UNIVERSITY OF
BATH

Citation for published version:
Johnson, M, Truong, T & Wheeler, M 2021, 'Solitary waves in a Whitham equation with small surface tension', *Studies in Applied Mathematics*.

Publication date:
2021

Document Version
Early version, also known as pre-print

Link to publication

University of Bath

Alternative formats
If you require this document in an alternative format, please contact:
openaccess@bath.ac.uk

Solitary waves in a Whitham equation with small surface tension

Mathew A. Johnson[*]   Tien Truong[†]   Miles H. Wheeler[‡]

March 5, 2021

Abstract

Using a nonlocal version of the center manifold theorem and a normal form reduction, we prove the existence of small-amplitude generalized solitary-wave solutions and modulated solitary-wave solutions to the steady gravity-capillary Whitham equation with weak surface tension. Through the application of the center manifold theorem, the nonlocal equation for the solitary wave profiles is reduced to a four-dimensional system of ODEs inheriting reversibility. Along particular parameter curves, relating directly to the classical gravity-capillary water wave problem, the associated linear operator is seen to undergo either a reversible $0^{2+}(ik_0)$ bifurcation or a reversible $(is)^2$ bifurcation. Through a normal form transformation, the reduced system of ODEs along each relevant parameter curve is seen to be well approximated by a truncated system retaining only second-order or third-order terms. These truncated systems relate directly to systems obtained in the study of the full gravity-capillary water wave equation and, as such, the existence of generalized and modulated solitary waves for the truncated systems is guaranteed by classical works, and they are readily seen to persist as solutions of the gravity-capillary Whitham equation due to reversibility. Consequently, this work illuminates further connections between the gravity-capillary Whitham equation and the full two-dimensional gravity-capillary water wave problem.

1   Introduction

In this paper, we consider the existence of small-amplitude solitary-wave solutions of the gravity-capillary Whitham equation

$$u_t + (\mathcal{M}_{g,d,T}u + u^2)_x = 0, \tag{1}$$

where here $\mathcal{M}_{g,d,T}$ is a Fourier-multiplier operator, acting on the spatial variable x, defined via its symbol

$$m_{g,d,T}(\xi) = \left((g + T\xi^2)\frac{\tanh(d\xi)}{\xi}\right)^{1/2}.$$

---

[*]Department of Mathematics, University of Kansas; `matjohn@ku.edu`
[†]Centre of Mathematical Sciences, Lund University; `tien.truong@math.lu.se`
[‡]Department of Mathematical Sciences, University of Bath; `mw2319@bath.ac.uk`

Here, $u(x,t)$ corresponds to the height of the fluid surface at position $x \in \mathbb{R}$ and time t, g is the gravitational constant, d is the undisturbed depth of the fluid, and $T \geq 0$ is the coefficient of surface tension. This symbol is precisely the phase speed for uni-directional waves in the full gravity-capillary water wave problem in [24, 32]. In the absence of surface tension, that is when $T = 0$, equation (1) is referred to as *the gravity Whitham equation*, or simply *the Whitham equation*, and was introduced by Whitham in [31, 32] as a full-dispersion generalization of the standard KdV equation. In the case $T = 0$, the bifurcation and dynamics of both periodic and solitary solutions of (1) have been studied intensively over the last decade by many authors. It has been found that many high-frequency phenomena in water waves, such as breaking, peaking and the famous Benjamin-Feir instability, which do not manifest in the KdV or other shallow water, are indeed manifested the Whitham equation. See for example [5, 9, 10, 12, 13, 18, 19] and references therein.

Given the success of the Whitham equation it is thus natural to consider the existence and dynamics of solutions when additional physical effects are incorporated. In this work, we will concentrate on the existence of solitary-wave solutions[1] of (1) with non-zero surface tension $T > 0$.

It is straightforward to see that the properties of $m_{g,d,T}$ depend on the non-dimensional ratio

$$\tau = \frac{T}{gd^2}, \tag{2}$$

which is referred to as the Bond number. In the full gravity-capillary water wave problem, it is known that the existence of solutions depends sensitively on whether $\tau \in (0, 1/3)$ or $\tau > 1/3$, referred to as the weak- and strong-surface tension cases, respectively. Indeed, in the case of strong surface tension the full gravity-capillary water wave problem admits subcritical[2] solitary waves of depression, i.e. asymptotically constant traveling wave solutions with a unique critical point corresponding to an absolute minimum. See, for instance, [1, 2]. In the small surface tension case, however, considerably less is known about the existence of truly localized (e.g. integrable) solitary waves. It is known, however, that for small surface tension there exist generalized solitary waves, sometimes referred to as *solitary waves with ripples*. These correspond to bounded solutions of (4) which are (roughly) a superposition of a solitary wave and a co-propagating periodic wave with significantly smaller amplitude[3]. See [4, 26–29]. In particular, note that generalized solitary waves are not, in fact, *solitary waves* in the traditional sense since they are not asymptotically constant at $x = \pm\infty$. It is also known that there exist *modulated solitary waves*, which are bounded solutions of (4) with a solitary-wave envelope multiplying a complex exponential. See, for instance, [21]. Illustrations of both generalized and modulated solitary waves can be seen in Figure 1.

Unfortunately, many of the existence proofs described above for the full gravity-capillary water wave problem rely fundamentally on classical dynamical systems techniques, requiring, in particular, that the equation governing the profile of the traveling wave be recast as a first-order system of ordinary differential equations. Such techniques seem at first glance to not be applicable to the gravity-capillary Whitham equation (1) due to the nonlocal operator $\mathcal{M}_{g,d,T}$. However, [14, 15] recently derived a generalization of the classical center-manifold theory that is applicable to a wide class of nonlocal problems, and this was further extended in [30] to an even wider class of nonlocal problems which, as we will show, includes (1). With this in mind, the primary goal of this paper is to use a nonlocal

---

[1]Note that the bifurcation and dynamics of periodic waves has been previously studied in [11, 20, 23].

[2]Here, "subcritical" means the speed of the traveling wave is strictly less than the long-wave speed $m_{g,d,T}(0) = \sqrt{gd}$. If the traveling wave's speed is greater than $\sqrt{gd}$, it is said to be supercritical.

[3]We allow for the possibility of an asymptotic phase shift in the periodic wave between $x = -\infty$ and $x = +\infty$.
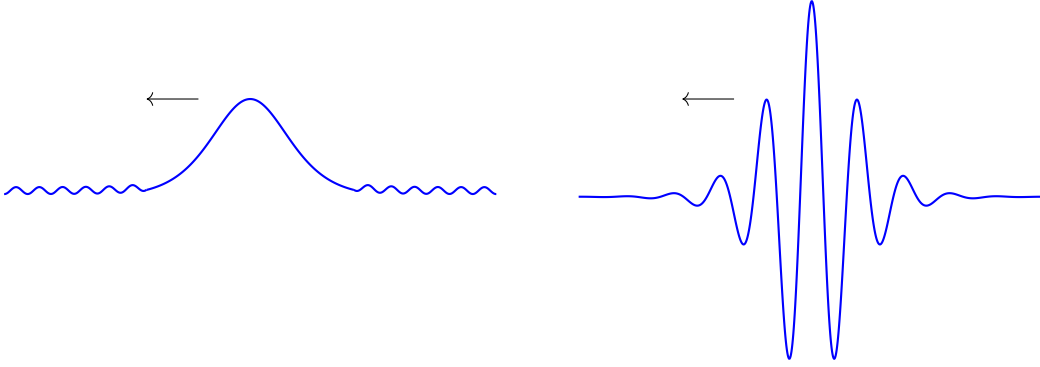
Figure 1: Illustration of a generalized solitary-wave solution (left) and a modulated solitary-wave (of elevation) solution (right). The arrow indicates the direction of travel, that is, $c > 0$.

version of the center manifold theorem and a corresponding normal form reduction to establish the existence of small amplitude generalized solitary and modulated solitary-wave solutions to the gravity-capillary Whitham equation (1) in the small surface tension case. While such solutions were recently shown to exist in [23], this work relies on direct implicit function theorem techniques. Our goal is to attempt to establish similar results using a center-manifold reduction technique.

To begin our search for solitary waves, we note that a straightforward nondimensionalization converts (1) to

$$u_t + \left(\mathcal{M}_\tau u + u^2 \right)_x = 0 \tag{3}$$

where now $\mathcal{M}_\tau$ is a Fourier multiplier with symbol

$$m_\tau(\xi) = \left((1 + \tau \xi^2) \frac{\tanh(\xi)}{\xi} \right)^{1/2},$$

and $\tau > 0$ is the Bond number defined in (2). Making the traveling wave ansatz $u(x, t) = \varphi(x - ct)$ in (3) and integrating yields the (nonlocal) stationary profile equation[4]

$$\mathcal{M}_\tau \varphi - c\varphi + \varphi^2 = 0. \tag{4}$$

The profile equation (4) has received several treatments in recent years and theoretical frameworks for studying them are expanding. Existence results for (4) include periodic waves by Hur & Johnson [20] in 2015 and Ehrnström, Johnson, Maehlen & Remonato [11] in 2019, solitary (e.g. integrable) waves for both strong and weak surface tension by Arnesen [3] in 2016, solitary waves of depression for strong surface tension $\tau > 1/3$ and subcritical wave speed $c < 1$ by Johnson & Wright [23] in 2018, as well as generalized solitary waves for weak surface tension $\tau \in (0, 1/3)$ and supercritical wave speed $c > 1$ also by [23]. Each of these known results use either the implicit function theorem and a Lyapunov-Schmidt reduction or appropriate variational methods.

---

[4]Note that thanks to Galilean invariance, one can without loss of generality take the constant of integration to be zero. See Remark 5.2 for more details.

In this work, we utilize instead an approach based on the recent nonlocal center manifold reduction technique developed by Faye & Scheel [14, 15] and further refined by Truong, Wahlén & Wheeler [30]. As we will see, this set of techniques provides a unified approach for proving existence of both periodic and solitary waves for (4). The nonlocal center manifold theorem bears resemblance to its classical local counterpart, that there exists a neighborhood in a uniform locally Sobolev space where the nonlocal equation is equivalent to a local finite-dimensional system of ODEs. After this reduction, tools for ODEs can be applied to find an approximate solution and then to investigate its persistence. Provided the solution is sufficiently small in the uniform local Sobolev norm it qualifies as a true solution to the original nonlocal equation. So, previously mentioned small-amplitude waves are likely to be included in the center manifold. However, this framework does not fit nonlocal equations from hydrodynamics. To remedy this, Truong, Wahlén & Wheeler [30] have extended this result to a larger class of nonlocal equations. They also demonstrate the strength of this reduction technique and exemplify how to extract qualitative information on the solutions from the reduced ODE, which they use to construct an extreme solitary wave for the gravity Whitham equation. This reduction technique is also available for local quasilinear problems [7].

Remark 1.1. As noted above, the nonlocal center manifold theorem developed in [14, 15] does not directly apply to the profile equation (4). Indeed, one of the hypotheses of Faye & Scheel's result is that both of the functions[5] $\mathcal{F}^{-1}(m_\tau^{-1})$ and $\partial_x \mathcal{F}^{-1}(m_\tau^{-1})$ are integrable and exhibit exponential decay, which are highly non-trivial properties. While the exponential decay and integrability of $\mathcal{F}^{-1}(m_\tau^{-1})$ was recently established in [11], this reference also unfortunately shows that $\partial_x \mathcal{F}^{-1}(m_\tau^{-1})$ is not an integrable function. By carefully considering the methodologies used in [14, 15], Truong, Wahlén and Wheeler were recently able to circumvent this difficulty in [30], where they present a refinement of the result in [14, 15] which does not rely on the integrability[6] of $\mathcal{F}^{-1}(m_\tau^{-1})'$. It is technically this refinement which we use in our analysis. For comparison, we note that such properties of $\mathcal{F}^{-1}(m_\tau^{-1})$ are not needed in the implicit function theorem approach used by Johnson & Wright [23].

We now provide an outline of the paper, as well as state the main results. We begin in Section 2 by inverting the operator $\mathcal{M}_\tau$ in (4), thereby recasting the profile equation into the form studied in [14, 15, 30]. We then study the equation

$$m_\tau(\xi) - c = \left((1 + \tau \xi^2) \frac{\tanh(\xi)}{\xi} \right)^{1/2} - c = 0,$$

which gives solutions to the linearized equation of (4) about the trivial solution $\varphi = 0$. By writing $c^{-2} = \alpha$, $\tau c^{-2} = \beta$ and rearranging the terms, the above equation is recognized as the well-known linear dispersion relation for purely imaginary eigenvalues in two-dimensional capillary-gravity water wave equations, modeling the motion of a perfect unit-density fluid with irrotational flow under the influence of gravity and surface tension in finite depth: see, for example, the works of Kirchgässner [25], Buffoni, Groves & Toland [6], Amick & Kirchgässner [2] and Diass & Iooss [8]. These classical bifurcation curves in the (β, α)-plane naturally guide us in selecting two parameter curves where, restricting ourselves now to

---

[5]Here and throughout, $\mathcal{F}$ denotes the Fourier transform. For the specific normalization used here, see equation (5) below.

[6]In [30], the authors explain the purpose of this hypothesis is to establish the Fredholmness of the linearized operator obtained by linearizing (4) about $\varphi = 0$. Fortunately, this can be checked directly by other means for equations of the form (4), which is one of the achievements of the refinement [30].

the case of small surface tension $\tau \in (0, 1/3)$, we expect generalized solitary-wave and modulated solitary-wave solutions could be found: see Figure 2 below. We further establish a key Fredholm property for the associated linearized operator in Section 3 which is required for the application of the center manifold theorem.

With this preliminary linear analysis completed, we then turn towards applying the nonlocal center manifold theorems from [14, 15, 30] to the profile equation (4). These results, as mentioned before, reduce the nonlocal profile equation considered here to a local ODE near the equilibrium and provide an algorithmic method of approximating the local ODE. We often refer to this local ODE as the reduced ODE. For completeness, we state the general center manifold theorem from [30] in Appendix B, and we apply it in Section 4 along both bifurcation curves in the (β, α)-plane of interest.

In Section 5 we approximate the reduced ODE near the bifurcation curve where generalized solitary-wave solutions are expected to be found as a result of an $0^{2+}(ik_0)$ reversible bifurcation. Up to a standard normal form reduction, rescaling and truncating the nonlinearity, this ends up being almost identical to the normal form equation obtained by Iooss & Kirchgässner in [22] in their analysis of the full gravity-capillary water wave equations. In particular, the truncated reduced ODE in this case admits an explicit family of small amplitude generalized solitary-wave solutions which are then shown to persist as solutions of (4) by a reversibility argument. Putting this all together establishes our first main result.

Theorem 1.2 (Existence of Generalized Solitary Waves) *For each sufficiently small $\mu \in \mathbb{R}$, there exists a family of generalized solitary waves to the gravity-capillary Whitham equation with wave speed $c = 1 + \mu$ and $\tau < 1/3$, given by*

$$\varphi(x) = \frac{3}{2}|\mu|\rho^{1/2}\text{sech}^2\left(\frac{\rho^{1/4}\sigma^{1/2}|\mu|^{1/2}x}{2\sqrt{2}}\right) + \frac{\mu}{2}(1 - \text{sgn}(\mu)\rho^{1/2})$$
$$+ |\mu|k^{1/2}\cos\left((k_0 + \mathcal{O}(\mu))\frac{x}{\sqrt{2}} + \Theta_* + \mathcal{O}(\mu)\right) + \mathcal{O}(\mu^2\rho^{1/2}),$$

where $\Theta_* \in \mathbb{R}/2\pi\mathbb{Z}$ is arbitrary, $\sigma = (1/3 - \tau)^{-1}$, $\rho = 1 + 6(1 + \sinh(k_0)/k_0)k$, $k_0 > 0$ is such that $m_\tau(k_0) = 1$, and $k = \mathcal{O}(|\mu|^{-1-2\kappa})$ for any $\kappa \in [0, 1/2)$.

It is interesting to note that the above allows for an asymptotic phase shift in the cosinus term between $x = -\infty$ and $x = \infty$ of order $\mathcal{O}(\rho^{1/4}|\mu|^{1/2})$. Further, after a Galilean change of variables, all the generalized solitary-wave solutions found above may be seen to have supercritical wave speed $c > 1$: see Remark 5.2 below for details. Note, however, this result does not capture waves whose asymptotic oscillations are exponentially small in relation to the solitary term as in Johnson & Wright [23]. On the other hand, we are able to allow for a more general asymptotic phase shift between $x = \pm\infty$.

In Section 6 we analogously treat the bifurcation curve in the (β, α)-plane where modulated solitary waves are expected to be found as a result of a Hamiltonian-Hopf bifurcation, also known as an $(is)^2$ bifurcation. By computing the necessary center manifold coefficients and performing the appropriate normal form reduction, we again find the results from [21] applicable, thus establishing our second main result.

