described in the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010), technology tools, such as calculators, spreadsheets, and computer algebra systems (CAS) "can provide ways for students to become better acquainted with . . . [and] understand the workings of matrix . . . algebra" (p. 58). Although we agree with and subscribe to such thinking, the first author (i.e., Maria) has observed, over multiple years of mathematics teaching at the secondary level, that yielding to technological tools too soon can constrain students' capacities to think and reason in ways productive to understanding: (1) matrix multiplication, (2) the use of matrices to solve systems of linear equations, and (3) the geometric interpretation of matrices as transformations of the plane. Hahn and Valentine (2019) assert, "Matrix multiplication is probably the most important matrix operation" (p. 127) due to its broad use in transformation of coordinate systems, network theory systems of linear equations, and population modeling. Furthermore, matrix multiplication is a main tool for computations in many scientific areas like seismic analysis, galactic simulations, signal and images processing, and aerodynamics (Snopce & Elmazi, 2008, p. 885).

The following two-day activity (for both block-schedule and shorter period classes), part of a larger unit on matrices, was created by Maria to help her students better understand the process of matrix multiplication. Although Maria utilizes the activity with Precalculus students, it would also be appropriate for Algebra 2, Advanced Algebra, or Integrated Mathematics 3 courses that include the study of matrices.

Prerequisite Student Knowledge and Skills

Student knowledge prerequisite to the activity includes the idea that a matrix is a way to organize and manage data (Ohio Department of Education [ODE], 2015), involves placing data (typically numbers, symbols, or expressions) in a rectangular configuration, or array, and basic matrix terminology (e.g., elements, dimension or order of a matrix). Students should also be proficient at adding and subtracting matrices and multiplying a matrix by a scalar. Such skills connect to future activities and lessons (not presented here) involving determinants, solving problems with applications, solving systems of equations, and areas of triangles. In addition to the prerequisite knowledge and skills identified above, the teacher and students, as part of an earlier lesson, should discuss careers that utilize matrices, including engineering (e.g., traffic flow problems), computer programming, purchasing or procurement agent, information processing, and business administration.

Activity Objectives and Standards

At the conclusion of the activity presented here, students will be able to: (1) better understand the process of matrix multiplication, (2) multiply matrices (of appropriate dimensions) proficiently, and (3) use matrix multiplication to solve problems. The mathematics content and practice standards addressed in the activity are displayed in Table 1 (NGA Center & CCSSO, 2010, p. 61).

Table 1: *Content and Practice Standards Addressed During Activity*

Conceptual Category	Number and Quantity
Domain	Vector and Quantities
Cluster (Statement)	Perform operations on matrices and use matrices in applications.
Content Standards	8.(+) Add, subtract, and multiply matrices of appropriate dimensions.
	9.(+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
	10.(+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
Standards for Mathematical Practice (SMPs)	SMP1—Make sense of problems and persevere in solving them. SMP3—Construct viable arguments and critique the reasoning of others. SMP5—Use appropriate tools strategically. SMP6—Attend to precision. SMP7—Look for and make use of structure.

Students engage in mathematical practices SMP5 and SMP6 relatively early in the activity; whereas mathematical practices SMP1, SMP3, and SMP7 are promoted and emphasized throughout a group discussion which occurs during the second day of the activity, as detailed below.

Activity

The activity begins with a 5-minute review of several key concepts covered in an earlier lesson on matrices (e.g., column, row, dimension of a matrix). In order to support an inclusive learning environment that attempts to reduce barriers to student success, the teacher assigns students to small heterogenous groups based on each student's academic, behavioral, and motivational strengths and challenges. Furthermore, the activity incorporates principles of Universal Design for Learning (UDL) by providing students with opportunities to activate their prior knowledge and explain their thinking verbally, visually, and with manipulatives (San Francisco Unified School District Mathematics Department, 2019). For example, concepts are reviewed using organizing clues, such as, "We have discussed several terms associated with the size of matrices, can someone provide an overview of some of these terms?"

Miller and Obara (2017) define a mathematical mnemonic as "a visual cue or verbal strategy used to aid initial memorisation and recall of a mathematical concept or procedure" (p. 13). The main portion of the activity presented here involves a mathematical mnemonic and is designed to assist students' initial memorization and recall of the process of matrix multiplication. Therefore, the activity involves visual cues used to aid initial memorization and recall of a mathematical procedure (i.e., matrix multiplication) and is comprised of two parts.

