A Further Extension of the KKMS Theorem

Yakar Kannai[*] Myrna H. Wooders[†]
Weizmann Institute of Science University of Toronto
Rehovot, Israel Toronto, Canada

This version: September 1999

Abstract

Recently Reny and Wooders ([23]) showed that there is some point in the intersection of sets in Shapley's ([24]) generalization of the Knaster-Kuratowski-Mazurkiwicz Theorem with the property that the collection of all sets containing that point is partnered as well as balanced. In this paper we provide a further extension by showing that the collection of all such sets can be chosen to be strictly balanced, implying the Reny-Wooders result. Our proof is topological, based on the Eilenberg-Montgomery fixed point Theorem. Reny and Wooders ([23]) also show that if the collection of partnered points in the intersection is countable, then at least one of them is minimally partnered. Applying degree theory for correspondences, we show that if this collection is only assumed to be zero dimensional (or if the set of partnered and strictly balanced points is of dimension zero), then there is at least one strictly balanced and minimally partnered point in the intersection. The approach presented in this paper sheds a new geometric-topological light on the Reny-Wooders results.

[*]Erica and Ludwig Jesselson Professor of Theoretical Mathematics. Parts of this paper were completed while this author was visiting the Institute for Mathematical Economics of the University of Bielefeld and the Fields Institute of Toronto. This research was partly supported by a MINERVA foundation (Germany) grant.

[†]The support of the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged. This author is indebted to Frank Page for helpful discussions and to Philip Reny for many interesting and stimulating discussions about partnership.

1 Introduction

A solution concept for a game (or economy) is said to be partnered if it exhibits no asymmetric dependencies between players. That is, whenever player i needs the cooperation of player j or is dependent upon the actions of player j then j similarly depends on i. Partnership is a natural property to require of a solution concept. If a solution concept is not partnered, there is an opportunity for one player to demand a larger share of the surplus from another player. Thus, a payo® that is not partnered exhibits a potential for instability. Consider, for example, the two-person divide the dollar game. If the two players can agree on the division, the dollar is divided between them according to the agreement. Any division giving the entire dollar to one player displays an asymmetric dependency since the player receiving the dollar needs the cooperation of the player getting nothing.

The de¯nition of partnership is based on the notion of partnered collections of subsets of a ¯nite set. Let N be a ¯nite set, whose members are called players. A collection of coalitions, consisting of subsets of N, is partnered if each player i in N is in some coalition in the collection and whenever i is in all the coalitions containing player j then j is in all the coalitions containing player i. If i is in all the coalitions containing j we think of this as a situation where j \needs" i. Thus, a collection of coalitions is partnered if and only if whenever a player i needs another player j then j similarly needs i:

Let x 2 R^N be an outcome of an jNj-person game. A coalition S ½ N is a supporting coalition for x if its part of x, x_S, can be achieved by cooperation of the membership of S alone. The supporting collection for x is the set of all supporting coalitions for x. The outcome x is partnered if it is feasible and if its supporting collection is partnered. To illustrate a partnered outcome for a game, we return to the divide the dollar example. An outcome in which one player receives the entire dollar is not partnered since the only coalition that can a®ord to give him the dollar is the two-player coalition, while the player getting nothing has an alternative coalition, the coalition consisting of himself alone. Thus, the player receiving the dollar needs the player receiving nothing but the player receiving nothing needs only himself.

A collection of subsets of a set N is minimally partnered if it is partnered and if for each player i there does not exist another player j such that j is in all the subsets containing player i. In other words, no one needs anyone else in particular. The only minimally partnered outcome for the divide the dollar game is that which assigns each player zero.

An outcome x is in the partnered core of a game if it is in the core (that is, it is feasible for the grand coalition and not in the interior of the feasible set for any coalition) and if, in addition, it is partnered. For the divide the dollar game, any division of the entire dollar which gives both players a positive share is in the partnered core. There are no outcomes in the partnered core that are minimally partnered since, to have a positive payo®, each player needs the other.

The partnership property was originally introduced to study solution concepts of games and economies and has now been applied in a number of papers; see, for example, ([12], [13], [14], [1], [3], [4], [21]). Recently, it has been shown that balanced games with and without side payments have nonempty partnered cores (see [20], [22]). As an outgrowth [22], Reny and Wooders ([23]) extend Shapley's ([24]) generalization of the Knaster-Kuratowski-Mazurkiwicz Theorem by showing that there is some point in the intersection (whose nonemptiness is assured by the Theorem) with the property that the supporting collection for that point is partnered as well as balanced. Reny and Wooders ([23]) also show that if the intersection of a balanced and partnered collection satisfying the conclusion of their extension of Shapley's generalization of the K-K-M Theorem contains at most countably many points, then at least one of these balanced collections is minimally partnered.

In this paper, we ¯rst obtain a further extension of Shapley's generalization of the K-K-M Theorem, showing that the collection of sets satisfying the conclusion of the Theorem can be chosen to be strictly balanced { the weights on the sets in the balanced collection are all positive. Our argument involves the Eilenberg-Montgomery ¯xed point Theorem for set-valued mappings. It is well-known (cf., Shapley and Vohra [25]) that properties of closed coverings indexed by coalitions may be inferred from ¯xed point theorems for convex-valued correspondences. Here we are dealing with partnerships of certain balanced collections of coalitions (indexing closed coverings) and make use of nonconvex-valued correspondences (satisfying the assumptions of Eilenberg-Montgomery). Assuming that the set of partnered and balanced points is zero dimensional (weaker than countable), we obtain a stronger result on minimal partnership than Reny and Wooders ([23]): There is at least one point in the intersection of a strictly balanced and partnered collection of sets that is minimally partnered. We use a version of degree theory valid for set-valued maps (correspondences), where the image of a point is not necessarily convex. In addition, we obtain the same conclusion under the assumption that the closure of the set of strictly balanced (and hence partnered) points is of zero dimension. We demonstrate, by examples, that the set of partnered and balanced points may be countably in¯nite with closure of positive dimension or may be uncountably in¯nite with dimension zero. Thus, our result showing the existence of a strictly balanced and partnered collection of sets that is minimally partnered provides a meaningful extension of the Reny and Wooders result on minimal partnership.

