March, the 4th, 2021

Numerics for Physics-Based PDEs with Boundary Control
The Partitioned Finite Element Method for PHs

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Outline

Outline

They have been used for simulating and controlling a variety of applications[1]:

- flexible thin beams;
- acoustic waves;
- stirred tank reactors;
- plasma in tokamak;
- shallow water equations;
- fluid-structure interactions.

Distributed port-Hamiltonian systems

$\partial_t \boldsymbol{\alpha} = \mathcal{J}\, \boldsymbol{e},$	$\boldsymbol{\alpha}$	Energy variables,	$\mathcal{J}$	Skew-symmetric operator,
$\boldsymbol{e} := \delta_{\boldsymbol{\alpha}} H,$	$\boldsymbol{e}$	Co-energy variables,		
$\boldsymbol{u}_\partial = \mathcal{B}_\partial\, \boldsymbol{e},$	$\boldsymbol{u}_\partial$	Control input,	$\mathcal{B}_\partial$	Input operator,
$\boldsymbol{y}_\partial = \mathcal{C}_\partial\, \boldsymbol{e},$	$\boldsymbol{y}_\partial$	Control output.	$\mathcal{C}_\partial$	Output operator.

[1] R. Rashad et al. "Twenty years of distributed port-Hamiltonian systems: a literature review". In: *IMA Journal of Mathematical Control and Information* (July 2020).

Discretization of port-Hamiltonian systems:

- Mixed finite elements for differential forms[2,3];
- Spectral methods[4];
- Finite differences[5].

This contribution

Mixed finite element for hyperbolic PDEs in port-Hamiltonian form under uniform or mixed boundary conditions.

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[2] G. Golo et al. "Hamiltonian discretization of boundary control systems". In: *Automatica* 40.5 (2004), pp. 757–771.

[3] P. Kotyczka, B. Maschke, and L. Lefèvre. "Weak form of Stokes-Dirac structures and geometric discretization of port-Hamiltonian systems". In: *Journal of Computational Physics* 361 (2018), pp. 442 –476.

[4] R. Moulla, L. Lefevre, and B. Maschke. "Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws". In: *Journal of computational Physics* 231.4 (2012), pp. 1272–1292.

[5] V. Trenchant et al. "Finite differences on staggered grids preserving the port-Hamiltonian structure with application to an acoustic duct". In: *Journal of Computational Physics* 373 (June 2018).

Outline

Structure preserving discretization

Infinite-dimensional pH system

PDE with boundary control:

$$\frac{\partial \boldsymbol{\alpha}}{\partial t}(\boldsymbol{x}, t) = \mathcal{J}\delta_{\boldsymbol{\alpha}} H.$$

Boundary conditions:

$$\boldsymbol{u}_\partial = \mathcal{B}_\partial \delta_{\boldsymbol{\alpha}} H, \qquad \boldsymbol{y}_\partial = \mathcal{C}_\partial \delta_{\boldsymbol{\alpha}} H.$$

Power balance (Stokes Theorem):

$$\dot{H} = \int_{\partial\Omega} \boldsymbol{u}_\partial \cdot \boldsymbol{y}_\partial \, \mathrm{d}S.$$

Structure-preserving discretization

Resulting ODE:

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_d &= \mathbf{J}\, \nabla H_d + \mathbf{B}_\partial \mathbf{u}_\partial, \\ \mathbf{y}_\partial &= \mathbf{B}_\partial^\top \nabla H_d. \end{aligned}$$

Discretized Hamiltonian:

$$H_d := H(\boldsymbol{\alpha} \equiv \boldsymbol{\alpha}_d).$$

Power balance:

$$\dot{H} = \mathbf{u}_\partial^\top \mathbf{y}_\partial.$$

Underlying hypotheses of the method

Assumption (Partitioned structure of the pH system)

