Option Values in Sequential Auctions with Time-Varying Valuations

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Abstract. We investigate second-price sequential auctions of unit-demand bidders with time-variable valuations under complete information. We describe how a bidder figures willingness to pay by calculating option values, and show that when bidders bid their option value, and a condition of *consistency* is fulfilled, a subgame-perfect equilibrium is the result. With no constraints on valuations, equilibria are not necessarily efficient, but we show that when bidder valuations satisfy a certain constraint, an efficient equilibrium always exists. This result may be extended to a model with arrivals of bidders. We show how the equilibrium allocation, bids, and bidder utilities are calculated in the general case. We prove constructively that a pure subgame-perfect equilibrium always exists, and show how all pure equilibria can be found by the method of option values

1 Introduction

This paper studies a setting where several identical items are sold sequentially via a second-price auction to a set of bidders. In each round, a bidder may have a different value for the item, and this value may change in arbitrary ways. This can capture, e.g., situations where the bidder is absent in some periods (indicated by having zero value), situations with discount factors, etc. Bidders may win at most a single item, and have complete information about the setting. This scenario is common in many computational settings in which resources are allocated periodically. For example, in Bitcoin, every 10 minutes a new block is mined and the transaction slots of this block are allocated to the users. In cloud computing, computing resources are sold periodically. And so on. In such settings, a bidder's value for the item is often time-dependent, e.g., different bidders have different arrival times, different departure times, different urgencies, and so on. Bidder values may decrease to reflect a preference to get the item sooner rather than later, or it may increase, e.g., when bidders value the flexibility to change their mind about needing the item.

We investigate strategic behavior in these settings. Even in a complete-information setting, when valuations change between rounds, it is unclear how a bidder should approach the bidding decision and how much should be his maximal willingness to pay in the current round. A natural approach that was suggested by Bernhardt and Scoones (1994) is the option-value approach: a player recursively determines his resulting utility in future rounds assuming that he sits out the current round (we term this the option utility). The player's maximal willingness to pay in the current round, the option value, is then his current value minus his option utility – paying more than that will result in a utility lower than the utility he can obtain in future rounds. In a sequential second-price auction, bids should apparently be the maximal willingness to pay, if the logic from the one-round setting serves as a guide. In this paper we examine the various aspects that stem from this strategic reasoning. In particular, we investigate several questions: How does one find an option-value equilibrium? Does one always exist? Is it necessarily efficient, in the sense

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that the sum of winner values is maximal? In a single round, the classic answer to the last two questions is *yes*. We seek a generalization to sequential auctions.

The problem is that the basic intuition, described above, cannot be implemented in a straightforward way. The option value is not well-defined since it depends on who wins the current auction if the player sits out. But, if all bidders make option-value calculations to determine their bid, an assumption about which bidder wins includes circular logic. This vicious circle will not be solved by backward induction, as in every round we have a simultaneous-move game.

In Bernhardt and Scoones (1994), who discuss incomplete-information two-round second-price auctions with unit demand, these complexities are circumvented by the adoption of an information structure in which bidders are *ex ante* indifferent to who wins any round.

Our results answer all these questions. Our solution to the cyclic nature of option values is in the concept of *consistency*: We start by *assuming* the highest (and second-highest) bidder, calculate option values based on that assumption, and finally discard assumptions that are found to be inconsistent, in the sense that the highest bidders are *not* who were assumed. As we show, assumptions that survive this consistency test are subgame-perfect equilibria (SPE) of the sequential auction.

We provide algorithms to calculate option-value bidding strategies, and to test them for consistency. These bidding strategies are pure and Markovian (i.e., depend only on the remainder of the sequential auction, regardless of how it was reached). Furthermore, we introduce the technique of *option-value matrices*, which efficiently and exhaustively discovers all option-value equilibria in a given valuation setting.

It is an intriguing empirical observation that option-value equilibria are often efficient, especially since, as we show, they not always are. We find a specific sufficient condition on bidders' valuations, which we call "ordered differences", for the existence of an *efficient* option-value SPE.

In the general case, these equilibria are not necessarily efficient, nor are they necessarily unique, nor do they always exist, though we show that there always exists an SPE in pure, Markovian strategies. Furthermore, we show that every pure SPE relies on option-value bidding, and our aforementioned technique of option-value matrices can be used to discover all pure equilibria.

In single-round second-price auctions, it is well-known that bidding one's value weakly dominates every other bidding strategy, and so must be played by every rational player. In a sequential second-price auction, this is untrue in the general case. Bidding one's value (option value in this case) remains a best response to any set of actions by other bidders, but it is not necessarily irrational to bid differently. I.e., the option-value bid does not necessarily weakly dominate all other bids.[3] However, we conjecture that, under our ordered-differences condition, the only equilibrium in weakly-undominated strategies is an option-value equilibrium.

Our emphasis on option-value bidding is justified by several considerations: First, as stated at the end of Section 5.2, bidding the option value is always a best response to other bids, even when not in equilibrium. Second, our construction of a pure equilibrium is derived from the option-value matrix (Definition 2), thus showing that the concept and methodology developed in this paper is highly relevant. Third, the logic behind option values comes naturally, and we believe that because of this it should be investigated and better understood.

Complete-information auctions have been recognized as an important category of auctions and have been justified many times. In their seminal paper on menu auctions, Bernheim and Whinston (1986) say that the assumption of complete information is often a good approximation and that in many settings "bidders are typically quite well informed about each others' costs...".

[3] Unlike in the single-round auction, the option value depends on other bids and is not a constant. It therefore does not amount to a weakly-undominated strategy, as such a strategy must be constant.

This view is continued to be held and in the last decade we have seen a significant interest in complete information auctions, see for example: Edelman et al. (2007), Varian and Harris (2014), Caragiannis et al. (2015), Roughgarden et al. (2017) and Christodoulou et al. (2016). The assumption of complete-information (a.k.a. the assumption of fully informed buyers) in various models of sequential auctions has been previously made by, e.g., Dudey (1992), Krishna (1993), Krishna (1999), Gale and Stegeman (2001) and Paes Leme et al. (2012). Narayan et al. (2019), in a very recent study on the well-documented declining price anomaly in sequential auctions, justify the assumption of complete information by remarking that "the restriction to full information is extremely useful ... [since] it allows one to focus on the strategic properties caused purely by the sequential sales of items and not by a lack of information".

In Bitcoin, all new transactions are stored in the so-called *mempool*, which is a distributed shared repository viewable by all. A transaction contains its transaction fee in addition to all other parameters including, e.g., the amount of funds to be transferred. Thus, users view most relevant details on other users which is an approximation to the complete information assumption. The transaction fees are, *de facto*, bids in a sequential auction which recurs approximately every 10 minutes. Miners naturally prioritize transactions with the highest fees. The transactions and their respective bids remain in the mempool for all recurring auctions (block compositions) until being included in a block (i.e., winning an auction). Moreover, Bitcoin's bidding protocol could be enhanced to allow transactions to specify different fees/bids for different time periods to allow transactions to express urgencies, deadlines, delayed transactions, installments, as well as many other temporal considerations. This will make the entire time sequence complete-information. Our model and methodology could be useful to determine transaction payments in such a context.

Moreover, we argue and demonstrate that the notion of option values that we formalize in this paper has an important role also for the analysis of incomplete information settings. Section 5 gives more details along with a formal model and some preliminary results.

To the best of our knowledge, the question of the existence of an efficient SPE in sequential auctions with general time-variable values was not previously considered. Most of the literature focuses on sequential auctions with incomplete information and stochastically-equivalent objects, i.e., bidders do not know their values for future periods, and all values are i.i.d.. Engelbrecht-Wiggans (1994) shows that in sequential auctions of a large-enough number of stochastically equivalent objects, with bounded values, prices will, on average, have a downwards trend. Said (2011) adds to the model the possibility of entry of new buyers, and the stochastic arrival of objects for sale. These two papers rely on the notion of option-value bidding, and make it tractable by the assumption of stochastic equivalence. Zeithammer (2006) conducts an empirical study that incorporates knowledge of future values, with three levels of sophistication, the lowest of which roughly corresponds to Engelbrecht-Wiggans (1994). In the highest level of sophistication, each bidder actually knows their value for future objects up for sale. His empirical results confirm that bidders engage in bidding sensitive to future values. The author, in his discussion, notes that this departs from previous models of sequential auctions and concludes: "These findings contribute to the auction theory literature and are relevant to bidders in sequential auctions on eBay and elsewhere." Earlier models usually assume constant values over time. For more details and references, see Krishna (2009).

In computer science's recent literature, Paes Leme et al. (2012) study the price of anarchy of complete information sequential auctions when bidders have values for subsets of items. *Inter alia*, they show that first-price sequential auctions have pure equilibria. We prove the existence of pure and Markovian equilibria of second-price sequential auctions, and furthermore show how to construct all such equilibria by calculating option values. We believe that our focus on Markovian strategies is important. Indeed, various counterexamples constructed in Paes Leme et al. (2012) involve various punishment techniques (as is common in repeated games) that rely on non-anonymity assumptions which are not so natural in online settings (which are our main motivation). In accordance, the resulting equilibrium outcomes of such strategies do not seem typical in our motivating online settings and applications. Another important difference from Paes Leme et al. (2012) is our definition of unit-demand bidders. In their definition, a unit-demand bidder that wins an item early on can continue to bid in subsequent auctions and (if winning) can gain additional value in case these later items are more valuable to the bidder. In contrast, we assume that once a bidder wins an item, all later items become worthless and therefore the bidder stops participating in future auctions. We believe that our assumption is more relevant to the online settings we consider as motivation. For example, someone who is interested in buying a ticket for a show might prefer a Friday ticket over a Tuesday ticket, but once a Tuesday ticket is bought and the show has been viewed, the value of a later ticket drops to a rewatch value, which is often close to zero. The same is true for buying a flight ticket. Similarly, someone who is interested in executing a program on the cloud, or submitting a transaction via Bitcoin, might prefer to perform this tomorrow, but if an earlier slot is purchased and the program/transaction has been completed, oftentimes it is not beneficial (or even possible) to repeat it the next day.

The rest of this paper is organized as follows. Section 2 describes our model. Section 3 discussed the option-value bidding strategy. In Section 4 we state and prove our results on efficient allocations. In Section 5 we discuss ramifications for the incomplete-information case, and in Section 6 we offer concluding remarks. Long proofs are to be found in the Appendix.

2 Model

Setting There are n individual bidders, labeled $1, 2, \ldots, n$. Each bidder can win only one item, one of which is auctioned in each of m sequential second-price auctions, which we also call *rounds*, held in order $1, 2, \ldots, m$.

Bidders may be absent in some of the rounds, in which case their value for the item in those rounds is 0. If present, a bidder's value is strictly positive. A bidder cannot be allocated a unit when absent.

Negative bids are invalid, as are bids by a bidder whose value is 0 (a bid of zero can conceivably win a round, but only if its bidder is present, i.e., has a positive value[4]). If no valid bid is submitted in a round, no unit is allocated.

The bidders' valuation of the item may vary over time. The $n \times m$ matrix

$$\boldsymbol{V} = \begin{pmatrix} v_1^1 \ldots v_1^m \\ \vdots \qquad \vdots \\ v_n^1 \ldots v_n^m \end{pmatrix}$$

details the value v_i^j of each bidder i in each round j. $\boldsymbol{V}$ is commonly known.

