Journal of Progressive Research in Mathematics
Vol 18,2 - 2021 — E-ISSN: 2395-0218
http://www.scitecresearch.com/journals/index.php/jprm

BCI/BCK-quantum algebra

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Received: May 22, 2021; Accepted: June 15, 2021; Published: July 19, 2021

Abstract

The paper contains an investigation of the notion of BCI-algebras and BCK-algebras. The concept of quantale was created in 1984 to develop a framework for non-commutative spaces and quantum mechanics with a view toward non-commutative logic. Implicational algebras like pseudo-BCK algebras are quantum B-algebras, and every quantum B-algebra can be recovered from its spectrum which is a quantale. [25].
A brief introduction to quantum mechanics is given. A new generalization of BCI/BCK-algebra and some there properties have been given. The BCI- quantum algebra and BCk- quantum algebra have been studied. The BCI- Lie algebra and BCK- Lie algebra are introduced. Various examples have been given.

Keywords:

BCI**-algebra,** BCK**-algebra,** B**-algebra, quantum.**

1. Introduction

The basic structure of quantum mechanics is quite different, the state of a system is given by a point in a space", It is can be thought of equivalently as the space of solutions of an equation of motion, or as the space of coordinates and momenta.
The quantities are just functions on this space. There is one distinguished observable, the energy or Hamiltonian. This functions determines how states evolve in time through Hamilton's equations.
BCI-algebra and BCK-algebra have been introduced by Y. Imai and K. Isé ([5, 7, 8]).
The former was raised in 1966 by Imai and Iseki.
Several generalizations of a BCI/BCK-algebra were introduced by many researchers, ([13, 14, 16, 17, 19, 20, 21, 23, 1]).
In the present paper we extend this work. We give a new generalization of BCI/BCK-algebra and some there properties. We define The BCI- Lie algebra and BCK- Lie algebra and study there properties.

How to cite this paper: M.Tabuni (2021) BCI/BCK-quantum algebra. Journal of Progressive Research in Mathematics, 18(2), 61-68. Retrieved from
http://scitecresearch.com/journals/index.php/jprm/article/view/2059

2 BCI/BCK-algebra

An algabra X with a consistent 0 and a binary operation "$*$," called a BCI-algebra if the following axioms satisfied
for all $x, y, z \in X$:

(BCI$_1$) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI$_2$) $x * 0 = x$,

(BCI$_3$) $x * y = 0$ and $y * x = 0$ imply that $x = y$,

Let X be a BCI-algebra, A partial order can be defined as $xleqy$ if and only if $x * y = 0$.

Proposition 1. *Let X be a BCI-algebra and let S be a subset of X, we call S sub algebra of X if the consistent 0 of X in S, and $(S, *, 0)$ satisfies a BCI-algebra axioms.*
suppose that $(X, *, 0)$ is a BCI-algebra. Define a binary relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$ for any $x, y \in X$ then $(x, \leq)$ is partially ordered set with 0 as a minimal element in the meaning that $x \leq 0$ implies $x = 0$ for any $x \in X$.

If a BCI-algebra X satisfies $0 * x = 0$, for all $x \in X$, then we say that X is a BCK-algebra.
Every BCI-algebra satisfying $0 * x = 0$ for all $x \in X$ is a BCK-algebra.
Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$:
(1) $(x * y) * z = (x * z) * y$
(2) $((x * z) * (y * z)) * (x * y) = 0$
(3) $x * 0 = x$
(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

A nonempty subset I of X is called an ideal of X if it satisfies
(I) $0 \in I$ and
(II) $x * y \in I$ and $y \in I$ imply $x \in I$.
A non-empty subset I of X is said to be an H-ideal of X if it satisfies (I) and
(II) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.
A BCI-algebra is said to be associative if $(x * y) * z = x * (y * z)$, for all $x, y, z \in X$.

Theorem 2. [15] Let $(X; *, 0)$ be a BCI-algebra. Then the following hold:

(i) $x * x = 0$,

(ii) $0 * (0 * x) = x$,

(iii) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$.

3 Quantum mechanics

In quantum mechanics, stats refers to physical state of quantum system. The state of a quantum mechanical system is given by a nonzero vector in a complex vector space H with Hermitian inner product $\langle ., . \rangle$.
H may be finite or infinite dimensional, we may want to require H to be a Hilbert space, A Hilbert space H consists of a set of vectors and a set of scalars. We will use the notation introduced by Dirac for vectors in the state space H such a vector with a label ψ is denoted ψ.