Theorem 1.3 (Existence of Modulated Solitary Waves) *For sufficiently small $s > 0$ set*

$$c_0^2 = \left(\frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)} \right)^{-1} = 1 - \frac{4}{45}s^4 + \mathcal{O}(s^6)$$

$$\tau_0 = c_0^2 \left(-\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)} \right) = \frac{1}{3} - \frac{2}{45}s^2 + \mathcal{O}(s^4)$$

so that $c_0^{-2}(1 + \tau_0)\sinh(s) = s\cosh(s)$. Then, for $\mu < 0$ sufficiently small, there exist two distinct modulated solitary-wave solutions to the gravity-capillary Whitham equation with amplitude of order $\mathcal{O}(|\mu|^{1/2})$, surface tension τ_0 and subcritical wave speed $c_0 + \mu < 1$. More precisely, the modulated solitary-wave solutions are described asymptotically via

$$\varphi(x) = \sqrt{\frac{-8q_0\mu}{q_1}} \operatorname{sech}(\sqrt{q_0\mu}\,x) \cos\left(st + \Theta_* + \mathcal{O}(|\mu|^{1/2}) \right) + \mathcal{O}(\mu^2), \quad \Theta_* \in \{0, \pi\}.$$

Here, $q_0 = -12 + \mathcal{O}(s^2)$, $q_1 = -32400s^{-8} + \mathcal{O}(s^{-6})$ and the solutions have an asymptotic phase shift between $x = -\infty$ and $x = \infty$ of order $\mathcal{O}(|\mu|^{1/2})$.

Remark 1.4. The restriction on s above, and hence the requirement that $0 < 1/3 - \tau_0 \ll 1$, comes from the need to determine the signs of the parameters q_0 and q_1. These coefficients arise in the forthcoming normal form calculations: see equations (35) and (36) in Section 6 below. Theorem 1.3 would continue to hold for all s such that the parameters q_0 and q_1 are both negative. While we can only analytically verify this for s sufficiently small, see Appendix D.2, calculations in Maple suggest that one has $q_0 < 0$ and $q_1 < 0$ valid for all $s > 0$. Analytically verifying this would establish the existence of such modulated solitary waves for all $\tau \in (0, 1/3)$.

The solutions constructed in Theorem 1.3 correspond to (distinct) modulated solitary waves of elevation ($\Theta_* = 0$) and depression ($\Theta_* = \pi$). Note that Figure 1 depicts a modulated solitary-wave solution of elevation. Further, even though the above theorem is for sufficiently small $s > 0$, calculations in Maple suggest that it is true for all $s > 0$.

Remark 1.5. As mentioned above, the center manifold methodology used here provides a unified approach for proving existence of both periodic and solitary waves for (4). Consequently, one could continue the above line of investigation to establish the existence of other classes of solutions as well including, for instance, the subcritical, small amplitude solitary waves of depression in the case of large surface tension $\tau > 1/3$ constructed in [23]. This specific case is very similar to the gravity Whitham equation studied by Truong, Wahlén and Wheeler [30] and is thus excluded here.

This paper provides further connections between the model equation (1) and the two-dimensional gravity-capillary water wave problem. It also exemplifies the application of nonlocal center manifold reduction in existence theory. A natural continuation of this paper could be to investigate the existence of multipulse modulated solitary-wave solutions, as well as bifurcation phenomena in other parameter regions.

Notation

The following notation will be used throughout this work.

– For $\sigma \in \mathbb{R}$, we define the *σ-weighted L^p spaces*

$$L^p_\sigma := \left\{ f\colon \mathbb{R} \to \mathbb{R} \ \Big| \ \int_{\mathbb{R}} |f|^p \omega_\sigma^p \, \mathrm{d}x \right\}.$$

Here the *weight function* $\omega_\sigma : \mathbb{R} \to \mathbb{R}$ is positive and smooth. Also, ω_σ is constantly 1 on $[-1,1]$ and equals $\exp(\sigma|x|)$ for $|x| \geq 2$.

– Similarly, we define the *weighted Sobolev spaces*

$$W_\sigma^{m,p} := \left\{ f \colon \mathbb{R} \to \mathbb{R} \,\middle|\, f^{(n)} \in L_\sigma^p, \text{ for } 0 \leq n \leq m \right\}.$$

We have the natural inclusions $W_{\sigma_2}^{m,p} \subset W_{\sigma_1}^{m,p}$ whenever $\sigma_1 < \sigma_2$. For $p = 2$, we denote the Hilbert space $W_\sigma^{m,2}$ by H_σ^m.

– The non-weighted Sobolev spaces are denoted by $W^{m,p}$ and $W^{m,2}$ by H^m.

– The uniform locally H^m space

$$H_{\mathrm{u}}^m := \left\{ f \colon \mathbb{R} \to \mathbb{R} \,\middle|\, \|f\|_{H_{\mathrm{u}}^m} < \infty \right\} \quad \text{with} \quad \|f\|_{H_{\mathrm{u}}^m} := \sup_{y \in \mathbb{R}} \|f(\,\cdot\, + y)\|_{H^m([0,1])}.$$

– We use the following scaling of the Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} f(x) \exp(-\mathrm{i}x\xi) \,\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}g(x) = \frac{1}{2\pi} \mathcal{F}g(-x). \quad (5)$$

2 The operator equation

In this section, we begin our study of the nonlocal profile equation (4). Observe that since m_τ is strictly positive on $\mathbb{R}$, the operator $\mathcal{M}_\tau$ is invertible on any Fourier based space. We denote the inverse of $\mathcal{M}_\tau$ by $\mathcal{L}_\tau$, defined via

$$\widehat{\mathcal{L}_\tau f}(\xi) = \ell_\tau(\xi)\hat{f}(\xi), \quad \ell_\tau(\xi) := m_\tau(\xi)^{-1}.$$

In particular, the profile equation (4) can be written in the "smoothing" form

$$\varphi - cK_\tau * \varphi + K_\tau * \varphi^2 = 0, \quad (6)$$

where here $K_\tau := \mathcal{F}^{-1}\ell_\tau$ denotes the convolution kernel corresponding to the operator $\mathcal{L}_\tau$. Observe that (6) is similar to the profile equation for the gravity Whitham equation (i.e. (1) with $T = 0$), but now with a *nonlocal* nonlinearity.

As we seek small amplitude solutions of (6), we begin by linearizing (6) about $\varphi = 0$ which, after applying the Fourier transform, yields the equation

$$(1 - c\ell_\tau(\xi))\hat{v}(\xi) = 0,$$

which we seek to solve for non-trivial $v \in L^2(\mathbb{R})$. This motivates considering the equation

$$1 - c\ell_\tau(\xi) = 0, \quad \text{i.e.} \quad \xi \cosh(\xi) = \left(\frac{1}{c^2} + \frac{\tau}{c^2}\xi^2 \right) \sinh(\xi). \quad (7)$$

By setting

$$\alpha = \frac{1}{c^2} \quad \text{and} \quad \beta = \frac{\tau}{c^2},$$

equation (7) is recognized as the well-known linear dispersion relation for purely imaginary eigenvalues in the two-dimensional water wave equations in finite depth: see, for example, [24, 32]. The importance of (7) in finding solutions to the gravity-capillary water wave equations was recognized by Kirchgässner [25], followed by a multitude of other papers (see e.g. [2, 6, 8, 22]). Looking at the bifurcation curves in the (β, α)-planes for the classical gravity-capillary water wave problem, it is natural to expect the following:

- that modulated solitary-wave solutions may be found as a result of a Hamiltonian–Hopf bifurcation, when crossing the curve

$$C_2 = \left\{ (\beta, \alpha) = \left(-\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)}, \frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)} \right) : s \in [0, \infty) \right\}$$

from below;

- that generalized solitary-wave solutions may be found as a result of an $0^{2+}(ik_0)$ bifurcation, when crossing the curve

$$C_3 = \left\{ (\beta, \alpha) \colon \beta \leq \frac{1}{3}, \alpha = 1 \right\}.$$

either from above or below. Here, $k_0 \in \mathbb{R}$ satisfies equation (7) for a fixed β along C_3;

- and that solitary-wave solutions of depression to be found as a result of an 0^{2+} bifurcation, when crossing the curve

$$C_4 = \left\{ (\beta, \alpha) \colon \beta \geq \frac{1}{3}, \alpha = 1 \right\}$$

from above.

There is an additional curve C_1 in the (β, α)-plane along which one may expect the existence of multi-pulse solitary waves [6]. Since we do not consider these waves in this work, we will not comment further regarding C_1. See Figure 2 for depictions of the curves C_1, C_2, C_3 and C_4 in the (β, α)-plane. It is illustrative to understand what these curves mean in terms of the physical (τ, c) parameters. For example, crossing $(\beta_0, \alpha_0) = (\beta_0, 1) \in C_3$ corresponds to studying (6) for $\beta = \beta_0$ and $\alpha = \alpha_0 - \tilde{\mu} = 1 - \tilde{\mu}$ for some $|\tilde{\mu}| \ll 1$ which, in terms of τ and c, gives

$$\tau = \beta_0 c_0^2 < \frac{1}{3} \quad \text{and} \quad c = c_0 + \mu \quad \text{with} \quad c_0 = 1 \quad \text{and} \quad |\mu| \ll 1. \tag{8}$$

Thus, crossing the curve C_3 correspond to waves with weak surface tension, while the wave speed is nearly critical. Likewise, crossing a point $(\beta_0, \alpha_0) \in C_2$ from below correspond to studying (6) with $\beta = \beta_0$ and $\alpha = \alpha_0 + \tilde{\mu}$ with $0 < \tilde{\mu} \ll 1$ which, in terms of τ and c gives

$$\tau = \beta_0 c_0^2 \quad \text{and} \quad c = c_0 + \mu, \quad \text{with} \quad c_0 = \alpha_0^{-1/2} \quad \text{and} \quad 0 < -\mu \ll 1. \tag{9}$$

Since $\alpha_0 > 1$, it follows that $\tau \in (0, 1/3)$, i.e. the surface tension is again weak, and the speed c is subcritical. By similar reasoning, crossing the curve C_4 from above correspond to strong surface tension, i.e. $\tau > 1/3$, and subcritical speeds. Note that in this work, we focus only on bifurcation phenomena connected to the curves C_2 and C_3. The 0^{2+} bifurcation along C_4, as one might expect, resembles the one already covered in [30] for the gravity-Whitham equation and is thus excluded here.

As mentioned previously, we approach the above bifurcation phenomena for the nonlocal profile equation (6) by following the center manifold reduction strategy in [14, 15, 30]. To this end, we rewrite (6) as

$$\mathcal{T}\varphi + \mathcal{N}(\varphi, \mu) = 0, \tag{10}$$

which is now of the structural form studied in Faye & Scheel [14, 15], where here

$$\mathcal{T} \colon \varphi \mapsto \varphi - c_0 K_\tau * \varphi \quad \text{and} \quad \mathcal{N} \colon (\varphi, \mu) \mapsto K_\tau * (\varphi^2 - \mu\varphi).$$

Since τ is fixed, the subscript τ will be dropped for notational convenience. Our goal is to study the operator equation (10) for parameters τ, c_0 corresponding to (8), corresponding to crossing C_3, as well as for τ, c_0 satisfying (9) corresponding to crossing C_2. The first step of this analysis is to understand the linear operator $\mathcal{T}$, which we now turn to studying.
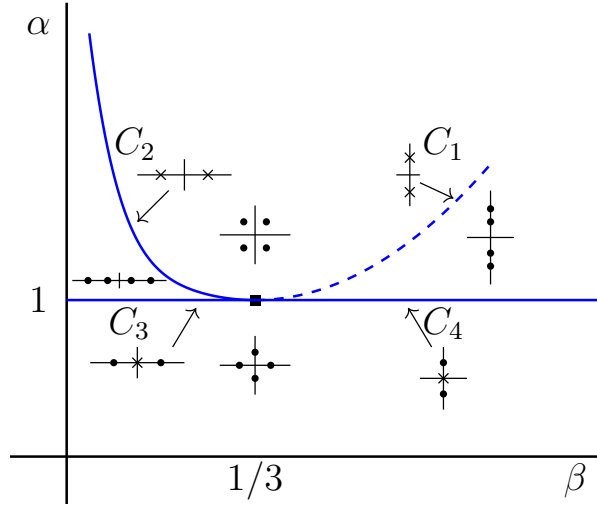
Figure 2: Sketch of the bifurcation curves C_1, C_2, C_3 and C_4 along with zeros of the function $1 - c\ell_\tau(\xi) = (\alpha + \beta\xi^2)\sinh(\xi) - \xi\cosh(\xi)$. Here, dots and crosses represent algebraic multiplicity one and two, respectively.

3 The linear operator $\mathcal{T}$

Fix $\tau \in (0, 1/3)$. As preparation for our forthcoming bifurcation analysis, and following the general strategy in [14, 15, 30], in this section we study the linear operator[7]

$$\mathcal{T} : \varphi \mapsto \varphi - c_0 K * \varphi, \quad H^5_{-\eta} \to H^5_{-\eta},$$

for $\eta > 0$ along the two parameter curves (8) and (9). Note that $\mathcal{T}$ is precisely the linearization of (10) about the trivial solution $\varphi = 0$ and, as such, it is crucial to understand the Fredholm and invertibility properties of $\mathcal{T}$ along the curves C_2 and C_3. Key to this analysis is an understanding of convolution kernel K. The relevant properties are detailed in the following result.

Proposition 3.1 *The convolution kernel K is even. Moreover, we have*

(i) the singularity of K as $|x| \to 0$ is

$$\lim_{x \to 0} \sqrt{|x|}K(x) = \frac{1}{\sqrt{2\pi\tau}},$$

(ii) K has exponential decay as $|x| \to \infty$, that is

$$|K(x)| \lesssim \exp(-\eta|x|) \quad for \quad |x| > 1,$$

where $0 < \eta < \eta^* := \min\{1/\sqrt{\tau}, \pi/2\}$.

For a proof, see Ehrnström, Johnson, Maehlen & Remonato [11, Theorem 2.7]. An immediate consequence is that $K \in L^1_{-\eta}$ for $\eta \in (0, \eta^*)$ and hence, by a straightforward application of Young's inequality, that for such η the linear operator

$$\mathcal{T} \colon H^5_{-\eta} \to H^5_{-\eta}$$

---

[7]Recall that since τ is fixed, for convenience the corresponding subscript will be dropped from K_τ and ℓ_τ.

is bounded regardless of the choice of c_0. Note that the works [14, 15] additionally require $K' \in L^1_{-\eta}$ which, by Proposition 3.1, does not hold in this case. Next, we follow Truong, Wahlén & Wheeler [30] and study the Fredholm properties of $\mathcal{T}$ using theory for pseudodifferential operators in non-weighted Sobolev spaces H^5 from Grushin [16] (see also Appendix A).

To this end, we fix $\eta \in (0, \eta^*)$ and consider the conjugated operator

$$\tilde{\mathcal{T}} := M^{-1} \circ \mathcal{T} \circ M, \quad H^5 \to H^5,$$

where $M : H^5 \to H^5_{-\eta}$ is multiplication with the strictly positive function $\cosh(\eta \cdot)$. Noting that conjugation by M preserves Fredholmness and the Fredholm index, we may establish the desired Fredholm properties of $\mathcal{T} = M \circ \tilde{\mathcal{T}} \circ M^{-1}$ acting on the weighted space $H^5_{-\eta}$ by studying the operator $\tilde{\mathcal{T}}$ acting on the non-weighted H^5. These latter properties are established by following the work [16], where the author relates to the pseudodifferential operator $\tilde{\mathcal{T}}$ acting on H^5 to a positively homogeneous function A and determining the winding number of A around the origin. The relevant details are summarized in Appendix A.

