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Armendariz Semirings and Semicommutative Semirings

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Abstract

In this paper we study Armendariz semiring, which has been introduced by V.Gupta and P.kumar, in the paper entitled 'Armendariz and qusi-Armendariz and PS-semirings' [8] .We extend some results of Armendariz rings and semi-commutative rings of [3] for semirings with $1 \neq 0$. (i)We obtain that for a semirings S, S is Armendariz if and only eS and $(1+e)S$ are Armendariz for every idempotent e of S if and only if eS and $(1+e)S$ are Armendariz for every central idempotent e of S. (ii) For a semiring S if S/I is an Armendariz semiring for some reduced ideal I of S then S is Armendariz.

Keywords:

Armendariz semiring, p.s Armendariz semiring, Abelian semiring, Reduced semiring, Right quotient semiring, Semicommutative semiring, k-ideal.

1. Introduction

In 1934, H.S. Vandiver published a paper [13] entitled "Note on a simple type of Algebra in which the cancellation law of addition does not hold" which opened a new horizon in the research of Advanced algebra. In this paper, he introduced a new type of algebraic system which is commonly known an Semiring. Semiring is a common generalization of the theory of associative rings and the theory of distributives lattices. A semiring is an algebraic system consisting of a nonempty set S together with two binary operations, called addition and multiplication, which forms a commutative semigroup relative to addition, a semigroup relative to multiplication and the left, right distributive laws hold. The set of natural numbers is a natural example

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of a semiring. Now a days there has been a remarkable growth of the theory of semiring. Many classical notions of the ring theory have been generalized to semiring.

The theory of semirings and related topics are scattered over diverse areas of mathematics. Semirings arise in combinatorics and graph theory, automata and formal language, commutative and noncommutative ring theory, Euclidean geometry and topology, functional analysis and mathematical modelling of quantum physics, probability theory and optimization theory and many other areas of mathematics. More information about semiring can be found in [6] written by J.S.Golan; in [9],written by U.Hebish and H.J.Weinert and in [4],[5] written by K.Glazek.
A ring R is said to be Armendariz if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. More precisely, if $f(x) = a_0+a_1x+....+a_mx^m$ and $g(x) = b_0+b_1x+....+b_nx^n \in R[x]$ be such that $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i=0,1,2....,m and j=0,1,2,...,n. We will refer to this as the Armendariz condition. This definition was given by Rega and chhawchharia in [11] using the name Armendariz since E.P. Armendariz had proved in [2] that reduced rings satisfied this condition.

2. Preliminaries

Definition 1. *A nonempty set S together with a binary addition $+$ and a multiplication $\cdot$ is called a semiring if*
(i) $(S, +)$ is commutative semigroup.
(ii) $(S, \cdot)$ is semigroup.
(iii)for any three elements $a, b, c \in S$
the left distributive law $a.(b + c) = a.b + a.c$ and
the right distributive law $(b + c).a = b.a + c.a$ both hold.

Example 1. *The set of all natural no $\mathbf{N}$, and the set $\mathbf{Z_0^+}$ are semirings.*

Definition 2. *An element $'0'$ in S is called a zero element of S if $a + 0 = 0 + a = a,\forall a \in S$ and $'0'$ is called an absorbing zero if $a.0 = 0.a = a \ \forall \ a \in S$.*

Example 2. *The set of all natural no $\mathbf{N}$, and the set $\mathbf{Z_0^+}$ are semirings.*

Definition 3. *An element $'0'$ in S is called a zero element of S if $a + 0 = 0 + a = a,\forall a \in S$ and $'0'$ is called an absorbing zero if $a.0 = 0.a = a \ \forall \ a \in S$.*

Definition 4. *An element $'1'$ in S is called an identity element of S if $a.1 = 1.a = a, \ \forall \ a \in S$.*

Definition 5. *A semiring S is called commutative if $a.b = b.a$. for all $a, b \in S$.*

Definition 6. *A subset T of a semiring S with zero is called a subsemiring of S if it contains 0 and is closed under the operations of addition and multiplication in S.*

Definition 7. *A nonempty subset I of a semiring S is called a left ideal of S if*
(i) $a, b \in I$ implies $a + b \in I$ and
(ii) $a \in I$, $s \in S$ implies $s.a \in I$
Similarly we can define a right ideal of a semiring. A nonempty subset I of a semiring S is called an ideal of S if it is a left ideal as well as a right ideal of S.