The pH system has the partitioned form

$$\partial_t \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} := \begin{pmatrix} \delta_{\boldsymbol{\alpha}_1} H \\ \delta_{\boldsymbol{\alpha}_2} H \end{pmatrix},$$

$$\begin{aligned} \boldsymbol{\alpha}_1 &\in L^2(\Omega, \mathbb{A}), \\ \boldsymbol{\alpha}_2 &\in L^2(\Omega, \mathbb{B}), \\ \boldsymbol{e}_1 &\in H^{\mathcal{L}} := \left\{ \boldsymbol{u}_1 \in L^2(\Omega, \mathbb{A}) |\ \mathcal{L}\boldsymbol{u}_1 \in L^2(\Omega, \mathbb{B}) \right\}, \\ \boldsymbol{e}_2 &\in H^{\mathcal{L}^*} := \left\{ \boldsymbol{u}_2 \in L^2(\Omega, \mathbb{B}) |\ \mathcal{L}^*\boldsymbol{u}_2 \in L^2(\Omega, \mathbb{A}) \right\}. \end{aligned}$$

The sets $\mathbb{A}, \mathbb{B}$ are Cartesian product of either scalar, vectorial or tensorial quantities.

Wave-like equations (e.g. linear elastic models) possess this structure[6].

[6] P. Joly. “Variational Methods for Time-Dependent Wave Propagation Problems”. In: *Topics in Computational Wave Propagation: Direct and Inverse Problems*. Ed. by M. Ainsworth et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003. Chap. 6, pp. 201–264.

Underlying hypotheses of the method

Assumption (Abstract integration by parts formula)

Assume that there exists two boundary operators $\mathcal{N}_{\partial,1}$, $\mathcal{N}_{\partial,2}$ such that a general integration by parts formula holds $\forall \boldsymbol{e}_1 \in H^{\mathcal{L}}$ and $\forall \boldsymbol{e}_2 \in H^{\mathcal{L}^*}$

$$\langle \boldsymbol{e}_2, \, \mathcal{L}\, \boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{B})} - \langle \mathcal{L}^* \, \boldsymbol{e}_2, \, \boldsymbol{e}_1 \rangle_{L^2(\Omega,\mathbb{A})} = \langle \mathcal{N}_{\partial,1} \boldsymbol{e}_1, \, \mathcal{N}_{\partial,2} \boldsymbol{e}_2 \rangle_{\partial\Omega} \, .$$

where $\langle \cdot, \, \cdot \rangle_{\partial\Omega}$ denotes an appropriate duality pairing.

Assumption (Uniform boundary condition)

The boundary operators $\mathcal{B}_\partial$, $\mathcal{C}_\partial$ are then assumed to verify, in an exclusive manner, either

$$\mathcal{B}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}, \qquad \mathcal{C}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix},$$

or

$$\mathcal{B}_\partial = \begin{bmatrix} \mathcal{N}_{\partial,1} & 0 \end{bmatrix}, \qquad \mathcal{C}_\partial = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2} \end{bmatrix}.$$

The partitioned finite element method

This discretization procedure represents the application of mixed finite elements to port-Hamiltonian systems:

1.
2.
3.

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2.
3.

The partitioned finite element method

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1. The system is written in weak form;
2. An integration by parts is applied to highlight the appropriate boundary control;
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The partitioned finite element method

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1. The system is written in weak form;
2. An integration by parts is applied to highlight the appropriate boundary control;
3. A Galerkin method is employed to obtain a finite-dimensional system. For the approximation basis the Finite Element Method FEM (large sparse matrices) is here employed but Spectral Methods SM (small full matrices) can be used as well.

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2. An integration by parts is applied to highlight the appropriate boundary control;
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The discretized system

Consider the causality

$$\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1, \qquad \boldsymbol{y}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2.$$

By integrating by parts $\mathcal{L}$ the appropriate causality is obtained for the discretized system.

Finite dimensional system for $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\mathcal{L}^*} \\ -\mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial,$$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{pmatrix},$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

The discretized system

Consider the causality

$$\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2, \qquad \boldsymbol{y}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1.$$

By integrating by parts $-\mathcal{L}^*$ the appropriate causality is obtained for the discretized system.

