NUMERICAL SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY SPECTRAL METHODS

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I would like to dedicate my Doctoral thesis to Prof. Madya Dr. Phang Chang and my beloved parents whose sincere prayers make it possible for me to fulfill their utmost desire. May Allah always bless them with more happiness and good health.

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ABSTRACT

Fractional partial differential equations (FPDEs) have become essential tool for the modeling of physical models by using spectral methods. In the last few decades, spectral methods have been developed for the solution of time and space dimensional FPDEs. There are different types of spectral methods such as collocation methods, Tau methods and Galerkin methods. This research work focuses on the collocation and Tau methods to propose an efficient operational matrix methods via Genocchi polynomials and Legendre polynomials for the solution of two and three dimensional FPDEs. Moreover, in this study, Genocchi wavelet-like basis method and Genocchi polynomials based Ritz- Galerkin method have been derived to deal with FPDEs and variable- order FPDEs. The reason behind using the Genocchi polynomials is that, it helps to generate functional expansions with less degree and small coefficients values to derive the operational matrix of derivative with less computational complexity as compared to Chebyshev and Legendre Polynomials. The results have been compared with the existing methods such as Chebyshev wavelets method, Legendre wavelets method, Adomian decomposition method, Variational iteration method, Finite difference method and Finite element method. The numerical results have revealed that the proposed methods have provided the better results as compared to existing methods due to minimum computational complexity of derived operational matrices via Genocchi polynomials. Additionally, the significance of the proposed methods has been verified by finding the error bound, which shows that the proposed methods have provided better approximation values for under consideration FPDEs.

ABSTRAK

Persamaan Pembezaan Separa Pecahan (PPSP) telah menjadi alat penting untuk pemodelan model fizikal dengan menggunakan kaedah spektral. Dalam beberapa dekad yang lalu, kaedah spektral telah dibangunkan untuk penyelesaian PPSP bagi terbitan dimensi masa dan ruang. Terdapat pelbagai jenis kaedah spektral seperti kaedah kolokasi, kaedah Tau dan kaedah Galerkin. Kajian ini memberi tumpuan kepada kaedah kolokasi dan kaedah Tau untuk mencadangkan kaedah matriks operasi yang berkesan melalui polinomial Genocchi dan polynomial Legendre untuk penyelesaian dua dan tiga dimensi PPSP. Tambahan pula, dalam kajian ini, kaedah asas seperti wavelet Genocchi dan kaedah Ritz-Galerkin berasaskan polynomial Gennochi telah diperolehi untuk menangani PPSP dan PPSP peringkat pembolehubah . Alasan di sebalik menggunakan polinomial Genocchi adalah bahawa ia membantu untuk menghasilkan kembangan fungsi dengan nilai pekali yang kecil dan cara memperoleh matriks operasi pembezaan yang kurang rumit pengiraannya berbanding dengan Polynomial Chebyshev dan Legendre. Hasilnya telah dibandingkan dengan kaedah yang sedia ada seperti kaedah wavelet Chebyshev, kaedah wavelet Legendre, kaedah penguraian Adomian, kaedah lelaran variasi, kaedah perbezaan terhingga dan kaedah unsur terhingga. Keputusan berangka telah mendedahkan bahawa kaedah yang dicadangkan telah memberikan hasil yang lebih baik berbanding dengan kaedah yang sedia ada disebabkan oleh pengiraan matriks operasi adalah kurang rumit dengan polinomial Genocchi. Selain itu, kepentingan kaedah yang dicadangkan telah dibukti dengan ralat sempadan, yang menunjukkan bahawa kaedah yang dicadangkan telah memberikan nilai anggaran yang lebih baik untuk PPSP.

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LIST OF SYMBOLS AND ABBREVIATIONS

α	–	Alpha
β	–	Beta
γ	–	Gamma
μ	–	Mu
λ	–	Lambda
η	–	Eta
ϖ	–	Pi Variant
ω	–	Omega
ζ	–	Zeta
θ	–	Theta
ϕ	–	Phi Variant
ψ	–	Psi
Δ	–	Delta
ρ	–	Rho
σ	–	Sigma
Φ	–	Phi
ξ	–	Xi
τ	–	Tau
ϑ	–	Theta Variant
$\otimes$	–	Kronecker product
H_*^σ	–	Operational matrix of fractional derivative
$U_{M\times M}^\sigma$	–	Genocchi polynomials operational matrix of fractional derivative
K	–	Coefficient vector or expansion coefficient vector for

