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Bootstrap Percolation on Random Geometric Graphs

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BOOTSTRAP PERCOLATION ON RANDOM GEOMETRIC GRAPHS

by

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BOOTSTRAP PERCOLATION ON RANDOM GEOMETRIC GRAPHS

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Bootstrap Percolation is a discrete-time process that models the spread of information or disease across the vertex set of a graph. It was introduced in 1979 by Chalupa, Leith and Reich as a simple model of dynamics of ferromagnetism. We consider the following version of this process: Initially, each vertex of the graph is set active with probability p or inactive otherwise. At each time step, every inactive vertex with at least k active neighbors becomes active. Active vertices will always remain active. The process ends when it reaches a stationary state. If all the vertices eventually become active, then we say we achieve percolation.

We analyze the Bootstrap Percolation process on a Random Geometric Graph. Random Geometric Graphs provide a simplified abstract model of spatial networks, and are particularly suitable to describe wireless ad-hoc networks.

More precisely, a Random Geometric Graph is obtained by choosing n vertices uniformly at random from the unit d-dimensional cube or torus, and joining any two vertices by an edge if they are within a certain distance, r, from each other. Until now, very little was known about Bootstrap Percolation on Random Geometric Graphs, other than some initial results in a paper by Bradonjić and Saniee (2012).

We obtain precise results that characterize the final state of the Bootstrap Percolation process in terms of the parameters p and r asymptotically almost surely as the number n of vertices tends to infinity. We show that, a.a.s., the process is either stationary from the very beginning (i.e. no inactive vertex ever changes to active) or

almost all vertices eventually become active. Moreover, we prove that in the latter case the only obstacle to achieve full percolation is the presence of vertices of degree less than k. Indeed, as soon as r is large enough to guarantee that the minimum degree is at least k, a.a.s., the process is either stationary or attains percolation. Finally, we study a version of the model with a restricted focus of infection (i.e. active points can initially occur in a small region of the torus), and obtain analogous results for that case.

DEDICATION

To my Dad, Grandma, and Mom.

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Chapter 1

Introduction

Bootstrap Percolation on Random Geometric Graphs combines two random processes: the Bootstrap Percolation Model and the model of Random Geometric Graphs. Here we informally describe each process, discuss their applications and mention some previously known results.

1.1 Random Geometric Graphs

A *Random Geometric Graph* is a type of random graph where the vertices are points in a metric space and adjacencies are determined by the distance between two points. They were first introduced by Gilbert in his 1961 paper "Random Plane Networks" [19]. In this paper, Gilbert used a Poisson point process to pick the points in the Euclidean plane to be the vertices of the graph (this gives a graph on infinitely many vertices). Then for a distance r, two vertices are adjacent if the Euclidean distance between the vertices is at most r. Typically, this distance r is a function of the number of vertices n of the graph. Since 1961, there have been extensive results involving random geometric graphs, including results concerning random geometric graphs on different metric spaces, with different probability distributions, and with varied adjacency rules (see [5] for examples of different models).