The results of the current paper induce similar game-theoretic results to those of Reny and Wooders ([22]). From our extension of Shapley's generalization of the K-K-M Theorem, it follows that for a balanced game there is a point in the core with the property that the supporting collection of sets for the said point is strictly balanced. From strict balancedness it follows that the point in the core is partnered. Our minimal partnership results on closed coverings also apply to partnered cores of games. We show by an example that the minimally partnered core of a game may be homeomorphic to the Cantor set.

Concerning mathematical methods, note that the Eilenberg-Montgomery ¯xed point Theorem for set-valued maps is deeper than the Kakutani ¯xed point theorem `customarily' used in game theory and economics. (Exceptions include Debreu ([6]), Mas-Colell ([15]), Keiding ([10]), and McLennan ([17]).) Degree theory for non-convex valued correspondences may appear not to be entirely standard; see, however, Borisovitch ([5]) for an exposition, Mas-Colell ([16]) for an accessible account of the key lemma required for development of the theory, and McLennan ([18]), where the lemma is applied for the construction of a Leftschetz ¯xed point index. Degree theory for (single-valued) functions, however, has been more extensively employed in the past in game theory and mathematical economics.

We are very much indebted to Philip Reny for pointing out an inaccuracy in an earlier version and for other suggestions which led to improvements in the present paper. We are also indebted to Andrew McLennan for references and to two anonymous referees for helpful comments.

2 Definitions and the Main Results

Let $N = \{1, 2, \ldots, n\}$ and let $\mathcal{P}$ be a collection of subsets of N. For each i in N let

$$\mathcal{P}_i = \{S \in \mathcal{P} : i \in S\}.$$

We say that $\mathcal{P}$ is *partnered* if for each i in N the set $\mathcal{P}_i$ is nonempty and for every i and j in N the following requirement is satisfied[1]:

$$\text{if } \mathcal{P}_i \subseteq \mathcal{P}_j \text{ then } \mathcal{P}_j \subseteq \mathcal{P}_i,$$

i.e. if all subsets in $\mathcal{P}$ that contain i also contain j then all subsets containing j also contain i. Let $\mathcal{P}[i]$ denote the set of those $j \in N$ such that $\mathcal{P}_i = \mathcal{P}_j$. We say that $\mathcal{P}$ is *minimally partnered* if it is partnered and for each $i \in N$, $\mathcal{P}[i] = \{i\}$.

Let $\mathcal{N}$ denote the set of nonempty subsets of N. For any $S \in \mathcal{N}$ let e^S denote the vector in $\Re^N$ whose i^{th} coordinate is 1 if $i \in S$ and 0 otherwise. For ease in notation we denote $e^{\{i\}}$ by e^i.

Let Δ denote the unit simplex in $\Re^N$. For every $S \in \mathcal{N}$ define

$$\Delta^S = \text{conv}\{e^i : i \in S\}, \text{ and}$$
$$m^S = \frac{e^S}{|S|},$$

where "conv" denotes the convex hull and $|S|$ denotes the number of elements in the set S.

Let $\mathcal{B}$ be a collection of subsets of N. The collection is *balanced* if there exist nonnegative weights $\{\lambda^S\}_{S \in \mathcal{B}}$ such that

$$\sum_{S \in \mathcal{B}} \lambda^S e^S = e^N$$

and the collection is *strictly balanced* if all weights λ^S can be chosen to be positive. It is easy to show that a strictly balanced collection of sets is partnered.

[1] The concept of a partnered collection of sets was introduced in Maschler and Peleg ([12], [13]). They used the term "separating collection." We follow the terminology of Bennett ([3]).

Proposition 1. Let Σ be a strictly balanced family of subsets of N. Then Σ is partnered.

Proof. Suppose that Σ is strictly balanced but not partnered. Then there exists $i, j \in N$ such that for all $S \in \Sigma$ with $i \in S$ it holds that $j \in S$, but there exists $T \in \Sigma$ with $j \in T$, $i \notin T$. Let $\{\omega_S : S \in \Sigma\}$ denote a set of strictly positive balancing weights for Σ. Since the weights ω_S on all the sets in Σ are strictly positive, $\sum_{S: i \in S} \omega_S < \sum_{S: j \in S} \omega_S = 1$. This is a contradiction. ∎

Observe that the collection $\mathcal{B}$ is balanced if and only if

$$m^N \in \text{conv}\{m^S : S \in \mathcal{B}\}.$$

Reny and Wooders ([23]) have obtained the following two results.

Theorem A. (Reny and Wooders ([23]) Let $\{C^S : S \in N\}$ be a collection of closed subsets of Δ such that

$$\bigcup_{S \subseteq T} C^S \supseteq \Delta^T \quad \text{for all } T \in N. \tag{1}$$

Then there exists $x^* \in \Delta$ such that the supporting collection for x^*, $S(x^*) \equiv \{S \in N : x^* \in C^S\}$, is balanced and partnered.

Remark 1. Observe that the supporting collection for the point x^* consists of all those coalitions S such that $x^* \in C^S$.

Theorem B. Reny and Wooders ([23]). Let $\{C^S : S \in N\}$ be a collection of closed subsets of Δ satisfying (1). If the set

$$\{x^* \in \Delta : S(x^*) \text{ is balanced and partnered}\}$$

is at most countable, then at least one $x^* \in \Delta$ renders the supporting collection $S(x^*)$ balanced and minimally partnered.

The next two Theorems, used in our extension of Reny and Wooders' results, are topological and essentially fixed point theorems for correspondences. Theorem 1 implies a strengthening of Theorem A of Reny and

Wooders ([23]). Under somewhat different assumptions, Theorem 2 yields a stronger conclusion than those of Theorem B of Reny and Wooders ([23]).

Theorem 1. Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ such that:

$$F \text{ is upper} - \text{hemicontinuous;} \tag{2}$$

$$\text{For all } x \in B\ (:= \partial\Delta);\ F(x) \subseteq B \text{ and } g(x) \notin F(x); \text{ where } g \text{ is the antipodal map; } g : B \to B; \tag{3}$$

and

$$F \text{ assumes finitely many values:} \tag{4}$$

Then there exists $x \in \Delta$ such that $m_N \in \text{rel int}(F(x))$.

