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FREE COMPLEXES OVER THE EXTERIOR ALGEBRA WITH SMALL HOMOLOGY

by

Erica Hopkins

A DISSERTATION

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FREE COMPLEXES OVER THE EXTERIOR ALGEBRA WITH SMALL HOMOLOGY

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University of Nebraska, 2021

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Let M be a graded module over a standard graded polynomial ring S. The Total Rank Conjecture by Avramov-Buchweitz predicts the total Betti number of M should be at least the total Betti number of the residue field. Walker proved this is indeed true in a large number of cases. One could then try to push this result further by generalizing this conjecture to finite free complexes which is known as the Generalized Total Rank Conjecture. However, Iyengar and Walker constructed examples to show this generalized conjecture is not always true.

In this thesis, we investigate other counterexamples of the Generalized Total Rank Conjecture and some of their properties. Under the BGG correspondence, a finite free graded complex over the exterior algebra with small homology corresponds to a free complex over the polynomial ring with a small total Betti number. Therefore, we focus on examples of finite free complexes over the exterior algebra with small homology. The main examples we consider are Koszul complexes of quadrics, and we show the Koszul complex of one general quadric and the Koszul complex of two general quadrics have the smallest possible homology among complexes over the exterior algebra with the same graded Poincaré series. Finally while analyzing these Koszul complexes, we notice the dimension of their total homology has a nice asymptotic behavior and investigate under what conditions other complexes have this same asymptotic behavior.

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Chapter 1

INTRODUCTION

One way to study a graded module, M, over a standard graded polynomial ring $S_n = k[x_1, \ldots, x_n]$, is to study invariants of M. A particular invariant of interest is the total Betti number of M. The i^{th} Betti number of M, denoted $\beta_i(M)$, is the rank of the i^{th} free module in the minimal graded free resolution of M (see Definition 2.16) and the total Betti number of M is $\sum_i \beta_i(M)$. For example, the minimal graded free resolution of k is the Koszul complex on the generators $x_1, \ldots, x_n$. The i^{th} free module in the Koszul complex is $S_n^{\binom{n}{i}}$ which implies

$$\beta_i(k) = \binom{n}{i}.$$

Thus, by the Binomial Theorem, the total Betti number of k is

$$\sum_i \beta_i(k) = \sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

After considering the residue field, the next logical step is to try to understand the finite length modules of S_n. Recall an S_n-module M has finite length if there exists a filtration

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

of S_n-modules such that $M_{i+1}/M_i \cong k$. Since we can think of finite length modules as essentially being built from k, then it is reasonable to predict the Koszul complex is the smallest possible resolution among all resolutions of finite length modules as stated in the following well-known conjecture (see [4, 1.4] and [11, Problem 24]):

Conjecture 1.1 (Buchsbaum, Eisenbud, Horocks Conjecture)**.** *If M is a non-zero S_n-module with finite length then*

$$\beta_i(M) \geq \binom{n}{i}.$$

This conjecture is known as the BEH conjecture and has been proven in a variety of cases. By the binomial theorem, the validity of the BEH conjecture would imply the validity of the following weaker conjecture (see [1, Page 148]):

Conjecture 1.2 (Total Rank Conjecture)**.** *If M is a non-zero S_n-module with finite length then*

$$\sum_i \beta_i(M) \geq 2^n.$$

In [17], Walker proved this conjecture when $\operatorname{char} k \neq 2$. One can then try to push this result further by considering a generalization of the Total Rank Conjecture to Betti numbers of finite free complexes.

Recall we have the following result (see Theorem 2.17):

$$\beta_i(M) = \dim_k \operatorname{Tor}_i^{S_n}(M, k)$$

where given a free resolution $\mathbf{G}$ of M over S_n we define

$$\operatorname{Tor}_i^{S_n}(M, k) = H_i(\mathbf{G} \otimes_{S_n} k).$$

We can generalize this description in order to define the i^{th} Betti number of a graded complex $\mathbf{F}$.

Definition 1.3. Let $\mathbf{F}$ be a free graded complex over S_n then

$$\beta_i(\mathbf{F}) = \dim_k \operatorname{Tor}_i^{S_n}(\mathbf{F}, k)$$

where

$$\operatorname{Tor}_i^{S_n}(\mathbf{F}, k) = H_i(\mathbf{F} \otimes_{S_n} k).$$

One could equivalently define the Betti numbers of a complex by defining $\beta_i(\mathbf{F})$ to be the rank of the i^{th} free module in a minimal free complex that is quasi-isomorphic to $\mathbf{F}$.

The Total Rank Conjecture (see Conjecture 1.2) can be generalized to finite free complexes as follows.

Conjecture 1.4 (Generalized Total Rank Conjecture)**.** *If $\boldsymbol{F}$ is a finite free graded complex over S_n such that the total homology $H(\boldsymbol{F})$ is non-zero and has finite length then*

$$\sum_i \beta_i(\boldsymbol{F}) \geq 2^n.$$

However, Iyengar and Walker [13] showed this conjecture is false in general by providing a counterexample. They posed the following question.

Question 1.5 ([13, page 11])**.** *Is there a real number $a > 1$ such that each finite free complex $\boldsymbol{F}$ over S_n with total homology $H(\boldsymbol{F})$ non-zero and of finite length satisfies*

$$\sum_i \beta_i(\boldsymbol{F}) \geq a^n?$$

In this thesis, we will investigate this question by analyzing other counterexamples to the Generalized Total Rank Conjecture that could provide insight on a choice of a. These counterexamples will arise by considering finite free graded complexes over the exterior algebra $E_n = k\langle e_1, \ldots, e_n\rangle$.

The BGG correspondence allows us to relate complexes of graded free E_n-modules to complexes of graded free S_n-modules. This correspondence is given by two functors $\mathbf{R}$ and $\mathbf{L}$ and we have the following useful corollary of the BGG correspondence (see Section 2.2 for more details).

Definition 1.6. A complex $\mathbf{F}$ of graded E_n-modules is called *perfect* if it is quasi-isomorphic to a finite free graded complex over E_n.

Corollary 1.7. *Under the BGG correspondence, a perfect complex* $\mathbf{F}$ *of graded* E_n*-modules corresponds to a complex* $\mathbf{G}$ *of graded* S_n*-modules with finite length total homology. In addition*

$$\dim_k H(\mathbf{F}) = \sum_i \beta_i(\mathbf{G}).$$

Therefore we can restate Question 1.5 into an equivalent question that is in terms of the homology of a complex over the exterior algebra.

Question 1.8. *Is there a real number* $a > 1$ *such that each non-exact perfect complex* $\mathbf{F}$ *over* E_n *satisfies*

$$\dim_k H(\mathbf{F}) \geq a^n$$

or equivalently

$$\left(\dim_k H(\mathbf{F})\right)^{1/n} \geq a?$$

In order to more easily discuss results related to this question, we introduce the following notation.

Definition 1.9. Let $\mathbf{F}$ be a finite free complex over the exterior algebra E_n then the *homological growth factor* of $\mathbf{F}$ is

$$HGF(\mathbf{F}) = \left(\sum_i \dim_k \left(H_i(\mathbf{F})\right)\right)^{1/n}.$$

Chapter 2 contains background on relevant topics for this thesis. Since we work over the exterior algebra which is a graded ring, Section 2.1 covers the necessary definitions and results on graded modules and graded complexes. Section 2.2 is devoted to the BGG correspondence. Even though the material in this section will be rarely referenced throughout the rest of this thesis, the BGG correspondence is crucial for motivating our interest in complexes over the exterior algebra with small homology. Therefore we carefully work out details of the correspondence, some examples, and proofs of important results including Corollary 1.7 stated above.

In Chapter 3, we begin investigating counterexamples to Conjecture 1.4. Section 3.1 focuses on the Koszul complex of one general quadric which, under BGG, corresponds to the original counterexample given by Iyengar and Walker [13]. Section 3.2 considers the Koszul complex of two generic quadrics. We show that both of these families of complexes gives counterexamples to the Generalized Total Rank Conjecture. Specifically, we demonstrate that, for some values of n, the homological growth factor is strictly less than 2. Finally, we notice both types of complexes have the same asymptotic behavior, namely the homological growth factor is asymptotically at least 2.

While analyzing these complexes, we note there exists a value of n such that the homological growth factor of the Koszul complex of two generic quadrics is smaller than every homological growth factor of the Koszul complex of one general quadric. It is then reasonable to ask, can we find a complex with an even smaller homological growth factor? Chapter 4 introduces the idea of minimal homology in order to assist in answering this question.

Definition 1.10. A finite free graded complex $\mathbf{F}$ over the exterior algebra has *minimal homology* if

$$\dim_k H(\mathbf{F}) \leq \dim_k H(\mathbf{G})$$

for any finite free graded complex $\mathbf{G}$ over the exterior algebra with the same graded Poincaré series as $\mathbf{F}$.

We show the Koszul complex of one general quadric and the Koszul complex of two general quadrics have minimal homology. Thus there are no complexes with the same graded Poincaré series as these complexes that have a smaller homological growth factor.

However this does not imply other types of complexes cannot also have minimal homology. Since every complex with graded Poincaré series $1+st^2$ is a Koszul complex, then clearly every complex with minimal homology having this graded Poincaré series is Koszul. However there are complexes with graded Poincaré series $1 + 2st^2 + st^4$ that have minimal homology and are not Koszul. We give a few examples of such complexes, and also give the following characterization when $n \geq 9$.

Theorem 1.11. *Let k be an infinite field and $\mathbf{F}$ be a graded complex over the exterior algebra with graded Poincaré series $1 + 2st^2 + st^4$. If $n \geq 9$ and $\mathbf{F}$ has minimal homology then $\mathbf{F}$ has the form*

$$\mathbf{F} : E_n(-4) \xrightarrow{\begin{pmatrix} -\lambda q_2 \\ \lambda q_1 \end{pmatrix}} E_n(-2)^2 \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n$$

for some nonzero scalar λ.

Finally in Chapter 5, we return to analyzing the asymptotic behavior of the homological growth factor of complexes. As previously mentioned, the homological growth factor of the Koszul complex of one general quadric and the Koszul complex of two generic quadrics is asymptotically at least 2. We then generalize this result and show in Section 5.2 that families of Koszul complexes of a fixed number of quadrics have this same asympotic behavior and in Section 5.3 that some families of Koszul complexes of a varying number of quadrics have this same asymptotic behavior.

Koszul complexes are not the only families of complexes whose homological growth factor is asymptotically at least 2. In Sections 5.2 and 5.3, we give conditions under which a family of complexes over the exterior algebra has this asymptotic behavior (see Theorem 5.9 and Proposition 5.12). However, not every complex has this behavior and Section 5.4 is devoted to giving an example of a family of complexes whose homological growth factor is asymptotically strictly less than 2.

Chapter 2

BACKGROUND

In this chapter, we cover the necessary definitions and results that are used in the rest of this thesis. The more experienced reader can skip this chapter and refer back to it when needed.

2.1 Graded rings

This section contains details about graded modules, complexes and Betti numbers. Unless stated otherwise, the details and results from this section are from [16].

2.1.1 Modules and complexes

Definition 2.1. A ring R is *graded* if there exists a collection of subgroups $\{R_i\}_{i\in\mathbb{Z}}$ such that

1. $R = \bigoplus_{i\in\mathbb{Z}} R_i$

2. $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$

Definition 2.2. Let A be a ring. A graded ring R is a *standard graded A-algebra* if $R_0 = A$ and R is generated by R_1 as an A-algebra.

Throughout we will assume the ring R is a standard graded k-algebra where k is a field. In this case, we also have

$$\mathbf{m} = R_+ = \bigoplus_{i \geq 1} R_i$$

is the unique homogeneous maximal ideal of R.

The polynomial ring $S_n = k[x_1, \ldots, x_n]$ with $\deg(x_i) = 1$ is an example of a standard graded k-algebra where the i^{th} subgroup of S_n is the k-vector space spanned by all the monomials of degree i. Another example is the exterior algebra $E_n = k\langle e_1, \ldots, e_n \rangle$ with $\deg(e_i) = 1$, and the i^{th} subgroup of E_n is the k-vector space spanned by all the wedge products of degree i.

Definition 2.3. An R-module M is called *graded* if there exists a collection of k-vector spaces $\{M_i\}_{i \in \mathbb{Z}}$ such that

1. $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as a k-vector space
2. $R_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$.

In addition, for any $p \in \mathbb{Z}$ we denote by $M(-p)$ the graded R-module such that $M(-p)_i = M_{i-p}$ for all $i \in \mathbb{Z}$.

If a graded module is finitely generated, then we can measure the size of the module by measuring the size of each of its components. In order to better study the size of a finitely generated module we can form the Hilbert series of the module.

Definition 2.4. Let $M = \bigoplus_i M_i$ be a finitely generated graded R-module. The function $i \mapsto \dim_k(M_i)$ is called the *Hilbert function* of M and the series

$$h_M(t) = \sum_i \dim_k(M_i) t^i$$

is called the *Hilbert series* of M. Note that the Hilbert series of a shifted module is given by

$$h_{M(-p)}(t) = t^p h_M(t).$$

Another way to study modules is to form complexes of modules.

Definition 2.5. A *graded complex* over R is a sequence of homomorphisms of graded R-modules

$$\mathbf{F} : \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \cdots$$

such that $d_{i-1}d_i = 0$ for all i and each d_i is a homomorphism of degree 0 (i.e. $\deg(d_i(f)) = \deg(f)$ for all $f \in F_i$). Note the complex $\mathbf{F}$ is bigraded as a k-vector space because each F_i is a graded module. Therefore for all i,

$$F_i = \bigoplus_{j \in \mathbb{Z}} F_{i,j}$$

and an element of $F_{i,j}$ is said to have homological degree i and internal degree j.

Example 2.6. Let $x_1, \ldots, x_r \in R$ and let $E_r = k\langle e_1, \ldots, e_r\rangle$. The Koszul complex of $x_1, \ldots, x_r$ over R is

$$\operatorname{Kos}_R(x_1, \ldots, x_r) : 0 \to F_r \xrightarrow{d_r} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

where F_i is a free R-module with basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq r\}$ and the differential is given by

$$d_i(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{p=1}^{i} (-1)^{p+1} x_{j_p} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_p}} \wedge \cdots \wedge e_{j_i}$$

where $\widehat{e_{j_p}}$ means e_{j_p} is omitted from the product.

Definition 2.7. The *homology in degree* i of a complex $\mathbf{F}$ is the graded R-module

$$H_i(\mathbf{F}) = \ker(d_i)/\operatorname{image}(d_{i+1})$$

and the *total homology* is the graded R-module

$$H(\mathbf{F}) = \bigoplus_i H_i(\mathbf{F}).$$

Definition 2.8. A graded complex $\mathbf{F}$ is *exact* if $H_i(\mathbf{F}) = 0$ for all i.

Proposition 2.9. *If*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of finitely generated graded R-modules with degree 0 homomorphisms then

$$h_M(t) = h_{M'}(t) + h_{M''}(t).$$

Given the above proposition, we can define the Hilbert series of a finite free graded complex.

Definition 2.10. A graded complex over R is *finite free* if it is a bounded graded complex of finite rank free R-modules.

Definition 2.11. Let $\mathbf{F}$ be a finite free graded complex over R

$$\mathbf{F} : 0 \to F_r \to F_{r-1} \to \cdots \to F_{s+1} \to F_s \to 0$$

where F_i is in homological degree i. The *Hilbert series* of $\mathbf{F}$ is

$$h_{\mathbf{F}}(t) = \sum_i (-1)^i h_{F_i}(t).$$

Another useful series used to study complexes is the graded Poincaré series.

Definition 2.12. Let $\mathbf{F}$ be a graded finite free complex over R. We define its *Poincaré series* to be

$$P_{\mathbf{F}}(t) = \sum_i \operatorname{rank}(F_i) t^i$$

where F_i is the free module in homological degree i. Since F_i is graded then

$$F_i = \bigoplus_{p \in \mathbb{Z}} R(-p)^{c_{i,p}}$$

for some $c_{i,p} \geq 0$. We define the *graded Poincaré series* to be

$$P_{\mathbf{F}}(s,t) = \sum_{i,p} c_{i,p} s^i t^p.$$

2.1.2 Resolutions and Betti numbers

Definition 2.13. A *graded free resolution* of a finitely generated graded R-module M is a finite free graded complex over R

$$\mathbf{F} : \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0$$

along with a map $\epsilon : F_0 \to M$ such that the augmented graded complex

$$\cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

is exact.

We are often interested in the smallest graded resolution of M in the sense that the ranks of each of its free modules is less than or equal to the rank of the corresponding free module in any other graded free resolution of M. As we will see in the next theorem, this notion is equivalent to a graded free resolution being minimal.

Definition 2.14. A graded free resolution $\mathbf{F}$ over R is *minimal* if it satisfies the condition

$$d_{i+1}(F_{i+1}) \subseteq \mathbf{m}F_i$$

for all i where $\mathbf{m} = R_+$ is the unique homogeneous maximal ideal of R.

Theorem 2.15. *Let M be a finitely generated graded R-module.*

1. *There exists a minimal graded free resolution of M.*

2. *Let $\mathbf{F}$ be a minimal graded free resolution of M. If $\mathbf{G}$ is a graded free resolution of M then $\mathbf{G} \cong \mathbf{F} \oplus \mathbf{T}$ for some trivial complex $\mathbf{T}$, and the direct sum is a direct sum of complexes. Recall a trivial complex is the direct sum of complexes of the form*
$$0 \to R(-p) \xrightarrow{1} R(-p) \to 0$$
that are possibly placed in different homological degrees.

3. *Up to isomorphism, there exists a unique minimal graded free resolution of M.*

Therefore we can define the following invariant of a graded module.

Definition 2.16. Let $\mathbf{F}$ be a minimal graded free resolution of a finitely generated graded R-module M. The i^{th} *Betti number* of M over R is
$$\beta_i^R(M) := \operatorname{rank}(F_i).$$

By Theorem 2.15, Betti numbers do not depend on the choice of the minimal graded free resolution of M, so this is well-defined.

There are also other ways to find the Betti numbers of a module as described in the following result.

Proposition 2.17. *Let M be a finitely generated graded R-module. Then*
$$\begin{aligned}\beta_i(M) &= \dim_k \operatorname{Tor}_i^R(M,k)\\ &= \dim_k \operatorname{Ext}_R^i(M,k)\end{aligned}$$

We can further refine the definition of Betti numbers to get the graded Betti numbers of a module.

Definition 2.18. Let $\mathbf{F}$ be a minimal graded free resolution of a graded finitely generated R-module M. Since $\mathbf{F}$ is graded, then each free module in the resolution is of the form

$$F_i = \bigoplus_{p \in \mathbb{Z}} R(-p)^{c_{i,p}}$$

for some $c_{i,p} \geq 0$. We define the *graded Betti numbers* of M to be

$$\beta^R_{i,p}(M) = c_{i,p}.$$

Remark 2.19. *Since* $\operatorname{Tor}^R_i(M,k)$ *and* $\operatorname{Ext}^i_R(M,k)$ *are graded R-modules we also have by Proposition 2.17*

$$\beta^R_{i,p}(M) = \dim_k(\operatorname{Tor}^R_i(M,k))_p = \dim_k(\operatorname{Ext}^i_R(M,k))_p.$$

2.2 The Bernstein, Gel'fand, Gel'fand correspondence

Let $S = k[x_1, \dots, x_n]$ be the polynomial ring with $\deg(x_i) = 1$ and let $E = k\langle e_1, \dots, e_n\rangle$ be the exterior algebra with $\deg(e_i) = -1$. This will be the only section where the exterior algebra will be negatively graded in order to better align with the literature on the BGG correspondence. Throughout this section, we will cover the main definitions and the important results related to the BBG correspondence needed to motivate this thesis. For more details on this correspondence, see [7] or the original paper [2].

2.2.1 Properties of the exterior algebra

The exterior algebra $E = k\langle e_1, \ldots, e_n\rangle$ is a graded commutative ring with multiplication denoted by $\wedge$. In particular

1. $e_i \wedge e_j = -e_j \wedge e_i$ for all i, j
2. $e_i \wedge e_i = 0$ for all i.

In addition, E is graded with $E = \bigoplus_i E_i$ where E_i is the k-vector space spanned by all wedge products of degree i. Recall we are assuming $\deg(e_i) = -1$ for all $1 \leq i \leq n$, so $E = \bigoplus_{i=0}^n E_{-i}$.

Remark 2.20. *E is a finite dimensional k-vector space with*

$$\dim_k E_{-i} = \binom{n}{i}.$$

Remark 2.21. $\mathrm{Hom}_k(E, k)$ *is a graded left E-module with* $\mathrm{Hom}_k(E, k) = \bigoplus_{i=0}^n \mathrm{Hom}_k(E_{-i}, k)$ *and E-module structure given by $xh(e) = h(e \wedge x)$ for all $x, e \in E$ and $h \in \mathrm{Hom}_k(E, k)$.*

Proposition 2.22. *Let $E = k\langle e_1, \ldots, e_n\rangle$ be the exterior algebra with* $\deg(e_i) = -1$. *Then*

$$\mathrm{Hom}_k(E, k) \cong E(-n)$$

as left graded E-modules.

Proof. First note the map $\langle -, - \rangle : E \times E \to k$ given by

$$\langle e, x\rangle = \text{ coefficient of } e_1 \wedge \cdots \wedge e_n \text{ in } e \wedge x$$

is k-bilinear, and thus gives a k-linear map $\varphi : E(-n) \to \mathrm{Hom}_k(E, k)$ defined by

$$\varphi(x) = [e \mapsto \langle e, x\rangle].$$

We claim φ is a graded left E-module isomorphism. Let $x, y \in E$ then notice

$$\left[x\varphi(y)\right](e) = \varphi(y)(e \wedge x) = \langle e \wedge x, y\rangle = \langle e, x \wedge y\rangle = \varphi(x \wedge y)(e).$$

Thus φ is a left E-module homomorphism.

Now supppose $x \in E$ is homogeneous of degree $-d$. Thus $x \in E_{-d} = E(-n)_{n-d}$. Note $\phi(x) \neq 0$ only on E_{d-n}, so $\phi(x) \in \operatorname{Hom}_k(E_{d-n}, k) = \operatorname{Hom}_k(E, k)_{n-d}$. Therefore φ is degree preserving, so φ is a left graded E-module homomorphism.

It remains to show φ is bijective. Since φ is a k-linear map and $\dim_k E(-n)) = \dim_k \operatorname{Hom}_k(E, k)$ then it sufficies to show φ is injective. First let us prove the following claim.

Claim 1: For all nonzero $x \in E$ there exists $e \in E$ such that $\langle e, x\rangle \neq 0$.