By direct calculation, the symbol of $\tilde{\mathcal{T}}$ is seen to be

$$\tilde{t}(x, \xi) = 1 - c_0 \phi_+(x) \ell(\xi - i\eta) - c_0 \phi_-(x) \ell(\xi + i\eta),$$

where $\phi_\pm(x) = \exp(\pm \eta x)/(2 \cosh(\eta x))$. In particular, note that

$$\begin{cases} \lim_{x \to \infty} \phi_+(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} \phi_+(x) = 0 \\ \lim_{x \to \infty} \phi_-(x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} \phi_-(x) = 1. \end{cases} \tag{11}$$

Following [16], we define the positive, homogeneous degree-zero function

$$A(x_0, x, \xi_0, \xi) := \tilde{t}\left(\frac{x}{x_0}, \frac{\xi}{\xi_0}\right)$$

for $x, \xi \in \mathbb{R}$ and $x_0, \xi_0 > 0$ and study A acting on[8]

$$\overline{\mathbb{S}^1_+} \times \overline{\mathbb{S}^1_+} := \left\{ (x_0, x, \xi_0, \xi) \in \mathbb{R}^4 : x_0^2 + x^2 = \xi_0^2 + \xi^2 = 1, \ x_0 \geq 0, \ \xi_0 \geq 0 \right\}.$$

According to Proposition A.1, the linear opertaor $\tilde{\mathcal{T}}$ is Fredholm provided that the function A is smooth in $\overline{\mathbb{S}^1_+} \times \overline{\mathbb{S}^1_+}$ and nowhere vanishing along the boundary Γ of $\overline{\mathbb{S}^1_+} \times \overline{\mathbb{S}^1_+}$, which can be decomposed into the arcs

$$\begin{aligned} \Gamma_1 &= \{ \ (0, 1, \xi_0, \xi) \mid \xi_0^2 + \xi^2 = 1, \xi_0 \geq 0 \} \\ \Gamma_2 &= \{ (0, -1, \xi_0, \xi) \mid \xi_0^2 + \xi^2 = 1, \xi_0 \geq 0 \} \\ \Gamma_3 &= \{ \ (x_0, x, 0, 1) \mid x_0^2 + x^2 = 1, x_0 \geq 0 \} \\ \Gamma_4 &= \{ (x_0, x, 0, -1) \mid x_0^2 + x^2 = 1, x_0 \geq 0 \}. \end{aligned}$$

Further, the Fredholm index of $\tilde{\mathcal{T}}$ is precisely the winding number of A as Γ is transversed in the counter-clockwise direction, that is

$$\begin{array}{ccc} (0, 1, 0, -1) & \xleftarrow{\;\;\Gamma_1\;\;} & (0, 1, 0, 1) \\ \downarrow{\scriptstyle \Gamma_3} & & {\scriptstyle \Gamma_4}\uparrow \\ (0, -1, 0, -1) & \xrightarrow{\;\Gamma_2\;} & (0, -1, 0, 1). \end{array}$$

---

[8]Note A may be extended by continuity down to $x_0 = 0$ and $\xi_0 = 0$.
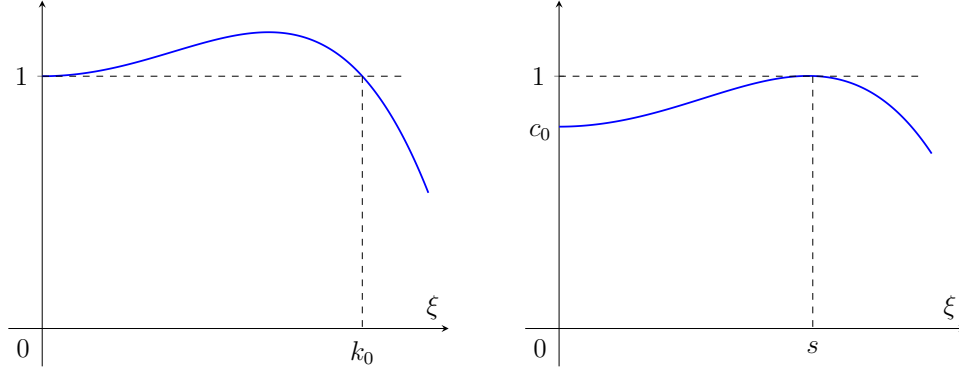
Figure 3: The multiplier $c_0\ell$ for (8) (left) and for (9) (right).

As such, it is important to locate the roots of the function $1 - c_0\ell$ when the parameters τ, c_0 correspond to crossing the bifurcation curves C_1 and C_2. This motivates the following Lemma.

Lemma 3.2 *The multiplier $\ell\colon \mathbb{C} \to \mathbb{C}$ is analytic in the complex strip $|\mathrm{Im}\, z| < \eta^*$, with η_* as in Proposition 3.1. Moreover, there exists a possibly smaller strip $|\mathrm{Im}\, z| < \tilde{\eta}$ in which the function $1 - c_0\ell\colon \mathbb{C} \to \mathbb{C}$ has precisely the zeros*

(i) $k_0, -k_0, 0$, and 0 counting multiplicities for some $k_0 > 0$, when τ_0 and c_0 are as in (8),

(ii) $s, s, -s$ and $-s$ counting multiplicities for some $s > 0$, when τ and c_0 are as in (9).

See Figure 3.

Proof. See Corollary 2.2 in [11] for the analyticity of ℓ. Item (i) can be found as Lemma 2 in [2] and item (ii) can be found in Section IV in [25]. □

With this preliminary result, we are now ready to prove the main result of this section.

Theorem 3.3 *For each $\eta \in (0, \min\{\eta^*, \tilde{\eta}\})$ and choice of parameters τ, c_0 satisfying either (8) or (9), the linear operator $\mathcal{T} : H^5_{-\eta} \to H^5_{-\eta}$ is Fredholm with Fredholm index four. For each set of parameters, its nullspace $\mathrm{Ker}\,\mathcal{T}$ is four-dimensional, given by*

$$\mathrm{Ker}\,\mathcal{T} = \mathrm{span}\{1, x, \cos(k_0 x), \sin(k_0 x)\} \tag{12}$$

if τ, c_0 satisfy (8), and

$$\mathrm{Ker}\,\mathcal{T} = \mathrm{span}\{\cos(sx), x\cos(sx), \sin(sx), x\sin(sx)\} \tag{13}$$

if τ, c_0 satisfy (9).

Proof. Let $\eta \in (0, \min\{\eta^*, \tilde{\eta}\})$ be fixed. Following the outline above, we first verify the Fredholmness of $\tilde{\mathcal{T}}$ by showing that A is non-vanishing on Γ. To this end, recall that

$$\ell(z) = \left(\frac{1}{1 + \tau z^2} \frac{z}{\tanh(z)}\right)^{1/2}, \quad z \in \mathbb{C}.$$

Along Γ_1 we have $\xi_0 = \sqrt{1 - \xi^2}$ and hence for $(0, 1, \xi_0, \xi) \in \Gamma_1$ with $\xi_0 \neq 0$, i.e. $\xi \neq \pm 1$, we have, recalling (11),

$$A(0, 1, \xi_0, \xi) = \lim_{x_0 \to 0^+} \tilde{t}\left(\frac{1}{x_0}, \frac{\xi}{\sqrt{1 - \xi^2}}\right) = 1 - c_0\ell\left(\frac{\xi}{\sqrt{1 - \xi^2}} - i\eta\right).$$
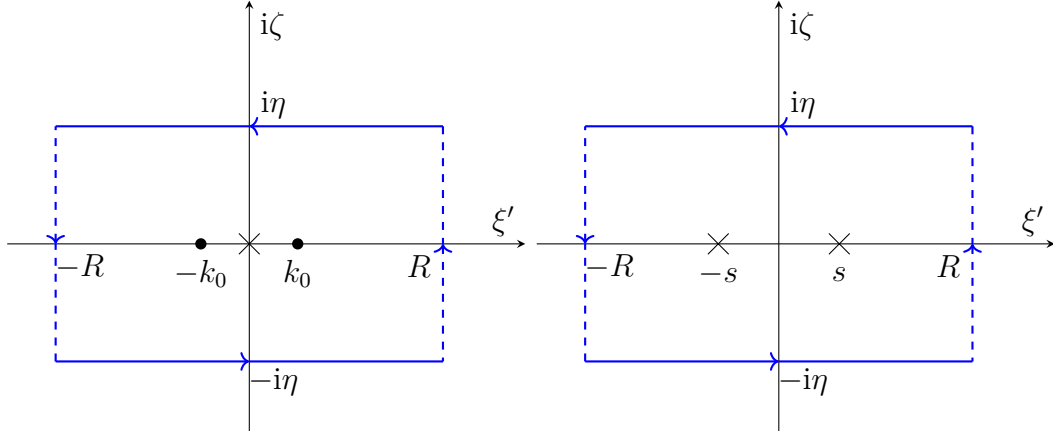
11

Figure 4: The rectangular contour Γ_R in Theorem 3.3. It consists of the arcs $\{\xi' \pm i\eta : |\xi'| \leq R\}$ and $\{\pm R + i\zeta : |\zeta| \leq \eta\}$. The left picture illustrates the case (8) and the right illustrates (9). Here, dots and crosses represent zeroes of $1 - c_0 \ell(\xi)$ with multiplicity one and two, respectively.

To evaluate at the end points $(0, 1, 0, 1)$ and $(0, 1, 0, -1)$, it is equivalent to compute the limit of $\ell(\xi' - i\eta)$ as $\xi' \to \infty$ and $\xi' \to -\infty$, respectively. A calculation gives

$$|\ell(\xi' \pm i\eta)|^4 = \frac{4(\xi')^2 + 4\eta^2}{(1 + \tau(\xi')^2 \mp \tau\eta^2)^2 + (2\tau\xi'\eta)^2} \cdot \frac{(\cosh^2(\xi') - 1 + \cos^2(\eta))^2}{\cosh^2(2\xi') - \cos^2(2\eta)},$$

which implies that $|\ell(\xi' \pm i\eta)| \to 0$ as $|\xi'| \to \infty$. Consequently, $A(0, 1, 0, \pm 1) = 1$. By similar calculations, we find

$$A(x_0, x, \xi_0, \xi) = \begin{cases} 1 - c_0 \ell\left(\frac{\xi}{\sqrt{1-\xi^2}} - i\eta\right), & \text{on } \Gamma_1 \setminus \{(0, 1, 0, \pm 1)\} \\ 1 - c_0 \ell\left(\frac{\xi}{\sqrt{1-\xi^2}} + i\eta\right), & \text{on } \Gamma_2 \setminus \{(0, -1, 0, \pm 1)\} \\ 1, & \text{on } \Gamma_3 \cup \Gamma_4. \end{cases}$$

In view of Lemma 3.2, A is smooth and nowhere vanishing on $\Gamma = \cup_{j=1}^4 \Gamma_j$ and hence Proposition A.1 implies that $\tilde{\mathcal{T}}$ is a Fredholm operator, as desired.

Next, we compute the Fredholm index of the operator $\tilde{\mathcal{T}}$ by computing the winding number of A along Γ transversed in the counter-clockwise direction (as described above). Setting $\xi' = \xi(1 - \xi^2)^{-1/2}$ for $|\xi| < 1$ we see that traversing from $(0, 1, 0, 1)$ to $(0, 1, 0, -1)$ along Γ_1 corresponds to considering $1 - c_0 \ell(\xi' - i\eta)$ as ξ' varies from $\xi' = -\infty$ to $\xi' = \infty$. while traversing from $(0, -1, 0, -1)$ to $(0, -1, 0, 1)$ along Γ_2 corresponds to considering $1 - c_0 \ell(\xi' + i\eta)$ as ξ' varies from $\xi' = \infty$ to $\xi' = -\infty$. Further, since A is constant along Γ_3 and Γ_4, traversing along these arcs does not contribute to the winding number of A along Γ. To compute the winding number, choose τ, c_0 to satisfy either (8) or (9). Let $R > 0$ be strictly larger than the corresponding values $k_0 > 0$ or $s > 0$ from Lemma 3.2, and consider the rectangular contour Γ_R with vertices at $(\pm R, \pm i\eta)$: see Figure 4. The winding number of A along Γ is in fact limit of the winding number of $1 - c_0 \ell$ along Γ_R as $R \to \infty$. The

latter is computed via

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{c_0 \ell'(z)}{1 - c_0 \ell(z)} \, dz = \frac{1}{2\pi i} \Big(- \int_{-R}^{R} \frac{c_0 \ell'(\xi' + i\eta)}{1 - c_0 \ell(\xi' + i\eta)} \, d\xi' + \int_{-R}^{R} \frac{c_0 \ell'(\xi' - i\eta)}{1 - c_0 \ell(\xi' - i\eta)} \, d\xi'$$
$$+ \int_{-\eta}^{\eta} \frac{c_0 \ell'(R + i\zeta)}{1 - c_0 \ell(R + i\zeta)} \, d\zeta - \int_{-\eta}^{\eta} \frac{c_0 \ell'(-R + i\zeta)}{1 - c_0 \ell(-R + i\zeta)} \, d\zeta \Big).$$

Since $1 - c_0 \ell$ is analytic, it follows by Lemma 3.2 and the residue theorem that the above integrals sum to exactly four. Noting that the last two integrals vanish as $R \to \infty$ since $|\ell(\pm R + i\zeta)| \to 0$ and $|\ell'(\pm R + i\zeta)| \to 0$ as $R \to \infty$ and $|\zeta| \leq \eta$, the winding number of A around Γ is four. Proposition A.1 now gives that the Fredholm index of $\tilde{\mathcal{T}}$, and hence that of $\mathcal{T}$, is indeed four.

Finally, it remains to characterize the kernel of $\mathcal{T}$ acting on $H^5_{-\eta}$ when τ and c_0 satisfy either (8) or (9). To this end, we argue along the same lines as [30, Proposition 2.9] and consider instead[9] the range of $\mathcal{T} \colon L^2_\eta \to L^2_\eta$, which is the adjoint of $\mathcal{T} \colon L^2_{-\eta} \to L^2_{-\eta}$ under the L^2-pairing. The Fourier transform of the range equation $\mathcal{T} f = g$ in L^2_η is precisely

$$(1 - c_0 \ell(\xi)) \mathcal{F} f(\xi) = \mathcal{F} g(\xi).$$

In view of Lemma 3.2, it follows that the range of $\mathcal{T}$ on L^2_η consists of functions g whose Fourier transforms vanish on the zero set of $1 - c_0 \ell(\xi)$. If τ, c_0 satisfy (8), it follows from Lemma 3.2 that the range of $\mathcal{T}$ acting on L^2_η consists of functions g that satisfy

$$\mathcal{F} g(0) = (\mathcal{F} g)'(0) = \mathcal{F} g(\pm k_0) = 0$$

or, equivalently,

$$\int_{\mathbb{R}} 1 \cdot g(x) \, dx = \int_{\mathbb{R}} x \cdot g(x) \, dx = \int_{\mathbb{R}} \exp(\mp i k_0 x) g(x) \, dx = 0.$$

By duality, such τ, c_0 the kernel of $\mathcal{T}$ acting on $L^2_{-\eta}$ is given by (12), which clearly also belongs to $H^5_{-\eta}$.

Similarly, if τ, c_0 satisfy (9) then Lemma 3.2 implies that the kernel of $\mathcal{T}$ acting on L^2_η consists of functions g satisfying

$$\mathcal{F} g(\pm s) = (\mathcal{F} g)'(\pm s) = 0$$

or, equivalently,

$$\int_{\mathbb{R}} \exp(\pm i s x) g(x) \, dx = \int_{\mathbb{R}} x \cdot \exp(\pm i s x) g(x) \, dx = 0.$$

Again by duality, this shows that the kernel of $\mathcal{T}$ on $L^2_{-\eta}$ is given by (13), which again belongs to $H^5_{-\eta}$. □

4 Center manifold reduction

We use a nonlocal center manifold theorem, originally introduced in [14] and later adapted in [30] to account for the non-integrability of K'. For completeness, the general result used

---

[9]As noted in [30], this is due to the fact that the equation $\mathcal{T} f = 0$ with $f \in L^2_{-\eta}$ can not be studied directly by the Fourier transform since the Fourier transform of such f is not a tempered distribution.

here is recorded in Appendix B. In this section, we apply this general result to the nonlocal profile equation (10) together with the modified equation

$$\mathcal{T}\varphi + \mathcal{N}^{\delta}(\varphi, \mu) = 0, \tag{14}$$

where

$$\mathcal{N}^{\delta}(\varphi, \mu) = \mathcal{N}(\chi^{\delta}(\varphi), \mu)$$

and $\chi^{\delta} : H^5_{-\eta} \to H^5_u$ is the nonlocal and translationally invariant cutoff operator defined in (38) in Appendix B. In particular, χ^{δ} maps $\varphi \in H^5_{-\eta}$ to a ball of radius $C\delta$ in H^5_u, the space of uniform locally H^5 functions, with norm

$$\|\varphi\|_{H^5_u} = \sup_{t \in \mathbb{R}} \|\varphi(\cdot + t)\|_{H^5([0,1])}.$$

More precisely, there exists a constant $C > 0$ such that

$$\chi^{\delta}(\varphi) = \begin{cases} \varphi & \text{if} \quad \|\varphi\|_{H^5_u} \leq C\delta \\ 0 & \text{if} \quad \|\varphi\|_{H^5_u} \quad \text{is sufficiently large} \end{cases}$$

and hence for $\|\varphi\|_{H^5_u} \leq C\delta$ we have $\mathcal{N}^{\delta}(\varphi, \mu) = \mathcal{N}(\varphi, \mu)$. Then, such small solutions of (14) are also solutions of the original profile equation (10). Furthermore, note that since H^5_u is continuously embedded in $H^5_{-\eta}$ for all $\eta > 0$, the operator χ^{δ} also serves as a cutoff in the $H^5_{-\eta}$ norm as well. For more details, see Appendix B.