		two variables
$\overline{K}$	–	Coefficient or expansion coefficient vector for three variables
$\varpi(x,t)$	–	Satisfier function
$\sigma_{i,j}$	–	Expansion coefficient of fractional derivative
D^{α}	–	Fractional order derivative
D_t^{α}	–	Caputo's fractional derivative
D_{b-}^{α}	–	Right-sided Riemann-Liouville fractional derivative
D_{a+}^{α}	–	Left-sided Riemann-Liouville fractional derivative
${}_{RL}D_{t,t_R}^{\gamma(t)}$	–	Right Riemann-Liouville fractional derivative of variable order
${}_{RL}D_{t_L,t}^{\gamma(t)}$	–	Left Riemann-Liouville fractional derivative of variable order
${}_{RL}D_{t,t_R}^{-\gamma(t)}$	–	Right Riemann-Liouville fractional integral of variable order
${}_{RL}D_{t_L,t}^{-\gamma(t)}$	–	Left Riemann-Liouville fractional integral of variable order
${}_{C}D_{t,t_R}^{\gamma(t)}$	–	Right Riemann-Liouville fractional derivative of variable order
${}_{C}D_{t_L,t}^{\gamma(t)}$	–	Left Riemann-Liouville fractional derivative of variable order
$I_{\mathrm{a}+}^{\alpha}$	–	Left Riemann-Liouville fractional integral
$I_{\mathrm{b}-}^{\alpha}$	–	Right Riemann-Liouville fractional integral
$G_r(x)$	–	Genocchi polynomials of degree r
$B_m(x)$	–	Bernoulli polynomials of degree m
$E_l(x)$	–	Euler polynomials of degree l
$\Gamma(.)$	–	Euler Gamma function
Ω	–	Bounded domain
M	–	Scale level of approximation
$\psi(x)$	–	An oscillatory function

DE	–	Differential equation
FC	–	Fractional calculus
FDEs	–	Fractional differential equations
PDE	–	Partial differential equation
FPDEs	–	Fractional partial differential equations
TFPDES	–	Time fractional partial differential equation
SFPDE	–	Space fractional partial differential equation
STFPDE	–	Space-time fractional partial differential equation
VOFPDEs	–	Variable-order fractional partial differential equations
ADM	–	Adomian decomposition method
LADM	–	Laplace Adomian decomposition method
VIM	–	Variational iteration method
LVIM	–	Laplace Variational iteration method
RVIM	–	Reconstruction of Variational iteration method
HPM	–	Homotopy perturbation method
HPTM	–	Homotopy perturbation transform method
HAM	–	Homotopy analysis method
CGL	–	Chebyshev Gauss Lobatto
SL-GL-C	–	Shifted Legendre Gauss-Lobatto collocation method
FIDEs	–	Fractional integro-differential equations
SLOM	–	Shifted Legendre operational matrix
NSFDM	–	Non-standard finite difference method
SFDM	–	Standard finite difference method
MWR	–	Method of weighted residuals
SLC	–	Shifted Legendre Collocation method
FBE	–	Fractional Burgers’ Equation
KDV	–	Korteweg de Vries equation
GFBWFs	–	Fractional-order Bernoulli wavelet functions
Q-SLT	–	Quadrature Shifted Legendre Tau method
KV	–	Kelvin-Voigt equation
GFLPs	–	Generalized fractional Legendre polynomials

NLP	–	Nonlinear programming problem

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LIST OF PUBLICATIONS

[1] Chang Phang**, Afshan Kanwal,** Loh Jian Rong and Abdul Nasir Isah (2016), "Legendre Operational Matrix for Solving Fractional Partial Differential Equations", *International Journal of Mathematical Analysis*, *volume 10*, number 19, page 903-911.

[2] **Afshan Kanwal,** Chang Phang and Hammad Khalil (2017), "A New Algorithm based on Shifted Legendre Polynomials for Fractional Partial Differential Equations", *Indian Journal of Science and Technology*, *volume 10*, number 12, page 1-7.

[3] Afshan Kanwal, Chang Phang and Umer Iqbal (2018), "Numerical Solution of Fractional Diffusion Wave Equation and Fractional Klein–Gordon Equation via Two-Dimensional Genocchi Polynomials with a Ritz–Galerkin Method", *Computation*, *volume 6*, number 3, doi:10.3390/computation6030040.

[4] Afshan Kanwal, Chang Phang and Loh Jian Rong (2019), "New collocation scheme for solving fractional partial differential equations", *Hacettepe Journal of Mathematics and Statistics*, (Accepted, **IF. 0.605**).

CHAPTER 1

INTRODUCTION

This chapter is comprised of some preliminaries given in section 1.1 that have been used in this research work. The reason to conduct this research is illustrated in section 1.3. To achieve the research aim, four objectives have been set in section 1.4. The scope of research and the main contribution are discussed in section 1.5 and section 1.6 respectively. Section 1.7 consists of thesis organization.