Let $x \in E$ be nonzero. For a subset $I \subseteq [n]$, set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_\ell}$ when $I = \{i_1, \ldots, i_\ell\}$ and $i_1 < \ldots < i_\ell$. Then

$$x = \sum_{I \subseteq [n]} a_I e_I$$

for some $a_I \in k$ with $a_I \neq 0$ for some $I \subseteq [n]$. Let $d = \min\left\{|I| \mid a_I \neq 0\right\}$ and let $J \subseteq [n]$ be such that $|J| = d$ and $a_J \neq 0$. Then notice

$$e_{[n]\setminus J} \wedge x = \pm a_J(e_1 \wedge \ldots \wedge e_n)$$

because $e_{[n]\setminus J} \wedge a_I e_I = 0$ for all $I \neq J$. Since $a_J \neq 0$, then $\langle e_{[n]\setminus J}, x\rangle \neq 0$ so we have proven our claim.

Now suppose $x_1, x_2 \in E$ such that $\varphi(x_1) = \varphi(x_2)$. Then $\langle x_1, e\rangle = \langle x_2, e\rangle$ for all $e \in E$ which implies $\langle x_1 - x_2, e\rangle = 0$ for all $e \in E$. Then by Claim 1, $x_1 - x_2 = 0$ so $x_1 = x_2$. Thus φ is injective, so we conclude φ is a left graded E-module isomorphism. □

Corollary 2.23. *E is injective as a left E-module.*

Proof. By [3, Lemma 3.1.6], $\mathrm{Hom}_k(E, k)$ is an injective left E-module. Thus by Proposition 2.22, E is an injective left E-module. □

2.2.2 Defining the functors

Definition 2.24. Suppose $P = \bigoplus_j P_j$ is a graded E-module and define $\mathbf{L}(P)$ to be the graded complex of S-modules

$$\mathbf{L}(P) : \cdots \xrightarrow{\psi} S \otimes_k P_j \xrightarrow{\psi} S \otimes_k P_{j-1} \xrightarrow{\psi} \cdots$$

where $\psi(f \otimes p) = \sum_i (x_i f \otimes e_i \wedge p)$ and $S \otimes_k P_j$ is in homological degree j.

Remark 2.25. *By the definition of $L(P)$, we have the following equality*

$$\mathbf{L}(P(-a)) = \Sigma^a \mathbf{L}(P).$$

Generalize this definition to a graded complex of E-modules in the following way.

Definition 2.26. Suppose we have a graded complex of E-modules

$$\mathbf{F} : \cdots \to F_{i+1} \to F_i \to \cdots .$$

Applying $\mathbf{L}$ to each module in $\mathbf{F}$ gives the following double complex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots \arrow[r] & S \otimes_k (F_{i+1})_j \arrow[r] \arrow[d] & S \otimes_k (F_{i+1})_{j-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & S \otimes_k (F_i)_j \arrow[r] \arrow[d] & S \otimes_k (F_i)_{j-1} \arrow[r] \arrow[d] & \cdots \\
 & {} & {} &
\end{tikzcd}$$

where the vertical maps are induced by the differential of $\mathbf{F}$ and the horizontal complexes are the complexes $\mathbf{L}(F_i)$ defined in Definition 2.24. Define $\mathbf{L}(\mathbf{F})$ to be the total complex of this double complex.

Proposition 2.27. $\mathbf{L}$ *is a functor from the category of graded complexes of E-modules to the category of graded complexes of S-modules.*

The functor $\mathbf{L}$ has a right adjoint functor $\mathbf{R}$ which can be defined in an analogous way.

Definition 2.28. Let $M = \bigoplus_d M_d$ be a graded S-module and define $\mathbf{R}(M)$ to be the complex of graded E-modules

$$\mathbf{R}(M) : \cdots \xrightarrow{\phi} \mathrm{Hom}_k(E, M_d) \xrightarrow{\phi} \mathrm{Hom}_k(E, M_{d+1}) \xrightarrow{\phi} \cdots$$

where $\phi(\alpha) = [e \mapsto \sum_i x_i \alpha(e_i \wedge e)]$ and $\mathrm{Hom}_k(E, M_d)$ is in homological degree $-d$.

Remark 2.29. *By the definition of* $\mathbf{R}(M)$*, we have the following equality*

$$\mathbf{R}(M(-a)) = \Sigma^a \mathbf{R}(M).$$

This definition generalizes to graded complexes of S-modules in the same way as $\mathbf{L}$ in Definition 2.26.

Definition 2.30. Let $\mathbf{G}$ be a graded complex of S-modules. Define $\mathbf{R}(\mathbf{G})$ to be the total complex of the double complex formed by applying $\mathbf{R}$ to each module in $\mathbf{G}$.

Proposition 2.31. $\mathbf{R}$ *is a functor from the category of graded complexes of S-modules to the category of graded complexes of E-modules.*

2.2.3 Examples

In order to better understand these functors, we have provided a few examples below.

Example 2.32. Let $E = k\langle e_1, e_2\rangle$ and $S = k[x_1, x_2]$. Then $\mathbf{L}(E)$ is the complex

$$0 \to S \otimes_k k \xrightarrow{\psi} S \otimes_k E_{-1} \xrightarrow{\psi} S \otimes_k E_{-2} \to 0.$$

Note each module in this complex is a free S-module and the differential acts on the basis elements in the following way

$$\psi(1 \otimes 1) = x_1 \otimes e_1 + x_2 \otimes e_2$$

$$\psi(1 \otimes e_1) = x_1 \otimes e_1 \wedge e_1 + x_2 \otimes e_2 \wedge e_1 = -x_2 \otimes e_1 \wedge e_2$$

$$\psi(1 \otimes e_2) = x_1 \otimes e_1 \wedge e_2 + x_2 \otimes e_2 \wedge e_2 = x_1 \otimes e_1 \wedge e_2$$

$$\psi(1 \otimes e_1 \wedge e_2) = 0.$$

Thus we have the following isomorphic complex

$$0 \to S \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} -x_2 & x_1 \end{pmatrix}} S \to 0$$

which is $\mathrm{Hom}_S(\mathrm{Kos}_S(x_1, x_2), S)$, and it is quasi-isomorphic to $\Sigma^{-2}k$.

We can also consider the dual of E as a k-vector space which we denote as $E^* := \mathrm{Hom}_k(E, k)$. Since $E^* \cong E(-2)$ by Proposition 2.22, then we have

$$\mathbf{L}(E^*) \cong \mathbf{L}(E(-2)) = \Sigma^2\mathbf{L}(E) \simeq \Sigma^2(\Sigma^{-2}k) = k.$$

Example 2.33. Let $E = k\langle e_1, e_2\rangle$ and $S = k[x_1, x_2]$. Then $\mathbf{R}(k)$ is the following complex

$$0 \to \mathrm{Hom}_k(E, k) \to 0.$$

Therefore $\mathbf{R}(k) = E^* := \mathrm{Hom}_k(E, k)$. By Example 2.32 we conclude

$$\mathbf{L}(\mathbf{R}(k)) = \mathbf{L}(E^*) \simeq k$$

and

$$\mathbf{R}(\mathbf{L}(E^*)) \simeq \mathbf{R}(k) = E^*.$$

Example 2.34. Let $E = k\langle e_1, e_2\rangle$, $S = k[x_1, x_2]$, and $w = e_1 \wedge e_2$. Consider the Koszul complex

$$\mathrm{Kos}_E(w) : E(2) \xrightarrow{w} E.$$

In order to find $\mathbf{L}(\mathrm{Kos}_E(w))$, we first consider the double complex

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & S \otimes_k k \arrow[r] \arrow[d, "id \otimes w"'] & S \otimes_k E_{-1} \arrow[r] \arrow[d] & S \otimes_k E_{-2} \arrow[r] & 0 \\
0 \arrow[r] & S \otimes_k k \arrow[r] & S \otimes E_{-1} \arrow[r] & S \otimes E_{-2} \arrow[r] & 0 & &
\end{tikzcd}$$

Using the calculations in Example 2.32 and the fact

$$id \otimes w(1 \otimes 1) = 1 \otimes e_1 \wedge e_2$$

we have the following isomorphic double complex

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & S \arrow[r, "\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}"] \arrow[d, "{\begin{bmatrix} 1 \end{bmatrix}}"'] & S^2 \arrow[r, "\begin{pmatrix} -x_2 & x_1 \end{pmatrix}"] \arrow[d] & S \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r, "\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}"'] & S^2 \arrow[r, "\begin{pmatrix} -x_2 & x_1 \end{pmatrix}"'] & S \arrow[r] & 0 & &
\end{tikzcd}$$

Thus $\mathbf{L}(\mathrm{Kos}_E(w))$ is isomorphic to the total complex of this double complex which is given by the following complex

$$0 \to S \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} -x_2 & x_1 & 1 \\ 0 & 0 & -x_1 \\ 0 & 0 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} 0 & x_2 & -x_1 \end{pmatrix}} S \to 0.$$

2.2.4 Relevant results

As seen in the above examples, the functors $\mathbf{L}$ and $\mathbf{R}$ do not define an equivalence of categories between the category of graded complexes of E-modules and the category of graded complexes of S-modules. However by refining these categories so that quasi-isomorphisms become isomorphisms, we have an equivalence of categories as desired.

Theorem 2.35 ([7, Corollary 2.7])**.** *The functors* $\mathbf{R}$ *and* $\mathbf{L}$ *define an equivalence of categories*

$$D^b(S\text{-}mod) \cong D^b(E\text{-}mod)$$

where $D^b(S\text{-}mod)$ *(or* $D^b(E\text{-}mod)$*) is the derived category of bounded complexes of finitely generated* S*-modules (or* E*-modules).*

In addition, we relate the homology and Betti numbers of these complexes using the following results.

Proposition 2.36. *If* $\mathbf{F}$ *is a complex of graded* E*-modules and* $\mathbf{G}$ *is a complex of graded* S*-modules then*

1. $H_i(\mathbf{R}(\mathbf{G}))_j = \mathrm{Tor}^S_{i+j}(k, \mathbf{G})_j$
2. $H_i(\mathbf{L}(\mathbf{F}))_j = \mathrm{Ext}^{i+j}_E(k, \mathbf{F})_j$

Proof. This proof is an extension of the proof of [7, Proposition 2.3] which is the equivalent result for modules.

Part (1):

Let $E^* := \mathrm{Hom}_k(E, k)$ denote the k-vector space dual of E. As demonstrated in Example 2.32, $\mathbf{L}(E^*)$ is isomorphic to the Koszul complex which is a minimal free resolution of k over S. Thus

$$\mathrm{Tor}^S_{i+j}(k, \mathbf{G})_j = H_{i+j}(\mathbf{L}(E^*) \otimes_S \mathbf{G})_j. \tag{2.2.1}$$

Now for all a, $\mathbf{R}(G_a)$ is the complex of E-modules

$$\mathbf{R}(G_a) : \cdots \xrightarrow{\phi} \mathrm{Hom}_k(E, (G_a)_b) \xrightarrow{\phi} \mathrm{Hom}_k(E, (G_a)_{b+1}) \xrightarrow{\phi} \cdots$$

where $\phi(\alpha) = [e \mapsto \sum_i x_i \alpha(e_i \wedge e)]$. Since

$$\mathrm{Hom}_k(E, (G_a)_b) \cong \mathrm{Hom}_k(E, k) \otimes_k (G_a)_b = E^* \otimes_k (G_a)_b$$

then $\mathbf{R}(G_a)$ is isomorphic to

$$\cdots \xrightarrow{\tilde{\phi}} E^* \otimes_k (G_a)_b \xrightarrow{\tilde{\phi}} E^* \otimes_k (G_a)_{b+1} \xrightarrow{\tilde{\phi}} \cdots$$

where $\tilde{\phi}(\alpha \otimes g) = \sum_i e_i \wedge \alpha \otimes x_i g$. Thus $\mathbf{R}(\mathbf{G})$ is isomorphic to the total complex of the following double complex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots \arrow[r] & E^* \otimes_k (G_{a+1})_b \arrow[r] \arrow[d] & E^* \otimes_k (G_{a+1})_{b+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & E^* \otimes_k (G_a)_b \arrow[r] \arrow[d] & E^* \otimes_k (G_a)_{b+1} \arrow[r] \arrow[d] & \cdots \\
 & {} & {} &
\end{tikzcd}$$

where the horizontal maps are $\tilde{\phi}$ and the vertical maps are $id \otimes d_G$. This implies

$$\mathbf{R}(\mathbf{G})_i \cong \bigoplus_a E^* \otimes_k (G_a)_{a-i}.$$

Also

$$(\mathbf{L}(E^*) \otimes_S \mathbf{G})_{i+j} = \bigoplus_a \mathbf{L}(E^*)_{i+j-a} \otimes_S G_a = \bigoplus_a (E^*)_{i+j-a} \otimes_k S$$

as S-modules, so

$$\mathbf{R}(\mathbf{G})_{i,j} \cong \bigoplus_a (E^*)_{i+j-a} \otimes_k (G_a)_{a-i} \cong (\mathbf{L}(E^*) \otimes_S \mathbf{G})_{i+j,j}$$

as S-modules. Note $\mathbf{L}(E^*) \otimes_S \mathbf{G}$ is isomorphic to the total complex of the following double complex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
\cdots \arrow[r] & (E^*)_{d-a} \otimes_k G_{a+1} \arrow[r] \arrow[d] & (E^*)_{d-a-1} \otimes_k G_{a+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (E^*)_{d-a} \otimes_k G_a \arrow[r] \arrow[d] & (E^*)_{d-a-1} \otimes_k G_a \arrow[r] \arrow[d] & \cdots \\
 & {} & {} &
\end{tikzcd}$$

where the horizontal maps are $\tilde{\phi}$ and the vertical maps are $id \otimes d_G$ which are the same maps as in the double complex that gives $\mathbf{R}(\mathbf{G})$. Therefore using (2.2.1) we conclude

$$H_i(\mathbf{R}(\mathbf{G}))_j \cong H_{i+j}(\mathbf{L}(E^*) \otimes_S \mathbf{G})_j = \mathrm{Tor}^S_{i+j}(k, \mathbf{G})_j$$

as S-modules.

Part (2):

Note $\mathbf{R}(S)$ is the following complex

$$0 \to \mathrm{Hom}_k(E, S_0) \to \mathrm{Hom}_k(E, S_1) \to \cdots .$$

Since $\mathrm{Hom}_k(E, S_i) \cong \mathrm{Hom}_k(E \otimes_k S_i^*, k)$ where $S_i^* = \mathrm{Hom}_k(S_i, k)$ then applying $\mathrm{Hom}_k(-, k)$ to $\mathbf{R}(S)$ gives the following complex

$$\cdots \to E \otimes_k S_1^* \to E \otimes_k S_0^* \to 0$$

which is the minimal free resolution of k over E. Therefore $\mathrm{Hom}_k(\mathbf{R}(S), k)$ is a minimal free resolution of k over E. Then by definition

$$\mathrm{Ext}^{i+j}_E(k, \mathbf{F}) = H^{i+j}(\mathrm{Hom}_E(\mathrm{Hom}_k(\mathbf{R}(S), k), \mathbf{F})_j. \tag{2.2.2}$$

Now for all a, $\mathbf{L}(F_a)$ is the complex of S-modules

$$\mathbf{L}(F_a) : \cdots \xrightarrow{\psi} S \otimes_k (F_a)_b \xrightarrow{\psi} S \otimes_k (F_a)_{b-1} \xrightarrow{\psi} \cdots$$

where $\psi(s \otimes f) = \sum_i x_i s \otimes e_i \wedge f$. Then by definition $\mathbf{L}(\mathbf{F})$ is the total complex of the following double complex

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\cdots \arrow[r] & S \otimes_k (F_{a+1})_b \arrow[r] \arrow[d] & S \otimes_k (F_{a+1})_{b-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & S \otimes_k (F_a)_b \arrow[r] \arrow[d] & S \otimes_k (F_a)_{b-1} \arrow[r] \arrow[d] & \cdots \\
 & \; & \; &
\end{tikzcd}$$

where the horizontal maps are ψ and the vertical maps are $id \otimes d_F$ which means

$$\mathbf{L}(\mathbf{F})_i = \bigoplus_a S \otimes_k (F_a)_{i-a}.$$

Also

$$\begin{aligned}
\mathrm{Hom}_E(\mathrm{Hom}_k(\mathbf{R}(S), k), \mathbf{F})^{i+j} &= \bigoplus_a \mathrm{Hom}_E(E \otimes_k S^*_{i+j-a}, F_a) \\
&\cong \bigoplus_a \mathrm{Hom}_E(E, \mathrm{Hom}_k(S^*_{i+j-a}, F_a)) \\
&\cong \bigoplus_a \mathrm{Hom}_k(S^*_{i+j-a}, F_a) \\
&\cong \bigoplus_a S_{i+j-a} \otimes_k F_a.
\end{aligned}$$

as E-modules, so

$$\mathbf{L}(\mathbf{F})_{i,j} = \bigoplus_a S_{i+j-a} \otimes_k (F_a)_{i-a} \cong \left(\mathrm{Hom}_E(\mathrm{Hom}_k(\mathbf{R}(S), k), \mathbf{F})^{i+j}\right)_j$$

as E-modules. Note $\mathrm{Hom}_E(\mathrm{Hom}_k(\mathbf{R}(S), k), \mathbf{F})$ is isomorphic to the total complex of the following double complex

$$
\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\cdots \arrow[r] & S_{d-a} \otimes_k F_{a+1} \arrow[r] \arrow[d] & S_{d-a+1} \otimes_k F_{a+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & S_{d-a} \otimes_k F_a \arrow[r] \arrow[d] & S_{d-a+1} \otimes_k F_a \arrow[r] \arrow[d] & \cdots \\
 & \; & \; &
\end{tikzcd}
$$

where the horizontal maps are ψ and the vertical maps are $id \otimes d_F$ which are the same maps as in the double complex that give $\mathbf{L}(\mathbf{F})$. Therefore using (2.2.2) we conclude

$$H_i(\mathbf{L}(\mathbf{F}))_j \cong H^{i+j}(\mathrm{Hom}_E(\mathrm{Hom}_k(\mathbf{R}(S), k), \mathbf{F}))_j = \mathrm{Ext}_E^{i+j}(k, \mathbf{F})_j.$$

$\square$

Definition 2.37. A complex $\mathbf{F}$ of graded E-modules is called *perfect* if it is quasi-isomorphic to a finite free graded complex over E.

Corollary 2.38. *Under BGG, a perfect complex* $\mathbf{F}$ *of graded E-modules corresponds to a complex* $\mathbf{G}$ *of graded S-modules with finite length total homology. In addition*

$$\dim_k H(\mathbf{F}) = \sum_i \beta_i(\mathbf{G}).$$

The following is a proof of only the forward direction because the other direction is not as relevant for this thesis.

Proof. Under BGG, quasi-isomorphic complexes map to the same complex. Therefore we may reduce to the case where $\mathbf{F}$ is a bounded complex of graded free E-modules of finite rank. Finite rank free E-modules are injective because E is injective as an E-module by Corollary 2.23. Thus

$$\mathrm{Ext}_E^{i+j}(k, \mathbf{F})_j = H^{i+j}(\mathrm{Hom}_E(k, \mathbf{F}))_j$$

which has finite dimension as a k-vector space for all i and j. Also since $\mathbf{F}$ is bounded, then $H_{i+j}(\mathrm{Hom}_E(k, \mathbf{F}))_j = 0$ for all but a finite number of pairs (i, j). Then by part (2) of Proposition 2.36, $H_i(\mathbf{L}(\mathbf{F}))_j = 0$ for all but a finite number of pairs (i, j). Thus $\mathbf{G} := \mathbf{L}(\mathbf{F})$ has finite length homology.

Moreover, part (1) of Proposition 2.36 gives

$$H_i(\mathbf{F})_j = H_i(\mathbf{R}(\mathbf{L}(\mathbf{F})))_j = \mathrm{Tor}^S_{i+j}(k, \mathbf{G})_j = \beta_{i+j,j}(\mathbf{G})$$

which implies

$$\dim_k H(\mathbf{F}) = \sum_i \beta_i(\mathbf{G}).$$

$\square$

Chapter 3

KOSZUL COMPLEXES OVER THE EXTERIOR ALGEBRA

We begin investigating Question 1.8 by considering counterexamples to Conjecture 1.4. The original counterexample given by Iyengar and Walker in [13] is a complex over the polynomial ring and, under BGG, it corresponds to the Koszul complex of a general quadric over the exterior algebra $E_n = k\langle e_1, \ldots, e_n\rangle$. Throughout the rest of this thesis, we are assuming $\deg(e_i) = 1$ and a quadric is a homogeneous element of degree 2. Section 3.1 is devoted to analyzing this original counterexample. In particular we will show the Koszul complex of a general quadric over E_n has homological growth factor (see Definition 1.9) less than 2 for large enough n.

We then produce a new counterexample in Section 3.2 by considering the Koszul complex of two generic quadrics over the exterior algebra E_n. We analyze these Koszul complexes by looking at a lower and upper bound of its homological growth factor. Using these bounds, we conclude that the homological growth factor of the Koszul complex of two generic quadrics is less than 2 for some n. In fact, we observe that the smallest homological growth factor of a Koszul complex of two generic quadrics appears to be less than the smallest homological growth factor of a Koszul complex of one general quadric.

3.1 Koszul complex of one quadric

Let k be a field and let $E_n = k\langle e_1, \ldots, e_n\rangle$. Consider the Koszul complex

$$\mathrm{Kos}_n(w) : E_n(-2) \xrightarrow{w} E_n$$

where $w \in E_n$ is a quadric.

Definition 3.1. Consider the affine space $\mathbb{A}_k^{\binom{n}{2}}$ over the field k (i.e. $\mathbb{A}_k^{\binom{n}{2}}$ is the set of all $\binom{n}{2}$-tuples of elements of k). Let $\Theta : \mathbb{A}_k^{\binom{n}{2}} \to (E_n)_2$ be the function defined by $\Theta((w_{ij})_{1\leq i<j\leq n}) = \displaystyle\sum_{1\leq i<j\leq n} w_{ij} e_i \wedge e_j$.

Remark 3.2. Θ *is bijective.*

Proposition 3.3. *Let k be a field. There exists a Zariski open set U of $\mathbb{A}_k^{\binom{n}{2}}$ such that for $w \in (E_n)_2$ the following are equivalent*

1. $w \in \Theta(U)$
2. *the map $\mu_i : (E_n(-2))_i \xrightarrow{w} (E_n)_i$ which is given by $\mu_i(x) = wx$ is injective for $i \leq \lfloor \frac{n}{2} \rfloor + 1$ and surjective for $i \geq \lfloor \frac{n}{2} \rfloor + 2$.*

Moreover, if n is even and $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k > \frac{n+1}{2}$ then U is nonempty and if n is odd and $\mathrm{char}\, k = 0$ then U is also nonempty.

Proof. Let $w \in (E_n)_2$. For each i, fix the standard bases for the vector spaces $(E_n(-2))_i$ and $(E_n)_i$. Let $M_i(w)$ denote the matrix representing the k-linear transformation

$$\mu_i : (E_n(-2))_i \xrightarrow{w} (E_n)_i$$

with respect to the chosen bases. Note the entries of $M_i(w)$ depend on the coefficients of w but not on any elements of E_n.