Definition 8. *Let I be a proper ideal of a semiring S. Then the congruence on S, denoted by ρ_I and defined by $s\rho_I s^{'}$ if and only if $s + a_1 = s^{'} + a_2$ for some $a_1, a_2 \in I$, is called the Bourne congruence on S defined by the ideal I.*
We denote the Bourne congruence (ρ_I) class of an element s of S by s/ρ_I or simply by s/I and denote the set of all such congruence classes of S by S/ρ_I or simply by S/I.
It should be noted that for any $s \in S$ and any proper ideal I of S, S/I is not necessarily equal to $S + I = \{s + a : a \in I\}$ but surely contains it.

Definition 9. *For any proper ideal I of S if the Bourne congruence ρ_I, defined by I, is proper i.e $0/I \neq S$ then we define the addition and multiplication on S/I by $a/I + b/I = (a+b)/I$ and $(a/I)(b/I) = (ab)/I$ for all $a, b \in S$. With these two operations S/I forms a semiring which is called the Bourne factor semiring or simply the factor semiring.*

Definition 10. *A proper ideal I of a semiring S is called a prime ideal if $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$, where A and B are ideals of S.*

Definition 11. *A semiring S is called a prime semiring if $\{0\}$ is a prime ideal of S.*

Definition 12. *A proper ideal I of a semiring S is called a semiprime ideal if $A^2 \subseteq I$ implies that $A \subseteq I$, where A is an ideal of S.*

Definition 13. *An element a of a semiring S is said to be nilpotent if there exists a positive integer n such that $a^n = 0$.*

Definition 14. *An ideal I of a semiring S is said to be nil ideal if each element of I is nilpotent.*

Definition 15. *An ideal I of a semiring S is said to be nilpotent if there exists a positive integer n such that $I^n = 0$.*

Definition 16. *Let A be a nonempty subset of a semiring S. Right annhilator of A, denoted by $ann_R(A)$, is defined by $ann_R(A) = \{s \in S : As = (0)\}$.*
Analogously we can define left annhilator $(ann_L(A))$ of A. Annhilator of a set A, denoted by $ann(A)$, is a left as well as a right annihilator of A.

Remark 1. *If S is a semiring with absorbing zero then $ann_R(A)$ is a right ideal of S and $ann_L(A)$ is left ideal of S. If A is an ideal of S then both annihilators are ideals of S.*

Definition 17. *A semiring S is called zerosumfree if $a + b = 0$ for some $a, b \in S$, implies that $a = b = 0$.*

Throughout this paper by a semiring S we shall always mean a semiring with zero and identity.

3. Armendariz semiring and Abelian semiring

Definition 18. *A semiring S is called Armendariz if $f = \sum_{i=0}^{m} a_i x^i$,$g = \sum_{j=0}^{n} b_j x^j \in S[x]$ be such that $fg = 0$ then $a_i b_j = 0$ for all i and j.*

Example 3. *Let S be a zerosumfree semiring. Also let $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j \in S[x]$ be such that $f(x)g(x) = 0$. Then we have $a_0b_0 = 0$, $a_0b_1 + a_1b_0 = 0$, $a_0b_2 + a_1b_1 + a_2b_0 = 0$,...., $a_0b_n + a_1b_{n-1} + ... + a_nb_0 = 0$. From the second equation we get $a_0b_1 = 0$ and $a_1b_0 = 0$(since S is zerosumfree semiring). Again from the third equation we get $a_0b_2 + a_1b_1 = 0 \Rightarrow a_0b_2 = a_1b_1 = 0$(since S is zerosumfree semiring) and $a_2b_0 = 0$(since S is a zerosumfree semiring). Continuing this process we get $a_ib_j = 0 \;\forall\; i, j$. Hence S is Armendariz semiring.*

Example 4. *Let Z^+ be the semiring of all positive integers. Let $\overline{f(x)}$, $\overline{g(x)} \in Z/9Z[x]$ be such that $\overline{f(x)g(x)} = 0$. This implies that $3^2 | f(x)g(x)$. Let $f(x) = 3^r f^{'}(x)$ and $g(x) = 3^s g^{'}(x)$ for some $f^{'}(x)$ and $g^{'}(x)$ such that the g.c.d of the coefficient of $f^{'}(x)$(also of $g^{'}(x)$) is not divisible by 3. Obviously $3^2 | 3^{r+s}$. So $r + s \geq 2$. It follows that $\overline{a_i}\overline{b_j} = 0$ for all i and j. Hence $Z/9Z$ is Armendariz semiring.*