We assume that ties are broken deterministically and consistently. W.l.o.g. ties are broken in favour of the bidder with the smallest label (row number). In other words, bidders are labeled in accordance with the tie-breaking order. Other than changing the tie-breaking order, permuting the order of rows in a game has no consequence. The columns are arranged chronologically, with the earliest round first.

[4] Some readers found this self-contradictory, but having a zero value is quite different from making a zero bid.

Subgames Let $I \subseteq [n]$ be a subset of bidders. Let j be a positive integer, and let $\boldsymbol{V}$ be a value matrix. The subgame $\boldsymbol{V}_I^j$ is a sequential auction with $m - j + 1$ rounds, if $j \leq m$, or with no rounds if $j > m$, in which every bidder $i \in I$ participates, with valuation $v_i^j, \dots, v_i^m$ in each of the rounds. The order of rows in the subgame is the order of labels (the order in $\boldsymbol{V}$). The valuation matrix of the subgame is a sub-matrix of the main matrix, in which some of the rows and columns are omitted, and is also labeled $\boldsymbol{V}_I^j$, i.e., we identify subgames with submatrices.

We use some shorthand notation: For a bidder k, $I \setminus k$ is shorthand for $I \setminus \{k\}$. For a set of bidders K, $\setminus K$ is shorthand for $[n] \setminus K$. We can combine both shorthand notations: $\setminus k$ stands for $[n] \setminus \{k\}$. Also, $\boldsymbol{V}^j$ is shorthand for $\boldsymbol{V}_{[n]}^j$, and $\boldsymbol{V}_i^j$ for $\boldsymbol{V}_{\{i\}}^j$.

The main game, for example, is $\boldsymbol{V}_{[n]}^1$, or $\boldsymbol{V}$ for short. If bidder i is allocated a unit in the first round, the remaining bidders continue playing the subgame $\boldsymbol{V}_{[n]\setminus i}^2$. This subgame is again a sequential auction, whose valuation matrix, $\boldsymbol{V}_{[n]\setminus i}^2$, is $\boldsymbol{V}$ with row i and column 1 omitted. Note that round numbers are retained in subgames, so that the first round of $\boldsymbol{V}_{[n]\setminus i}^2$ is 2, not 1.

$\mathcal{G}(\boldsymbol{V})$ is the set of all subgames of $\boldsymbol{V}$.

Allocations and Matchings An *allocation* of the sequential auction is a vector in $\{0, 1, \dots, n\}^m$, $\boldsymbol{A} = (A_1, A_2, \dots, A_m)$, where, for every $j \in [m]$, $A_j = 0$ indicates that round j has no unit allocated. Otherwise A_j is the bidder to which a unit is allocated in round j. An allocation is *feasible* if

(i) No bidder is matched to a round with zero valuation, i.e., for every $j \in [m]$, if $A_j > 0$, $v_{A_j}^j > 0$.

(ii) Every bidder appears in it at most once, i.e., for every $j \neq k$, $A_j \neq A_k$, unless $A_j = A_k = 0$. Every allocation corresponds to a partial or maximal matching of bidders to rounds, and vice versa.

The *social welfare* of an allocation is the sum of values of allocated items, for the winning bidders at the time won.

$$SW(\boldsymbol{A}) := \sum_{j \in [m], A_j > 0} v_{A_j}^j \tag{1}$$

An *efficient* allocation is one whose social welfare is maximal. I.e., $\boldsymbol{A}$ is efficient, iff for every allocation $\boldsymbol{A'}$, $SW(\boldsymbol{A}) \geq SW(\boldsymbol{A'})$. Clearly every game has an efficient allocation, but it may not be unique.

An efficient allocation may be found by calculating a maximal weighted matching of bidders to rounds, e.g., by the Hungarian Algorithm, which solves the problem in the order of $[\max(m, n)]^3$ steps.

We define the *social welfare* of a valuation matrix $\boldsymbol{V}$, marked $SW(\boldsymbol{V})$, as the social welfare of any efficient allocation. It follows that for every allocation $\boldsymbol{A}$ of $\boldsymbol{V}$,

$$SW(\boldsymbol{V}) \geq SW(\boldsymbol{A})$$

Strategies and Equilibrium We consider *Markovian* strategies, as defined by Maskin and Tirole (2001). In our context, this means that bidder i's strategy is a function $\sigma_i : \mathcal{G}(\boldsymbol{V}) \mapsto \mathbb{R}$, depending only on the current subgame, regardless of how it was reached. This excludes, *inter alia*, various forms of signalling and punishment strategies by bidders (as in, e.g., Paes Leme et al. (2012) Appendix D).

Given a profile of strategies, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, and outcomes determined by the auction allocation rule, every bidder i has utility $u_i^{\boldsymbol{\sigma}}(\boldsymbol{U}) = v_i^j - M_j$ for every subgame $\boldsymbol{U}$ of $\boldsymbol{V}$, if j is a round in subgame $\boldsymbol{U}$ where i was allocated a unit, and M_j is the amount paid to win round j, or 0 if i wins no round.

Every bidder i maximizes his utility for the main game $u_i^{\boldsymbol{\sigma}}(\boldsymbol{V})$.

We seek a subgame-perfect equilibrium in Markovian strategies. A subgame-perfect equilibrium is a profile of strategies $\boldsymbol{\sigma}$ in which no bidder i, given the strategies of bidders other than himself $\boldsymbol{\sigma_{-i}}$, can deviate from his strategy to improve his utility $u_i^{\boldsymbol{\sigma}}(\boldsymbol{U})$ for any subgame $\boldsymbol{U}$ of $\boldsymbol{V}$.

3 The Option-Value Bidding Strategy

3.1 Allocation Profiles

In this section we will define a profile of bidding strategies for the bidders. For this purpose, we first define *allocation profiles.*

Definition 1. *An* allocation profile *for $\boldsymbol{V}$ is a pair (A, S) of functions $A, S : \mathcal{G}(\boldsymbol{V}) \mapsto \{0, 1, \ldots, n\}$ that specifies a tentative outcome of every possible bidding round. For each subgame $\boldsymbol{U}$ of $\boldsymbol{V}$,*

- *$A(\boldsymbol{U})$ is the bidder with the highest valid bid in the first round of $\boldsymbol{U}$, or 0 indicating there are no valid bids.*
- *$S(\boldsymbol{U})$ is the bidder with the second-highest valid bid in the first round of $\boldsymbol{U}$, or 0 indicating there are less than two valid bids.*

Given an allocation profile $\mathcal{P} = (A, S)$ for $\boldsymbol{V}$, we specify below (Algorithm 1) a complete strategy profile that is induced by it, $\boldsymbol{\sigma}^{\mathcal{P}} = (\sigma_1^{\mathcal{P}}, \ldots, \sigma_n^{\mathcal{P}})$ for every bidder, in every subgame that is reachable under $\mathcal{P}$.

An allocation profile $\mathcal{P} = (A, S)$ is *consistent* if, under the induced strategy profile $\boldsymbol{\sigma}^{\mathcal{P}}$, for every subgame $\boldsymbol{U}$, $A(\boldsymbol{U})$ is the highest bidder in the earliest round of $\boldsymbol{U}$, if there is one, or is 0 if there is none, and $S(\boldsymbol{U})$ is the second-highest bidder in the earliest round of $\boldsymbol{U}$, if there is one, or is 0 if there is none.

We will show that if an allocation profile is consistent, its induced strategy profile is in subgame-perfect equilibrium.

We provide algorithms to compute the bidding strategies for all bidders, and to determine whether the allocation profile is consistent.

Utilities, Bids and Option Values Given an allocation profile $\mathcal{P} = (A, S)$, we define the induced strategy profile $\boldsymbol{\sigma}^{\mathcal{P}}$, constructed by algorithms specified below in Algorithms 1 and 2. We precede them by the following informal description, and follow with a simple 2-round, 2-bidder example auction.

The strategy will be defined recursively, and along the way we will define the *utility* $u_i^{\mathcal{P}}(\boldsymbol{U})$, and *bid* $b_i^{\mathcal{P}}(\boldsymbol{U})$ of bidder i in subgame $\boldsymbol{U}$, induced by $\mathcal{P}$. When the implied allocation profile is unambiguous, we use the shorthand $u_i(\boldsymbol{U})$ and $b_i(\boldsymbol{U})$.

The basis for calculating bids and utilities for each bidder is the calculation of a bidder's *option* utility. The *option utility* of a bidder is the utility he expects to make, in what remains of the sequential auction, had he, hypothetically, sat out the current round (by not submitting a valid bid). A bidder i whose option utility at round j is ω would not rationally agree to pay more than $v_i^j - \omega$, which we call his *option value*, to win the round, as his utility in such a case would then be $< v_i^j - (v_i^j - \omega) = \omega$, which is what he could get by not bidding in the current round. Note that the option utility and option value of a bidder needs an assumption of who wins the current round (and all future rounds) in the bidder's hypothetical absence, so is well-defined only subject to an allocation profile.

We define the induced strategy of a bidder (under the given allocation profile) to bid *exactly* this option value, the maximal price he is willing to pay in the current round, in every round of every subgame. In Algorithm 1, it is computed in step 3a for the proposed winning bidder ($A(\boldsymbol{U})$), and in step 2b for every other bidder. We call such an induced strategy an *option-value strategy*, and a subgame-perfect equilibrium in option-value strategies (all induced by the same allocation profile) an *option-value equilibrium*. Note that, by their definition, option-value strategies and equilibria are pure and Markovian.

Using the above logic, and the second-price auction mechanism of individual rounds, we get the following recursive definition/algorithm of bids and utilities.

Algorithm 1 (Utilities and Bids) *Parameters:*

- $\boldsymbol{U} = \boldsymbol{V}_I^j$*: Subgame of main game* $\boldsymbol{V}$*, whose first round bids and utilities are calculated.*
- $\mathcal{P} = (A, S)$*: The (candidate) allocation profile.*
- $I \subseteq [n]$*: Set of remaining bidders in the subgame.*
- $j > 0$*: The round number.*

Returns:

- $\boldsymbol{u}(\boldsymbol{U}) \equiv \boldsymbol{u}(\boldsymbol{U}, \mathcal{P}, I, j)$*: Vector of bidder utilities in* $\boldsymbol{U}$*.*
- $\boldsymbol{b}(\boldsymbol{U})$ *: Vector of bids in first round of* $\boldsymbol{U}$*.*

1. *If* $j > m$*, set* $\boldsymbol{u}(\boldsymbol{U}) := \{0\}^n, \boldsymbol{b}(\boldsymbol{U}) := \{0\}^n$ *and return.*
2. *For every* $i \in I \setminus A(\boldsymbol{U})$
   (a) *set* $u_i(\boldsymbol{U}) := u_i(\boldsymbol{V}_{I \setminus A(\boldsymbol{U})}^{j+1}, \mathcal{P}, I \setminus A(\boldsymbol{U}), j+1)$
   (b) *set* $b_i(\boldsymbol{U}) := v_i^j - u_i(\boldsymbol{U})$
3. *If* $A(\boldsymbol{U}) \neq 0$
   (a) *set* $b_{A(\boldsymbol{U})}(\boldsymbol{U}) := v_{A(\boldsymbol{U})}^j - u_{A(\boldsymbol{U})}(\boldsymbol{V}_{I \setminus S(\boldsymbol{U})}^{j+1}, \mathcal{P}, I \setminus S(\boldsymbol{U}), j+1)$
   (b) *If* $S(\boldsymbol{U}) = 0$
      i. *set* $u_{A(\boldsymbol{U})}(\boldsymbol{U}) := v_i^j$
   (c) *else*
      i. *set* $u_{A(\boldsymbol{U})}(\boldsymbol{U}) := v_i^j - b_{S(\boldsymbol{U})}(\boldsymbol{U})$

There follows a recursive algorithm, for an *a posteriori* consistency check of the allocation profile calculated by Algorithm 1 for subgame $\boldsymbol{U}$. Invoking it to calculate $C(\boldsymbol{V}, \mathcal{P}, [n], 1)$ returns the (*true/false*) consistency of allocation profile $\mathcal{P}$ for the main game $\boldsymbol{V}$.