Finite dimensional system for $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\boldsymbol{\alpha}}_{d,1} \\ \dot{\boldsymbol{\alpha}}_{d,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_\mathcal{L}^\top \\ \mathbf{D}_\mathcal{L} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,$$

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{bmatrix} \partial_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d) \\ \partial_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d) \end{bmatrix},$$

$$\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.$$

Discrete power balance

The power balance

$$\dot{H}_d = \partial^\top_{\boldsymbol{\alpha}_{d,1}} H_d(\boldsymbol{\alpha}_d)\dot{\boldsymbol{\alpha}}_{d,1} + \partial^\top_{\boldsymbol{\alpha}_{d,2}} H_d(\boldsymbol{\alpha}_d)\dot{\boldsymbol{\alpha}}_{d,2}$$

mimics the continuous one.

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1}\boldsymbol{e}_1$

$$\begin{aligned}\dot{H}_d &= \mathbf{e}_1^\top \mathbf{D}_{-\mathcal{L}^*}\mathbf{e}_2 - \mathbf{e}_2^\top \mathbf{D}_{-\mathcal{L}^*}^\top \mathbf{e}_1 + \mathbf{e}_2^\top \mathbf{B}_2 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial\end{aligned}$$

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2}\boldsymbol{e}_2$

$$\begin{aligned}\dot{H}_d &= -\mathbf{e}_1^\top \mathbf{D}_{\mathcal{L}}^\top \mathbf{e}_2 + \mathbf{e}_2^\top \mathbf{D}_{\mathcal{L}} \mathbf{e}_1 + \mathbf{e}_1^\top \mathbf{B}_1 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial.\end{aligned}$$

Outline

The linear case

Assumption (Quadratic separable Hamiltonian)

The Hamiltonian is assumed to be a positive quadratic separable functional in $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$

$$H = \frac{1}{2}\langle \boldsymbol{\alpha}_1, \, \mathcal{Q}_1 \boldsymbol{\alpha}_1 \rangle_{L^2(\Omega, \mathbb{A})} + \frac{1}{2}\langle \boldsymbol{\alpha}_2, \, \mathcal{Q}_2 \boldsymbol{\alpha}_2 \rangle_{L^2(\Omega, \mathbb{B})},$$

where $\mathcal{Q}_1, \mathcal{Q}_2$ *are positive symmetric bounded operators*

$$m_1 \boldsymbol{I}_{\mathbb{A}} \leq \mathcal{Q}_1 \leq M_1 \boldsymbol{I}_{\mathbb{A}}, \qquad m_2 \boldsymbol{I}_{\mathbb{B}} \leq \mathcal{Q}_2 \leq M_2 \boldsymbol{I}_{\mathbb{B}}, \qquad m_1 > 0, \; m_2 > 0, \; M_1 > 0, \; M_2 > 0.$$

PH linear system

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}, \qquad \begin{aligned} \boldsymbol{e}_1 &\in H^{\mathcal{L}}, \\ \boldsymbol{e}_2 &\in H^{-\mathcal{L}^*}, \end{aligned}$$

where $\mathcal{M}_1 := \mathcal{Q}_1^{-1}$, $\mathcal{M}_2 := \mathcal{Q}_2^{-1}$. **Constitutive laws** have been included in the dynamics.

The linear discretized system

Finite dimensional system for $u_\partial = \mathcal{N}_{\partial,1} e_1, \; y_\partial = \mathcal{N}_{\partial,2} e_2$

$$
\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\mathcal{L}^*} \\ -\mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_\partial,
$$

$$
\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{0} & \mathbf{B}_2^\top \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.
$$

Finite dimensional system for $u_\partial = \mathcal{N}_{\partial,2} e_2, \; y_\partial = \mathcal{N}_{\partial,1} e_1$

$$
\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathcal{M}_2} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\mathcal{L}}^\top \\ \mathbf{D}_{\mathcal{L}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} \mathbf{u}_\partial,
$$

$$
\mathbf{M}_\partial \mathbf{y}_\partial = \begin{bmatrix} \mathbf{B}_1^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}.
$$

Power balance

The power balance

$$\dot{H}_d = \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \dot{\mathbf{e}}_1 + \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \dot{\mathbf{e}}_2$$

mimics the continuous one.

