THE DENSITY OF SETS CONTAINING LARGE SIMILAR COPIES OF FINITE SETS

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ABSTRACT. We prove that if $E \subseteq \mathbb{R}^d$ $(d \geq 2)$ is a Lebesgue-measurable set with density larger than $\frac{n-2}{n-1}$, then E contains similar copies of every n-point set P at all sufficiently large scales. Moreover, 'sufficiently large' can be taken to be uniform over all P with prescribed size, minimum separation and diameter. On the other hand, we construct an example to show that the density required to guarantee all large similar copies of n-point sets tends to 1 at a rate $1 - O(n^{-1/5} \log n)$.

1. INTRODUCTION

In this paper a finite subset P of $\mathbb{R}^d$ with at least two distinct points will be called a *pattern*. There are many ways of viewing the question of finding necessary or sufficient conditions on a set $E \subseteq \mathbb{R}^d$ to contain some, or many, similar (or alternatively homothetic or congruent) copies of a given pattern P. Here we will be concerned with finding conditions that guarantee that E contains scaled similar copies of P for all sufficiently large scalings. We will assume throughout that $d \geq 2$ and that E is $\mathcal{L}^d$-measurable, where $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$. The Euclidean norm of $x \in \mathbb{R}^d$ will simply be written as $\|x\|$.

The most basic result of this kind is for 2-point patterns: for every Lebesgue-measurable set $E \subseteq \mathbb{R}^2$ with positive upper density (or positive upper Banach density, see (2.1) and (2.2)) there exists $R > 0$ such that all distances greater than R are realised between the points of E. This problem was posed by Székely [29] and several proofs were given in the 1980s, by Falconer and Marstrand [10] with a geometric proof, by Bourgain [2] for $\mathbb{R}^d$ with $d \geq 2$ using harmonic analysis and by Furstenberg, Katznelson and Weiss [11] using ergodic theory. More recently Quas [27] gave a more combinatorial proof.

Rice [28] showed that the positive density requirement cannot be weakened. For all $d \geq 1$ and any function $f\colon (0,\infty) \to [0,1]$ with $\lim_{r\to\infty} f(r) = 0$ he constructed a measurable set $E \subseteq \mathbb{R}^d$ and a sequence $r_n \to \infty$ such that $\|x - y\| \neq r_n$ for all $x, y \in E$, with $\mathcal{L}^d(E \cap B_{r_n}(0))/\mathcal{L}^d(B_{r_n}(0)) \geq f(r_n)$ for all $n \in \mathbb{N}$, where $B_r(x)$ is the ball of radius r centered at x.

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It is natural to consider analogous questions for patterns with more than two points. Indeed, Bourgain's paper also showed that a set of positive upper density $E \subseteq \mathbb{R}^d$ contains all sufficiently large similar copies of every d-point pattern provided that the points span a $(d-1)$-dimensional hyperplane, see [16,23,24] for various other proofs. On the other hand, he showed by the following example, which relies on the parallelogram identity, that this spanning condition is necessary.

Example 1.1. [2]. *Let $0 < s < \frac{1}{4}$ and let $E := \{x \in \mathbb{R}^d : \|x\|^2 \in [0,s] + (\mathbb{N} \cup \{0\})\}$, so that E is a union of annuli and has density s. Then there are arbitrarily large values of r such that E contains no congruent copy of $\{0, r, 2r\}$.*

Subsequently, Graham showed that a similar conclusion holds for any non-spherical set.

Example 1.2. [12]. *Let $P \subseteq \mathbb{R}^d$ be a finite set of points that do not all lie on the surface of any $(d-1)$-sphere. Then there is a set E of positive upper density and arbitrarily large values of r such that E does not contain a congruent copy of rP .*

It is an open question whether every plane set of positive upper density contains all large copies of every non-degenerate triangle. However, Furstenberg, Katznelson and Weiss [11, Theorem B] showed that if $E \subseteq \mathbb{R}^2$ has positive upper density, then every δ-neighbourhood of E contains all sufficiently large similar copies of every triangle, and Ziegler [32] extended this to larger patterns (i.e., simplices, possibly degenerate) in $\mathbb{R}^d$ for $d \geq 2$.

As far as other configurations go, Morris [26] showed that in any set of positive density one can find triangles with all sufficiently large (compatible) perimeters and areas. Lyall and Magyar [23] considered products of k- and k'-simplices in $\mathbb{R}^d$ where $k + k' + 6 \leq d$ and in particular showed that, given the vertices of a rectangle P, any subset of $\mathbb{R}^d$ ($d \geq 4$) with positive upper Banach density contains all sufficiently large similar copies of P. Generalizations to multiple products were discussed in [8] and [25]; the latter paper successfully handled arbitrary products of non-degenerate simplices. Moreover, Lyall and Magyar [24] showed that sets of positive upper Banach density in $\mathbb{R}^d$ contain all large enough copies of 'proper k-degenerate distance graphs' if $d \geq k+1$; for example when $k = 1$ these include finite trees and chains with prescribed edge lengths. Here the position of the vertices of the graphs in the large copies is immaterial provided that the scaled distances between vertices are realised. It is also possible to study analogous questions for anisotropic patterns, that is, for families of point configurations with power-type dependence on a real parameter which might be thought of as their size, see [20]. Finally, several authors have got around Example 1.1 by investigating these questions when $\mathbb{R}^d$ is endowed with the ℓ^p-norm for $1 \leq p \leq \infty$, $p \neq 2$, see [5, 7–9, 19].

Given such conclusions it is natural to seek general sufficient conditions that ensure that a set contains all sufficiently large similar copies of a given pattern. For homothetic copies (i.e., when we do not allow rotations), a measure-theoretic pigeonholing argument easily establishes the following statement.

Proposition 1.3. *Let $E \subseteq \mathbb{R}^d$ have upper density $\rho > \frac{n-1}{n}$ and let P be an n-point pattern in $\mathbb{R}^d$. Then there exists $R > 0$ such that if $r > R$, then E contains a homothetic copy of P scaled by a factor r.*

An aim of this paper is to obtain a quantitatively stronger result, that sets of density greater than $\frac{n-2}{n-1}$ contain all sufficiently large similar copies of n-point patterns in a sense that is uniform over certain patterns of a fixed size. For a pattern $P = \{x_0, \dots, x_{n-1}\} \subseteq \mathbb{R}^d$ we write $\operatorname{sep} P = \min_{i \neq j} \|x_i - x_j\|$ for the *minimum separation* of P and $\operatorname{diam} P = \max_{i \neq j} \|x_i - x_j\|$ for the *diameter* of P. By allowing rotations, the density of E that guarantees similar copies of P does not have to be as large as for homothetic copies.