(As usual, rel int(K) means the interior of K in the affine submanifold spanned by K.)

Recall the definition of zero (topological) dimension (Hocking and Young [8], Spanier [26], Arkhangel'skiĭ and L.S. Pontryagin [2]): A topological space X has dimension zero if for every $p \in X$ there is an arbitrarily small open set with empty boundaries containing p. It is well known (cf., Hocking and Young [8], p. 147 or Arkhangel'skiĭ and L.S. Pontryagin[2], p. 106-109) that among compact spaces the zero-dimensional spaces and the totally disconnected spaces are identical.

Theorem 2. Let $F(x)$ be a correspondence from Δ into the closed convex subsets of Δ satisfying (2), (4) and:

$$\text{For all } x \in B;\ x \in \Delta^S \Rightarrow F(x) \subseteq \Delta^S\ (S \subseteq N): \tag{5}$$

Assume also that:

$$\text{The closure of the set } \{x : m_N \in \text{rel int}(F(x)\} \text{ is zero} - \text{dimensional:} \tag{6}$$

Then there exists $x \in \Delta$ such that $F(x)$ has non-empty $(n-1)$-dimensional interior and $m_N \in \text{int}(F(x))$:

(By “int” we mean the “interior in the topology on the hyperplane $\sum_{i=1}^{n} x_i = 1$”.)

The following Proposition, establishing a link between correspondences and closed coverings, is derived from the Shapley and Vohra ([25]) proof of the KKMS Theorem. A proof of the Proposition is provided in an appendix to this paper.

Proposition 2. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ satisfying (1). Then there is a homeomorphism φ of Δ into the interior of Δ and a correspondence F from Δ into the closed convex subsets of Δ satisfying (2), (4), (5) and such that

$$F(y) = \text{conv}\{m_S : \varphi^{-1}(y) \in C^S\} \text{ for all } y \in \varphi(\Delta); \tag{7}$$

and

$$\text{if } m_N \in F(x) \text{ then } x \in \varphi(\Delta). \tag{8}$$

The following Theorem shows that the point satisfying the conclusion of the statement of the KKMS Theorem can be chosen so that its supporting collection is strictly balanced. By Proposition 1, strict balancedness implies partnership so the Theorem implies Theorem A of Reny and Wooders ([23]). To show that our Theorem is a strengthening of the prior result, we must exhibit a collection of sets which is partnered but not strictly balanced. It is well known (Maschler, Peleg, and Shapley ([14])) that there exist partnered collections which are not balanced. Let Σ be such a collection for an n-person game. Then $\{N\} \cup \Sigma$ is balanced and partnered, but $m_N \notin \text{rel int}[\text{conv}\{m_S\}_{S \in \Sigma \cup \{N\}}]$. One may even choose Σ to be minimally partnered. Then $\Sigma \cup \{N\}$ is balanced and minimally partnered, but again $m_N \notin \text{rel int}[\text{conv}\{m_S\}_{S \in \Sigma \cup \{N\}}]$. As a concrete example, take $N = \{1, 2, 3, 4, 5\}$ and $\Sigma = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}\}.$ The collection Σ is partnered and, in fact, minimally partnered. The collection $\Sigma \cup \{N\}$ is balanced and minimally partnered, but the only possible collection of balancing weights λ_S must assign zero weight to all sets $S \neq N$ and weight 1 to N.

The next Theorem follows from Theorem 1 and Proposition 2:

Theorem 3. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ such that (1) is satisfied. Then there exists $x \in \Delta$ such that the supporting collection $S(x) = \{S : x \in C^S\}$ is strictly balanced.

Proof: Let F be the map whose existence is stated in Proposition 2. Note that since F satisfies condition (5) it also satisfies condition (3). By Theorem 1 there exists $y \in \Delta$ such that $m_N \in \text{rel int}(F(y))$, and by (7) and (8) there exists $x \in \Delta$ ($x = \varphi^{-1}(y)$) such that

$$m_N \in \text{rel int}[\text{conv}\{m_S : x \in C^S\}]. \tag{9}$$

Clearly, $\Sigma := \{S : x \in C^S\}$ is balanced. Moreover, it is strictly balanced. In fact, let $S \in \Sigma;\ S \neq N$ (without loss of generality, $\Sigma \neq \{N\}$) and let ℓ_S denote the line joining m_N and m_S. Then m_N is contained in the interior of the interval $\ell_S \cap \text{conv}\{m_S\}_{S \in \Sigma}$. Hence there exists an $a_S \in \text{conv}\{m_S\}_{S \in \Sigma}$ and positive numbers $\alpha_S; \beta_S$ such that $\alpha_S + \beta_S = 1$ and $m_N = \alpha_S m_S + \beta_S a_S$. We may average these equations with positive weights over $S \in \Sigma;\ S \neq N$ and obtain m_N as a convex combination of the points $m_S;\ S \in \Sigma$, with positive weights for each $S \neq N$. ∎

The following consequence of Theorem 2 is related to Theorem B of Reny and Wooders ([23]).

Theorem 4. Let $\{C^S : S \subseteq N\}$ be a family of closed subsets of Δ such that (1) is satisfied. Assume that the closure of the set

$$\{x : \{S : x \in C^S\} \text{ is strictly balanced}\}$$

is zero-dimensional. Then there exists $x \in \Delta$ such that the collection $\{S : x \in C^S\}$ is minimally partnered and strictly balanced.

Proof. Let F be the map whose existence is stated in Proposition 2. Note in particular that F satisfies (5), and the assumptions imply that (6) is satisfied as well. Hence there exists (by Theorem 2) $x \in \Delta$ such that $m_N \in \text{int}(D(x))$ [where $D(x) = \text{conv}(m_S : x \in C^S)$]. If $\Sigma = \{S : x \in C^S\}$ is not minimally partnered, then there exists a pair $i; j$ such that for every $S \in \Sigma$ either i and

j both belong to S, or neither belongs. Hence for all $y \in D(x)$; $y_i = y_j$. Thus $\text{int}(D(x))$ is empty, a contradiction. ∎

Comparing the assumptions of Theorem 4 with those of Reny and Wooders ([23]) Theorem B, it appears that neither is stronger than the other. On the one hand a countable set (as assumed in Theorem B) may be dense and hence have closure of positive dimension; on the other hand, a set of dimension zero (as assumed in Theorem 4) may be uncountable (for example, a Cantor set on a line). Example 1 below illustrates a situation covered by Theorem B but not by Theorem 4 while Example 2 illustrates a situation covered by Theorem 4 but not by Theorem B.