A central ingredient of the center manifold reduction is the construction of a bounded projection $\mathcal{Q} : H^5_{-\eta} \to H^5_{-\eta}$ onto $\text{Ker}\,\mathcal{T}$, which could be any bounded projection having a continuous extension to $H^4_{-\eta}$ and commuting with the inclusion map from $H^5_{-\eta}$ to $H^5_{-\eta'}$ for all $0 < \eta' < \eta$. Since the nonlocal profile equation (10) is invariant under all spatial translations, a specific choice of $\mathcal{Q}$ simplifies the computations significantly. Indeed, from Theorem 3.3 we know $\text{Ker}\,\mathcal{T}$ has dimension four and hence, keeping generality for the moment, we may take

$$\text{Ker}\,\mathcal{T} = \text{span}\,\{e_1, e_2, e_3, e_4\}$$

for appropriately chosen, linearly independent functions e_j. Follow the recommendation in [14], we aim to choose a projection $\mathcal{Q} \colon H^5_{-\eta} \to \text{Ker}\,\mathcal{T}$

$$\mathcal{Q} \colon \varphi \mapsto A e_1 + B e_2 + C e_3 + D e_4,$$

which relates the coefficients A, B, C and D to $\varphi(0), \varphi'(0), \varphi''(0)$ and $\varphi'''(0)$ via a transition matrix $\mathcal{T}$

$$\mathcal{T} \colon (\varphi(0), \varphi'(0), \varphi''(0), \varphi'''(0)) \mapsto (A, B, C, D).$$

Using that $\mathcal{Q}\varphi = \mathcal{Q}^2\varphi$, a straightforward computation yields

$$\mathcal{T} = \begin{pmatrix} e_1(0) & e_2(0) & e_3(0) & e_4(0) \\ e_1'(0) & e_2'(0) & e_3'(0) & e_4'(0) \\ e_1''(0) & e_2''(0) & e_3''(0) & e_4''(0) \\ e_1'''(0) & e_2'''(0) & e_3'''(0) & e_4'''(0) \end{pmatrix}^{-1}.$$

When the parameters τ, c_0 satisfy (8), $\text{Ker}\,\mathcal{T} = \text{span}\{1, x, \cos(k_0 x), \sin(k_0 x)\}$ according to Theorem 3.3 and the transition matrix with respect to these basis functions is

$$\mathcal{T}_1 = \begin{pmatrix} 1 & 0 & k_0^{-2} & 0 \\ 0 & 1 & 0 & k_0^{-2} \\ 0 & 0 & -k_0^{-2} & 0 \\ 0 & 0 & 0 & -k_0^{-3} \end{pmatrix}, \tag{15}$$

14

which gives the explicit choice

$$\mathcal{Q}_1 \colon \varphi \mapsto \left(\varphi(0) + k_0^{-2} \varphi''(0) \right) + \left(\varphi'(0) + k_0^{-2} \varphi'''(0) \right) x$$
$$- k_0^{-2} \varphi''(0) \cos(k_0 x) - k_0^{-3} \varphi'''(0) \sin(k_0 x). \tag{16}$$

Similarly, when τ, c_0 satisfy (9) we have $\operatorname{Ker} \mathcal{T} = \operatorname{span}\{\cos(sx), x\cos(sx), \sin(sx), x\sin(sx)\}$, and the transition matrix with respect to these basis functions is

$$\mathcal{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3(2s)^{-1} & 0 & (2s^3)^{-1} \\ 0 & -1/2 & 0 & -(2s^2)^{-1} \\ s/2 & 0 & (2s)^{-1} & 0 \end{pmatrix}, \tag{17}$$

which gives the explicit choice

$$\mathcal{Q}_2 \colon \varphi \mapsto \varphi(0) \cos(sx) + \left(\frac{3}{2s} \varphi'(0) + \frac{1}{2s^3} \varphi'''(0) \right) x \cos(sx)$$
$$- \left(\frac{1}{2} \varphi'(0) + \frac{1}{2s^2} \varphi'''(0) \right) \sin(sx) + \left(\frac{s}{2} \varphi(0) + \frac{1}{2s} \varphi''(0) \right) x \sin(sx). \tag{18}$$

Remark 4.1. Our analysis up until this point holds in any space $H^m_{-\eta}$ with $m \geq 1$ and the choice of space $H^5_{-\eta}$ is made here. The projections $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are required to have a continuous extension to $H^{m-1}_{-\eta}$. Because these involve pointwise evaluation of φ''', we need at least $m - 1 = 4$ which explains the choice $H^5_{-\eta}$.

Lastly, the shift operator $\varphi \mapsto \varphi(\cdot + t)$ will be denoted by S_t. We are in position to apply the nonlocal center manifold theorem to equation (10).

Theorem 4.2 *There exist a neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$, a cutoff radius $\delta > 0$, a weight $\eta < \min\{\eta^*, \tilde{\eta}\}$ and a map*

$$\Psi \colon \mathbb{R}^4 \times \mathcal{V} \to \operatorname{Ker} \mathcal{Q} \subset H^5_{-\eta}$$

with the center manifold

$$\mathcal{M}_0^\mu = \{ Ae_1 + Be_2 + Ce_3 + De_4 + \Psi(A, B, C, D, \mu) \,|\, (A, B, C, D) \in \mathbb{R}^4 \}$$

as its graph for each $\mu \in \mathcal{V}$. Here, $\operatorname{Ker} \mathcal{T} = \operatorname{span}\{e_j\}_{j=1}^4$ and functions e_j are taken to be as in Theorem 3.3 for the given choices of τ, c_0. The following statements hold:

(i) *(smoothness) Ψ is $\mathscr{C}^4$;*

(ii) *(tangency) $\Psi(0,0,0,0,0) = 0$ and $\mathrm{D}_{(A,B,C,D)}\Psi(0,0,0,0,0) = 0$;*

(iii) *(global reduction) $\mathcal{M}_0^\mu$ consists precisely of solutions $\varphi \in H^5_{-\eta}$ with parameter μ to the modified equation (14);*

(iv) *(local reduction) any φ solving (10) with parameter μ and $\|\varphi\|_{H^5_u} \lesssim \delta$ is contained in $\mathcal{M}_0^\mu$;*

(v) *(correspondence) $\varphi \in \mathcal{M}_0^\mu$ if and only if it solves the ODE*

$$\varphi''''(t) = g(\mathcal{T}(\varphi(t), \varphi'(t), \varphi''(t), \varphi'''(t)), \mu), \tag{19}$$

where

$$g(A, B, C, D, \mu)$$
$$= \frac{\mathrm{d}^4}{\mathrm{d}x^4} \left(Ae_1(x) + Be_2(x) + Ce_3(x) + De_4(x) + \Psi(A, B, C, D, \mu)(x) \right)\Big|_{x=0}$$

and $\mathcal{T}$ is the transition matrix $\mathcal{T}_1$ in (15), $\mathcal{T}_2$ in (17) for (8) and (9), respectively;

(vi) (reversibility) equation (10) *possesses the translation symmetries* S_t *and a reflection symmetry* $R\varphi(x) := \varphi(-x)$, *meaning*

$$\mathcal{T}S\varphi = S\mathcal{T}\varphi \quad and \quad \mathcal{N}(S\varphi, \mu) = S\mathcal{N}(\varphi, \mu)$$

if S *is* S_t *or* R. *Equation* (10) *is thus reversible; so is the modified equation* (14) *and the reduced ODE* (19) *in item (v).*

Proof. We use the nonlocal center manifold theorem from [30], which for completeness is stated in Theorem B.5. The Hypothesis B.1 on the linear operator $\mathcal{T}$ has been verified in the previous section and, further, the Hypothesis B.3 for the nonlinearity $\varphi \mapsto \varphi^2$ in $H^m_{-\eta}$ for $m \geq 1$ was verified in[10] [15, 30]. Using that convolution with K is a bounded linear mapping on $H^5_{-\eta}$ it follows that Hypothesis B.3 holds for our nonlocal nonlinearity $\varphi \mapsto K * \varphi^2$ as well. Note that the regularity $k \geq 2$ in Hypothesis B.3 is arbitrary for $\mathcal{N}^\delta$ in (14), possibly at the cost of a smaller cutoff radius δ and $\eta > 0$. Forthcoming computations with the reduced ODEs motivate the choice of $k = 4$, which we now take. Finally, the symmetries in item (vi) above are easily checked, using that equation (10) is steady, that the cutoff χ^δ commutes with both R and S_t, and that K is an even function. It follows that Theorem B.5 applies directly to the present case, giving statements (i)–(iv) and (vi).

It remains to prove the claim in (v) above. To this end, let $\varphi \in \mathscr{M}_0^\mu$ and note that, since Theorem B.5(vi) implies that $\mathscr{M}_0^\mu$ is invariant under translation symmetries, we have $S_t\varphi \in \mathscr{M}_0^\mu$ for all $t \in \mathbb{R}$. Consequently, there exists functions $A(t)$, $B(t)$, $C(t)$ and $D(t)$ defined for all $t \in \mathbb{R}$ such that

$$\begin{aligned} S_t\varphi(x) =& A(t)e_1(x) + B(t)e_2(x) + C(t)e_3(x) + D(t)e_4(x) \\ &+ \Psi(A(t), B(t), C(t), D(t), \mu)(x). \end{aligned} \tag{20}$$

for each $t \in \mathbb{R}$. Noting that the left-hand side of (19) can be rewritten as

$$\varphi''''(t) = \varphi''''(x+t)\Big|_{x=0} = \frac{\mathrm{d}^4}{\mathrm{d}x^4}S_t\varphi\Big|_{x=0},$$

differentiating the identity (20) four times in x and evaluating at $x = 0$ yields precisely the right-hand side in (19) with $A(t)$, $B(t)$, $C(t)$ and $D(t)$. Statement (v) is now proved by using the transition matrix $\mathscr{T}$ to rewrite (19) in terms of $\varphi(t)$, $\varphi'(t)$, $\varphi''(t)$ and $\varphi'''(t)$. $\square$

Remark 4.3. The above proof shows that the projection coefficients $A(t), B(t), C(t)$ and $D(t)$, defined by the shift action S_t on $\varphi \in \mathscr{M}_0^\mu$, are in fact $H^5_{-\eta}$ functions in t because

$$(A(t), B(t), C(t), D(t)) = \mathscr{T}(\varphi(t), \varphi'(t), \varphi''(t), \varphi'''(t)), \quad t \in \mathbb{R}.$$

Here, $\mathscr{T}$ is the transition matrix $\mathscr{T}_1$ for (8) and $\mathscr{T}_2$ for (9).

In the next sections, we will verify our main results Theorem 1.2 and Theorem 1.3 by studying the reduced ODE equations for (14) for appropriate values of of the parameters τ and c_0.

---

[10]Technically, the hypothesis was verified in [15] in the case $m = 1$, and then later extended to $m \geq 1$ in [30].

5 Existence of generalized solitary waves

We now establish Theorem 1.2 by deriving and studying the reduced ODE for equation (14) for τ, c_0 satisfying (8). First, we assume that $\varphi \in \mathscr{M}_0^\mu$ is so small in the H_u^5 norm that it is a solution of (10). Expanding the reduced function Ψ in A, B, C, D and μ, and then substituting into (10) gives the reduced ODE up to second-order terms. We observe from the linear part of the truncated ODE that we have a reversible $0^{2+}(ik_0)$ bifurcation and then apply normal form theory for this bifurcation phenomenon. It turns out that equation (19) at leading orders is almost identical to the reduced ODE for the two-dimensional gravity-capillary water wave equations in this parameter region. Theorem 1.2 is then established after a persistence argument.

5.1 The reduced system

Recall that Remark 4.3 highlights how the projection coefficients A, B, C and D may be interpreted as differentiable functions and, further, Theorem 4.2(v) suggests working with these rather than the $\varphi(t), \varphi'(t), \varphi''(t)$ and $\varphi'''(t)$ directly. Next, we Taylor expand the function Ψ up to second-order terms to obtain the following truncated system of ODEs.

Proposition 5.1 *Equation* (19) *in terms of A, B, C and D is*

$$
\begin{cases}
\dfrac{\mathrm{d}A}{\mathrm{d}t} = B \\[2mm]
\dfrac{\mathrm{d}B}{\mathrm{d}t} = \dfrac{1}{k_0^2}\Psi(A, B, C, D, \mu)''''(0) \\[2mm]
\dfrac{\mathrm{d}C}{\mathrm{d}t} = k_0 D \\[2mm]
\dfrac{\mathrm{d}D}{\mathrm{d}t} = -k_0 C - \dfrac{1}{k_0^3}\Psi(A, B, C, D, \mu)''''(0).
\end{cases}
\tag{21}
$$

Moreover, with $\sigma = \ell''(0)^{-1} = 1/(1/3 - \tau_0)$, we have

$$
\begin{aligned}
\Psi(A, B, C, D, \mu)''''(0) ={}& 2\sigma k_0^2 \mu A - \frac{2k_0^3}{\ell'(k_0)}\mu C - 2\sigma k_0^2 A^2 + \frac{4k_0^3}{\ell'(k_0)}AC \\
& - \left(\frac{3\sigma^{-2} - 4\sigma^{-1} - 4/15}{3}\sigma^2 k_0^2 - 4\sigma\right)B^2 + \left(2k_0^3 \frac{\ell''(k_0) - 2\ell'(k_0)^2}{\ell'(k_0)^2} - \frac{10k_0^2}{\ell'(k_0)}\right)BD \\
& + \left(\frac{6\ell(2k_0)k_0^4}{\ell(2k_0) - 1} - \sigma k_0^2\right)C^2 - \left(\frac{6\ell(2k_0)k_0^4}{\ell(2k_0) - 1} + \sigma k_0^2\right)D^2 \\
& + \mathcal{O}\left(|(A, B, C, D)|\left(\mu^2 + |A|^2 + |B|^2 + |C|^2 + |D|^2\right)\right).
\end{aligned}
\tag{22}
$$

Proof. Deriving equation (21) from (19) using the transition matrix $\mathscr{T}_1$ is straightforward. Indeed, simply note that (19) is equivalent to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \mathscr{T}_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathscr{T}_1^{-1} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ g(A, B, C, D, \mu) \end{pmatrix}.
$$

Noting that, in this case,

$$
g(A, B, C, D, \mu) = k_0^4 C + \Psi(A, B, C, D, \mu)''''(0),
$$

a direct calculation shows that the above is precisely (21).

It remains to compute the asymptotic expansion (22). Specifically, we focus on computing the function $\Psi(A, B, C, D, \mu)''''$ evaluated at $x = 0$ up to order two in A, B, C, D and μ. According to Theorem 4.2(i), Ψ is $\mathscr{C}^4$ in (A, B, C, D, μ). Together with item (ii) in Theorem 4.2 and the fact that $\varphi \equiv 0$ is a solution to (10) for all $\mu \in \mathbb{R}$, it follows that the Taylor expansion of Ψ must be of the form

$$\Psi(A, B, C, D, \mu)(x) = \sum_{\substack{2 \leq p+q+l+m+n \leq 3 \\ n \geq 1}} \Psi_{pqlmn}(x) \cdot A^p B^q C^l D^m \mu^n + \cdots$$

where each $\Psi_{pqlmn} \colon \mathbb{R} \to \mathbb{R}$ belongs to $\operatorname{Ker} \mathcal{Q}_1 \subset H^5_{-\eta}$. It thus remains to compute $\Psi''''_{pqlmn}(0)$ for $p + q + l + m + n = 2$ and $n \geq 1$. To this end, let $\varphi \in \mathscr{M}^\mu_0$ and $\|\varphi\|_{H^5_u} \lesssim \delta$ and note by Theorem 4.2(iv) that φ solves (10). To conveniently group the terms, we rewrite the left-hand side of equation (10) to have

$$\mathcal{T}\varphi + (\operatorname{Id} - \mathcal{T})(\varphi^2 - \mu\varphi) = 0.$$

Since φ belongs to $\mathscr{M}^\mu_0$, we know that $\varphi(x) = \mathcal{Q}_1\varphi(x) + \Psi(A, B, C, D, \mu)(x)$, and plugging this into the above equation gives

$$\mathcal{T}(\mathcal{Q}_1\varphi + \Psi) + (\operatorname{Id} - \mathcal{T})\big((\mathcal{Q}_1\varphi + \Psi)^2 - \mu(\mathcal{Q}_1\varphi + \Psi)\big) = 0.$$