Theorem 1.4. *Let $E \subseteq \mathbb{R}^d$ have upper Banach density $\rho > \frac{n-2}{n-1}$. Then there exists a number $R := R(E, S, D, n) > 0$ such that, for every n-point pattern $P \subseteq \mathbb{R}^d$ satisfying $S \leq \operatorname{sep} P \leq \operatorname{diam} P \leq D$, if $r \geq R$, then there exist $z_r \in \mathbb{R}^d$ and a rotation $Q_r \in SO(d)$ such that $rQ_r(P) + z_r \subseteq E$, i.e., E contains a similar copy of P at all scales at least R.*

To prove Theorem 1.4 we develop a quantitative version of the argument by Falconer and Marstrand [10], which we extend to $\mathbb{R}^d$ for $d \geq 2$. Note that, because in the proofs of Corollary 2.9 and Lemma 3.1 we choose x and Q to be any points in certain sets of positive $\mathcal{L}^d$-measure and σ-measure respectively, there will be a set of isometries of positive $(\sigma \times \mathcal{L}^d)$-measure under which copies of P at a (large) given scale will be contained in E.

It is natural to ask for the minimum upper density required in Theorem 1.4: to what extent can the value $\frac{n-2}{n-1} = 1 - \frac{1}{n-1}$ be reduced, and how does the density required to guarantee the presence of all sufficiently large copies of all n-point patterns behave as $n \to \infty$? Using arithmetic sequences we show that this density must approach 1 as n gets large, indeed at a rate $1 - O(n^{-1/5} \log n)$. The logarithm function is understood to have the number e as its base.

Theorem 1.5. *For all $n \in \mathbb{N}$ $(n \geq 2)$ and $d \in \mathbb{N}$ there exists a measurable set $E = E(d, n) \subseteq \mathbb{R}^d$ of density at least*

$$1 - \frac{10 \log n}{n^{1/5}} \tag{1.1}$$

such that there are arbitrarily large values of r for which E contains no congruent copy of $\{0, r, 2r, \dots, (n-1)r\}$.

Theorems 1.4 and 1.5 leave open the following question.

Question 1.6. *What is the smallest $0 \leq \rho_{\min}(d, n) < 1$ such that every measurable set $E \in \mathbb{R}^d$ of upper density $\rho > \rho_{\min}(d, n)$ contains all sufficiently large scale similar copies of all n-point patterns? Theorems 1.4 and 1.5 give*

$$1 - \frac{10 \log n}{n^{1/5}} \leq \rho_{\min}(d, n) \leq 1 - \frac{1}{n-1}.$$

Is it possible to improve either one of the two asymptotic bounds $1 - O(n^{-1/5} \log n)$ and $1 - O(n^{-1})$ as $n \to \infty$?

We remark in passing that problems of a similar nature are widely studied in the context of null Lebesgue measure, where one seeks conditions on the Hausdorff dimension or thickness of a set to guarantee that it contains a similar copy of a pattern. In particular, Łaba and Pramanik [22] gave conditions on fractal sets in the real line that ensure the existence of an arithmetic progression of length 3. Then Henriot, Łaba and Pramanik [15] and Chan, Łaba and Pramanik [4] improved the hypotheses and obtained results for more general patterns in $\mathbb{R}^d$. Iosevich and Liu [17] made a further improvement in $\mathbb{R}^4$ for copies of triangles. See also [13, 14, 18], where patterns are guaranteed in sets of sufficiently large Hausdorff dimension, and [31] for sets of large enough thickness.

The proof of Theorem 1.4 will span over Sections 2 and 3, while the proof of Theorem 1.5 will be given in Section 4.

2. Key estimates

We denote by $B_r(x) \subseteq \mathbb{R}^d$ the closed ball of centre x and radius r; we will abbreviate this to B_r for any ball of radius r when the centre is not relevant. The *upper Banach density* of a Lebesgue-measurable $E \subseteq \mathbb{R}^d$ is defined by

$$\rho := \rho(E) := \limsup_{r\to+\infty} \sup_{x\in\mathbb{R}^d} \frac{\mathcal{L}^d(E \cap B_r(x))}{\mathcal{L}^d(B_r(x))} \tag{2.1}$$

and the usual *upper density* by

$$\overline{d}(E) := \limsup_{r\to+\infty} \frac{\mathcal{L}^d(E \cap B_r(0))}{\mathcal{L}^d(B_r)} \tag{2.2}$$

Note that the last definition is invariant under changing the centre of the ball, that is replacing $B_r(0)$ by $B_r(x)$ for any other $x \in \mathbb{R}^d$. Then $\rho(E) \geq \overline{d}(E)$ and inequality can be strict; in fact, there exists a set $E \subseteq \mathbb{R}^d$ with $\rho(E) = 1$ and $\overline{d}(E) = 0$.

The following lemma shows that E has mean density not much more than ρ in all sufficiently large balls but also there exist balls of all large radii where E has mean density close to ρ.

Lemma 2.1. *Let $E \subseteq \mathbb{R}^d$ be Lebesgue-measurable with upper Banach density $\rho > 0$ and let $\alpha > 0$. Then we may find $s_1 := s_1(\alpha, E)$ such that*

$$\frac{\mathcal{L}^d(E \cap B)}{\mathcal{L}^d(B)} < \rho(1 + \alpha), \tag{2.3}$$

for all closed balls B of radii grater than s_1. Furthermore, for all $s > 0$, there exists a closed ball B_s such that

$$\frac{\mathcal{L}^d(E \cap B_s)}{\mathcal{L}^d(B_s)} > \rho(1 - \alpha). \tag{2.4}$$

Proof. Inequality (2.3) is clear from the definition of ρ.

For (2.4), given $s > 0$, we may find $r > s$ such that $1 - (1 - \frac{s}{r})^d < \frac{1}{2}\rho\alpha$ and $x \in \mathbb{R}^d$ satisfying

$$\frac{\mathcal{L}^d(E \cap B_r(x))}{\mathcal{L}^d(B_r)} > \rho\Big(1 - \frac{\alpha}{2}\Big).$$

Then,

$$\int_{B_r(x)} \mathcal{L}^d(B_s(y) \cap E)\,dy \geq \mathcal{L}^d(E \cap B_{r-s}(x))\mathcal{L}^d(B_s)$$

and

$$\mathcal{L}^d(E \cap B_{r-s}(x)) \geq \mathcal{L}^d(E \cap B_r(x)) - \Big(1 - \Big(1 - \frac{s}{r}\Big)^d\Big)\mathcal{L}^d(B_r).$$

Hence,

$$\begin{aligned}\frac{1}{\mathcal{L}^d(B_r)} \int_{B_r(x)} \frac{\mathcal{L}^d(E \cap B_s(y))}{\mathcal{L}^d(B_s)}\,dy &\geq \frac{\mathcal{L}^d(E \cap B_{r-s}(x))}{\mathcal{L}^d(B_r)} \\ &\geq \frac{\mathcal{L}^d(E \cap B_r(x))}{\mathcal{L}^d(B_r)} - \Big(1 - \Big(1 - \frac{s}{r}\Big)^d\Big) \\ &> \rho\Big(1 - \frac{\alpha}{2}\Big) - \rho\frac{\alpha}{2} = \rho(1 - \alpha).\end{aligned}$$

