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Asymptotic behaviour of a system modelling rigid structures floating in a viscous fluid ⋆

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Abstract: The PDE system introduced in Maity et al. (2019) describes the interaction of surface water waves with a floating solid, and takes into account the viscosity μ of the fluid. In this work, we study the Cummins type integro-differential equation for unbounded domains, that arises when the system is linearized around equilibrium conditions. A proof of the input-output stability of the system is given, thanks to a diffusive representation of the generalized fractional operator $\sqrt{1+\mu\, s}$. Moreover, relying on Matignon (1996) stability result for fractional systems, explicit solutions are established both in the frequency and the time domains, leading to an explicit knowledge of the decay rate of the solution. Finally, numerical evidence is provided of the transition between different decay rates as a function of the viscosity μ.

Keywords: Fluid-Structure Interaction, Fractional Differential Equations, Asymptotic behaviour.

1. INTRODUCTION

In this work we consider the return to the equilibrium problem of a model describing the vertical motion of a solid floating at the free surface of a viscous fluid with finite depth and flat bottom. This problem concerns a particular configuration of the system coupling the free surface motion of a fluid and a floating strucuture. More precisely, it consists of releasing a partially submerged solid body in a fluid initially at rest and letting it evolve towards its equilibrium position. The return to equilibrium problem (also called *decay test*) consists in describing the large time behavior of the oscillation amplitude of the solid. The interest of this problem is that it can easily be used experimentally and is useful to determine important characteristics of floating objects, from an engineering point of view.

For inviscid fluids filling an unbounded domain, the motion of the solid is often described in the literature by a linear integro-differential equation, known as the Cummins equation, which has been obtained empirically by Cummins (1962). In his paper the Cummins equation for vertical displacements of a floating structure reads as

$$(M + a_\infty)\, \ddot{h}_S(t) = c\, h_S(t) + K * \dot{h}_S(t), \qquad (1)$$

where $h_S(t)$ denotes the displacement of the structure from the equilibrium position, M denotes the mass of the structure, a_∞ denotes the added mass at infinite frequency, c is the hydrostatic stiffness, and $K(t)$ denotes the radiation force impulse response function. An equation with similar characteristics but including non-linear effects has been developed in Lannes (2017).

As far as we know, the only work using a viscous model for the fluid is Maity et al. (2019), where an equation of Cummins type is obtained, even in cases in which the fluid could be bounded by vertical walls; here, we are interested in describing the model of Cummins type in an unbounded viscous domain. More precisely, we study the correct version of this model for vertical displacements of a floating structure, which now reads:

$$\left(1 + \frac{(b-a)^3}{12}\right) \ddot{h}_S(t) = -\frac{(b-a)^2}{2}\, F * \dot{h}_S(t) - \mu(b-a)\, \dot{h}_S(t) - (b-a)\, h_S(t), \qquad (2)$$

where μ is the viscosity coefficient of the fluid, $(b-a)$ is the width of the interval $\mathcal{I} = [a,b]$ obtained by projecting the floating object (supposed symmetric around the axis $x = \frac{1}{2}(a+b)$) on the flat horizontal bottom, and $\mathcal{E} = \mathbb{R} \setminus [a,b]$ denotes the viscous fluid domain. Moreover, F is the causal distribution with Laplace transform $\widehat{F}(s) = \sqrt{1+\mu\, s}$.

The novelties brought in by this work are:

- the correct form of an equation of Cummins type for an unbounded viscous domain,
- the proof of stability of the system, and its asymptotic behaviour at infinity, using a diffusive representation,

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- an explicit form of the solutions of the system in the time domain, and the large time behaviour of these, recovering another stability proof,
- numerical evidence of the transition between the differents decay rates of the system as a function of the viscosity coefficient μ of the fluid.

This work is a companion paper to Vergara-Hermosilla et al. (2020), where this system with force as input, and distance from the solid to the sea bottom as output was first recast as a linear well-posed system and second proved to be input-to-output stable.

The outline of the paper is as follows: in § 2, the physical model and its linearization around a steady state are recalled; in § 3, an equivalent diffusive representation of the system is provided, which helps prove stability and even compute refined asymptotics in some cases; in § 4, the analytical solution of the system is provided thanks to Mittag-Leffler special functions, the asymptotic behaviour are provided in full generality, helping to recover the previous stability property; finally a conclusion is drawn and future works are investigated in § 5.

2. RECALLS ON THE LINEARIZED PHYSICAL MODEL

In this Section the floating solid is supposed, without loss of generality, to have mass $\mathcal{M} = 1$ and it is constrained to move only in the vertical direction. Given $t > 0$, we denote by $h(t,x)$ the height of the free surface of the fluid, by $q(t,x)$ the flux of viscous fluid in the direction x and by $h_S(t)$ the distance from the bottom of the rigid body to the bottom of the fluid, supposed to be horizontal, as described in Fig. 1.

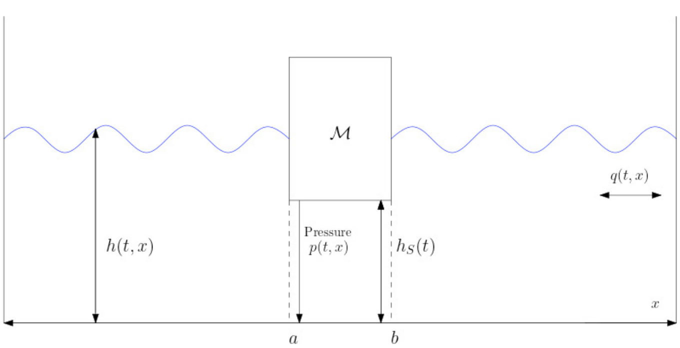


Fig. 1. Graphical sketch of the model. The function $h(t,x)$ denote the height of the free surface of the fluid, $q(t,x)$ denote the flux of viscous fluid, $h_S(t)$ is the function which describes the distance from the bottom of the rigid body to the bottom of the fluid and $p(t,x)$ is a pressure term.