Proposition 1. *[8] Subsemiring of an Armendariz semiring is Armendariz.*

Proposition 2. *Suppose S is an Armendariz semiring. If $f_1, f_2, ...f_n \in S[x]$ are such that $f_1f_2......f_n = 0$, then $a_1a_2a_3......a_n = 0$, where a_i is a coefficient of f_i.*

Proof. We shall prove the proposition by induction on n. If $n = 1$, proof is obvious. Next suppose that $n = 2$ i.e $f_1f_2 = 0$. Since S is Armendariz, $a_1a_2 = 0$, where a_1 is any coefficient of f_1 and a_2 is any coefficient of f_2. Suppose that the proposition is true for all $k < n$. Suppose that $f_1f_2f_3...f_n = 0$. Then $f_1(f_2f_3.....f_n) = 0$. By our induction hypothesis $a_1b = 0$ where a_1 is any coefficient of f_1 and b is any coefficient $f_2f_3...f_n$.Then we have $a_1(f_2f_3.....f_n) = 0$, i.e $(a_1f_2)(f_3f_4.....f_n) = 0$. By our induction hypothesis $a_1a_2......a_n = 0$, where a_i is any coefficient of f_i. □

Theorem 3. *[8] A semiring S is Armendariz if and only if $S[x]$ is Armendariz.*

Definition 19. *A Semiring S is called abelian if every idempotent e of S central, i.e $es = se$ $\forall s \in S$.*

Theorem 4. *An additive cancellative semiring S is abelian if and only if $S[x]$ is abelian.*

Proof. Suppose that S is abelian. Then every idempotent of S is central. Let $f \in S[x]$ be idempotent. Then $f^2 = f$. Let $f = e_0 + e_1x + e_2x^2 + ... + e_nx^n$ where $e_i \in S, i = 0, 1, 2, ..., n$. Now $f^2 = f$ implies that
$e_0^2 = e_0$............(1)
$e_0e_1 + e_1e_0 = e_1$...........(2)
$e_0e_2 + e_1e_1 + e_2e_0 = e_2$.........(3)
....
$e_0e_n + e_1e_{n-1} + ... + e_ne_0 = e_n$.........(n)
(1) yields e_0 is idempotent; so it is central. If we multiply equation (2) on the left side by e_0,we get $e_0e_1 + e_0e_1e_0 = e_0e_1$. But $e_0e_1e_0 = e_0e_1$ because e_0 is central and since S is additively cancellative, $e_0e_1 = 0$ and from (2)we get $e_1 = 0$. Hence equation (3) becomes $e_0e_2 + e_2e_0 = e_2$, If we multiply equation (3) on the left side by e_0 we get $e_0e_2 + e_0e_2e_0 = e_0e_2$. But $e_0e_2e_0 = e_0e_2$. Since S is additively cancellative, $e_0e_2 = 0$ and from (3) $e_2 = 0$. Proceeding in this way we can see that $e_n = 0$. Thus $f = e_0$ is an idempotent of S and hence it is central. So $S[x]$ is abelian. Conversely assume that $S[x]$ is abelian. Since every idempotent of S is an idempotent of $S[x]$, every idempotent of S is central. So S is abelian. □

Theorem 5. *For a semiring S the following statements are equivalent:*
(1) S is an Armendariz semiring.
(2) eS and $(1 + e)S$ are Armendariz for every idempotent e of S.
(3) eS and $(1 + e)S$ are Armendariz for every central idempotent e of S.

Proof. (1) ⇒ (2). Obviously $0 = e.0 \in eS$. So eS is non-empty. Let ex, $ey \in eS$ where $x, y \in S$ and e is an idempotent of S. Then $ex + ey = e(x + y) \in eS$ and also $ex.ey \in eS$. Thus eS is a subsemiring of S, and hence it is *Armendariz*. Similarly we can prove that $(1 + e)S$ is *Armendariz* for every idempotent e of S.
(2) ⇒ (3) is obvious.
(3) ⇒ (1) Let $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ where $a_i, b_j \in S$ for $i = 0, 1, 2....., n$ and $j = 0, 1, 2, ..., m$. Let $f(x)g(x) = 0$. Let e be a central idempotent of S. Let $f_1(x) = ef(x)$, $f_2(x) = (1+e)f(x)$, $g_1(x) = eg(x)$ and $g_2(x) = (1 + e)g(x)$. Now $f_1(x)g_1(x) = ef(x).eg(x) = e^2f(x)g(x) = ef(x)g(x) = 0$,(since e is a central idempotent) and $f_2(x)g_2(x) = (1 + e)f(x).(1 + e)g(x) = (f(x) + ef(x))(g(x) + eg(x)) = f(x)g(x) + 3ef(x)g(x)$(since e is a central idempotent)$= f(x)g(x) + 3f_1(x)g_1(x) = 0 + 0 = 0$. Since $f_1(x)$, $g_1(x) \in eS[x]$ and eS is Armendariz,$f_1(x)g_1(x) = 0$ implies that $ea_i.eb_j = 0$ i.e $ea_ib_j = 0$(since e is central idempotent). Again $f_2(x)$, $g_2(x) \in (1 + e)S[x]$ and $(1 + e)S$ is Armendariz, $f_2(x)g_2(x) = 0$ implies that $(1 + e)a_i.(1 + e)b_j = 0$ i.e $a_ib_j + 3ea_ib_j = 0$ which implies that $a_ib_j = 0$(since $ea_ib_j = 0$). Thus S is Armendariz. □