Part 1—A Focus on Multiplying Rows by Columns

For the first part of the activity, the teacher uses visual manipulatives she created (small colored foam rectangles cut from foamboard) to illustrate a $2 \cdot 3$ matrix being multiplied by a $3 \cdot 2$ matrix to produce a $2 \cdot 2$ matrix. The teacher can color code the visual to support her students in creating an image of rows being multiplied by columns (Figure 1).

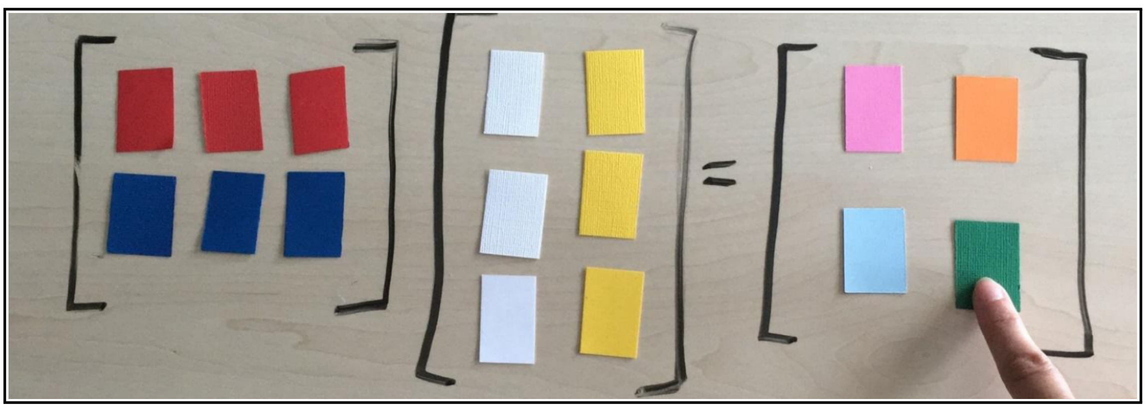

Figure 1: *Multiplying rows by columns.*

The first matrix is color coded by its rows (red and blue, respectively) and the second matrix by its columns (white and yellow, respectively) so that when the matrices are multiplied together (row by column), the elements of the resultant $2 \cdot 2$ product matrix are made up of four different colors (Figure 1).

The teacher should have her students create the same visual at their desks (in groups of four), using their own colored foam rectangle manipulatives, to represent matrix elements, and mini whiteboards and dry erase markers to draw the matrix brackets and equal sign. The teacher next illustrates moving the elements in R_1 (row 1) of the first matrix to line up with the entries in C_1 (column 1) of the second matrix. These elements then join (mix) together, as R_1C_1, to form a new colored entry in the product matrix. Using their own foam rectangle 'elements,' students are able to manipulate their own entries to visualize the resultant entry (R_1C_1) produced by the red row in the first matrix and the white column in the second matrix; resulting in the pink entry in the product matrix. The teacher should continue to guide students through this row by column multiplication process, so students see how each entry of the $2 \cdot 2$ product matrix is produced. This phase of the activity allows students to focus on coordinating rows with columns, without having to think about specific operations with numeric matrix entries.

If no student inquires as to a potential relationship between the number of entries in a row and its corresponding column—Maria has observed that such inquiries occur roughly 75% of the time—the teacher should ask students to think about and discuss (in their groups) anything they notice regarding these entries. Maria has also observed that many students are better able to understand the process of multiplying two matrices together using this activity, because the activity is connected to primary colors and color mixing—something students have had familiarity with for years. Although color blindness affects five to 10 percent of the U.S. population (CBS Interactive, 2014), Maria has yet to have a student who did not benefit from the activity. To address potential issues with color blind students, the teacher may write out (make explicit) the names of the colors utilized and assign color-blind and non-color-blind students to work together as needed (Colour Blind Awareness, 2010). After students complete the first part of the activity, they next explore the process of matrix multiplication using numeric matrix entries.

Part 2—Color-Coded Matrix Multiplication

For the second part of the activity, students next receive a bag with colored foam numbers. The teacher should have a similar set of magnetic foam numbers that she places on her whiteboard to help guide her students through the process of multiplying matrices. In their groups of four, students are instructed to construct the same $2 \cdot 3$ and $3 \cdot 2$ matrices (denoted by A and B, respectively) the teacher illustrates on her board (Figure 2) using the foam numbers from their bags, mini whiteboards, and dry erase markers.