So, there exists $y \in B_r(x)$ such that $\frac{\mathcal{L}^d(E \cap B_s(y))}{\mathcal{L}^d(B_s)} > \rho(1 - \alpha)$. ∎

We will need to estimate the $(d-1)$-dimensional measure of the intersection of $(d-1)$-spheres with the set E. To facilitate this we approximate such spheres by annuli. Let $A_{r_1,r_2}(x) := B_{r_2}(x) \setminus B_{r_1}(x)$ be the *d-dimensional annulus* of centre s, inner radius r_1 and outer radius r_2. The intersection of pairs of such annuli is key to our calculations, and for $v \in \mathbb{R}^d$ and $\delta > 0$ we define

$$\phi_\delta^{(d)}(v) := \delta^{-2}\mathcal{L}^d(A_{r,r+\delta}(0) \cap A_{r,r+\delta}(v)). \tag{2.5}$$

We will check that the limit as $\delta \to 0$ of $\phi_\delta^{(d)}(v)$ exists pointwise and in L^1 and equals the following function $K_r^{(d)}$ which may be thought of as a potential kernel on $\mathbb{R}^d$.

Definition 2.2. *For $r > 0$ define $K_r^{(d)} : \mathbb{R}^d \to \mathbb{R}$ by*

$$K_r^{(d)}(v) := \begin{cases} \dfrac{2r^2\pi^{(d-1)/2}(r^2 - \frac{\|v\|^2}{4})^{(d-3)/2}}{\Gamma\left(\frac{d-1}{2}\right)\|v\|} & \text{if } \|v\| < 2r \text{ and } v \neq 0 \\ 0 & \text{if } \|v\| > 2r \\ +\infty & \text{if } \|v\| = 2r \text{ or } v = 0 \end{cases},$$

where Γ is the gamma function.

Throughout we will write A_r^d for the $(d-1)$-dimensional surface area of a ball $B_r \subseteq \mathbb{R}^d$, given by

$$A_r^d := \frac{d\, r^{d-1}\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}. \tag{2.6}$$

Lemma 2.3. *For all $r > 0$, $\phi_\delta^{(d)} \to K_r^{(d)}$ pointwise and in $L^1(\mathbb{R}^d)$. Furthermore,*

$$\int K_r^{(d)}(v)\ dv = (A_r^d)^2. \tag{2.7}$$

Proof. Pointwise convergence is trivial if $v = 0$ or $\|v\| \geq 2r$.

For $0 < \|v\| < 2r$ first consider the case when $d = 2$. The circles $C_r(0)$ and $C_r(v)$ intersect at angle θ where $\sin\frac{\theta}{2} = \|v\|/2r$. Then for small $\delta > 0$, $A_{r,r+\delta}(0) \cap A_{r,r+\delta}(v)$ is a pair of regions, each close to a rhombus of side $\delta/\sin\theta$ and height δ, so of area $\delta^2/\sin\theta$. Hence,

$$\begin{aligned}
\mathcal{L}^2\{A_{r,r+\delta}(0) \cap A_{r,r+\delta}(v)\} &= 2\frac{\delta^2}{\sin\theta} + O(\delta^3) = \frac{2\delta^2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} + O(\delta^3) \\
&= \frac{\delta^2}{\frac{\|v\|}{2r}\left(1 - \frac{\|v\|^2}{(2r)^2}\right)^{1/2}} + O(\delta^3) = \frac{2\delta^2 r^2}{\|v\|\left(r^2 - \frac{\|v\|^2}{4}\right)^{1/2}} + O(\delta^3) \\
&= \delta^2 K_r^{(2)}(v) + O(\delta^3),
\end{aligned}$$

noting that $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, so pointwise convergence at v when $d = 2$ follows noting (2.5).

For $d \geq 3$, we use the half of the estimate when $d = 2$, rotating one of the two approximate rhombii. Let $G_r := rS^{d-1} \cap (rS^{d-1} + v)$ where S^{d-1} is the unit $(d-1)$-sphere centred at 0, so G_r is the $(d-2)$-sphere $\tilde{r}S^{d-2} + \frac{1}{2}v$ of radius $\tilde{r} := \left(r^2 - \frac{\|v\|^2}{4}\right)^{1/2}$ which is contained in the hyperplane $\langle v\rangle^\perp + \frac{1}{2}v$. Then

$$\mathcal{L}^{d-2}(G_r) = \mathcal{L}^{d-2}(\tilde{r}S^{d-2}) = \frac{2\pi^{(d-1)/2}\,\tilde{r}^{d-2}}{\Gamma\left(\frac{d-1}{2}\right)}.$$

For $0 < \|v\| < 2r$,

$$\begin{aligned}
\mathcal{L}^d\{A_{r,r+\delta}(0) \cap A_{r,r+\delta}(v)\} &= \left(\delta^2\frac{K_r^{(2)}(v)}{2} + O(\delta^3)\right)\mathcal{L}^{d-2}(G_r) \\
&= \delta^2\frac{K_r^{(2)}(v)}{2}\frac{2\pi^{(d-1)/2}\,\tilde{r}^{d-2}}{\Gamma\left(\frac{d-1}{2}\right)} + O(\delta^3) \\
&= \delta^2\frac{K_r^{(2)}(v)}{2}\frac{2\pi^{(d-1)/2}\left(r^2 - \frac{\|v\|^2}{4}\right)^{(d-2)/2}}{\Gamma\left(\frac{d-1}{2}\right)} + O(\delta^3) \\
&= \delta^2 K_r^{(d)}(v) + O(\delta^3)
\end{aligned}$$

again giving convergence at v.

Pointwise convergence of $\phi_\delta^{(d)}(v)$ is not uniform, but to establish L^1 convergence it is enough to check that $\|\phi_\delta^{(d)}\|_1 \to \|K_r^{(d)}\|_1$. Noting that $\int \mathcal{L}^d(A\cap(B+v))dv = \mathcal{L}^d(A)\mathcal{L}^d(B)$ for measurable $A, B \subseteq \mathbb{R}^d$, from (2.5)

$$\begin{aligned}\int \phi_\delta^{(d)}(v)\ dv &= \delta^{-2}\mathcal{L}^d(A_{r,r+\delta}(0))^2\\ &= \delta^{-2}(\delta A_r^d + O(\delta^2))^2\\ &= (A_r^d)^2 + O(\delta). \qquad (2.8)\end{aligned}$$