We consider the model introduced in Maity et al. (2019), with the particularity that the fluid is supposed to be infinite in the horizontal direction, denoting $\mathcal{I} := [a,b]$ the projection on the fluid bottom of the solid domain and setting $\mathcal{E} := \mathbb{R} \setminus [a,b]$. Then, following Maity et al. (2019), we have

$$\overline{h} = \overline{h}_S + \frac{1}{b-a},$$

and for simplicity, we assume that

$$\overline{h} = 1, \quad g = 1, \quad \overline{p} = \frac{1}{b-a}.$$

Hence, linearizing around the trajectory $(h_S, h, q, p) = (\overline{h}_S, \overline{h}, 0, \overline{p})$ we obtain the equations

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x \in \mathcal{E}), \tag{3}$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0, \quad (x \in \mathcal{E}), \tag{4}$$

$$h(t,a^-) - \mu \frac{\partial q}{\partial x}(t,a^-) = p(t,a^+) + h_S(t) - \mu \frac{\partial q}{\partial x}(t,a^+), \tag{5}$$

$$h(t,b^+) - \mu \frac{\partial q}{\partial x}(t,b^+) = p(t,b^-) + h_S(t) - \mu \frac{\partial q}{\partial x}(t,b^-), \tag{6}$$

$$\dot{h}_S(t) + \frac{\partial q}{\partial x} = 0 \qquad (x \in \mathcal{I}), \tag{7}$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \qquad (x \in \mathcal{I}), \tag{8}$$

$$\ddot{h}_S(t) = \int_a^b p(t,x)dx \qquad (t > 0), \tag{9}$$

where p is a Lagrange multiplier, similar to a pressure term (which is obtained in the Hamiltonian modelling process),

Remark 1. In particular, for initial data satisfying

$$q_0(x) = -q_0(a+b-x), \quad h_0(x) = h_0(a+b-x) \quad (x \in \mathcal{E}), \tag{10}$$

we have

$$q(t,a) = -q(t,b), \qquad h(t,a) = h(t,b) \qquad (t \geqslant 0).$$

To this aim, if we first write the pressure term p in system (3)-(9) as $p = p_1 + p_2$, where p_1 and p_2 solve

$$\frac{\partial^2 p_1}{\partial x^2} = \ddot{h}_S, \quad p_1(t,a) = p_1(t,b) = 0, \tag{11}$$

$$\frac{\partial^2 p_2}{\partial x^2} = 0, \quad p_2(t,a) = p_a(t), \quad p_2(t,b) = p_b(t), \tag{12}$$

respectively, with

$$p_a(t) := h\left(t,a^-\right) - \mu \frac{\partial q}{\partial x}\left(t,a^-\right) - h_{sol}(t) - \mu \dot{h}_{sol}(t), \tag{13}$$

$$p_b(t) := h\left(t,b^+\right) - \mu \frac{\partial q}{\partial x}\left(t,b^+\right) - h_{sol}(t) - \mu \dot{h}_{sol}(t), \tag{14}$$

then, by solving equations (11) and (12), it follows that

$$p_1(t,x) = \ddot{h}_S \left(\frac{x^2}{2} - \frac{b+a}{2}x + \frac{ab}{2} \right), \tag{15}$$

and

$$p_2(t,x) = p_a(t) + (p_b(t) - p_a(t)) \frac{x-a}{l}, \tag{16}$$

where $l := b - a$. Substituting these values of p_1 and p_2 in (9), we obtain

$$\left(1 + \frac{l^3}{12}\right) \ddot{h}_S(t) = p_a(t) l + (p_b(t) - p_a(t)) \frac{l}{2}$$
$$= \frac{l}{2}\left(p_a(t) + p_b(t)\right).$$

Considering the values of p_a and p_b from (13) and (14) respectively, the equation above can be rewritten as

$$\left(1 + \frac{l^3}{12}\right) \ddot{h}_S(t) = -l\left(h_S(t) + \mu \dot{h}_S(t)\right)$$
$$+ \frac{l}{2}\left(h\left(t,a^-\right) - \mu \frac{\partial q}{\partial x}\left(t,a^-\right) + h\left(t,b^+\right) - \mu \frac{\partial q}{\partial x}\left(t,b^+\right)\right). \tag{17}$$

We first express $h\left(t,a^{-}\right)-\mu\frac{\partial q}{\partial x}(t,a^{-})$ in terms of h_S and $\dot{h}_S$. To this end, for $x\in\mathcal{I}$, we first note that

$$q(t,b)-q(t,a)=-l\dot{h}_S(t). \tag{18}$$

Moreover, using Remark 1 we obtain

$$h(t,a^{-})=h(t,b^{+}),\quad -q(t,a^{-})=q(t,b^{+}), \tag{19}$$

thus

$$q(t,a)=\frac{l}{2}\dot{h}_S,\quad q(t,b)=-\frac{l}{2}\dot{h}_S. \tag{20}$$

For $t\geq 0$, we set $q_a(t):=q(t,a)$ and $q_b(t):=q(t,b)$. Since (7) implies that q is a linear function of x on $\mathcal{I}$, for every $t\geq 0$ and $x\in\mathcal{I}$,

$$\dot{h}_S(t)=-\frac{q_b(t)-q_a(t)}{b-a}\qquad (t\geq 0). \tag{21}$$

From equations (21) and (3) it follows that for $x\in(-\infty,a]$ we have

$$\begin{aligned}&\frac{\partial^2 q}{\partial t^2}-\frac{\partial^2 q}{\partial x^2}-\mu\frac{\partial^3 q}{\partial t\partial x^2}=0,\\ &q(t,x)\to 0 \text{ as } x\to-\infty,\quad q(t,a)=\frac{b-a}{2}\dot{h}_S(t),\\ &q(0,x)=\frac{\partial q}{\partial t}(0,x)=0.\end{aligned} \tag{22}$$

For $f\in L^1[0,\infty]$, let $\widehat{f}$ the Laplace transform of f. Applying the Laplace transform to both sides of (22), we obtain

$$\begin{aligned}&s^2\widehat{q}-(1+s\mu)\frac{\partial^2\widehat{q}}{\partial x^2}=0,\\ &\widehat{q}(s,x)\to 0 \text{ as } x\to-\infty,\quad \widehat{q}(s,a)=\frac{b-a}{2}\widehat{\dot{h}}_S,\ \Re(s)>0.\end{aligned} \tag{23}$$

Hence we can conclude that

$$\widehat{q}(s,x)=\frac{b-a}{2}e^{\frac{-sa}{\sqrt{1+s\mu}}}e^{\frac{sx}{\sqrt{1+s\mu}}}\widehat{\dot{h}}_S(s), \tag{24}$$

and

$$\begin{aligned}\widehat{h}\left(s,a^{-}\right)-\mu\frac{\partial\widehat{q}}{\partial x}\left(s,a^{-}\right)&=-\frac{l}{2}\left(\frac{1}{s}+\mu\right)\frac{s}{\sqrt{1+s\mu}}\widehat{\dot{h}}_S(s)\\ &=-\frac{l}{2}(\sqrt{1+\mu s})\widehat{\dot{h}}_S(s).\end{aligned} \tag{25}$$