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,1} \boldsymbol{e}_1$

$$\begin{aligned}\dot{H}_d &= \mathbf{e}_1^\top \mathbf{D}_{-\mathcal{L}^*} \mathbf{e}_2 - \mathbf{e}_2^\top \mathbf{D}_{-\mathcal{L}^*}^\top \mathbf{e}_1 + \mathbf{e}_2^\top \mathbf{B}_2 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial\end{aligned}$$

Causality $\boldsymbol{u}_\partial = \mathcal{N}_{\partial,2} \boldsymbol{e}_2$

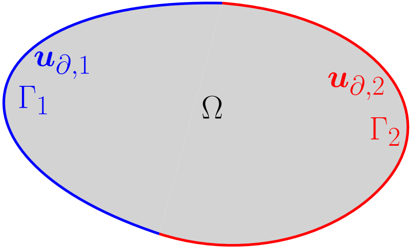
$$\begin{aligned}\dot{H}_d &= -\mathbf{e}_1^\top \mathbf{D}_{\mathcal{L}}^\top \mathbf{e}_2 + \mathbf{e}_2^\top \mathbf{D}_{\mathcal{L}} \mathbf{e}_1 + \mathbf{e}_1^\top \mathbf{B}_1 \mathbf{u}_\partial, \\ &= \mathbf{y}_\partial^\top \mathbf{M}_\partial \mathbf{u}_\partial.\end{aligned}$$

Outline

Mixed boundary conditions (linear system)

Consider now the following boundary-controlled linear pH system in co-energy form

$$\begin{bmatrix} \mathcal{M}_1 & 0 \\ 0 & \mathcal{M}_2 \end{bmatrix} \partial_t \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\mathcal{L}^* \\ \mathcal{L} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix},$$
$$\begin{pmatrix} \boldsymbol{u}_{\partial,1} \\ \boldsymbol{u}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathcal{N}_{\partial,1}^{\Gamma_1} & 0 \\ 0 & \mathcal{N}_{\partial,2}^{\Gamma_2} \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix},$$
$$\begin{pmatrix} \boldsymbol{y}_{\partial,1} \\ \boldsymbol{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} 0 & \mathcal{N}_{\partial,2}^{\Gamma_1} \\ \mathcal{N}_{\partial,1}^{\Gamma_2} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}.$$



The operator $\mathcal{N}_{\partial,*}^{\Gamma_\circ}$ with $*, \circ \in \{1,2\}$ represents the restriction of operator $\mathcal{N}_{\partial,*}$ over the subset $\Gamma_\circ \subset \partial\Omega$.

Lagrange multiplier method

A Lagrange multiplier can be introduced to include the input that does not explicitly appear in the weak formulation, i.e. to enforce the essential boundary condition.

Integration by parts of $-\mathcal{L}^*$ ($\boldsymbol{\lambda}_{\partial,1} = \boldsymbol{y}_{\partial,1}$)

$$
\mathrm{Diag}\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,1} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\mathcal{L}}^\top & \mathbf{B}_{1,\Gamma_1} \\ \mathbf{D}_{\mathcal{L}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}_{1,\Gamma_1}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_{1,\Gamma_2} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{\partial,1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,1} \\ \mathbf{B}_{1,\Gamma_2}^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,1} \end{pmatrix}.
$$

A pH differential-algebraic system is obtained in this case (pHDAE).

Lagrange multiplier method

A Lagrange multiplier can be introduced to include the input that does not explicitly appear in the weak formulation, i.e. to enforce the essential boundary condition.