Example 1. For a two-dimensional simplex, let m denote the barycenter and let $C_{\{i\}} = \{e^i\}$ for $i = 1;2$; $C_{\{i,j\}} = \text{conv}\{e^i; e^j; m\}$ for $i;j = 1;2,3$; and $C_{\{1;2;3\}} = \text{conv}\{e^1; e^2; e^3\}$. For $C_{\{3\}}$ first select a sequence Q in the interior of $C_{\{1;2\}}$ such that the set of limit points of Q is the interval $[e^2; m]$. Then set $C_{\{3\}}$ to be the union of $\{e^3\}$ with the closure of Q. The set of partnered points in the intersection, $\cap_S C_S$ consists of Q and m, a countable infinite set, whose closure is 1-dimensional.

Example 2. Let $\mathcal{C}$ denote the Cantor set. Denote by E the union of intervals removed in an even step (that is, numbers for which the first "1" in the ternary expansion appears in an even place) and let O be the union of intervals removed in an odd step. Let C_1 denote the union of E and $\mathcal{C}$, let C_2 denote the union of O and $\mathcal{C}$ and let C_{12} be $\mathcal{C}$. Note that the sets C_S satisfy the conditions of the KKMS theorem. The set of partnered points $\mathcal{C}$ is neither countable nor a (topological) continuum.

Remark 2. Note that the statement "$m_N \in \text{int}(D(x))$;" established in the proof of Theorem 4, is stronger than the conclusion of the Theorem. The statement means that every hyperplane through m_N (except for $\sum_{i=1}^n x_i = 1$) has vectors e^S with $x \in C^S$ on both sides.

Our final result is a proper strengthening of Theorem B of Reny and Wooders ([23]). As a formulation for correspondences (similar to Theorems 1 and 2) is cumbersome, we state here the result only for closed coverings.

Theorem 5. Let $\{C^S\}_{S \subseteq N}$ be a closed covering of Δ such that (1) is satisfied. If the set $\{x^* \in \Delta : S(x^*)$ is balanced and partnered$\}$ is zero dimensional, then at least one $x^* \in \Delta$ renders $S(x^*)$ strictly balanced and minimally partnered. Moreover, $m_N \in \text{int}\,[\text{conv}\{m_S\}_{S \in S(x^*)}]$.

3 Partnered cores of games

In this section we obtain, as a corollary to Theorem 3, the Reny-Wooders result that a balanced game has a nonempty partnered core. We also present an example showing that the partnered core may be homeomorphic to the Cantor set. This resolves the question raised in Reny and Wooders ([23]) whether it is possible that the set of points in the partnered core is either countable or zero-dimensional but not finite. We use standard notation, definitions and terminology – see, for example, Shapley and Vohra [25] and Reny and Wooders ([22]).

Let $(N;V)$ be a game and let $x \in \Re^N$ be a payoff for $(N;V)$. A coalition S is said to support the payoff x if $x \in V(S)$. Let $S(x)$ denote the set of coalitions supporting the payoff x. The payoff x is called a partnered payoff if the collection $S(x)$ has the partnership property. The payoff x is minimally partnered if it is partnered and if the set of supporting coalitions is minimally partnered. Note that partnered payoffs need not be feasible.

Let $P(N;V)$ denote the set of all partnered payoffs for the game $(N;V)$. The partnered core is denoted by $C^*(N;V)$ and is defined by

$$C^*(N;V) = P(N;V) \bigcap C(N;V)$$

where $C(N;V)$ denotes the core of the game $(N;V)$.

A game is balanced if for any balanced collection β

$$\bigcap_{S \in \beta} V(S) \subseteq V(N).$$

With Theorem 3 in hand, it is easy to prove that there is a point in the core of a balanced game whose supporting collection is strictly balanced. From Proposition 1, this implies the Reny-Wooders result that a balanced game has a nonempty partnered core. For simplicity, our proof makes use of two properties of games as defined by Shapley and Vohra [25] and Reny and Wooders ([22], namely, (i) for each $S \subseteq N$, $V(S)$ is bounded from above and (ii) for each $i \in N$; $V(\{i\}) > 0$. With some additional work, another version of Scarf's proof, not requiring these two properties, as in Kannai ([9]), p. 376-377, could be used.

Theorem 6. Let $(N; V)$ be a balanced game. Then there is a point y in the core whose supporting collection $S(y) = \{S \subset N : y \in V(S)\}$ is strictly balanced.

Proof. Recall that in the proof of Scarf's Theorem on nonemptiness of the core as in Shapley and Vohra ([25]), p. 111, (compare Kannai ([9]), p. 376-377) a certain function $f : \Delta \to \Re^N$ and certain closed subsets C_S of a simplex are constructed with the properties that

(a) $f(x) \notin \operatorname{int} V(S)$ for any $S \subseteq N$ and

(b) if $f(x) >> 0$ then for each coalition S the statements $x \in C_S$ and $f(x) \in V(S)$ are equivalent.

The sets C_S satisfy the assumptions of Theorem 3 and thus x can be chosen so that its supporting collection $S(x) = \{S : x \in C_S\}$ is strictly balanced. Let $y = f(x)$. It follows that the collection of sets $S(y) = \{S : y \in V(S)\}$ is strictly balanced, and since $(N; V)$ is a balanced game and $y \notin \operatorname{int} V(S)$ for each $S \subseteq N$; y is in the core. ∎

Theorems 4 and 5 similarly induce theorems on the set of minimally partnered core outcomes of games.

The following example illustrates a balanced game whose partnered core is homeomorphic to the Cantor set.