In a similar way, we obtain

$$\widehat{h}\left(s,b^{+}\right)-\mu\frac{\partial\widehat{q}}{\partial x}\left(s,b^{+}\right)=-\frac{l}{2}\left(\sqrt{1+\mu s}\right)\widehat{\dot{h}}_S(s). \tag{26}$$

By considering Remark 1 and the inverse of Laplace transform of eqs. (25) and (26), we obtain the following result:

Proposition 2. The vertical movement of a floating object, in an unbounded viscous fluid that is initially at rest, is described by the following integro-differential equation

$$\begin{aligned}\left(1+\frac{l^3}{12}\right)\ddot{h}_S(t)=&-\frac{l^2}{2}\int_0^t F(\sigma)\dot{h}_S(t-\sigma)d\sigma\\ &-l\left(h_S(t)+\mu\dot{h}_S(t)\right).\end{aligned} \tag{27}$$

with initial conditions

$$h_S(0)=h_0,\quad \dot{h}_S(0)=0,$$

and where F is the causal distribution, such that $\widehat{F}(s)=\sqrt{1+\mu s}$ in $\Re(s)>-1/\mu$.

3. DIFFUSIVE REPRESENTATION, STABILITY PROOF AND ASYMPTOTIC BEHAVIOUR

The main idea of this section is to get rid of the F term. First since its Laplace transform is not bounded in any right-half plane, it does not correspond to a causal function, but rather a causal distribution: indeed, when $\mu\to\infty$, the term $\sqrt{s}$ appears, which is related to the fractional derivative of order 1/2, see e.g. Matignon (2009) and references therein. On the contrary, $1/\sqrt{s}$ is bounded and corresponds to the fractional integration of order 1/2, this is the reason why we shall be interested rather in

$$\hat{G}(s):=\frac{\hat{F}(s)-1}{s}=\frac{\mu}{1+\sqrt{1+\mu s}}\,,\ for\ \Re(s)>-1/\mu\,. \tag{28}$$

This extra transfer function is of so-called *diffusive type*, and enjoys nice properties, see e.g. Matignon and Prieur (2005): it is a completely monotone function, i.e. $G(t):=\int_0^\infty g(\xi)\,\exp(-\xi\,t)\,\mathrm{d}\xi$ for some appropriate positive and real-valued weight function g to be computed, or equivalently $\hat{G}(s):=\int_0^\infty g(\xi)\,(s+\xi)^{-1}\,\mathrm{d}\xi$, for $\Re(s)>0$. Following e.g. Matignon (1998b), we can compute g explicitely as:

$$\begin{aligned}g(\xi):=&\lim_{\epsilon\to 0^+}\frac{1}{2i\pi}(\hat{G}(-\xi-i\epsilon)-(\hat{G}(-\xi+i\epsilon))\,, &(29)\\ =&\frac{1}{\pi}\frac{\sqrt{\mu\xi-1}}{\mu\xi}\,,\quad for\ \xi>1/\mu\,. &(30)\end{aligned}$$

This weight is indeed real valued, positive, and fulfills the well-posedness condition

$$\int_{\mu^{-1}}^{\infty}\frac{g(\xi)}{1+\xi}\,\mathrm{d}\xi<\infty \tag{31}$$

that is required for the functional setting to make sense.

3.1 Extended diffusive representation

For the G transfer function alone with input $v:=\dot{h}$ and output $y:=G*v$, a diffusive realization is of the form:

$$\begin{aligned}\partial_t\phi(t,\xi)&=-\xi\,\phi(t,\xi)+v(t),\quad \phi(0,\xi)=0 &(32)\\ y(t)&=\int_{\mu^{-1}}^{\infty}g(\xi)\,\phi(t,\xi)\,\mathrm{d}\xi\,. &(33)\end{aligned}$$

The formal proof is straightforward and relies on the fact that $\partial_t(e^{-\xi\,t}*v)=-\xi\,(e^{-\xi\,t}*v)+v(t)$.

Now for the F transfer function with input v and new output $z:=F*v$, since $\hat{F}(s)=1+s\,\hat{G}(s)$, the following extended diffusive realisation can be proposed:

$$\begin{aligned}\partial_t\varphi(t,\xi)&=-\xi\,\varphi(t,\xi)+v(t),\quad \varphi(0,\xi)=0 &(34)\\ z(t)&=\int_{\mu^{-1}}^{\infty}g(\xi)\,\partial_t\varphi(t,\xi)\,\mathrm{d}\xi+1\,v(t)\,. &(35)\end{aligned}$$

Indeed, defining as energy $\mathcal{E}_\varphi(t):=\frac{1}{2}\int_{\mu^{-1}}^{\infty}\xi g(\xi)\,|\varphi(t,\xi)|^2\,\mathrm{d}\xi$, one can easily compute the following balance:

$$\frac{d}{dt}\mathcal{E}_\varphi(t)=+v(t)\,z(t)-1\,(v(t))^2-\int_{\mu^{-1}}^{\infty}g(\xi)\,|\partial_t\varphi(t,\xi)|^2\,\mathrm{d}\xi\,. \tag{36}$$

This latter energy balance will play a key role when analyzing the stability of the coupled system. Note that the whole rigourous functional analytic setting needed to address this problem is fully detailed in Matignon and Prieur (2005), both for standard and extended diffusive realizations.

3.2 Energy balance and new stability proof

Consider the original system (27), set $\dot{h} := v$ and $z := F * \dot{h}$, it can then be viewed as a coupled system

$$\begin{cases} \left(1+\frac{l^3}{12}\right)\ddot{h}_S + z(t) + l\dot{h}_S + l\mu h_S = 0 \\ v(t) = \dot{h}_S(t) \\ \partial_t\varphi(t,\xi) = -\xi\varphi(t,\xi) + v(t);\ \varphi(0,\xi) = 0 \\ z(t) = \int_{\mu^{-1}}^{\infty} g(\xi)\partial_t\varphi(t,\xi)\,\mathrm{d}\xi + 1v(t). \end{cases} \tag{37}$$

The mechanical energy of the oscillator is

$$E(t) := \frac{1}{2}\left(1+\frac{l^3}{12}\right)(\dot{h}_S)^2(t) + \frac{1}{2}l\mu\,(h_S)^2(t)\,.$$

Its energy balance reads

$$\frac{d}{dt}E(t) = -l(\dot{h}_S)^2(t) - \dot{h}_S(t)\,(F*\dot{h}_S)(t)\,;$$

while the first term is indeed negative, the second has no definite sign; however it reads $-v(t)\,z(t)$ and compensates exactly with $+v(t)\,z(t)$ in (36).