1.1 Preliminaries

In this section, some basic definitions of FPDEs, mathematical solution, solution methods and method of weighted residuals (MWR) are explained.

1.1.1 Fractional partial differential equations

FPDEs are the generalization of classical partial differential equations (PDEs) with the fractional order derivatives D^{α}. The general form of FPDEs (Al-Khaled, 2015) can be written as

$$D_t^{\alpha}u(x,t) = Lu(x,t) + Nu(x,t) + g(x,t),\ m-1 < \alpha \leqslant m, \tag{1.1}$$

where $u(x,t)$ is the unknown function, L is the linear operator, N is the general nonlinear operator and $g(x,t)$ is the source term. Similarly if the fractional order derivative is replaced with the variable order derivative then the equation would be

REFERENCES

Abbasbandy, S., Kazem, S., Alhuthali, M. S., and Alsulami, H. H. (2015). Application of the operational matrix of fractional-order Legendre functions for solving the time- fractional convection–diffusion equation. *Applied Mathematics and Computation*, 266, 31-40.

Agoh, T., and Dilcher, K. (2014). Higher-order convolutions for Bernoulli and Euler polynomials. *Journal of Mathematical Analysis and Applications*, 419(2), 1235-1247.

Ahmadian, A., Ismail, F., Salahshour, S., Baleanu, D., and Ghaemi, F. (2017). Uncertain viscoelastic models with fractional order: A new spectral tau method to study the numerical simulations of the solution. *Communications in Nonlinear Science and Numerical Simulation*, 53, 44-64.

Ahmed, H. F., Bahgat, M. S., and Zaki, M. (2017). Numerical approaches to system of fractional partial differential equations. *Journal of the Egyptian Mathematical Society*, 25(2), 141-150.

Al-Khaled, K. (2015). Numerical solution of time-fractional partial differential equations using Sumudu decomposition method. *Romanian. Journal of. Physics*, 60(1-2), 99-110.

Al-Smadi, M., Freihat, A., Khalil, H., Momani, S., and Ali Khan, R. (2017). Numerical multistep approach for solving fractional partial differential equations. *International Journal of Computational Methods*, 14(03), 1750029.

Albuohimad, B., Adibi, H., and Kazem, S. (2017). A numerical solution of time-fractional coupled Korteweg-de Vries equation by using spectral collection method. *Ain Shams Engineering Journal*.

Aldroubi, A. (2017). Wavelets in medicine and biology. Routledge.

Araci, S. (2014). Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. *Applied Mathematics and Computation*, 233, 599-607.

Atabakzadeh, M. H., Akrami, M. H., and Erjaee, G. H. (2013). Chebyshev operational matrix method for solving multi-order fractional ordinary differential equations. *Applied Mathematical Modelling*, 37(20-21), 8903-8911.

Baeumer, B., Kovács, M., and Sankaranarayanan, H. (2018). Fractional partial differential equations with boundary conditions. *Journal of Differential Equations*, 264(2), 1377-1410.

Balaji, S. (2015). Legendre wavelet operational matrix method for solution of fractional order Riccati differential equation. *Journal of the Egyptian Mathematical Society*, 23(2), 263-270.

Barikbin, Z. (2017). Two-dimensional Bernoulli wavelets with satisfier function in the Ritz–Galerkin method for the time fractional diffusion-wave equation with damping. *Mathematical Sciences*, 11(3), 195-202.

Barnes, B., Boateng, F. O., Ackora-Prah, J., and Osei-Frimpong, E. (2016). A two-dimensional Chebyshev wavelet method for solving partial differential equations.

Baseri, A., Abbasbandy, S., and Babolian, E. (2018). A collocation method for fractional diffusion equation in a long time with Chebyshev functions. *Applied Mathematics and Computation*, 322, 55-65.

Bhrawy, A. H. (2014). A new legendre collocation method for solving a two-dimensional fractional diffusion equation. In *Abstract and Applied Analysis*. Hindawi.

Bhrawy, A. H. (2016). A new spectral algorithm for time-space fractional partial differential equations with subdiffusion and superdiffusion. In *Proceedings of the .Romanian. Academy, Series* A(17(1)), 39-47.

Bhrawy, A. H., and Abdelkawy, M. A. (2015b). A fully spectral collocation Approximation for multi-dimensional fractional Schrödinger equations. *Journal of Computational Physics*, 294, 462-483.

Bhrawy, A. H., Alofi, A. S., and Ezz-Eldien, S. S. (2011). A quadrature tau method for fractional differential equations with variable coefficients. *Applied Mathematics Letters*, 24(12), 2146-2152.

Bhrawy, A. H., Doha, E. H., Alzaidy, J. F., and Abdelkawy, M. A. (2016). A space–time spectral collocation algorithm for the variable order fractional wave equation. *SpringerPlus*, 5(1), 1220.