Example 3. A marriage and adoption game. We consider a game with twelve players. To make clearer the roles of players in the game, we'll provide some interpretation. Players 1 and 3 are adult males and players 2 and 4 are adult females. The remaining players are children, who may be adopted by male-female pairs who marry. Any married couple has the opportunity to adopt either one of two children. However, there are complicated adoption rules so that different pairs of married players cannot adopt the same children. Players 1 and 2, if they marry, may only adopt a child from the set $\{5,6\}$, players 3 and 4 may only adopt a child from the set $\{7,8\}$, players 1 and

4 may only adopt from the set $\{9,10\}$ and 2 and 3 players may only adopt from the set $\{11,12\}$. Let us call coalition consisting of a male-female pair and one of their potential children a family.

In the following, as in Example 2 above, C denotes the Cantor set, E denotes the union of intervals removed in an even step and O denotes the union of intervals removed in an odd step. Each player alone may only realize a outcome of 0. That is, for all $i \in N = \{1, 2, ..., 12\}$;

$$V(\{i\}) = \{x \in \Re^{12} : x_i \leq 0\}.$$

Also,

$$\begin{aligned} V(1,2,c) = \{x \in \Re^{12} : &\text{ there exists } y \in C \textstyle\bigcup O \text{ with } x_1 \leq y,\ x_2 \leq 1 - y \\ &\text{and } x_c \text{ satisfies } x_1 + x_2 + x_c \leq 1\} \text{ for } c \in \{5, 6\}. \end{aligned}$$

$$\begin{aligned} V(3,4,c) = \{x \in \Re^{12} : &\text{ there exists } y \in C \textstyle\bigcup E \text{ with } x_3 \leq y,\ x_4 \leq 1 - y \\ &\text{and } x_c \text{ satisfies } x_3 + x_4 + x_c \leq 1\} \text{ for } c \in \{7, 8\}. \end{aligned}$$

$$\begin{aligned} V(1,4,c) = \{x \in \Re^{12} : &\text{ there exists } y \in C \textstyle\bigcup O \text{ with } x_1 \leq y,\ x_4 \leq 1 - y \\ &\text{and } x_c \text{ satisfies } x_1 + x_4 + x_c \leq 1\} \text{ for } c \in \{9, 10\}. \end{aligned}$$

$$\begin{aligned} V(2,3,c) = \{x \in \Re^{12} : &\text{ there exists } y \in C \textstyle\bigcup E \text{ with } x_3 \leq y,\ x_2 \leq 1 - y \\ &\text{and } x_c \text{ satisfies } x_2 + x_3 + x_c \leq 1\} \text{ for } c \in \{11, 12\}. \end{aligned}$$

Let us call the above coalitions consisting of individual players and families, basic coalitions. For any nonbasic coalition S, define $V(S)$ as the minimal set which renders the game (N, V) superadditive and comprehensive. Equivalently, define

$$V(S) = \bigcup_{P(S)} \bigcap_{S' \in P(S)} V(S')$$

where $P(S)$ denotes a partition of S into basic coalitions and the union is taken over all such partitions.

We claim that the game (N, V) is balanced. To show this, without loss of generality we can restrict attention to balanced collections containing only basic coalitions. Observe that for any balanced collection β that is a partition, it follows from the definition of $V(N)$ that $\bigcap_{S\in\beta} V(S) \subset V(N)$. If the balanced collection β is not a partition, then at least one basic coalition $S \in \beta$ must have a positive weight ω less than 1. Let us suppose that, for a given balanced collection β, not a partition, there is a outcome x for which $x \in \bigcap_{S\in\beta} V(S)$ and $x \notin V(N)$. Since the outcome giving each player zero is in $V(N)$, it follows that $x \neq 0$. Since $x \neq 0$ there is at least one family, say F, in the balanced collection β and for at least one member i of the family, $x_i > 0$. If all families in the collection β have balancing weights equal to 1, then the balanced collection contains a partition and we have a contradiction to the definition of $V(N)$. Thus, let us suppose that the family F has balancing weight ω where $0 < \omega < 1$. This implies that the child in F must be in another coalition in β. The only possibility is the coalition consisting of that child alone. Thus, the child must receive an outcome of at most zero. It follows that $x_i \leq 0$ for $i = 5, 6, ..., 12$. Also, since the parents in F must each be in at least two different basic coalitions in the collection β, it follows that x_i must satisfy either $x \leq (d, 1-d, d', 1-d', 0, ..., 0)$ or $x \leq (d, 1-d', d', 1-d, 0, ..., 0)$ for some d and d' in the Cantor set C. But then $x \in V(N)$, a contradiction, so the game (N, V) is balanced.

We now claim that a payoff $x \in \Re^{12}$ is in the core if and only if x is of the form $\mathbf{d} = (d, 1-d, d, 1-d, 0, 0, ..., 0)$ for d in the Cantor set C and that every outcome in the core is minimally partnered. First, it is easy to see that for the case $d \in C$, $\mathbf{d}$ is in the core. In fact, $\mathbf{d}$ is in the minimally partnered core; for example, for $0 < d < 1$ the supporting collection $\{\{1,2,c\}_{c\in\{5,6\}}, \{3,4,c\}_{c\in\{7,8\}}, \{1,4,c\}_{c\in\{9,10\}}, \{2,3,c\}_{c\in\{11,12\}}\}$ is minimally partnered.