Definition 20. *An ideal I of a semiring S is called a k-ideal if a, $b \in S$, $a + b \in I$ and $a \in I$ implies $b \in I$.*

Definition 21. *A semiring S is called the reduced if it has no non zero nilpotent elements.*

Theorem 6. *[8] Let S be a semiring. Let S/I be an Armendariz semiring for some k-ideal I of S. If I is reduced then S is Armendariz.*

Definition 22. *An element a of a semiring S is called regular if it is neither a left nor a right zero divisor. Following the definition of right quotient ring [12] we define right quotient semiring as follows.*

Definition 23. *A semiring Q is said to be a right quotient semiring of a semiring S with respect to a set T of regular elements of S if*
(i) $S \subseteq Q$
(ii) The elements of T are units in Q.
(iii) The elements of Q have the form ac^{-1} where $c \in T$, $a \in S$.

Theorem 7. *Suppose that there exists a right quotient semiring Q of a semiring S. Then S is Armendariz if and only if Q is Armendariz.*

Proof. Suppose that S is *Armendariz*. Consider two polynomials $f(x) = \sum_{i=0}^{m} \alpha_i x^i$, $g(x) = \sum_{j=0}^{n} \beta_j x^j$ of $Q[x]$, such that $\alpha_i, \beta_j \in Q$. We may assume that $\alpha_i = a_i u^{-1}$, $\beta_j = b_j v^{-1}$ with $a_i, b_j, u, v \in S$, and u, v regular. Again for each j there exists $c_j, w \in S$ with w regular such that $u^{-1}b_j = c_j w^{-1}$. Now $f_1(x) = \sum_{i=0}^{m} a_i x^i$,$g_1(x) = \sum_{j=0}^{n} b_j x^j \in S[x]$. Again we have $0 = f(x)g(x) = \sum_{i=0}^{m}\sum_{j=0}^{n}(\alpha_i\beta_j)x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n}(a_i u^{-1})(b_j v^{-1})x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i(u^{-1}b_j)v^{-1}x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i c_j w^{-1}v^{-1}x^{i+j} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_i c_j (vw)^{-1}x^{i}$ $f_1(x)g_1(x)(vw)^{-1}$. Hence $f_1(x)g_1(x) = \sum_{i=0}^{m}\sum_{j=0}^{n}(a_i c_j x^{i+j}) = 0$ in $S[x]$. Since S is Armendariz, $a_i c_j = 0$ $\forall i, j$ and so $\alpha_i\beta_j = (a_i u^{-1})(b_j v^{-1}) = a_i(u^{-1}b_j)v^{-1} = a_j c_j w^{-1}v^{-1} = 0 \ \forall\ i, j$. Therefore Q is Armendariz. Converse follows, since subring of Armendariz semiring is Armendariz. □

Definition 24. *Let S be a semiring and $S[[x]]$ denote the set of all sequences $\{a_n\} = \{a_0, a_1, ...\}$ of elements of S. Then $S[[x]]$ is a semiring with addition and multiplication defined by $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ and $\{a_n\}\{b_n\} = \{c_n\}$ where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. This semiring $S[[x]]$ is called the semiring of formal power series over S.*
Obviously $S[x]$ is a subsemiring of $S[[x]]$. Any element of $S[[x]]$ will be written as $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Definition 25. *A semiring S is called a semiprime semiring if $\{0\}$ is a semiprime ideal of S.*

Definition 26. *A semiring S is called power-serieswise quasi-Armendariz if whenever $f = \sum_{i=0}^{\infty} a_i x^i$, $g = \sum_{j=0}^{\infty} b_j x^j \in S[[x]]$ be such that $fSg = 0$ then $a_i S b_j = 0$ for all i and j.*