Bhrawy, A. H., Doha, E. H., Baleanu, D., and Ezz-Eldien, S. S. (2015). A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations. *Journal of Computational Physics*, 293, 142-156.

Bhrawy, A. H., Zaky, M. A., and Baleanu, D. (2015a). New numerical approximations for space-time fractional Burgers' equations via a Legendre spectral-collocation method. *Romanian Reports in Physics*, 67(2), 340-349.

Bhrawy, A. H., and Zaky, M. A. (2015c). A method based on the Jacobi tau approximation for solving multi-term time–space fractional partial differential equations. *Journal of Computational Physics*, 281, 876-895.

Bhrawy, A. H., Zaxy, M. A., Baleanu, D., and Abdelkawy, M. A. (2015). A novel spectral approximation for the two-dimensional fractional sub-diffusion problems.

Bicer, K. E. and Yalcinbas, S. (2016). A matrix approach to solving hyperbolic partial differential equations using Bernoulli polynomials. Filomat. 30(4), 993–1000.

Bildik, N., and Konuralp, A. (2006). The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 7(1), 65-70.

Bos, L., Narayan, A., Levenberg, N., and Piazzon, F. (2017). An orthogonality property of the legendre polynomials. *Constructive Approximation*, 45(1), 65-81.

Buckingham, M. J. (2007). On pore-fluid viscosity and the wave properties of saturated granular materials including marine sediments. *The Journal of the Acoustical Society of America*, 122(3), 1486-1501.

Canuto, C., Hussaini, M. Y., Quarteroni, A., and Thomas Jr, A. (2012). *Spectral methods in fluid dynamics*. Springer Science and Business Media.

Chen, F., Xu, Q., and Hesthaven, J. S. (2015). A multi-domain spectral method for time-fractional differential equations. *Journal of Computational Physics*, 293, 157-172.

Chen, N., Huang, J., Wu, Y., and Xiao, Q. (2017). Operational matrix method for the variable order time fractional diffusion equation using Legendre polynomials approximation. *IAENG International Journal of Applied Mathematics*, 47(3), 282-286.

Chen, Y., Sun, Y., and Liu, L. (2014). Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order Legendre functions. *Applied Mathematics and Computation*, 244, 847-858.

Chen, Y., Wu, Y., Cui, Y., Wang, Z., and Jin, D. (2010). Wavelet method for a class of fractional convection-diffusion equation with variable coefficients. *Journal of Computational Science*, 1(3), 146-149.

Chui, C. K. (2016). An introduction to wavelets. Elsevier.

Dabiri, A. and Butcher, E. A. (2017). Efficient modified Chebyshev differentiation matrices for fractional differential equations. Communications in Nonlinear Science and Numerical Simulation. 50, 284–310.

Das, S., Vishal, K., Gupta, P. K., and Ray, S. S. (2011). Homotopy analysis method for solving fractional diffusion equation. *Int. J. Appl. Math. Mech*, 7(9), 28-37.

Dehghan, M., Abbaszadeh, M., and Mohebbi, A. (2016). Analysis of two methods based on Galerkin weak form for fractional diffusion-wave: Meshless interpolating element free Galerkin (IEFG) and finite element methods. *Engineering Analysis with Boundary Elements*, 64, 205-221.

Dehghan, M., Abbaszadeh, M., and Mohebbi, A. (2016). Legendre spectral element method for solving time fractional modified anomalous sub-diffusion equation. *Applied Mathematical Modelling*, 40(5-6), 3635-3654.

Dehghan, M., and Taleei, A. (2010). Numerical solution of nonlinear Schrödinger equation by using time-space pseudo-spectral method. *Numerical Methods for Partial Differential Equations: An International Journal*, 26(4), 979-992.

Dhaigude, D. B., and Dhaigude, C. D. (2012). Linear initial value problems for fractional partial differential equations. *Bulletin of Marathwada Mathematical Society*, 13(2), 17-38.

Dhaigude, D. B., and Birajdar, G. A. (2012). Numerical solution of system of fractional partial differential equations by discrete Adomian decomposition method. *Journal of Fractional Calculus and Applications*, 3(12), 1-11.

Diethelm, K. (2010). The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type. *Springer Science and Business Media*.

Diethelm, K., and Walz, G. (1997). Numerical solution of fractional order differential equations by extrapolation. *Numerical Algorithms*, 16(3-4), 231-253.

Doha, E. H., Bhrawy, A. H., and Ezz-Eldien, S. S. (2011). Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. *Applied Mathematical Modelling*, 35(12), 5662-5672.