Using that $\mathcal{T}\mathcal{Q}_1\varphi = 0$ by definition, the above can be rearranged as

$$\mathcal{T}\Psi + (\mathcal{Q}_1\varphi)^2 - \mu\mathcal{Q}_1\varphi - \mathcal{T}(\mathcal{Q}_1\varphi)^2 = -(\operatorname{Id} - \mathcal{T})\left(2(\mathcal{Q}_1\varphi)\Psi + \Psi^2 - \mu\Psi\right),$$

where we note the right hand side above consists of all terms that are at least cubic in (A, B, C, D, μ). Linear equations for the functions Ψ_{pqlmn} can now be read off easily, and are recorded in Appendix C.1. Note that by the condition $\mathcal{Q}_1\Psi_{pqlmn} = 0$, these coefficient functions Ψ_{pqlmn} are uniquely determined.[11] Further, we observe that symmetries can be used to greatly simplify the necessary computations. Indeed, note that the basis functions $1, x, \cos(k_0 x)$ and $\sin(k_0 x)$ are either even or odd and that the operators $\mathcal{T}$, $\operatorname{Id} - \mathcal{T}$ and $\mathcal{Q}_1$ map even to even and odd to odd functions. Consequently, as seen in Appendix C.1 the linear equations for Ψ_{pqlmn} involve either only even or odd functions and hence the solutions Ψ_{pqlmn} are also necessarily either even or odd functions. Since only even functions Ψ_{pqlmn} contribute to $\Psi(A, B, C, D, \mu)''''$ evaluated at 0, equations for odd Ψ_{pqlmn} may be disregarded. The computations for Ψ_{pqlmn} are detailed in Appendix C.1 and these give equation (22). $\qquad\square$

5.2 Normal form reduction

We use normal form theory to study the reduced system (21), which can be written as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathbf{L}U + \mathbf{R}(U, \mu), \tag{23}$$

where $U = (A, B, C, D)$, and $\mathbf{L}$ is precisely the linearization of (21) about the origin, that is,

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_0 \\ 0 & 0 & -k_0 & 0 \end{pmatrix},$$

---

[11] Indeed, as seen in Appendix C.1, if there are two solutions Ψ_{pqlmn} and $\tilde{\Psi}_{pqlmn}$, then their difference must belong to $\operatorname{Ker}\mathcal{T} \cap \operatorname{Ker}\mathcal{Q}_1$, and hence must be zero.

and $\mathbf{R}$ is $\mathscr{C}^4$ in a neighborhood of $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$, satisfying $\mathbf{R}(0,0) = 0$ and $\mathrm{D}_U \mathbf{R}(0,0) = 0$. The spectrum of $\mathbf{L}$ consists of the algebraically double and geometrically simple eigenvalue 0, as well as the pair of simple purely imaginary eigenvalues $\pm i k_0$. Further, we note that the reflection symmetry $R\varphi(x) = \varphi(-x)$ on $\operatorname{Ker} \mathcal{T}$ with respect to the basis functions $1, x, \cos(k_0 x)$ and $\sin(k_0 x)$ is

$$A + B(-x) + C\cos(-k_0 x) + D\sin(-k_0 x) = A - Bx + C\cos(k_0 x) - D\sin(k_0 x).$$

This shows that R restricted to $\operatorname{Ker} \mathcal{T}$ is a linear mapping on $\mathbb{R}^4$, given by

$$R\colon (A, B, C, D) \mapsto (A, -B, C, -D)$$

and, clearly, $R^2 = \operatorname{Id}$. Further, by Theorem 4.2(vi), R anticommutes with $\mathbf{L}$ and $\mathbf{R}$, that is, $R\mathbf{L}U = -\mathbf{L}RU$ and $R\mathbf{R}(U, \mu) = -\mathbf{R}(RU, \mu)$. Taken together, it follows that it is natural to expect that the origin undergoes a reversible $0^{2+}(ik_0)$ bifurcation for parameters μ sufficiently small. In this section, we use the correspoding normal form theory for such bifurcations from [17, Chapter 4.3.1] to study (23) near the origin for μ sufficiently small.

To begin, we note that the eigenvectors and generalized eigenvectors of $\mathbf{L}$ are given by

$$\xi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and } \zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix},$$

which are readily seen to satisfy

$$\begin{aligned} \mathbf{L}\xi_0 &= 0, & \mathbf{L}\xi_1 &= \xi_0, & \mathbf{L}\zeta &= i k_0 \zeta, \\ R\xi_0 &= \xi_0, & R\xi_1 &= -\xi_1, & R\zeta &= \overline{\zeta}. \end{aligned} \tag{24}$$

Based on the structure of $\mathbf{L}$, throughout the remainder of this section $\mathbb{R}^4$ will be identified with $\mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$ where $\widetilde{\mathbb{R}^2} := \{(C, \overline{C}) : C \in \mathbb{C}\}$. We are now in the position to directly apply the normal form result [17, Lemma 3.5]. This result implies that there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of $0 \in \mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$ and $0 \in \mathbb{R}$, respectively, and a polynomial change of variables

$$U = \mathbf{A}\xi_0 + \mathbf{B}\xi_1 + \mathbf{C}\zeta + \overline{\mathbf{C}}\overline{\zeta} + \Phi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu) \tag{25}$$

defined in $\mathcal{V}_1$ and $\mathcal{V}_2$, which transforms the reduced system (21) into the normal form

$$\begin{cases} \dfrac{d\mathbf{A}}{dt} = \mathbf{B} \\ \dfrac{d\mathbf{B}}{dt} = P(\mathbf{A}, |\mathbf{C}|^2, \mu) + \rho_{\mathbf{B}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu) \\ \dfrac{d\mathbf{C}}{dt} = i k_0 \mathbf{C} + i\mathbf{C}Q(\mathbf{A}, |\mathbf{C}|^2, \mu) + \rho_{\mathbf{C}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu), \end{cases} \tag{26}$$

where P and Q are polynomials of degree two and one in $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})$, respectively. Here, the function Φ is $\mathscr{C}^4$, satisfying

$$\Phi(0, 0, 0, 0, 0) = 0, \quad \partial_{(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})} \Phi(0, 0, 0, 0, 0) = 0$$

$$\Phi(\mathbf{A}, -\mathbf{B}, \overline{\mathbf{C}}, \mathbf{C}, \mu) = R\Phi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)$$

while the remainders $\rho_{\mathbf{B}}$ and $\rho_{\mathbf{C}}$ are $\mathscr{C}^4$ with

$$|\rho_{\mathbf{B}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)| + |\rho_{\mathbf{C}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)| = o((|\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}|)^2).$$

19

For proof and more details, see [17, Chapter 4.3.1].

Let

$$P(\mathbf{A}, |\mathbf{C}|^2, \mu) = p_0\mu + p_1\mu\mathbf{A} + p_2\mathbf{A}^2 + p_3|\mathbf{C}|^2$$

$$Q(\mathbf{A}, |\mathbf{C}|^2, \mu) = q_0\mu + q_1\mathbf{A}.$$

The scalar coefficients p_0, p_1, p_2, p_3, q_0 and q_1 are computed in Appendix D.1. Setting $\sigma = (1/3 - \tau)^{-1}$ these calculations yield the normal form of (21) as

$$\begin{cases} \dfrac{d\mathbf{A}}{dt} = \mathbf{B} \\[2mm] \dfrac{d\mathbf{B}}{dt} = 2\sigma\mu\mathbf{A} - 2\sigma\mathbf{A}^2 - 2\sigma|\mathbf{C}|^2 + \mathcal{O}(|\mu|^2 + (|\mu| + |\mathbf{A}| + |\mathbf{C}|)^2)) \\[2mm] \dfrac{d\mathbf{C}}{dt} = ik_0\mathbf{C} - \dfrac{i}{\ell'(k_0)}\mu\mathbf{C} - \dfrac{4i}{\ell'(k_0)}\mathbf{A}\mathbf{C} + \mathcal{O}(|\mathbf{C}|(|\mu| + |\mathbf{A}| + |\mathbf{C}|^2)^2). \end{cases} \tag{27}$$

5.3 Generalized solitary waves

Next, we consider the normal form system (27) truncated at second-order terms, i.e.

$$\begin{cases} \dfrac{d\mathbf{A}}{dt} = \mathbf{B} \\[2mm] \dfrac{d\mathbf{B}}{dt} = 2\sigma\mu\mathbf{A} - 2\sigma\mathbf{A}^2 - 2\sigma|\mathbf{C}|^2 \\[2mm] \dfrac{d\mathbf{C}}{dt} = ik_0\mathbf{C} - \dfrac{i}{\ell'(k_0)}\mu\mathbf{C} - \dfrac{4i}{\ell'(k_0)}\mathbf{A}\mathbf{C}. \end{cases} \tag{28}$$

The change of variables

$$w = \frac{1}{\sqrt{2}}t, \quad \mathbf{A}(t) = -\frac{3}{2}\widetilde{\mathbf{A}}(w), \quad \mathbf{B}(t) = -\frac{3}{\sqrt{2}}\widetilde{\mathbf{B}}(w), \tag{29}$$

$$\mathbf{C}(t) = |\mu|k^{1/2}\exp(i\Theta(t))$$

transforms (28) into the system (3.14) studied by Iooss & Kirchgässner in [22] in their search for generalized solitary waves in the context of the full gravity-capillary water wave problem.[12] Here, k is fixed but arbitrary. Observe that

$$\Theta'(t) = k_0 - \frac{\mu}{\ell'(k_0)} - \frac{4}{\ell'(k_0)}\mathbf{A}(t).$$

Equations (3.17)–(3.19) in [22] provide us with a one-parameter family of explicit solutions, parametrized by k, of the rescaled truncated system given by

$$\widetilde{\mathbf{A}}(w) = -\frac{\mu}{3}(1 - \mathrm{sgn}(\mu)\rho^{1/2}) - |\mu|\rho^{1/2}\mathrm{sech}^2\left(\frac{\rho^{1/4}|\mu|^{1/2}\sigma^{1/2}w}{2}\right), \tag{30}$$

$$\widetilde{\mathbf{B}}(w) = \widetilde{\mathbf{A}}'(w).$$

Then, substituting $\mathbf{A}(t) = -3\widetilde{\mathbf{A}}(w)/2$ into the differential equation for $\Theta(t)$ gives

$$\Theta(t) = \Theta_* + \left(k_0 - \frac{\mu}{\ell'(k_0)} - \frac{2\mu}{\ell'(k_0)}\left(1 - \mathrm{sgn}(\mu)\rho^{1/2}\right)\right)t$$

$$+ \frac{8\sqrt{2}\,\rho^{1/4}|\mu|^{1/2}}{\sigma^{1/2}\ell'(k_0)}\tanh\left(\frac{\rho^{1/4}|\mu|^{1/2}\sigma^{1/2}t}{2^{3/2}}\right),$$

---

[12]The only difference between our rescaled system and that studied in [22] is the ODE for $\mathbf{C}$, which is inconsequential. Note in [22] that the small parameter used is $\frac{1}{c^2} - 1$, which corresponds to $-\mu$ in our case.

where $\Theta_* \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ is arbitrary and $\rho = 1 + 6\left(1 + \sinh(k_0)/k_0 + \mathcal{O}(|\mu|)\right)k$.

It remains to see if the above family of solutions of the rescaled truncated normal-form system persist as solutions of the full rescaled normal-form system. Luckily, the persistence of (30) under reversible perturbations has received considerable treatment (see, for example, the work of Iooss & Kirchgässner in [22]). In particular, these persistence results are summarized for $\mathscr{C}^m$ vector fields in [17, Theorem 3.10] and, in the present context, this work guarantees that the family of explicit solutions (30) persists provided that

$$r = |\mu|k^{1/2} > r_*(\mu) = \mathcal{O}(|\mu|^{1/2}). \tag{31}$$

In particular, note that since μ is small the persistence condition (31) is effectively a lower bound on the frequency k, corresponding to high-frequency oscillation in $\tilde{\mathbf{C}}$.

Finally, we undo the above variable changes to return to the original unknown function φ. Undoing (29) in (5.3) yields

$$\mathbf{A}(t) = \frac{\mu}{2}(1 - \mathrm{sgn}(\mu)\rho^{1/2}) + \frac{3}{2}|\mu|\rho^{1/2}\mathrm{sech}^2\left(\frac{\rho^{1/4}\sigma^{1/2}|\mu|^{1/2}t}{2\sqrt{2}}\right),$$

$$\mathbf{B}(t) = \mathbf{A}'(t),$$

$$\mathbf{C}(t) = |\mu|k^{1/2}\exp\left(\mathrm{i}(k_0 + \mathcal{O}(\mu))\frac{t}{\sqrt{2}} + \Theta_0 + \mathcal{O}(\mu)\right),$$

while undoing the polynomial change of variables (25) in the above normal form analysis yields

$$A(t) = \mathbf{A}(t) + \mathcal{O}(\mu^2\rho^{1/2}), \quad B(t) = \mathbf{B}(t) + \mathcal{O}(\mu^2\rho^{1/2}),$$

$$C(t) = \frac{1}{2}(\mathbf{C} + \overline{\mathbf{C}}) + \mathcal{O}(\mu^2\rho^{1/2}). \tag{32}$$

Recalling now that (16) implies $A(t) = \varphi(t) + k_0^{-2}\varphi''(t)$, $C(t) = -k_0^{-2}\varphi''(t)$ and switching back to the original variable x, it follows that

$$\varphi(x) = A(x) + C(x)$$
$$= \frac{3}{2}|\mu|\rho^{1/2}\mathrm{sech}^2\left(\frac{\rho^{1/4}\sigma^{1/2}|\mu|^{1/2}x}{2\sqrt{2}}\right) + \frac{\mu}{2}(1 - \mathrm{sgn}(\mu)\rho^{1/2})$$
$$+ |\mu|k^{1/2}\cos\left((k_0 + \mathcal{O}(\mu))\frac{x}{\sqrt{2}} + \Theta_* + \mathcal{O}(\mu)\right) + \mathcal{O}(\mu^2\rho^{1/2}).$$

Here, $\Theta_* \in \mathbb{R}/2\pi\mathbb{Z}$ is an arbitrary integration constant. Due to the hyperbolic tangent in Θ, there is an asymptotic phase shift in the cosinus term between $x = -\infty$ and $x = \infty$ of order $\mathcal{O}(\rho^{1/4}|\mu|^{1/2})$.

Provided the persistence condition (31) holds, the function ϕ above solves the modified profile equation (14). For φ to be a solution to the original profile equation (10) with parameter μ, it must additionally satisfy the smallness assumption $\|\varphi\|_{H_\mathrm{u}^5} \lesssim \delta$. This can be achieved by setting, for example,

$$k = k'|\mu|^{-1-2\kappa}, \quad \text{for some } \kappa \in [0, 1/2) \text{ and some constant } k' > 0.$$

Indeed, under this condition the persistence condition (31) is clearly met and the functions $\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}$ have amplitude $\mathcal{O}(|\mu|^{1/2-\kappa})$ which, in turn, implies that $\varphi, \varphi', \varphi'', \varphi'''$ are also $\mathcal{O}(|\mu|^{1/2-\kappa})$ via (32) and (16). This bound is carried over to the fourth and fifth derivatives by differentiating (19) twice (see [30, Theorem 3.3]). It follows from choosing μ sufficiently small that the H_u^5 norm of φ is small, and hence that φ is a solution to (10) with parameter μ. This establishes Theorem 1.2.

Remark 5.2. When $\mu > 0$, (28) features an orbit which is homoclinic to the saddle equilibrium $(\mathbf{A}, \mathbf{B}) = (0,0)$ once projected onto the $(\mathbf{A}, \mathbf{B})$-plane. When $\mu < 0$, it is homoclinic to $(\mathbf{A}, \mathbf{B}) = (\frac{\mu}{2}(1 + \rho^{1/2}), 0)$, which is close to $(0,0)$. In the latter case, we point out that this solution has supercritical wave speed. Indeed, equation (1) is invariant under a Galilean change of variable

$$\varphi \mapsto \varphi + v, \quad c \mapsto c - 2v, \quad (1 - c)^2 b \mapsto (1 - c)^2 b + (1 - c)v + v^2,$$

where b is an integration constant which doesn't affect the critical wavespeed: see [20]. Putting $v = \mu/2(1 + \rho^{1/2})$, the new wave speed is

$$c - 2v = 1 + \mu - 2 \cdot \frac{\mu}{2}(1 + \rho^{1/2}) = 1 + |\mu|\rho^{1/2} > 1.$$

To summarize, all generalized solitary-wave solutions in Theorem 1.2 have supercritical wave speed $c > 1$.

6 Existence of modulated solitary waves

The aim of this section is to prove existence of modulated solitary waves in (9). As for the classical two-dimensional gravity-capillary water wave equations, the signs of two terms in the normal form are to be determined – one of those terms will be of cubic order. Instead of deriving the full reduced ODE as in Section 5, we only determine it roughly using the symmetries. We then perform a normal form reduction and determine linear equations for the relevant normal form coefficients. From these, it will be clear which center manifold coefficients are necessary. Throughout this section, we assume that the parameters τ and c_0 satisfy (9).