Integration by parts of $\mathcal{L}$ ($\boldsymbol{\lambda}_{\partial,2} = \boldsymbol{y}_{\partial,2}$)

$$
\mathrm{Diag}\begin{bmatrix} \mathbf{M}_{\mathcal{M}_1} \\ \mathbf{M}_{\mathcal{M}_2} \\ \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \\ \dot{\boldsymbol{\lambda}}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{D}_{-\mathcal{L}^*} & \mathbf{0} \\ -\mathbf{D}_{-\mathcal{L}^*}^\top & \mathbf{0} & \mathbf{B}_{2,\Gamma_2} \\ \mathbf{0} & -\mathbf{B}_{2,\Gamma_2}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{2,\Gamma_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\partial,1} \\ \mathbf{u}_{\partial,2} \end{bmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\partial,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{y}_{\partial,1} \\ \mathbf{y}_{\partial,2} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{B}_{2,\Gamma_1}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{\partial,2} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \boldsymbol{\lambda}_{\partial,2} \end{pmatrix}.
$$

A pH differential-algebraic system is obtained in this case (pHDAE).

Power balance

The energy balance

$$\dot{H}_d = \mathbf{e}_1^\top \mathbf{M}_{\mathcal{M}_1} \dot{\mathbf{e}}_1 + \mathbf{e}_2^\top \mathbf{M}_{\mathcal{M}_2} \dot{\mathbf{e}}_2$$

mimics the continuous counterpart.

Integration by parts of $-\mathcal{L}^*$ ($\boldsymbol{\lambda}_{\partial,1} = \boldsymbol{u}_{\partial,1}$)

$$\begin{aligned}\dot{H}_d &= -\mathbf{e}_1^\top \mathbf{D}_{\mathcal{L}}^\top \mathbf{e}_2 + \mathbf{e}_2^\top \mathbf{D}_{\mathcal{L}} \mathbf{e}_1 + \mathbf{e}_1^\top (\mathbf{B}_{1,\Gamma_1} \boldsymbol{\lambda}_{\partial,1} + \mathbf{B}_{1,\Gamma_2} \mathbf{u}_{\partial,2}),\\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}.\end{aligned}$$

Integration by parts of $\mathcal{L}$ ($\boldsymbol{\lambda}_{\partial,2} = \boldsymbol{u}_{\partial,2}$)

$$\begin{aligned}\dot{H}_d &= \mathbf{e}_1^\top \mathbf{D}_{-\mathcal{L}^*}^\top \mathbf{e}_2 - \mathbf{e}_2^\top \mathbf{D}_{-\mathcal{L}^*} \mathbf{e}_1 + \mathbf{e}_2^\top (\mathbf{B}_{2,\Gamma_2} \boldsymbol{\lambda}_{\partial,2} + \mathbf{B}_{2,\Gamma_1} \mathbf{u}_{\partial,1}),\\ &= \mathbf{y}_{\partial,1}^\top \mathbf{M}_{\partial,1} \mathbf{u}_{\partial,1} + \mathbf{y}_{\partial,2}^\top \mathbf{M}_{\partial,2} \mathbf{u}_{\partial,2}.\end{aligned}$$

Outline

Irrotational shallow water equations

The Hamiltonian is a non-quadratic and non-separable functional

$$H(\alpha_h, \boldsymbol{\alpha}_v) = \frac{1}{2}\int_\Omega \left\{\frac{1}{\rho}\alpha_h \left\|\boldsymbol{\alpha}_v\right\|^2 + \rho g \alpha_h^2\right\} \, \mathrm{d}\Omega.$$

Variables:

- α_h the fluid height;
- $\boldsymbol{\alpha}_v$ the linear momentum;

Parameters:

- ρ density;
- g gravity acceleration

Dynamics:

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha_h \\ \boldsymbol{\alpha}_v\end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \mathbf{0}\end{bmatrix}\begin{pmatrix} e_h \\ \boldsymbol{e}_v\end{pmatrix}, \qquad (x, y) \in \Omega = \{x^2 + y^2 \leq R\},$$

$$\begin{pmatrix} e_h \\ \boldsymbol{e}_v\end{pmatrix} = \begin{pmatrix}\delta_{\alpha_h} H \\ \delta_{\boldsymbol{\alpha}_v} H\end{pmatrix} = \begin{pmatrix}\frac{1}{2\rho}\left\|\boldsymbol{\alpha}_v\right\|^2 + \rho g \alpha_h \\ \frac{1}{\rho}\alpha_h \boldsymbol{\alpha}_v\end{pmatrix},$$