Theorem 8. *[8] Let S be a semiprime semiring. Then S is a p.s-quasi Armendariz semiring.*

Definition 27. *A semiring S is called quasi-Armendariz if whenever $f = \sum_{i=0}^{m} a_i x^i$,$g = \sum_{j=0}^{n} b_j x^j \in S[x]$ be such that $fSg = 0$ implies that $a_i S b_j = 0$ for all i and j.*

Theorem 9. *[8] Let S be a p.s quasi-Armendariz semiring. Then matrix semiring $T_n(S)$ of all $n \times n$ matrices over S is also a p.s quasi-Armendariz semiring.*

Corollary 1. *[8] Let S be a quasi-Armendariz semiring. Then $T_n(S)$ is also a quasi-Armendariz semiring.*

Theorem 10. *[8] Let S be a p.s quasi-Armendariz semiring. Then eSe is also a p.s quasi-Armendariz semiring for any non-zero idempotent e in S.*

Corollary 2. *[8] Let S be a quasi-Armendariz semiring. Then eSe is also a quasi-Armendariz semiring for any non-zero idempotent e in S.*

4. Armendariz semiring and Semicommutative semiring

Definition 28. *A semiring S is called semicommutative if for every $a \in S$, $\{b \in R : ab = 0\}$ is an ideal of S. i.e the right annhilator of a in S is an ideal of S.*

Theorem 11. *For a semiring S the following statements are equivalent:*
(1) *S is semi-commutative.*
(2) *Any right annihilator over S is an ideal of S.*
(3) *Any left annihilator over S is an ideal of S.*
(4) *For any $a, b \in S$, $ab = 0$ implies $aSb = 0$.*

Proof. we shall show that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Leftrightarrow$ (3) and finally (4) $\Rightarrow$ (1)
(1) $\Rightarrow$ (2)
Since S is semi-commutative, from the definition of semicommutative semiring $rann_S(x)$ is an ideal of S; Hence (2).
(2) $\Rightarrow$ (4) Let $a, b \in S$ be such that $ab = 0$. Then $b \in rann_S(a)$. Since by (2), $rann_S(a)$ is an ideal of S, $sb \in rann_S(a)\ \forall\ s \in S$. Hence $asb = 0\ \forall\ s \in S$,i.e $aSb = 0$.
(4) $\Rightarrow$ (3) We define left annihilator of an element $x \in S$, denoted by $lann_S(x)$, as follows: $lann_S(x) = \{b \in S : bx = 0\}$. It can be readily seen that $lann_S(x)$ is a left ideal of S. Let $b \in lann_S(x)$. Then $bx = 0$. By (4) $bsx = 0 \forall s \in S$. This implies that $bs \in lann_S(x)$. So $lann_S(x)$ is a right ideal and hence an ideal of S.
(3) $\Rightarrow$ (4) Let $a, b \in S$, be such that $ab = 0$. Then $a \in lann_S(b)\ \forall\ s \in S$. By (3), $as \in lann_S(b)\ \forall\ s \in S$. So $aSb = 0$
(4) $\Rightarrow$ (1) The proof is similar to the proof of (4) $\Rightarrow$ (3). □

Theorem 12. *Subsemiring of semicommutative semiring is semicommutative.*

Proof. Suppose S is a semicommuataive semiring and T be a subsemiring of S. Let $a, b \in T$ be such that $ab = 0$. This implies that $aSb = 0$, since S is semicomutative and hence $aTb = 0$. So T is semicommutaive. □

Corollary 3. *Let S be a semiring such that $S[x]$ is semi-commutative. Then S is semi-commutative.*

Proof. The proof follows from the theorem 12, since S is a subsemiring of $S[x]$. □

Let S be a semiring and Ω be a subsemigroup of S consisting of central regular elements of the semigroup $(S, .)$. Let $\Omega^{-1}S = \{\alpha^{-1}s :$ for all $\alpha \in \Omega$ and for all $s \in S\}$. Now we have the following theorem.