Now suppose $x \in \Re^{12}$ is a payoff in the core. It is immediate that the supporting collection for x must contain at least two nonintersecting families. Also, for any male m and female f in the same family it must hold that $x_m + x_f > 0$; this is immediate since at least one child available to that couple for adoption must receive zero and that child, along with the two parents m and f, could improve. In addition, $x_m \geq 0$ and $x_f \geq 0$. Now let us suppose, for the purpose of obtaining a contradiction, that $x_m + x_f < 1$. There are three possibilities:

1. For some y in O, $x_m \leq y$ and $x_f \leq 1 - y$.

2. For some y in E, $x_m \leq y$ and $x_f \leq 1 - y$.

3. For some $y \in C$, $x_m \le y$ and $x_f \le 1 - y$.

Suppose that $y \in O$. Since O is the union of open intervals and since $x_m + x_f < 1$ there are points x'_m and x'_f in O satisfying $x_m < x'_m$, $x_f < x'_f$, and $x'_m + x'_f < 1$. Thus, a family consisting of m and f and one of the children available to them for adoption can improve upon x, a contradiction. Therefore $x_m + x_f = 1$. Similarly, if $y \in E$, it follows that $x_m + x_f = 1$. If $y \in C$, since every point in C is an accumulation point of C, the points x'_m and x'_f can be chosen to be in C. Thus, for any pair of parents m and f in the same family, it must hold that $x_m + x_f = 1$. Now suppose that m and f are members of two different families. Consideration of all possibilities as above leads to the conclusion that $x_m + x_f = 1$. The cases where both families have outcomes dominated by points in the same set, C, E or O, can be treated as the above cases. If for one set of parents, say m_1 and f_1, it holds that for some $y_1 \in O$, $x_{m_1} \le y_1$ and $x_{f_1} \le 1 - y_1$ and for the other set of parents, say m_2 and f_2, it holds that for some $y_2 \in E \cup C$, $x_{m_2} \le y_2$ and $x_{f_2} \le 1 - y_2$, then there is a point y_3 in C such that, for $x_m = \min\{x_{m_1}, x_{m_2}\}$ and $x_f = \min\{x_{f_1}, x_{f_2}\}$, $x_m < y_3$ and $x_f < 1 - y_3$. It follows that the male and female who are receiving the smallest payoffs can, along with a child, improve upon x. This proves that a payoff x is in the core if and only if x is of the form $d = (d, 1 - d, d, 1 - d, 0, \ldots, 0)$ for d in the Cantor set.

4 Proofs of Theorems 1, 2 and 5

To prove Theorem 1, note that if y is not in the relative interior of a convex set K, then removing an open ball $B(y; \delta)$ of radius δ centered at y from K results in a nonempty closed contractible set (i.e., a set homeomorphic to a simplex of a certain dimension) for $\delta > 0$ sufficiently small. It follows from (2) and (4) that if $m_N \in$ rel int$(F(x))$ for no $x \in \Delta$, then there exists a $\delta > 0$ such that $F(x) \setminus B(m_N; \delta)$ is nonempty and contractible for all $x \in \Delta$. Moreover the openness of $B(m_N; \delta)$ implies that the correspondence $x \to F(x) \setminus B(m_N; \delta)$ is upper-hemicontinuous. Let h denote the usual radial retraction of the punctured simplex $\Delta \setminus \{m_N\}$ onto B. Then $h(F(x) \setminus B(m_N; \delta))$ is contractible for all x, and the same is true of $g(h(F(x) \setminus B(m_N; \delta)))$, where g is the antipodal map as given in (3). Clearly the correspondence $x \to g(h(F(x) \setminus B(m_N; \delta)))$ is upper hemicontinuous. By the Eilenberg-Montgomery fixed point theorem ([7]) every upper-hemicontinuous correspondence mapping the simplex into the collection of its non-empty, closed, and contractible subsets has a fixed point. Hence there exists a point $x^* \in \Delta$ such that $x^* \in g(h(F(x^*) \setminus B(m_N, \delta)))$. In particular, $x^* \in B$. By assumption (3) $F(x^*) \subseteq B$. But on B, h is the identity. Hence, $x^* \in g(F(x^*) \setminus B(m_N, \delta)) \subseteq g(F(x^*))$, contradicting (3). ∎

For the proof of Theorem 2 we need degree theory as extended for correspondences (see, for example, Lloyd ([11]), 115–120). Actually, a stronger version is needed, where the values are not necessarily convex (see, for example, Borisovich ([5])). In our case the values assumed by the correspondence are contractible and compact, so that the Begle-Vietoris mapping theorem ([7], [8], [26]) is applicable and may serve as a basis for degree theory. See also Mas-Colell ([16]) for an elementary proof of the key lemma required for development of the theory and McLennan ([18]) where the lemma is applied for the construction of a Leftschetz fixed point index.

It follows from (5) and a simple homotopy argument that

$$d(F, \mathrm{int}(\Delta), m_N) = 1. \tag{10}$$

Denote by $\overline{X}$ the closure of the set $\{x : m_N \in \mathrm{rel\ int}(F(x))\}$. By assumption, $\overline{X}$ is zero-dimensional. This means that for every $\epsilon > 0$, the set $\overline{X}$ may be covered by a finite number of disjoint open sets whose diameter is less than ϵ. Let $\{D_{i,m}\}_{i=1}^{P_m}$ denote such a collection of sets with diam $(D_{i,m}) < \frac{1}{m}$, $\overline{X} \subseteq \cup_{i=1}^{P_m} D_{i,m}$ and $D_{i,m} \cap D_{j,m} = \emptyset$ for $i \neq j$. Then $D_{i,m} \cap \overline{X}$ is both open and closed in $\overline{X}$, so that $\partial D_{i,m} \cap \overline{X} = \emptyset$.

With δ as in the proof of Theorem 1, set $\delta(x) = \min[\mathrm{dist}(x, \overline{X}), \delta]$, and define an upper-hemicontinuous correspondence G by

$$G(x) = F(x) \setminus B(m_N, \delta(x)). \tag{11}$$

Then $m_N \notin G(x)$ if $x \notin \overline{X}$. It follows from (5) (compare (11)) that

$$d(G, \mathrm{int}(\Delta), m_N) = 1. \tag{12}$$

By construction, $m_N \notin G(y)$ for all $y \in \partial D_{i,m}$, $1 \leq i \leq P_m$. Hence $d(G, D_{i,m}, m_N)$ is well defined and

$$\sum_{i=1}^{P_m} d(G, D_{i,m}, m_N) = d(G, \mathrm{int}(\Delta), m_N). \tag{13}$$