Proportional control law

Consider a uniform Neumann bc

$$u_\partial = -\boldsymbol{e}_v \cdot \boldsymbol{n}|_{\partial\Omega}.$$

Conjugated output

$$y_\partial = e_h|_{\partial\Omega}.$$

Proportional control: La Salle argument

Proportional control for stabilization around a given fluid height h^{des}

$$u_\partial = -k(y_\partial - y_\partial^{\text{des}}), \qquad y_\partial^{\text{des}} = \rho g h^{\text{des}}, \quad k > 0.$$

The control law ensures that the Lyapunov functional

$$V = \frac{1}{2}\int_\Omega \left\{ \frac{1}{2}\rho g(\alpha_h - h^{\text{des}})^2 + \frac{1}{2\rho}\alpha_h \left\|\boldsymbol{\alpha}_v\right\|^2 \right\} d\Omega \geq 0,$$

has negative semi definite time derivative

$$\dot{V} = -k\int_{\partial\Omega} \left(y_\partial - y_\partial^{\text{des}}\right)^2 \, \mathrm{d}\Gamma \leq 0.$$

Discretization strategy

- The div operator is integrated by parts to highlight the appropriate the Neumann boundary control.
- FEniCS is used to generate the matrices.

Parameters	
ρ	$1000\ [\mathrm{kg}\cdot\mathrm{m}^3]$
g	$10\ [\mathrm{m/s}^2]$
R	$1\ [\mathrm{m}]$
h^{des}	$1\ [\mathrm{m}]$

Simulation Settings	
Integrator	Runge-Kutta 45
$\mathrm{N}^\circ_{\text{dof}}$	3973
FE spaces	$(\alpha_h \approx \mathrm{CG}_1) \times (\boldsymbol{\alpha}_v \approx \mathrm{DG}_0) \times (u_\partial \approx \mathrm{DG}_0)$
t_{end}	$3\ [\mathrm{s}]$

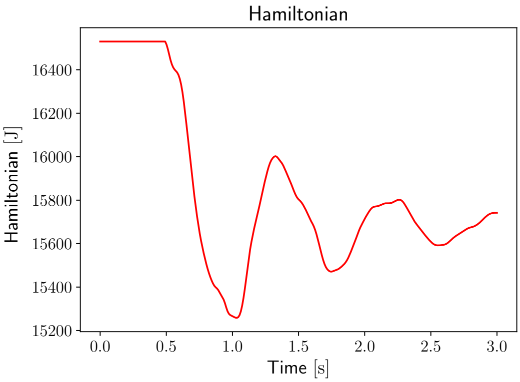
Control parameter $$k = \begin{cases} 0, & \forall t < 0.5\ [\mathrm{s}], \\ 10^{-3}, & \forall t \geq 0.5\ [\mathrm{s}]. \end{cases}$$

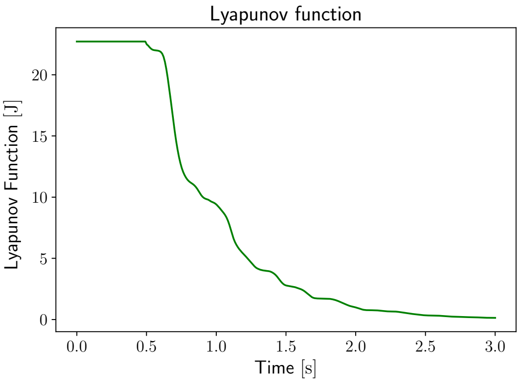
Results irrotational SWE

Play/Pause

Control parameter $$k = \begin{cases} 0, & \forall t < 0.5\,[\mathrm{s}], \\ 10^{-3}, & \forall t \geq 0.5\,[\mathrm{s}]. \end{cases}$$