Using spherical coordinates,

$$\begin{aligned}\int K_r^{(d)}(v)\ dv &= \frac{2r^2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\left(\int_0^{2r}\left(r^2-\frac{\rho^2}{4}\right)^{(d-3)/2}\rho^{d-2}\,d\rho\right)A_1^d\\ &= \frac{2r^2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}2^{d-2}r^{2d-4}\left(\int_0^1(1-t)^{(d-3)/2}\,t^{(d-3)/2}\,dt\right)A_1^d \qquad (2.9)\\ &= \frac{2r^2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}2^{d-2}r^{2d-4}\left(\frac{\Gamma(\frac{d-1}{2})^2}{(d-2)!}\right)A_1^d\\ &= \frac{d\pi^{d-1/2}2^{d-1}r^{2d-2}}{\Gamma(\frac{d}{2}+1)}\frac{\Gamma(\frac{d-1}{2})}{(d-2)!} \qquad (2.10)\\ &= \frac{d^2r^{2d-2}\pi^d}{\left(\Gamma(\frac{d}{2}+1)\right)^2} = (A_r^d)^2, \qquad (2.11)\end{aligned}$$

where we have used the substitution $\rho = 2rt^{1/2}$ to get the integral form of the beta function $\beta(\frac{d-1}{2},\frac{d-1}{2})$ at (2.9), followed by (2.6) at (2.10), and the factorial form of the gamma function at multiples of $\frac{1}{2}$ to get (2.11). From (2.8), $\|\phi_\delta^{(d)}\|_1 \to \|K_r^{(d)}\|_1$ which, together with pointwise convergence, implies that $\phi_\delta^{(d)} \to K_r^{(d)}$ in L^1. $\square$

For $r > 0$ write

$$g_r(x) := \mathcal{L}^{d-1}(E\cap S_r(x)) \quad (x\in\mathbb{R}^d)$$

for the measure of intersection of the set $E \subseteq \mathbb{R}^d$ with the sphere $S_r(x)$. The next lemma enables us to find the mean and mean square of g_r.

Lemma 2.4. *Let E be a bounded Lebesgue-measurable subset of $\mathbb{R}^d$ and let $r > 0$. Then*

$$\int g_r(x)\,dx = A_r^d\,\mathcal{L}^d(E). \qquad (2.12)$$

and

$$\int g_r(x)^2\ dx = \int_E\int_E K_r(y-z)\,dy\,dz. \qquad (2.13)$$

Proof. Let $g \in L^1(\mathbb{R}^d)$ be continuous and of compact support. Then,

$$\begin{aligned}\int \Big(\int_{A_{r,r+\delta}(x)} g(v)\ dv \Big) dx &= \int\int \chi_{A_{r,r+\delta}(0)}(v-x) g(v)\, dv\, dx \\ &= \int\int \chi_{A_{r,r+\delta}(0)}(u) g(x+u)\, du\, dx \\ &= \int \chi_{A_{r,r+\delta}(0)}(u)\, du \int g(y)\, dy \\ &= \big(\delta A_r^d + O(\delta^2)\big) \int g(y)\, dy.\end{aligned}$$

Dividing by δ and letting $\delta \to 0$,

$$\int \Big(\int_{S_r(x)} g(v)\, d\mathcal{L}^{d-1}(v) \Big)\, dx = A_r^d \int g(y)\, dy,$$

where the left-hand side inner integral is with respect to $(d-1)$-dimensional Lebesgue measure on the sphere. Identity (2.12) follows on approximating χ_E by continuous functions g.

Now let $g, h \in L^1(\mathbb{R}^d)$ be bounded and of compact support with g continuous. Then,

$$\begin{aligned}\int \Big[\int g(y) h(y-x)\ dy \Big]^2\ dx &= \int\int\int g(y)h(y-x)g(z)h(z-x)\ dx\ dy\ dz \\ &= \int\int g(y)g(z) \Big(\int h(u)h(u-y+z)\ du \Big)\, dy\, dz\end{aligned}$$

Taking $h(u) := \delta^{-1}\chi_{A_{r,r+\delta}(0)}(u)$,

$$\begin{aligned}\int \Big[\delta^{-1} \int_{A_{r,r+\delta}(x)} g(y)\, dy \Big]^2\ dx &= \int\int g(y)g(z)\, \delta^{-2}\mathcal{L}^d\{A_{r,r+\delta}(0) \cap A_{r,r+\delta}(y-z)\}\, dy\, dz \\ &= \int\int g(y)g(z)\phi_\delta(y-z)\, dy\, dz.\end{aligned}$$

Letting $\delta \searrow 0$ then, as g is continuous, $\frac{1}{\delta}\int_{A_{r,r+\delta}(x)} g \to \int_{S_r(x)} g$ and $\phi_\delta \to K_r^{(d)}$ in $L^1(\mathbb{R}^d)$ by Lemma 2.3,

$$\int \Big(\int_{S_r(x)} g(y)\, dy \Big)^2 dx = \int\int g(y)g(z)K_r^{(d)}(y-z)\, dy\, dz.$$

Again, approximating χ_E by continuous g gives (2.13). $\square$

The next lemma provides a good upper bound for the right-hand integral of (2.13) when E is reasonably uniformly distributed across a region.

Lemma 2.5. *Let $\delta > 0, 0 < \varepsilon_0 < 1$ and $0 < \xi \le 1$ be given. Then there exists $\lambda := \lambda(\varepsilon_0, \xi, \delta) \in (0, \varepsilon_0)$ such that if $B_s \subseteq \mathbb{R}^d$ is any ball of radius $s > 0$ and $E \subseteq B_s$ is any measurable set such that*

$$\frac{\mathcal{L}^d(E \cap B)}{\mathcal{L}^d(B)} < \rho(1+\alpha) \tag{2.14}$$

for all balls $B \subseteq B_s$ of radius at least λs, then for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$,

$$\int_E \int_E K_{\varepsilon s}^{(d)}(x-y)\ dx\ dy < (A_{\varepsilon s}^d)^2 (1+\varepsilon)^d \big((1+\alpha)^2 \rho^2 + \delta\big) \mathcal{L}^d(B_s).$$

Proof. By applying a similarity transformation it is enough to prove the lemma in the special case $s = 1$ and $B_s = B_1(0)$. For each $0 < \lambda < 1$ and $u \in \mathbb{R}^d$ we define $h_\lambda(u) := \frac{1}{\mathcal{L}^d(B_\lambda)} \chi_{B_\lambda(0)}(u)$.

Let $\eta := \delta(A_{\xi\varepsilon_0}^d)^2 (1+\xi\varepsilon_0)^d \mathcal{L}^d(B_1(0))$. Choose $\lambda := \lambda(\varepsilon_0, \xi, \delta) \in (0, \varepsilon_0)$ sufficiently small so that for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$,

$$\int_{B_1(0)} \int_{B_1(0)} \left| K_\varepsilon^{(d)}(x-y) - \int\int K_\varepsilon^{(d)}(z-w) h_\lambda(x-z) h_\lambda(y-w)\, dz\, dw \right| dx\, dy < \eta. \tag{2.15}$$

To achieve this, note that the double integral is continuous in ε, for example using that $K_{\varepsilon'}^{(d)}$ converges to $K_\varepsilon^{(d)}$ as $\varepsilon' \to \varepsilon$ pointwise almost everywhere and in $L^1(B_1(0) \times B_1(0))$. We can find a value of λ such that (2.15) is satisfied for each $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$ so compactness enables a choice of λ valid for all such ε.