Algorithm 2 (Consistency) *Parameters:*

- $\boldsymbol{U} = \boldsymbol{V}_I^j$*: Subgame of main game* $\boldsymbol{V}$*, whose first round bids and utilities are calculated.*
- $\mathcal{P} = (A, S)$*: The (candidate) allocation profile.*
- $I \subseteq [n]$*: Set of remaining bidders in the subgame.*
- $j > 0$*: The round number.*

Returns:

- $C(\boldsymbol{U}, \mathcal{P}, I, j)$*:* True *if allocation profile is consistent,* false *otherwise.*

1. If $j > m$, return true.
2. If $A(\boldsymbol{U}) \neq 0$
   (a) If $S(\boldsymbol{U}) \neq 0$
      i. if $b_{S(\boldsymbol{U})}(\boldsymbol{U}) < \max_{i \in I \setminus A(\boldsymbol{U})} b_i(\boldsymbol{U})$ return false
      ii. if $b_{A(\boldsymbol{U})} < b_{S(\boldsymbol{U})}$ or $b_{S(\boldsymbol{U})} < 0$ return false
   (b) else
      i. if $\max_{i \in I \setminus A(\boldsymbol{U})} b_i(\boldsymbol{U}) \geq 0$ return false
      ii. if $b_{A(\boldsymbol{U})} < 0$ return false
   (c) if not $C(\boldsymbol{V}^{j+1}_{I \setminus S(\boldsymbol{U})}, \mathcal{P}, I \setminus S(\boldsymbol{U}), j+1)$ return false
3. else
   (a) if $\max_{i \in I} b_i(\boldsymbol{U}) \geq 0$ return false
4. return $C(\boldsymbol{V}^{j+1}_{I \setminus A(\boldsymbol{U})}, \mathcal{P}, I \setminus A(\boldsymbol{U}), j+1)$

Example 1.

$$\boldsymbol{V} = \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$$

Set $A(\boldsymbol{V}) = 2, S(\boldsymbol{V}) = 0$, $A(\boldsymbol{V}^2_{[2]}) = 1, S(\boldsymbol{V}^2_{[2]}) = 2$, and $A(\boldsymbol{V}^2_1) = 1, S(\boldsymbol{V}^2_1) = 0$.

The allocation profile for the single-round subgames (i.e., for $\boldsymbol{V}^2_{[2]}$ and $\boldsymbol{V}^2_1$) is the outcome of the standard second-price auction, which is trivially consistent. For the first round we calculate bids by Algorithm 1:

$$\begin{aligned} b_1(\boldsymbol{V}) &= v^1_1 - u_1(\boldsymbol{V}^2_1) = 2 - 6 = -4 \\ b_2(\boldsymbol{V}) &= v^2_1 - u_2(\boldsymbol{V}^2_{[2]}) = v^2_1 - u_2(\boldsymbol{V}^3_2) = 1 - 0 = 1 \end{aligned}$$

The bidder utilities are

$$u_1(\boldsymbol{V}) = u_1(\boldsymbol{V}^2_1) = v^2_1 = 6 \qquad\qquad u_2(\boldsymbol{V}) = v^1_2 = 1$$

Since there is only one valid bid (2's) in the first round, (A, S) is consistent. It induces the efficient allocation $(2, 1)$.

The bids are in equilibrium: Bidder 1 can change his bid to win round 1, and then his utility will be $v^1_1 - b_2(\boldsymbol{V}) = 1$. But this is less than his current utility 6. Bidder 2 can avoid winning round 1 by not submitting a valid bid. But then no unit will be allocated in round 1, and in round 2 he will lose to bidder 1 and get 0 utility.

3.2 Consistent Profiles Induce Equilibria

To recap, the definition of an option-value strategy requires an allocation profile, and a consistent allocation profile induces bids that agree with its proposed allocation. Example 1 demonstrated such a consistent allocation profile, in the context of a 2-player, 2-round auction. It was observed that its induced bids form an SPE of the sequential auction. Our first main result states that this is true in general, justifying the emphasis on consistency.

Theorem 1. *Every consistent allocation profile induces an option-value bidding strategy profile which is in subgame-perfect equilibrium, with non-negative utilities for all bidders, and vice versa.*

Proof. See Appendix.

3.3 On the Existence, Uniqueness and Efficiency of Equilibria

Since consistent profiles induce option-value strategies that, as Theorem 1 shows, are in (pure) SPE, we ask

1. Is an option-value equilibrium of a sequential auction necessarily unique?
2. Is an option-value equilibrium necessarily efficient?
3. Is an efficient allocation necessarily an equilibrium?
4. Does every sequential auction have an option-value equilibrium?
5. Does every sequential auction have a pure equilibrium?

Our answer to questions 1-4 is: No, and we provide a counterexample for each. On the other hand, the answer to question 5 is Yes, as we prove later.

Example 2.

$$\boldsymbol{V} = \begin{pmatrix} \underline{8} & 6 & 0 \\ 6 & \underline{5} & 0 \\ 7 & 3 & \underline{2} \end{pmatrix}$$

The unique efficient allocation $SW(\boldsymbol{V}) = 8 + 5 + 2 = 15$ is underlined.

It is easily verified that the uniquely-efficient allocation profile (A, S) is consistent. It is (omitting the trivial last-round allocations)

$$A(\boldsymbol{V}) = 1 \qquad A(\boldsymbol{V}^2_{\{1,2\}}) = 1 \qquad A(\boldsymbol{V}^2_{\{2,3\}}) = 2$$
$$S(\boldsymbol{V}) = 3 \qquad S(\boldsymbol{V}^2_{\{1,2\}}) = 2 \qquad S(\boldsymbol{V}^2_{\{2,3\}}) = 3$$

inducing consistent first-round bids of $(7, 2, 5)$ for bidders $1, 2, 3$, respectively. This is an equilibrium.

However, another, inefficient equilibrium allocation $(2, 1, 3)$ is induced by the following allocation profile (A, S)

$$A(\boldsymbol{V}) = 2 \qquad A(\boldsymbol{V}^2_{\{1,2\}}) = 1 \qquad A(\boldsymbol{V}^2_{\{1,3\}}) = 1$$
$$S(\boldsymbol{V}) = 3 \qquad S(\boldsymbol{V}^2_{\{1,2\}}) = 2 \qquad S(\boldsymbol{V}^2_{\{1,3\}}) = 3$$

inducing first-round bids of $(3, 6, 5)$ for bidders $1, 2, 3$, respectively. Since all bids are consistent with the allocation profile, they are in equilibrium, which is inefficient: The social welfare $6+6+2 = 14$ is not maximal.

The above example shows that the equilibrium is not unique, and that there exist inefficient equilibria. In the next example, the only efficient allocation is *not* in equilibrium, and the sole option-value equilibrium is inefficient.

Example 3.

$$\boldsymbol{V} = \begin{pmatrix} \underline{9} & 1 & 9 \\ 4 & \underline{1} & 7 \\ 3 & 1 & \underline{8} \end{pmatrix}$$

The efficient allocation $(1, 2, 3)$ is underlined, but it does not induce consistency. The induced first-round bids, $(0, 3, -5)$, are inconsistent with bidder 1 winning it. On the other hand, the allocation $(2, 3, 1)$ is consistent, inducing first-round bids of $(0, 3, 2)$. This is the only option-value equilibrium, and its social welfare, $4 + 1 + 9 = 14$, is less than the maximal $9 + 1 + 8 = 18$.

Example 4. The following sequential auction has no option-value equilibrium, not even an inefficient one.

$$\boldsymbol{V} = \begin{pmatrix} 16 & 8 & 8 \\ 9 & 3 & 2 \\ 12 & 10 & 0 \\ 16 & 16 & 13 \end{pmatrix}$$

To prove this, we first define the *option-value matrix*.

Definition 2 (Option-Value Matrix). *Assume some bidding equilibrium (not necessarily an option-value one) for all proper subgames of $\boldsymbol{V}$, so that every bidder has a well-defined option utility and option value for every possible outcome of the first round of $\boldsymbol{V}$.*

The option-value matrix $\boldsymbol{\omega}(\boldsymbol{V})$ of $\boldsymbol{V}$ with n bidders, is the $n \times n$ matrix, where the element in row i, column j, ω_i^j is $v_i^1 - u_i(\boldsymbol{V}_{\setminus j}^2)$, where $i \neq j$ (ω_i^i is undefined).

It is easily verified that in all second-round subgames $\boldsymbol{V}_{\setminus i}^2, i \in [4]$, the efficient allocation induces the unique option-value equilibrium, which determines a unique option utility for all bidders. Using this fact, the option-value matrix for $\boldsymbol{V}$ is[5]

$$\boldsymbol{\omega}(\boldsymbol{V}) = \begin{bmatrix} - & 8 & \mathbf{10} & \mathbf{10} \\ \mathbf{9} & - & 9 & 9 \\ 7 & \mathbf{12} & - & 5 \\ 5 & 10 & 3 & - \end{bmatrix}$$

Lemma 1. *Given an option-value matrix $\omega(\boldsymbol{V}) = \{\omega_j^i\}$, an allocation profile with $A(\boldsymbol{V}) = i$ and $S(\boldsymbol{V}) = j$ is consistent (and so induces an option-value equilibrium) iff ω_j^i is the largest entry in column i, and it is smaller than ω_i^j (subject to the tie-breaking rule for bidders, i.e., rows).*

Proof. Fix the stated first-round allocation profile. Then the column i entries are the induced bids under this allocation profile (see Algorithm 1) for every bidder except i, while ω_j^i is i's induced bid. The lemma states the condition for these induced bids to be consistent with the allocation profile (i's bid is highest and j's bid is second-highest). □

No allocation profile of $\boldsymbol{V}$ is consistent. To see this, it is enough to inspect the highest element of each column of $\boldsymbol{\omega}(\boldsymbol{V})$ (in **bold**), and verify that it is not smaller than the element in the symmetric position across the main diagonal. E.g., $\omega_1^4 > \omega_4^1$. This gives the negative answer to Question 4.

Nevertheless, first-round bids of $(8, 9, 12, 3)$ are in equilibrium, albeit not an option-value one, since bidder 1's option value is 10, not 8. In general, a pure, but not necessarily an option-value equilibrium always exists.

Remark 1. The following Theorem 2 is essentially Paes Leme et al. (2012)'s Theorem 2.1, where it is stated for first-price auctions. A pure second-price equilibrium is easily derived from a pure first-price equilibrium. They non-constructively show that an ascending price auction must terminate at a pure equilibrium. Narayan et al. (2019) give a constructive version of the proof. Our proof explicitly uses the notion of option values, resulting in a very simple constructive proof. By Example 2, showing that equilibria are not unique, the two proofs do not necessarily demonstrate the same equilibrium.