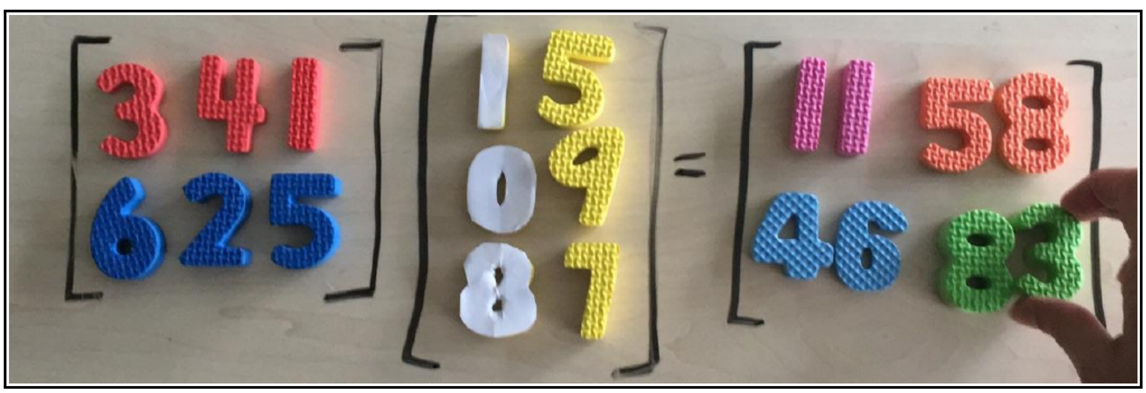

Figure 2: *Color-coded matrix multiplication.*

Students physically pick up the entries in R_1 (row 1) of the first matrix and line them up with their corresponding entries in C_1 (column 1) of the second matrix (line up each multiplicand and its corresponding multiplier) to see how they combine to create R_1C_1; that is, entry p_{11} of the product matrix. The teacher should continue to guide the class through this row entry by column entry multiplication process so that students see how each entry of the $2 \cdot 2$ product matrix is determined (Figure 3).

$$A \cdot B = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + 4 \cdot 0 + 1 \cdot 8 & 3 \cdot 5 + 4 \cdot 9 + 1 \cdot 7 \\ 6 \cdot 1 + 2 \cdot 0 + 5 \cdot 8 & 6 \cdot 5 + 2 \cdot 9 + 5 \cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 0 + 8 & 15 + 36 + 7 \\ 6 + 0 + 40 & 30 + 18 + 35 \end{bmatrix} = \begin{bmatrix} 11 & 58 \\ 46 & 83 \end{bmatrix}$$

Figure 3: *Matrix multiplication color illustration.*

After completing the product $A \cdot B$, students are asked to attempt the products $A \cdot A$ and $B \cdot A$. These products allow for discussions regarding which matrices are able to be multiplied (i.e., the number of columns of the 1st matrix must equal the number of rows of the 2nd matrix) and the inherent limitations to this method of using colors to represent matrix entries (e.g., the product $B \cdot A$ requires 13 colors).

Throughout the second part of the activity, the teacher should hold students accountable for articulating their meanings, thinking, and reasoning. In addition, the teacher should require, as a general classroom policy, that her students hold her and one another accountable to this same convention. In doing so, the teacher will have established an environment where "students try to communicate precisely to others" (i.e., SMP6) (NGA Center & CCSSO, 2010, p. 7). Finally, throughout the second part of the activity, students utilize what we believe to be "appropriate tools" (NGA Center & CCSSO, 2010, p. 7) for supporting the development of their initial understanding of matrix multiplication (i.e., SMP5)—the physical tools of colored foam numbers, mini whiteboards, and dry erase markers.

Group Problem Sets

At the conclusion of the hands-on portion of the activity, the teacher and her students work on Problem #1 (see Appendix) as a whole class. After completing Problem #1, students are asked to continue to work on Problems #2 through #5 (see Appendix) in their groups. As groups work on the problems, the teacher should circulate throughout the room to provide group and individual student support. Such support includes engaging students in conversations about the concepts and skills and suggesting students: (1) create a diagram or flowchart depicting matrix multiplication or (2) write an explanation of the matrix multiplication process in their own words (Fleming & Baume, 2006). For both block-schedule and shorter period courses, the class will end before students complete all problems. Therefore, students should be asked to continue their work at home and informed they will be provided additional time (as needed) during the next class. Finally, students are allowed to use their manipulatives to help them through the matrix multiplication process if needed.