Theorem 13. *Suppose that S is a semiring and Ω is a subsemigroup of S consisting of central regular elements of the semigroup $(S, \cdot)$. Then $\Omega^{-1}S$ is a semiring.*

Proof. Let $a = \alpha^{-1}s_1$ and $b = \beta^{-1}s_2$ where $\alpha, \beta \in \Omega$ and $s_1, s_2 \in S$. Now $a + b = \alpha^{-1}s_1 + \beta^{-1}s_2 = \alpha^{-1}\beta^{-1}(\beta s_1 + \alpha s_2) = (\beta\alpha)^{-1}(\beta s_1 + \alpha s_2) \in \Omega^{-1}S$. Let $a = \alpha^{-1}s_1, b = \beta^{-1}s_2$ and $c = \gamma^{-1}s_3$, where $\alpha, \beta, \gamma \in \Omega$ and $s_1, s_2, s_3 \in S$. Now $(a + b) + c = (\alpha^{-1}s_1 + \beta^{-1}s_2) + \gamma^{-1}s_3 = \alpha^{-1}\beta^{-1}(\beta s_1 + \alpha s_2) + \gamma^{-1}s_3 = \alpha^{-1}\beta^{-1}\gamma^{-1}(\gamma(\beta s_1 + \alpha s_2) + \beta\alpha s_3) = (\gamma\beta\alpha)^{-1}(\gamma(\beta s_1 + \alpha s_2) + \beta\alpha s_3)$=$(\gamma\beta\alpha)^{-1}((\gamma\beta s_1 + \gamma\alpha s_2) + \beta\alpha s_3) = (\gamma\beta\alpha)^{-1}((\gamma\beta s_1 + \alpha\gamma s_2) + \alpha\beta s_3)$ since α, β, γ are central regular elements. Again $a + (b + c) = \alpha^{-1}s_1 + (\beta^{-1}s_2 + \gamma^{-1}s_3) = \alpha^{-1}s_1 + \beta^{-1}\gamma^{-1}(\gamma s_2 + \beta s_3) = \alpha^{-1}\beta^{-1}\gamma^{-1}(\gamma\beta s_1 + \alpha(\gamma s_2 + \beta s_3))$=$(\gamma\beta\alpha)^{-1}(\gamma\beta s_1 + (\alpha\gamma s_2 + \alpha\beta s_3))$. Therefore $a + (b + c) = (a + b) + c\ \forall\ a, b, c \in \Omega^{-1}S$. Now $a + b = \alpha^{-1}s_1 + \beta^{-1}s_2 = \alpha^{-1}\beta^{-1}(\beta s_1 + \alpha s_2) = (\beta\alpha)^{-1}(\beta s_1 + \alpha s_2)$ and $b + a = \beta^{-1}s_2 + \alpha^{-1}s_1 = \beta^{-1}\alpha^{-1}(\alpha s_2 + \beta s_1) = (\beta\alpha)^{-1}(\alpha s_2 + \beta s_1) = (\beta\alpha)^{-1}(\beta s_1 + \alpha s_2)$. So $a + b = b + a\ \forall$ a,b Thus $(\Omega^{-1}S, +)$ is a commutative semigroup. Also $a \cdot b = \alpha^{-1}s_1 \cdot \beta^{-1}s_2 = (\alpha\beta)^{-1}s_1s_2 \in \Omega^{-1}S$ and $a.(b.c) = \alpha^{-1}s_1.(\beta^{-1}s_2.\gamma^{-1}s_3) = \alpha^{-1}s_1.\beta^{-1}\gamma^{-1}(s_1s_2) = (\gamma\beta\alpha)^{-1}(s_1.(s_2s_3))$, since $\alpha, \beta, \gamma \in \Omega$.In a similar fashion we can show that $(a.b).c = (\gamma\beta\alpha)^{-1}((s_1s_2)s_3)$,so $a.(b.c) = (a.b).c\ \forall\ a, b, c \in \Omega^{-1}S$, since S is semiring.Thus $(\Omega^{-1}S, .)$ is a semigroup.Finally we shall show that the both distributive laws holds. $a.(b + c) = \alpha^{-1}s_1.(\beta^{-1}s_2 + \gamma^{-1}s_3) = \alpha^{-1}s_1.\beta^{-1}s_2 + \alpha^{-1}s_1.\gamma^{-1}s_3 = a.b + a.c$. Similarly we can show that $(a + b).c = a.c + b.c$. Hence $\Omega^{-1}S$ is a semiring. □

Theorem 14. *Suppose that S is a semiring and Ω is subsemigroup of S consisting of central regular elements of the semigroup $(S, \cdot)$. Then S is semicommutative if and only if $\Omega^{-1}S$ is semicommutative.*