Doha, E. H., Bhrawy, A. H., and Ezz-Eldien, S. S. (2012). A new Jacobi operational matrix: an application for solving fractional differential equations. *Applied Mathematical Modelling*, 36(10), 4931-4943.

Doha, E. H., Bhrawy, A. H., and Ezz-Eldien, S. S. (2011). A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order. *Computers and Mathematics with Applications*, 62(5), 2364-2373.

Duan, J. S., Rach, R., Baleanu, D., and Wazwaz, A. M. (2012). A review of the Adomian decomposition method and its applications to fractional differential equations. *Communications in Fractional Calculus*, 3(2), 73-99.

El-Baghdady, G. I., and El-Azab, M. (2016). Chebyshev-Gauss–Lobatto pseudo-spectral method for one-dimensional advection-diffusion equation with variable coefficients, Sohag Journal of Mathematics, 3, 1-8.

El-Sayed, A. M. A., Behiry, S. H., and Raslan, W. E. (2010). Adomian's decomposition method for solving an intermediate fractional advection–dispersion equation. *Computers and Mathematics with Applications*, 59(5), 1759-1765.

El-Sayed, A. M. A., and Gaber, M. (2006). The Adomian decomposition method for solving partial differential equations of fractal order in finite domains. *Physics Letters A*, 359(3), 175-182.

Ezz-Eldien, S. S., Hafez, R. M., Bhrawy, A. H., Baleanu, D., and El-Kalaawy, A. A. (2017).New numerical approach for fractional variational problems using shifted Legendre orthonormal polynomials. *Journal of Optimization Theory and Applications*, 174(1), 295-320.

Fan, W., Liu, F., Jiang, X., and Turner, I. (2018). Some novel numerical techniques for an inverse problem of the multi-term time fractional partial differential equation. *Journal of Computational and Applied Mathematics*, 336, 114-126.

Ghandehari, M. A. M., and Ranjbar, M. (2013). A numerical method for solving a fractional partial differential equation through converting it into an NLP problem. *Computers and Mathematics with Applications*, 65(7), 975-982.

Graps, A. (1995). An introduction to wavelets. *IEEE computational science and engineering*, 2(2), 50-61.

Gupta, A. K., and Ray, S. S. (2014). Wavelet methods for solving fractional order differential equations. *Mathematical Problems in Engineering*, *2014*.

Gupta, A. K., and Ray, S. S. (2015). Numerical treatment for the solution of fractional fifth-order Sawada–Kotera equation using second kind Chebyshev wavelet method. *Applied Mathematical Modelling*, 39(17), 5121-5130.

Hesameddini, E., Rahimi, A., and Asadollahifard, E. (2016). On the convergence of a new reliable algorithm for solving multi-order fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation*, 34, 154-164.

Heydari, M. H., Hooshmandasl, M. R., Ghaini, F. M., and Fereidouni, F. (2013). Two-dimensional Legendre wavelets for solving fractional Poisson equation with Dirichlet boundary conditions. *Engineering Analysis with Boundary Elements*, 37(11), 1331-1338.

Heydari, M. H. (2016). A new approach of the Chebyshev wavelets for the variable-order time fractional mobile-immobile advection-dispersion model. *arXiv preprint arXiv:1605.06332*.

Hoang, N. (2016). On node distributions for interpolation and spectral methods. *Mathematics of Computation*, 85(298), 667-692.

Hramov, A. E., Koronovskii, A. A., Makarov, V. A., Pavlov, A. N., and Sitnikova, E. (2015). *Wavelets in neuroscience*. Springer Berlin Heidelberg.

Isah, A., and Phang, C. (2017). Operational matrix based on Genocchi polynomials for solution of delay differential equations. *Ain Shams Engineering Journal*.

Jafari, H., Nazari, M., Baleanu, D., and Khalique, C. M. (2013). A new approach for solving a system of fractional partial differential equations. *Computers and Mathematics with Applications*, 66(5), 838-843.

Jassim, H. K. (2015). Analytical solutions for system of fractional partial differential equations by homotopy perturbation transform method. *International Journal of Advances in Applied Mathematics and Mechanics*, 3(1), 36-40.

Karatay, I., and Kale, N. (2014). Finite difference method of fractional parabolic partial differential equations with variable coefficients. *International Journal of Contemporary Mathematical Sciences*, 9(16), 767-776.

Keshavarz, E., Ordokhani, Y., and Razzaghi, M. (2014). Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *Applied Mathematical Modelling*, 38(24), 6038-6051.

Khader, M. M., Sweilam, N. H., and Mahdy, A. M. S. (2015). Two computational algorithms for the numerical solution for system of fractional differential equations. *Arab Journal of Mathematical Sciences*, 21(1), 39-52.