The map μ_i is injective for $i \leq \lfloor \frac{n}{2} \rfloor + 1$ and surjective for $i \geq \lfloor \frac{n}{2} \rfloor + 2$ if and only if the rank of M_i is maximal; that is $\operatorname{rank} M_i = \min\{\dim_k(E_n(-2))_i, \dim_k(E_n)_i\}$. Set

$$r_i = \min\{\dim_k(E_n(-2))_i, \dim_k(E_n)_i\}$$

and let $I_{r_i}(M)$ be the ideal generated by the $r_i \times r_i$ minors of M for any matrix M. Also let X_i be the matrix obtained from $M_i(w)$ by replacing each coefficient w_{ij} of w with the variable x_{ij}. Then

$$\begin{aligned} w \text{ satisfies (2)} &\Leftrightarrow \operatorname{rank} M_i(w) = r_i \text{ for all } 0 \leq i \leq n \\ &\Leftrightarrow I_{r_i}(M_i(w)) \neq 0 \text{ for all } 0 \leq i \leq n \\ &\Leftrightarrow \Theta^{-1}(w) \notin V(I_{r_i}(X_i)) \text{ for all } 0 \leq i \leq n \\ &\Leftrightarrow \Theta^{-1}(w) \notin V\left(\bigcap_{i=0}^{n} I_{r_i}(X_i)\right) \end{aligned}$$

Set $U = \mathbb{A}_k^{\binom{n}{2}} \setminus V\left(\bigcap_{i=0}^{n} I_{r_i}(X_i)\right)$ then U is a Zariski open set in $\mathbb{A}_k^{\binom{n}{2}}$ and by the argument above w satisfies (2) if and only if $\Theta^{-1}(w) \in U$. Thus w satisfies (2) if and only if $w \in \Theta(U)$.

Now suppose n is even, so $n = 2m$ for some m. By [5, Proposition A.2], if $\operatorname{char} k = 0$ or $\operatorname{char} k > \frac{n+1}{2}$ then $\tilde{w} = \sum_{i=1}^{m} e_{2i-1}e_{2i}$ satisfies (2). Therefore $\tilde{w} \in \Theta(U)$, so U is nonempty.

Finally suppose n is odd. By [15, proof of Theorem 5.2], if $\operatorname{char} k = 0$ then the element $\tilde{w}$ which is the sum of all monomials in E_n of degree 2 satisfies (2). Therefore $\tilde{w} \in U$, so U is also non-empty in this case. $\square$

Definition 3.4. If n is even then assume $\operatorname{char} k = 0$ or $\operatorname{char} k > \frac{n+1}{2}$ and if n is odd then assume $\operatorname{char} k = 0$. An element $w \in (E_n)_2$ is *general* if $w \in \Theta(U)$ where U is the non-empty Zariski open set given in Proposition 3.3.

Proposition 3.5. *Let $w \in E_n$ be a general quadric. If n is even, then $n = 2m$ for some m and*

$$\dim_k H(\mathrm{Kos}_{2m}(w)) = \binom{2m+2}{m+1}.$$

If n is odd, then $n = 2m+1$ for some m and

$$\dim_k H(\mathrm{Kos}_{2m+1}(w)) = 2\binom{2m+2}{m+1}.$$

Proof. This proof is an extension of the proof of [13, Proposition 2.1] which proves the even case for $w = \sum_{i=1}^{n} e_{2i}e_{2i-1}$.

By definition, $w \in \Theta(U)$ where U is the non-empty Zariski open set given in Proposition 3.3. Therefore $\mu_i : (E_n(-2))_i \xrightarrow{w} (E_n)_i$ is injective for $i \leq \lfloor \frac{n}{2} \rfloor + 1$ and surjective for $i \geq \lfloor \frac{n}{2} \rfloor + 2$ which implies

$$\dim_k \ker(\mu_i) = \begin{cases} 0 & i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ \binom{n}{i-2} - \binom{n}{i} & i \geq \lfloor \frac{n}{2} \rfloor + 2 \end{cases}$$

and

$$\dim_k \mathrm{coker}(w)_i = \begin{cases} \binom{n}{i} - \binom{n}{i-2} & i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ 0 & i \geq \lfloor \frac{n}{2} \rfloor + 2. \end{cases}$$

First suppose n is even, so $n = 2m$ for some m. Then

$$\begin{aligned} \dim_k H(\mathrm{Kos}_{2m}(w)) &= \sum_i \dim_k \mathrm{coker}(\mu_i) + \dim_k \ker(\mu_i) \\ &= \sum_{i=0}^{m+1} \binom{2m}{i} - \binom{2m}{i-2} + \sum_{i=m+1}^{2m} \binom{2m}{i-2} - \binom{2m}{i} \\ &= \binom{2m}{m-1} + 2\binom{2m}{m} + \binom{2m}{m+1} \\ &= \binom{2m+2}{m+1}. \end{aligned}$$

Now suppose n is odd so $n = 2m+1$ for some m. Then

$$\begin{aligned}
\dim_k H(\mathrm{Kos}_{2m+1}(w)) &= \sum_i \dim_k \operatorname{coker}(\mu_i) + \dim_k \ker(\mu_i) \\
&= \sum_{i=0}^{m+1} \binom{2m+1}{i} - \binom{2m+1}{i-2} + \sum_{i=m+2}^{2m+1} \binom{2m+1}{i-2} - \binom{2m+1}{i} \\
&= 2\left(\binom{2m+1}{m} + \binom{2m+1}{m+1}\right) \\
&= 2\binom{2m+2}{m+1}.
\end{aligned}$$

□

Corollary 3.6. *Let $n \geq 8$ and $w \in E_n$ be a general quadric then*

$$HGF(\mathrm{Kos}_n(w)) < 2.$$

Proof. This proof is an extension of the ideas in [13, Remark 2.5] which prove the case when n is even.

First notice for $n = 8$ and $n = 9$, Proposition 3.5 gives the equalities

$$\dim_k H(\mathrm{Kos}_8(w)) = \binom{10}{5} = 252 < 2^8$$

and

$$\dim_k H(\mathrm{Kos}_9(w)) = 2\binom{10}{5} = 504 < 2^9.$$

Now suppose $n \geq 10$. If $n = 2m$ for some m then by Proposition 3.5 and Stirling's approximation

$$\dim_k H(\mathrm{Kos}_{2m}(w)) = \binom{2m+2}{m+1} < 2^{2m}\frac{4}{\sqrt{\pi(m+1)}} < 2^{2m}. \tag{3.1.1}$$

Now if $n = 2m+1$ for some m then we also have by Proposition 3.5 and (3.1.1) that

$$\dim_k H(\mathrm{Kos}_{2m+1}(w)) = 2\binom{2m+2}{m+1} < 2^{2m+1}.$$

We conclude for $n \geq 8$

$$HGF(\mathrm{Kos}_n(w)) = \left(\dim_k H(\mathrm{Kos}_n(w))\right)^{1/n} < 2.$$

□

Therefore Corollary 3.6 shows $\mathrm{Kos}_n(w)$ is a counterexample to Conjecture 1.4 for $n \geq 8$. Also, computations performed using Macaulay2 [9] indicate the smallest homological growth factor occurs when $n = 24$ and in this case

$$HGF(\mathrm{Kos}_{24}(w)) = \binom{26}{13}^{1/24} \approx 1.961. \tag{3.1.2}$$

Even though this family of complexes has homology smaller than predicted by Conjecture 1.4, its homological growth factor is eventually increasing. In particular, the homological growth factor will asymptotically approach 2.

Corollary 3.7. *For all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w)$ for any general quadric $w \in E_n$ then*

$$\lim_{n\to\infty} HGF(\mathbf{K}_n) = 2.$$

Proof. First note by Corollary 3.6, $HGF(\mathbf{K}_n) < 2$ for $n \geq 8$. Thus

$$\limsup_{n\to\infty} HGF(\mathbf{K}_n) \leq 2.$$

Now suppose n is even, so $n = 2m$ for some m. Proposition 3.5 gives

$$\left(\dim_k H(\mathbf{K}_{2m})\right)^{1/2m} = \binom{2m+2}{m+1}^{1/2m}$$

and by using the bounds given by Stirling's approximation we have the inequality

$$\binom{2m+2}{m+1}^{1/2m} \geq \left(\frac{8\sqrt{\pi}2^{2m}}{e^2\sqrt{m+1}}\right)^{1/2m} = 2\left(\frac{8\sqrt{\pi}}{e^2\sqrt{m+1}}\right)^{1/2m}.$$

Since $\lim_{n\to\infty}(m+1)^{1/m} = 1$, then

$$\liminf_{m\to\infty}\binom{2m+2}{m+1}^{1/2m} \geq \lim_{m\to\infty} 2\left(\frac{8\sqrt{\pi}}{e^2\sqrt{m+1}}\right)^{1/2m} = 2. \tag{3.1.3}$$

Thus

$$\liminf_{m\to\infty} HGF(\mathbf{K}_{2m}) \geq 2.$$

Now suppose n is odd, so $n = 2m+1$ for some m. Proposition 3.5 gives

$$\left(\dim_k H(\mathbf{K}_{2m+1})\right)^{1/2m+1} = 2^{1/(2m+1)}\binom{2m+2}{m+1}^{1/(2m+1)}.$$

Then (3.1.3) implies

$$\liminf_{m\to\infty}\binom{2m+2}{m+1}^{1/(2m+1)} = \lim_{m\to\infty}\left(\binom{2m+2}{m+1}^{1/2m}\right)^{(2m+1)/2m} \geq 2.$$

Thus

$$\liminf_{m\to\infty} HGF(\mathbf{K}_{2m+1}) = \liminf_{m\to\infty} 2^{1/(2m+1)}\binom{2m+2}{m+1}^{1/(2m+1)} \geq 2.$$

Therefore in both cases we have shown

$$\liminf_{n\to\infty} HGF(\mathbf{K}_{2m+1}) \geq 2$$

so by the Squeeze Theorem, we conclude

$$\lim_{n\to\infty} HGF(\mathbf{K}_{2m+1}) = 2.$$

$\square$

3.2 Koszul complex of two quadrics

Let $w_1, w_2 \in E_n = k\langle e_1, \ldots, e_n\rangle$ be quadrics and consider the Koszul complex

$$\mathrm{Kos}_n(w_1, w_2) : E_n(-4) \xrightarrow{\begin{pmatrix} -w_2 \\ w_1 \end{pmatrix}} E_n(-2)^2 \xrightarrow{\begin{pmatrix} w_1 & w_2 \end{pmatrix}} E_n.$$

Recall by Definition 1.9 the homological growth factor of this complex is given by

$$HGF(\mathrm{Kos}_n(w_1, w_2)) = \left(\dim_k H(\mathrm{Kos}_n(w_1, w_2))\right)^{1/n}.$$

In order to analyze the homological growth factor of these complexes, we will focus on the dimension of the total homology of $\mathrm{Kos}_n(w_1, w_2)$. First consider the following useful results about the homology of $\mathrm{Kos}_n(w_1, w_2)$.

Lemma 3.8. *Let $w_1, w_2 \in E_n$ be quadrics. Consider the maps*

$$\alpha : E_n(-2)^2 \xrightarrow{\begin{pmatrix} w_1 & w_2 \end{pmatrix}} E_n$$

$$\beta : E_n(-4) \xrightarrow{\begin{pmatrix} -w_2 \\ w_1 \end{pmatrix}} E_n(-2)^2$$

then $\dim_k \ker(\beta) = \dim_k \mathrm{coker}(\alpha)$.

Proof. Consider the map of vector spaces

$$\gamma : E_n^2 \xrightarrow{\begin{pmatrix} -w_2 & w_1 \end{pmatrix}} E_n.$$

The following is an exact sequence of vector spaces

$$0 \to \ker(\gamma) \to E_n^2 \xrightarrow{\gamma} E_n \to \mathrm{coker}(\gamma) \to 0. \tag{3.2.1}$$

Since every short exact sequence of vector spaces is split exact, applying $\mathrm{Hom}_k(-, k)$ to (3.2.1) gives the following exact sequence of k-vector spaces

$$0 \leftarrow \mathrm{Hom}_k(\ker(\gamma), k) \leftarrow \mathrm{Hom}_k(E_n^2, k) \xleftarrow{\mathrm{Hom}_k(\gamma,k)} \mathrm{Hom}_k(E_n, k) \leftarrow \mathrm{Hom}_k(\mathrm{coker}(\gamma), k) \leftarrow 0$$

which is isomorphic to

$$0 \leftarrow \mathrm{Hom}_k(\ker(\gamma), k) \leftarrow E_n^2 \xleftarrow{\beta} E_n \leftarrow \mathrm{Hom}_k(\mathrm{coker}(\gamma), k) \leftarrow 0$$

because E_n is a finite-dimensional k-vector space. Therefore $\ker(\beta) = \mathrm{Hom}_k(\mathrm{coker}(\gamma), k)$, and

$$\dim_k \ker(\beta) = \dim_k \mathrm{Hom}_k(\mathrm{coker}(\gamma), k) = \dim_k \mathrm{coker}(\gamma) = \dim_k \mathrm{coker}(\alpha).$$

□

Proposition 3.9. *Let $w_1, w_2 \in E_n$ be quadrics then*

$$\dim_k H(\mathrm{Kos}_n(w_1, w_2)) = 4 \dim_k \frac{E_n}{(w_1, w_2)}.$$

Proof. Let α and β be the maps given in Lemma 3.8. Then

$$\dim_k H(\mathrm{Kos}_n(w_1, w_2)) = \dim_k \mathrm{coker}(\alpha) + \dim_k \ker(\alpha) - \dim_k \mathrm{image}(\beta) + \dim_k \ker(\beta).$$

By the Rank-Nullity Theorem, we have the equalities

$$\dim_k \ker(\alpha) + \dim_k \mathrm{image}(\alpha) = \dim_k E_n^2 = 2 \dim_k E_n \tag{3.2.2}$$

and

$$\dim_k \ker(\beta) + \dim_k \mathrm{image}(\beta) = \dim_k E_n. \tag{3.2.3}$$

By subtracting (3.2.3) from (3.2.2) we obtain

$$\dim_k \ker(\alpha) - \dim_k \mathrm{image}(\beta) = \dim_k \mathrm{coker}(\alpha) + \dim_k \ker(\beta),$$

which implies

$$\dim_k H(\mathrm{Kos}_n(w_1, w_2)) = 2\left(\dim_k \mathrm{coker}(\alpha) + \dim_k \ker(\beta)\right).$$

Then by Lemma 3.8, we conclude

$$\dim_k H(\mathrm{Kos}_n(w_1, w_2)) = 4\dim_k \mathrm{coker}(\alpha) = 4\dim_k \frac{E_n}{(w_1, w_2)}.$$

$\square$

Therefore finding the dimension of the total homology of $\mathrm{Kos}_n(w_1, w_n)$ can be reduced to finding the dimension of $E_n/(w_1, w_2)$. In order to find the dimension, we can focus on finding the Hilbert series (see Definition 2.4) of $E_n/(w_1, w_2)$. However, the Hilbert series of $E_n/(w_1, w_2)$ is not known and challenging to compute (see [6]), so we instead consider coefficient-wise bounds on the Hilbert series of $E_n/(w_1, w_2)$.

3.2.1 Lower bound

First let us consider a coefficient-wise lower bound of the Hilbert series of $E_n/(w_1, w_2)$ where $w_1, w_2 \in E_n$ are quadrics. This lower bound is the exterior algebra version of a lower bound given by Fröberg in the commutative setting (see [8]).

Definition 3.10. Given a polynomial $g(t) = a_0 + a_1 t + \cdots + a_m t^m$ let $[g(t)]$ mean truncation before the first non-positive term. Therefore $[g(t)] = a_0 + a_1 t + \cdots + a_{j-1} t^{j-1}$ where $a_j \leq 0$ and $a_i > 0$ for all $i < j$.

Example 3.11. Let $g(t) = 1 + 3t + t^2 - 4t^3 + 8t^4$ then $[g(t)] = 1 + 3t + t^2$.

Lemma 3.12. *Let $w \in E_n$ be a general quadric as in Definition 3.4 then*

$$h_{E_n/(w)}(t) = [(1+t)^n(1-t^2)].$$

Proof. Consider the map

$$\mu_i : (E_n(-2))_i \xrightarrow{w} (E_n)_i.$$

Since w is general then, by Definition 3.4, μ_i is injective for $i \leq \lfloor \frac{n}{2} \rfloor + 1$ and surjective for $i \geq \lfloor \frac{n}{2} \rfloor + 2$. Thus

$$\dim_k \operatorname{coker}(\mu_i) = \begin{cases} \binom{n}{i} - \binom{n}{i-2} & i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ 0 & i \geq \lfloor \frac{n}{2} \rfloor + 2. \end{cases}$$

Since

$$\dim_k \left(\frac{E_n}{(w_1)} \right)_i = \dim_k \operatorname{coker}(\mu_i)$$

we conclude $h_{E_n/(w_1)}(t) = [(1+t)^n(1-t^2)]$ because $\binom{n}{i} - \binom{n}{i-2} < 0$ for $i \geq \lfloor \frac{n}{2} \rfloor + 2$. □

Definition 3.13. Suppose $f(t) = \sum_i a_i t^i$ and $g(t) = \sum_i b_i t^i$ with $a_i, b_i \in \mathbb{Z}$. Let $f(t) \succeq g(t)$ if $a_i \geq b_i$ for all i.

Proposition 3.14. *Let $w_1, w_2 \in E_n$ be quadrics with w_1 general as in Definition 3.4. Then we have the coefficient-wise lower bound*

$$h_{E_n/(w_1,w_2)} \succeq [(1+t)^2(1-t^2)^2].$$

Proof. Consider the following exact sequence

$$0 \to \operatorname{ann}(w_2)(-2) \hookrightarrow \frac{E_n}{(w_1)}(-2) \xrightarrow{w_2} \frac{E_n}{(w_1)} \to \frac{E_n}{(w_1, w_2)} \to 0.$$

Then Proposition 2.9 gives the equality

$$h_{E_n/(w_1,w_2)}(t) = h_{E_n/(w_1)}(t) - h_{E_n/(w_1)}(t)t^2 + h_{\operatorname{ann}(w_2)}(t)t^2 = h_{E_n/(w_1)}(t)(1-t^2) + h_{\operatorname{ann}(w_2)}(t)t^2.$$

Since $h_{\mathrm{ann}(w_2)}(t) \succeq 0$, then $h_{E_n/(w_1,w_2)}(t) \succeq h_{E_n/(w_1)}(t)(1-t^2)$ which implies

$$h_{E_n/(w_1,w_2)}(t) \succeq \left[h_{E_n/(w_1)}(t)(1-t^2)\right].$$

Then Lemma 3.12 gives

$$h_{E_n/(w_1,w_2)}(t) \succeq \left[[(1+t)^n(1-t^2)](1-t^2)\right]. \tag{3.2.4}$$

Therefore it remains to show

$$\left[[(1+t)^n(1-t^2)](1-t^2)\right] = \left[(1+t)^n(1-t^2)^2\right].$$

Let

$$(1+t)^n(1-t^2) = \sum_i a_i t^i$$

then

$$(1+t)^n(1-t^2)^2 = \sum_i (a_i - a_{i-2}) t^i$$

so

$$[(1+t)^n(1-t^2)^2] = \sum_i c_i t^i$$

where $c_i = a_i - a_{i-2}$ if $a_j > a_{j-2}$ for all $j \leq i$ and 0 otherwise. In addition

$$[(1+t)^n(1-t^2)] = \sum_i b_i t^i$$

where $b_i = a_i$ if $a_j > 0$ for all $j \leq i$ and 0 otherwise. Thus

$$[(1+t)^n(1-t^2)](1+t^2) = \sum_i (b_i - b_{i-2}) t^i$$

and

$$\left[[(1+t)^n(1-t^2)](1-t^2)\right] = \sum_i d_i t^i$$

where $d_i = b_i - b_{i-2} = a_i - a_{i-2}$ if $a_j > a_{j-2}$ and $a_j > 0$ for all $j \leq i$ and 0 otherwise. Let i_0 be the first index where $a_{i_0} \leq 0$, then notice we also have $a_{i_0} < a_{i_0-2}$. Therefore

$d_i = a_i - a_{i-2}$ if $a_j > a_{j-2}$ for all $j \leq i$ which implies $d_i = c_i$ for all i. Thus we conclude

$$\left[[(1+t)^n(1-t^2)](1-t^2)\right] = \left[(1+t)^n(1-t^2)^2\right]$$

which means (3.2.4) gives

$$h_{E_n/(w_1,w_2)}(t) \succeq \left[(1+t)^n(1-t^2)^2\right].$$

□

Definition 3.15. Let $b(n,s)$ be the s^{th} coefficient of $[(1+t)^n(1-t^2)^2]$.

The following table contains the values of $b(n,s)$ for $n \leq 20$. These numbers are lower bounds for the coefficients of the Hilbert series of $E_n/(w_1,w_2)$.

n/s	0	1	2	3	4	5	6	7	8	9
3	1	3	1	0	0	0	0	0	0	0
4	1	4	4	0	0	0	0	0	0	0
5	1	5	8	0	0	0	0	0	0	0
6	1	6	13	8	0	0	0	0	0	0
7	1	7	19	21	0	0	0	0	0	0
8	1	8	26	40	15	0	0	0	0	0
9	1	9	34	66	55	0	0	0	0	0
10	1	10	43	100	121	22	0	0	0	0
11	1	11	53	143	221	143	0	0	0	0
12	1	12	64	196	364	364	0	0	0	0
13	1	13	76	260	560	728	364	0	0	0
14	1	14	89	336	820	1288	1092	0	0	0
15	1	15	103	425	1156	2108	2380	884	0	0
16	1	16	118	528	1581	3264	4488	3264	0	0
17	1	17	134	646	2109	4845	7752	7752	1938	0
18	1	18	151	780	2755	6954	12597	15504	9690	0
19	1	19	169	931	3535	9709	19551	28101	25194	3230
20	1	20	188	1100	4466	13244	29260	47652	53295	28424

Table 3.1: The values of $b(n,s)$ for $n \leq 20$

Definition 3.16. Let $k = \mathbb{C}$. A list of elements $w_1, w_2, \ldots, w_c \in E_n$ are *generic* if all the coefficients of $w_1, w_2, \ldots, w_c$ with respect to the standard basis are algebraically independent over $\mathbb{Q}$.

Proposition 3.17. *The lower bound of Proposition 3.14 is not tight. In fact, if $k = \mathbb{C}$ and $w_1, w_2 \in E_n$ are generic quadrics then for $n \geq 11$ the Hilbert series of $E_n/(w_1, w_2)$ is not equal to $[(1+t)^n(1-t^2)^2]$.*

Proof. Note $(1+t)^n(1-t^2)^2 = (1-2t^2+t^4)\sum_{k=0}^{n}\binom{n}{k}t^k$. We will consider when n is odd and when n is even separately.