Proof. Suppose that S is a semicommutative semiring. Let $a, b \in \Omega^{-1}S$ be such that $ab = 0$. Then $a = \alpha^{-1}s_1$ and $b = \beta^{-1}s_2$ where $\alpha, \beta \in \Omega$ and $s_1, s_2 \in S$. Now $0 = ab = (\alpha^{-1}s_1)(\beta^{-1}s_2) = \alpha^{-1}\beta^{-1}s_1s_2$[Since Ω is contained in the center of S]=$(\beta\alpha)^{-1}s_1s_2$. This implies $s_1s_2 = 0$, it follows that $s_1Ss_2 = 0$, since S is semicommutative. Let $\gamma = \omega^{-1}s \in \Omega^{-1}S$, where $\omega \in \Omega$ and $s \in S$. Now $a\gamma b = \alpha^{-1}s_1\omega^{-1}s\beta^{-1}s_2 = \alpha^{-1}\omega^{-1}\beta^{-1}(s_1ss_2) = 0$. Hence $\Omega^{-1}S$ is semicommutative. Converse is obvious. □

Proposition 15. *The semiring of Laurent polynomials in x with coefficients in a semiring S, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in S$ and k, n are integers(not necessarily positive). We denote this semiring by $S[x; x^{-1}]$*

Theorem 16. *For a semiring S, $S[x]$ is semicommutative if and only if $S[x;x^{-1}]$ is semicommutative.*

Proof. Suppose that $S[x]$ is semicommutative. Let $\Omega = \{1, x, x^2, ...\}$. Obviously Ω is a subsemigroup of $S[x]$ and closed under multiplication. Since $S[x;x^{-1}] = \Omega.S[x]$, it follows that $S[x;x^{-1}]$ is semicommutative by proposition 15. Converse follows from theorem 12. □

Theorem 17. *Let S be a semiring and I be a $k-ideal$ of S such that S/I is semicommutative. Now if I is reduced then S is semicommutative.*

Proof. Let $ab = 0$ with $a, b \in S$. Now $bIa \subseteq I$. Also $(bIa)^2 = bIabIa = 0$(Since $ab = 0$). Since I is reduced, $bIa = 0$. Again $((aSb)I)^2 = aSbIaSbI = 0$,since $bIa = 0$. So $(aSb)I = 0$,since I is reduced and $(aSb)I \subseteq I$. Now $(a/I)(b/I) = ab/I = 0/I = I$. Since S/I is semicommutative, then $(a/I)S/I(b/I) = 0/I$, i.e $aSb/I = I$. This implies that $aSb \subseteq I$, since I is a $k-ideal$ of S.Now $(aSb)^2 \subseteq (aSb)I = 0$ which implies that $(aSb) = 0$. Thus S is semicommutative. □

Theorem 18. *Let S be a semicommutative semiring which is also an Armendariz semiring. Then $S[x]$ is a semicommutative semiring.*

Proof. Let $f(x), g(x)$ be two polynomials in $S[x]$ be such that $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ and $a_i, b_j \in S$ and $i, j \in \{0, 1, 2...\}$. Let $h(x) = \sum_{k=0}^{t} c_k x^k \in S[x]$. Since S is Armendariz and $f(x)g(x) = 0$, $a_i b_j = 0$ for all i and j. Since S is semicommutative, $a_i S b_j = 0$. This implies that $a_i c_k b_j = 0$ for each i,j and k. Hence $f(x)h(x)g(x) = 0$. Thus $S[x]$ is a semicommutative semiring. □

5 Armendariz semiring and Reduced semiring

Proposition 19. *[8] Subsemiring of a reduced semiring is reduced.*

Theorem 20. *[8] A semiring S is reduced if and only if $S[x]$ is reduced.*

Theorem 21. *Every reduced semiring is an Armendariz semiring.*

Proof. Let S be a reduced semiring and $f, g \in S[x]$ with $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ where $a_i, b_j \in S$, $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$. Let $fg = 0$. We can assume that $n = m$. Then we have $a_0 b_0 = 0$, $a_1 b_0 + a_0 b_1 = 0$,......., $a_n b_0 + ... + a_0 b_n = 0$. Now $(b_0 a_0)^2 = b_0 a_0 b_0 a_0 = 0$ which implies that $b_0 a_0 = 0$, since S is reduced. Hence left multiplying the second equation by b_0 from the left we get $b_0 a_1 b_0 + b_0 a_0 b_1 = 0$. i.e $b_0 a_1 b_0 = 0$. Again $(a_1 b_0)^2 = a_1 b_0 a_1 b_0 = 0$ which implies that $a_1 b_0 = 0$ since S is reduced. Similarly we get $a_i b_0 = 0$ for $1 \leqslant i \leqslant n$. Then we get $a_0 b_1 = 0$, $a_1 b_1 + a_0 b_2 = 0$,.......,$a_{n-1} b_1 + ... + a_0 b_n = 0$. Now $(b_1 a_0)^2 = b_1 a_0 b_1 a_0 = 0$ which implies that $b_1 a_0 = 0$ since S is reduced. Again we multiply the second equation by b_1. We get $b_1 a_1 b_1 = 0$; $(a_1 b_1)^2 = a_1 b_1 a_1 b_1 = 0$ which implies that $a_1 b_1 = 0$ since S is reduced. Continuing this process we get $a_i b_1 = 0 \ \forall \ 1 \leqslant i \leqslant n$. Again continuing the above process we get $a_i b_j = 0 \ \forall \ 1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ as desired.
The converse is obvious since if $a_i b_j = 0 \ \forall \ 0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$, then $fg = 0$. □