Let $E \subseteq B_1(0)$ be a measurable set such that (2.14) holds for all balls $B \subseteq B_1(0)$ of radius at least λ. Then, for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$, restricting the domain of integration in (2.15) to $E \times E \subseteq B_1(0) \times B_1(0)$, we get

$$\begin{aligned}
\int_E \int_E K_\varepsilon^{(d)}(x-y)\ dx\ dy &< \eta + \int_E \int_E \int \int K_\varepsilon^{(d)}(z-w) h_\lambda(x-z) h_\lambda(y-w)\, dz\, dw\, dx\, dy \\
&< \eta + \rho^2 (1+\alpha)^2 \int_{B(0,1+\lambda)} \int_{B(0,1+\lambda)} K_\varepsilon^{(d)}(z-w)\, dz\, dw \\
&\le \eta + \rho^2 (1+\alpha)^2 \int_{B(0,1+\varepsilon)} (A_\varepsilon^d)^2\, dw \\
&\le (A_\varepsilon^d)^2 (1+\varepsilon)^d ((1+\alpha)^2 \rho^2 + \delta) \mathcal{L}^d(B_1(0)),
\end{aligned}$$

where we used (2.14) with the definition of h_λ, that $\lambda \le \varepsilon$, the integral (2.7), and the definition of η. □

The next lemma shows that we can find a ball B_s in which E has mean density close to ρ but also with good estimates for proportions of the surfaces of smaller spheres that intersect E. We will then use (2.17) and (2.18) to show that $g_{\varepsilon s}$ is nearly constant across B_s. Recall that

$$g_r(x) := \mathcal{L}^{d-1}(E \cap S_r(x)) \quad (x \in \mathbb{R}^d).$$

Lemma 2.6. *Let $E \subseteq \mathbb{R}^d$ be a Lebesgue-measurable set of upper Banach density $\rho > 0$, and let $0 < \xi \le 1$. Then, given $\eta \in (0,1)$, we can find $\varepsilon_0 > 0$ and $s_0 > 0$ such that for each $s > s_0$ there is a ball $B_s \subseteq \mathbb{R}^d$ of radius s satisfying*

$$\frac{\mathcal{L}^d(E \cap B_s)}{\mathcal{L}^d(B_s)} > \rho(1-\eta), \tag{2.16}$$

$$\frac{\int_{B_s} g_{\varepsilon s}(x)\,dx}{\mathcal{L}^d(B_s)} > A^d_{\varepsilon s}\,\rho(1-\eta) \tag{2.17}$$

and

$$\frac{\int_{B_s} g_{\varepsilon s}(x)^2\,dx}{\mathcal{L}^d(B_s)} < (A^d_{\varepsilon s})^2\rho^2(1+\eta), \tag{2.18}$$

for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$.

Proof. Given $\eta \in (0,1)$, we choose positive numbers α, δ, $\varepsilon_0 \in (0,1)$ small enough to ensure that

$$(1+\varepsilon_0)^d((1+\alpha)^2\rho^2+\delta) + (1-(1-\varepsilon_0)^d) < \rho^2(1+\eta), \tag{2.19}$$

and

$$\rho\alpha + (1-(1-\varepsilon_0)^d) < \rho\eta. \tag{2.20}$$

Let λ be given by Lemma 2.5 for these δ, ε_0 and ξ. Let $s_1 := s_1(\alpha, E)$ from Lemma 2.1 and let $s_0 := s_1/\lambda$. If $s > s_0$ then $s > s_1$ as $\lambda < 1$, and there is a ball B_s of radius s such that

$$\frac{\mathcal{L}^d(E \cap B_s)}{\mathcal{L}^d(B_s)} > \rho(1-\alpha). \tag{2.21}$$

By (2.20) $\alpha < \eta$ giving (2.16).

We now establish (2.18). Let $f_{\varepsilon s}(x) := \mathcal{L}^{d-1}((E \cap B_s) \cap S_{\varepsilon s}(x))$. By Lemma 2.4 applied to $E \cap B_s$,

$$\int f_{\varepsilon s}(x)^2\,dx = \int_{E\cap B_s}\int_{E\cap B_s} K_{\varepsilon s}(y-z)\,dy\,dz. \tag{2.22}$$

By Lemma 2.5 (which hypotheses are satisfied by definition of s_1 and that $s > s_1/\lambda = s_0$), we get that for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$,

$$\int_{E\cap B_s}\int_{E\cap B_s} K_{\varepsilon s}(x-y)\,dx\,dy < (A^d_{\varepsilon s})^2(1+\varepsilon)^d((1+\alpha)^2\rho^2+\delta)\mathcal{L}^d(B_s). \tag{2.23}$$

Writing $B_{s-\varepsilon s}$ for the ball concentric with B_s and of radius $s-\varepsilon s$, then $f_{\varepsilon s}(x) = g_{\varepsilon s}(x)$ for $x \in B_{s-\varepsilon s}$ and $\mathcal{L}^d(B_s \setminus B_{s-\varepsilon s}) = (1-(1-\varepsilon)^d)\mathcal{L}^d(B_s)$. By (2.22) and (2.23),

$$\int_{B_s} g_{\varepsilon s}(x)^2\,dx = \int_{B_{s-\varepsilon s}} g_{\varepsilon s}(x)^2\,dx + \int_{B_s\setminus B_{s-\varepsilon s}} g_{\varepsilon s}(x)^2\,dx$$

$$
\begin{aligned}
&\le \int_{B_s} f_{\varepsilon s}(x)^2\,dx + (A^d_{\varepsilon s})^2(1-(1-\varepsilon)^d)\mathcal{L}^d(B_s)\\
&\le (A^d_{\varepsilon s})^2\Big[(1+\varepsilon)^d((1+\alpha)^2\rho^2+\delta)+(1-(1-\varepsilon)^d)\Big]\mathcal{L}^d(B_s)\\
&< (A^d_{\varepsilon s})^2\rho^2(1+\eta)\mathcal{L}^d(B_s),
\end{aligned}
$$

using (2.19) since $\varepsilon \le \varepsilon_0$.