Khalil, H., Khan, R. A., Al-Samdi, M., Friehat, A., and Shawagfeh, N. (2015a). New operational matrix for shifted Legendre polynomials and Fractional differential equations with variable coefficients. *Journal of Mathematics*, 47(1).

Khalil, H., Khan, R. A., Baleanu, D., and Rashidi, M. M. (2016). Some new operational matrices and its application to fractional order Poisson equations with integral type boundary constrains. *Computers and Mathematics with Applications*.

Khalil, H., Khan, R. A., Smadi, M. A., and Freihat, A. (2015b). A generalized algorithm based on Legendre polynomials for numerical solutions of coupled system of fractional order differential equations. *Journal of Fractional Calculus and Applications*, 6(2), 123-143.

Khalil, H., and Khan, R. A. (2014). A new method based on Legendre polynomials for solutions of the fractional two-dimensional heat conduction equation. *Computers and Mathematics with Applications*, 67(10), 1938-1953.

Khan, Y., and Wu, Q. (2011). Homotopy perturbation transform method for nonlinear equations using He's polynomials. *Computers and Mathematics with Applications*, 61(8), 1963-1967.

Khan, R. A., and Khalil, H. (2014). A new method based on legendre polynomials for solution of system of fractional order partial differential equations. *International Journal of Computer Mathematics*, 91(12), 2554-2567.

Kilbas, A. A. A., Srivastava, H. M., and Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*.

Li, C., and Zeng, F. (2012). Finite difference methods for fractional differential equations. *International Journal of Bifurcation and Chaos*, 22(04), 1230014.

Li, Z., Wang, H., Xiao, R., and Yang, S. (2017). A variable-order fractional differential equation model of shape memory polymers. *Chaos, Solitons and Fractals*, 102, 473-485.

Liu, F., Meerschaert, M. M., McGough, R. J., Zhuang, P., and Liu, Q. (2013). Numerical methods for solving the multi-term time-fractional wave-diffusion equation. *Fractional Calculus and Applied Analysis*, 16(1), 9-25.

Loh, J. R., and Phang, C. (2018). A new numerical scheme for solving system of Volterra integro-differential equation. *Alexandria Engineering Journal*, 57(2), 1117-1124.

Loh, J. R., Phang, C., and Isah, A. (2017). New operational matrix via Genocchi polynomials for solving Fredholm-Volterra fractional integro-differential equations. *Advances in Mathematical Physics*, *2017*.

Lubich, C. (1986). Discretized fractional calculus. *SIAM Journal on Mathematical Analysis*, 17(3), 704-719.

Ma, X., and Huang, C. (2014). Spectral collocation method for linear fractional integro-differential equations. *Applied Mathematical Modelling*, 38(4), 1434-1448.

Maleknejad, K., Hashemizadeh, E., and Basirat, B. (2012). Computational method based on Bernstein operational matrices for nonlinear Volterra–Fredholm–Hammerstein integral equations. *Communications in Nonlinear Science and Numerical Simulation*, 17(1), 52-61.

Mashayekhi, S., and Razzaghi, M. (2016). Numerical solution of the fractional Bagley-Torvik equation by using hybrid functions approximation. *Mathematical Methods in the Applied Sciences*, 39(3), 353-365.

Meerschaert, M. M., and Tadjeran, C. (2004). Finite difference approximations for fractional advection–dispersion flow equations. *Journal of Computational and Applied Mathematics*, 172(1), 65-77.

Meerschaert, M. M., and Tadjeran, C. (2006). Finite difference approximations for two-sided space-fractional partial differential equations. *Applied numerical mathematics*, 56(1), 80-90.

Mohamed, A. S., and Mahmoud, R. A. (2016). Picard, Adomian and predictor–corrector methods for an initial value problem of arbitrary (fractional) orders differential equation. *Journal of the Egyptian Mathematical Society*, 24(2), 165-170.

Mohammadi, F., and Cattani, C. (2018). A generalized fractional-order Legendre wavelet Tau method for solving fractional differential equations. *Journal of Computational and Applied Mathematics*, 339, 306-316.

Mohammadi, F., and Hosseini, M. M. (2011). A comparative study of numerical methods for solving quadratic Riccati differential equations. *Journal of the Franklin Institute*, 348(2), 156-164.

Momani, S., and Odibat, Z. (2006). Analytical approach to linear fractional partial differential equations arising in fluid mechanics. *Physics Letters A*, 355(4-5), 271-279.

Momani, S., and Odibat, Z. (2007). Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Physics Letters A*, 365(5-6), 345-350.

Morgado, M. L., Rebelo, M., Ferras, L. L., and Ford, N. J. (2017). Numerical solution for diffusion equations with distributed order in time using a Chebyshev collocation method. *Applied Numerical Mathematics*, 114, 108-123.