First suppose $n = 2m$. Then the m^{th} coefficient of $(1+t)^{2m}(1-t^2)^2$ is

$$\binom{2m}{m} - 2\binom{2m}{m-2} + \binom{2m}{m-4}.$$

By rewriting the binomial coefficients as factorials and simplifying we have the following equality

$$\begin{aligned}\binom{2m}{m} - 2\binom{2m}{m-2} + \binom{2m}{m-4} &= \frac{(2m)!}{m!(m+4)!}\left(-8m^3 + 36m^2 + 68m + 24\right) \\ &= \frac{(2m)!}{m!(m+4)!}(-4)(2m+1)(m+1)(m-6).\end{aligned}$$

Therefore the m^{th} coefficient is negative if $m > 6$ and zero if $m = 6$. This implies the m^{th} coefficient of $[(1+t)^{2m}(1-t^2)^2]$ will be 0 for $m \geq 6$. However the m^{th} coefficient of $h_{E_{2m}/(w_1,w_2)}(t)$ is 2^m by [6, Proposition 6].

Now suppose $n = 2m+1$. Then the $(m+1)^{st}$ coefficient of $(1+t)^{2m+1}(1-t^2)^2$ is

$$\binom{2m+1}{m+1} - 2\binom{2m+1}{m-1} + \binom{2m+1}{m-3}.$$

By rewriting the binomial coefficients as factorials and simplifying we have the following equality

$$\binom{2m+1}{m+1} - 2\binom{2m+1}{m-1} + \binom{2m+1}{m-3} = \frac{(2m+1)!}{(m+1)!(m+4)!}\left(-8m^3 - 4m^2 + 28m + 24\right)$$
$$= \frac{(2m+1)!}{(m+1)!(m+4)!}(-4)(2m+3)(m+1)(m-2).$$

Therefore the $(m+1)^{st}$ coefficient is negative if $m > 2$ and zero if $m = 2$. This implies the $(m+1)^{st}$ coefficient of $[(1+t)^{2m+1}(1-t^2)^2]$ will be 0 for $m \geq 2$. However the $(m+1)^{st}$ coefficient of $h_{E_{2m+1}/(w_1,w_2)}(t)$ is 1 by [6, Proposition 6]. $\square$

Remark 3.18. *Using a similar argument, one can show when $n = 2m$ then the $(m-j)^{th}$ coefficient of $(1+t)^{2m}(1-t^2)^2$ will be*

$$\frac{(2m)!}{(m-j)!(m+j+4)!}(-4)(2m+1)(m+1)(m-2(j^2+4j+3))$$

which is zero or negative when $m \geq 2(j^2+4j+3)$. Also when $n = 2m+1$ then the $(m+1-j)^{th}$ coefficient of $(1+t)^{2m+1}(1-t^2)^2$ will be

$$\frac{(2m+1)!}{(m-j+1)!(m+j+4)!}(-4)(2m+3)(m+1)(m-2(j^2+3j+1))$$

which is zero or negative when $m \geq 2(j^2+3j+1)$.

Finally, consider the following result about the asymptotic behavior of the lower bound.

Proposition 3.19. *Let $b(n,s)$ be the s^{th} coefficient of $[(1+t)^n(1-t^2)^2]$. Then*

$$\liminf_{n\to\infty}\left(\sum_s b(n,s)\right)^{1/n} \geq 2.$$

Proof. Note $(1+t)^n(1-t^2)^2 = (1-2t^2+t^4)\sum_{k=0}^{n}\binom{n}{k}t^k$ so the s^{th} coefficient of $(1+t)^n(1-t^2)^2$ is

$$\binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}.$$

Now let us consider the even and odd case separately. First suppose $n = 2m$. As demonstrated in Remark 3.18, the $(m-j)^{th}$ coefficient of $(1+t)^{2m}(1-t^2)^2$ is negative or zero for $m \geq 2(j^2+4j+3)$. This implies

$$\begin{aligned}
\sum_s b(2m,s) &\geq \sum_{s=0}^{m} \left(\binom{2m}{s} - 2\binom{2m}{s-2} + \binom{2m}{s-4} \right) \\
&= \binom{2m}{m} + \binom{2m}{m-1} - \binom{2m}{m-2} - \binom{2m}{m-3} \\
&= \frac{(2m)!}{m!(m+3)!} 6(2m+1)(m+1).
\end{aligned}$$

By using the bounds given by Stirling's approximation, $\sqrt{2\pi} m^{m+1/2} e^{-m} \leq m! \leq e m^{m+1/2} e^{-m}$, we have the following inequalities

$$\begin{aligned}
6(2m+1)(m+1)\frac{(2m)!}{m!(m+3)!} &\geq 6(2m+1)(m+1)\frac{\sqrt{2\pi}(2m)^{2m+1/2}e^{-2m}}{em^{m+1/2}e^{-m}e(m+3)^{m+7/2}e^{-m-3}} \\
&\geq (2m+1)(m+1)\frac{2^{2m}m^m}{(m+3)^{m+7/2}}.
\end{aligned}$$

Therefore

$$\left(\sum_s b(2m,s) \right)^{1/2m} \geq (2m+1)^{1/2m}(m+1)^{1/2m} \frac{2m^{1/2}}{(m+3)^{1/2}(m+3)^{7/4m}}.$$

Since $\lim_{m\to\infty} m^{1/m} = 1$, we conclude

$$\liminf_{m\to\infty} \left(\sum_s b(2m,s) \right)^{1/2m} \geq 2.$$

Now suppose $n = 2m+1$. As demonstrated in Remark 3.18, the $(m+1-j)^{th}$ coefficient of $(1+t)^{2m+1}(1-t^2)^2$ is negative or zero for $m \geq 2(j^2+3j+1)$. This implies

$$\begin{aligned}
\sum_s b(2m+1,s) &\geq \sum_{s=0}^{m+1}\left(\binom{2m+1}{s} - 2\binom{2m+1}{s-2} + \binom{2m+1}{s-4}\right) \\
&= \binom{2m+1}{m+1} + \binom{2m+1}{m} - \binom{2m+1}{m-1} - \binom{2m+1}{m-2} \\
&= \frac{(2m+1)!}{m!(m+3)!}4(2m+3).
\end{aligned}$$

By using the bounds given by Stirling's approximation, we have the following inequalities

$$\begin{aligned}
4(2m+3)\frac{(2m+1)!}{m!(m+3)!} &\geq 4(2m+3)\frac{\sqrt{2\pi}(2m+1)^{2m+3/2}e^{-2m-1}}{em^{m+1/2}e^{-m}e(m+3)^{m+7/2}e^{-m-3}} \\
&\geq (2m+3)\frac{(2m+1)^{2m+3/2}}{m^{m+1/2}(m+3)^{m+7/2}} \\
&\geq (2m+3)\frac{2^{2m+1}m^{m+1/2}}{(m+3)^{m+7/2}}.
\end{aligned}$$

Therefore

$$\left(\sum_s b(2m+1,s)\right)^{1/(2m+1)} \geq (2m+3)^{1/(2m+1)}\frac{2m^{1/2}}{(m+3)^{1/2}(m+3)^{3/(2m+1)}}.$$

Since $\lim_{m\to\infty} m^{1/m} = 1$, we conclude

$$\liminf_{m\to\infty}\left(\sum_s b(2m+1,s)\right)^{1/(2m+1)} \geq 2.$$

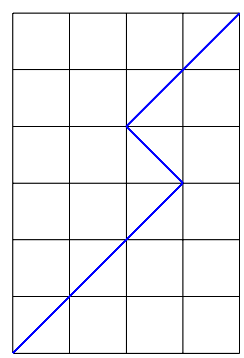
□

3.2.2 Upper bound

Now let us consider a coefficient-wise upper bound for the Hilbert series of $E_n/(w_1, w_2)$ which is found by relating the coefficients of the Hilbert series to lattice paths.

Definition 3.20. Let $a(n, s)$ be the number of lattice paths inside a $(n + 2 - 2s) \times (n + 2)$ rectangle that start at the bottom left corner and end at the top right corner with moves of two types: $(x, y) \to (x + 1, y + 1)$ or $(x, y) \to (x - 1, y + 1)$.

Example 3.21. Let $n = 4, s = 1$ then the following is an example of a valid lattice path counted by $a(4, 1)$.



Theorem 3.22 ([6, Theorem 5])**.** *Let $k = \mathbb{C}$ and let $w_1, w_2 \in E_n$ be generic quadrics (see Definition 3.16). The dimension of the s^{th} graded component of $E_n/(w_1, w_2)$ is at most $a(n, s)$. Therefore we have a coefficient-wise upper bound*

$$h_{E_n/(w_1,w_2)}(t) \preceq \sum_s a(n, s)t^s.$$

In order to calculate $a(n, s)$, we form a bijection between the lattice paths described above to another type of lattice paths whose cardinality is easier to compute.

Definition 3.23. A lattice path stays *weakly below* (or *weakly above*) the line $y = mx + b$ if for each point (x_0, y_0) on the lattice path, $y_0 \leq mx_0 + b$ (or $y_0 \geq mx_0 + b$).

Proposition 3.24. *Let P_1 be the lattice paths which are inside a $(n+2-2s)\times(n+2)$ rectangle and start at the bottom left corner and end at the top right corner with moves of two types: $(x,y)\to(x+1,y+1)$ or $(x-1,y+1)$. Let P_2 be the lattice paths which start at $(0,0)$ and end at $(s,n-2+s)$ and stay weakly below the line $y=x+n+2-2s$ and weakly above $y=x$ with moves of two types: $(x,y)\to(x+1,y)$ or $(x,y)\to(x,y+1)$. Then P_1 and P_2 have the same cardinality.*

The following example demonstrates how to transform a P_1 path into a P_2 path, and provides intuition for the proof of this proposition.

Example 3.25. Suppose $n=6, s=2$ and consider the P_1 lattice path in Figure 3.1 which is inside a 4×8 rectangle.

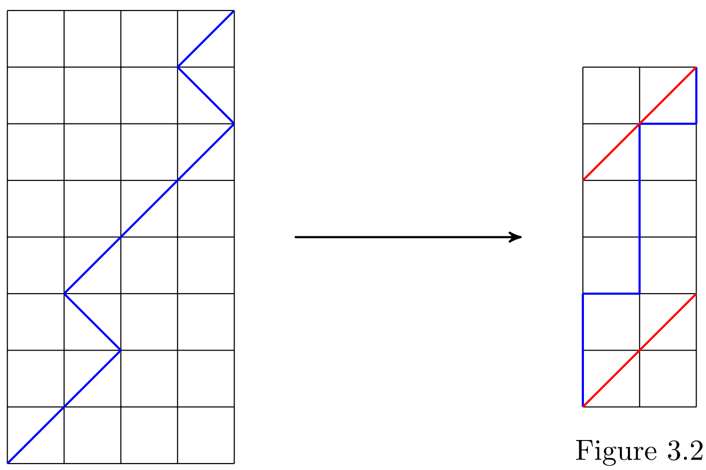


Figure 3.1

Figure 3.2

By changing every move of type $(x,y)\to(x+1,y+1)$ to $(x,y)\to(x,y+1)$ and every move of type $(x,y)\to(x-1,y+1)$ to $(x,y)\to(x+1,y)$ we get the P_2 path in Figure 3.2 which is drawn in blue. Note the path stays between the diagonal lines $y=x+4$ and $y=x$ which are drawn in red, so it is indeed a valid P_2 path.

Proof of Proposition 3.24. Throughout this proof, we will use the following notation. Let R moves be of type $(x,y)\to(x+1,y+1)$, L moves be of type $(x,y)\to(x-1,y+1)$,

U moves be of type $(x, y) \to (x, y+1)$ and O moves be of type $(x, y) \to (x+1, y)$. Therefore any P_1 path has just R and L moves and any P_2 path has just U and O moves.

Consider the function $f : P_1 \to P_2$ given by taking a P_1 path and changing all R moves to U moves and all L moves to O moves as demonstrated in Example 3.25. We first show f is well-defined.

Suppose P is a P_1 path. Note in any P_1 path the number of L moves is s and the number of R moves is $n+2-2s+s = n+2-s$. Since P stays inside the $(n+2-2s) \times (n+2)$ rectangle, while traveling along P, for $0 \leq i \leq n-2$ the number of R moves is always greater than or equal to the number of L moves after a total of i moves. Also the number of R moves is always less than or equal to $n+2-2s$ plus the number of L moves after a total of i moves.

Now consider $f(P)$ which is a lattice path that starts at $(0, 0)$ and ends at $(s, n-2+s)$. While traveling along $f(P)$, for $0 \leq i \leq n-2$ the number of U moves is greater than or equal to the number of O moves after a total of i moves which means $f(P)$ stays weakly above the line $y = x$. Also the number of U moves is less than or equal to $n+2-2s$ plus the number of O moves after a total of i moves which means $f(P)$ stays weakly below the line $y = x+n+2-2s$. Therefore $f(P)$ is a P_2 path, so f is well-defined.

Now define the function $g : P_2 \to P_1$ by taking a P_2 path and changing all R moves to U moves and all L moves to O moves. By a similar argument, g is also well-defined. Since f and g are clearly inverses, we have demonstrated a bijection between P_1 paths and P_2 paths. $\square$

Given this bijection, we may now use the following results to calculate $a(n, s)$.

Theorem 3.26 ([14, Theorem 10.3.3]). *Let $a+t \geq b \geq a+r$ and $c+t \geq d \geq c+r$. The number of all paths from (a,b) to (c,d) staying weakly below the line $y = x+t$ and weakly above the line $y = x+r$ is given by*

$$\sum_{k\in\mathbb{Z}} \left(\binom{c+d-a-b}{c-a-k(t-r+2)} - \binom{c+d-a-b}{c-b-k(t-r+2)+t+1} \right).$$

Corollary 3.27.

$$a(n,s) = \sum_{k\in\mathbb{Z}} \left(\binom{n+2}{s-k(n+4-2s)} - \binom{n+2}{n+3-s-k(n+4-2s)} \right)$$

or equivalently

$$a(n,s) = \sum_{k\in\mathbb{Z}} \left(\binom{n+2}{s-k(n+4-2s)} - \binom{n+2}{s-1+k(n+4-2s)} \right).$$

Proof. Let $a = b = 0, c = s, d = n+2-s, t = n+2-2s$ and $r = 0$. Then the lattice paths described in Theorem 3.26 are the P_2 lattice paths. By Proposition 3.24, $a(n,s)$ is the number of P_2 lattice paths. Thus the result follows immediately from Theorem 3.26. □

Therefore we now have an upper bound for the Hilbert series of $E_n/(w_1, w_2)$ that is easily calculated.

Remark 3.28. *It is conjectured the upper bound given in Theorem 3.22 is an equality (see [6, Conjecture 1]).*

This conjecture is currently still an open problem, but it is known that $a(n,s)$ is equal to the dimension of the s^{th} graded component of $E_n/(w_1, w_2)$ for certain values of s.

Proposition 3.29 ([6, Proposition 7]). *Let $k = \mathbb{C}$ and let $w_1, w_2 \in E_n$ be generic quadrics. If $s \leq \lfloor \frac{n}{3} \rfloor + 1$ then*

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s = a(n, s) = \binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}.$$

We provide a different proof of this result.

Proof. First note by Proposition 3.14,

$$h_{E_n/(w_1,w_2)} \succeq [(1+t)^n(1-t^2)^2]$$

and when $s \leq \lfloor \frac{n}{3} \rfloor + 1$ then the s^{th} coefficient of $[(1+t)^n(1-t^2)^2]$ is $\binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}$. Thus

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \geq \binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}.$$

By Theorem 3.22, we know $\dim_k(E_n/(w_1, w_2))_s \leq a(n, s)$ so it remains to show

$$a(n, s) = \binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}.$$

By Corollary 3.27,

$$a(n, s) = \sum_{k \in \mathbb{Z}} \left(\binom{n+2}{s - k(n+4-2s)} - \binom{n+2}{s-1+k(n+4-2s)} \right). \tag{3.2.5}$$

Notice if $k \geq 2$ then

$$s - k(n+4-2s) \leq s - n - 2 + s = 3s - n - 4 \leq n + 3 - n - 4 = -1 < 0.$$

and

$$s - 1 + k(n+4-2s) \geq s - 1 + 2(n+4-2s) = 2n + 7 - 3s \geq 2n + 7 - n - 3 > n + 2.$$

This implies that if $k \geq 2$ then

$$\binom{n+2}{s-k(n+4-2s)} - \binom{n+2}{s-1+k(n+4-2s)} = 0.$$

Also if $k \leq -2$ then

$$s-k(n+4-2s) \geq s+2n+8-4s = 2n+8-3s \geq 2n+8-n-3 > n+2.$$

and

$$s-1+k(n+4-2s) \leq s-1-n-4+2s = 3s-n-5 \leq n+3-n-5 < 0.$$

This implies if $k \leq -2$ then

$$\binom{n+2}{s-k(n+4-2s)} - \binom{n+2}{s-1+k(n+4-2s)} = 0.$$

Therefore when using 3.2.5 we only need to consider when $-2 \leq k \leq 2$. Now by Pascal's identity the following equality holds

$$\begin{aligned}\binom{n+2}{s-k(n+4-2s)} - \binom{n+2}{s-1+k(n+4-2s)} \\ = \binom{n}{s-k(n+4-2s)} + 2\binom{n}{s-1-k(n+4-2s)} \\ + \binom{n}{s-2-k(n+4-2s)} - \binom{n}{s-1+k(n+4-2s)} \\ - 2\binom{n}{s-2+k(n+4-2s)} - \binom{n}{s-3+k(n+4-2s)}\end{aligned}$$

Using this equality and the fact we may restrict to $-2 \leq k \leq 2$, (3.2.5) becomes the desired equality

$$a(n,s) = \binom{n}{s} - 2\binom{n}{s-2} + \binom{n}{s-4}.$$

$\square$

The following table contains values of $a(n,s)$ calculated using the formula for $a(n,s)$ given in Corollary 3.27. Note these numbers are upper bounds for the coefficients of the Hilbert series of $E_n/(w_1, w_2)$.

n/s	0	1	2	3	4	5	6	7	8	9	10
3	1	3	1	0	0	0	0	0	0	0	0
4	1	4	4	0	0	0	0	0	0	0	0
5	1	5	8	1	0	0	0	0	0	0	0
6	1	6	13	8	0	0	0	0	0	0	0
7	1	7	19	21	1	0	0	0	0	0	0
8	1	8	26	40	16	0	0	0	0	0	0
9	1	9	34	66	55	1	0	0	0	0	0
10	1	10	43	100	121	32	0	0	0	0	0
11	1	11	53	143	221	144	1	0	0	0	0
12	1	12	64	196	364	364	64	0	0	0	0
13	1	13	76	260	560	728	377	1	0	0	0
14	1	14	89	336	820	1288	1093	128	0	0	0
15	1	15	103	425	1156	2108	2380	987	1	0	0
16	1	16	118	528	1581	3264	4488	3280	256	0	0
17	1	17	134	646	2109	4845	7752	7753	2584	1	0
18	1	18	151	780	2755	6954	12597	15504	9841	512	0
19	1	19	169	931	3535	9709	19551	28101	25213	6765	1
20	1	20	188	1100	4466	13244	29260	47652	53296	29524	1024

Table 3.2: The entries are the values of $a(n,s)$ for $n \leq 20$.

3.2.3 Bounding the homological growth factor

Throughout this section let $k = \mathbb{C}$ and let $w_1, w_2 \in E_n$ be generic quadrics. Recall the homological growth factor of $\mathrm{Kos}_n(w_1, w_2)$ is defined to be

$$HGF(\mathrm{Kos}_n(w_1,w_2)) = \left(\dim_k H(\mathrm{Kos}_n(w_1,w_2))\right)^{1/n}$$

and Proposition 3.9 states

$$\dim_k H(\mathrm{Kos}_n(w_1,w_2)) = 4 \dim_k \frac{E_n}{(w_1,w_2)}.$$

Theorem 3.22 and Proposition 3.14 gives

$$\sum_s b(n,s) \leq \dim_k \frac{E_n}{(w_1,w_2)} \leq \sum_s a(n,s)$$

which implies

$$\left(4\sum_s b(n,s)\right)^{1/n} \leq HGF(\mathrm{Kos}_n(w_1,w_2)) \leq \left(4\sum_s a(n,s)\right)^{1/n}. \tag{3.2.6}$$

The following table contains the values of these bounds on the homological growth factor of $\mathrm{Kos}_n(w_1,w_2)$ which can be calculated using the values of $a(n,s)$ given in Table 3.2 and the values of $b(n,s)$ given in Table 3.1.

n	$(4\sum_s b(n,s))^{1/n}$	$(4\sum_s a(n,s))^{1/n}$
3	2.714	2.714
4	2.449	2.449
5	2.237	2.268
6	2.196	2.196
7	2.119	2.126
8	2.087	2.09
9	2.057	2.059
10	2.03	2.037
11	2.02	2.021
12	1.996	2.007
13	1.997	1.998
14	1.983	1.988
15	1.98	1.982
16	1.974	1.976
17	1.969	1.972
18	1.967	1.968
19	1.961	1.965
20	1.961	1.963

Table 3.3: The entries in the middle column give a lower bound and the entries in the right column give an upper bound on the homological growth factor of $\mathrm{Kos}_n(w_1,w_2)$.

As seen in this table, there are values of n such that the upper bound is smaller than 2 which implies there are values of n such that $HGF(\mathrm{Kos}_n(w_1,w_2)) < 2$. This immediately gives the following result.

Theorem 3.30. *Let $k = \mathbb{C}$ and let $w_1, w_2 \in E_n$ be generic quadrics. Then* $\mathrm{Kos}_n(w_1, w_2)$ *is a counterexample to Conjecture 1.4 for* $13 \leq n \leq 20$.

In addition, computationally the smallest value for the upper bound appears to happen when $n = 37$ and in this case $(4\sum_s a(n,s))^{1/n} = 1.9507$. In addition, the smallest value for the lower bound appears to happen when $n = 36$ and in this case $(4\sum_s b(n,s))^{1/n} = 1.9489$. These calculations along with further computations of the upper bound of $HGF(\mathrm{Kos}_n(w_1, w_2))$ lead to the following conjecture.

Conjecture 3.31. *Let $k = \mathbb{C}$ and let $w_1, w_2 \in E_n$ be generic quadrics. Then*

$$HGF(\mathrm{Kos}_n(w_1, w_2)) < 2$$

when $n \geq 15$ and

$$HGF(\mathrm{Kos}_n(w_1, w_2)) > 1.9489$$

for all n.

Finally we observe the homological growth factor of $\mathrm{Kos}_n(w_1, w_2)$ has a similar asymptotic behavior as $\mathrm{Kos}_{2n}(w)$ discussed in Section 3.1 in the sense that the homological growth factor of $\mathrm{Kos}_n(w_1, w_2)$ is asymptotically at least 2.

Proposition 3.32. *For all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w_1, w_2)$ for any quadrics $w_1, w_2 \in E_n$ with w_1 general (see Definiton 3.4). Then*

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq 2.$$

Proof. The homological growth factor of $\mathbf{K}_n$ has the following lower bound

$$HGF(\mathbf{K}_n) \geq \left(4\sum_s b(n,s)\right)^{1/n}$$

which was given in (3.2.6). By Proposition 3.19

$$\liminf_{n\to\infty}\left(4\sum_s b(n,s)\right)^{1/n} = \liminf_{n\to\infty} 4^{1/n}\left(\sum_s b(n,s)\right)^{1/n} \geq 2.$$

Therefore

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq \liminf_{n\to\infty}\left(4\sum_s b(n,s)\right)^{1/n} \geq 2.$$

$\square$

Chapter 4

MINIMAL HOMOLOGY

In order to answer Question 1.8, one would like to find the smallest possible total homology of a graded complex over the exterior algebra. This is a difficult task, so we proceed by fixing a particular graded Poincaré series and determine which complexes have the smallest possible total homology amongst those complexes having that Poincaré series.