This is the reason why we shall define a global energy functional $\mathcal{E}(t) := E(t) + \mathcal{E}_\varphi(t)$ for the augmented system with state variables (h_S,ω,φ) in the state space $\mathbb{R}\times\mathbb{R}\times\tilde{H}$, where $\tilde{H} = \left\{\varphi \in L^2_{\mathrm{loc}}(\mathbb{R}^+,\mathrm{d}g)\,, \int_0^\infty \xi|\varphi|^2\mathrm{d}g(\xi) < \infty\right\}$. Indeed, the global energy balance reads, at least formally:

$$\frac{d}{dt}\mathcal{E}(t) = -(1+l)(\dot{h}_S)^2(t) - \int_{\mu^{-1}}^{\infty} g(\xi)\,|\partial_t\varphi(t,\xi)|^2\,\mathrm{d}\xi \le 0\,.$$

This decay of the global energy is the starting point to the following asymptotic stability result.

Proposition 3. For all $(h_{S,0},\omega_0)\in\mathbb{C}^2$, the solution of the coupled system (37), with initial condition $(h_{S,0},\omega_0,0)$, satisfies

$$(h_S,\dot{h}_S,\varphi)(t) \to_{t\to\infty} 0 \quad \text{in } \mathbb{C}^2\times\tilde{H}.$$

Proof. Indeed, since the weight $g(\xi)$ is positive and satisfies the well-posedness condition (31), Theorem 3.7 in Matignon and Prieur (2005) applies directly to our problem.

3.3 Asymptotic behaviour (special case)

Thanks to the diffusive representation of $\hat{F}$, involving a branch cut on $(-\infty,-\frac{1}{\mu}]$ on $\mathbb{R}^-$, following e.g. Matignon (1998b), it is known thanks to the Watson lemma that the branchpoint at $s=-\frac{1}{\mu}$ with local behaviour $\frac{\sqrt{\mu}}{\pi}\sqrt{\xi-\frac{1}{\mu}}$ translates into $\frac{\sqrt{\mu}}{\pi}\Gamma(\frac{3}{2})\,e^{-\frac{t}{\mu}}\,t^{-3/2}$ as $t\to+\infty$ by inverse Laplace transform.

But as usual, apart from the branchcut, other singularities of the transfer function like poles s_k can appear, giving rise to $r_k\,e^{s_k\,t}$ terms in the time domain. At this stage however, we are not in a position to state whether or not $\Re(s_k)\le-\frac{1}{\mu}$, so our result is only a partial one.

Proposition 4. If all the poles s_k of the transfer function lie in the left halfplane $\Re(s) < -\frac{1}{\mu}$, then the asymptotic behaviour of the solution h_S of the system (37) reads

$$h_S(t) \sim K\,e^{-\frac{t}{\mu}}\,t^{-3/2}\,,\quad \text{as}\quad t\to+\infty\,.$$

Hence, there is a need to inspect the location of the poles more thoroughly in order to analyze the asymptotic behaviour of the solution in the general case, i.e. whatever the location of those poles.

4. ANALYTICAL SOLUTION AND ASYMPTOTIC BEHAVIOUR

For simplicity in this Subsection, we use the following notations: $l=b-a$, $A=1+\frac{l^3}{12}$, $B=\frac{l^2}{2}$ and $C=l\mu$; all are positive constants. In § 4.1, the case of the inviscid fluid $\mu=0$ is recalled, while in § 4.2, the general case of the viscous fluid $\mu>0$ is examined. Finally in § 4.3, numerical evidence is provided of the possible transition between different asymptotic regimes, as the viscosity μ increases.

4.1 Case $\mu = 0$

If we consider $\mu=0$ in (27), the model reduces to an ODE:

$$\begin{cases} A\ddot{h}_S + B\dot{h}_S + lh_S = 0, \\ h_S(0) = h_0, \quad \dot{h}_S(0) = \dot{h}_0. \end{cases} \tag{38}$$

This model has the form of a simple mechanical oscillator, free of external forces, which we shall call *free oscillation.*

Applying Laplace transform to the equation (38), and after simplifications, we get

$$\left[As^2 + Bs + l\right]\,\hat{h}_S(s) = \left[As+B\right]\,h_0 + A\,\dot{h}_0. \tag{39}$$

By calculating the inverse of the Laplace transform of the rational function appearing implicitely in (39), we obtain that the solution for the model (38) is given by

$$h_S(t) = (C_1\cos(\omega_d t) + C_2\sin(\omega_d t))\,e^{-\delta t}, \tag{40}$$

when $B^2 < 4Al$, where

$$\delta = \frac{B}{2A},\quad \omega_0 = \sqrt{\frac{l}{A}},\quad \omega_d = \sqrt{\omega_0^2-\delta^2} = \frac{\sqrt{4Al-B^2}}{2A}, \tag{41}$$

are the damping coefficient, the undamped natural angular frequency and the damped angular frequency, respectively. The constants C_1 and C_2, are given by

$$C_1 = h_0,\quad C_2 = \frac{\dot{h}_0 + h_0\delta}{\omega_d} = \frac{h_0B + 2\dot{h}_0A}{\sqrt{4Al-B^2}}. \tag{42}$$

Remark 5. If $(b-a) > \sqrt[3]{6}$, the free oscillation is *overdamped*, that is, if $\delta > \omega_0$, then ω_d is imaginary. In this situation, $B^2 > 4Al$, the general solution for the model (38) is a linear combination of two real, decaying exponential functions, with explicit form given by

$$h_S(t) = (C_1\cosh(\overline{\omega}_d t) + \overline{C}_2\sinh(\overline{\omega}_d t))\,e^{-\delta t}, \tag{43}$$

where $\overline{C}_2 = \frac{h_0B+2\dot{h}_0A}{\sqrt{|4Al-B^2|}}$ and $\overline{\omega}_d = \frac{\sqrt{|Al-B^2|}}{2A}$.