Converse of the above result may not be true which follows from the fact that an Armendariz ring which is evidently Armendariz seming may not be reduced [4].

Theorem 22. *Let S be a semiring and $Q(S)$ be its right quotient semiring of S. Then S is reduced if and only if $Q(S)$ is reduced.*

Proof. Let $Q(S)$ be reduced. Then S is reduced, since subsemiring of a reduced semiring is again a reduced semiring. Conversely let S be a reduced semiring. Now we shall show that $Q(S)$ is a reduced semiring. Let $q = ab^{-1} \in Q(S)$ where $a \in S$ and b is regular such that and $q^2 = 0 \Rightarrow ab^{-1}ab^{-1} = 0$. Obviously $b^{-1}a \in Q(S)$. So there exists elements $c, d \in S$ with d regular such that $b^{-1}a = cd^{-1}$. Now $ac(bd)^{-1} = acd^{-1}b^{-1} = ab^{-1}ab^{-1} = 0$. This implies $ac = 0$. Now $(ca)^2 = caca = 0$. Since S is reduced, $ca = 0$. Now from $b^{-1}a = cd^{-1}$, we get $ad = bc$, which implies that $ada = bca = 0$. Hence $(ad)^2 = adad = 0$. Since S is reduced, $ad = 0$. Again $a = (ad)d^{-1}$ which implies that $a = 0$. Thus $q = ab^{-1} = 0$. Hence $Q(S)$ is reduced. □

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References

[1] Anderson, D. D., Camillo, V. (1998). Armendariz rings and Gaussian rings. *Communications in Algebra.* 26(7): 2265-2272. DOI:10.1080/00927879808826274.

[2] Armendariz, E. P. (1974). A note on Extensions of Bear and p.p. rings, *J.Austral. Math.Soc.* 18: 470-473. DOI:10.1017/S1446788700029190.

[3] Chan, H., Yang, L., Smoktunowicz, A. (2002). Armendariz rings and semicommutative rings, *Communications in Algebra.* 30(2): 751-761. DOI:10.1081/AGB-120013179.

[4] Glazek, K. (2002). A Guide to the Literature on semirings and their Application in Mathematics and Information science; Kluwer Academic Publishers; Dorrecht.

[5] Glazek, K. (2002). Linear Algebra over Semirings; A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences pp 25-42. DOI:10.1007/978-94-015-9964-1-4.

[6] Golan, J. S. (1999). Semirings and their Applications, Kluwer Academic Publishers; Dorrecht.

[7] Goldie, A. W. (1972). Struture of noetherian rings, In book: Lecture on rings and modules in Mathematics, 246: pp.213-321. DOI:10.1007/BFb0059567.

[8] Gupta, V., Kumar, P. (2011). Armendariz and Quasi-Armendariz semirings and PS-semirings,*Asia-Euro J. of Math,world scintific* . 4(1): 81-94. DOI:10.1142/S1793557111000071.

[9] Hebisch, U., Weinert, H. J. (1998). Semirings-Algebric Theory and applications in computer science; World scientific publication co. Inc; River Edge; NJ.

[10] Kim, N. K., Lee, Y. (2000). Armendariz rings and reduced rings. *J.Algebra.* 223: 477-488. DOI:10.1006/jabr.1999.8017.

[11] Rega, M. B., Chhawcharia, S. (1997). Armendariz rings proc. *Japan Acad.ser. A Math.Sci.* 73: 14-17. DOI: 10.3792/pjaa.73.14.

[12] Reid, A. (2009). Twisted group rings which are semi-prime Goldie rings. Cambridge University Press. DOI:10.1017/S0017089500002433.

[13] Vandiver, H. S. (1934). Note on a Simple Type of Algebra in which the cancellation Laws of Addition does not Hold. *Bull. Amr. Math. Soc.* 40(12),914-920.