Time evolution of states $\psi(t) \in H$ is given by the Schrodinger equation

$$\frac{d}{dt}\psi(t) = \frac{i}{h}H\psi(t)$$

The Hamiltonian observable H will have a physical interpretation in terms of energy, with the boundlessness condition necessary in order to assure the existence of a stable lowest energy state. h is a dimensional constant, called Planck's constant.

3.1 Group representations

The mathematical framework of quantum mechanics is closely related to what mathematicians describe as the theory of group representations.
A standard definition of a Lie group is as a smooth manifold, with group laws given by smooth maps.
Most of the finite dimensional Lie groups of interest are matrix Lie groups, which can be defined as closed subgroups of the group of invertible matrices of some fixed dimension.

A Lie algebra is a vector space g over a field F with an operation

$$[\cdot,\cdot] : g \times g \to g$$

which we call a Lie bracket, such that the following axioms are satisfied
◇ It is bi linear.
◇ It is skew symmetric $[x, x] = 0$ which implies $[x, y] = [y, x]$ for all $x, y \in g$.
◇ It satisfies the Jacobi Identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

An action of a group G on a set M is given by a map

$$(g, x) \in G \times M \to g \cdot x \in M. \tag{1}$$

such that

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot (x)$$

and

$$e \cdot x = x$$

where e is the identity element of G

An action of the group $G_1 = R_3$ on R_3 by translations.

An action of the group $G_2 = O(3)$ of three dimensional orthogonal transformations of R_3. These are the rotations about the origin (possibly combined with a reflection).

Given a group action of G on M, functions on M come with an action of G by linear transformations, given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x)$$

where f is some function on M.

A group G is a set with an associative multiplication, such that the set contains an identity element, as well as the multiplicative inverse of each element.

4 BCI- quantum algebra

By a BCI- quantum algebra we mean a vector space V over a field F with an operation "$*$" satisfying the following axioms for all $u, v, w \in V$:

(BCI$_1$) $((u*v)*(x*w))*(w*v) = 0$,

(BCI$_2$) $u*0 = u$,

(BCI$_3$) $u*v = 0$ and $u*v = 0$ imply that $u = v$,

.

for all $u, v, w \in V$.

If a BCI*- quanum algebra* V *satisfies* $0*x = 0$*, for all* $x \in V$*, then we say that* V *is a* BCK*- quantum algebra.*
Every BCI- quntum algebra satisfying $0*x = 0$ *for all* $x \in V$ *is a BCK- quantum algebra.*
Any BCK*- quantum algebra* V *satisfies the following axioms for all* $x, y, z \in V$*:*
(1) $(x*y)*z = (x*z)*y$
(2) $((x*z)*(y*z))*(x*y) = 0$
(3) $x*0 = x$
(4) $x*y = 0 \Rightarrow (x*z)*(y*z) = 0, (z*y)*(z*x) = 0.$

A nonempty subset I *of* V *is called an ideal of* V *if it satisfies*
(I) $0 \in I$ *and*
(II) $x*y \in I$ *and* $y \in I$ *imply* $x \in I$.
A non-empty subset I *of* V *is said to be an* H*-ideal of* V *if it satisfies* (I) *and*
(II) $x*(y*z) \in I$ *and* $y \in I$ *imply* $x*z \in I$*, for all* $x, y, z \in X$.
A BCI- quantum algebra is said to be associative if $(x*y)*z = x*(y*z)$*, for all* $x, y, z \in V$.

Theorem 3. Let $(V;*,0)$ be a BCI-quantum algebra. Then the following hold:

(i) $x*x = 0$,

(ii) $0*(0*x) = x$,

(iii) $(x*y)*z = (x*z)*y$,

for all $x, y, z \in V$.

4.1 BCI- Lie algebra

A Lie group is as a smooth manifold, with group laws given by smooth (infinitely differentiable) maps.
By a BCI- Lie algebra we mean a Lie algebra with an operation "$*$" satisfying the following axioms for all $u, v, w \in V$:

(BCI$_1$) $((u*v)*(x*w))*(w*v) = 0$,

(BCI$_2$) $u*0 = u$,

(BCI$_3$) $u*v = 0$ and $u*v = 0$ imply that $u = v$,

.

If a BCI*- Lie algebra* V *satisfies* $0*x = 0$*, for all* $x \in V$*, then we say that* V *is a* BCK*- Lie algebra.*
Every BCI- Lie algebra satisfying $0*x = 0$ *for all* $x \in V$ *is a BCK- quantum algebra.*

Any BCK- Lie algebra V satisfies the following axioms for all $x, y, z \in V$:
(1) $(x * y) * z = (x * z) * y$
(2) $((x * z) * (y * z)) * (x * y) = 0$
(3) $x * 0 = x$
(4) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0.$

Let $V = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $ab \neq 0$ be the real invertible matrix and a, b take the following value:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

.