Finally we apply (2.12) to $E\cap B_{s-\varepsilon s}$ to get (2.17).

$$
\begin{aligned}
\int_{B_s} g_{\varepsilon s}(x)\,dx &= \int_{B_s} \mathcal{L}^{d-1}(E\cap S_{\varepsilon s}(x))\,dx\\
&\ge \int_{\mathbb{R}^d} \mathcal{L}^{d-1}((E\cap B_{s-\varepsilon s})\cap S_{\varepsilon s}(x))\,dx\\
&= A^d_{\varepsilon s}\,\mathcal{L}^d(E\cap B_{s-\varepsilon s})\\
&\ge A^d_{\varepsilon s}\left[\mathcal{L}^d(E\cap B_s)-\mathcal{L}^d(B_s\setminus B_{s-\varepsilon s})\right]\\
&\ge A^d_{\varepsilon s}\left[\rho(1-\alpha_0)-\big(1-(1-\varepsilon s)^d\big)\right]\mathcal{L}^d(B_s)\\
&> A^d_{\varepsilon s}\,\rho(1-\eta)\mathcal{L}^d(B_s),
\end{aligned}
$$

using (2.21) and (2.20). □

The following general lemma bounds the deviation of a function from its mean in terms of its second moment.

Lemma 2.7. *Let $D\subseteq\mathbb{R}^d$ be measurable with $0<\mathcal{L}^d(D)<\infty$, let $g\colon D\to\mathbb{R}_{\ge 0}$ be measurable and not identically 0, and let $\theta>0$. Then*

$$
\begin{aligned}
\mathcal{L}^d\bigg\{x\in D: \bigg|g(x)-\frac{1}{\mathcal{L}^d(D)}\int_D g(y)dy\bigg| \ge \theta\frac{1}{\mathcal{L}^d(D)}\int_D g(y)dy\bigg\}&\\
\le \frac{1}{\theta^2}\,\mathcal{L}^d(D)\bigg[\frac{\mathcal{L}^d(D)\int_D g^2}{(\int_D g)^2}-1\bigg]&.
\end{aligned}
$$

Proof. Identically

$$
\int_D\bigg(g(x)-\frac{1}{\mathcal{L}^d(D)}\int_D g(y)\,dy\bigg)^2dx = \int_D g(x)^2\,dx - \frac{(\int_D g)^2}{\mathcal{L}^d(D)},
$$

so by Chebyshev's inequality

$$
\begin{aligned}
\mathcal{L}^d\bigg\{x\in D: \bigg|g(x)-\frac{1}{\mathcal{L}^d(D)}\int_D g(y)dy\bigg| \ge \theta\frac{1}{\mathcal{L}^d(D)}\int_D g(y)dy\bigg\}&\\
\le \frac{1}{\theta^2}\frac{\mathcal{L}^d(D)^2}{(\int_D g)^2}\bigg[\int_D g^2-\frac{(\int_D g)^2}{\mathcal{L}^d(D)}\bigg]&\\
= \frac{1}{\theta^2}\,\mathcal{L}^d(D)\bigg[\frac{\mathcal{L}^d(D)\int_D g^2}{(\int_D g)^2}-1\bigg]&.
\end{aligned}
$$

□

Using Lemma 2.7 with the estimates of Lemma 2.6 we now show that there is a ball B_s such that 'most' $(d-1)$-spheres of radius εs centred inside B_s intersect E in a proportion of the sphere 'close to' ρ, the Banach density of E, for a suitable range of ε.

Proposition 2.8. *Let $E \subseteq \mathbb{R}^d$ be a Lebesgue-measurable set of upper Banach density $\rho > 0$ and let $0 < \rho' < \rho$. Let $0 < \xi \leq 1$ and $\delta > 0$. Then there exist $s_0 > 0$ and $\varepsilon_0 > 0$ such that for all $s \geq s_0$ there is a ball $B_s \subseteq \mathbb{R}^d$ such that*

$$\mathcal{L}^d(E \cap B_s) > \rho' \mathcal{L}^d(B_s) \tag{2.24}$$

and

$$\mathcal{L}^d\big\{x \in B_s : g_{\varepsilon s}(x) \leq \rho' A^d_{\varepsilon s}\big\} < \delta \mathcal{L}^d(B_s) \tag{2.25}$$

for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$.

Proof. Let $\rho' = (1-\theta)\rho$ where $0 < \theta < 1$. Choose $\eta > 0$ small enough so that

$$\frac{4}{\theta^2}\left[\frac{1+\eta}{(1-\eta)^2} - 1\right] < \delta \quad \text{and} \quad \eta < \tfrac{1}{2}\theta \tag{2.26}$$

By Lemma 2.6, given these ρ, ξ and η, there exist ε_0 and s_0 such that for all $s > s_0$ there is a ball B_s satisfying (2.24) by (2.16) and (2.26), and also for all $\varepsilon \in [\xi\varepsilon_0, \varepsilon_0]$,

$$\begin{aligned}
\mathcal{L}^d\big\{x \in B_s : g_{\varepsilon s}(x) \leq & \rho(1-\theta)A^d_{\varepsilon s}\big\} \\
&\leq \mathcal{L}^d\big\{x \in B_s : g_{\varepsilon s}(x) \leq \rho(1-\tfrac{1}{2}\theta)(1-\eta)A^d_{\varepsilon s}\big\} \\
&\leq \mathcal{L}^d\left\{x \in B_s : g_{\varepsilon s}(x) \leq (1-\tfrac{1}{2}\theta)\frac{\int_{B_s} g_{\varepsilon s}(x)\,dx}{\mathcal{L}^d(B_s)}\right\} \\
&= \mathcal{L}^d\left\{x \in B_s : \frac{\int_{B_s} g_{\varepsilon s}(x)\,dx}{\mathcal{L}^d(B_s)} - g_{\varepsilon s}(x) \geq \tfrac{1}{2}\theta\,\frac{\int_{B_s} g_{\varepsilon s}(x)\,dx}{\mathcal{L}^d(B_s)}\right\} \\
&\leq \frac{4}{\theta^2}\,\mathcal{L}^d(B_s)\left[\frac{\mathcal{L}^d(B_s)\int_{B_s} g_{\varepsilon s}(x)^2}{(\int_{B_s} g_{\varepsilon s}(x))^2} - 1\right] \\
&\leq \frac{4}{\theta^2}\,\mathcal{L}^d(B_s)\left[\frac{1+\eta}{(1-\eta)^2} - 1\right] \\
&< \delta\mathcal{L}^d(B_s),
\end{aligned}$$

where we have used (2.26), (2.17), Lemma 2.7, (2.17) and (2.18), and (2.26). □

The following corollary shows that if E has upper Banach density ρ and $\rho' < \rho$ then any given family of a finite number of concentric spheres can be scaled and translated so that a proportion at least ρ' of each spherical surface is in E, for all sufficiently large scalings.

Corollary 2.9. *Let $E \subseteq \mathbb{R}^d$ be measurable and let $0 < \rho' < \rho(E)$ and $0 < S \leq D$. Then there is an $s_0 := s_0(E, S, D, m) > 0$ such that, for every set of numbers $\{r_i\}_{i=1}^m$ with $r_i \in [S, D]$ for all i, for all $s \geq s_0$ there exists $x \in E$ such that*

$$\mathcal{L}^{d-1}(E \cap S_{r_i s}(x)) > \rho' A^d_{r_i s} \tag{2.27}$$

for all $1 \leq i \leq m$.