It follows from (13) and (12) that there exists $i_0 = i_0(m)$ such that $d(G; D_{i_0(m);m}; m_N) \neq 0$. By compactness there exists $\bar{x} \in \bar{X}$ and a sequence $D_{i_0(m);m}$ of neighborhoods (with $D_{i_0(m);m} \cap \bar{X}$ compact) such that $\bar{x} = \cap_{m=1}^{\infty} D_{i_0(m);m}$. For each m, $d(G; D_{i_0(m);m}; m_N) \neq 0$ implies the existence of an $(n-1)$-dimensional ball B_m, centered at m_N, such that

$$B_m \subseteq \bigcup_{x \in D_{i_0(m);m}} G(x) \subseteq \bigcup_{x \in D_{i_0(m);m}} F(x). \tag{14}$$

Set $\underline{a}_i = e^i - m_N$, $1 \le i \le n$. Fix for a moment $\underline{a}_j$ for an index $1 \le j \le n$. By (4) there exists a positive number δ_j such that if $m_N + \varepsilon \underline{a}_j \in F(y)$ for a certain $y \in \Delta$ and a positive ε (no matter how small), then $m_N + \delta_j \underline{a}_j \in F(y)$. By (14) there exists a sequence x^m converging to $\bar{x}$ and a sequence of positive real numbers ε_m such that $m_N + \varepsilon_m \underline{a}_j \in F(x^m)$. Hence $m_N + \delta_j \underline{a}_j \in F(x^m)$. By the upper-hemicontinuity $m_N + \delta_j \underline{a}_j \in F(\bar{x})$. The convexity of $F(\bar{x})$ and the spanning property of $\underline{a}_1, \ldots, \underline{a}_n$ imply that m_N is an interior point of $F(\bar{x})$. ∎

Remark 2: Theorems 1 and 2 may be generalized to contractible non-convex sets. Inspection of the proof of Theorem 1 shows that the condition (4) may be replaced by the assumption that there exists a positive number δ such that the sets $F(x) \setminus B(m_N; \delta)$ are nonempty and contractible for all $x \in \Delta$. Similarly, Theorem 2 is true if (4) is replaced by the property that for every $\underline{a} \neq 0$ there exists a $\delta > 0$ such that if both m_N and $m_N + \varepsilon \underline{a}$ are in $F(x)$ for any $x \in \Delta$ and $\varepsilon > 0$ then $m_N + \delta \underline{a} \in F(x)$.

The proof of Theorem 5 runs parallel to the previous proofs, with several essential refinements. Thus, let F and φ denote the correspondence and homeomorphism introduced in Proposition 1 and let the positive number δ be chosen so that the set $F(x) \setminus B(m_N; \delta)$ is nonempty and contractible for all $x \in \Delta$, (compare the proof of Theorem 1), and if $m_N \notin F(x)$, then $B(m_N; \delta) \cap F(x) = \emptyset$. We follow Reny-Wooders ([23]) (inspired by [4]) and set

$$c_{ij}(x) = \min_{\{S: i \notin S, j \in S\}} \operatorname{dist}(x; C^S) \tag{15}$$

for $x \in \Delta$, $1 \le i < j \le n$,

$$c_{ii}(x) = 0 \quad \text{for} \quad x \in \Delta;\ 1 \le i \le n; \tag{16}$$

$$\eta_i(x) = \sum_{j=1} [c_{ij}(x) - c_{ji}(x)] \quad \text{for} \quad x \in \Delta;\ 1 \le i \le n; \tag{17}$$

$$\delta(x) = \min[\sum_{i=1} |\eta_i(x)|; \delta] \quad \text{for} \quad x \in \Delta; \tag{18}$$

Then $\delta(x)$ is a non-negative continuous function on Δ. Define the correspondence $H(x)$ by

$$H(x) = F(x) \setminus B(m_N; \delta(\varphi^{-1}(x))) \quad \text{for} \quad x \in \varphi(\Delta); \tag{19}$$

$$H(x) = F(x) \quad \text{for} \quad x \notin \varphi(\Delta): \tag{20}$$

(Contrast with the definition of $G(x)$ in (11).) The choice of δ and (8) imply that $H(x)$ is upper-hemicontinuous. Let X denote the set $\{x^* \in \Delta : S(\varphi^{-1}(x^*))\}$ is balanced and partnered.

We claim that if $x \notin X$ then $m_N \notin H(x)$. In fact, if $m_N \in H(x)$ then $m_N \in F(x)$. Thus (8) implies that $x \in \varphi(\Delta)$. Hence $H(x)$ is given by (19). It follows that $S(\varphi^{-1}(x))$ is balanced and $\delta(\varphi^{-1}(x)) = 0$. But according to a Lemma of Bennett-Zame ([4]), as adapted by Reny-Wooders ([23]), if $\eta_i(y) = 0$ for all $i = 1; \ldots; n$ (as implied by $\delta(y) = 0$), then $S(y)$ is partnered. Hence $x = \varphi(y) \in X$.

Set now $Y = \{x \in \Delta : m_N \in H(x)\}$. Then Y is a closed subset of X, hence a closed zero-dimensional set. Note that the correspondence $H(x)$ satisfies the conditions of Theorem 1, except for (4). But by Remark 2, the conclusion of Theorem 1 holds for the correspondence H (see also (18)). Hence there exists a point $x \in \Delta$ such that $m_N \in$ rel int$(H(x))$, and in particular Y is not empty.

We can now continue the proof as in the proof of Theorem 2, with Y replacing $\overline{X}$ and H replacing G. We conclude that there exists $\bar{x} \in Y$ such that m_N is an interior point of $F(\bar{x})$. As in the proof of Theorem 4, this implies the existence of $x \in \Delta$ $(x = \varphi^{-1}(\bar{x}))$ such that $m_N \in \text{int}(D(x))$, from which all the assertions of Theorem 5 follow. ∎

Remark 3: Note that $\bar{x} \in Y$ implies $\delta(\varphi^{-1}(\bar{x})) = 0$ or $\eta_i(\varphi^{-1}\bar{x})) = 0$ for all $1 \le i \le n$. Thus the net credits (defined in ([4], [22])) of each player at x are zero. However, one does not need all the assumptions of Theorem 5 for the non-emptiness of Y. For this the assumptions of Theorem 3 suffice. (This observation was made in response to a suggestion by Philip Reny.)

5 Appendix

Proof of Proposition 2.