[5] For example, $\omega_3^1 = v_3^1 - u_3 \begin{pmatrix} 3 & 2 \\ 10 & 0 \\ 16 & 13 \end{pmatrix} = 12 - 5 = 7$

Theorem 2. *Every sequential auction has an equilibrium in pure strategies.*

Proof. We prove by induction on m, the number of rounds. For $m = 1$, bidding one's value is a pure equilibrium. Assume the theorem for up to $m - 1$ rounds. Assume a strategy profile for all bidders, not necessarily an option-value one, that induces a pure equilibrium in all subgames of up to $m - 1$ rounds, so that all such subgames have a well-defined equilibrium value for all bidders.

Construct the option-value matrix $\boldsymbol{\omega}(\boldsymbol{V}) = \{\omega_i^j\}$, as defined in Definition 2. Let ω_i^j at row i, column j be the highest value in $\boldsymbol{\omega}(\boldsymbol{V})$, subject to the tie-breaking rule for bidders, i.e., rows. Then complete the strategy profile by setting the first-round bids for all players as follows: i bids ω_i^j, j bids ω_j^i, and all other bidders bid any value $< \omega_j^i$.

We show that these bids are in equilibrium:

For bidder i, suppose first that $\omega_i^j \geq 0$. Since his option value is not negative, he cannot gain by deviating and *not* winning the round. Alternatively, if $\omega_i^j < 0$, all bidders make negative, invalid bids, and no unit is allocated. Bidder i can change the outcome by bidding 0 or more, but since his option value is negative, this is not to his advantage.

Every bidder $k \neq i$ cannot gain by deviating, since his option value $\omega_k^i \leq \omega_i^j$ is insufficient to win the round. □

4 Efficient Allocations

In this section we shall consider profiles inducing *efficient* allocations. We define tie breaking that renders one of them, called the orderly efficient allocations, unique.[6]

Definition 3. *An* early *efficient allocation is an efficient allocation in which rounds are allocated as early as possible for efficient allocations. I.e., if $\boldsymbol{A}$ is an early efficient allocation, $\boldsymbol{A'}$ an efficient allocation, j a round, and $\boldsymbol{A}$ and $\boldsymbol{A'}$ have the same rounds unallocated before round j, then if $\boldsymbol{A'}$ allocates a unit in round j, so does $\boldsymbol{A}$. E.g., if $(1, 2, 0)$, $(1, 0, 2)$ and $(0, 1, 2)$ are efficient allocations, only $(1, 2, 0)$ is early.*

An orderly *efficient allocation is an early allocation in which, in all cases where the order of allocation of bidders can be permuted and remain efficient, the smaller label is allocated first. E.g., if both $(1, 2)$ and $(2, 1)$ are efficient allocations, only $(1, 2)$ is orderly. Every subgame has a unique orderly efficient allocation.*

With these definitions, there is a unique allocation profile that induces an orderly efficient allocation. We call it the *orderly allocation profile.*

Definition 4. *Given a game $\boldsymbol{V}$, a orderly allocation profile is an allocation profile $\mathcal{E} = (A, S)$ such that for every subgame $\boldsymbol{V}_I^j$ with $I \subseteq [n], j \in [m]$*

- *$i := A(\boldsymbol{V}_I^j)$ is matched to round j in the orderly efficient allocation of $\boldsymbol{V}_I^j$.*
- *If $i > 0$, $S(\boldsymbol{V}_I^j)$ is matched to round j in the orderly efficient allocation of $\boldsymbol{V}_{I\setminus i}^j$.*

We shall show that, under some restrictions on the valuations, orderly allocation profiles are consistent. It will follow, by Theorem 1, that orderly allocation profiles induce bidding strategies that are in subgame-perfect equilibrium.

[6] This definition is relevant only if there exist multiple efficient allocations. In the generic case, where there exists a unique efficient allocation, that allocation is, *ipso facto*, orderly.

Ordered differences We adopt an assumption on the valuations, called *ordered differences*, which, when satisfied, will enable us to reach results that are not true in the general case. For simplicity and brevity, assume at least as many bidders as rounds ($n \geq m$). Assume w.l.o.g. that every round has an allocation, since if $n \geq m$ an unallocated round occurs only if no bidder is present in it, in which case the round can be deleted without materially changing the auction.

Briefly, the ordered-differences condition assumes an order satisfied by bidder valuation differences from one round to the next. As bidder labels are arbitrary, the condition also assumes bidders are numbered so that these differences are non-increasing with the bidder labels $1, \ldots, n$. Specifically, let $\boldsymbol{E} = (E_1, \ldots, E_m)$ be the orderly efficient allocation of $\boldsymbol{V}$. $\boldsymbol{V}$ fulfils the *ordered-differences* condition if the valuations satisfy

Criterion 1 (Ordered-Differences) *For every round $j < m$, and bidders $g, h \in [n]$ where $g < h$, and $g \leq E_j$,*

$$v_g^j - v_g^{j+1} \geq v_h^j - v_h^{j+1}$$

Example 5. For example, a valuation matrix that fulfils ordered-differences is

$$\boldsymbol{V} = \begin{pmatrix} \mathbf{11} & \mathbf{0} & \mathbf{6} & \mathbf{0} & \mathbf{2} \\ \underline{\mathbf{15}} & \mathbf{5} & \mathbf{11} & \mathbf{6} & \mathbf{9} \\ 0 & \underline{\mathbf{9}} & \mathbf{15} & \mathbf{10} & \mathbf{13} \\ 0 & 2 & \mathbf{9} & \mathbf{5} & \mathbf{8} \\ 0 & 0 & \underline{\mathbf{17}} & \mathbf{13} & \mathbf{16} \\ 0 & 3 & 11 & \mathbf{7} & \mathbf{10} \\ 0 & 0 & 11 & \underline{\mathbf{7}} & \mathbf{11} \\ 0 & 11 & 20 & 16 & \underline{\mathbf{20}} \end{pmatrix}$$

Its orderly efficient allocation, $\boldsymbol{E} = (2, 3, 5, 7, 8)$, is underlined. Above, we marked in **bold** all elements v_i^j for which $i \leq E_j$. These are the elements on or *above* a boundary defined by $\boldsymbol{E}$.

The condition, in words, states that $\boldsymbol{V} = \{v_i^j\}$ is divided into top and bottom parts by an *allocation boundary* $i \leq E_j$. The order of round difference in valuations of the bidders match the label order, with differences above the boundary ordered by label, while differences below the boundary are smaller than differences above the boundary, but have no particular order among themselves. For example, Example 2 does *not* fulfil ordered differences, because $7 - 3 = v_3^1 - v_3^2 > v_1^1 - v_1^2 = 8 - 6$. However, if v_3^1 is changed to 5, it would be an ordered-differences matrix.

The ordered-differences condition holds, for example

- For constant valuations.
- For every two-round sequential auction. (Label the bidders by the order of valuation differences).
- For uniformly-discounted valuations (i.e., where there exists $\delta \geq 0$ such that $v_i^j = \delta^{j-1} v_i^1$ for every $i \in [n], j \in [m]$).
- Every ordered-differences valuation matrix, that is modified to include *arrival* of bidders (we say that bidder i arrives in round j iff $v_i^j > 0$ but $v_i^k = 0$ for every $k < j$), subject to a restriction that every bidder $i \in [m]$ arrives before (left of) the allocation boundary. (Since allocated bidders are necessarily present when allocated, this only restricts the time of arrival of *unallocated* bidders).

We can now formulate our result for ordered differences:

Theorem 3. *Given* $\boldsymbol{V}$ *satisfying the ordered-differences condition, its orderly allocation profile* $\mathcal{E} = (A, S)$ *is consistent.*

Proof. See Appendix.

It follows, by Theorem 1

Corollary 1. *Every sequential auction with ordered-differences valuations has an option-value equilibrium, inducing an efficient allocation.*

The *principal* ordered-differences condition places an additional requirement.

Criterion 2 (Principal Ordered-Differences) *A* principal *ordered-differences valuation is an ordered-differences valuation whose orderly efficient allocation allocates a unit to every bidder* $\in [m]$*.*

The allocation boundary of a principal ordered-differences valuation is simply the major diagonal v_1^1 to v_m^m. Example 5 is an ordered-differences valuation that is *not* principal. Note that all constant and uniformly-discounted valuation matrices have principal ordered differences.

If the ordered-differences valuation is also principal, we can say more: The orderly efficient allocation is now the *unique* option-value equilibrium. It follows that, under other tie-breaking rules, all option-value equilibria are essentially the same (the social welfare and payments are the same, and any differences in allocation are due only to different tie breaking).

Theorem 4. *Given* $\boldsymbol{V}$ *satisfying the principal ordered-differences condition, the option-value equilibrium induced by the orderly allocation profile is the unique option-value equilibrium.*

Proof. See Appendix.

Corollary 2. *Given* $\boldsymbol{V}$ *satisfying the principal ordered-differences condition, the unique option-value equilibrium is equivalent to the option-value equilibria under other tie-breaking orders, in the sense that all subgames are won by equal bids, and pay equal amounts.*

Furthermore, we conjecture that the unique option-value equilibrium is the only one that is not weakly-dominated, thus making it, like the single-round second-price auction, the only outcome that is reached by rational bidders. Like Paes Leme et al. (2012), our definition of a strategy that is *weakly-undominated* is a strategy that cannot be eliminated by *any* sequence of elimination of strategies.

Conjecture 1 *Given* $\boldsymbol{V}$ *satisfying the principal ordered-differences condition, all equilibria in weakly-undominated strategies have the same outcome as the option-value outcome, in the sense that all subgames are won by the same bidder, who pays the same amount.*

5 Incomplete-Information Sequential Auctions

5.1 Outline

We now briefly consider incomplete-information sequential auctions, and show how our considerations for complete-information auctions generalize to incomplete information.

A natural, but faulty generalization, would be to define option utilities, and option values, as expectations of their complete-information counterparts over the bidder's information set (under some generalized consistency restriction), and have the bidders bid their thus-defined option values. In fact, as we will show, there is no reason for expectations of option values to form a bidding equilibrium. Conditions for equilibrium indeed involve all the complete-information option values of the bidder's information set, but are more complex than mere averaging of expectations.

In this section, we aim to show the correct generalization to incomplete information. While it is surely interesting to investigate this further, this would go beyond the scope of this paper.

For simplicity and brevity, we consider auctions where

1. Bidders know their current and future values.
2. All bidders know which bidder, if any, was allocated in each previous round.
3. There are two rounds ($m = 2$).
4. There is a finite number of bidder valuations with non-zero probability.[7]

We seek bidding strategies that are in *perfect Bayesian equilibrium* (PBE), which means that bidders hold consistent beliefs and are sequentially rational in every round of the sequential auction. By limiting the number of rounds to two, we restrict the problem to finding the first-round bidding strategy. This is because, as is well-known, in the second and last round, the sole weakly-undominated strategy is to bid one's current value, regardless of beliefs.

5.2 Model and Analysis

The *type* of bidder i is the vector of his valuations for all rounds, denoted $\boldsymbol{v_i} := (v_i^1, \ldots, v_i^m)$, where $v_i^j \in \mathbb{R}_{\geq 0}$ is bidder i's valuation in round j.