Proof. Given E, choose $0 < \delta < \rho'/m$ and set $\xi = S/D$. Let s_0 and ε_0 be given by Proposition 2.8 for these values. Thus for all $s \geq s_0$ there is a ball B_s such that (2.24) and (2.25) hold for all $\varepsilon \in [\varepsilon_0 S/D, \varepsilon_0]$. By scaling by a factor ε_0/D it is enough to assume that $r_i \in [\varepsilon_0 S/D, \varepsilon_0]$ for all i. Then

$$\begin{aligned}
\mathcal{L}^d\big\{x \in E \cap B_s : g_{r_i s}(x) \geq \rho' A^d_{r_i s} \text{ for all } 1 \leq i \leq m\big\} \\
&\geq \mathcal{L}^d(E \cap B_s) - \sum_{i=1}^m \mathcal{L}^d\big\{x \in E \cap B_s : g_{r_i s}(x) \leq \rho' A^d_{r_i s}\big\} \\
&\geq \rho' \mathcal{L}^d(B_s) - m\delta\, \mathcal{L}^d(B_s) > 0.
\end{aligned}$$

Thus for all $s \geq s_0$ we may choose $x \in E \cap B_s$ such that (2.27) is satisfied for all i. □

3. Finite patterns in $\mathbb{R}^d$

In this section we will apply Corollary 2.9 to prove Theorem 1.4, that is to show that a set E contains similar copies of a pattern P at all sufficiently large scalings provided that the Banach density of E is sufficiently large. We will also see that 'sufficiently large' can be taken to be uniform over the patterns P satisfying $\operatorname{card} P = n$ and $S \leq \operatorname{sep} P \leq \operatorname{diam} P \leq D$.

For a pattern $P = \{x_0, \ldots, x_{n-1}\} \subseteq \mathbb{R}^d$ write $r_i := \|x_i - x_0\| > 0$ for $1 \leq i \leq n-1$. Let $SO(d)$ be the special orthogonal group of rotations of $\mathbb{R}^d$ about the origin and let σ be normalised Haar measure on $SO(d)$.

Lemma 3.1. *Let $E \subseteq \mathbb{R}^d$ be measurable and let $P := \{x_0, \ldots, x_{n-1}\} \subseteq \mathbb{R}^d$ be a pattern. Suppose that $x_0 \in E$ and*

$$\mathcal{L}^{d-1}(E \cap S_{r_i}(x_0)) > \Big(\frac{n-2}{n-1}\Big) A^d_{r_i} \quad (1 \leq i \leq n-1). \tag{3.1}$$

Then there exists $Q \in SO(d)$ such that $Q(P - x_0) + x_0 \subseteq E$, i.e., P may be rotated about x_0 so that $x_i \in E$ for all $0 \leq i \leq n-1$.

Proof. Without loss of generality take $x_0 = 0$ so that $x_i \in S_{r_i}(0)$ for $1 \leq i \leq n-1$. From (3.1),

$$\sigma\big\{Q \in SO(d) : Q(x_i) \in E \cap S_{r_i}(0)\big\} = \frac{\mathcal{L}^{d-1}(E \cap S_{r_i}(0))}{\mathcal{L}^{d-1}(S_{r_i}(0))} > \frac{n-2}{n-1}.$$

Hence

$$\sigma\big\{Q \in SO(d) : Q(x_i) \in E \cap S_{r_i}(0) \text{ for all } 1 \leq i \leq n-1\big\}$$

$$\begin{aligned}&\geq \sigma(SO(d)) - \sum_{i=1}^{n-1} \sigma\big\{Q \in SO(d) : Q(x_i) \notin E \cap S_{r_i}(0)\big\}\\&> 1 - (n-1)\Big(1 - \frac{n-2}{n-1}\Big) = 0.\end{aligned}$$

Hence there is a set of rotations Q of positive σ-measure such that $Q(x_i) \in E$ for all $1 \leq i \leq n-1$, as required. (Note that this argument remains valid if the r_i are not all distinct.) □

Our main theorem, stating that sets of density greater than $\frac{n-2}{n-1}$ contain all sufficiently large copies of n point patterns, now follows easily.

Proof of Theorem 1.4. Taking $\rho' = \frac{n-2}{n-1}$ and $m = n-1$ in Corollary 2.9 there is a number $s_0(E,S,D,m)$ such that for all $s \geq s_0$ there exists $x_0 \in E$ such that

$$\mathcal{L}^{d-1}(E \cap S_{r_i s}(x_0)) > \Big(\frac{n-2}{n-1}\Big) A^d_{r_i s}.$$

for all $1 \leq i \leq n-1$, noting that $S \leq r_i \leq D$. Thus for all $s \geq s_0$, by Lemma 3.1 there is a $Q \in SO(d)$ such that $sQ(P) + x_0 - sQ(x_0) = Q\big(s(P - x_0)\big) + x_0 \subseteq E$. □

4. Lower bound

In this section we will prove a lower bound claimed in Theorem 1.5, that is, construct a set of density at least that stated in (1.1) that does not contain all sufficiently large n-term arithmetic progressions.

Proof of Theorem 1.5. Note that the claim is void unless $10 \log n / n^{1/5} < 1$. Thus, we assume that n is large enough so that this holds and denote

$$\varepsilon := \frac{10 \log n}{n^{1/5}} \in (0,1).$$

In particular, we will have $n > 10^5$ throughout the proof.

The set E will come from Bourgain's construction in [2], that is, it will be a 'thin' version of the set from Example 1.1. We define

$$\begin{aligned}E &:= \bigcup_{m=0}^{\infty} \Big\{x \in \mathbb{R}^d : m - \frac{1-\varepsilon}{2} < \|x\|^2 < m + \frac{1-\varepsilon}{2}\Big\}\\&= \Big\{x \in \mathbb{R}^d : \operatorname{dist}(\|x\|^2, \mathbb{Z}) < \frac{1-\varepsilon}{2}\Big\}.\end{aligned}$$

It is easy to see that E has (the most usual type of) density equal to $1-\varepsilon$.

Take some $r > 0$ and suppose that there exists an isometry $\mathbb{R} \to E \subseteq \mathbb{R}^d$ mapping $kr \mapsto x_k$ for $k = 0, 1, \ldots, n-1$, where $x_0, x_1, \ldots, x_{n-1}$ are some points in the set E. Let $a_k := \|x_k\|^2$. The parallelogram law gives

$$2\big(\|x_k\|^2 + \|x_{k+2}\|^2\big) = \|2x_{k+1}\|^2 + \|x_{k+2} - x_k\|^2,$$

i.e.,

$$a_{k+2} - 2a_{k+1} + a_k = 2r^2.$$

Solving this recurrence relation easily gives

$$a_k = r^2k^2 + Ak + B; \quad k = 0, 1, 2, \dots \tag{4.1}$$

for some constants A and B. Note that we are constrained to indices $k \le n-1$ only, but the above formula extends and defines an infinite sequence $(a_k)_{k=0}^{\infty}$. For now we only assume that r^2 is an irrational number; later we will refine this choice.