4.2 Case $\mu > 0$

Considering $\mu > 0$ and applying Laplace transform to the equation (27), setting $\overline{B} := \sqrt{\mu}B$, and $\varepsilon := \frac{1}{\mu}$, we obtain after simplifications

$$\left[As^2 + \overline{B}\, s\sqrt{s+\varepsilon} + Cs + l\right]\, \hat{h}_S(s) = \left[As + \overline{B}\sqrt{s+\varepsilon} + C\right]\, h_0 + A\,\dot{h}_0, \quad (44)$$

Remark 6. In the case $\varepsilon = 0$, which corresponds to an infinitely viscous fluid, the above equation is a Fractional Differential Equation (FDE) of order 1/2.

When $\varepsilon > 0$, this is a Generalized Fractional Differential Equation (GFDE), originally studied in Matignon (1998a); to tackle this, we proceed in 4 steps:

(1) perform a change of variables in order to work with polynomials,
(2) decompose the rational functions of interest into simple elements,
(3) apply the inverse Laplace transform, using Mittag-Leffler special functions of fractional calculus,
(4) make use of the adapted algebraic stability criterion to get the asymptotic behaviour of the solution, and conclude to stability.

Change of variables Let us denote $\sigma := \sqrt{s+\varepsilon}$, then the pseudo polynomials appearing in (44) can be equivalently transformed thanks to the algebraic relation $s = -\varepsilon + \sigma^2$.

$$\begin{aligned} n_0(s) &:= As + \overline{B}\sqrt{s+\varepsilon} + C \\ &= A\sigma^2 + \overline{B}\sigma + (C - \varepsilon\, A) := N_0(\sigma)\,, \end{aligned} \quad (45)$$

$$\begin{aligned} d(s) &:= As^2 + \overline{B}\, s\sqrt{s+\varepsilon} + Cs + l\,, \\ &= A\sigma^4 + \overline{B}\sigma^3 + (C - 2\varepsilon\, A)\,\sigma^2 - \varepsilon\,\overline{B}\,\sigma + \varepsilon^2\, A\,, \\ &:= P_T(\sigma)\,. \end{aligned} \quad (46)$$

The viscous polynomial P_T is real valued, of degree 4, and has 4 complex roots, called λ_i, which can be found analytically in Appendix A, $P_T(\sigma) = A\prod_{i=1}^4(\sigma - \lambda_i)$; alternatively, they can be computed numerically as in § 4.3 to study their parametric depence w.r.t. μ.

Remark 7. One has to be careful with this change of variables. Indeed, as is usual with multivalued complex functions, a cut has to be performed first on the branch cut $]-\infty, -\varepsilon]$, then $\forall s \in \mathbb{C}\backslash]-\infty,-\varepsilon]$, $\exists!\,\sigma \in \mathbb{C}_0^+$, defined by $\sigma := \sqrt{s+\varepsilon}$, that is with positive real part. But care must be taken that a complex number σ with negative real part has no counterpart s in the Laplace plane $\mathbb{C}\backslash]-\infty,-\varepsilon]$ given by this relation.

Decomposition into simple elements From (44), we get

$$\begin{aligned} \hat{h}_S(s) &= \frac{n_0(s)}{d(s)}\, h_0 + \frac{A}{d(s)}\,\dot{h}_0\,, \\ &= \frac{N_0(\sigma)}{P_T(\sigma)}\, h_0 + \frac{A}{P_T(\sigma)}\,\dot{h}_0\,, \\ &= \left(\sum_{i=1}^4 \frac{r_i}{\sigma - \lambda_i}\right) h_0 + \left(\sum_{i=1}^4 \frac{\tilde{r}_i}{\sigma - \lambda_i}\right) \dot{h}_0\,. \end{aligned}$$

Each r_i and $\tilde{r}_i$ are to the residues of the rational function of interest at the pole λ_i: they correspond either to the response to initial displacement h_0, or to the response to initial velocity $\dot{h}_0$. Their algebraic expression can be found in Appendix B.

Time-domain solution The key issue here is to identify $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s+\varepsilon}-\lambda}\right)$ in some right-half plane to be determined later, for $\epsilon \geq 0$ and $\lambda \in \mathbb{C}$. The easiest way to proceed is to use the shift theorem for Laplace transform, and identify the eigenfunctions of the fractional derivative operators, which are Mittag-Leffler functions.

Definition 8. Let us denote $\mathcal{E}_\alpha(\lambda, t)$ the function for which

$$\mathcal{L}[\mathcal{E}_\alpha(\lambda,.)](s) = \frac{1}{s^\alpha - \lambda}, \quad \text{for } \Re(s) > a_\lambda\,. \quad (47)$$

This special function is related to the so-called two parametric Mittag-Leffler functions,

$$\mathcal{E}_\alpha(\lambda,t) := t^{\alpha-1}\, E_{\alpha,\alpha}(z = \lambda\, t^\alpha)\,,$$

where we have used

Definition 9. The two-parametric Mittag-Leffler function is the complex-valued function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (48)$$

where $\alpha > 0, \beta \in \mathbb{C}$ and $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Euler Gamma function.

See for instance Podlubny (1998) or Matignon (2009) for many useful properties of these functions.

Thanks to the shift theorem for Laplace transforms, we are now in a position to identify the useful elementary functions,

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s+\varepsilon}-\lambda}\right) = \exp(-\varepsilon\, t)\,\mathcal{E}_{\frac{1}{2}}(\lambda, t)\,,$$

and state the following result in the time domain:

Theorem 10. The solution of the GFDE (44) is given by

$$h_S(t) = \exp(-\varepsilon\, t)\left(\sum_{i=1}^4 \Theta_i\,\mathcal{E}_{\frac{1}{2}}(\lambda_i, t)\right), \quad (49)$$

with constants $\Theta_i := r_i\, h_0 + \tilde{r}_i\,\dot{h}_0$.

Thanks to this explicit solution, we are now in a position to examine the asymptotic behaviour more in depth.