Mousavi, A., and Asemani, D. (2017, May). Signal detection improvement using Unbalanced Haar Wavelet transform for digital receivers. In *2017 Iranian Conference on Electrical Engineering (ICEE)*, 551-556. IEEE.

Nagy, A. M. (2017). Numerical solution of time fractional nonlinear Klein–Gordon equation using Sinc–Chebyshev collocation method. *Applied Mathematics and Computation*, 310, 139-148.

Odibat, Z., and Momani, S. (2008). Numerical methods for nonlinear partial differential equations of fractional order. *Applied Mathematical Modelling*, 32(1), 28-39.

Odibat, Z. M., and Momani, S. (2006). Application of variational iteration method to nonlinear differential equations of fractional order. *International Journal of Nonlinear Sciences and Numerical Simulation*, 7(1), 27-34.

Odibat, Z., and Momani, S. (2008). A generalized differential transform method for linear partial differential equations of fractional order. *Applied Mathematics Letters*, 21(2), 194-199.

Oldham, K., and Spanier, J. (1974). *The fractional calculus theory and applications of differentiation and integration to arbitrary order*,111, Elsevier.

Orszag, S. A. (1972). Comparison of pseudospectral and spectral approximation. *Studies in Applied Mathematics*, 51(3), 253-259.

Patel, V. K., Singh, S., and Singh, V. K. (2017). Two-dimensional wavelets collocation method for electromagnetic waves in dielectric media. *Journal of Computational and Applied Mathematics*, 317, 307-330.

Rahimkhani, P., Ordokhani, Y., and Babolian, E. (2016). Fractional-order Bernoulli wavelets and their applications. *Applied Mathematical Modelling*, 40(17-18), 8087-8107.

Rahimkhani, P., Ordokhani, Y., and Babolian, E. (2017). A new operational matrix based on Bernoulli wavelets for solving fractional delay differential equations. *Numerical Algorithms*, 74(1), 223-245.

Ray, S. S., and Gupta, A. K. (2017). Two-dimensional Legendre wavelet method for travelling wave solutions of time-fractional generalized seventh order KdV equation. *Computers and Mathematics with Applications*, 73(6), 1118-1133.

Reutskiy, S. Y. (2017). A new semi-analytical collocation method for solving multi-term fractional partial differential equations with time variable coefficients. *Applied Mathematical Modelling*, 45, 238-254.

Rim, S. H., Park, K. H., and Moon, E. J. (2008, January). On Genocchi Numbers and Polynomials. In *Abstract and Applied Analysis*.

Saadatmandi, A., Dehghan, M., and Azizi, M. R. (2012). The Sinc–Legendre collocation method for a class of fractional convection–diffusion equations with variable coefficients. *Communications in Nonlinear Science and Numerical Simulation*, 17(11), 4125-4136.

Saadatmandi, A., and Dehghan, M. (2010). A new operational matrix for solving fractional-order differential equations. *Computers and mathematics with applications*, 59(3), 1326-1336.

Sabeg, D. J., Ezzati, R., and Maleknejad, K. (2017). A new operational matrix for solving two-dimensional nonlinear integral equations of fractional order. *Cogent Mathematics and Statistics*, 4(1), 1347017.

Saker, M. A., Ezz-Eldien, S. S., and Bhrawy, A. H. (2017). A Pseudospectral Method For Solving The Time-Fractional Generalized Hirota–Satsuma Coupled Korteweg–De Vries System. *Romanian Journal of Physics*, 62, 105.

Samko, S. (2013). Fractional integration and differentiation of variable order: an overview. *Nonlinear Dynamics*, 71(4), 653-662.

Samko, S. G., and Ross, B. (1993). Integration and differentiation to a variable fractional order. *Integral transforms and special functions*, 1(4), 277-300.

Singh, H., and Singh, C. S. (2016). Stable numerical solutions of fractional partial differential equations using Legendre scaling functions operational matrix. *Ain Shams Engineering Journal*.

Singh, I., and Kumar, S. (2017). Wavelet methods for solving three-dimensional partial differential equations. *Mathematical Sciences*, 11(2), 145-154.

Srivastava, H. M., Kurt, B., and Simsek, Y. (2012). Some families of Genocchi type polynomials and their interpolation functions. *Integral Transforms and Special Functions*, 23(12), 919-938.

Sun, H., Chen, D., Zhang, Y., and Chen, L. (2015). Understanding partial bed-load transport: Experiments and stochastic model analysis. *Journal of Hydrology*, 521, 196-204.

Sun, H., Chen, W., and Chen, Y. (2009). Variable-order fractional differential operators in anomalous diffusion modeling. *Physica A: Statistical Mechanics and its Applications*, 388(21), 4586-4592.