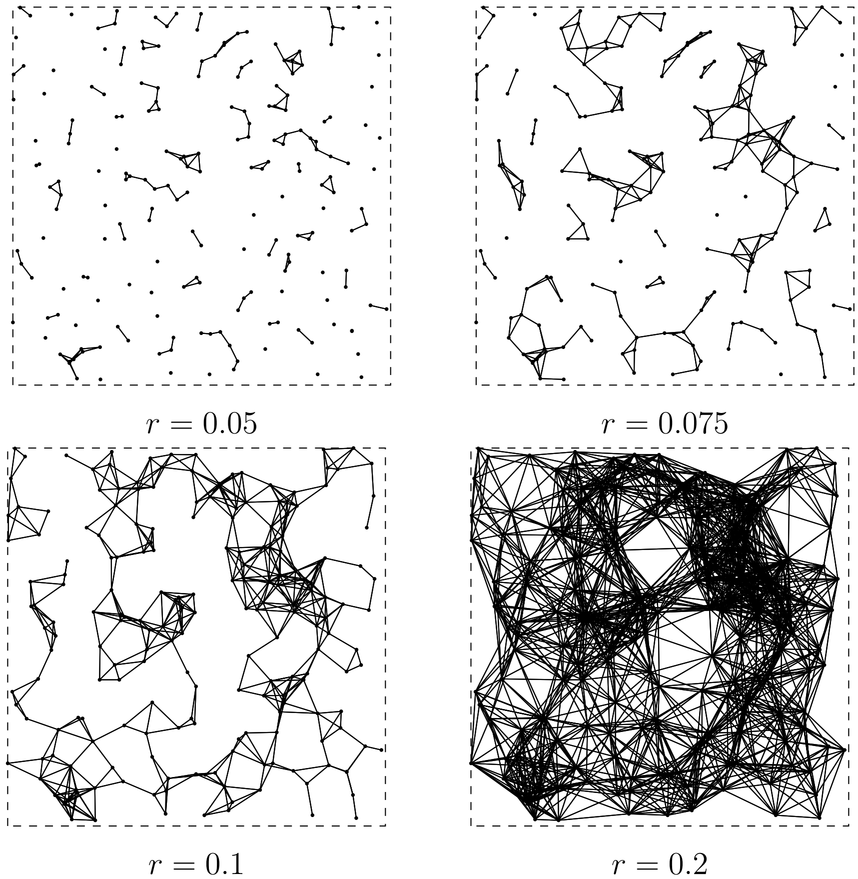


Figure 1.1: Random geometric graphs on the same set of $n = 200$ randomly placed vertices in $[0, 1]^2$ with the euclidean norm and different values of r.

There are several applications of random geometric graphs which has cemented them as a prevailing object of research in mathematics, computer science, engineering, and more. Random geometric graphs can be used as mathematical models for large spacial networks, specifically when the network relies on the geometry of its space. Examples of applications of these graphs include wireless networks (such as ad-hoc networks, cell phone networks and wireless internet), transportation networks, power grids, social networks, neural networks, and uses in statistics. In particular, random geometric graphs can help decide whether a collection of data comes from single or multiple distributions by analyzing the data as points in a space and deciding if their layout is "typical" (see [39, 20]). Computer scientists and engineers also use these graphs as theoretical models of wireless ad-hoc networks to better design communication protocols (see [28, 40, 22, 26, 37, 32, 41]). Random geometric graphs are also used frequently in percolation theory since they can be used as a continuous version of site percolation on the lattice.

We may think of the random geometric graph as a process where, given a random set of points, the radius that decides adjacency, r, starts at 0 and grows larger, so that one edge is in increasing order of length. In the 1999 paper "On k-Connectivity for a Geometric Random Graph" [33], Penrose proves an important result that states asymptotically almost surely (see Definition 2.1), the random geometric graph becomes k-connected at the same time that the minimum degree becomes k. In particular, in [33, 35] and more recently in [36], Penrose discusses the existence of isolated vertices (the case where $k = 1$) in random geometric graphs (and in the case of [36], *soft random geometric graphs*, a similar type of random graph where two vertices are adjacent with some probability if their distance is at most r; note that for random geometric graphs, this probability is 1).

Frequently, the metric space for the graph is taken to be the unit cube in $\mathbb{R}^d$ with general norm or ℓ_p norms; however, our focus will be on random geometric graphs on the d-dimensional unit torus with a distance given by a slight adjustment of any given norm on $\mathbb{R}^d$. Focusing on the torus allows us to avoid complexities created by the edges and corners of the cube, affecting some graph properties such as the minimum degree of the set of vertices.

Other important results may be found in books and surveys such as [30, 34, 42].

1.2 Bootstrap Percolation

Bootstrap Percolation is a model of the spread of information or disease within a network. It has been particularly seen as a model of a certain type of cellular automata. Let G be a graph and let $k \in \mathbb{N}$. Designate a subset of the vertices, $A_0 \subseteq V(G)$ (the *initial seed*) to begin as *active* (or infected). Then in each time step, a vertex becomes activated if, in the last time step, at least k of its neighbors were active. Once a vertex is active, it remains active forever. We want to know: if time goes on, will all of the graph's vertices become activated? If that happens, we say A_0 *percolates* G.

There are a few ways in which an initial seed A_0 may be chosen: (1) A_0 can be chosen in a non-random way where each vertex in A_0 is explicitly selected, or (2) A_0 can be chosen randomly. We can choose A_0 randomly by either randomly choosing a subset of vertices of a specific size or by choosing each vertex to be in A_0 independently with some probability p. These two methods of choosing A_0 randomly lead to similar models by choosing an appropriate size of A_0 or and appropriate p value. In the case of choosing A_0 of a certain size, one might be interested in minimal percolating sets. In the case of choosing each vertex in A_0 independently with probability p, one might be interested in finding threshold results with respect to p.