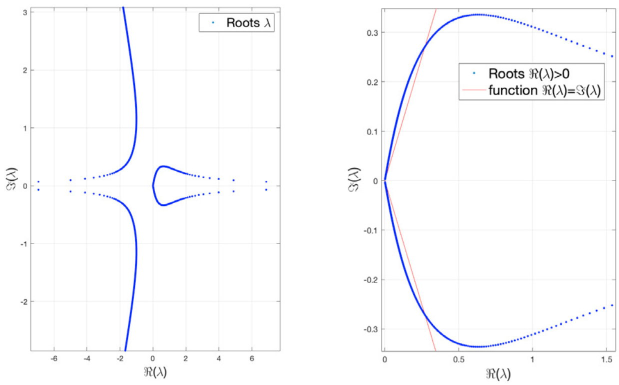
Asymptotic behaviour (general case) Indeed, let us recall the following seminal results about the long time behaviour of the Mittag-Leffler functions:

Theorem 11. (Matignon (1996)). We have the following asymptotic equivalents for $\mathcal{E}_\alpha(\lambda, t)$ as t reaches $+\infty$:

- for $|\arg(\lambda)| \leq \alpha\frac{\pi}{2}$,
$$\mathcal{E}_\alpha(\lambda,t) \sim \frac{1}{\alpha}\lambda^{\frac{1}{\alpha}-1}e^{\lambda^{\frac{1}{\alpha}}t}, \quad (50)$$
- for $|\arg(\lambda)| > \alpha\frac{\pi}{2}$,
$$\mathcal{E}_\alpha(\lambda,t) \sim \frac{\alpha}{\Gamma(1-\alpha)}\lambda^{-2}t^{-1-\alpha}, \quad (51)$$

which belongs to $L^r([1,+\infty),\mathbb{R})$, for all $r \geq 1$.

Recently, some higher order asymptotics have been provided to all sorts of Mittag-Leffler functions, see (Popov and Sedletskii, 2013, Section 1.4).



(a) (b)

Fig. 2. Evolution of the four roots λ_i in the σ-plane, as a function of μ. (a): global picture with 4 trajectories. (b): zoom in the right-half plane $\Re(\sigma) > 0$, 2 trajectories crossing the segment $Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

For our purpose, the following asymptotics are needed:

Theorem 12. (Matignon (1998b)). We have the following asymptotic equivalents for $\exp(-\varepsilon\, t)\, \mathcal{E}_{1/2}\,(\lambda, t)$ as t reaches $+\infty$:

- for $|\arg(\lambda)| \leq \frac{\pi}{4}$,

$$e^{-\varepsilon\, t}\, \mathcal{E}_{1/2}\,(\lambda, t) \sim 2\,\lambda \exp((\lambda^2 - \varepsilon)\, t), \tag{52}$$

- for $|\arg(\lambda)| > \frac{\pi}{4}$,

$$e^{-\varepsilon\, t}\, \mathcal{E}_{1/2}\,(\lambda, t) \sim \frac{1}{2\Gamma(1/2)} \lambda^{-2} t^{-\frac{3}{2}} \exp(-\varepsilon\, t)\,. \tag{53}$$

Indeed, with (53), the case of Proposition 4 is recovered as a special case, which occurs if and only if *all* the roots λ_i fulfill $|\arg(\lambda_i)| > \frac{\pi}{4}$.

Otherwise, if *but one* λ_0 lies in the sector $|\arg(\lambda)| < \frac{\pi}{4}$, then a very different asymptotic behaviour is to be found, namely a purely exponentially decaying one, with decay rate $\delta := \varepsilon - \Re(\lambda^2) > 0$ (it must be positive indeed, since asymptotic stability has already been proved in Proposition 3). To be more specific from a geometric viewpoint, by decomposing λ into its real and imaginary parts, the new zone of interests lies between the sector $\Re(\lambda) > |\Im(\lambda)|$ and the hyperbola $\Re(\lambda)^2 < \varepsilon + \Im(\lambda)^2$.

We are now in a position to state the general stability theorem:

Theorem 13. For the solution (49) of the GFDE (44), for a given value of the viscosity μ, two cases may occur, depending of the location of the four roots λ_i of the viscous polynomial P_T:

- either there is at least one root with $\Re(\lambda_j) > |\Im(\lambda_j)|$ then the asymptotics is of exponential type, with rate $\delta(\mu) := \frac{1}{\mu} - \Re(\lambda^2) > 0$

$$h_S(t) \sim \sum_j C_j \exp((\lambda_j^2 - \frac{1}{\mu})\, t), \tag{54}$$

- or all the four roots lie in $|\arg(\lambda)| > \frac{\pi}{4}$, then the asymptotics is of mixed type,

$$h_S(t) \sim C\, t^{-\frac{3}{2}} \exp(-\frac{1}{\mu}\, t)\,. \tag{55}$$

Proof. Using the explicit solution (49), and the asymptotic results of Theorem 12 for one root λ_i, upon selecting between these roots, we obtain the desired asymptotic result.

4.3 Evolution of the asymptotic behaviour with viscosity

The goal of this last part is to provide numerical evidence that both situations stated by Theorem 13 may occur in practise. In particular, we shall illustrate the transition between the two possible regimes, as the viscosity μ of the fluid increases.

In Figure 2 the trajectory of the four roots λ_i is drawn as a function of μ in the σ-plane: two roots belong to the left half-plane and will have no couterpart in the Laplace plane; the two other roots belong to the right half-plane and will give rise to a pole in the Laplace plane; picture (b) provides a zoom on these two, which cross the segments $\Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

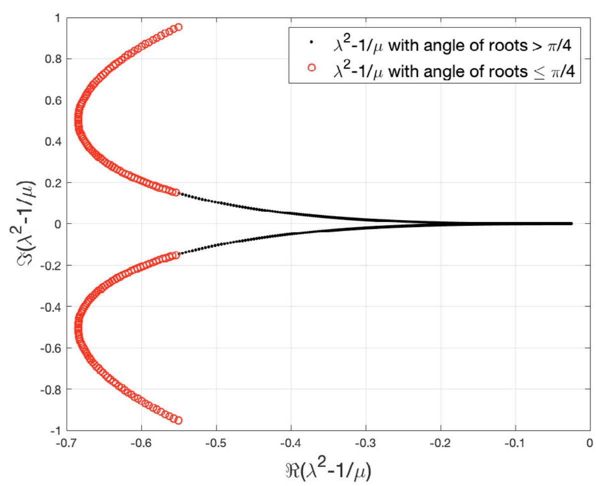


Fig. 3. Plot of the poles $s_j = \lambda_j^2 - \frac{1}{\mu}$ in the Laplace plane.

Figure 3 shows the 2 conjugate poles $s_j = \lambda_j^2 - \frac{1}{\mu}$ in the Laplace corresponding to the 2 roots $\lambda_{1,2}$: starting from the case $\mu = 0$, and increasing μ, there is some more damping up to some value μ^*, then the damping reduces monotonically towards 0.