Definition 4.1. A finite free graded complex $\mathbf{F}$ over the exterior algebra has *minimal homology* if

$$\dim_k H(\mathbf{F}) \leq \dim_k H(\mathbf{G})$$

for any finite free graded complex $\mathbf{G}$ over the exterior algebra with $P_{\mathbf{G}}(s,t) = P_{\mathbf{F}}(s,t)$. Recall $P_{\mathbf{F}}(s,t)$ is the graded Poincaré series of $\mathbf{F}$ (see Definition 2.12).

In Section 4.1, we show Koszul complexes of one general quadric have minimal homology. In fact, Proposition 4.3 demonstrates these are the only complexes with graded Poincaré series $1 + st^2$ that have minimal homology. Then in Section 4.2, we show Koszul complexes of two general quadrics also have minimal homology. However, there are other complexes with Poincaré series $1 + 2st^2 + s^2t^4$ that have minimal homology and we give a characterization of these complexes when n is large enough.

4.1 Koszul complex of one quadric

Let k be a field and let $E_n = k\langle e_1, \ldots, e_n\rangle$. Consider the Koszul complex

$$\mathrm{Kos}_n(w) : E_n(-2) \xrightarrow{w} E_n$$

where $w \in E_n$ is a quadric.

Remark 4.2. $\mathrm{Kos}_n(w)$ *has the following graded Poincaré series*

$$P_{\mathrm{Kos}_n(w)}(s,t) = 1 + st^2.$$

Proposition 4.3. $\mathrm{Kos}_n(w)$ *has minimal homology if and only if w is a general quadric as in Definition 3.4.*

Proof. Let $w \in E_n$ be a quadric and consider

$$\mathrm{Kos}_n(w) : E_n(-2) \xrightarrow{w} E_n.$$

For all i, consider the map

$$\mu_i : (E_n(-2))_i \xrightarrow{w} (E_n)_i.$$

Note

$$\dim_k(H(\mathrm{Kos}_n(w)))_i = \dim_k \ker(\mu_i) + \dim_k \operatorname{coker}(\mu_i)$$

then by the Rank-Nullity Theorem

$$\begin{aligned}\dim_k \ker(\mu_i) + \dim_k \operatorname{coker}(\mu_i) &= \binom{n}{i-2} - \operatorname{rank}(\mu_i) + \binom{n}{i} - \operatorname{rank}(\mu_i)\\ &= \binom{n}{i-2} + \binom{n}{i} - 2\operatorname{rank}(\mu_i).\end{aligned}$$

Therefore

$$\dim_k(H(\mathrm{Kos}_n(w)))_i = \binom{n}{i-2} + \binom{n}{i} - 2\operatorname{rank}(\mu_i)$$

which implies the smallest possible value of $\dim_k(H(\mathrm{Kos}_{2n}(w)))_i$ occurs when the rank of μ_i is as large as possible, namely when μ_i is injective for $i \leq \lfloor \frac{n}{2} \rfloor + 1$ and surjective for $i \geq \lfloor \frac{n}{2} \rfloor + 2$. By Proposition 3.3, this occurs if and only if w is general. Thus $\dim_k H(\mathrm{Kos}_n(w))$ is as small as possible if and only if w is general.

Note that any finite free graded complex over E_n with graded Poincaré series $1 + st^2$ is a Koszul complex of the form $\mathrm{Kos}_n(w)$ for some quadric $w \in E_n$. Thus we conclude $\mathrm{Kos}_n(w)$ has minimal homology if and only if w is a general quadric.

□

4.2 Koszul complex of two quadrics

Now let $w_1, w_2 \in E_n = k\langle e_1, \ldots, e_n \rangle$ be quadrics and consider the Koszul complex

$$\mathrm{Kos}_n(w_1, w_2) : E_n(-4) \xrightarrow{\begin{pmatrix} -w_2 \\ w_1 \end{pmatrix}} E_n(-2)^2 \xrightarrow{\begin{pmatrix} w_1 & w_2 \end{pmatrix}} E_n.$$

Remark 4.4. $\mathrm{Kos}_n(w_1, w_2)$ *has the following graded Poincaré series*

$$P_{\mathrm{Kos}_n(w_1,w_2)}(s,t) = 1 + 2st^2 + s^2t^4.$$

Lemma 4.5. *For any integers r and s, there exists a Zariski open set U_s of $\mathbb{A}_k^{\binom{n}{2}} \times \mathbb{A}_k^{\binom{n}{2}}$ such that*

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \leq r$$

if and only if $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U_s$ where $\Theta : \mathbb{A}_k^{\binom{n}{2}} \to (E_n)_2$ is the bijection given in Definition 3.1.

Proof. Let $w_1, w_2 \in (E_n)_2$. For each i, fix the standard bases for the vector spaces $(E_n(-2))_s^2$ and $(E_n)_s$. Let $M_s(w_1, w_2)$ denote the matrix representing the k-linear transformation

$$\mu_s : (E_{2n}(-2))_s^2 \xrightarrow{\begin{pmatrix} w_1 & w_2 \end{pmatrix}} (E_{2n})_s$$

with respect to the chosen bases. Note the entries of $M_s(w_1, w_2)$ depend on the coefficients of w_1, w_2 but not on any other elements of E_{2n}. Then

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \leq r$$

if and only if $\dim_k \operatorname{coker}(\mu_s) \leq r$ if and only if $\dim_k \operatorname{coker}(M_s(w_1, w_2)) \leq r$. Since

$$\dim_k \operatorname{coker}(M_s(w_1, w_2)) = \binom{n}{s} - \operatorname{rank}(M_s(w_1, w_2))$$

then $\dim_k \operatorname{coker}(M_s(w_1, w_2)) \leq r$ if and only if $\operatorname{rank}(M_s(w_1, w_2)) \geq \binom{n}{s} - r$. Let $I_t(M)$ be the ideal generated by the $t \times t$ minors of M for any matrix M. Also let X_s be the matrix obtained from $M_s(w_1, w_2)$ by replacing each coefficient of w_1, w_2 with a variable. Then

$$\begin{aligned} \dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \leq r &\Leftrightarrow \operatorname{rank}(M_s(w_1, w_2)) \geq \binom{n}{s} - r \\ &\Leftrightarrow I_{\binom{n}{s}-r}(M_s(w_1, w_2)) \neq 0 \\ &\Leftrightarrow (\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \notin V\left(I_{\binom{n}{s}-r}(X_s) \right) \end{aligned}$$

Set $U_s = \mathbb{A}_k^{2\binom{n}{2}} \setminus V\left(I_{\binom{n}{s}-r}(X_s) \right)$ then U_s is a Zariski open set in $\mathbb{A}_k^{2\binom{n}{2}}$ and by the argument above

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \leq r$$

if and only if $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U_s$. $\square$

Proposition 4.6. *Let k be an infinite field and let*

$$r = \min \left\{ \dim_k \frac{E_n}{(q_1, q_2)} : q_i \in (E_n)_2 \right\}.$$

Then there exists a non-empty Zariski open set U of $\mathbb{A}_k^{\binom{n}{2}} \times \mathbb{A}_k^{\binom{n}{2}}$ such that

$$\dim_k \frac{E_n}{(w_1, w_2)} = r$$

for all $w_1, w_2 \in (E_n)_2$ with $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U$.

Proof. Let

$$r_s = \min \left\{ \dim_k \left(\frac{E_n}{(q_1, q_2)} \right)_s : q_i \in (E_n)_2 \right\}$$

then by Lemma 4.5 there exists a Zariski open set U_s of $\mathbb{A}_k^{\binom{n}{2}} \times \mathbb{A}_k^{\binom{n}{2}}$ such that

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \leq r_s$$

for all $w_1, w_2 \in (E_n)_2$ with $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U_s$. Note by definition of r_s, we also have

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \geq r_s$$

for all $w_1, w_2 \in (E_n)_2$, so

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s = r_s$$

for all $w_1, w_2 \in (E_n)_2$ with $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U_s$.

Then let $U = \bigcap_{s=0}^{n} U_s$. Note U_s is non-empty for $0 \leq s \leq n$ by definition of r_s. Therefore U is the finite intersection of Zariski open sets and thus a Zariski open set. Also U is non-empty because k being infinite implies a finite intersection of non-empty Zariski open sets is non-empty. Finally the definition of U implies that for all

$0 \leq s \leq n$ we have the following equality

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s = r_s$$

for all $w_1, w_2 \in (E_n)_2$ with $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U$. Since

$$r = \min \left\{ \dim_k \frac{E_n}{(q_1, q_2)} : q_i \in (E_n)_2 \right\} = \sum_{s=0}^{n} r_s$$

because U is nonempty, then

$$\dim_k \frac{E_n}{(w_1, w_2)} = r$$

for all $w_1, w_2 \in (E_n)_2$ with $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U$. □

Definition 4.7. Let k be an infinite field. A pair of elements $w_1, w_2 \in (E_n)_2$ are said to be *general* if $(\Theta^{-1}(w_1), \Theta^{-1}(w_2)) \in U$ where U is the non-empty Zariski open set given in Proposition 4.6.

Proposition 4.8. *Let $w_1, w_2 \in E_n$ be general quadrics, then* $\mathrm{Kos}_n(w_1, w_2)$ *has minimal homology.*

Proof. Suppose $\mathbf{F}$ is a graded complex of the form

$$\mathbf{F} : 0 \to E_n(-4) \xrightarrow{d_1} E_n(-2)^2 \xrightarrow{d_0} E_n \to 0.$$

Let

$$r = \min \left\{ \dim_k \frac{E_n}{(q_1, q_2)} : q_i \in (E_n)_2 \right\}$$

and let $q_1, q_2 \in E_n$ be quadrics such that $\mathrm{image}(d_0) = (q_1, q_2)$. Then

$$\dim_k H_0(\mathbf{F}) = \dim_k \frac{E_n}{(q_1, q_2)} \geq r.$$

Now consider the k-vector space dual of $\mathbf{F}$

$$\mathbf{F}^* : 0 \to (E_n)^* \xrightarrow{d_0^*} (E_n(-2)^2)^* \xrightarrow{d_1^*} (E_n(-4))^* \to 0$$

where $(-)^*$ denotes the k-vector space dual. Since $(E_n)^* = \operatorname{Hom}_k(E_n, k) \cong E_n(n)$ (by the positive graded exterior algebra version of Proposition 2.22), then $\mathbf{F}^*$ is isomorphic to the complex

$$0 \to (E_n)(n) \xrightarrow{d_0^*} E_n(n+2)^2 \xrightarrow{d_1^*} E_n(n+4) \to 0.$$

The graded shifts of the modules in $\mathbf{F}^*$ imply d_1^* can be represented as a matrix of two quadrics say $q_3, q_4 \in E_n$. Then $\operatorname{image}(d_1^*) = (q_3, q_4)$ which implies

$$\dim_k H_2(\mathbf{F}) = \dim_k H_0(\mathbf{F}^*) = \dim_k \frac{E_n}{(q_3, q_4)} \geq r.$$

Thus using the fact $\dim_k H_1(\mathbf{F}) = \dim_k H_0(\mathbf{F}) + \dim_k H_2(\mathbf{F})$ we conclude

$$\dim_k H(\mathbf{F}) = 2(\dim_k H_0(\mathbf{F}) + \dim_k H_2(\mathbf{F})) \geq 4r.$$

Now suppose $w_1, w_2 \in E_n$ are general quadrics, so $\dim_k E_n/(w_1, w_2) = r$ by Definition 4.7. Then Proposition 3.9 gives

$$\dim_k H(\operatorname{Kos}_n(w_1, w_2)) = 4 \dim_k \frac{E_n}{(w_1, w_2)} = 4r.$$

Therefore

$$\dim_k H(\operatorname{Kos}_n(w_1, w_2)) \leq \dim_k H(\mathbf{F})$$

so by definition $\operatorname{Kos}_n(w_1, w_2)$ has minimal homology. □

As discussed in Section 3.2, $\operatorname{Kos}_n(w_1, w_2)$ produces a counterexample to Conjecture 1.4 for certain values of n when w_1, w_2 are generic. Since $\operatorname{Kos}_n(w_1, w_2)$ has minimal homology, then there does not exist a complex with graded Poincaré series

$1 + st^2 + s^2t^4$ that will have smaller total homology and thus produce a stronger counterexample. However, Koszul complexes are not the only complexes with minimal homology.

Example 4.9. Let $q_1, q_2 \in E_n$ be quadrics and let λ be a nonzero scalar. Consider the graded complex

$$\mathbf{F} : E_n(-4) \xrightarrow{\begin{pmatrix} -\lambda q_2 \\ \lambda q_1 \end{pmatrix}} E_n(-2)^2 \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n.$$

Since $\mathbf{F}$ is isomorphic to $\mathrm{Kos}_n(q_1, q_2)$, the homology of $\mathbf{F}$ is the same as the homology of $\mathrm{Kos}_n(q_1, q_2)$. Thus if q_1, q_2 are general then $\mathbf{F}$ will have minimal homology.

Definition 4.10. A complex is called *Koszul up to a scalar* if it has the form as in Example 4.9 with $\lambda \neq 0$.

We now show that for large enough n, all complexes over E_n with minimal homology are Koszul up to a scalar. We first need the following lemma.

Lemma 4.11. *Let q_1, q_2, q_3 and q_4 be quadrics such that the sequence of maps*

$$0 \to E_n(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_n^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n \to 0$$

form a graded complex. If

$$\dim_k \left(\frac{E_n}{(q_1, q_2)} \right)_4 = \binom{n}{4} - 2\binom{n}{2} + 1$$

then $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$ for some scalar λ (i.e. the complex is Koszul up to a scalar or $q_3 = q_4 = 0$).

Proof. Suppose

$$0 \to E_n(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_n^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n \to 0$$

is a complex. Then $q_1q_3 + q_2q_4 = 0$. Consider the following exact complex

$$0 \to \ker(\phi) \to (E_n^2)_2 \xrightarrow{\phi} (E_n)_4 \to \big(E_n/(q_1, q_2)\big)_4 \to 0$$

where $\phi = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$. By assumption $\dim_k \left(\frac{E_n}{(q_1,q_2)}\right)_4 = \binom{n}{4} - 2\binom{n}{2} + 1$, so

$$\begin{aligned}
\dim_k(\ker(\phi)) &= \dim_k \big(E_n/(q_1, q_2)\big)_4 - \dim(E_n)_4 + \dim(E_n^2)_2 \\
&= \binom{n}{4} - 2\binom{n}{2} + 1 - \binom{n}{4} + 2\binom{n}{2} \\
&= 1.
\end{aligned}$$

Since $q_1q_3 + q_2q_4 = 0$ then $(q_3, q_4) \in \ker(\phi)$. However we also have $(-q_2, q_1) \in \ker(\phi)$ and $\dim_k \ker(\phi) = 1$. Thus $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$ for some scalar λ. $\square$

Remark 4.12. *Let $q_1, q_2, q_3, q_4 \in E_n$ be quadrics. In the case, $q_3 = q_4 = 0$ the complex*

$$\mathbf{F} : 0 \to E_n(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_n^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n \to 0$$

will not have minimal homology because for any general quadrics $w_1, w_2 \in E_n$,

$$\dim_k H_2(\mathbf{F}) = \dim_k E_n > \dim_k \frac{E_n}{(w_1, w_2)}$$

and

$$\dim_k H_0(\mathbf{F}) = \dim_k \frac{E_n}{(q_1, q_2)} \geq \dim_k \frac{E_n}{(w_1, w_2)}.$$

Thus

$$\dim_k H(\mathbf{F}) = 2\left(\dim_k H_0(\mathbf{F}) + \dim_k H_2(\mathbf{F})\right) > 4\dim_k \frac{E_n}{(w_1, w_2)} = \dim_k \mathrm{Kos}_n(w_1, w_2).$$

Theorem 4.13. *Let k be an infinite field and $\mathbf{F}$ be a graded complex over E_n with graded Poincaré series $1 + 2st^2 + st^4$. If $n \geq 9$ and $\mathbf{F}$ has minimal homology, then $\mathbf{F}$ is Koszul up to a scalar.*

Proof. Since $P_{\mathbf{F}}(s,t) = 1 + 2st^2 + st^4$, $\mathbf{F}$ is a graded complex of the form

$$0 \to E_n(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_n^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n \to 0$$

for some $q_1, q_2, q_3, q_4 \in E_n$. Using a similar argument as in the proof of Proposition 3.9, we can show

$$\dim_k H(\mathbf{F}) = 2\left(\dim_k H_0(\mathbf{F}) + \dim_k H_2(\mathbf{F})\right).$$

Note $H_0(\mathbf{F}) = E_n/(q_1, q_2)$ and Lemma 3.8 gives

$$\dim_k H_2(\mathbf{F}) = \dim_k \ker \begin{pmatrix} q_3 \\ q_4 \end{pmatrix} = \dim_k \mathrm{coker} \begin{pmatrix} q_4 & -q_3 \end{pmatrix} = \dim_k \frac{E_n}{(q_3, q_4)}.$$

Therefore

$$\dim_k H(\mathbf{F}) = 2\left(\dim_k \frac{E_n}{(q_1, q_2)} + \dim_k \frac{E_n}{(q_3, q_4)}\right). \tag{4.2.1}$$

Since $\mathbf{F}$ has minimal homology, then

$$\dim_k H(\mathbf{F}) \leq \dim_k H(\mathrm{Kos}_n(w_1, w_2))$$

where $w_1, w_2 \in E_n$ are general quadrics as in Definition 4.7. Then by Proposition 3.9

and (4.2.1), we have the inequality

$$\dim_k \frac{E_n}{(q_1, q_2)} + \dim_k \frac{E_n}{(q_3, q_4)} \leq 2 \dim_k \frac{E_n}{(w_1, w_2)}. \tag{4.2.2}$$

Since $w_1, w_2 \in E_n$ are general quadrics,

$$\dim_k \frac{E_n}{(w_1, w_2)} \leq \dim_k \frac{E_n}{(q_1, q_2)}$$

and

$$\dim_k \frac{E_n}{(w_1, w_2)} \leq \dim_k \frac{E_n}{(q_3, q_4)}.$$

These inequalities, along with (4.2.2), imply the following equalities must hold:

$$\dim_k \frac{E_n}{(w_1, w_2)} = \dim_k \frac{E_n}{(q_1, q_2)} \tag{4.2.3}$$

and

$$\dim_k \frac{E_n}{(w_1, w_2)} = \dim_k \frac{E_n}{(q_3, q_4)}.$$

In addition, notice the proof of Proposition 4.6 gives

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s \leq \dim_k \left(\frac{E_n}{(q_1, q_2)} \right)_s$$

for all $0 \leq s \leq n$. Thus (4.2.3) implies

$$\dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_s = \dim_k \left(\frac{E_n}{(q_1, q_2)} \right)_s$$

for all $0 \leq s \leq n$. In particular when $n \geq 9$

$$\dim_k \left(\frac{E_n}{(q_1, q_2)} \right)_4 = \dim_k \left(\frac{E_n}{(w_1, w_2)} \right)_4 = \binom{n}{4} - 2\binom{n}{2} + 1$$

by Proposition 3.29. Therefore Lemma 4.11 determines $\mathbf{F}$ is Koszul up to a scalar or $q_3 = q_4 = 0$. However if $q_3 = q_4 = 0$ then $\mathbf{F}$ would not have minimal homology by

Remark 4.12. Thus $\mathbf{F}$ must be Koszul up to a scalar. $\square$

Notice the above result holds only when $n \geq 9$. When $n \leq 8$, there exist complexes that are not Koszul up to a scalar and have minimal homology.

Example 4.14. Let $n = 6$ and consider the sequence of maps

$$0 \to E_6(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_6^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_6 \to 0$$

where

$$\begin{aligned} q_1 &= e_1e_2 + e_3e_4 + e_5e_6 \\ q_2 &= e_1e_2 + \frac{1}{2}e_3e_4 - e_5e_6 \\ q_3 &= e_1e_4 + e_2e_5 + e_3e_6 \\ q_4 &= e_1e_4 - 2e_2e_5 - e_3e_6. \end{aligned}$$

Notice $q_1q_3 + q_2q_4 = 0$ so this is in fact a complex. Using Macaulay2 [9], we verify the dimension of the total homology of this complex is the same as the $\dim_k \operatorname{Kos}_6(w_1, w_2)$ for generic w_1, w_2 which can be calculated using Tables 3.1 and 3.2. Thus the complex described above has minimal homology.

Example 4.15. Let $n = 8$ and consider the sequence of maps

$$0 \to E_8(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_8^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_8 \to 0$$

where

$$
\begin{aligned}
q_1 &= e_1e_2 + e_3e_4 + e_5e_6 + e_7e_8 \\
q_2 &= e_1e_2 + 2e_3e_4 + 3e_5e_6 + 4e_7e_8 \\
q_3 &= e_1e_2 - 9e_3e_4 - 11e_5e_6 - 11e_7e_8 \\
q_4 &= 2e_1e_2 + 4e_3e_4 + 4e_5e_6 + 2e_7e_8
\end{aligned}
$$

Note $q_1q_3 + q_2q_4 = 0$ so this is in fact a complex. Using Macaulay2 [9], we verify the dimension of the total homology of this complex is the same as the $\dim_k \operatorname{Kos}_8(w_1, w_2)$ for generic w_1, w_2 which can be calculated using Tables 3.1 and 3.2 and [6, Proposition 6]. Thus the complex described above has minimal homology.

Recall by Theorem 4.13, that for $n \geq 9$ if the complex

$$0 \to E_n(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_n^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n \to 0$$

has minimal homology then it is Koszul up to a scalar. If we restrict to the case that q_1, q_2 are generic quadrics, then we determine a characterization for when the above sequence of maps is even a complex.

Proposition 4.16. *Let $k = \mathbb{C}$ and let $q_1, q_2 \in E_n$ be generic quadrics. If $n \geq 9$ then the sequence of maps*

$$0 \to E_n(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_n^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_n \to 0$$

is a complex if and only if it is Koszul up to a scalar or $q_3 = q_4 = 0$.

In order to prove this result, we consider the cases when n is even and when n

is odd separately. We will first consider the case when n is odd. Before proving the desired result, we need the following technical lemma.

Lemma 4.17. *Let $k = \mathbb{C}$ and let $q_1, q_2 \in E_n$ be generic quadrics. If $n = 2m+1$ for some $m \geq 4$ and the sequence of maps*

$$0 \to E_{2m+1}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2m+1}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2m+1} \to 0$$

form a graded complex, then there is a change of variables such that

$$q_1 = \sum_{i=1}^{m} e_i e_{m+i+1} \qquad\qquad q_3 = \sum_{i=1}^{m} b_i e_i e_{m+i}$$
$$q_2 = \sum_{i=1}^{m} e_i e_{m+i} \qquad\qquad q_4 = \sum_{i=1}^{m} c_i e_i e_{m+i+1}.$$

for some scalars b_i, c_i.