Results irrotational SWE





Outline

The Hamiltonian is a quadratic functional (linear case), hence a co-energy formulation is used

$$H(e_w, \boldsymbol{E}_\kappa) = \frac{1}{2}\int_\Omega \left\{\rho h e_w^2 + \boldsymbol{\mathcal{D}}_b^{-1}(\boldsymbol{E}_\kappa) : \boldsymbol{E}_\kappa\right\} \, \mathrm{d}\Omega, \qquad \text{where} \qquad \boldsymbol{A} : \boldsymbol{B} = \sum_{ij} A_{ij} B_{ij}.$$

Variables:

- e_w the vertical velocity;
- $\boldsymbol{E}_\kappa$ the bending stress tensor;

Parameters:

- ρ density, h plate thickness;
- $\boldsymbol{\mathcal{D}}_b^{-1}$ the bending compliance tensor

$$\begin{bmatrix} \rho h & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{D}}_b^{-1} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\operatorname{Div} \\ \operatorname{Hess} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_w \\ \boldsymbol{E}_\kappa \end{pmatrix} \qquad (x, y) \in \Omega = [0, 1] \times [0, 1],$$

Consider mixed Dirichlet homogeneous conditions and Neumann boundary control

$$\begin{aligned}\partial_t e_w|_{\Gamma_D} &= 0, \\ \partial_x e_w|_{\Gamma_D} &= 0,\end{aligned} \qquad \Gamma_D = \{x = 0\}, \qquad \begin{aligned} u_{\partial,q} &= \widetilde{q}_n|_{\Gamma_N}, \\ u_{\partial,m} &= M_{nn}|_{\Gamma_N}.\end{aligned} \qquad \Gamma_N = \{y = 0 \cup x = 1 \cup y = 1\}.$$

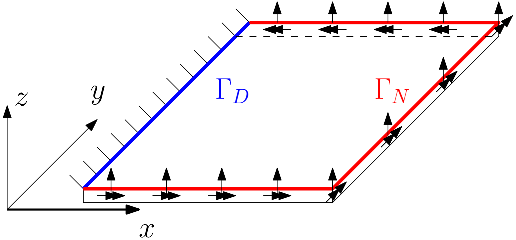
where M_{nn} is the flexural moment and $\widetilde{q}_n$ is the effective shear force.

The corresponding boundary outputs read

$$\begin{aligned} y_{\partial,q} &= e_w|_{\Gamma_N}, \\ y_{\partial,m} &= \partial_{\boldsymbol{n}} e_w|_{\Gamma_N}.\end{aligned}$$

The following control law stabilizes the system[6]

$$\begin{aligned} u_{\partial,q} &= -k y_{\partial,q}, \\ u_{\partial,m} &= -k y_{\partial,m},\end{aligned} \qquad k > 0.$$



[6] J.E. Lagnese. *Boundary Stabilization of Thin Plates*. Society for Industrial and Applied Mathematics, 1989.

Discretization strategy

- The $\operatorname{div}\operatorname{Div}$ operator is integrated by parts twice to enforce weekly the Neumann bc.
- The FIREDRAKE library is used to generate the matrices.
- The Dirichlet condition is imposed weakly through a Lagrange multiplier (strong imposition of boundary conditions for H^2 conforming elements is not trivial[7]).

Plate Parameters	
E	70 [GPa]
ρ	2700 [kg $\cdot$ m^3]
ν	0.35
h/L	0.05
$L_x = L_y$	1 [m]

Simulation Settings	
Integrator	Störmer-Verlet
Δt	1 [μs]
$\mathrm{N}^{\circ}_{\mathrm{dof}}$	2574
FE spaces	$(e_w \approx \text{Argyris}) \times (\boldsymbol{E}_\kappa \approx \text{DG}_3) \times (\boldsymbol{\lambda} \approx \text{CG}_2)$
t_{end}	5 [s]

Control parameter $$k = \begin{cases} 0, & \forall t < 1 \,[\mathrm{s}], \\ 10, & \forall t \geq 1 \,[\mathrm{s}]. \end{cases}$$