Following Shapley and Vohra (1991) set

$$\Delta' = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = 1;\ x_i \geq -1 \text{ for all } i \in N\}$$

and let $\pi : \Delta' \to \Delta$ be defined by

$$\pi_i(y) = \frac{\max(y_i; 0)}{\sum_{j \in N} \max(y_j; 0)} \text{ for all } i \in N:$$

In addition, (not found in Shapley and Vohra) define $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\varphi(x) = (\frac{1 + x_1}{n + 1}; \ldots; \frac{1 + x_n}{n + 1}):$$

Then $\pi : \Delta' \to \Delta;\ \varphi : \Delta \to \varphi(\Delta) \subset \text{int}(\Delta)$ are homeomorphisms. Note that

$$\varphi^{-1}(y) = [(n + 1)y_1 - 1; \ldots; (n + 1)y_n - 1]$$

and that $\varphi(\Delta)$ is an $(n - 1)$-simplex whose vertices are

$$e_i = (\frac{1}{n + 1}; \ldots; \frac{2}{n + 1}; \ldots; \frac{1}{n + 1})$$

where $\frac{2}{n+1}$ occurs in the i^{th} position.

Similarly to Shapley and Vohra, we define the following labelling function in Δ':

$$L'(y) = \{S : \pi(y) \in C^S \text{ and } y_i \geq 0 \text{ for all } i \in S\};$$

and a correspondence

$$G'(y) = \text{conv}\{m_S : S \in L'(y)\}:$$

We now set

$$F(y) = G'(\varphi^{-1}(y)):$$

(Note that $G'(y)$ and $F(y)$ are set-valued mappings and can assume only a finite number of distinct values. Note also that the domain of G' is Δ' while the domain of F is Δ.) For $y \in \varphi(\Delta);\ \varphi^{-1}(y) \in \Delta \Rightarrow \pi(y) = y$ and (7) follows.

The boundary behavior and upper-semi-continuity of F follow as in Shapley and Vohra. ■

References

[1] Albers, W. (1979) \Core and kernel variants based on imputations and demand pro¯les" in Game Theory and Related Topics, O. Moeschlin and D. Pallaschke, eds. North Holland, Amsterdam.

[2] Arkhangel'ski¸³ and L.S. Pontryagin (1988) General Topology 1; Basic Concepts and Constructions Dimension Theory, Springer-Verlag Berlin/Heidelberg/New York.

[3] Bennett, E. (1983) \The aspiration approach to predicting coalition formation and payo® distribution in sidepayment games," International Journal of Game Theory 12, 1{28.

[4] Bennett, E. and W.R. Zame (1988) \Bargaining in cooperative games," International Journal of Game Theory 17, 279{300.

[5] Borosovitch, Y.G., B.D. Gelman, A.D. Myshkis, and V.V. Obukhovski (1980) \Topological methods in the ¯xed-point theory of multi-valued maps," Russian Math. Surveys 35 (1), 65{143.

[6] Debreu, G. (1952) \A social equilibrium existence theorem," Proceedings of the National Academy of Sciences of the U.S.A. 38, 886-893.

[7] Eilenberg, S. and D. Montgomery (1946) \Fixed point theorems for multi-valued transformations," American Journal of Mathematics 68, 214{22.

[8] Hocking, J.G. and G.S. Young (1961) Topology, Addison-Wesley, Reading MA.

[9] Kannai, Y. (1992) \The core and balancedness" Handbook of Game Theory, Volume 1, eds. R. Aumann and S. Hart, Elsevier Science Publishers B.V., 355-395.

[10] Keiding, H. (1985) \On the existence of equilibrium in social systems with coordination," Journal of Mathematical Economics 14, 105{111.

[11] Lloyd, N.G. (1978) \Degree Theory," Cambridge University Press, Cambridge.

[12] Maschler, M. and B. Peleg (1966) \A characterization, existence proof and dimension bounds for the kernel of a game," Paci¯c Journal of Mathematics 18, 289{328.

[13] Maschler, M. and B. Peleg (1967) \The structure of the kernel of a cooperative game," SIAM Journal of Applied Mathematics 15, 569{604.

[14] Maschler, M., B. Peleg and L.S. Shapley (1972) \The kernel and bargaining set for convex games," International Journal of Game Theory 1, 73{93.

[15] Mas-Colell, A. (1974) \An equilibrium existence theorem without complete or transitive preferences," Journal of Mathematical Economics 1, 237-246.

[16] Mas-Colell, A. (1974), \A note on a theorem of F. Browder," Mathematical Programming 6, 229-233.

[17] McLennan, A. (1989) \Consistent conditional systems in noncooperative game theory," International Journal of Game Theory 18, 141-174.

[18] McLennan, A. (1989) \Fixed points of contractible valued correspondences," International Journal of Game Theory 18, 175-184.

[19] Page, F.H., Jr. and M.H. Wooders (1996) \The partnered core of an economy and the partnered competitive equilibrium," Economics Letters 52, 143-152.

[20] Reny, P.J., E. Winter and M.H. Wooders (1993) \The partnered core of a game with side payments," University of Western Ontario Discussion Paper.

[21] Reny, P.J. and M.H. Wooders (1995) \Credible threats of secession, partnerships, and commonwealths," Understanding Strategic Interaction; Essays in Honor of Reinhard Selten, W. Albers, W. GÄuth, P. Hammerstein, B. Moldovanu, and E. van Damme, eds. Spinger -Verlag Berlin/Heidelberg/New York/Tokyo, 305-312.

[22] Reny, P.J. and M.H. Wooders (1996) \The partnered core of a game without side payments," Journal of Economic Theory 70, 298-311.

[23] Reny, P.J. and M.H. Wooders (1998) \An extension of the KKMS Theorem," Journal of Mathematical Economics 28, 125-134.

[24] Shapley, L.S. (1973) \On balanced games without side payments," in Mathematical Programming, Hu, T.C. and S.M. Robinson (eds), Academic Press, New York, 261-290.

[25] Shapley, L.S. and R. Vohra (1991) \On Kakutani's ¯xed point Theorem, the KKMS Theorem and the core of a balanced game," Economic Theory 1, 108{116.

[26] Spanier, E.H. (1966) \Algebraic Topology," McGraw Hill, New York.