Figure 4 shows the damping rate $\delta(\mu)$ as a function of μ, as can be forecast from Figure 3. Note that above the critical value μ^c, $\delta(\mu) = 1/\mu$, meaning that we are in the mixed type regime. Indeed, the two roots $\lambda_{1,2}$ now fulfill $|\arg(\lambda_{1,2})| > \frac{\pi}{4}$

5. CONCLUSION

In this work, we have given an analytic solution and computed the refined asymptotic estimates of solution the return to the equilibrium problem of a model describing the vertical motion of a solid floating at the free surface of a viscous fluid with finite depth and flat bottom. Moreover, numerical evidence has been provided of the possible

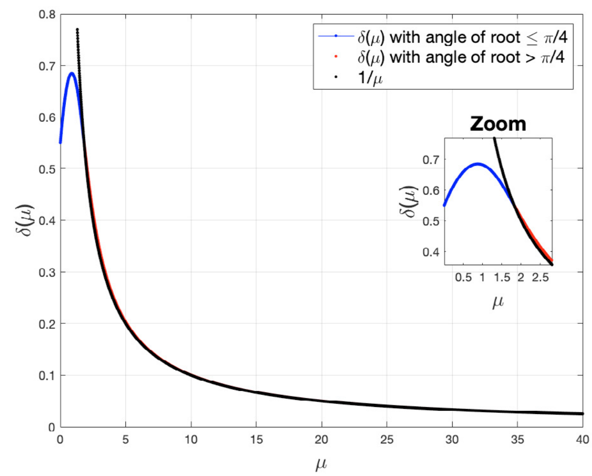


Fig. 4. Damping rate $\delta(\mu) = \Re(\lambda^2) - \frac{1}{\mu}$ as a function of viscosity μ

transition between different asymptotic regimes when the viscosity of the fluid increases.

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REFERENCES

Cummins, W. (1962). The impulse response function and ship motions. Technical report, David Taylor Model Basin Washington DC.

Katkova, O.M. and Vishnyakova, A.M. (2008). A sufficient condition for a polynomial to be stable. *Journal of Mathematical Analysis and Applications*, 347(1), 81–89.

Lannes, D. (2017). On the dynamics of floating structures. *Annals of PDE*, 3(1), 11.

Maity, D., San Martín, J., Takahashi, T., and Tucsnak, M. (2019). Analysis of a simplified model of rigid structure floating in a viscous fluid. *Journal of Nonlinear Science*, 29(5), 1975–2020.

Matignon, D. (1996). Stability results for fractional differential equations with applications to control processing. In *Computational engineering in systems applications*, volume 2, 963–968. Lille, France.

Matignon, D. (1998a). Generalized fractional differential and difference equations: stability properties and modelling issues. In *Proc. Mathematical Theory of Networks and Systems*, 4 p. MTNS, Padova, Italy.

Matignon, D. (1998b). Stability properties for generalized fractional differential systems. In *ESAIM: proceedings*, volume 5, 145–158. EDP Sciences.

Matignon, D. (2009). An introduction to fractional calculus. In P. Abry, P. Gonçalvès, and J. Lévy-Véhel (eds.), *Scaling, Fractals and Wavelets*, volume 1 of *Digital signal and image processing series*, 237–277. ISTE - Wiley, London : ISTE ; Hoboken, NJ : Wiley. URL `http://oatao.univ-toulouse.fr/4854/`.

Matignon, D. and Prieur, C. (2005). Asymptotic stability of linear conservative systems when coupled with diffusive systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 11(3), 487–507.

Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, volume 198. Elsevier.

Popov, A.Y. and Sedletskii, A. (2013). Distribution of roots of Mittag-Leffler functions. *Journal of Mathematical Sciences*, 190(2), 209–409.

Vergara-Hermosilla, G., Matignon, D., and Tucsnak, M. (2020). Input-output stability for a system modelling rigid structures floating in a viscous fluid. In *Proc. World Congress*, 8 p. IFAC, Berlin, Germany. Submitted (invited session).

Vergara-Hermosilla, G. (2021). On a dual to the properties of hurwitz polynomials i. *American Journal of Computational Mathematics*, 11(01), 31.

Appendix A. ANALYSIS OF THE VISCOUS POLYNOMIAL

In this Appendix we develop explicit formulas for the roots and its distribution in the complex plane of the so-called Viscous polynomial in the variable λ, which is given by

$$P_T(\lambda) = (1 + \frac{l^3}{12})\lambda^4 + l^2\sqrt{\mu}\lambda^3 + (l\mu - \frac{2}{\mu}(1 + \frac{l^3}{12}))\lambda^2 - \frac{l^2}{\sqrt{\mu}}\lambda + \frac{1}{\mu^2}(1 + \frac{l^3}{12}), \quad (A.1)$$

where l and μ are positive numbers.

A.1 Roots

By combine terms in the Viscous polynomial (A.1), we obtain an equivalent form given by

$$P_T(\lambda) = \left(1 + \frac{l^3}{12}\right)\left(\lambda^2 - \frac{1}{\mu}\right)^2 + l\sqrt{\mu}\lambda\left(l\lambda^2 + \sqrt{\mu}\lambda - \frac{l}{\mu}\right). \quad (A.2)$$

Multiplying this by $1/\lambda^2$, denoting by $y = \lambda - \frac{1}{\lambda\mu}$ and suppose that $\lambda\mu \neq 0$, we conclude that solve the equation $P_T(\lambda) = 0$ is equivalent to solve

$$Q(y) = \left(1 + \frac{l^3}{12}\right)y^2 + l^2\sqrt{\mu}y + l\mu = 0. \quad (A.3)$$

This means that we can actually compute the roots via a nested sequence of two quadratic equations. In fact, the roots of $Q(y) = 0$ are given by

$$y_{1,2} = \frac{-l^2\sqrt{\mu} \pm \sqrt{l\mu(l^3 - (1 + l^3/12))}}{2(1 + l^3/12)}, \quad (A.4)$$

then, like $y_{1,2} = \frac{\lambda^2\mu - 1}{\lambda\mu}$, we see that the roots of the equation (A.1) follows of solve

$$\mu\lambda^2 - y_{1,2}\mu\lambda - 1 = 0. \quad (A.5)$$

Therefore, the explicit roots of eq. (A.1) are given by

$$\lambda_{1,2,3,4} = \frac{y_{1,2}\mu \pm \sqrt{y_{1,2}^2\mu^2 + 4\mu}}{2\mu} \in \mathbb{C}. \quad (A.6)$$

Remark 14. If $l \geq \sqrt[3]{6}$ then all the roots of Viscous polynomial are reals. In fact, the discriminat of the polynomial (A.3) is given by $\Delta_Q = l^4\mu - 4l\mu(1 + (l^3/12))$. We see that Δ_Q is positive if and only if $l \geq \sqrt[3]{6}$. If the roots of equation

(A.3) are reals, then the discriminant of equation (A.5) is positive, and hence the result follow.