The *type matrix* $\boldsymbol{V}$ is an $n \times 2$ matrix in which row i is bidder i's type.

$$\boldsymbol{V} = \begin{pmatrix} v_1^1 & v_1^2 \\ \vdots & \vdots \\ v_n^1 & v_n^2 \end{pmatrix}$$

Let $\Omega \subseteq \mathbb{R}^{2n}_{\geq 0}$ be a finite set of all type matrices with non-zero prior probability.

There is a commonly-known probability function P on the elements and subsets of Ω, where $\sum_{\boldsymbol{w} \in \Omega} P(\boldsymbol{w}) = P(\Omega) = 1$.

For every bidder i, $\Omega_i(\boldsymbol{v_i})$ is the set of all type matrices in Ω which agree with i's type, i.e., $\Omega_i(\boldsymbol{v_i}) := \{\boldsymbol{w} \in \Omega | \boldsymbol{w_i} = \boldsymbol{v_i}\}$.

A strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ specifies the strategy of each bidder $i \in [n]$. A strategy of bidder $i \in [n]$, σ_i, specifies the bid $b_i(\boldsymbol{U}; \boldsymbol{v_i})$ in every sub-auction $\boldsymbol{U}$ when the bidder's type is $\boldsymbol{v_i}$. In practice, only $b_i(\boldsymbol{V}; \boldsymbol{v_i})$, the first-round bid, needs to be determined, since the equilibrium second-round strategy is to bid one's value.

Consider now the first-round bids. Suppose we are given bids by each player, and we are tasked with confirming or denying that these bids are in equilibrium. In an equilibrium, for every bidder i, given bids by other bidders, his utility expectation with every other bid b does not exceed his utility expectation in with the suggested equilibrium bid.

Define $G_i(b; \boldsymbol{v_i})$ as the set of all type matrices in bidder i's information set (that is, $\Omega_i(\boldsymbol{v_i})$) which he wins by bidding b in the first round, when all other bidders bid their equilibrium bids. Then $G_i(b; \boldsymbol{v_i}) \subseteq \Omega_i(\boldsymbol{v_i})$ and $b < b'$ entails $G_i(b; \boldsymbol{v_i}) \subseteq G_i(b'; \boldsymbol{v_i})$.

[7] Relaxing these restrictions would require, respectively (1) defining bidder types differently, and replacing bidder values with expectations; (2) replacing values with their expectation in some formulas; (3) implementing sequential rationality by updating beliefs and information sets from round to round, and (4) defining a probability measure for the valuation probability space and replacing several sums by integrals.

For every type matrix $\boldsymbol{w}$, bidder i's *alternate winner* is the bidder that is allocated if i does not participate, or is 0 if no bidder is allocated. $M_i(\boldsymbol{w})$ is i's payment if he wins the first round of $\boldsymbol{w}$.[8] By the second-price auction allocation rule, these are

$$AW_i(\boldsymbol{w}) := \mathbb{1}_{\max_{k\in[n]\setminus i} b_k(\boldsymbol{V};\boldsymbol{w_k})\geq 0} \arg\max_{k\in[n]\setminus i} b_k(\boldsymbol{V};\boldsymbol{w_k})$$
$$M_i(\boldsymbol{w}) := \mathbb{1}_{AW_i(\boldsymbol{w})\neq 0} b_{AW_i(\boldsymbol{w})}(\boldsymbol{V};\boldsymbol{w_{AW_i(w)}})$$

Given $\boldsymbol{w}$ and i, assuming i does not participate in the first round, and goes on to win the second round, he will pay the second-round bid (= value) of some other bidder $s := \arg\max_{k\in[n]\setminus\{i,AW_i(\boldsymbol{w})\}} w_k^2$. Bidder i's option utility and option value, marked $OU_i(\boldsymbol{w})$ and $OV_i(\boldsymbol{w})$ respectively, are defined

$$OU_i(\boldsymbol{w}) := \max(w_i^2 - w_s^2, 0)$$
$$OV_i(\boldsymbol{w}) := w_i^1 - OU_i(\boldsymbol{w})$$

The following theorem characterizes bidding equilibria.

Theorem 5. *Let an incomplete-information sequential auction be given by type matrix $\boldsymbol{V}$, set of type matrices Ω and probability function P. Then a first-round bidding strategy given by $b_i(\boldsymbol{V};\boldsymbol{v_i})$ for each $i \in [n]$ is in equilibrium iff, for every bid b by every bidder i with every type $\boldsymbol{v_i}$*

$$MU_i(b_i(\boldsymbol{V};\boldsymbol{v_i})) \geq MU_i(b)$$

where

$$MU_i(b) := \sum_{\boldsymbol{w}\in G_i(b;\boldsymbol{v_i})} \Big\{OV_i(\boldsymbol{w}) - M_i(\boldsymbol{w})\Big\}P(\boldsymbol{w}) \tag{2}$$

Proof. Define $U_i(b)$ to be bidder i's utility from bidding b. Then (noting that $\boldsymbol{v_i} = \boldsymbol{w_i}$ in all relevant information sets):

$$\begin{aligned} U_i(b) &= \sum_{\boldsymbol{w}\in G_i(b;\boldsymbol{v_i})} \Big\{v_i^1 - M_i(\boldsymbol{w})\Big\}P(\boldsymbol{w}) + \sum_{\boldsymbol{w}\in \Omega_i(\boldsymbol{v_i})\setminus G_i(b;\boldsymbol{v_i})} OU_i(\boldsymbol{w})P(\boldsymbol{w}) \\ &= \sum_{\boldsymbol{w}\in \Omega_i(\boldsymbol{v_i})} OU_i(\boldsymbol{w})P(\boldsymbol{w}) + \sum_{\boldsymbol{w}\in G_i(b;\boldsymbol{v_i})} \Big\{v_i^1 - M_i(\boldsymbol{w})\Big\}P(\boldsymbol{w}) - \sum_{\boldsymbol{w}\in G_i(b;\boldsymbol{v_i})} OU_i(\boldsymbol{w})P(\boldsymbol{w}) \\ &= \sum_{\boldsymbol{w}\in \Omega_i(\boldsymbol{v_i})} OU_i(\boldsymbol{w})P(\boldsymbol{w}) + \sum_{\boldsymbol{w}\in G_i(b;\boldsymbol{v_i})} \Big\{OV_i(\boldsymbol{w}) - M_i(\boldsymbol{w})\Big\}P(\boldsymbol{w}) \end{aligned} \tag{3}$$

Now, as the first term of (3) does not depend on b, the equilibrium bid $b_i(\boldsymbol{V};\boldsymbol{v_i})$ maximizes the second term, which equals $MU_i(b)$. The theorem follows. □

Note that $MU_i(b)$ depends on b only via the information set $G_i(b;\boldsymbol{v_i})$, and so is a step function for a finite set of type matrices. For negative b, $MU_i(b) = 0$, as $G_i(b;\boldsymbol{v_i})$ is empty.

Note also that for $b > v_i^1$, $MU_i(b) \leq MU_i(v_i^1)$, with equality only if $G_i(b;\boldsymbol{v_i}) = G_i(v_i^1;\boldsymbol{v_i})$, so it is never necessary to bid more than one's value. This is because $OV_i(\boldsymbol{w}) \leq w_i^1 = v_i^1$ always, while for every $\boldsymbol{w} \in G_i(b;\boldsymbol{v_i}) \setminus G_i(v_i^1;\boldsymbol{v_i})$, $M_i(\boldsymbol{w}) > v_i^1$.

In the complete-information case, Ω is a singleton set $\{\boldsymbol{w}\}$, and $G_i(b;\boldsymbol{v_i})$ is non-empty for every $b > M_i(\boldsymbol{w})$, in which case $MU_i(b)$ is positive iff $OV_i(\boldsymbol{w}) > M_i(\boldsymbol{w})$. Therefore bidding the

[8] We use the notation $\mathbb{1}_z$, instead of the Iverson bracket, for an expression whose value is 1 if condition z is true, and 0 if false.

option value in complete-information valuations always achieves optimal utility. This is true even when the equilibrium is not an option-value equilibrium.

An example incomplete information auction, with its equilibrium bidding strategy, is provided in the Appendix.

5.3 Discussion

Much remains to be done here, but it would take us far beyond the scope of our paper. For example, how to find the bidding equilibrium in the first place? Best response dynamics may work, i.e., start with a hypothesis, say, $\boldsymbol{b}(\boldsymbol{V};\cdot) = \begin{pmatrix} 0|0 \\ 0|0 \\ 0|0 \end{pmatrix}$ and then calculate optimal bids for each bidder, round robin, replacing them in the hypothesis, until a fixed point is reached. There is no guarantee that such a procedure will converge, though.

It is also noteworthy that, in the example, the bidding equilibrium for the incomplete-information auction is also an equilibrium for each constituent complete-information $\boldsymbol{w} \in \Omega$. We are not aware of any reason why this should be so. But since it is, and since in each $\boldsymbol{w}$ the equilibrium is also efficient (in the main part of the paper we noted that two-round auctions always have an efficient equilibrium), the incomplete-information auction is *ex post* efficient.

6 Discussion

6.1 Conclusion

We analyzed sequential auctions with time-varying valuations, including arrival of bidders, in a context where bidders have complete information. There always exists a pure bidding subgame-perfect equilibrium, and one based on option values iff there exists a consistent allocation profile. We showed how to compute such bidding strategies.

For general valuations, we show via examples that sequential auctions are not necessarily efficient, and can have multiple equilibria. However, when the ordered-differences condition is satisfied, an efficient equilibrium always exists.

We showed the path to generalize our work to incomplete information.

6.2 Future Work

Our work suggests several interesting directions for further investigation.

Many settings that do not satisfy the ordered-differences condition have efficient equilibrium allocations, and the challenge is to formulate less-demanding sufficient conditions on the valuation for efficiency, or even to formulate sufficient *and* necessary conditions. Generalizing our results to sequential first-price and other auctions seems attainable. The possible inefficiency of equilibrium allocations raises the question of bounding the price of anarchy.

For incomplete information, many open questions remain, some of which were discussed in Section 5.3.

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A Proof of Theorem 1

Proof. From allocation profile to equilibrium, we prove this by backward induction. Assume that the induced strategy profile for all subgames of $\boldsymbol{V}$ are in subgame-perfect equilibrium, and all utilities are non-negative. To complete the backward induction, we need only prove that if the bids $b_1(\boldsymbol{V}), \ldots, b_n(\boldsymbol{V})$ are consistent with $A(\boldsymbol{V})$ and $S(\boldsymbol{V})$, then no bidder can change his first-round bid to his advantage, and all bidders have non-negative utilities.

Let $i \in [n]$ be a bidder. There are two cases:

- $i = A(\boldsymbol{V})$: The bidder wins the current round, so can deviate by *not* winning it. In this case, his utility will be, by definition, his option utility. Since he bids his option value, his option utility is $\omega = v_i^1 - b_i(\boldsymbol{V})$. But this is not more than his current utility $u_i(\boldsymbol{V})$, because
  - If $S(\boldsymbol{V}) \neq 0$, $u_i(\boldsymbol{V}) = v_i^1 - b_{S(\boldsymbol{V})}(\boldsymbol{V}) \geq v_i^1 - b_i(\boldsymbol{V})$, because i makes the highest bid.
  - If $S(\boldsymbol{V}) = 0$, $u_i(\boldsymbol{V}) = v_i^1 \geq v_i^1 - b_i(\boldsymbol{V})$, because i's bid is non-negative.