We will consider the sequence

$$\mathbf{a} = \big(a_k \bmod 1\big)_{k=0}^{\infty} \tag{4.2}$$

on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \equiv [0, 1)$, so that we can apply quantitative results on uniform distribution of sequences. These results belong to the realm of *discrepancy theory* [1,6,21], also known as *single-scale equidistribution theory* [30, §1.1.2]. The main idea is the following:

- On the one hand, by the construction of E, the first n terms of the sequence $\mathbf{a}$ completely avoid the interval $[(1-\varepsilon)/2, (1+\varepsilon)/2] \subseteq \mathbb{T}$ of length ε.
- On the other hand, for sufficiently large n the first n terms of the sequence $\mathbf{a}$ should be 'sufficiently uniformly distributed' over $\mathbb{T}$.

These two claims will lead to a contradiction. For the second claim we could use some result on quantitative equidistribution of polynomial sequences on $\mathbb{T}$, such as Exercise 1.1.21 from Tao's book [30]. However, since our sequence (4.1) is very special (i.e., it is only quadratic), and since we want to be entirely quantitative (i.e., with a precise exponent $1/5$ and an explicit constant, such as 10), we prefer to redo some of the theory from scratch.

The *discrepancy* of the first n terms of the sequence (4.2) is the number

$$D_n(\mathbf{a}) := \sup_{[\alpha,\beta)\subseteq[0,1)} \Big| \frac{\operatorname{card}\big\{k \in \{0, 1, \dots, n-1\} : a_k \bmod 1 \in [\alpha, \beta)\big\}}{n} - (\beta - \alpha) \Big|,$$

which quantifies how uniformly the a_k are distributed over $\mathbb{T}$. Once we can guarantee

$$D_n(\mathbf{a}) < \varepsilon, \tag{4.3}$$

we will arrive at a contradiction by taking $[\alpha, \beta] = [(1-\varepsilon)/2, (1+\varepsilon)/2]$. The famous Erdős–Turán inequality (see [21, Chapter 2, Theorem 2.5]) gives an explicit upper bound for the discrepancy in terms of exponential sums:

$$D_n(\mathbf{a}) \le \frac{6}{M+1} + \frac{4}{\pi} \sum_{m=1}^{M} \frac{1}{m} \Big| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i m a_k} \Big| \tag{4.4}$$

for any positive integer M (to be chosen later).

Next, we use an explicit version of van der Corput's trick for exponential sums (see [30, Lemma 1.1.6]) to get for yet another positive integer H (to be chosen later):

$$\Big|\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i m a_k} - \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i m a_{k+h}}\Big| \leq \frac{2H}{n}. \tag{4.5}$$

We now estimate the above double sum. We use the Cauchy-Schwarz inequality for the sum in k, expand out the square, take into account the explicit formula (4.1), and sum up a few finite geometric sequences:

$$\begin{aligned}
\Big|\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i m a_{k+h}}\Big|^2 &\leq \frac{1}{n}\sum_{k=0}^{n-1}\Big|\frac{1}{H}\sum_{h=1}^{H} e^{2\pi i m a_{k+h}}\Big|^2 \\
&= \frac{1}{H} + \frac{2}{H^2 n}\operatorname{Re}\sum_{\substack{0\leq k\leq n-1\\ 1\leq h<h'\leq H}} e^{2\pi i m (a_{k+h'}-a_{k+h})} \\
&\quad [\,\text{substitute } j=k+h,\ l=h'-h\,] \\
&= \frac{1}{H} + \frac{2}{H^2 n}\operatorname{Re}\sum_{l=1}^{H-1} e^{2\pi i l m (l r^2 + A)} \sum_{h=1}^{H-l}\sum_{j=h}^{n+h-1} e^{4\pi i j l m r^2} \\
&\leq \frac{1}{H} + \frac{4}{Hn}\sum_{l=1}^{H-1}\frac{1}{|1-e^{4\pi i l m r^2}|}. \qquad (4.6)
\end{aligned}$$

The final ingredient comes from the theory of *Diophantine approximations* [3]. Let us choose a *badly approximable* $z\in[0,1)$, which means that

$$\Big|z-\frac{p}{q}\Big| \geq \frac{c}{q^2} \tag{4.7}$$

for some $c=c(z)>0$ and all $p,q\in\mathbb{Z}$, $q\neq 0$. One such choice is the golden ratio

$$z = \frac{-1+\sqrt{5}}{2}, \tag{4.8}$$

in which case we can take

$$c = \frac{1}{3}. \tag{4.9}$$

This can be seen in an entirely elementary way, by using Viète's formulae and writing

$$\begin{aligned}
\frac{1}{q^2} \leq \frac{|p^2+pq-q^2|}{q^2} &= \Big|\frac{p}{q}-\frac{-1+\sqrt{5}}{2}\Big|\Big|\frac{p}{q}-\frac{-1-\sqrt{5}}{2}\Big| \\
&\leq \Big|\frac{p}{q}-z\Big|\bigg(\Big|\frac{p}{q}-z\Big|+\sqrt{5}\bigg).
\end{aligned}$$

A consequence of (4.7)–(4.9) is

$$\operatorname{dist}(qz,\mathbb{Z}) \geq \frac{1}{3q} \tag{4.10}$$

for every positive integer q. It is interesting to remark that uncountably many choices of z would work out here, provided that we were willing to lower the constant (4.9) to 2^{-15}, see [3, Theorem 7.8].

Now let $r>0$ be any number such that $r^2-z\in\mathbb{Z}$, where z was given in (4.8). The set of such numbers is unbounded. From (4.10) we get

$$|1-e^{4\pi i l m r^2}| = |1-e^{4\pi i l m z}| \geq 4 \operatorname{dist}(2lmz,\mathbb{Z}) \geq \frac{2}{3lm}$$

for positive integers l and m, so

$$\frac{4}{Hn}\sum_{l=1}^{H-1}\frac{1}{|1-e^{4\pi i l m r^2}|} \leq \frac{3Hm}{n}. \tag{4.11}$$

Combining (4.4), (4.5), (4.6), and (4.11) we end up with

$$D_n(\mathbf{a}) \leq \frac{6}{M+1} + \frac{4}{\pi}(1+\log M)\Big(\frac{2H}{n}+\frac{1}{H^{1/2}}\Big) + \frac{8\sqrt{3}}{\pi}\Big(\frac{HM}{n}\Big)^{1/2},$$

so choosing

$$H=\lfloor (1/25)n^{2/5}\rfloor, \quad M=\lfloor 4n^{1/5}\rfloor$$

we obtain

$$D_n(\mathbf{a}) < \frac{10\log n}{n^{1/5}}.$$

This is precisely (4.3) and it leads to the desired contradiction. □

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