Proof. By [6, Theorem 3], there is a change of variables such that $q_1 = \sum_{i=1}^{n} e_i e_{m+i+1}$ and $q_2 = \sum_{i=1}^{n} e_i e_{m+i}$. Suppose $q_3 = \sum_{i<j} b_{i,j} e_i e_j$ and $q_4 = \sum_{i<j} c_{i,j} e_i e_j$ and note the sequence of maps

$$0 \to E_{2n}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2n}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2n} \to 0$$

forms a complex if and only if $q_1 q_3 + q_2 q_4 = 0$.

Claim 1: If $1 \leq i < j \leq 2m+1$ such that $e_i e_j$ is not of the forms $e_i e_{m+i+1}$ or $e_i e_{m+i}$ then $b_{i,j} = c_{i,j} = 0$.

Case $m \geq 5$: Let $1 \leq i < j \leq 2m+1$ such that e_ie_j is not of the forms e_ie_{m+i+1} or e_ie_{m+i}. Then choose $1 \leq \ell \leq m$ such that $\ell \notin \{i-1, i, j-1, j, i-1-m, i-m, j-1-m, j-m\}$. Note we can always choose such an ℓ because at most 4 choices for $1 \leq \ell \leq m$ are eliminated by the second condition and $m \geq 5$. Then consider the monomial $e_\ell e_{m+\ell+1}e_ie_j$. The coefficient of this monomial in q_1q_3 is $b_{i,j}$ and the coefficient in q_2q_4 is 0. Thus in order for $q_1q_3 + q_2q_4 = 0$ we must have $b_{i,j} = 0$.

Let $1 \leq i < j \leq 2m+1$ such that e_ie_j is not of the forms e_ie_{m+i+1} or e_ie_{m+i}. Then choose $1 \leq \ell \leq m$ such that $\ell \notin \{i, i+1, j, j+1, i-m, i-m+1, j-m, j-m+1\}$. Note we can always choose such an ℓ because at most 4 choices for $1 \leq \ell \leq m$ are eliminated by the second condition and $m \geq 5$. Then consider the monomial $e_\ell e_{m+\ell}e_ie_j$. The coefficient of this monomial in q_1q_3 is 0 and the coefficient in q_2q_4 is $c_{i,j}$. Thus in order for $q_1q_3 + q_2q_4 = 0$ we must have $c_{i,j} = 0$.

Case $m = 4$: Let $1 \leq i < j \leq 9$ such that e_ie_j is not of the form e_ie_{m+i+1} or of the form e_ie_{m+i}. If there is $1 \leq \ell \leq 4$ such that $\ell \notin \{i-1, i, j-1, j, i-5, i-4, j-5, j-4\}$ we obtain $b_{i,j} = 0$ by the same argument as above. However there are cases where the second condition eliminates all 4 choices for ℓ, namely $(i,j) \in \{(2,4), (2,8), (4,6), (6,8)\}$. Therefore when $(i,j) \notin \{(2,4), (2,8), (4,6), (6,8)\}$ then $b_{i,j} = 0$. Similarly we know $c_{i,j} = 0$ when $(i,j) \notin \{(1,3), (1,8), (3,6), (6,8)\}$.

Thus it remains to show $b_{i,j} = 0$ when $(i,j) \in \{(2,4), (2,8), (4,6), (6,8)\}$ and $c_{i,j} = 0$ when $(i,j) \in \{(1,3), (1,8), (3,6), (6,8)\}$. First suppose $(i,j) \in \{(2,8), (4,6), (6,8)\}$. Then consider the monomial $e_{j-4}e_{j+1}e_ie_j$. The coefficient of this monomial in q_1q_3 is $b_{i,j}$ and the coefficient in q_2q_4 is $-c_{i,j+1}$. Since $(i,j) \in \{(2,8), (4,6), (6,8)\}$, then $(i, j+1) \in \{(2,9), (4,7), (7,9)\}$ and in all these cases we know $c_{i,j+1} = 0$. Therefore in order for $q_1q_3 + q_2q_4 = 0$ we must have $b_{i,j} = 0$.

Next let us show $b_{2,4} = 0$. Consider the monomial $e_1e_6e_2e_4$. The coefficient of this monomial in q_1q_3 is $b_{2,4}$ and the coefficient in q_2q_4 is $-c_{1,4}$. We know $c_{1,4} = 0$ so in order for $q_1q_3 + q_2q_4 = 0$ we must have $b_{2,4} = 0$.

Now suppose $(i, j) \in \{(1, 8), (3, 6), (6, 8)\}$ and consider the monomial $e_{j-4-1}e_{j-1}e_ie_j$. The coefficient of this monomial in q_1q_3 is $-b_{i,j-1}$ and the coefficient in q_2q_4 is $c_{i,j}$. Since $(i, j) \in \{(1, 8), (3, 6), (6, 8)\}$, then $(i, j-1) \in \{(1, 7), (3, 5), (6, 7)\}$ and in all these cases we know $b_{i,j+1} = 0$ by our argument above. Therefore in order for $q_1q_3+q_2q_4 = 0$ we must have $c_{i,j} = 0$.

Finally let us show $c_{1,3} = 0$. Consider the monomial $e_4e_8e_1e_3$. The coefficient of this monomial in q_1q_3 is $-b_{1,4}$ and the coefficient in q_2q_4 is $c_{1,3}$. We know $b_{1,4} = 0$ so in order for $q_1q_3 + q_2q_4 = 0$ we must have $b_{1,3} = 0$.

Thus we have proven Claim 1.

Claim 2: $b_{i,j} = 0$ when $1 \leq i < j \leq 2m + 1$ and e_ie_j is of the form e_ie_{m+i+1} and $c_{i,j} = 0$ when $1 \leq i < j \leq 2m + 1$ and e_ie_j is of the form e_ie_{m+i}.

Let $1 \leq i \leq m$ and choose $1 \leq \ell \leq m$ such that $\ell \notin \{i, i - 1\}$. Consider the monomial $e_\ell e_{m+\ell+1}e_ie_{m+i+1}$. The coefficient of this monomial in q_1q_3 is $b_{i,m+i+1} + b_{\ell,m+\ell+1}$ and the coefficient in q_2q_4 is $-c_{i,m+\ell+1}$ if $\ell = i + 1$ and 0 otherwise. By Claim 1, $c_{i,m+\ell+1} = 0$ because $\ell \notin \{i, i-1\}$. Therefore in order for $q_1q_3 + q_2q_4 = 0$ we must have

$$b_{i,m+i+1} + b_{\ell,m+\ell+1} = 0.$$

Next consider the monomial $e_{\ell+1}e_{m+\ell+2}e_ie_{m+i+1}$. The coefficient of this monomial in q_1q_3 is $b_{i,m+i+1} + b_{\ell+1,m+\ell+2}$ and the coefficient in q_2q_4 is $-c_{i,m+\ell+2}$ if $\ell = i - 2$ and 0 otherwise. By Claim 1, $c_{i,m+\ell+2} = 0$ because $\ell \notin \{i, i - 1\}$. Therefore in order for

$q_1q_3 + q_2q_4 = 0$ we must have

$$b_{i,m+i+1} + b_{\ell+1,m+\ell+2} = 0.$$

Finally consider the monomial $e_\ell e_{m+\ell+1} e_{\ell+1} e_{m+\ell+2}$. The coefficient of this monomial in q_1q_3 is $b_{\ell,m+\ell+1} + b_{\ell+1,m+\ell+2}$ and the coefficient in q_2q_4 is $-c_{\ell,\ell+m+2}$. By Claim 1, $c_{\ell,\ell+m+2} = 0$. Therefore in order for $q_1q_3 + q_2q_4 = 0$ we must have

$$b_{\ell,m+\ell+1} + b_{\ell+1,m+\ell+2} = 0.$$

Thus we have the following system of equations

$$\begin{aligned} b_{i,m+i+1} + b_{\ell,m+\ell+1} &= 0 \\ b_{i,m+i+1} + b_{\ell+1,m+\ell+2} &= 0 \\ b_{\ell,m+\ell+1} + b_{\ell+1,m+\ell+2} &= 0. \end{aligned}$$

The tivial solution is the only solution to this system, so we conclude $b_{i,m+i+1} = 0$ for all $1 \leq i \leq m$.

By a very similar argument we also conclude $c_{i,m+i} = 0$ for all $1 \leq i \leq m$ which proves Claim 2.

Therefore Claim 1 and Claim 2 imply q_3 and q_4 are of the following form

$$\begin{aligned} q_3 &= \sum_{i=1}^{m} b_i e_i e_{m+i} \\ q_4 &= \sum_{i=1}^{m} c_i e_i e_{m+i+1}. \end{aligned}$$

which proves the desired result. $\square$

Proof of Proposition 4.16 when n is odd. Let $n = 2m+1$ for some $m \geq 4$. By Lemma 4.17, we may assume q_1, q_2, q_3 and q_4 have the following form

$$q_1 = \sum_{i=1}^{m} e_i e_{m+i+1} \qquad q_3 = \sum_{i=1}^{m} b_i e_i e_{m+i}$$
$$q_2 = \sum_{i=1}^{m} e_i e_{m+i} \qquad q_4 = \sum_{i=1}^{m} c_i e_i e_{m+i+1}.$$

for some scalars b_i, c_i. Note the sequence of maps

$$0 \to E_{2n}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2n}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2n} \to 0$$

forms a complex if and only if $q_1q_3 + q_2q_4 = 0$.

We claim $b_i = b_j$ for all $1 \leq i < j \leq m$. We will first consider when $m \geq 5$. Let $1 \leq i < j \leq m$ and choose $1 \leq \ell \leq m$ such that $\ell \notin \{i, i-1, j, j-1\}$. Note we can choose such an ℓ because $m \geq 5$. Consider the monomial $e_\ell e_{m+\ell+1} e_i e_{m+i}$. The coefficient of this monomial in q_1q_3 is b_i and the coefficient in q_2q_4 is c_ℓ. Therefore in order for $q_1q_3 + q_2q_4 = 0$ we must have $b_i + c_\ell = 0$ which implies $b_i = -c_\ell$. Next consider the monomial $e_\ell e_{m+\ell+1} e_j e_{m+j}$. The coefficient of this monomial in q_1q_3 is b_j and the coefficient in q_2q_4 is c_ℓ. Therefore in order for $q_1q_3 + q_2q_4 = 0$ we must have $b_j + c_\ell = 0$ which implies $b_j = -c_\ell = b_i$. Thus we conclude $b_i = b_j$ for all $1 \leq i, j \leq m$ with $i \neq j$, and thus $q_3 = b_1q_2$.

Now let us consider when $m = 4$. Let $1 \leq i < j \leq 4$. If there exists $1 \leq \ell \leq 4$ such that $\ell \notin \{i, i-1, j, j-1\}$ then by the same argument as above we obtain $b_i = b_j$. The only case when we cannot choose such an ℓ is when $i = 2$ and $j = 4$. For every other pair i, j, we may use the same argument as when $m \geq 5$ to conclude $b_i = b_j$. In particular this means $b_2 = b_3$ and $b_3 = b_4$, so $b_2 = b_4$. Therefore $b_i = b_j$ for all $1 \leq i < j \leq 4$, and thus $q_3 = b_1q_2$.

By a similar argument, we can also show $c_i = c_j$ for all $1 \leq i < j \leq m$ so $q_4 = c_1 q_1$. In addition notice our argument above further proves $b_i = -c_\ell$ for all $1 \leq i, \ell \leq m$. Therefore there exists a scalar $\lambda = -b_1 = c_1$ such that $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$.

□

Now let us consider the case when n is even. Before proving the desired result in this case, we need the following technical lemma.

Lemma 4.18. *Let $k = \mathbb{C}$ and let $q_1, q_2 \in E_n$ be generic quadrics. If $n = 2m$ for some $m \geq 4$ and the sequence of maps*

$$0 \to E_{2m}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2m}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2m} \to 0$$

form a graded complex, then there is a change of variables such that

$$q_1 = \sum_{i=1}^{m} e_{2i-1}e_{2i} \qquad\qquad q_3 = \sum_{i=1}^{m} b_i e_{2i-1}e_{2i}$$
$$q_2 = \sum_{i=1}^{m} a_i e_{2i-1}e_{2i} \qquad\qquad q_4 = \sum_{i=1}^{m} c_i e_{2i-1}e_{2i}.$$

for some scalars a_i, b_i, c_i where $a_i \neq 0$ for all i and $a_i \neq a_j$ for all $i \neq j$.

Proof. By [6, Theorem 3], there is a change of variables such that $q_1 = \sum_{i=1}^{m} e_{2i-1}e_{2i}$ and $q_2 = \sum_{i=1}^{m} a_i e_{2i-1}e_{2i}$ where $a_i \neq 0$ and $a_i \neq a_j$ for all $i \neq j$. Suppose $q_3 = \sum_{i<j} b_{i,j} e_i e_j$ and $q_4 = \sum_{i<j} c_{i,j} e_i e_j$ and note the sequence of maps

$$0 \to E_{2m}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2m}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2m} \to 0$$

forms a complex if and only if $q_1 q_3 + q_2 q_4 = 0$.

Let $1 \leq i < j \leq 2m$ such that e_ie_j is not of the form $e_{2i-1}e_{2i}$ and choose $1 \leq \ell_1, \ell_2 \leq m$ such that $\ell_1 \neq \ell_2$ and $2\ell_1, 2\ell_2 \notin \{i, i+1, j, j+1\}$. Note that such a pair $1 \leq \ell_1, \ell_2 \leq m$ exists because the last condition eliminates at most 2 options from the m choices and $m \geq 4$.

Consider the monomial $e_{2\ell_1-1}e_{2\ell_1}e_ie_j$. The coefficient of this monomial in q_1q_3 is $b_{i,j}$ and the coefficient in q_2q_4 is $a_{\ell_1}c_{i,j}$. Thus, in order for $q_1q_3 + q_2q_4 = 0$, we must have

$$b_{i,j} + a_{\ell_1}c_{i,j} = 0.$$

Now consider the monomial $e_{2\ell_2-1}e_{2\ell_2}e_ie_j$. The coefficient of this monomial in q_1q_3 is $b_{i,j}$ and the coefficient in q_2q_4 is $a_{\ell_2}c_{i,j}$. Thus, in order for $q_1q_3 + q_2q_4 = 0$, we must have

$$b_{i,j} + a_{\ell_2}c_{i,j} = 0.$$

If $c_{i,j} \neq 0$ then $a_{\ell_1} = \frac{-b_{i,j}}{c_{i,j}} = a_{\ell_2}$ which contradicts that $a_{\ell_1} \neq a_{\ell_2}$. Therefore $c_{i,j} = 0$ which implies $b_{i,j} = 0$.

Thus we have shown $b_{i,j} = c_{i,j} = 0$ for all $1 \leq i < j \leq 2m$ such that e_ie_j is not of the form $e_{2i-1}e_{2i}$. This implies q_3 and q_4 are of the form

$$q_3 = \sum_{i=1}^{m} b_ie_{2i-1}e_{2i}$$
$$q_4 = \sum_{i=1}^{m} c_ie_{2i-1}e_{2i}.$$

which proves the desired result. $\square$

Proof of Proposition 4.16 when n is even. Let $n = 2m$ for some $m \geq 5$. By Lemma 4.18, we may assume q_1, q_2, q_3 and q_4 have the following form

$$q_1 = \sum_{i=1}^{m} e_{2i-1}e_{2i} \qquad q_3 = \sum_{i=1}^{m} b_i e_{2i-1}e_{2i}$$
$$q_2 = \sum_{i=1}^{m} a_i e_{2i-1}e_{2i} \qquad q_4 = \sum_{i=1}^{m} c_i e_{2i-1}e_{2i}.$$

where $a_i \neq 0$ for all i and $a_i \neq a_j$ for all $i \neq j$. Therefore the sequence of maps

$$0 \to E_{2m}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2m}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2m} \to 0$$

forms a complex if and only if $q_1q_3 + q_2q_4 = 0$ if and only if

$$b_i + b_j + a_i c_j + a_j c_i = 0 \tag{4.2.4}$$

for all $1 \leq i < j \leq m$. Then let M_m be the coefficient matrix for this linear system. Note there are $\binom{m}{2}$ equations and $2m$ unknowns which are the b_i's and c_i's. Therefore M_m has $\binom{m}{2}$ rows and $2m$ columns. In particular the entry of M_m in the row (i, j) and column l is

$$\begin{cases} 1 & \text{if } l \leq m \text{ and } l = j \text{ or } i \\ a_i & \text{if } l > m \text{ and } l = i \\ a_j & \text{if } l > m \text{ and } l = j \\ 0 & \textit{otherwise} \end{cases}$$

We prove the rank of M_m is $2m - 1$ by induction. First suppose $m = 5$ and consider the matrix M_5 shown below.

$$M_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & a_3 & 0 & a_1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & a_4 & 0 & 0 & a_1 & 0 \\ 1 & 0 & 0 & 0 & 1 & a_5 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 1 & 0 & 0 & 0 & a_3 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & a_4 & 0 & a_2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & a_5 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & a_4 & a_3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & a_5 & 0 & a_3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & a_5 & a_4 \end{pmatrix} \tag{4.2.5}$$

The matrix M_5 does not have full rank because there are an infinite number of solutions to the corresponding linear system, namely for any scalar λ we know $b_i = \lambda a_i$ and $c_i = -\lambda$ for all i is a solution. This corresponds to the case when $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$. Therefore the rank of M_5 is at most 9. Using Macaulay2 [9], we calculate the minor given by deleting the first column and last row, which is

$$\begin{aligned} -4a_1^3a_2a_4 + 4a_1^2a_2^2a_4 + 4a_1^3a_3a_4 - 4a_1a_2^2a_3a_4 - 4a_1^2a_3^2a_4 + 4a_1a_2a_3^2a_4 \\ + 4a_1^3a_2a_5 - 4a_1^2a_2^2a_5 - 4a_1^3a_3a_5 + 4a_1a_2^2a_3a_5 + 4a_1^2a_3^2a_5 - 4a_1a_2a_3^2a_5 \end{aligned}$$

and this factors as

$$-4a_1(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)(a_4 - a_5).$$

Since the a_i's are all distinct and nonzero, the above polynomial is nonzero which implies M_5 has a nonzero 9×9 minor and thus has rank 9.

Now let $m > 5$ and suppose M_{m-1} has rank $2m-3$. First note that, similar to the $m = 5$ case, for any scalar λ we have $b_i = \lambda a_i$ and $c_i = -\lambda$ for all i is a solution and this corresponds to the case when $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$. Therefore M_m has rank at most $2m-1$.

Now notice M_{m-1} is a submatrix of M_m because the linear system that corresponds to M_m contains the equations that make up the linear system that corresponds to M_{m-1}. One can obtain the matrix M_{m-1} from M_m by deleting every row that has a nonzero entry in columns $m, 2m$ and by deleting columns $m, 2m$. For example, notice we can obtain M_4 from M_5 (see (4.2.5) for M_5) by deleting columns $5, 10$ and rows $4, 7, 9, 10$.

Let $\mathbf{r}_1, \ldots, \mathbf{r}_\ell$ be the rows of M_m that correspond to the rows that make up the submatrix M_{m-1}. Since M_{m-1} has rank $2m-3$ by assumption then some subset of size $2m-3$ of these rows are linearly independent. Without loss of generality we may assume $\mathbf{r}_1, \ldots, \mathbf{r}_{2m-3}$ are linearly independent. Next let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the two rows of M_m that correspond to the equations

$$b_1 + b_m + a_m c_1 + a_1 c_m = 0$$

$$b_2 + b_m + a_m c_2 + a_2 c_m = 0$$

We want to show $\mathbf{r}_1, \ldots, \mathbf{r}_{2m-3}, \mathbf{s}_1, \mathbf{s}_2$ form a linearly independent set. Suppose

$$\gamma_1 \mathbf{r}_1 + \ldots + \gamma_{2m-3}\mathbf{r}_{2m-3} + \gamma_{2m-2}\mathbf{s}_1 + \gamma_{2m-1}\mathbf{s}_2 = 0 \tag{4.2.6}$$

for some γ_i. Note this sum is a vector of length $2m$ and the m^{th} and the $(2m)^{th}$ entries of this vector are

$$\gamma_{2m-2} a_m + \gamma_{2m-1} a_m = 0$$

$$\gamma_{2m-2} a_1 + \gamma_{2m-1} a_2 = 0$$

because the m^{th} and $(2m)^{th}$ entries of $\mathbf{r}_i$ are 0 for all $1 \leq i \leq \ell$. From the first equation, since $a_m \neq 0$ we conclude $\gamma_{2m-2} = -\gamma_{2m-1}$ so our second equation becomes

$$\gamma_{2m-1}(-a_1 + a_2) = 0.$$

Since $a_1 \neq a_2$ then $\gamma_{2m-1} = \gamma_{2m-2} = 0$. Therefore (4.2.6) becomes

$$\gamma_1\mathbf{r}_1 + \ldots + \gamma_{2m-3}\mathbf{r}_{2m-3} + \gamma_{2m-2}\mathbf{s}_1 + \gamma_{2m-1}\mathbf{s}_2 = \gamma_1\mathbf{r}_1 + \ldots + \gamma_{2m-3}\mathbf{r}_{2m-3} = 0.$$

Since $\mathbf{r}_1, \ldots, \mathbf{r}_{2n-3}$ are linearly independent, we conclude $\gamma_i = 0$ for all i. Thus $\mathbf{r}_1, \ldots, \mathbf{r}_{2n-3}, \mathbf{s}_1, \mathbf{s}_2$ form a linearly independent set which implies M_m has rank $2m-1$.

Since M_m has rank $2m-1$ then the solution set of the linear system of equations (4.2.4) has dimension 1. Thus the only solutions correspond to $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$ for some constant λ. $\square$

The following also proves Proposition 4.16 in the even case but only when $n \geq 12$. It has been included because it utilizes a different approach.