Bootstrap percolation was first introduced by Chalupa, Leath, and Reich in 1979 [16], analyzing the percolation model on a Bethe lattice as a simplified model of ferromagnetism. Since then, this percolation model has been studied on various random and deterministic families of graphs.

One of the most frequently studied graphs with respect to bootstrap percolation is the m-dimensional grid, denoted $[n]^m$. In [1], Aizenman and Lebowitz analyze the number of vertices on an *m-dimensional grid* that are eventually activated. In [25],

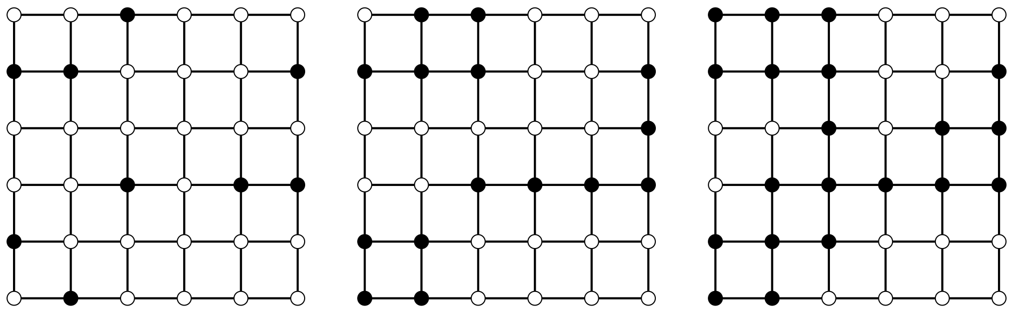


Figure 1.2: The first two time steps of the bootstrap percolation process on $[6]^2$ with $k = 2$, where filled vertices are active.

Holroyd found a percolation threshold on the 2-dimensional grid for $k = 2$, which was then improved by Gravner, Holroyd, and Morris in 2010 [21], and then further improved by Hartarsky and Morris in 2019 [23]. In [14], Cerf and Cirillo study finite size scaling on the 3 dimensional grid for $k = 3$. In [8], Balogh, Bollobás, and Morris give a percolation threshold for m-dimensional grids for fixed m and k. Further work on the m-dimensional grid can also be seen in [10, 15, 7].

Balogh, Peres, and Pete studied bootstrap percolation on infinite trees in 2006 [11], finding a threshold relating to the branching number of the tree. In the same paper, they also looked at bootstrap percolation on non-amenable Cayley graphs. In 2006, Balogh and Bollobás [6] found a sharp threshold on the n-dimensional hypercube. These results were then expanded in 2009 by Balogh, Bollobás, and Morris [9] to find results on a hypercube for *majority bootstrap percolation*, a related percolation model where vertices become activated if more than half of their neighbors are active. Sausset, Toninelli, Biroli, and Tarjus studied bootstrap percolation on hyperbolic lattices in 2009 [38].

In 2012, Janson, Łuczak, Turova, and Vallier gave a sharp threshold for bootstrap percolation on the *Erdős-Rényi graph* $\mathscr{G}(n, p)$, the graph with n vertices and each pair of vertices are adjacent independently with probability p. They analyzed how many steps the process takes before it stops and showed that asymptotically almost surely the barrier between full percolation and near percolation is the existence of vertices of degree less than the percolation parameter, k. A threshold for bootstrap percolation on a d-regular graph for $k < d$ was found in 2007 by Balogh and Pittel [12]. In 2010, Amini [4] looked at bootstrap percolation on graphs with given vertex degrees, where the degree of a vertex determines whether it is initially active and its own percolation parameter. Amini and Fountoulakis analyzed bootstrap percolation on power-law random graphs to find how large the initial seed must be for percolation to occur. In 2016, Candellero and Fountoulakis give a percolation threshold for hyperbolic random graphs. Amin Abdullah and Fountoulakis showed in 2014 [3] that there is a percolation threshold in the preferential attachment graph of the size of the initial seed that depends on the number of vertices in the graph.