6.1 Normal form reduction

As in Section 5, we will work with projection coefficients A, B, C, and D rather than $\varphi, \varphi', \varphi''$ and φ'''. Using the transition matrix $\mathscr{T}_2$ from Section 4 and proceeding along the same lines as the proof of Proposition 5.1, we find that (19) in this case is equivalent to the system

$$\begin{cases} \dfrac{\mathrm{d}A}{\mathrm{d}t} = Bs + C \\ \dfrac{\mathrm{d}B}{\mathrm{d}t} = -As + D + \dfrac{1}{2s^3}\Psi(A,B,C,D,\mu)''''(0) \\ \dfrac{\mathrm{d}C}{\mathrm{d}t} = Ds - \dfrac{1}{2s^2}\Psi(A,B,C,D,\mu)''''(0) \\ \dfrac{\mathrm{d}D}{\mathrm{d}t} = -Cs \end{cases} \tag{33}$$

Letting $U = (A, B, C, D)$, (33) can be rewritten as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathbf{L}U + \mathbf{R}(U, \mu) \tag{34}$$

where here

$$\mathbf{L} = \begin{pmatrix} 0 & s & 1 & 0 \\ -s & 0 & 0 & 1 \\ 0 & 0 & 0 & s \\ 0 & 0 & -s & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{R}(U, \mu) = \frac{1}{2s^3} \begin{pmatrix} 0 \\ \Psi(A,B,C,D,\mu)''''(0) \\ -s\Psi(A,B,C,D,\mu)''''(0) \\ 0 \end{pmatrix}.$$

22

In particular, in view of Theorem 4.2(ii) the matrix $\mathbf{L}$ is precisely the linearization of (33) about the trivial solution $0 \in \mathbb{R}^4$. The spectrum of $\mathbf{L}$ is readily seen to consist of a pair of algebraically double and geometrically simple eigenvalues s and $-s$ with corresponding eigenvectors and generalized eigenvectors

$$\zeta_0 = (1, \mathrm{i}, 0, 0) \quad \text{and} \quad \zeta_1 = (0, 0, 1, \mathrm{i})$$

that satisfy

$$(\mathbf{L} - \mathrm{i}s)\zeta_0 = 0, \quad (\mathbf{L} - \mathrm{i}s)\zeta_1 = \zeta_0,$$
$$(\mathbf{L} + \mathrm{i}s)\overline{\zeta_0} = 0, \quad (\mathbf{L} + \mathrm{i}s)\overline{\zeta_1} = \overline{\zeta_0}.$$

As such, it is natural to expect that the system undergoes an $(\mathrm{i}s)^2$ bifurcation.

To analyze this bifurcation, observe that the set $\{\zeta_0, \zeta_1, \overline{\zeta_0}, \overline{\zeta_1}\}$ spans $\widetilde{\mathbb{R}^2} \times \widetilde{\mathbb{R}^2} \sim \mathbb{R}^4$, and that the reversible symmetry $R\varphi(x) = \varphi(-x)$ restricted on $\operatorname{Ker}\mathcal{T}$ takes the form

$$R\colon (A, B, C, D) \mapsto (A, -B, -C, D)$$

with respect to the basis $\{\cos(sx), \sin(sx), x\cos(sx), x\sin(sx)\}$. Furthermore, the vectors ζ_0 and ζ_1 also satisfy

$$R\zeta_0 = \overline{\zeta_0} \quad \text{and} \quad R\zeta_1 = -\overline{\zeta_1}.$$

Normal form theory for $(\mathrm{i}s)^2$ bifurcations now asserts that there exists a polynomial change of variable

$$U = \mathbf{A}\zeta_0 + \mathbf{B}\zeta_1 + \overline{\mathbf{A}\zeta_0} + \overline{\mathbf{B}\zeta_1} + \Phi(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu),$$

where Φ is a polynomial in $(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}})$ of degree 3, that transforms (34) into the normal form

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathrm{i}s\mathbf{A} + \mathbf{B} + \mathrm{i}\mathbf{A}P\left(|\mathbf{A}|^2, \dfrac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) + \rho_{\mathbf{A}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu) \\[2mm] \dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \mathrm{i}s\mathbf{B} + \mathrm{i}\mathbf{B}P\left(|\mathbf{A}|^2, \dfrac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) \\[2mm] \quad + \mathbf{A}Q\left(|\mathbf{A}|^2, \dfrac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) + \rho_{\mathbf{B}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu). \end{cases} \tag{35}$$

Here, the polynomials P and Q have degree 2 in $(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}})$. For details, see [17, Section 4.3.3] and, specifically, Lemma 3.17 in that reference.

6.2 Modulated solitary waves

We now aim to consider the normal form system (35) truncated at second-order terms. Let

$$\begin{aligned} P\left(|\mathbf{A}|^2, \frac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) &= p_0\mu + p_1|\mathbf{A}|^2 + \frac{\mathrm{i}p_2}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B}), \\ Q\left(|\mathbf{A}|^2, \frac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) &= q_0\mu + q_1|\mathbf{A}|^2 + \frac{\mathrm{i}q_2}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B}). \end{aligned} \tag{36}$$

In Appendix C.2 and D.2, the coefficients q_0 and q_1 are computed and for small $s > 0$ are found to satisfy

$$\begin{aligned} q_0 &= -12 + \mathcal{O}(s^2)) < 0, \\ q_1 &= -\frac{32400}{s^8} + \mathcal{O}(s^{-6}) < 0. \end{aligned}$$

Recalling that $\mu < 0$ in this case, the above puts (35) into the subcritical case considered by Iooss & Pèrouéme in [21, Section IV3]. Through the change of variables

$$\mathbf{A}(t) = r_0(t)\exp(\mathrm{i}(st + \Theta_0(t))), \quad \mathbf{B}(t) = r_1(t)\exp(\mathrm{i}(st + \Theta_1(t))),$$

the normal form truncated at third order terms has explicit homoclinic solutions

$$r_0(t) = \sqrt{\frac{-2q_0\mu}{q_1}}\operatorname{sech}(\sqrt{q_0\mu}\,t),$$

$$r_1(t) = |r_0'|,$$

$$\Theta_0(t) = p_0\mu t - \frac{2p_1\sqrt{q_0\mu}}{q_1}\tanh(\sqrt{q_0\mu}\,t) + \Theta_*,$$

$$\Theta_1 - \Theta_0 \in \{0, \pi\},$$

(see [17, pp.217–223]). Here, p_0, p_1 are as in (36) and $\Theta_* \in \mathbb{R}$ is an arbitrary integration constant, resulting in a full circle of homoclinic solutions. However, only two distinct homoclinic solutions persist under reversible perturbation, when $\Theta_* = 0$ and $\Theta_* = \pi$. Tracing back to the original unknown φ and variable x, we get

$$\varphi(x) = \sqrt{\frac{-8q_0\mu}{q_1}}\operatorname{sech}(\sqrt{q_0\mu}\,x)\cos\left(st + \mathcal{O}\left(|\mu|^{1/2}\right)\right) + \mathcal{O}(\mu^2),$$

and

$$\varphi(x) = -\sqrt{\frac{-8q_0\mu}{q_1}}\operatorname{sech}(\sqrt{q_0\mu}\,x)\cos\left(st + \mathcal{O}\left(|\mu|^{1/2}\right)\right) + \mathcal{O}(\mu^2).$$

The first solution φ is often referred to as a modulated solitary wave of elevation and the latter is a modulated solitary wave of depression. We illustrate the elevation case in Figure 1. Due to the hyperbolic tangent, there is an asymptotic phase shift of order $\mathcal{O}(|\mu|^{1/2})$ between $x = -\infty$ and $x = \infty$. Lastly, it can be shown that $\varphi, \varphi', \varphi''$ and φ''' are of order $\mathcal{O}(|\mu|^{1/2})$ by arguing as in the previous section. The uniform locally Sobolev norm $\|\varphi\|_{H_{\mathrm{u}}^5}$ can thus be made arbitrarily small, qualifying these as solutions to (10) with parameters (9). This establishes Theorem 1.3.

Acknowledgements

A Fredholm theory for pseudodifferential operators

In this appendix, we review a Fredholm theory for pseudodifferential operators developed by Grushin in [16]. This theory is applied in Section 3 to determine the Fredholm properties of the linear operator $\mathcal{T}$.

Let $x^* = (x_0, x) \in \mathbb{R} \times \mathbb{R}^n$ and

$$X^* = \{x^* \in \mathbb{R}^{n+1} \,|\, x_0 \geq 0, x^* \neq 0\}.$$

24

Similarly, let $\xi^* = (\xi_0, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$E^* = \{\xi^* \in \mathbb{R}^{n+1} \mid \xi_0 \geq 0, \xi^* \neq 0\}.$$

Let $\mathcal{A}$ be the class of functions $A(x^*, \xi^*) \in C^\infty(X^* \times E^*)$ such that A is positive-homogeneous of degree 0 in x^* and ξ^*, that is,

$$A(\lambda x^*, \xi^*) = A(x^*, \lambda \xi^*) = A(x^*, \xi^*), \quad \lambda > 0.$$

Let $\mathbb{S}_+^n$ denote the hemisphere $|x^*| = 1$ and $x_0 > 0$, or $|\xi^*| = 1$ and $\xi_0 > 0$. Let $\overline{\mathbb{S}_+^n}$ denote the relative closure of $\mathbb{S}_+^n$ in X^*, or in E^*, that is, $\overline{\mathbb{S}_+^n}$ is the hemisphere $|x^*| = 1$ (or $|\xi^*| = 1$), $x_0 \geq 0$ (or $\xi_0 \geq 0$). Clearly, each $A \in \mathcal{A}$ is uniquely determined by its values on $\mathbb{S}_+^n \times \mathbb{S}_+^n$. Conversely, each function $\tilde{A} \in C^\infty(\overline{\mathbb{S}_+^n} \times \overline{\mathbb{S}_+^n})$ can be uniquely homogeneously extended to $X^* \times E^*$. So, $\mathcal{A} \cong C^\infty(\overline{\mathbb{S}_+^n} \times \overline{\mathbb{S}_+^n})$. By $S_{\mathcal{A}}^0$, we denote the set of symbols $p_A(x, \xi)$ which are given by

$$p_A(x, \xi) = A(1, x; 1, \xi),$$

for some $A \in \mathcal{A}$. For $p_A \in S_{\mathcal{A}}^0$, we have the following result, which combines Theorems 4.1 and 4.2 in Grushin [16].

Theorem A.1 *If $p_A(x, \xi) \in S_{\mathcal{A}}^0$ and $A(x^*, \xi^*) \neq 0$ on Γ, then*

$$p_A(x, D) : H^s \to H^s$$

is Fredholm and the index is

$$\operatorname{ind} p_A(x, D) = \frac{1}{2\pi} \Big(\arg \det A(x^*, \xi^*) \big|_\Gamma \Big),$$

where Γ is the boundary of $\overline{\mathbb{S}_+^n} \times \overline{\mathbb{S}_+^n}$, and $\arg \det A(x^*, \xi^*)|_\Gamma$ is the increase in the argument of $\det A(x^*, \xi^*)$ around Γ oriented counterclockwise.

B A nonlocal center manifold theorem

In this section, we record a version due to Truong, Wahlén & Wheeler [30] of the nonlocal center manifold theorem originally developed by Faye & Scheel [14, 15]. This result is the main analytical tool used throughout Section 5 and Section 6.

We consider nonlocal nonlinear parameter-dependent problems of the form

$$\mathcal{T}v + \mathcal{N}(v, \mu) = 0, \tag{37}$$

where

$$\mathcal{T}v = v + \mathcal{K} * v,$$

in the weighted Sobolev spaces $H_{-\eta}^m$ for some $\eta > 0$ and positive integer m. $\mathcal{T}$ is referred to as the linear part and $\mathcal{N}$ as the nonlinear part of (37). Before introducing the modified equation, we define a cutoff operator χ which is invariant under all translations and reversible symmetries. The translation map by $t \in \mathbb{R}$, that is $\varphi \mapsto \varphi(\cdot + t)$, is denoted by S_t. First, let $\underline{\chi} : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function satisfying $\underline{\chi} = 1$ for $|x| < 1$, 0 for $|x| > 2$ and $\sup_{x \in \mathbb{R}} |\underline{\chi}'(x)| \leq 2$. Secondly, let $\theta : \mathbb{R} \to \mathbb{R}$ be an even and smooth function with

$$\sum_{j \in \mathbb{Z}} \theta(x - j) = 1, \quad \operatorname{supp} \theta \subset [-1, 1], \quad \theta\left(\left[0, \tfrac{1}{2}\right]\right) \subset \left[\tfrac{1}{2}, 1\right],$$

for all $x \in \mathbb{R}$. Define

$$\chi \colon v \mapsto \int_{\mathbb{R}} \underline{\chi}(\|S_y \theta \cdot v\|_{H^m}) \theta(x-y) v(x) \, \mathrm{d}y \quad \text{and} \quad \chi^\delta \colon v \mapsto \delta \cdot \chi\left(\frac{v}{\delta}\right), \quad \delta > 0. \tag{38}$$

It has been shown [15] $\chi \colon H^m_{-\eta} \to H^m_{\mathrm{u}}$ is well-defined, Lipschitz continuous, invariant under all S_t and reversible symmetries on $\mathbb{R}$, and its image is contained in a ball in H^m_{u}. As a consequence, the scaled cutoff χ^δ inherits all these properties except for its image, which will be contained in a ball of radius δ in H^m_{u}. The modified equation is

$$\mathcal{T}v + \mathcal{N}^\delta(v, \mu) = 0 \quad \text{where} \quad \mathcal{N}^\delta(v, \mu) := \mathcal{N}(\chi^\delta(v), \mu). \tag{39}$$

Also, let $\mathcal{Q} \colon H^m_{-\eta} \to H^m_{-\eta}$ be a bounded projection on the nullspace $\mathrm{Ker}\,\mathcal{T}$ of $\mathcal{T}$ with a continuous extension to $H^{m-1}_{-\eta}$, such that $\mathcal{Q}$ commutes with the inclusion map from $H^m_{-\eta}$ to $H^m_{-\eta'}$, for all $0 < \eta' < \eta$.

HYPOTHESIS B.1 (The linear part $\mathcal{T}$)

(i) There exists $\eta_0 > 0$ such that $\mathcal{K} \in L^1_{\eta_0}$.

(ii) The operator

$$\mathcal{T} \colon v \mapsto v + \mathcal{K} * v, \quad \mathcal{T} \colon H^m_{-\eta} \to H^m_{-\eta}$$

is Fredholm for $\eta \in (0, 2\eta_0)$, its nullspace $\mathrm{Ker}\,\mathcal{T}$ is finite-dimensional and $\mathcal{T}$ is onto.

Remark B.2. A straightforward application of Young's inequality shows that Hypothesis B.1(i) implies the operator $\mathcal{T} \colon H^m_{-\eta} \to H^m_{-\eta}$ is bounded for each $\eta \in (0, \eta_*)$ for each choice of c_0. In the works [14,15] the authors additionally assumed that $K' \in L^1_{-\eta}$ which, as seen from Proposition 3.1, does not hold for the current case. This assumption, however, is used to guarantee Hypothesis B.1(ii) which, here, we instead require directly.

HYPOTHESIS B.3 (The nonlinear part $\mathcal{N}$) There exist $k \geq 2$, a neighborhood $\mathcal{U}$ of $0 \in H^m_{-\eta}$ and $\mathcal{V}$ of $0 \in \mathbb{R}$, such that for all sufficiently small $\delta > 0$, we have

(a) $\mathcal{N}^\delta \colon H^m_{-\eta} \times \mathcal{V} \to H^m_{-\eta}$ is $\mathscr{C}^k$. Moreover, for all non-negative pairs (ζ, η) such that $0 < k\zeta < \eta < \eta_0$, $\mathrm{D}^l_v \mathcal{N}^\delta(\,\cdot\,, \mu) \colon (H^j_{-\zeta})^l \to H^j_{-\eta}$ is bounded for all $0 < l\zeta \leq \eta < \eta_0$ and $0 \leq l \leq k$, and is Lipschitz in v for $1 \leq l \leq k-1$ uniformly in $\mu \in \mathcal{V}$.

(b) $\mathcal{N}^\delta$ commutes with translations of v,

$$\mathcal{N}^\delta(S_t v, \mu) = S_t \mathcal{N}(v, \mu), \quad \text{for all } t \in \mathbb{R}.$$

(c) $\mathcal{N}^\delta(0, 0) = 0$, $\mathrm{D}_v \mathcal{N}^\delta(0, 0) = 0$ and as $\delta \to 0$, the Lipschitz constant

$$\mathrm{Lip}_{H^m_{-\eta} \times \mathcal{V}} \mathcal{N}^\delta = \mathcal{O}(\delta + |\mu|).$$

Recall that $v \colon \mathbb{R} \to \mathbb{R}$. A symmetry is a triple $(\rho, S_t, \kappa) \in \mathbf{O}(1) \times (\mathbb{R} \times \mathbf{O}(1))$, acting on v in the following way: the orthogonal linear transformation $\rho \in \mathbf{O}(1)$ acts on the value $v(x) \in \mathbb{R}$, while S_t and κ act on the variable $x \in \mathbb{R}$. A symmetry (ρ, S_t, κ) is equivariant if $\kappa = \mathrm{Id}$, and reversible otherwise.