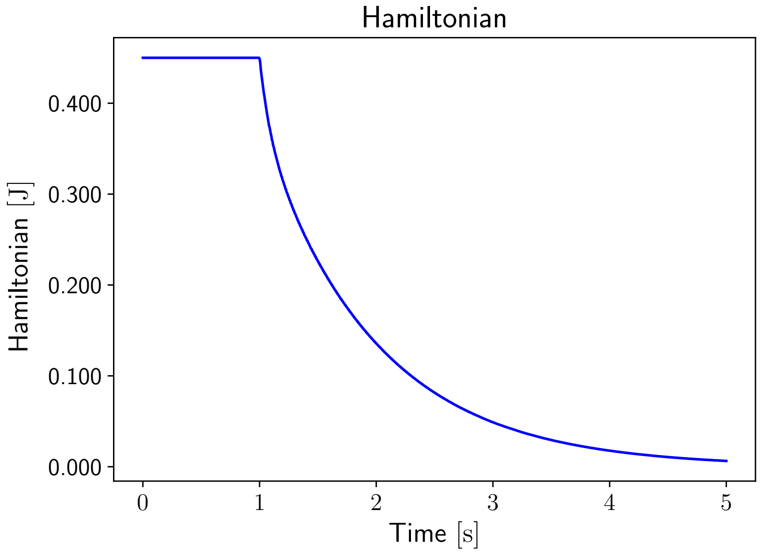
[7] R.C. Kirby and L. Mitchell. "Code Generation for Generally Mapped Finite Elements". In: *ACM Trans. Math. Softw.* 45.4 (Dec. 2019).

Results cantilever Kirchoff plate

$\boxed{\text{Play/Pause}}$

Control parameter $\quad k = \begin{cases} 0, & \forall t < 1 \, [\mathrm{s}], \\ 10, & {\color{red}\forall t \geq 1} \, [\mathrm{s}]. \end{cases}$

Results cantilever Kirchoff plate



Outline

Open problems and Outlook

Open problem

Possible developments:

[8] L. Chen and X. Huang. "Finite elements for divdiv-conforming symmetric tensors". In: *arXiv preprint arXiv:2005.01271* (2020).

[9] H. Egger et al. "On Structure-Preserving Model Reduction for Damped Wave Propagation in Transport Networks". In: *SIAM Journal on Scientific Computing* 40.1 (2018), A331–A365.

[10] J. Toledo et al. "Observer-based boundary control of distributed port-Hamiltonian systems". In: *Automatica* 120 (2020).

[11] Y. Wu et al. "Reduced Order LQG Control Design for Infinite Dimensional Port Hamiltonian Systems". In: *IEEE Transactions on Automatic Control* (2020).

Open problems and Outlook

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- Finite element space for the Lagrange multiplier, satisfying the inf-sup condition;

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- Discretization: efficient finite element for the Kirchhoff plate based on div-div conforming elements[8];

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- Observer based boundary control[10] and reduced LQG design for distributed control[11].

[8] L. Chen and X. Huang. “Finite elements for divdiv-conforming symmetric tensors”. In: *arXiv preprint arXiv:2005.01271* (2020).

[9] H. Egger et al. “On Structure-Preserving Model Reduction for Damped Wave Propagation in Transport Networks”. In: *SIAM Journal on Scientific Computing* 40.1 (2018), A331–A365.

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[11] Y. Wu et al. “Reduced Order LQG Control Design for Infinite Dimensional Port Hamiltonian Systems”. In: *IEEE Transactions on Automatic Control* (2020).

Additional informations

Jupiter Notebooks for the wave and heat equation and the Mindlin plate model are available:

A. Brugnoli et al. *Supplementary material for "Numerical approximation of port-Hamiltonian systems for hyperbolic or parabolic PDEs with boundary control"*. https://doi.org/10.5281/zenodo.3938600. Dataset on Zenodo. 2020.

Flexible multibody dynamics for pHs based on the proposed discretization:

A. Brugnoli et al. “Port-Hamiltonian flexible multibody dynamics”. In: *Multibody System Dynamics* (2020). https://doi.org/10.1007/s11044-020-09758-6.

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