In 2012, Bradonjić and Saniee [13] studied bootstrap percolation on random geometric graphs in the supercritical regime of connectivity (above the connectivity threshold) and where the percolation parameter k depends on n. In 2020, Koch and Lengler [29] analyze bootstrap percolation on geometric inhomogeneous random graphs with a power-law degree sequence, giving a threshold with respect to p and finding the amount of time it takes for a constant fraction of the vertices to become active.

In this dissertation, we study bootstrap percolation on the random geometric graph with a constant bootstrap percolation parameter, k, giving percolation and near percolation thresholds for p, the probability each vertex is initially active, for varied values of r, the upper bound of edge lengths in the random geometric graphs.

Chapter 2

Preliminaries

Bootstrap percolation on random geometric graphs introduces two sources of randomness, one from the random geometric graph and one from the choice of initially active set in the bootstrap percolation process. We will begin this chapter by formally describing the random geometric graph, continue by defining the bootstrap percolation process, and finally combining the two. We end this chapter with a list of probability results we will use throughout this dissertation.

We will frequently use the phrase "asymptotically almost surely" (abbreviated as a.a.s.).

Definition 2.1. A family of events $\{E_n\}_{n\in\mathbb{N}}$ happens a.a.s. if and only if $\mathbb{P}(E_n) \to 1$ as $n \to \infty$.

We wish to remind the readers of the following definitions:

- $f_n = O(g_n)$ as $n \to \infty$ if there exist constants $C > 0$ and $n_0 \in \mathbb{N}$ so that $|f_n| \leq C\,|g_n|$ for all $n \geq n_0$.
- $f_n = \Omega(g_n)$ as $n \to \infty$ if $g_n = O(f_n)$.
- $f_n = \Theta(g_n)$ as $n \to \infty$ if $f_n = O(g_n)$ and $f_n = \Omega(g_n)$.

- $f_n = o(g_n)$ (or $f_n \ll g_n$) as $n \to \infty$ if for every constant $\varepsilon_0 > 0$, there is an $n_0 \in \mathbb{N}$ so that $|f_n| \le \varepsilon_0 |g_n|$ for all $n \ge n_0$.

- $f_n = \omega(g_n)$ (or $f_n \gg g_n$) as $n \to \infty$ if for every constant $K > 0$, there is an $n_0 \in \mathbb{N}$ so that $|f_n| \ge K |g_n|$ for all $n \ge n_0$.

We will later consider $\varepsilon > 0$ to be a sufficiently small constant. We will use the notation O_ε, Ω_ε, and Θ_ε to express that the constants in the definitions of the O, Ω, and Θ notations depend on ε.

2.1 Random Geometric Graphs

We will consider our vertices to be placed in a metric space $\mathcal{S}$ and distance to be given by distance function $\operatorname{dist}(\cdot, \cdot)$. A *geometric graph* is a graph in which n vertices $v_1, v_2, \ldots, v_n$ are placed in $\mathcal{S}$ so that $v_i \sim v_j$ if and only if $\operatorname{dist}(v_i, v_j) \le r$ for a given distance r. We will consider $r = r(n)$, but will frequently hide the dependency on n. If we place the n vertices randomly, this graph is said to be a *random geometric graph.*

Let $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_n)$ be an n-tuple of points in $\mathcal{S}$. The version of the random geometric graph that we will be analyzing is on $\mathcal{S} = \mathcal{T}_d$, the d-dimensional unit torus, $[0,1)^d$, for a fixed $d \ge 2$. Since our space is the torus, we will adapt norms from $\mathbb{R}^d$ into distance functions on $\mathcal{T}_d$ in the following way:

Given any norm $\|\cdot\|$ defined on $\mathbb{R}^d$ and any two points $x, y \in \mathcal{T}_d$, define

$$\operatorname{dist}(x, y) = \inf\{\|x + z - y\| : z \in \mathbb{Z}^d\}. \tag{2.1}$$

This dist function defines a pseudometric on $\mathcal{T}_d$. We will see soon that there are only a finite amount of $z \in \mathbb{Z}^d$ that need to be considered, so we can change the inf to a min in the definition of dist. For a given norm $\|\cdot\|$ and for $\mathscr{A} \subseteq \mathcal{T}_d$, define

$$\operatorname{diam}(\mathscr{A}) := \sup\{\operatorname{dist}(x, y) : x, y \in \mathscr{A}\}. \tag{2.2}$$

In both cases of dist and diam, we may find it useful to specify the norm, and thus we will use notation such as dist_* and diam_* to signify we are using the norm $\|\cdot\|_*$ in the definition of dist_* and diam_*. We will specifically define

$$\operatorname{dist}_\infty(x, y) := \min\{\|x + z - y\|_\infty : z \in \mathbb{Z}^d\},$$

and

$$\operatorname{diam}_\infty(\mathscr{A}) := \sup\{\operatorname{dist}_\infty(x, y) : x, y \in \mathscr{A}\},$$

where $\|\cdot\|_\infty$ is the ℓ^∞ norm.