Sweilam, N. H., Nagy, A. M., and El-Sayed, A. A. (2015). Second kind shifted Chebyshev polynomials for solving space fractional order diffusion equation. *Chaos, Solitons and Fractals*, 73, 141-147.

Sweilam, N. H., Nagy, A. M., and El-Sayed, A. A. (2016). On the numerical solution of space fractional order diffusion equation via shifted Chebyshev polynomials of the third kind. *Journal of King Saud University-Science*, 28(1), 41-47.

Sweilam, N. H., and Assiri, T. A. (2015). Numerical scheme for solving the space–time variable order nonlinear fractional wave equation. *Progress in Fractional Differentiation and Applications*, 1, 269-280.

Talib, I., Belgacem, F. B. M., Asif, N. A., and Khalil, H. (2017, January). On mixed derivatives type high dimensional multi-term fractional partial differential equations approximate solutions. In *AIP Conference Proceedings*, 1798(1), AIP Publishing.

Talib, I., Tunc, C., and Noor, Z. A. (2019). New operational matrices of orthogonal Legendre polynomials and their operational. *Journal of Taibah University for Science*, 13(1), 377-389.

Tohidi, E., Bhrawy, A. H., and Erfani, K. (2013). A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. *Applied Mathematical Modelling*, 37(6), 4283-4294.

Toutounian, F., and Tohidi, E. (2013). A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis. *Applied Mathematics and Computation*, 223, 298-310.

Urooj, S., and Singh, S. P. (2017). Wavelet Transform-Based Soft Computational

Techniques and Applications in Medical Imaging. In *Biometrics: Concepts, Methodologies, Tools, and Applications*, 969-993. IGI Global.

Vanani, S. K., and Aminataei, A. (2011). Tau approximate solution of fractional partial differential equations. *Computers and Mathematics with Applications*, 62(3), 1075-1083.

Venkatesh, S. G., Ayyaswamy, S. K., and Balachandar, S. R. (2013). Legendre wavelets based approximation method for solving advection problems. *Ain Shams Engineering Journal*, 4(4), 925-932.

Xu, Q., and Hesthaven, J. S. (2014). Stable multi-domain spectral penalty methods for fractional partial differential equations. *Journal of Computational Physics*, 257, 241-258.

Yang, Y., and Xue, D. (2016). Continuous fractional-order grey model and electricity prediction research based on the observation error feedback. *Energy*, 115, 722-733.

Yang, X. J., Srivastava, H. M., and Cattani, C. (2015a). Local fractional homotopy perturbation method for solving fractal partial differential equations arising in mathematical physics. *Romanian Reports in Physics*, 67(3), 752-761.

Yang, Y., Chen, Y., Huang, Y., and Wei, H. (2017). Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis. *Computers and Mathematics with Applications*, 73(6), 1218-1232.

Yang, Y., Ma, Y., and Wang, L. (2015b). Legendre polynomials operational matrix method for solving fractional partial differential equations with variable coefficients. *Mathematical Problems in Engineering*, *2015*.

Yi, M., Huang, J., and Wei, J. (2013). Block pulse operational matrix method for solving fractional partial differential equation. *Applied Mathematics and Computation*, 221, 121-131.

Zaky, M. A., Ezz-Eldien, S. S., Doha, E. H., Machado, J. T., and Bhrawy, A. H. (2016). An efficient operational matrix technique for multidimensional variable-order time fractional diffusion equations. *Journal of Computational and Nonlinear Dynamics*, 11(6), 061002.

Zhang, J., Wei, Z., and Xiao, L. (2014). A fast adaptive reweighted residual-feedback iterative algorithm for fractional-order total variation regularized multiplicative noise removal of partly-textured images. *Signal processing*, 98, 381-395.

Zhao, F., Huang, Q., Xie, J., Li, Y., Ma, L., and Wang, J. (2017). Chebyshev polynomials approach for numerically solving system of two-dimensional fractional PDEs and convergence analysis. *Applied Mathematics and Computation*, 313, 321-330.

Zhou, F., and Xu, X. (2016). The third kind Chebyshev wavelets collocation method for solving the time-fractional convection diffusion equations with variable coefficients. *Applied Mathematics and Computation*, 280, 11-29.

Zogheib, B., and Tohidi, E. (2016). A new matrix method for solving two-dimensional time-dependent diffusion equations with Dirichlet boundary conditions. *Applied Mathematics and Computation*, 291, 1-13.

Zogheib, B., Tohidi, E., and Shateyi, S. (2017). Bernoulli Collocation Method for Solving Linear Multidimensional Diffusion and Wave Equations with Dirichlet Boundary Conditions. *Advances in Mathematical Physics*.