HYPOTHESIS B.4 (Symmetries) There exists a symmetry group S, under which the equation is invariant, that is

$$\gamma(\mathcal{T}v) = \mathcal{T}(\gamma v), \quad \mathcal{N}(\gamma v, \mu) = \gamma \mathcal{N}(v, \mu), \quad \text{for all } \gamma \in S,$$

such that S contains all translations on the real line.

Theorem B.5 *Assume Hypotheses B.1, B.3 and B.4 are met for the* (37). *Then, by possibly shrinking the neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$, there exists a cutoff radius $\delta > 0$, a weight $\eta > 0$ and a map*

$$\Psi \colon \operatorname{Ker}\mathcal{T} \times \mathcal{V} \subset H_{-\eta}^m \times \mathbb{R} \to \operatorname{Ker}\mathcal{Q} \subset H_{-\eta}^m$$

with the center manifold

$$\mathcal{M}_0^\mu = \left\{ v_0 + \Psi(v_0, \mu) \,\middle|\, v_0 \in \operatorname{Ker}\mathcal{T}, \mu \in \mathcal{V} \right\} \subset H_{-\eta}^m,$$

as its graph for each μ. The following statements hold:

(i) *(smoothness) $\Psi \in \mathscr{C}^k$, where k is as in Hypothesis B.3;*

(ii) *(tangency) $\Psi(0,0) = 0$ and $\mathrm{D}_{v_0}\Psi(0,0)$;*

(iii) *(global reduction) $\mathcal{M}_0^\mu$ consists precisely of functions v such that $v \in H_{-\eta}^m$ is a solution of the modified equation* (39) *with parameter μ;*

(iv) *(local reduction) any function v solving* (37) *with $\|v\|_{H_{\mathrm{u}}^m} \lesssim \delta$ is contained in $\mathcal{M}_0^\mu$;*

(v) *(translation invariance) the shift S_t, $t \in \mathbb{R}$ acting on $\mathcal{M}_0^\mu$ induces a μ-dependent flow*

$$\Phi_t \colon \operatorname{Ker}\mathcal{T} \to \operatorname{Ker}\mathcal{T}$$

through $\Phi_t = \mathcal{Q} \circ S_t \circ (\operatorname{Id} + \Psi)$;

(vi) *(reduced vector field) the reduced flow $\Phi_t(v_0, \mu)$ is of class $\mathscr{C}^k$ in v_0, μ, t and is generated by a reduced parameter dependent vector field f of class $\mathscr{C}^{k-1}$ on the finite-dimensional $\operatorname{Ker}\mathcal{T}$;*

(vii) *(correspondence) any element $v = v_0 + \Psi(v_0, \mu)$ of $\mathcal{M}_0^\mu$ corresponds one-to-one to a solution of*

$$\frac{\mathrm{d}v_0}{\mathrm{d}t} = f(v_0) := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}(S_t v)\Big|_{t=0};$$

(viii) *(equivariance) $\operatorname{Ker}\mathcal{T}$ is invariant under Γ and $\mathcal{Q}$ can be chosen to commute with all $\gamma \in S$. Consequently, Ψ commutes with $\gamma \in S$ and $\mathcal{M}_0^\mu$ is invariant under Γ. Finally, the reduced vector field f in item (vi) commutes with all translations S_t and anticommutes with reversible symmetries in S.*

C Coefficients in center manifold reduction

In this appendix, we compute the center manifold coefficients $\psi_{pqlmn} := \Psi'''''_{pqlmn}(0)$ up to second-order terms in Proposition 5.1 as well as the coefficients

$$\psi_{10001}, \quad \psi_{20000}, \quad \psi_{02000}, \quad \psi_{10010}, \quad \psi_{01100}, \quad \psi_{12000}, \quad \psi_{30000}$$

from Section 6. The proof Proposition 5.1 observes that $\mathcal{T}$, $\operatorname{Id} - \mathcal{T}$ and $\mathcal{Q}$ map even to even, and odd to odd functions. Also, the basis functions of $\operatorname{Ker}\mathcal{T}$ are either even or odd functions. Using these, vanishing ψ_{rslmn} are identified and excluded. Then, linear equations

for non-vanishing ψ_{rslmn} are written down. To compute ψ_{rslmn}, we will extensively use

$$
\mathbf{m}(\mathrm{D})\colon x^{2k}\cos(yx) \mapsto \sum_{j=0}^{k}\binom{2k}{2j}(-1)^{j}\mathbf{m}^{(2j)}(y)\cdot x^{2(k-j)}\cos(yx)
$$

$$
+\sum_{j=0}^{k-1}\binom{2k}{2j+1}(-1)^{j}\mathbf{m}^{(2j+1)}(y)\cdot x^{2(k-j)-1}\sin(yx)
$$

(40)

$$
\mathbf{m}(\mathrm{D})\colon x^{2k+1}\sin(yx) \mapsto \sum_{j=0}^{k}\binom{2k+1}{2j}(-1)^{j}\mathbf{m}^{(2j)}(y)\cdot x^{2(k-j)+1}\sin(yx)
$$

$$
+\sum_{j=0}^{k}\binom{2k+1}{2j+1}(-1)^{j+1}\mathbf{m}^{(2j+1)}(y)\cdot x^{2(k-j)}\cos(yx),
$$

where $\mathbf{m}\colon \mathbb{R}\to\mathbb{R}$ is an even multiplier and $y\in\mathbb{R}$. Finally, we observe that if f satisfies $\mathcal{T}f = g$, then $h := f - \mathcal{Q}f$ satisfies $\mathcal{T}h = g$ and $\mathcal{Q}h = 0$.

C.1 For generalized solitary waves

Here, let τ, c_0 satisfy (8) and note, specifically, that $c_0 = 1$ here. Equation (10) with $\varphi = \mathcal{Q}_1\varphi + \Psi$ is
$$
\mathcal{T}\Psi - \mu\mathcal{Q}_1\varphi + (\mathrm{Id}-\mathcal{T})(\mathcal{Q}_1\varphi + \Psi)^2 = 0.
$$

By noting that second-order μ-inhomogeneous terms come from $\mathcal{T}\Psi$ and $(\mathrm{Id}-\mathcal{T})(\mathcal{Q}_1\varphi)^2$, the linear equations from grouping A^2, B^2, C^2, D^2, AB, AC, AD, BC, BD and CD terms are given by

$$
\begin{aligned}
&\mathcal{T}\Psi_{20000} + (\mathrm{Id}-\mathcal{T})1 = 0, && \mathcal{T}\Psi_{00200} + (\mathrm{Id}-\mathcal{T})\cos^2(k_0x) = 0,\\
&\mathcal{T}\Psi_{02000} + (\mathrm{Id}-\mathcal{T})\,x^2 = 0, && \mathcal{T}\Psi_{00020} + (\mathrm{Id}-\mathcal{T})\sin^2(k_0x) = 0,\\
&\mathcal{T}\Psi_{11000} + 2(\mathrm{Id}-\mathcal{T})\,x = 0, && \mathcal{T}\Psi_{10100} + 2(\mathrm{Id}-\mathcal{T})\cos(k_0x) = 0,\\
&\mathcal{T}\Psi_{10010} + 2(\mathrm{Id}-\mathcal{T})\sin(k_0x) = 0, && \mathcal{T}\Psi_{01100} + 2(\mathrm{Id}-\mathcal{T})\,x\cos(k_0x) = 0,\\
&\mathcal{T}\Psi_{00110} + (\mathrm{Id}-\mathcal{T})\sin(2k_0x) = 0,
\end{aligned}
$$

and, by noting that the μ-homogeneous terms come from $\mathcal{T}\Psi$ and $-\mu\mathcal{Q}_1\varphi$,

$$
\begin{aligned}
&\mathcal{T}\Psi_{10001} - 1 = 0, && \mathcal{T}\Psi_{00101} - \cos(k_0x) = 0,\\
&\mathcal{T}\Psi_{01001} - x = 0, && \mathcal{T}\Psi_{00011} - \sin(k_0x) = 0.
\end{aligned}
$$

Note that equations arising from grouping $AB, AD, BC, CD, \mu B$ and μD terms are excluded here since they involve only odd functions. Using (40) with $\mathbf{m} = \ell$ and $y = 0$ or $y = k_0$, we arrive at

$$
\begin{aligned}
&\mathcal{T}\Psi_{10001} = 1, && \mathcal{T}\Psi_{00101} = \cos(k_0x),\\
&\mathcal{T}\Psi_{20000} = -1, && \mathcal{T}\Psi_{10100} = -2\cos(k_0x),\\
&\mathcal{T}\Psi_{02000} = -x^2 - \nu && \mathcal{T}\Psi_{00200} = \tfrac{1}{2}(1 - \ell(2k_0))\cos(2k_0x) + \tfrac{1}{2},\\
&\mathcal{T}\Psi_{01010} = -2\ell'(k_0)\cos(k_0x), && \mathcal{T}\Psi_{00020} = -\tfrac{1}{2} - \tfrac{1}{2}(1 - \ell(2k_0))\sin(2k_0x),
\end{aligned}
$$

all subjected to the condition $\mathcal{Q}_1 \Psi_{pqlmn} = 0$, which ensures uniqueness.

Let $\sigma = \ell''(0)^{-1} = (1/3 - \tau)^{-1}$. Lengthy but straightforward calculations employing (40) with $\mathbf{m} = 1 - \ell$ now yield

$$\psi_{10001} = -\psi_{20000} = 2\sigma k_0^2, \qquad \psi_{00101} = -2\psi_{10100} = -\frac{2k_0^3}{\ell'(k_0)},$$

$$\psi_{02000} = -\frac{\ell''''(0) - 6\sigma^{-2}}{3}\sigma^2 k_0^2 - 4\sigma, \quad \psi_{01010} = 2\frac{\ell''(k_0) - 2\ell'(k_0)^2}{\ell'(k_0)^2}k_0^3 - \frac{10}{\ell'(k_0)}k_0^2,$$

$$\psi_{00200} = \frac{6\ell(2k_0)}{\ell(2k_0) - 1}k_0^4 - \sigma k_0^2, \qquad \psi_{00020} = -\frac{6\ell(2k_0)}{\ell(2k_0) - 1}k_0^4 - \sigma k_0^2.$$

C.2 For modulated solitary waves

Now, let τ, c_0 satisfy (9) and note in this case that both c_0 and τ_0 are parametrized by $s \in (0, \infty)$. Applying (40) with $\mathbf{m} = \ell$, $y = 0, s$ or $2s$ gives

$$\mathcal{T}\Psi_{10001} = \frac{1}{c_0}\cos(sx),$$

$$\mathcal{T}\Psi_{20000} = -\frac{1}{2}(1 + \ell(2s)\cos(2sx)),$$

$$\mathcal{T}\Psi_{11000} = -\ell(2s)\sin(2sx),$$

$$\mathcal{T}\Psi_{02000} = -\frac{1}{2}(1 - \ell(2s)\cos(2sx)),$$

$$\mathcal{T}\Psi_{10010} = \mathcal{T}\Psi_{01100} = \ell'(2s)\cos(2sx) - \ell(2s)x\sin(2sx),$$

$$\mathcal{T}\Psi_{30000} = -\frac{2}{c_0}(\mathrm{Id} - \mathcal{T})\cos(sx)\Psi_{20000},$$

$$\mathcal{T}\Psi_{12000} = -\frac{2}{c_0}(\mathrm{Id} - \mathcal{T})(\cos(sx)\Psi_{02000} + \sin(sx)\Psi_{11000}),$$

where all Ψ_{pqlmn} are subject to $\mathcal{Q}_2\Psi_{pqlmn} = 0$ and $\mathcal{Q}_2$ is given in (18). Using (40) with $\mathbf{m} = 1 - c_0\ell$, $y = 0, s, 2s$ or $3s$ again, then evaluating $\Psi'''_{pqlmn}(0) = \psi_{pqlmn}$ gives

$$\psi_{10001} = \frac{16s^2}{c_0^2\ell''(s)},$$

$$\psi_{20000} = -\frac{s^4}{2}\left(\frac{1}{1 - c_0} + \frac{15\ell(2s)}{1 - c_0\ell(2s)}\right),$$

$$\psi_{02000} = -\frac{s^4}{2}\left(\frac{1}{1 - c_0} - \frac{15\ell(2s)}{1 - c_0\ell(2s)}\right),$$

$$\psi_{10010} = \psi_{01100} = 9s^4\frac{(1 - 2c_0\ell(2s))\ell'(2s)}{(1 - c_0\ell(2s))^2} - 8s^3\frac{\ell(2s)}{1 - c_0\ell(2s)},$$

$$\psi_{30000} = a(s^4 - 8s^2) - 14bs^2 + 21c(s^4 - 8s^2) + 91d(s^4 - 8s^2) - 6es^3,$$

$$\psi_{12000} = \tilde{a}(s^4 - 8s^2) - 14\tilde{b}s^2 + 21\tilde{c}(s^4 - 8s^2) + 91\tilde{d}(s^4 - 8s^2) - 6\tilde{e}s^3,$$

with

$$u = -\frac{1}{2}\frac{1}{1-c_0}, \qquad v = -\frac{1}{2}\frac{\ell(2s)}{1-c_0\ell(2s)},$$

$$a = -2u(u+v), \qquad b = \frac{2u+v}{c_0^2\ell''(s)},$$

$$c = -2v(u+v) + 2v\frac{\ell'(2s)}{\ell(2s)}\left(\frac{s}{2}(u+v) - 2sv\right)(1-2vc_0),$$

$$d = -\frac{\ell(3s)}{1-c_0\ell(3s)}v, \qquad e = -2v\left(\frac{s}{2}(u+v) - 2sv\right),$$

and

$$\tilde{a} = -2u(u-v) - 14u\ell(2s), \qquad \tilde{b} = \frac{2u-v-\ell(2s)}{c_0^2\ell''(s)},$$

$$\tilde{c} = 2v(u-v) - 2v\frac{\ell'(2s)}{\ell(2s)}\left(\frac{s}{2}(u-v) + 2sv\right)(1+2vc_0) + 14v - 6s\ell'(2s)v(1-2v),$$

$$\tilde{d} = v\frac{\ell(3s)}{1-c_0\ell(3s)} + \frac{\ell(2s)\ell(3s)}{1-c_0\ell(3s)}, \qquad \tilde{e} = 2v\left(\frac{s}{2}(u-v) + 2sv\right) + 6\ell(2s)sv.$$

Since they are useful in the calculations in Section 6, we identify the dominant terms from the Taylor expansions of $c_0(s), \tau_0(s), \ell$ for small $s > 0$ and obtain

$$\psi_{10001} = 48s^2 - \frac{176}{5}s^4 + \mathcal{O}(s^6), \quad \psi_{20000} = -45 - \frac{165}{56}s^2 + \mathcal{O}(s^4),$$

$$\psi_{02000} = -45 - \frac{795}{56}s^2 + \mathcal{O}(s^4), \quad \psi_{10010} = \psi_{01100} = 6 - \frac{27}{4}s + \frac{21}{4}s^2 - \frac{297}{16}s^3 + \mathcal{O}(s^4),$$

$$\psi_{30000} = 4 \cdot 90^2\frac{1}{s^6} + \mathcal{O}(s^{-4}), \qquad \psi_{12000} = 4 \cdot 90^2\frac{1}{s^6} + \mathcal{O}(s^{-4}).$$

D   Coefficients in normal form reduction

D.1   For generalized solitary waves

Our goal here is to compute coefficients p_0, p_1, p_2, p_3, q_0 and q_1 in Section 5.2. The expansion of Φ in $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})$ and μ up to second-order terms is

$$\Phi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)$$
$$= \phi_0\mu + \phi_{1000}\mu\mathbf{A} + \phi_{0100}\mu\mathbf{B} + \phi_{0010}\mu\mathbf{C} + \phi_{0001}\mu\overline{\mathbf{C}} + \phi_{2000}\mathbf{A}^2 + \phi_{1100}\mathbf{AB}$$
$$+ \phi_{1010}\mathbf{AC} + \phi_{1001}\mathbf{A}\overline{\mathbf{C}} + \phi_{0200}\mathcal{B}^2 + \phi_{0110}\mathbf{BC} + \phi_{0101}\mathbf{B}\overline{\mathbf{C}} + \phi_{0020}\mathbf{C}^2$$
$$+ \phi_{0011}|\mathbf{C}|^2 + \phi_{0002}\overline{\mathbf{C}}^2 + \mathcal{O}(|\mu^2| + (|\mu| + |(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})|)^3)).$$