Alternate proof of Proposition 4.16 when n is even. Let $n = 2m$ for some $m \geq 6$. By Lemma 4.18 we may assume

$$q_1 = \sum_{i=1}^{m} e_{2i-1}e_{2i} \qquad q_3 = \sum_{i=1}^{m} b_i e_{2i-1}e_{2i}$$
$$q_2 = \sum_{i=1}^{m} a_i e_{2i-1}e_{2i} \qquad q_4 = \sum_{i=1}^{m} c_i e_{2i-1}e_{2i}.$$

where $a_i \neq 0$ for all i and $a_i \neq a_j$ for all $i \neq j$. Therefore the sequence of maps

$$0 \to E_{2m}(-4) \xrightarrow{\begin{pmatrix} q_3 \\ q_4 \end{pmatrix}} E_{2m}^2(-2) \xrightarrow{\begin{pmatrix} q_1 & q_2 \end{pmatrix}} E_{2m} \to 0$$

forms a complex if and only if $q_1q_3 + q_2q_4 = 0$ if and only if

$$b_i + b_j + a_ic_j + a_jc_i = 0 \tag{4.2.7}$$

for all $1 \leq i < j \leq m$. These equations form a linear system of $\binom{m}{2}$ equations and $2m$ unknowns which are the b_i's and c_i's. Let $C = \begin{pmatrix} B & A \end{pmatrix}$ be the coefficient matrix where the entry of B in row (i,j) and column l is

$$\begin{cases} 1 & \text{if } l = j \text{ or } l = i \\ 0 & \text{otherwise} \end{cases}$$

and the entry of A in row (i,j) and column l is

$$\begin{cases} a_i & \text{if } l = j \\ a_j & \text{if } l = i \\ 0 & \text{otherwise} \end{cases}$$

We claim the rank of C is $2m-1$. Consider $R = k[x_1, \dots, x_m]/(x_1^2, \dots, x_m^2)$ and let

$$\ell_1 = \sum_{i=1}^{m} x_i$$
$$\ell_2 = \sum_{i=1}^{m} a_ix_i$$

Then B represents the map given by multiplication on R by ℓ_1 and A represents the map given by multiplication on R by ℓ_2 with respect to the monomial basis of R. Moreover since $a_i \neq 0$ and $\dim_k R_2 = \binom{m}{2} \geq m = \dim_k R_1$ because $m \geq 6$ then by [10, Corollary 3.28] these maps are injective. Therefore

$$\begin{aligned} \operatorname{rank}(C) &= \dim_k(\ell_1R_1 + \ell_2R_1) \\ &= \dim_k(\ell_1R_1) + \dim_k(\ell_2R_1) - \dim_k(\ell_1R_1 \cap \ell_2R_1). \end{aligned}$$

Since $\dim_k(\ell_i R_1) = m$,

$$\operatorname{rank}(C) = m + m - \dim_k(\ell_1 R_1 \cap \ell_2 R_1). \tag{4.2.8}$$

Note $\ell_1\ell_2 \in \ell_1 R_1 \cap \ell_2 R_1$, so the dimension of $\ell_1 R_1 \cap \ell_2 R_1$ is at least one. We will show the dimension is exactly one by proving $\ell_1 R_1 \cap \ell_2 R_1 = \operatorname{span}_k(\ell_1\ell_2)$.

Consider $S = R/\ell_1$ then $S \cong k[x_1, \ldots x_{m-1}]/(x_1^2, \ldots, x_{m-1}^2, L^2)$ where $L = x_1 + x_2 + \ldots + x_{m-1}$. Let $K = \ker(S_1 \xrightarrow{\ell_2} S_2)$ and consider the complex

$$0 \to K(-1) \to S(-2) \xrightarrow{\ell_2} S \to S/(\ell_2) \to 0$$

which is exact. The Socle Lemma [12, Corollary 3.11] says $\alpha(K(-1)) > \alpha(\operatorname{soc}(S/\ell_2))$ where, for a graded module M, $\alpha(M) = \min\{d \mid M_d \neq 0\}$ is the initial degree of M and $\operatorname{soc}(M) = \{w \in M \mid x_i w = 0 \text{ for all } i\}$.

Suppose $\alpha(K) = 1$. Then $\alpha(K(-1)) = 2$ which implies $\alpha(\operatorname{soc}(S/\ell_2)) = 1$, so there exists $\ell \in S_1$ such that $\ell x_i = 0$ for all i in S/ℓ_2. Note

$$S/\ell_2 \cong k[x_1, \ldots, x_{m-2}]/(x_1^2, \ldots, x_{m-2}^2, L^2, \tilde{L}^2)$$

for some linear forms $L, \tilde{L} \in k[x_1, \ldots, x_{m-2}]$. Thus

$$\ell x_i \in (x_1^2, \ldots, x_{m-2}^2, L^2, \tilde{L}^2)$$

for all $1 \leq i \leq m-2$. Since $\{\ell x_1, \ldots, \ell x_{m-2}\}$ is linearly independent in $k[x_1, \ldots, x_{m-2}]$, we extend it to a minimal generating set $\{\ell x_1, \ldots, \ell x_{m-2}, f, g\}$ of $(x_1^2, \ldots, x_{m-2}^2, L^2, \tilde{L}^2)$. Thus

$$(x_1^2, \ldots, x_{m-2}^2, L^2, \tilde{L}^2) = (\ell x_1, \ldots, \ell x_{m-2}, f, g) \subseteq (\ell, f, g).$$

However $\operatorname{ht}(x_1^2, \ldots, x_{m-2}^2, L^2, \tilde{L}^2) = m - 2$ because $x_1^2, \ldots, x_{m-2}^2$ is a regular sequence and $\operatorname{ht}(\ell x_1, \ldots, \ell x_{m-2}, f, g) \leq \operatorname{ht}(\ell, f, g) \leq 3$. Therefore when $m \geq 6$ we have reached a contradiction. Thus we must have $\alpha(K) > 1$ which implies $\ker(S_1 \xrightarrow{\ell_2} S_2) = 0$. Note if $\ell \in R_1$ such that $\ell\ell_2 \in \ell_1 R_1$, then $\bar{\ell} \in \ker(S_1 \xrightarrow{\ell_2} S_2) = 0$. Therefore $\ell \in \operatorname{span}(\ell_1)$ which implies $\ell_1 R_1 \cap \ell_2 R_1 = \operatorname{span}(\ell_1 \ell_2)$.

Thus the dimension of $\ell_1 R_1 \cap \ell_2 R_1$ is exactly one, so by (4.2.8) the rank of C is $2m - 1$. This implies the solution set of the linear system of equations (4.2.7) has dimension 1. In particular the only solutions correspond to when $q_3 = -\lambda q_2$ and $q_4 = \lambda q_1$ for some constant λ. $\square$

Chapter 5

ASYMPTOTIC BEHAVIOR OF HOMOLOGICAL GROWTH FACTORS OF FAMILIES OF COMPLEXES

Recall both the Koszul complex of one quadric (see Corollary 3.7) and the Koszul complex of two quadrics (see Proposition 3.32) have a homological growth factor that is asymptotically at least 2. In this chapter, we will continue to investigate the asymptotic behavior of Koszul complexes of quadrics over the exterior algebra utilizing a different technique. In Section 5.1, we outline the general ideas of this technique. Then in Sections 5.2 and 5.3 we use this technique to argue that various families of Koszul complexes of quadrics have a homological growth factor that is asymptotically at least 2.

Koszul complexes are not the only complexes that have this asymptotic behavior. In Sections 5.2 and 5.3, we also discuss some conditions under which a family of complexes over the exterior algebra has homological growth factor asymptotically at least 2. However, not every family of complexes has this asymptotic behavior and we give an example that illustrates a different behavior in Section 5.4.

5.1 Lower bound on total homology via Hilbert series and complex norm

In order to discuss the behavior of the homological growth factor of certain complexes, we will consider a lower bound of the total homology of a complex via the complex norm of its Hilbert series. The Hilbert series of a complex is defined in Definition 2.11.

Suppose $\mathbf{F}$ is a finite free graded complex over E_n. We relate the Hilbert series of $\mathbf{F}$ to the Hilbert series of its homology in the following way.

Lemma 5.1. *Given a finite free graded complex* $\mathbf{F}$ *over* E_n *(or more generally a bounded complex of finite dimensional* k*-vector spaces), we have the following equality*

$$h_{\mathbf{F}}(t) = \sum_i (-1)^i h_{H_i(\mathbf{F})}(t).$$

Proof. Let $\mathbf{F}$ be a finite free graded complex so $\mathbf{F}$ is of the form

$$\mathbf{F} : 0 \to F_r \xrightarrow{d_m} F_{r-1} \xrightarrow{d_{r-1}} \dots \xrightarrow{d_{s+1}} F_{s+1} \xrightarrow{d_{s+1}} F_s \to 0$$

for some $r \geq s$. For all $s \leq i \leq r$, let $Z_i = \ker(d_i)$ and $B_i = \text{image}(d_{i+1})$, so $H_i(\mathbf{F}) = Z_i/B_i$ (see Definition 2.7). Consider the following short exact sequences

$$0 \to B_i \hookrightarrow Z_i \twoheadrightarrow H_i(\mathbf{F}) \to 0$$

and

$$0 \to Z_i \hookrightarrow F_i \xrightarrow{d_i} B_{i-1} \to 0.$$

By Proposition 2.9, Hilbert series is additive on short exact sequences. Thus

$$h_{Z_i}(t) = h_{B_i}(t) + h_{H_i(\mathbf{F})}(t)$$

and

$$h_{F_i}(t) = h_{Z_i}(t) + h_{B_{i-1}}(t).$$

This implies

$$h_{F_i}(t) = h_{B_i}(t) + h_{B_{i-1}}(t) + h_{H_i(\mathbf{F})}(t). \tag{5.1.1}$$

Then by Definition 2.11 and (5.1.1), we conclude

$$\begin{aligned} h_{\mathbf{F}}(t) &= \sum_i (-1)^i h_{F_i}(t) \\ &= \sum_i (-1)^i \left(h_{B_i}(t) + h_{B_{i-1}}(t) + h_{H_i(\mathbf{F})}(t) \right) \\ &= \sum_i (-1)^i h_{H_i(\mathbf{F})}(t). \end{aligned}$$

□

The previous lemma yields a lower bound on the total homology of a finite free graded complex. In the following, $\|\cdot\| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ denotes the norm of a complex number.

Lemma 5.2. *Let* $\mathbf{F}$ *be a finite free graded complex over* E_n *then*

$$\dim_k H(\mathbf{F}) \geq \left\| h_{\mathbf{F}}(z) \right\|$$

for all $z \in S^1 = \{z \in \mathbb{C} : \|z\| = 1\}$.

Proof. For $f(t) = \sum_{i=0}^m a_i t^i \in \mathbb{Q}[t]$, let $\mathcal{L}(f(t)) := \sum_{i=0}^m |a_i|$. The triangle inequality gives

$$\left\| f(z) \right\| \leq \sum_{i=0}^m \left\| a_i z^i \right\| = \sum_{i=0}^m |a_i| \|z\|^i = \sum_{i=0}^m |a_i| = \mathcal{L}(f(t)) \tag{5.1.2}$$

for any $z \in S^1$.

Now suppose $\mathbf{F}$ is a finite free graded complex over E_n. Then Lemma 5.2 and the

Triangle Inequality yield

$$\left\|h_{\mathbf{F}}(z)\right\| = \left\|\sum_i (-1)^i h_{H_i(\mathbf{F})}(z)\right\| \leq \sum_i \left\|(-1)^i h_{H_i(\mathbf{F})}(z)\right\|.$$

By (5.1.2), there is an inequality

$$\left\|(-1)^i h_{H_i(\mathbf{F})}(z)\right\| \leq \mathcal{L}(h_{H_i(\mathbf{F})}(t)) = \dim_k H_i(\mathbf{F})$$

so we conclude

$$\left\|h_{\mathbf{F}}(z)\right\| \leq \sum_i \dim_k H_i(\mathbf{F}) = \dim_k H(\mathbf{F}).$$

□

Example 5.3. Consider the Koszul complex of one quadric $w \in E_n$

$$\mathrm{Kos}_n(w) : 0 \to E_n(-2) \to E_n \to 0.$$

Since $h_{E_n}(t) = (1+t)^n$, then

$$h_{\mathrm{Kos}_n(w)}(t) = h_{E_n}(t) - h_{E_n}(t)t^2 = (1-t^2)(1+t)^n.$$

Therefore Lemma 5.2 gives

$$\dim_k H(\mathrm{Kos}_n(w)) \geq \left\|h_{\mathrm{Kos}_n(w)}(z)\right\| = \left\|1 - z^2\right\| \|1+z\|^n$$

for all $z \in S^1$.

Example 5.4. Consider the Koszul complex of two quadrics $w_1, w_2 \in E_n$

$$\mathrm{Kos}_n(w_1, w_2) : 0 \to E_n(-4) \to E_n(-2)^2 \to E_n \to 0.$$

Since $h_{E_n}(t) = (1+t)^n$, then

$$h_{\mathrm{Kos}_n(w_1,w_2)}(t) = h_{E_n}(t) - 2h_{E_n}(t)t^2 + h_{E_n}(t)t^4 = (1 - 2t^2 + t^4)(1+t)^n.$$

Therefore Lemma 5.2 gives

$$\dim_k H(\mathrm{Kos}_n(w_1, w_2)) \geq \left\| h_{\mathrm{Kos}_n(w_1,w_2)}(z) \right\| = \left\| 1 - 2z^2 + z^4 \right\| \| 1 + z \|^n$$

for all $z \in S^1$.

In the above examples, we explicitly state the lower bound on the total homology. Notice the lower bounds have similarities because the Hilbert series of both have a factor of $(1+t)^n$ and the other factor is a polynomial that records the ranks and the degrees of the generators of the free modules in the complex. In other words, the second factor is the graded Poincaré series (see Definition 2.12) evaluated at $s = -1$. This remains true for all bounded graded free complexes over E_n which gives a more explicit lower bound on the total homology.

Definition 5.5. Let $\mathbf{F}$ be a bounded graded finite free complex over E_n, and define

$$g_{\mathbf{F}}(t) := P_{\mathbf{F}}(-1, t)$$

where $P_{\mathbf{F}}(s,t)$ is the graded Poincaré series of $\mathbf{F}$ (see Definition 2.12).

Proposition 5.6. *Given a finite free graded complex* $\mathbf{F}$ *over* E_n*,*

$$\dim_k H(\mathbf{F}) \geq \left\| g_{\mathbf{F}}(z) \right\| \| 1 + z \|^n$$

for all $z \in S^1$*.*

Proof. Let $\mathbf{F}$ be a finite free graded complex over E_n

$$\mathbf{F} : 0 \to F_m \to \ldots \to F_1 \to F_0 \to 0.$$

Each F_i is free, so $F_i = \bigoplus_{p \in \mathbb{N}} E_n^{c_{i,p}}(-p)$ where $c_{i,p} = 0$ for all but finitely many $p \in \mathbb{N}$.

Since $h_{E_n} = (1+t)^n$,

$$h_{F_i}(t) = \sum_{p\in\mathbb{N}} h_{E_n^{c_{i,p}}}(t)t^p = (1+t)^n \sum_{p\in\mathbb{N}} c_{i,p}t^p$$

which implies

$$h_{\mathbf{F}}(t) = \sum_{i=0}^{m}(-1)^i h_{F_i}(t) = (1+t)^n \sum_{i=0}^{m}\sum_{p\in\mathbb{N}}(-1)^i c_{i,p}t^p = g_{\mathbf{F}}(t)(1+t)^n.$$

Then Lemma 5.2 gives

$$\dim_k H(\mathbf{F}) \geq \left\| h_{\mathbf{F}}(z)\right\| = \left\| g_{\mathbf{F}}(z)\right\| \|1+z\|^n$$

for all $z \in S^1$. □

5.2 Koszul complex of a fixed number of quadrics

Let $\mathrm{Kos}_n(w_1,\dots,w_c)$ be the Koszul complex of c quadrics $w_1,\dots,w_c \in E_n$. First let us analyze families of Koszul complexes where the number of quadrics remains fixed, while the number of generators, n, of the exterior algebra E_n varies.

Lemma 5.7. *Let c be an integer. Then*

$$g_{\mathrm{Kos}_n(w_1,\dots,w_c)}(t) = (1-t^2)^c.$$

Proof. The i^{th} module of $\mathrm{Kos}_n(w_1,\dots,w_c)$ is $F_i = (E(-2i))^{\binom{\ell n}{i}}$. Thus

$$g_{\mathrm{Kos}_n(w_1,\dots,w_c)}(z) = \sum_i (-1)^i \binom{c}{i} t^{2i} = \sum_i \binom{c}{i}(-t^2)^i = (1-t^2)^c.$$

□

Note since c is a fixed integer and independent of n, then $g_{\mathrm{Kos}_n(w_1,\dots,w_c)}(t)$ is the same for any value of n. Using this fact, we argue the asymptotic behavior of the homological growth factor of $\mathrm{Kos}_n(w_1, \ldots, w_c)$ is consistent with the cases $c = 1$ (see Corollary 3.7) and $c = 2$ (see Proposition 3.32).

Proposition 5.8. *Let c be a fixed integer and for all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w_1, \ldots, w_c)$ for any list of quadrics $w_1, \ldots, w_c \in E_n$. Then*

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq 2$$

Proof. Recall

$$HGF(\mathbf{K}_n) = \left(\dim_k H(\mathbf{K}_n)\right)^{1/n}.$$

Then Proposition 5.6 and Lemma 5.7 yield

$$\begin{aligned} HGF(\mathbf{K}_n) &\geq \left(\left\|1 - z^2\right\|^c \|1 + z\|^n\right)^{1/n} \\ &= \left\|1 - z^2\right\|^{c/n} \|1 + z\| \\ &= \|1 - z\|^{c/n} \|1 + z\|^{(c/n)+1} \end{aligned}$$

for all $z \in S^1$. Thus

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq \lim_{n\to\infty} \|1 - z\|^{c/n} \|1 + z\|^{(c/n)+1} = \|1 + z\|$$

for all $z \in S^1$. Since the maximum value of $\|1 + z\|$ is 2 for $z \in S^1$, we conclude

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq 2.$$

□

The key fact in the above argument is that $g_{\mathrm{Kos}_n(w_1,\ldots,w_c)}(t)$ does not depend on n. Therefore, we can generalize Proposition 5.8 to any family of complexes such that $g_{\mathbf{F}}(t)$ is the same for any $\mathbf{F}$ in the family and, in particular, it does not depend on n.

Theorem 5.9. *Suppose for each n that $\mathbf{F}_n$ is a finite free graded complex over E_n, and for all n that $h_{\mathbf{F}_n}(t) = g(t)(1+t)^n$ for some fixed non-zero $g(t)$. Then*

$$\liminf_{n\to\infty} HGF(\mathbf{F}_n) \geq 2.$$

Proof. Since $h_{\mathbf{F}_n}(t) = g(t)(1+t)^n$ then $g_{\mathbf{F}_n}(t) = g(t)$ for all n. By Proposition 5.6,

$$\dim_k H(\mathbf{F}_n) \geq \left\|g(z)\right\| \|1+z\|^n$$

for all $z \in S^1$. This gives a bound on the homological growth factor of $\mathbf{F}_n$

$$HGF(\mathbf{F}_n) = (\dim_k H(\mathbf{F}_n))^{1/n} \geq \left\|g(z)\right\|^{1/n} \|1+z\|$$

for all $z \in S^1$. Since $g(t) \neq 0$, for all $\epsilon > 0$ there exists $z \in S^1$ sufficiently close to 1 such that $\|1+z\| > 2-\epsilon$ and $g(z) \neq 0$. Thus $\left\|g(z)\right\|^{1/n} > 1-\epsilon$ for n sufficiently large which implies

$$HGF(\mathbf{F}_n) > (1-\epsilon)(2-\epsilon)$$

for n sufficiently large. Since this holds for all $\epsilon > 0$, we conclude

$$\liminf_{n\to\infty} HGF(\mathbf{F}_n) \geq 2.$$

□

In addition to analyzing the asymptotic behavior of the homological growth factor of $\mathrm{Kos}_n(w_1, \ldots, w_c)$, we determine when the homological growth factor could be less than 2 and thus possibly produce a counterexample to Conjecture 1.4. We also find a lower bound on the homological growth factor of $\mathrm{Kos}_n(w_1, \ldots, w_c)$.

Proposition 5.10. *Let c and n be fixed integers, let $w_1, \ldots, w_c \in E_n$ be quadrics, and let $\ell = c/n$. If $\ell > .22763$, then*

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq 2$$

and for all $\ell > 0$,

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq 1.93185.$$

Proof. Let $z \in S^1$ then Proposition 5.6 and Lemma 5.7 gives the following lower bound

$$\begin{aligned} HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) &\geq \left(\left\|1 - z^2\right\|^c \|1 + z\|^n\right)^{1/n} \\ &= \left\|1 - z^2\right\|^\ell \|1 + z\| \\ &= \|1 - z\|^\ell \|1 + z\|^{\ell+1} \end{aligned}$$

Let $s = \|1 - z\|$ and $r = \|1 + z\|$. Since $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$. In addition, $a^2 + b^2 = 1$ because $z \in S^1$. Thus

$$r^2 = \|1 + a + bi\|^2 = (1 + a)^2 + b^2 = 1 + 2a + a^2 + b^2 = 2 + 2a$$

and similarly

$$s^2 = \|1 - a + bi\|^2 = (1 - a)^2 + b^2 = 1 - 2a + a^2 + b^2 = 2 - 2a.$$

Then $r^2 = 4 - s^2$ which implies

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq s^\ell (4 - s^2)^{(\ell+1)/2}.$$

Let $f(s) = s^\ell (4 - s^2)^{(\ell+1)/2}$ where $s \in [0, 2]$ because $z \in S^1$. Therefore

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq f(s) \tag{5.2.1}$$

for all $s \in [0,2]$. We are interested in finding the maximum value of $f(s)$ on $s \in [0,2]$, so we find the critical points of $f(s)$. Note

$$\begin{aligned} f'(s) &= \ell s^{\ell-1}(4-s^2)^{(\ell+1)/2} - (\ell+1)s^{\ell+1}(4-s^2)^{(\ell-1)/2} \\ &= s^{\ell-1}(4-s^2)^{(\ell-1)/2}(\ell(4-s^2)-(\ell+1)s^2) \\ &= s^{\ell-1}(4-s^2)^{(\ell-1)/2}(4\ell-(2\ell+1)s^2) \end{aligned}$$

Thus the critical points are $0, 2, 2\sqrt{\frac{\ell}{2\ell+1}}$. Since $f(s) \geq 0$ on $[0,2]$ and $f(0) = f(2) = 0$, f has a global maximum at $2\sqrt{\frac{\ell}{2\ell+1}}$, and hence the maximum value of f is $f(2\sqrt{\frac{\ell}{2\ell+1}})$.

Let $g(\ell) = f(2\sqrt{\frac{\ell}{2\ell+1}}) - 2$ then using Mathematica [18], we graph g and find g has a root at $\ell \approx 0.227627$. Moreover, from the graph below we determine $g(\ell) < 0$ if $\ell \in (0, 0.22762)$ and $g(\ell) > 0$ if $\ell \in (0.22763, 2)$.