Although each group receives a different set of problems, each set focuses on the same main ideas. Specifically, the problems focus on potentially problematic, but significant ideas and promote and support reflective discourse (Cobb et al., 1997) during subsequent whole-class discussions. Problems include the existence and roles of the zero and identity matrices and exploring whether matrices satisfy the commutative and associative properties of multiplication and the distributive property of multiplication over addition. The second day of the activity should begin by providing groups with additional time, as needed, to complete Problems #2 through #5 (see Appendix).

During the last 15 minutes of the activity, groups are asked to come together to discuss and reflect on the activity and problems. Throughout this discussion, the teacher should continue to engage students in the mathematical practices by promoting an environment where students: explain the meaning of a problem to themselves and their classmates (i.e., SMP1); step back for an overview of a problem and shift perspective, if necessary (i.e., SMP7); and construct arguments using definitions and previously established results, justify their conclusions, and critique the reasoning of others (i.e., SMP3) (NGA Center CCSSO, 2010, pp. 6–7). Finally, the teacher should manage the whole-class discussion in ways that support students in developing generalized conceptions regarding the zero and identity matrices, the commutative and associative property of multiplication (of matrices), and the distributive property of matrix multiplication over matrix addition.

Conclusion

The study of matrices is identified as "additional mathematics" in the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010); that is, mathematics students should learn to be successful in advanced courses (e.g., calculus, advanced statistics, discrete mathematics). An understanding of matrices is also requisite for popular college entrance exams—matrices are included in the Intermediate Algebra content area of the ACT test (which comprises 15-20% of mathematics portion of the ACT) and the Number and Operations topic area of the SAT Mathematics Level 1 and Mathematics Level 2 subject tests (which make up 10-14% of each exam).

Although the concept of matrix multiplication can be difficult for students to understand and apply, the use of hands-on manipulatives and visual cues, as presented here, along with the sequence of matrix multiplications problems and subsequent reflective discourse, has helped Maria's students develop an understanding for the process of multiplication. Finally, such activities can help students develop images that support productive interpretations of what it means to solve systems of linear equations with matrices and perform geometric transformations in the plane with matrices not provided with the use of technology (e.g., graphing calculator, spreadsheet).

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Appendix

(1) For the following matrices:

$$W = \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ -2 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} 6 & 2 & 9 \\ 7 & 1 & 8 \end{bmatrix}, \quad Y = \begin{bmatrix} 7 & -6 \\ -3 & 2 \end{bmatrix}$$

- Determine the given products, if possible. If a product cannot be calculated, then indicate what the problem is; what makes it so that a product cannot be calculated?

  a) $Y \cdot W$ b) $Y \cdot X$

(2) For the following matrices:

$$A = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ -1 & 7 \end{bmatrix}$$

- Calculate the given products, if possible. Does it appear the Commutative Property of Multiplication holds true for matrices? Explain why or why not.

  a) $A \cdot B$ b) $B \cdot A$

(3) For the following matrices:

$$D = \begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -1 & 4 \\ -5 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$$

- Calculate the given products, if possible. Does it appear the Associative Property of Multiplication holds true for matrices? Explain why or why not.

  a) $D \cdot (E \cdot F)$ b) $(D \cdot E) \cdot F$

- Perform the given operations, if possible. Does it appear the Distributive Property of Multiplication Over Addition holds true for matrices? Explain why or why not.

  c) $D \cdot (E + F)$ d) $(D \cdot E) + (D \cdot F)$

(4) For the following matrices:

$$C = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$$

- Determine if there exists a matrix, O, such that $C + O = O + C = C$ and $C \cdot O = O \cdot C = O$. If such a matrix does exist, what are its dimensions and entries? If such a matrix does not exist, explain why not.

(5) For the following matrix:

$$U = \begin{bmatrix} -1 & 2 \\ -2 & 0 \end{bmatrix}$$

- Determine if there exists a matrix, I, such that $U \cdot I = I \cdot U = U$. If such a matrix does exist, what are its dimensions and entries? If such a matrix does not exist, explain why not.