Denote the coefficients in front of $(A, B, C, D, \mu)^\alpha$ in (21) by ψ_α. We also Taylor expand the nonlinear term

$$\mathbf{R}(U, \mu) = \mu\mathbf{R}_1(U) + \mathbf{R}_2(U, U) + \mathcal{O}(|\mu|^2 + |(U, \mu)|^3),$$

where

$$\mathbf{R}_1(x, y, z, w) = \frac{1}{k_0^3}\begin{pmatrix} 0 \\ k_0\psi_{10001}x + k_0\psi_{00101}z \\ 0 \\ -\psi_{10001}x - \psi_{00101}z \end{pmatrix} \quad \text{and} \quad \mathbf{R}_2(U, \tilde{U}) = \frac{1}{k_0^3}\begin{pmatrix} 0 \\ k_0H(U, \tilde{U}) \\ 0 \\ -H(U, \tilde{U}) \end{pmatrix}$$

30

with

$$H(U, \tilde{U}) = H((x, y, z, w), (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}))$$

$$= \psi_{20000} x\tilde{x} + \frac{\psi_{10100}}{2}(x\tilde{z} + z\tilde{x}) + \psi_{20000} z\tilde{z} + \psi_{02000} y\tilde{y} + \frac{\psi_{01010}}{2}(y\tilde{w} + w\tilde{y}) + \psi_{00020} w\tilde{w}.$$

Plugging $U = \mathbf{A}\xi_0 + \mathbf{B}\xi_1 + \mathbf{C}\zeta + \overline{\mathbf{C}\zeta} + \Phi$ into (23), relevant linear equations are identified

$$\begin{aligned}
\mathcal{O}(\mu): & \quad p_0\xi_1 = \mathbf{L}\phi_0 \\
\mathcal{O}(\mu\mathbf{A}): & \quad p_1\xi_1 + p_0\phi_{1100} = \mathbf{L}\phi_{1000} + \mathbf{R}_1(\xi_0) \\
\mathcal{O}(\mathbf{A}^2): & \quad p_2\xi_1 = \mathbf{L}\phi_{2000} + \mathbf{R}_2(\xi_0, \xi_0) \\
\mathcal{O}(|\mathbf{C}|^2): & \quad p_3\xi_1 = \mathbf{L}\phi_{0011} + 2\mathbf{R}_2(\zeta, \overline{\zeta}) \\
\mathcal{O}(\mu\mathbf{C}): & \quad iq_0\zeta + p_0\phi_{0110} + ik_0\phi_{0010} = \mathbf{L}\phi_{0010} + \mathbf{R}_1(\zeta) \\
\mathcal{O}(\mathbf{A}\mathbf{C}): & \quad iq_1\zeta + ik_0\phi_{1010} = \mathbf{L}\phi_{1010} + 2\mathbf{R}_2(\xi_0, \zeta).
\end{aligned}$$

Since ξ_1 is not in the range of $\mathbf{L}$, $p_0 = 0$. Similarly, the equation from $\mathcal{O}(\mu\mathbf{A})$ terms

$$\mathbf{L}\phi_{1000} = p_1\xi_1 - \mathbf{R}_1(\xi_0) = \frac{1}{k_0^3}\begin{pmatrix} 0 \\ k_0^3 p_1 - k_0\psi_{10001} \\ 0 \\ -\psi_{10001} \end{pmatrix}$$

is solvable if and only if $k_0^3 p_1 - k_0\psi_{10001} = 0$. Equations for c and d are handled in the same fashion. To solve q_0, we note that

$$(\mathbf{L} - ik_0)\phi_{0010} = iq_0\zeta - \mathbf{R}_1(\zeta)$$

Writing $\phi_{0010} = x_0\xi_0 + x_1\xi_1 + x_3\zeta + \overline{x_3\zeta}$, equation (24) can be used to show that $x_3\zeta = 0$ on the left-hand side. The right-hand side in the basis $\{\xi_0, \xi_1, \zeta, \overline{\zeta}\}$ is

$$iq_0\zeta - \mathbf{R}_1(\zeta) = \frac{1}{k_0^2}\psi_{00101}\xi_1 + \left(iq_0 + \frac{i}{2k_0^3}\psi_{00101}\right)\zeta - \frac{i}{2k_0^3}\overline{\zeta},$$

which gives $q_0 = -(2k_0^3)^{-1}\psi_{00101}$. Using results from Appendix C.1 and writing $\sigma = (1/3 - \tau)^{-1}$, we get

$$p_0 = 0, \quad p_1 = 2\sigma, \quad p_2 = p_3 = -2\sigma, \quad q_0 = -\frac{1}{\ell'(k_0)}, \quad q_1 = -\frac{4}{\ell'(k_0)}.$$

D.2 For modulated solitary waves

As before, the Taylor expansion of $\Phi(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu)$ is

$$\Phi(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu) = \sum_{\substack{2 \leq p+q+l+m+n \leq 3 \\ n \geq 1}} \phi_{pqlmn}\mathbf{A}^p\mathbf{B}^q\overline{\mathbf{A}}^l\overline{\mathbf{B}}^m\mu^n + \cdots.$$

Our goal here is to determine the Taylor expansion of $\mathbf{R}(U, \mu)$ from (34) up to order three. Using the symmetries as in the proof of Proposition 5.1, contributing terms are

$$\begin{aligned}
& \mathcal{O}(\mu A), \mathcal{O}(\mu D), \mathcal{O}(\mu^2 A), \mathcal{O}(\mu^2 D), \mathcal{O}(\mu A^2), \mathcal{O}(\mu D^2), \\
& \mathcal{O}(A^2), \mathcal{O}(AD), \mathcal{O}(B^2), \mathcal{O}(BC), \mathcal{O}(C^2), \mathcal{O}(D^2), \\
& \mathcal{O}(A^3), \mathcal{O}(A^2 D), \mathcal{O}(AB^2), \mathcal{O}(AC^2), \mathcal{O}(AD^2), \mathcal{O}(B^2 D), \mathcal{O}(C^2 D), \mathcal{O}(D^3).
\end{aligned}$$

In short, if a multiplication between a pair or a triple of $\cos(sx), \sin(sx), x\cos(sx), x\sin(sx)$ is an even function in x, the product of their coefficients A, B, C, D will contribute to $\Psi(A, B, C, D, \mu)''''(0)$. Denote the coefficients of $A^p B^q C^l D^m \mu^n$ by ψ_{pqlmn}. The Taylor expansion of $\mathbf{R}(U, \mu)$ is

$$\mathbf{R}(U, \mu) = \mu\mathbf{R}_{11}(U) + \mu^2\mathbf{R}_{12}(U) + \mathbf{R}_{20}(U, U) + \mu\mathbf{R}_{21}(U, U) + \mathbf{R}_{30}(U, U, U),$$

where

$$\mathbf{R}_{11}(U) = \frac{1}{2s^3}\begin{pmatrix} 0 \\ H_1(U) \\ -sH_1(U) \\ 0 \end{pmatrix}, \quad \mathbf{R}_{20}(U, \tilde{U}) = \frac{1}{2s^3}\begin{pmatrix} 0 \\ H_2(U, \tilde{U}) \\ -sH_2(U, \tilde{U}) \\ 0 \end{pmatrix},$$

$$\mathbf{R}_{30}(U, \tilde{U}, \hat{U}) = \frac{1}{2s^3}\begin{pmatrix} 0 \\ H_3(U, \tilde{U}, \hat{U}) \\ -sH_3(U, \tilde{U}, \hat{U}) \\ 0 \end{pmatrix},$$

with

$H_1(x, y, z, w)$
$\quad = \psi_{10001}x + \psi_{00011}w,$
$H_2((x, y, z, w), (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}))$
$\quad = \psi_{20000}x\tilde{x} + \psi_{02000}y\tilde{y} + \psi_{00200}z\tilde{z} + \psi_{00020}w\tilde{w} + \frac{1}{2}\psi_{10010}(x\tilde{w} + \tilde{x}w) + \frac{1}{2}\psi_{01100}(y\tilde{z} + \tilde{y}z),$
$H_3((x, y, z, w), (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}), (\hat{x}, \hat{y}, \hat{z}, \hat{w}))$
$\quad = \psi_{30000}x\tilde{x}\hat{x} + \frac{\psi_{20010}}{3}(x\tilde{x}\hat{w} + x\hat{x}\tilde{w} + \tilde{x}\hat{x}w) + \frac{\psi_{12000}}{3}(x\tilde{y}\hat{y} + \tilde{x}y\hat{y} + \hat{x}y\tilde{y})$
$\quad + \frac{\psi_{10200}}{3}(x\tilde{z}\hat{z} + \tilde{x}z\hat{z} + \hat{x}z\tilde{z}) + \frac{\psi_{10020}}{3}(x\tilde{w}\hat{w} + \tilde{x}w\hat{w} + \hat{x}w\tilde{w}) + \frac{\psi_{02010}}{3}(y\tilde{y}\hat{w} + y\hat{y}\tilde{w} + \tilde{y}\hat{y}w)$
$\quad + \frac{\psi_{00210}}{3}(z\tilde{z}\hat{w} + z\hat{z}\tilde{w} + \tilde{z}\hat{z}w) + \psi_{00030}w\tilde{w}\hat{w}.$

Let $\zeta_1^* = \frac{1}{2}(0, 0, 1, -i)$. It is a vector orthogonal to the range of $\mathbf{L} - is$ and satisfies

$$\langle \zeta_0, \zeta_1^* \rangle = 0, \quad \langle \overline{\zeta_0}, \zeta_1^* \rangle = 0, \quad \langle \zeta_1, \zeta_1^* \rangle = 0, \quad \langle \overline{\zeta_1}, \zeta_1^* \rangle = 0,$$

and $R\zeta_1^* = -\overline{\zeta_1^*}$. Equations (D.45) and (D.47), [17, Appendix D.2], give

$$q_0 = \langle \mathbf{R}_{11}\zeta_0 + 2\mathbf{R}_{20}(\zeta_0, \phi_{00001}), \zeta_1^* \rangle,$$
$$q_1 = \langle 2\mathbf{R}_{20}(\zeta_0, \phi_{10100}) + 2\mathbf{R}_{20}(\overline{\zeta_0}, \phi_{20000}) + 3\mathbf{R}_{30}(\zeta_0, \zeta_0, \overline{\zeta_0}), \zeta_1^* \rangle,$$

respectively. Here, $\phi_{00001}, \phi_{10100}, \phi_{20000}$ satisfy

$$\mathbf{L}\phi_{00001} + \mathbf{R}_{01} = 0, \quad \mathbf{L}\phi_{10100} + 2\mathbf{R}_{20}(\zeta_0, \overline{\zeta_0}) = 0, \quad (\mathbf{L} - 2is)\phi_{20000} + \mathbf{R}_{20}(\zeta_0, \zeta_0) = 0.$$

A computation gives

$$\phi_{00001} = 0, \quad \phi_{10100} = \frac{\psi_{20000} + \psi_{02000}}{s^3}\begin{pmatrix} 2/s \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \phi_{20000} = \frac{\psi_{20000} - \psi_{02000}}{6s^3}\begin{pmatrix} -2/(3s) \\ 2i/(3s) \\ -2i \\ 1 \end{pmatrix},$$

which in turn yields

$$q_0 = -\frac{1}{4s^2}\psi_{10001},$$

$$q_1 = -\frac{17}{18s^6}\psi_{20000}^2 - \frac{10}{9s^6}\psi_{20000}\psi_{02000} + \frac{1}{18s^6}\psi_{02000}^2 - \frac{1}{12s^5}\psi_{01100}(\psi_{20000} - \psi_{02000})$$

$$+ \frac{1}{24s^5}\psi_{10010}(5\psi_{20000} + 7\psi_{02000}) - \frac{3}{4s^2}\left(\psi_{30000} + \frac{\psi_{12000}}{3}\right).$$

References

[1] C. J. Amick and K. Kirchgässner, *Solitary water-waves in the presence of surface tension*, Dynamical problems in continuum physics (Minneapolis, Minn., 1985), 1987, pp. 1–22. MR893395

[2] J. C. Amick and K. Kirchgässner, *A theory of solitary water-waves in the presence of surface tension*, Arch. Rational Mech. Anal. **105** (1989), 1–49.

[3] M. N. Arnesen, *Existence of solitary-wave solutions to nonlocal equations*, Discrete Contin. Dyn. Syst. **36** (2016), no. 7, 3483–3510.

[4] J. T. Beale, *Exact solitary water waves with capillary ripples at infinity*, Comm. Pure Appl. Math. **44** (1991), no. 2, 211–257. MR1085829

[5] G. Bruell, M. Ehrnström, and L. Pei, *Symmetry and decay of traveling wave solutions to the Whitham equation*, J. Differential Equations **262** (2017), no. 8, 4232–4254.

[6] B. Buffoni, M. D. Groves, and J. F. Toland, *A plethora of solitary gravity-capillary water waves with nearly critical bond and froude numbers*, Philos. Trans. R. Soc. London. Ser. A. Math. Phys. Sci. Eng. **354** (1996), 575–607.

[7] R. M. Chen, S. Walsh, and M. H. Wheeler, *Center manifolds without a phase space for quasilinear problems in elasticity, biology, and hydrodynamics* (2019).

[8] F. Dias and G. Iooss, *Capillary-gravity solitary waves with damped oscillations*, Phys. D **65** (1993), 399–423.

[9] M. Ehrnström, M. D. Groves, and E. Wahlén, *On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type*, Nonlinearity **25** (2012), no. 10, 2903–2936.

[10] M. Ehrnström, M. A. Johnson, and K. M. Claassen, *Existence of a highest wave in a fully dispersive two-way shallow water model*, Arch. Ration. Mech. Anal. **231** (2019), no. 3, 1635–1673. MR3902471

[11] M. Ehrnström, M. A. Johnson, O. I. H. Maehlen, and F. Remonato, *On the bifurcation diagram of the capillary-gravity Whitham equation*, Water Waves **1** (2019), 217–313.

[12] M. Ehrnström and H. Kalisch, *Traveling waves for the Whitham equation*, Differential Integral Equations **22** (2009), no. 11-12, 1193–1210.

[13] M. Ehrnström and E. Wahlén, *On Whitham's conjecture of a highest cusped wave for a nonlocal dispersive equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2019), no. 6, 1603–1637.

[14] G. Faye and A. Scheel, *Center manifolds without a phase space*, Trans. Amer. Math. Soc. **370** (2018), no. 8, 5843–5885.

[15] _____, *Corrigendum to center manifolds without a phase space* (2020). Preprint `https://arxiv.org/pdf/2007.14260.pdf`.

[16] V. V. Grušin, *Pseudodifferential operators in R^n with bounded symbols*, Funkcional. Anal. i Priložen **4** (1970), no. 3, 37–50.

[17] M. Haragus and G. Iooss, *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems*, Springer, London, 2011.

[18] V. M. Hur, *Wave breaking in the Whitham equation*, Adv. Math. **317** (2017), 410–437.

[19] V. M. Hur and M. A. Johnson, *Modulational instability in the Whitham equation for water waves*, Stud. Appl. Math. **134** (2015), no. 1, 120–143. MR3298879

[20] ———, *Modulational instability in the Whitham equation with surface tension and vorticity*, Nonlinear Anal. **129** (2015), 104–118.

[21] G. Iooss and M. C. Pérouème, *Perturbed homoclinic solutions in reversible 1:1 resonance vector fields*, J. Diff. Equ. **102** (1993), 62–88.

[22] G. Iooss and K. Kirchgässner, *Water waves for small surface tension: an approach via normal form*, Proc. Roy. Soc. Edinburg **122** (1991), 267–299.

[23] M. A. Johnson and D. J. Wright, *Generalized solitary waves in the gravity-capillary Whitham equation* (2018).

[24] R. S. Johnson, *A modern introduction to the mathematical theory of water waves*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1997. MR1629555

[25] K. Kirchgässner, *Nonlinearly resonant surface waves and homoclinic bifurcation*, 1988, pp. 135–181.

[26] E. Lombardi, *Orbits homoclinic to exponentially small periodic orbits for a class of reversible systems. application to water waves.*, Arch. Rational Mech. Anal. **137** (1997), 227–304.

[27] ———, *Oscillatory integrals and phenomena beyond all algebraic orders. with applications to homoclinic orbits in reversible system*, Lecture Notes in Mathematics, vol. 1741, Springer-Verlag, Berlin, 2002.

[28] S. M. Sun, *Existence of a generalized solitary wave solution for water with positive Bond number less than 1/3*, J. Math. Anal. Appl. **156** (1991), 471–504.

[29] S. M. Sun and M. C. Shen, *Exponentially small estimate for the amplitude of capillary ripples of a generalized solitary waves*, J. Math. Anal. Appl. **172** (1993), 533–566.

[30] T. Truong, E. Wahlén, and M. H. Wheeler, *Global bifurcation of solitary waves for the Whitham equation* (2020).

[31] G. B. Whitham, *Variational methods and applications to water waves*, Hyperbolic equations and waves (Rencontres, Battelle Res. Inst., Seattle, Wash., 1968), 1970, pp. 153–172.

[32] ———, *Linear and nonlinear waves*, John Wiley & Sons, 1999.