  Since by the induction hypothesis, ω is non-negative, $u_i(\boldsymbol{V}) \geq \omega$ is non-negative.
- $i \neq A(\boldsymbol{V})$: The bidder does not win the current round, so can deviate by making a winning bid. His current utility is his option utility, which, as he bids his option value, is $u_i(\boldsymbol{V}) = v_i^1 - b_i(\boldsymbol{V})$. If he changes his bid to win the current round
  - If $A(\boldsymbol{V}) \neq 0$, he will pay $b_{A(\boldsymbol{V})}$, but then his utility will be $v_i^1 - b_{A(\boldsymbol{V})} \leq v_i^1 - b_i(\boldsymbol{V}) = u_i(\boldsymbol{V})$.
  - If $A(\boldsymbol{V}) = 0$, he will pay 0, but then his utility will be $v_i^1 < v_i^1 - b_i(\boldsymbol{V}) = u_i(\boldsymbol{V})$, because if there is no winner, we must have $b_i(\boldsymbol{V}) < 0$.

  Since $u_i(\boldsymbol{V})$ is the bidder's option utility, it is, by the induction hypothesis, non-negative.

In all cases, a deviation does not improve utility, and the bidder utility is non-negative.

In the other direction, suppose we have option-value bidding strategies for a sequential second-price auction, that are in subgame-perfect equilibrium. It is trivial to construct an allocation profile from these bids, by setting, in each subgame $\boldsymbol{U}$, $A(\boldsymbol{U})$ to be the highest bidder, and $S(\boldsymbol{U})$ to be the second-highest bidder. We then have an induced allocation profile $\mathcal{P} = (A, S)$. As we have shown, by construction this allocation profile uniquely leads to a bidding strategy profile by Algorithm 1. Therefore, if our bidding strategy is in equilibrium, it must be the same as that induced by $\mathcal{P}$, which, by the same token, is consistent. □

B Lemmas

We use the following lemmas in proofs. The first lemma considers the effect of removing a row or column on the efficient allocation.

Lemma 2. *Let $\boldsymbol{A}$ be the orderly efficient allocation of $\boldsymbol{V}$ and i a bidder in $[n]$, $\boldsymbol{A'}$ the orderly efficient allocation of $\boldsymbol{V}_{\setminus i}$, and $\boldsymbol{A^*}$ the orderly efficient allocation of $\boldsymbol{V}^2$. Then (i) At most one bidder from $\boldsymbol{A}$ is not in $\boldsymbol{A'}$: i if $i \in \boldsymbol{A}$. (ii) At most one bidder from $\boldsymbol{A}$ is not in $\boldsymbol{A^*}$: A_1 if $A_1 > 0$.*

Proof. (i) Consider the following round-deletion procedure. (E.g., let $i = 2$, $\boldsymbol{A} = (1, 2, 3)$, $\boldsymbol{A'} = (3, 5, 4)$). If i is not allocated in $\boldsymbol{A}$, procedure terminates. Otherwise, let $A_k = i$. ($k = 2$ in the example). Delete round k from both allocations. Let $i' := A'_k$. ($i' = 5$ in the example). If i' is not allocated in $\boldsymbol{A}$, procedure terminates (as in the example). Otherwise, say $i' = A_{k'}$, delete round k' from both allocations, and repeat. This procedure necessarily stops, and the bidders deleted from the allocations differ by at most one (bidder i). The procedure never deletes an unallocated round from $\boldsymbol{A}$. Furthermore, the remaining undeleted bidders in each allocation have no bidder in common with the deleted bidders in *both* allocations. ($(1,3)$ and $(3,4)$ are undeleted in the example, while 2 and 5 are deleted). Therefore, the bidders in the undeleted rounds may be swapped between allocations without disturbing feasibility. (In the example, creating allocations $(1,5,3)$ and $(3,2,4)$). It follows that the undeleted rounds must be identical in the allocations. Otherwise, one of them cannot be an orderly efficient allocation, as it can be improved by the swapping. This proves the claim.

(ii) By item (i) of the lemma, the orderly efficient allocation of $\boldsymbol{V}^2$ and $\boldsymbol{V}^2_{\setminus A_1}$ differ by at most one bidder: A_1. But the latter is $(A_2,\dots,A_m)$, which also differs from $\boldsymbol{A}$ by at most one bidder, A_1 again. This proves the claim.

□

The following is a useful inequality about efficient allocations, regardless of bidding strategies.

Lemma 3. *Let bidder $i \in [n]$ be the round-k bidder in an efficient allocation $\boldsymbol{A}$ of a subgame $\boldsymbol{V}^j_I$, where $j \in [m]$, $k \in \{j,\dots,m\}$ and $I \subseteq [n]$. Let $K := I \setminus \{A_j,\dots,A_{k-1}\}$ be the remaining bidders at round k. Then*

$$SW(\boldsymbol{V}^k_K) - SW(\boldsymbol{V}^k_{K\setminus i}) \geq SW(\boldsymbol{V}^j_I) - SW(\boldsymbol{V}^j_{I\setminus i})$$

Proof. $(A_k,\dots,A_m)$ must be an efficient allocation of $\boldsymbol{V}^k_K$, because if there is an allocation with greater social welfare, it could improve the social welfare of $\boldsymbol{A}$ by replacing the round k to m allocations, contradicting the assumption that $\boldsymbol{A}$ is efficient. So we have

$$SW(\boldsymbol{V}^j_I) - SW(\boldsymbol{V}^k_K) = \sum_{j\leq l<k, A_l>0} v^l_{A_l}$$

Let $\boldsymbol{A'}$ be an efficient allocation of $\boldsymbol{V}^k_{K\setminus i}$. Then $(A_j,\dots,A_{k-1},A'_k,\dots,A'_m)$ is a feasible allocation of $\boldsymbol{V}^j_{I\setminus i}$, so its social welfare is not higher than $SW(\boldsymbol{V}^j_{I\setminus i})$. But this social welfare is $[SW(\boldsymbol{V}^j_I) - SW(\boldsymbol{V}^k_K)] + SW(\boldsymbol{V}^k_{K\setminus i})$. The lemma follows. □

The following lemma lays ground for the main result.

Lemma 4. *Let $\boldsymbol{V}$ fulfil the ordered-differences condition, and let $\boldsymbol{E}$ be its orderly efficient allocation. Let $\boldsymbol{U} = \boldsymbol{V}^j_I$ be a subgame, and let $\boldsymbol{A}$ be its orderly efficient allocation.*

1. *The first bidder allocated in $\boldsymbol{A}$, A_j, is the lowest-label bidder i in $\boldsymbol{A}$ with $i \leq E_j$, if there are any.*
2. *If $|I| \geq n-j+1$, the bidders in $\boldsymbol{A}$ are allocated in order of their labels.*

Proof. 1. Let the lowest-label bidder in $\boldsymbol{A}$ be $g = A_s$ and assume $g \leq E_j$. Suppose the first bidder allocated in $\boldsymbol{A}$, $h := A_j$ does *not* have the lowest label in $\boldsymbol{A}$. Now as $g < h$ and $g \leq E_j$, by the ordered-differences criterion

$$v^j_g - v^s_g = \sum_{k=j}^{s-1}[v^k_g - v^{k+1}_g] \geq \sum_{k=j}^{s-1}[v^k_h - v^{k+1}_h] = v^j_h - v^s_h$$
$$\Rightarrow v^s_g + v^j_h \leq v^j_g + v^s_h$$

This contradicts the orderly efficiency of $\boldsymbol{A}$. Therefore $g = A_j$ is the lowest-label bidder in $\boldsymbol{A}$.

2. Applying the first item of this lemma to $\boldsymbol{V}$, $E_1 = \min\{E_1, \ldots, E_m\}$. In particular, $E_1 < E_2$. Similarly, successively applying the first item to $\boldsymbol{V}^2_{\setminus E_1}, \boldsymbol{V}^3_{\setminus \{E_1, E_2\}}, \ldots$, in which $E_2, E_3, \ldots$ are first allocated, we conclude $E_1 < E_2 < \ldots < E_m$. Therefore, for every $j \in [m]$, $E_j \geq j$.

   There are E_j bidders in $[n]$ with label $\leq E_j$, and in subgame V^j_I at least $E_j - (j-1)$ such bidders remain. Now $E_j - (j-1) \geq j - (j-1) \geq 1$, so at least one such bidder remains. Therefore, by the previous item, the first item allocated is the lowest-label bidder. Since the second item (A_{j+1}) is the first allocated in the orderly efficient allocation of $\boldsymbol{V}^{j+1}_{I \setminus A_j}$, it is the second-lowest bidder in $\boldsymbol{A}$, and so on. So the bidders in $\boldsymbol{A}$ are in ascending order of label.

□

C Proof of Theorem 3

We shall prove the theorem by proving the following, stronger proposition by induction on the number of rounds m.

Proposition 1. *Given $\boldsymbol{V}$ satisfying the ordered-differences condition, and an orderly allocation profile $\mathcal{E} = (A, S)$*

1. *$\mathcal{E}$ is consistent.*
2. *For every subgame $\boldsymbol{V}^j_I$ with $|I| \geq n - j + 1$, let $\boldsymbol{A'}$ be its orderly efficient allocation. In the strategy profile induced by $\mathcal{E}$, the utility of every bidder is at least the "contribution" his presence made to the social welfare of the subgame, with equality if i is allocated first or is unallocated. I.e., for every $i \in [n]$, $j \in [m]$ and $I \subseteq [n]$ s.t. $|I| \geq n - j + 1$*

$$u_i(\boldsymbol{V}^j_I) = SW(\boldsymbol{V}^j_I) - SW(\boldsymbol{V}^j_{I \setminus i}) \qquad i = A(\boldsymbol{V}^j_I) \vee i \notin \boldsymbol{A'} \tag{4}$$

$$u_i(\boldsymbol{V}^j_I) \geq SW(\boldsymbol{V}^j_I) - SW(\boldsymbol{V}^j_{I \setminus i}) \qquad i \neq A(\boldsymbol{V}^j_I) \wedge i \in \boldsymbol{A'} \tag{5}$$

Proof. Our induction hypothesis is that the proposition is true for up to $m-1$ rounds. We shall prove the induction step that it is true for m rounds. But first, we demonstrate the induction base $m = 1$.

C.1 Proof of Induction Base

For $m = 1$, we have a single second-bid auction round. By Algorithm 1 $b^1_i = v^1_i$ for every $i \in [n]$, so the outcome is the result of a second-round auction of the bidders' values. The winner has the highest value and the second highest-bid is by the second-highest value, so this is consistent with the efficient allocation.

The only subgame is $\boldsymbol{V}$ itself. Now every bidder $i \neq A(\boldsymbol{V})$ has zero utility. In this case $SW(\boldsymbol{V}) = SW(\boldsymbol{V}_{\setminus i}) = v^1_{A(\boldsymbol{V})}$. Therefore $u_i(\boldsymbol{V}) = SW(\boldsymbol{V}) - SW(\boldsymbol{V}_{\setminus i})$, as claimed. For $i = A(\boldsymbol{V})$, utility is i's value, less the amount he pays, which is the second-highest bid, or 0 if there is no such other bid. Since $SW(\boldsymbol{V}) = v^1_i$ and $SW(\boldsymbol{V}_{\setminus i}) = v^1_{S(\boldsymbol{V})}$ when $S(\boldsymbol{V}) > 0$, while otherwise $SW(\boldsymbol{V}_{\setminus i}) = 0$, we again have

$$u_i(\boldsymbol{V}) = SW(\boldsymbol{V}) - SW(\boldsymbol{V}_{\setminus i})$$

as claimed.