The following Lemma is a well-known fact that, for instance, can be found in Section 6.2 (p. 249) in Hoffman's book *Analysis in Euclidean Space* [24]:

Lemma 2.1. Equivalence of Norms

For any two norms $\|\cdot\|$ and $\|\cdot\|_*$ on $\mathbb{R}^d$, there exists a constant $\beta > 0$ so that for any point $x \in \mathbb{R}^d$,

$$\frac{1}{\beta}\|x\| \leq \|x\|_* \leq \beta\|x\|.$$

We may also switch the role of $\|\cdot\|$ and $\|\cdot\|_\infty$ for the same $\beta > 0$.

Remark 2.1. Using Lemma 2.1, for any norm on $\mathbb{R}^d$ and two points in $\mathcal{T}_d$, we may equivalently define dist as:

$$\operatorname{dist}(x, y) = \min\{\|x + z - y\| : z \in \mathbb{Z}^d\}. \tag{2.3}$$

To see that these definitions are indeed the same, let $\beta > 0$ be the constant so that for any $x \in \mathcal{T}_d$, $\frac{1}{\beta}\|x\|_\infty \leq \|x\| \leq \beta\|x\|_\infty$ and suppose $\|x - y + z\|_\infty > \beta^2$. Then we must have that $\|x - y + z\| > \frac{1}{\beta}\beta^2 = \beta$. But $\operatorname{dist}(x, y) \leq \|x - y\| \leq \beta\|x - y\|_\infty \leq \beta$ since $x, y \in [0, 1)^d$. Thus, we only need to consider $z \in \mathbb{Z}^d$ so that $\|x - y + z\| \leq \beta^2$, or equivalently (using the reverse triangle inequality and Lemma 2.1), $\|z\| \leq \beta^2 + \beta$, of which there are finitely many, so the infimum in equation (2.1) is actually a minimum. This also gives us that $\operatorname{dist}(\cdot, \cdot)$ on $\mathcal{T}_d$ is a metric and the topology on $\mathcal{T}_d$ that we consider is the quotient topology of the usual topology on $\mathbb{R}^d$.

Lemma 2.2. Consider the general norms $\|\cdot\|$ and $\|\cdot\|_*$, and $\operatorname{dist}(\cdot, \cdot)$, $\operatorname{dist}_*(\cdot, \cdot)$, $\operatorname{diam}(\cdot)$, and $\operatorname{diam}_*(\cdot)$ defined as in equations (2.2) and (2.3).

(i) **Strong Equivalence of Metrics**

There exists a $\beta > 0$ so that for all $x, y \in \mathcal{T}_d$,

$$\frac{1}{\beta}\operatorname{dist}_*(x, y) \leq \operatorname{dist}(x, y) \leq \beta\operatorname{dist}_*(x, y).$$

We may also switch the roles of dist and dist_* in this equation for the same $\beta > 0$.

(ii) There exists a $\beta > 0$ so that for a subset $\mathscr{A} \subseteq \mathcal{T}_d$,

$$\frac{1}{\beta}\operatorname{diam}_*(\mathscr{A}) \leq \operatorname{diam}(\mathscr{A}) \leq \beta\operatorname{diam}_*(\mathscr{A}).$$

Remark 2.2. We will particularly consider the case in Lemma 2.2 where $\|\cdot\|_* = \|\cdot\|_\infty$, which tells us:

(i) There exists a $\beta > 0$ so that for all $x, y \in \mathcal{T}_d$,

$$\frac{1}{\beta} \operatorname{dist}_\infty(x,y) \leq \operatorname{dist}(x,y) \leq \beta \operatorname{dist}_\infty(x,y).$$

(ii) There exists a $\beta > 0$ so that for a subset $\mathscr{A} \subseteq \mathcal{T}_d$,

$$\frac{1}{\beta} \operatorname{diam}_\infty(\mathscr{A}) \leq \operatorname{diam}(\mathscr{A}) \leq \beta \operatorname{diam}_\infty(\mathscr{A}).$$

Proof of Lemma 2.2.