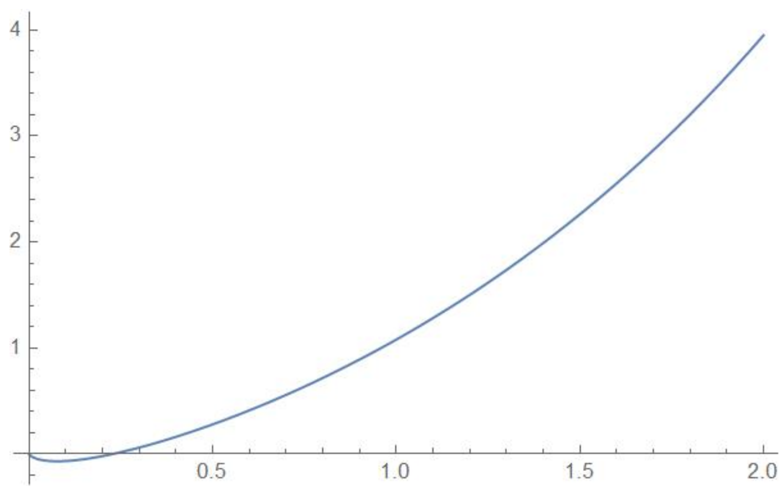


Figure 5.1: Graph of $g(\ell) = f(2\sqrt{\frac{\ell}{2\ell+1}}) - 2$

Thus $f(2\sqrt{\frac{\ell}{2\ell+1}}) \geq 2$ if $\ell \in (0.22763, 2)$, so it remains to show $f\left(2\sqrt{\frac{\ell}{2\ell+1}}\right) \geq 2$ for $\ell \geq 2$. Notice

$$\begin{aligned}
f\left(2\sqrt{\frac{\ell}{2\ell+1}}\right) &= 2^{2\ell+1}\left(\frac{\ell}{2\ell+1}\right)^{\ell/2}\left(\frac{\ell+1}{2\ell+1}\right)^{(\ell+1)/2} \\
&= 2^{2\ell+1}\left(\frac{\ell}{2\ell+1}\frac{\ell+1}{2\ell+1}\right)^{\ell/2}\left(\frac{\ell+1}{2\ell+1}\right)^{1/2} \\
&\geq 2^{2\ell+1}\left(\frac{\ell^2}{(2\ell+2)^2}\right)^{\ell/2}\left(\frac{\ell+1}{2(\ell+1)}\right)^{1/2} \\
&= 2^{2\ell+1}\left(\frac{\ell}{2\ell+2}\right)^{\ell}\left(\frac{1}{2}\right)^{1/2} \\
&= \sqrt{2}\left(\frac{2\ell}{\ell+1}\right)^{\ell}.
\end{aligned}$$

Since $\left(\frac{2\ell}{\ell+1}\right)^{\ell}$ is increasing and $\ell \geq 2$, we conclude

$$f\left(2\sqrt{\frac{\ell}{2\ell+1}}\right) \geq \sqrt{2}\left(\frac{2\ell}{\ell+1}\right)^{\ell} \geq \sqrt{2}\left(\frac{2\cdot 2}{3}\right)^{2} \geq 2.$$

Therefore (5.2.1) implies

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq f\left(2\sqrt{\frac{\ell}{2\ell+1}}\right) \geq 2$$

for $\ell > .22763$.

Now let us consider when $\ell \leq .22763$. Using the Mathematica [18] FindMinimum function, we find the minimum value of $f(2\sqrt{\frac{\ell}{2\ell+1}})$ occurs when $\ell = \frac{1}{\sqrt{3}} - \frac{1}{2} \approx 0.07735$. In addition, $f(2\sqrt{\frac{\ell}{2\ell+1}}) \geq 1.93185$ when $\ell = \frac{1}{\sqrt{3}} - \frac{1}{2}$. Thus (5.2.1) implies

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq f\left(2\sqrt{\frac{\ell}{2\ell+1}}\right) \geq 1.93185.$$

for all $\ell > 0$. $\square$

5.3 Koszul complex of a varying number of quadrics

Let $\mathrm{Kos}_n(w_1, \ldots, w_{c_n})$ be the Koszul complex of c_n quadrics $w_1, \ldots, w_{c_n} \in E_n$. Now let us analyze families of Koszul complexes where the number of quadrics is dependent on the number of variables of the exterior algebra. We divide our analysis into three cases: the number of quadrics grows super-linearly (see Proposition 5.11), sub-linearly (see Corollary 5.13), or linearly (see Remark 5.14) with respect to the number of variables of the exterior algebra.

First consider the case when the number of quadrics grows super-linearly, that is there exists some fixed $\ell > 0$ and $r > 1$ such that $c_n \geq \ell n^r$ for all n.

Proposition 5.11. *Let $\ell > 0$ and $r > 1$ be fixed. For all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w_1, \ldots, w_{c_n})$ for any list of quadrics $w_1, \ldots, w_{c_n} \in E_n$ where $c_n \geq \ell n^r$. Then*

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) = \infty.$$

Proof. Recall

$$HGF(\mathbf{K}_n) = \left(\dim_k H(\mathbf{K}_n)\right)^{1/n}.$$

By Proposition 5.6 and Lemma 5.7 we have the lower bound

$$HGF(\mathbf{K}_n) \geq \left(\left\|1 - z^2\right\|^{c_n} \|1 + z\|^n\right)^{1/n}$$

for all $z \in S^1$. In particular this is true when $z = i$, and, in this case, we have

$$HGF(\mathbf{K}_n) \geq \left(\left\|1 - i^2\right\|^{c_n} \|1 + i\|^n\right)^{1/n} = 2^{c_n/n}\sqrt{2}$$

Since $c_n \geq \ell n^r$ then

$$HGF(\mathbf{K}_n) \geq 2^{\ell n^{r-1}}\sqrt{2} \geq 2^{\ell n^{r-1}}.$$

Note $r > 1$, so $r - 1 > 0$ which implies

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq \liminf_{n\to\infty} 2^{\ell n^{r-1}} = \infty.$$

□

Next consider the case when the number of quadrics grows sub-linearly, that is $c_n \leq \ell n^r$ for some $\ell > 0$ and $0 < r < 1$. This case can be considered more generally and is addressed in the following result.

Proposition 5.12. *Suppose* $\mathbf{F}_n$ *is a finite free graded complex over* E_n *with*

$$h_{\mathbf{F}_n}(t) = (1-t)^{u(n)} g_n(t)(1+t)^n$$

where $u : \mathbb{N} \to \mathbb{N}$ *is a function. Assume*

1. *coefficients of* $g_n(t)$ *are positive and*

2. $\liminf_{n\to\infty} \frac{u(n)}{n^{1-\epsilon}} < \infty$ *for some* $\epsilon > 0$.

Then

$$\liminf_{n\to\infty} HGF(\mathbf{F}_n) \geq 2.$$

Proof. Suppose $g_n(t) = a_0 + a_1 t + \ldots + a_d t^d$. Note if $z \in S^1$ is such that $-\pi/2d < \arg(z) < \pi/2d$ then $Re(z^i) \geq 0$ for all $0 \leq i \leq d$, and

$$\left\| g_n(z) \right\| \geq Re(g_n(z)) \geq a_0 \geq 1. \tag{5.3.1}$$

By assumption there exists $\epsilon > 0$ such that $\liminf_{n\to\infty} \frac{u(n)}{n^{1-\epsilon}} < \infty$. Fix $z_n \in S^1$ such that $Re(z_n) = 1 - 2^{-n^{\epsilon/2}}$. Then $\lim_{n\to\infty} Re(z_n) = 1$, so $-\pi/2d < \arg(z_n) < \pi/2d$ for large enough n. Since $\arg(z_n^i) = i \arg(z_n)$, then $-\pi/2 < \arg(z_n^i) < \pi/2$ for large enough n and for all $0 \leq i \leq d$. Therefore $Re(z_n^i) > 0$ for large enough n and for all

$0 \le i \le d$, so Proposition 5.6 and (5.3.1) gives

$$HGF(\mathbf{F}_n) \ge \|1 - z_n\|^{u(n)/n} \left\|g(z_n)\right\|^{1/n} \|1 + z_n\| \ge \|1 - z_n\|^{u(n)/n} \|1 + z_n\|$$

for large enough n. Then

$$\log_2 HGF(\mathbf{F}_n) \ge \frac{u(n)\log_2\|1 - z_n\|}{n} + \frac{n \log_2\|1 + z_n\|}{n} \tag{5.3.2}$$

for large enough n. Thus (5.3.2) implies

$$\begin{aligned}
\log_2 HGF(\mathbf{F}_n) &\ge \frac{u(n)\log_2\|1 - z_n\|}{n} + \frac{n \log_2\|1 + z_n\|}{n} \\
&\ge \frac{u(n)\log_2(1 - Re(z_n))}{n} + \frac{n \log_2(1 + Re(z_n))}{n} \\
&= \frac{u(n)(-n^{\epsilon/2})}{n} + \log_2(2 - 2^{-n^{\epsilon/2}}) \\
&= \frac{-u(n)}{n^{1-\epsilon/2}} + \log_2(2 - 2^{-n^{\epsilon/2}})
\end{aligned}$$

for large enough n. Then

$$\begin{aligned}
\liminf_{n\to\infty} \log_2 HGF(\mathbf{F}_n) &\ge \liminf_{n\to\infty} \left(\frac{-u(n)}{n^{1-\epsilon/2}} + \log_2(2 - 2^{-n^{\epsilon/2}}) \right) \\
&= \liminf_{n\to\infty} \left(\frac{-u(n)}{n^{1-\epsilon}} \frac{1}{n^{\epsilon/2}} + \log_2(2 - 2^{-n^{\epsilon/2}}) \right).
\end{aligned}$$

Since $\liminf_{n\to\infty} \frac{u(n)}{n^{1-\epsilon}} < \infty$ and $\liminf_{n\to\infty} \frac{1}{n^{\epsilon/2}} = 0$ we conclude

$$\liminf_{n\to\infty} \frac{-u(n)}{n^{1-\epsilon}} \frac{1}{n^{\epsilon/2}} = 0.$$

Also

$$\liminf_{n\to\infty} \log_2(2 - 2^{-n^{\epsilon/2}}) = 1$$

so we conclude

$$\liminf_{n\to\infty} \log_2 HGF(\mathbf{F}_n) \ge 1$$

which implies

$$\liminf_{n\to\infty} HGF(\mathbf{F}_n) \geq 2.$$

$\square$

Corollary 5.13. *Let $\ell > 0$ and $0 < r < 1$ be fixed. For all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w_1, \ldots, w_{c_n})$ for any list of quadrics $w_1, \ldots, w_{c_n} \in E_n$ where $c_n \leq \ell n^r$. Then*

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq 2.$$

Proof. Proposition 3.14 and Lemma 5.7 gives

$$h_{\mathbf{K}_n}(t) = (1-t^2)^{c_n}(1+t)^n = (1-t)^{c_n}(1+t)^{c_n}(1+t)^n.$$

Let $u : \mathbb{N} \to \mathbb{N}$ be given by $u(n) = c_n$ and let $g_n(t) = (1+t)^{c_n}$. Since $0 < r < 1$, there exists $\epsilon > 0$ such that $1 - \epsilon > r$. Thus

$$\liminf_{n\to\infty} \frac{u(n)}{n^{1-\epsilon}} = \liminf_{n\to\infty} \frac{c_n}{n^{1-\epsilon}} \leq \lim_{n\to\infty} \frac{\ell n^r}{n^{1-\epsilon}} = 0.$$

Then apply Proposition 5.12 to attain the desired result. $\square$

Finally consider when the number of quadrics grows linearly, that is $c_n = \ell n$ for some $\ell > 0$.

Remark 5.14. *Let $\ell > .22763$ be fixed. For all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w_1, \ldots, w_{c_n})$ for any list of quadrics $w_1, \ldots, w_{c_n} \in E_n$ where $c_n = \ell n$. By Proposition 5.10, we know*

$$HGF(\mathbf{K}_n) \geq 2.$$

which implies

$$\lim_{n\to\infty} HGF(\mathbf{K}_n) \geq 2.$$

The case when $c_n = \ell n$ for some $0 < \ell \leq .22763$ is still open.

5.4 Family where the homological growth factor is asymptotically strictly less than 2

As seen in the previous sections, there are many families of graded complexes over E_n where the homological growth factor is asymptotically at least 2. However this is not true for every family of graded complexes over E_n. Before discussing an example to demonstrate this fact, we first prove a couple results.

Lemma 5.15. *For all $0 \leq i \leq c$, let $\mathbf{C}_i$ be a finite free graded complex over the exterior algebra $E_{i,n} = k\langle e_{i,1}, \ldots, e_{i,n}\rangle$. Then $\mathbf{C}_1 \otimes_k \ldots \otimes_k \mathbf{C}_c$ is a finite free graded complex over the exterior algebra $E_{1,n} \otimes_k \ldots \otimes_k E_{c,n}$ with cn variables.*

Proof. By induction we just need to show this is true for $c = 2$.

First note $E_{1,n} \otimes_k E_{2,n} \cong k\langle e_{1,1}, \ldots, e_{1,n}, e_{2,1}, \ldots, e_{2,n}\rangle$ is an exterior algebra with $2n$ variables.

Claim 1: Suppose F_1 is a graded free $E_{1,n}$-module and F_2 is a graded free $E_{2,n}$-module then $F_1 \otimes_k F_2$ is a graded free module over $E_{1,b} \otimes_k E_{2,n}$.

We have $F_1 = \bigoplus_{a \in \mathbb{Z}} E_{1,n}(-a)^{c_a}$ for some $c_a \in \mathbb{N}$ and $F_2 = \bigoplus_{b \in \mathbb{Z}} E_{2,n}(-b)^{d_b}$ for some $d_b \in \mathbb{N}$. Therefore

$$\begin{aligned}
F_1 \otimes_k F_2 &= \left(\bigoplus_{a \in \mathbb{Z}} E_{1,n}(-a)^{c_a}\right) \otimes_k \left(\bigoplus_{b \in \mathbb{Z}} E_{2,n}(-b)^{d_b}\right) \\
&\cong \bigoplus_{a,b \in \mathbb{Z}} \left(E_{1,n}(-a)^{c_a} \otimes_k E_{2,n}(-b)^{d_b}\right) \\
&\cong \bigoplus_{a,b \in \mathbb{Z}} \left(E_{1,n}(-a) \otimes_k E_{2,n}(-b)\right)^{c_a d_b} \\
&\cong \bigoplus_{p \in \mathbb{Z}} \bigoplus_{a+b=p} \left(E_{1,n} \otimes_k E_{2,n}\right)(-p)^{c_a d_b}
\end{aligned}$$

Since $\left(E_{1,n} \otimes_k E_{2,n}\right)(-p)^{c_a d_b}$ is a graded free module over $E_{1,n} \otimes_k E_{2,n}$ and the direct sum of graded free modules is a graded free module then we conclude $F_1 \otimes_k F_2$ is a graded free module over $E_{1,b} \otimes_k E_{2,n}$. Thus we have proven Claim 1.

Now suppose $F_{1,i}$ is the i^{th} free module of $\mathbf{C}_1$ and $F_{2,j}$ is the j^{th} free module of $\mathbf{C}_2$. By definition of tensor product of complexes, the i^{th} module of the complex $\mathbf{C}_1 \otimes_k \mathbf{C}_2$ is

$$\bigoplus_{\ell+m=i} F_{1,\ell} \otimes_k F_{2,m}.$$

Claim 1 gives $F_{1,\ell} \otimes_k F_{2,m}$ is a graded free module over $E_{1,n} \otimes_k E_{2,n}$ which implies

$$\bigoplus_{\ell+m=i} F_{1,\ell} \otimes_k F_{2,m}$$

is a graded free module over $E_{1,n} \otimes_k E_{2,n}$.

Now let $d_{\mathbf{C}_1}$ and $d_{\mathbf{C}_2}$ be the differentials of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively. By definition of the tensor product of complexes the differential of $\mathbf{C}_1 \otimes \mathbf{C}_2$ is given by

$$d_{\mathbf{C}_1 \otimes \mathbf{C}_2}(x, y) = (d_{\mathbf{C}_1}(x), y) + (-1)^{\deg(x)}(x, d_{\mathbf{C}_2}(y).$$

Since $d_{\mathbf{C}_1}$ and $d_{\mathbf{C}_2}$ both have degree 0, then $d_{\mathbf{C}_1 \otimes \mathbf{C}_2}$ also has degree 0. Finally since $\mathbf{C_1}$ and $\mathbf{C_2}$ are finite complexes then $\mathbf{C_1} \otimes_k \mathbf{C_2}$ will clearly be a finite complex. Therefore $\mathbf{C}_1 \otimes_k \mathbf{C}_2$ is a finite free graded complex over $E_{1,n} \otimes_k E_{2,n}$. $\square$

Lemma 5.16. *In the setting of the previous lemma, the following equality holds*

$$\dim_k H(\mathbf{C}_1 \otimes_k \mathbf{C}_2 \otimes_k \ldots \otimes_k \mathbf{C}_c) = \dim_k H(\mathbf{C}_1) \dim_k H(\mathbf{C}_2) \cdots \dim_k H(\mathbf{C}_c).$$

Proof. By induction we just need to show this is true for $c = 2$.

Recall the Künneth Theorem gives

$$H_j(\mathbf{C}_1 \otimes_k \mathbf{C}_2) \cong \bigoplus_i (H_i(\mathbf{C}_1) \otimes_k H_{j-i}(\mathbf{C}_2))$$

of k-vector spaces. Thus

$$\begin{aligned}
\dim_k H(\mathbf{C}_1 \otimes_k \mathbf{C}_2) &= \sum_j \dim_k H_j(\mathbf{C}_1 \otimes_k \mathbf{C}_2) \\
&= \sum_j \sum_i \dim_k H_i(\mathbf{C}_1) \dim_k H_{j-i}(\mathbf{C}_2) \\
&= \sum_i \dim_k H_i(\mathbf{C}_1) \left(\sum_j \dim_k H_{j-i}(\mathbf{C}_2) \right) \\
&= \dim_k H(\mathbf{C}_1) \dim_k H(\mathbf{C}_2).
\end{aligned}$$

□

Proposition 5.17. *Let* $\mathbf{F}$ *be a bounded finite free graded complex over the exterior algebra* E_n*. Then for any integer* c

$$HGF(\mathbf{F}^{\otimes c}) = HGF(\mathbf{F}).$$

Proof. By Lemma 5.15

$$\mathbf{F}^{\otimes c} = \underbrace{\mathbf{F} \otimes_k \mathbf{F} \otimes_k \ldots \otimes_k \mathbf{F}}_{c \text{ times}}$$

is a graded complex over the exterior algebra with cn variables and Lemma 5.16 yields

$$\dim_k H(\mathbf{F}^{\otimes c}) = \prod_{i=0}^{c} \dim_k H(\mathbf{F}) = (\dim_k H(\mathbf{F}))^c.$$

Therefore

$$
\begin{aligned}
HGF(\mathbf{F}^{\otimes c}) &= \big(\dim_k H(\mathbf{F}^{\otimes c})\big)^{1/cn} \\
&= \big((\dim_k H(\mathbf{F}))^c\big)^{1/cn} \\
&= \big(\dim_k H(\mathbf{F})\big)^{1/n} \\
&= HGF(\mathbf{F}).
\end{aligned}
$$

□

Finally let us consider an example of a family of complexes whose homological growth factor is less than 2.

Example 5.18. Let $\mathbf{K} = \mathrm{Kos}_{24}(w)$ for any general quadric $w \in E_n$ (see Definition 3.4). By (3.1.2) in Section 3.1, $HGF(\mathbf{K})$ is at most 1.97. Then by Proposition 5.17, $HGF(\mathbf{K}^{\otimes c})$ is also at most 1.97 for all c. Therefore

$$\liminf_{c\to\infty} HGF(\mathbf{K}^{\otimes c}) \leq 1.97 < 2.$$

Chapter 6

OPEN PROBLEMS

Recall one of the main goals of this thesis is to analyze examples of finite free graded complexes that correspond to counterexamples of the Generalized Total Rank Conjecture (see Conjecture 1.4). In particular, we are interested in being able to answer the following open question.

Question 6.1. *Is there a real number $a > 1$ such that each non-exact perfect complex $\mathbf{F}$ over E_n satisfies*

$$\dim_k H(\mathbf{F}) \geq a^n$$

or equivalently

$$HGF(\mathbf{F}) \geq a?$$

This thesis focuses mostly on the analysis of Koszul complexes of quadrics over the exterior algebra, and there are still many open questions about these complexes. As discussed in Section 3.2, one can find bounds on the Hilbert series of $E_n/(w_1, w_2)$ which give bounds on the homological growth factor of the Koszul complex of two generic quadrics.

Conjecture 6.2 ([6, Conjecture 1]). *Let $w_1, w_2 \in E_n$ be generic quadrics. The upper bound on the Hilbert series of $E_n/(w_1, w_2)$ given in Theorem 3.22 is an equality.*

In addition, computations of the bounds of the homological grwoth factor of the Koszul complex of two generic quadrics suggests the following conjecture is true, but the validity is still not known.

Conjecture 6.3. *Let $k = \mathbb{C}$ and let $w_1, w_2 \in E_n$ be generic quadrics. Then*

$$HGF(\mathrm{Kos}_n(w_1, w_2)) < 2$$

when $n \geq 15$ and

$$HGF(\mathrm{Kos}_n(w_1, w_2)) > 1.951$$

for all n.

However by Proposition 5.10, we can use the lower bound via Hilbert series and the complex norm discussed in Section 5.1 to determine that if $w_1, \ldots, w_c \in E_n$ are quadrics then

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq 1.93185.$$

Therefore, $a = 1.93185$ answers Question 6.1, but only for Koszul complexes of quadrics. It is possible that this lower bound could be improved which leads to the following question.

Question 6.4. *Let $w_1, \ldots, w_c \in E_n$ be quadrics. Is the lower bound*

$$HGF(\mathrm{Kos}_n(w_1, \ldots, w_c)) \geq 1.93185$$

given in Proposition 5.10 tight or can one find a better lower bound?

In order to further investigate Question 6.1, one could first further investigate the minimal homology of Koszul complexes. By results in Section 4.1 and 4.2, we know the Koszul complex of one general quadric and the Koszul complex of two general quadrics has minimal homology, but Koszul complexes of a larger number of quadrics have not yet been considered.

Question 6.5. *Let $w_1, \ldots, w_c \in E_n$ be quadrics. Is there a generality condition on $w_1, \ldots, w_c$ such that $\mathrm{Kos}_n(w_1, \ldots, w_c)$ will have minimal homology?*

If $\mathrm{Kos}_n(w_1, \ldots, w_c)$ has minimal homology, then $a = 1.93185$ would answer Question 6.1, but only for any finite free graded complex that has the same graded Poincare series as a Koszul complex of quadrics.

Finally, this thesis contains analysis of the asymptotic behavior of the homological growth factor of families of Koszul complexes. Section 5.2 focuses on families of the Koszul complexes where the number of quadrics is independent of the number of variables, and Proposition 5.8 gives

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq 2$$

where c is fixed and $\mathbf{K}_n = \mathrm{Kos}_n(w_1, \ldots, w_c)$ for any list of quadrics $w_1, \ldots, w_c \in E_n$. Then Section 5.3 contains analysis of families of Koszul complexes where the number of quadrics is dependent on the number of variables, and in most cases we show these families have similar asymptotic behavior. However, there is one case still open.

Question 6.6. *Let $0 < \ell \leq .22763$ be fixed. For all n, let $\mathbf{K}_n = \mathrm{Kos}_n(w_1, \ldots, w_{\ell n})$ for any list of quadrics $w_1, \ldots, w_{\ell n} \in E_n$. Does the following lower bound hold*

$$\liminf_{n\to\infty} HGF(\mathbf{K}_n) \geq 2?$$

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