Then $(V, *, 0)$ it is not BCI- Lie algebra because the axiom BCI_1 does not satisfied.

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

consider the flowing table

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

.

Let $V = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$.
And will defined

$$AB = \begin{bmatrix} A_{11}B_{11} & A_{12}B_{12} \\ A_{21}B_{21} & A_{22}B_{22} \end{bmatrix}$$

Then $(V, *, 0)$ satisfies BCI_1,

$$\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also $(V, *, 0)$ satisfies BCI_2,

$$= \left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \left(\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix},$$

Also $(V, *, 0)$ satisfies BCI_3,

$$\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

0r

$$\left(\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

this means $xy = 0$ or $yx = 0$ from the table $x = y$.
Then $(V, *, 0)$ is BCI- Lie algebra

4.2 Representations and quantum mechanics

A representation (π, V) *of a group* G *is a homomorphism*

$$\pi \in G \to \pi(g) \in GL(V)$$

where $GL(V)$ *is the group of invertible linear maps* $V \to V$*, with* V *a vector space.*
Saying the map π *is a homomorphism means*

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2)$$

for all $g_1, g_2 \in G$*.*
We call a map $f : (X_1; *_1, S_1) \longrightarrow (X_2; *_2, S_2)$ between two BCK-algebras an homomorphism, if f is for any $x, y \in X_1$

$$f(x * y) = f(x) * f(y).$$

If the mapping f is onto and one- to-one then f called isomorphism.

Consider the flowing table

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

; $V = 0, 1, a, b, c$

Here $(V, *, 0)$ is BCK- Lie algebra
We defined

$$\pi(x) = e^A, A = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

for all $x \in V$.
The map π is a homomorphism

$$\pi(x_1)\pi(x_2) = \pi(x_1 x_2)$$

for all $x_1, x_2 \in V$.
Also the mapping f is onto and one- to-one, hence π is isomorphism.
The set of all homomorphisms from X_1 to X_2 denoted by $Hom(X, X_1)$.

4.3 Unitary group representations

A representation (π, V) on a complex vector space V with Hermitian inner product $\langle \cdot, \cdot \rangle$ is a unitary representation if it preserves the inner product, i.e.,

$$\langle \pi(g)v_1 \cdot \pi(g)v_1 \rangle = \langle v_1, v_2 \rangle$$

for all $g \in G$ and $v_1, v_2 \in V$.
Consider the group in the flowing table

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

; $V = 0, 1, a, b, c,$

and consider a unitary representation of V on the space of states H. The corresponding self-adjoint operator is the Hamiltonian operator H (divided by h) and the representation is given by

$$t \in V \to \pi(t) = e^{\frac{-i}{h}Ht}$$

This is a group homomorphism from the BCI- algebra $(V, *, 0)$ to a group of unitary operators.

Given representations π_1 and π_2 of dimensions n_1 and n_2, there is a representation of dimension $n_1 + n_2$ called the direct sum of the two representations, denoted by $\pi_1 \otimes \pi_2$. This representation is given by the homomorphism

$$\pi_1 \otimes \pi_2 P g \in G \to \begin{bmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{bmatrix}$$

Any unitary representation π can be written as a direct sum

$$\pi = \pi_1 \otimes \pi_2 \otimes ... \otimes \pi_m$$

where the π_j are irreducible.

The elements of the group U(1) are points on the unit circle, which can be labeled by a unit complex number $e^{i\theta}$, or an angle $\theta \in R$ with θ and $\theta + 2\pi N$ l abeling the same group element for $N \in Z$. Multiplication of group elements is complex multiplication, which by the properties of the exponential satisfies

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

so in terms of angles the group law is addition (mod 2π).

5 Discussion

We have introduced the quantum mechanics. The basic structure of quantum mechanics is quite different, the state of a system is given by a point in a space",It is can be thought of equivalently as the space of solutions of an equation of motion, or as the space of coordinates and momenta.The Group representations has been , the definition of Lie algebra is given. The Representations and quantum mechanics have been discussed.
a definition of BCI/BCK-algebra are given.A new generalization of BCI/BCK-algebra has been introduced.
The BCI- quantum algebra and BCk- quantum algebra have been defined. The BCI- Lie algebra and BCK- Lie algebra are introduced. Various examples have been given.

Acknowledgments

We thank the Editor and the referee for their comments.

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