C.2 Induction Step

Since the induction hypothesis applies to all proper subgames, to complete the proof, it is enough to prove the proposition for $\boldsymbol{V}$. Let $\boldsymbol{E}$ be the orderly efficient allocation of $\boldsymbol{V}$.

Proof of 1

- If $A(\boldsymbol{V}) = 0$, for consistency we need to show that there are no valid bids, i.e., that all first-round option values are negative.
  Assume not, and that for some bidder i, $b_i(\boldsymbol{V}) \geq 0$.

$$\begin{aligned} 0 &\leq b_i(\boldsymbol{V}) \\ &= v_i^1 - u_i(\boldsymbol{V}^2) \\ &\leq v_i^1 - SW(\boldsymbol{V}^2) + SW(\boldsymbol{V}^2_{\setminus i}) \end{aligned}$$

  With the last equality following from (4), (5) and the induction hypothesis. But $SW(\boldsymbol{V}^2) = SW(\boldsymbol{V})$. It follows that

$$v_i^1 + SW(\boldsymbol{V}^2_{\setminus i}) \geq SW(\boldsymbol{V})$$

  The left-hand side is the social welfare of an alternative allocation of $\boldsymbol{V}$, so the inequality cannot be strict, as it would have better-than-maximal social welfare. Equality is also not possible, since it displays an earlier efficient allocation than induced by $\mathcal{E}$. A contradiction.
- $A(\boldsymbol{V}) > 0$. Let $i := A(\boldsymbol{V})$. Then $i = E_1$.

  Lemma 5. *Let $k \in \{0\} \cup [n] \setminus i$, and let $\boldsymbol{E'}$ be the orderly efficient allocation of $\boldsymbol{V}^2_{\setminus k}$. Then, E_1 is either the first bidder allocated in $\boldsymbol{E'}$, or is not allocated in $\boldsymbol{E'}$.*

  Proof. By Lemma 4, $i = E_1$ is the lowest-label bidder in $\boldsymbol{E}$. Consider first the subgame $\boldsymbol{V}^1_{\setminus k}$, and let its orderly efficient allocation be $\boldsymbol{E}^*$. If $k \notin \boldsymbol{E}$, $\boldsymbol{E}$ and $\boldsymbol{E}^*$ are identical. Otherwise, by Lemma 2, $\boldsymbol{E}$ and $\boldsymbol{E}^*$ differ by at most one bidder, which is necessarily k. In either case i is also the lowest-label bidder in $\boldsymbol{E}^*$.
  Now by Lemma 2, $\boldsymbol{E}^*$ and $\boldsymbol{E'}$ differ by at most one bidder. If it is i, i is not allocated in $\boldsymbol{E'}$. Otherwise, being the lowest-label bidder in $\boldsymbol{E}^*$, it is also the lowest-label bidder in $\boldsymbol{E'}$. By Lemma 4, the first unit allocated in $\boldsymbol{E'}$ is the lowest-label bidder $\leq E_2$, if there is one. By Lemma 4 again, $i = E_1 < E_2$, so i is the first unit allocated in $\boldsymbol{E'}$. □

  - $S(\boldsymbol{V}) = 0$. Then consistency requires that i makes the only valid bid.
    By Lemma 5 (for $k = 0$) and the induction hypothesis, (4) applies to i in subgame $\boldsymbol{V}^2$, i.e., $u_i(\boldsymbol{V}^2) = SW(\boldsymbol{V}^2) - SW(\boldsymbol{V}^2_{\setminus i})$. Therefore

$$\begin{aligned} b_i(\boldsymbol{V}) &= v_i^1 - u_i(\boldsymbol{V}^2) \\ &= v_i^1 - \left[SW(\boldsymbol{V}^2) - SW(\boldsymbol{V}^2_{\setminus i})\right] \\ &= \left[v_i^1 + SW(\boldsymbol{V}^2_{\setminus i})\right] - SW(\boldsymbol{V}^2) \\ &= SW(\boldsymbol{V}) - SW(\boldsymbol{V}^2) \geq 0 \end{aligned}$$

    With the last inequality following from the fact that $\boldsymbol{V}^2$ is a submatrix of $\boldsymbol{V}$.
    From $S(\boldsymbol{V}) = 0$ we infer that there is no efficient allocation of $\boldsymbol{V}^1_{\setminus i}$ that allocates a unit in the first round. Hence, reiterating the proof of the case $A(\boldsymbol{V}) = 0$ above, no bidder other than i makes a valid bid.

- $S(\boldsymbol{V}) > 0$. Let $s := S(\boldsymbol{V})$. For consistency, we need to show that i's bid is highest, and s's second highest, and that both are valid bids.
  i's bid, by Algorithm 1, is

$$b_i(\boldsymbol{V}) = v_i^1 - u_i(\boldsymbol{V}^2_{\setminus s})$$

By Lemma 5 (for $k = s$) and the induction hypothesis, (4) applies to i in subgame $\boldsymbol{V}^2_{\setminus s}$, i.e.,

$$u_i(\boldsymbol{V}^2_{\setminus s}) = SW(\boldsymbol{V}^2_{\setminus s}) - SW(\boldsymbol{V}^2_{\setminus \{i,s\}})$$

Therefore

$$b_i(\boldsymbol{V}) = v_i^1 - SW(\boldsymbol{V}^2_{\setminus s}) + SW(\boldsymbol{V}^2_{\setminus \{i,s\}}) \tag{6}$$

For every other bidder $k \neq i$, suppose first that $k \in \boldsymbol{E}$, say $k = E_l$. Then by Algorithm 1

$$b_k(\boldsymbol{V}) = v_k^1 - u_k(\boldsymbol{V}^l_{\setminus \{E_1,\ldots,E_{l-1}\}})$$

By the induction hypothesis and (4) (remember $k = E_l$)

$$b_k(\boldsymbol{V}) = v_k^1 - SW(\boldsymbol{V}^l_{\setminus \{E_1,\ldots,E_{l-1}\}}) + SW(\boldsymbol{V}^l_{\setminus \{E_1,\ldots,E_l\}}) \tag{7}$$

Let $\boldsymbol{E'}$ be the orderly efficient allocation of $\boldsymbol{V}^2_{\setminus \{i,k\}}$. By Lemma 2, every bidder $\in \boldsymbol{E} \setminus \{i,k\}$ is in $\boldsymbol{E'}$. If $l > 2$ this includes E_2, in which case, by Lemma 4, $E'_2 = E_2$. Similarly, by induction $E'_3 = E_3, \ldots, E'_{l-1} = E_{l-1}$. I.e.,

$$\begin{aligned} SW(\boldsymbol{V}^2_{\setminus \{i,k\}}) &= [v^2_{E_2} + \ldots + v^{l-1}_{E_{l-1}}] + SW(\boldsymbol{V}^l_{\setminus \{E_1,\ldots,E_l\}}) \\ &= [SW(\boldsymbol{V}) - v_i^1 - SW(\boldsymbol{V}^l_{\setminus \{E_1,\ldots,E_{l-1}\}})] + SW(\boldsymbol{V}^l_{\setminus \{E_1,\ldots,E_l\}}) \\ &= SW(\boldsymbol{V}) - v_i^1 + b_k(\boldsymbol{V}) - v_k^1 \end{aligned}$$

The last equality following from (7). Rewrite this

$$b_k(\boldsymbol{V}) = v_k^1 + SW(\boldsymbol{V}^2_{\setminus \{i,k\}}) - SW(\boldsymbol{V}) + v_i^1 \tag{8}$$

So far we considered the case $k \in \boldsymbol{E}$. Alternatively, if $k \notin \boldsymbol{E}$, we have $b_k(\boldsymbol{V}) = v_k^1$ and $SW(\boldsymbol{V}^2_{\setminus \{i,k\}}) = SW(\boldsymbol{V}^2_{\setminus i}) = SW(\boldsymbol{V}) - v_i^1$, so (8) holds for this case too, and so for every $k \neq i$.
Now, by Definition 1 for the alternate winner $s = S(\boldsymbol{V})$, for every $k \neq i$

$$SW(\boldsymbol{V}^1_{\setminus i}) = v_s^1 + SW(\boldsymbol{V}^2_{\setminus \{i,s\}}) \geq v_k^1 + SW(\boldsymbol{V}^2_{\setminus \{i,k\}}) \tag{9}$$

We conclude from (8) and (9) that $b_s(\boldsymbol{V}) \geq b_k(\boldsymbol{V})$, for every bidder k other than i.
It remains to be shown that $b_i(\boldsymbol{V}) \geq b_s(\boldsymbol{V}) \geq 0$. From Algorithm 1, (5) and the induction hypothesis,

$$b_s(\boldsymbol{V}) = v_s^1 - u_s(\boldsymbol{V}^2_{\setminus i}) \leq v_s^1 - SW(\boldsymbol{V}^2_{\setminus i}) + SW(\boldsymbol{V}^2_{\setminus \{i,s\}}) \tag{10}$$

From (6) and (10)

$$
\begin{aligned}
b_i(\boldsymbol{V}) - b_s(\boldsymbol{V}) &\geq \left[v_i^1 - SW(\boldsymbol{V}_{\setminus s}^2) + SW(\boldsymbol{V}_{\setminus\{i,s\}}^2)\right] - \left[v_s^1 - SW(\boldsymbol{V}_{\setminus i}^2) + SW(\boldsymbol{V}_{\setminus\{i,s\}}^2)\right] \\
&= \left[v_i^1 + SW(\boldsymbol{V}_{\setminus i}^2)\right] - \left[v_s^1 + SW(\boldsymbol{V}_{\setminus s}^2)\right] \\
&= SW(\boldsymbol{V}) - \left[v_s^1 + SW(\boldsymbol{V}_{\setminus s}^2)\right] \\
&\geq 0
\end{aligned}
$$

Because $v_s^1 + SW(\boldsymbol{V}_{\setminus s}^2)$ is the social welfare of an alternate allocation of $\boldsymbol{V}$, which by definition does not exceed $SW(\boldsymbol{V})$.

Substituting $s = k$ in (8) and using (9)

$$
\begin{aligned}
b_s(\boldsymbol{V}) &= v_s^1 + SW(\boldsymbol{V}_{\setminus\{i,s\}}^2) - SW(\boldsymbol{V}) + v_i^1 \\
&= SW(\boldsymbol{V}_{\setminus i}^1) - SW(\boldsymbol{V}) + v_i^1 \qquad (11)
\end{aligned}
$$

As $SW(\boldsymbol{V}) = v_i^1 + SW(\boldsymbol{V}_{\setminus i}^2)$ and $\boldsymbol{V}_{\setminus i}^2$ is a submatrix of $\boldsymbol{V}_{\setminus i}^1$, we conclude from (11)

$$
b_s(\boldsymbol{V}) = SW(\boldsymbol{V}_{\setminus i}^1) - SW(\boldsymbol{V}_{\setminus i}^2) \geq 0
$$

Proof of 2 For the induction step, we need only prove that, for every every bidder $k \in [n]$

$$
u_k(\boldsymbol{V}) \geq SW(\boldsymbol{V}) - SW(\boldsymbol{V}_{\setminus k}) \qquad (12)
$$

with equality for $k = A(\boldsymbol{V})$ and for bidders k unallocated in $\boldsymbol{E}$.