A.2 Distribution of roots

In this Section, we study how the roots of the Viscous polynomial (A.1) are distributed on $\mathbb{C}$. To this end, we introduce the notion of anti-Hurwitz polynomial. Morevover, we denote by $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ the set of roots of the polynomial $P_T(\lambda)$ defined in eq. (46) and we consider the following set

$$\mathcal{L}_{1/2} = \{z \in \mathbb{C} \,:\, z \neq 0, \, |\arg z| \leq \pi/2\}.$$

Definition 15. A real polynomial $f(X)$ in the complex variable X is said to be *Hurwitz* if the real part of all its roots is negative, that is $\Re(u) < 0$ for all $u \in \mathbb{C}$ such that $f(u) = 0$.

The following result attributed to A. Stodola (see for instance pp. 81 in Katkova and Vishnyakova (2008) or Theorem 1 in Vergara-Hermosilla (2021)), is a well-known necessary condition for a real polynomial to be Hurwitz.

Theorem 16. (Stodola condition). If a polynomial with real coeficients is Hurwitz, then all its coeficients are of the same sign.

Remark 17. Since there are different signs in the coefficients of the Viscous polynomial $p_T(\lambda)$, we conclude that it is not Hurwitz, i.e. $\Lambda \cap \mathcal{L}_{1/2} \neq \emptyset$.

Definition 18. A real polynomial $f(X)$ in the complex variable X is a *anti-Hurwitz polynomial* if and only if, the real part of all its complex roots is posititive, that is; $\Re(u) > 0$ for all $u \in \mathbb{C}$ such that $f(u) = 0$.

Lemma 19. A real polynomial $f(X)$ is anti-Hurwitz if and only if $f(-X)$ is Hurwitz.

Proof. If $f(X)$ is anti-Hurwitz and u is a complex root of $f(-X)$, since $f(-u) = 0$, we conclude that $\Re(-u) > 0$. Hence $\Re(u) < 0$ and therefore $f(-X)$ is Hurwitz. Similarly, if $f(-X)$ is Hurwitz and u is a root of $f(X)$, since $f(u) = f(-(-u)) = 0$ we conclude that $\Re(-u) < 0$. Hence $\Re(u) > 0$ and then $f(X)$ is an anti-Hurwitz polynomial.

Remark 20. Since there are different signs in the coefficients of the polynomial $p_T(-\lambda)$, by Lemma 19 we conclude that $p_T(\lambda)$ is not anti-Hurwitz, i.e. $\Lambda \cap \mathbb{C} \setminus \mathcal{L}_{1/2} \neq \emptyset$.

Appendix B. RESIDUES

In this appendix, our aim is show the explicit form of each r_i and each $\tilde{r}_i$ corresponds to the partial-fraction decompositions of the rational functions present in the equation (46). To this end, if Λ is the set of the four roots of P_T, $\lambda_i \in \Lambda$ and $d(s) = A\prod_{i=1}^{4}(\sqrt{s+\varepsilon} - \lambda_i)$, then

$$\frac{A}{P_T(\sigma)} = \sum_{i=1}^{4} \frac{\tilde{r}_i}{(\sigma - \lambda_i)},$$

where

$$\tilde{r}_1 = -\left(\lambda_1^2\,\lambda_2 + \lambda_1^2\,\lambda_3 + \lambda_1^2\,\lambda_4 - \lambda_1^3 - \lambda_1\,\lambda_2\,\lambda_3 - \lambda_1\,\lambda_2\,\lambda_4 - \lambda_1\,\lambda_3\,\lambda_4 + \lambda_2\,\lambda_3\,\lambda_4\right)^{-1}, \quad \text{(B.1)}$$

$$\tilde{r}_2 = -A\left(\lambda_1\,\lambda_2^2 + \lambda_2^2\,\lambda_3 + \lambda_2^2\,\lambda_4 - \lambda_2^3 - \lambda_1\,\lambda_2\,\lambda_3 - \lambda_1\,\lambda_2\,\lambda_4 + \lambda_1\,\lambda_3\,\lambda_4 - \lambda_2\,\lambda_3\,\lambda_4\right)^{-1}, \quad \text{(B.2)}$$

$$\tilde{r}_3 = -A\left(\lambda_1\,\lambda_3^2 + \lambda_2\,\lambda_3^2 + \lambda_3^2\,\lambda_4 - \lambda_3^3 - \lambda_1\,\lambda_2\,\lambda_3 + \lambda_1\,\lambda_2\,\lambda_4 - \lambda_1\,\lambda_3\,\lambda_4 - \lambda_2\,\lambda_3\,\lambda_4\right)^{-1}, \quad \text{(B.3)}$$

$$\tilde{r}_4 = -A\left(\lambda_1\,\lambda_4^2 + \lambda_2\,\lambda_4^2 + \lambda_3\,\lambda_4^2 - \lambda_4^3 + \lambda_1\,\lambda_2\,\lambda_3 - \lambda_1\,\lambda_2\,\lambda_4 - \lambda_1\,\lambda_3\,\lambda_4 - \lambda_2\,\lambda_3\,\lambda_4\right)^{-1}. \quad \text{(B.4)}$$

Moreover,

$$\frac{A\sigma^2 + B\mu^{1/2}\sigma + (C - A/\mu)}{P_T(\sigma)} = \sum_{i=1}^{4} \frac{r_i}{(\sigma - \lambda_i)},$$

where

$$r_1 = (\mu A)^{-1}(C\,\mu - A + \lambda_1^2\,A\,\mu + \lambda_1\,B\,\mu^{3/2})\tilde{r}_1^{-1}, \quad \text{(B.5)}$$

$$r_2 = A\mu^{-1}(C\,\mu - A + \lambda_2^2\,A\,\mu + \lambda_2\,B\,\mu^{3/2})\tilde{r}_2^{-1}, \quad \text{(B.6)}$$

$$r_3 = A\mu^{-1}(C\,\mu - A + \lambda_3^2\,A\,\mu + \lambda_3\,B\,\mu^{3/2})\tilde{r}_3^{-1}, \quad \text{(B.7)}$$

$$r_4 = A\mu^{-1}(C\,\mu - A + \lambda_4^2\,A\,\mu + \lambda_4\,B\,\mu^{3/2})\tilde{r}_4^{-1}. \quad \text{(B.8)}$$