Let $i = A(\boldsymbol{V})$. For every bidder k except i, $u_k(\boldsymbol{V}) = u_k(\boldsymbol{V}_{\setminus i}^2)$. By (5) and the induction hypothesis

$$
u_k(\boldsymbol{V}_{\setminus i}^2) \geq SW(\boldsymbol{V}_{\setminus i}^2) - SW(\boldsymbol{V}_{\setminus\{i,k\}}^2)
$$

and (12) follows by Lemma 3. If k is unallocated in $\boldsymbol{E}$, $u_k(\boldsymbol{V}) = 0$. Also $SW(\boldsymbol{V}) = SW(\boldsymbol{V}_{\setminus k})$. So (12) holds with equality.

It remains to prove for i.

When $S(\boldsymbol{V}) = 0$: By the definition of $S(\boldsymbol{V})$, there is no efficient allocation of $\boldsymbol{V}_{\setminus i}^1$ with a first-round allocation, i.e., $SW(\boldsymbol{V}_{\setminus i}^1) = SW(\boldsymbol{V}_{\setminus i}^2)$. But

$$
SW(\boldsymbol{V}) = v_i^1 + SW(\boldsymbol{V}_{\setminus i}^2)
$$

Therefore

$$
u_i(\boldsymbol{V}) = v_i^1 = SW(\boldsymbol{V}) - SW(\boldsymbol{V}_{\setminus i}^2) = SW(\boldsymbol{V}) - SW(\boldsymbol{V}_{\setminus i}^1)
$$

As claimed.

If $s = S(\boldsymbol{V}) > 0$, then using (11)

$$
u_i(\boldsymbol{V}) = v_i^1 - b_s(\boldsymbol{V}) = SW(\boldsymbol{V}) - SW(\boldsymbol{V}_{\setminus i}^1)
$$

as claimed.

□

D Proof of Theorem 4

Proof. Corollary 1 guarantees an option-value equilibrium induced by the orderly allocation profile, so we must prove that there are no other option-value equilibria. We prove the theorem by induction on the number of rounds. It is trivial for one round. Assume it true for up to $m-1$ rounds. We shall complete the induction by proving it for m rounds.

Let (A, S) be an allocation profile of $\boldsymbol{V}$ which is consistent, therefore inducing an option-value equilibrium. By the induction hypothesis, (A, S) induces the orderly efficient allocation in all proper subgames of $\boldsymbol{V}$, leaving only $A(\boldsymbol{V})$ and $S(\boldsymbol{V})$ to be determined. The orderly allocation profile has $A(\boldsymbol{V}) = 1$, by Lemma 4. So, to prove uniqueness, we must show that $A(\boldsymbol{V}) \neq 1$ is *not* consistent.

Construct the option-value matrix $\boldsymbol{\omega}(\boldsymbol{V}) = \{\omega_i^k\}$, as defined in Definition 2. We show that for every column k s.t. $2 \leq k \leq n$, ω_1^k is maximal, i.e., $\omega_1^k \geq \omega_i^k$ for every $i \in [n], i \neq k$. Because, consider the subgame $\boldsymbol{V}_{\setminus k}$. By Lemma 2, its orderly efficient allocation differs by at most one bidder from the orderly efficient allocation of $\boldsymbol{V}$. I.e., it contains all bidders in $[m] \setminus k$, including 1. By Lemma 4, bidder 1 is allocated in the first round. I.e.,

$$SW(\boldsymbol{V}_{\setminus k}) = v_1^1 + SW(\boldsymbol{V}^2_{\setminus\{1,k\}})$$

Now

$$\omega_1^k - \omega_i^k = \left[v_1^1 - u_1(\boldsymbol{V}^2_{\setminus k})\right] - \left[v_i^1 - u_i(\boldsymbol{V}^2_{\setminus k})\right]$$

If bidder 1 is in the orderly efficient allocation of $\boldsymbol{V}^2_{\setminus k}$, by Lemma 4 it must be allocated in its first round. Therefore by Proposition 1 (4) $u_1(\boldsymbol{V}^2_{\setminus k}) = SW(\boldsymbol{V}^2_{\setminus k}) - SW(\boldsymbol{V}^2_{\setminus\{1,k\}})$. By Proposition 1 (5) $u_i(\boldsymbol{V}^2_{\setminus k}) \geq SW(\boldsymbol{V}^2_{\setminus k}) - SW(\boldsymbol{V}^2_{\setminus\{i,k\}})$. Substituting

$$\begin{aligned}
\omega_1^k - \omega_i^k &\geq \left[v_1^1 - SW(\boldsymbol{V}^2_{\setminus k}) + SW(\boldsymbol{V}^2_{\setminus\{1,k\}})\right] - \left[v_i^1 - SW(\boldsymbol{V}^2_{\setminus k}) + SW(\boldsymbol{V}^2_{\setminus\{i,k\}})\right] \\
&= \left[v_1^1 + SW(\boldsymbol{V}^2_{\setminus\{1,k\}})\right] - \left[v_i^1 + SW(\boldsymbol{V}^2_{\setminus\{i,k\}})\right] \\
&= SW(\boldsymbol{V}_{\setminus k}) - \left[v_i^1 + SW(\boldsymbol{V}^2_{\setminus\{i,k\}})\right] \\
&\geq 0
\end{aligned}$$

since $v_i^1 + SW(\boldsymbol{V}^2_{\setminus\{i,k\}})$ is the social welfare of *particular* allocation of $\boldsymbol{V}_{\setminus k}$.

We conclude that if $A(\boldsymbol{V}) = k \neq 1$, bidder 1 is the second-highest bidder, i.e., $S(\boldsymbol{V}) = 1$, and the winner k bids ω_k^1. To be consistent (and therefore in equilibrium), we must have $\omega_k^1 > \omega_1^k$. But

$$\begin{aligned}
\omega_1^k - \omega_k^1 &\geq \left[v_1^1 - SW(\boldsymbol{V}^2_{\setminus k}) + SW(\boldsymbol{V}^2_{\setminus\{1,k\}})\right] - \left[v_k^1 - SW(\boldsymbol{V}^2_{\setminus 1}) + SW(\boldsymbol{V}^2_{\setminus\{1,k\}})\right] \\
&= \left[v_1^1 - SW(\boldsymbol{V}^2_{\setminus k})\right] - \left[v_k^1 - SW(\boldsymbol{V}^2_{\setminus 1})\right] \\
&= SW(\boldsymbol{V}) - \left[v_k^1 + SW(\boldsymbol{V}^2_{\setminus k})\right] \\
&\geq 0
\end{aligned}$$

Therefore the sole option-value equilibrium is for $A(\boldsymbol{V}) = 1$. □

E Example of Incomplete-Information Sequential Auction for Section 5

Example 6. There are three bidders and two rounds. Every bidder has two types, each occurring with marginal probability 1/2. Every bidder's valuations are independent of the other bidders.

Bidder 1's first type is $(4, 0)$ and second, $(5, 0)$.

Bidder 2's first type is $(9, 2)$ and second, $(4, 4)$.

Bidder 3's first type is $(8, 6)$ and second, $(4, 0)$.

I.e.,

$$\Omega = \left\{ \begin{pmatrix} 4\,0 \\ 9\,2 \\ 8\,6 \end{pmatrix}, \begin{pmatrix} 4\,0 \\ 9\,2 \\ 4\,0 \end{pmatrix}, \begin{pmatrix} 4\,0 \\ 4\,4 \\ 8\,6 \end{pmatrix}, \begin{pmatrix} 4\,0 \\ 4\,4 \\ 4\,0 \end{pmatrix}, \begin{pmatrix} 5\,0 \\ 9\,2 \\ 8\,6 \end{pmatrix}, \begin{pmatrix} 5\,0 \\ 9\,2 \\ 4\,0 \end{pmatrix}, \begin{pmatrix} 5\,0 \\ 4\,4 \\ 8\,6 \end{pmatrix}, \begin{pmatrix} 5\,0 \\ 4\,4 \\ 4\,0 \end{pmatrix} \right\}$$

with $P(K) = \frac{|K|}{8}$ for every $K \subseteq \Omega$.

In a slight abuse of notation (*1st type value* | *2nd type value*), which we shall also use later, denote this

$$\boldsymbol{V} = \begin{pmatrix} 4|5 & 0|0 \\ 9|4 & 2|4 \\ 8|4 & 6|0 \end{pmatrix}$$

We seek bidding strategies in perfect Bayesian equilibrium. In the last round, all bidders weakly-dominant strategy is to bid their value. What should they bid in the first round?

We shall verify that the following bidding strategies are in equilibrium.

$$\boldsymbol{b}(\boldsymbol{V}; \cdot) = \begin{pmatrix} 4|5 \\ 8|2 \\ 6|4 \end{pmatrix}$$

E.g., bidder 2 bids 8 when his type is $(9, 2)$.

The alternate winners and payments for each bidder, and for each type matrix (in corresponding order to the order of Ω):

$$\boldsymbol{AW}(\cdot) = \left\{ \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\boldsymbol{M}(\cdot) = \left\{ \begin{pmatrix} 8 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix} \right\}$$

The option values for each bidder, and for each type matrix (in corresponding order to the order of Ω):

$$\boldsymbol{OV}(\cdot) = \left\{ \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} \right\}$$

So

$$\boldsymbol{OV}(\cdot) - \boldsymbol{M}(\cdot) = \left\{ \begin{pmatrix} -4 \\ 1 \\ -6 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ -6 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ -6 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix} \right\}$$

Note that the minimum bid that wins bidder 1 any type matrices is 4, while bidders 2 and 3 must bid *above* 4 (due to tie-breaking rules) to win any type matrices.

Therefore, for $\boldsymbol{v_1} = (4, 0)$, the only step is $MU_1(4) = 0$ (as noted, there is no need to check $b > v_1^1$). Similarly for $\boldsymbol{v_1} = (5, 0)$, the only step below v_i^1 is $MU_1(4) = 1/8$, so $b_1(\boldsymbol{V}) = (4|5)$ is confirmed as optimal.

For $\boldsymbol{v_2} = (9, 2)$, note that $\boldsymbol{OV}(\cdot) - \boldsymbol{M}(\cdot)$ is positive for all type matrices in the relevant information set $\Omega_2(\boldsymbol{v_2})$. A bid of 8 wins all of them, and so is optimal. For $\boldsymbol{v_2} = (4, 4)$, there are no steps $\leq v_2^1$, so any bid in $[0, 4]$ is optimal. So $b_2(\boldsymbol{V}) = (8|2)$ is confirmed as optimal.

For $\boldsymbol{v_3} = (8, 6)$, the steps are $MU_2(4 + \epsilon) = 2/8$, $MU_2(5 + \epsilon) = 2/8 + 1/8 = 3/8$, and there are no other steps $\leq v_3^1$. For $\boldsymbol{v_3} = (4, 0)$, there are no steps $\leq v_3^1$, so $b_3(\boldsymbol{V}) = (6, 4)$ is confirmed as optimal.