(i) By Lemma 2.1, there exists a $\beta > 0$ so that for every $x \in \mathbb{R}^d$, $\frac{1}{\beta}\|x\|_* \leq \|x\| \leq \beta\|x\|_*$. Let $x, y \in \mathcal{T}_d$. There exists a z_0 so that $\|z_0\| \leq \beta^2 + \beta$ and $\operatorname{dist}(x,y) = \|x + z_0 - y\|$. By Lemma 2.1, we have that

$$\operatorname{dist}(x,y) = \|x + z_0 - y\| \geq \frac{1}{\beta}\|x + z_0 - y\|_* \geq \frac{1}{\beta} \operatorname{dist}_*(x,y).$$

By a symmetric argument, we have that

$$\operatorname{dist}_*(x,y) \geq \frac{1}{\beta} \operatorname{dist}(x,y).$$

Therefore,

$$\frac{1}{\beta} \operatorname{dist}_*(x,y) \leq \operatorname{dist}(x,y) \leq \beta \operatorname{dist}_*(x,y).$$

(ii) Let $\mathscr{A} \subseteq \mathcal{T}_d$ and $\mathscr{A}'$ the closure of $\mathscr{A}$ under the usual quotient topology, which coincides with the topology given by dist. If $\mathscr{A} = \mathscr{A}'$, then $\mathscr{A}$ is closed, and hence compact since $\mathcal{T}_d$ is compact. Thus $\mathscr{A} \times \mathscr{A}$ is also compact under the product topology. Note that $\operatorname{dist}(\cdot,\cdot)$ and $\operatorname{dist}_\infty(\cdot,\cdot)$ are continuous since they are metrics. So then the sets $\{\operatorname{dist}(x,y) : x, y \in \mathscr{A}\}$ and $\{\operatorname{dist}_*(x,y) : x, y \in$

$\mathscr{A}\}$ attain their supremum, which is thus the maximum. Thus $\operatorname{diam}(\mathscr{A}) = \max\{\operatorname{dist}(x,y) : x,y \in \mathscr{A}\}$ and $\operatorname{diam}_*(\mathscr{A}) = \max\{\operatorname{dist}(x,y) : x,y \in \mathscr{A}\}$. Then there must exist $x_1, y_1 \in \mathscr{A}$ so that $\operatorname{diam}(\mathscr{A}) = \operatorname{dist}(x_1, y_1)$. By Lemma 2.2(i), there exists a $\beta > 0$ so that for all $x, y \in \mathscr{A}$, $\frac{1}{\beta}\operatorname{dist}_*(x,y) \leq \operatorname{dist}(x,y) \leq \beta \operatorname{dist}_*(x,y)$. Thus, we have

$$\operatorname{diam}(\mathscr{A}) = \operatorname{dist}(x_1, y_1) \leq \beta \operatorname{dist}_*(x_1, y_1) \leq \beta \operatorname{diam}_*(\mathscr{A}).$$

Similarly, we also have that there exists $x_2, y_2 \in \mathscr{A}$ so that $\operatorname{diam}_*(\mathscr{A}) = \operatorname{dist}_*(x_2, y_2)$. Then we have

$$\operatorname{diam}_*(\mathscr{A}) = \operatorname{dist}_*(x_2, y_2) \leq \beta \operatorname{dist}(x_2, y_2) \leq \beta \operatorname{diam}(\mathscr{A}).$$

Therefore we have

$$\frac{1}{\beta}\operatorname{diam}_*(\mathscr{A}) \leq \operatorname{diam}(\mathscr{A}) \leq \beta \operatorname{diam}_*(\mathscr{A}).$$

If $\mathscr{A} \neq \mathscr{A}'$, then $\operatorname{diam}(\mathscr{A}) \leq \operatorname{diam}(\mathscr{A}')$ for any norm. Let $\varepsilon > 0$. Since $\mathscr{A}' \times \mathscr{A}'$ is compact, we know that there exists $x_0, y_0 \in \mathscr{A}'$ so that $\operatorname{diam}(\mathscr{A}') = \operatorname{dist}(x_0, y_0)$. Thus for every $x, y \in \mathscr{A}$, we must have that $\operatorname{dist}(x,y) \leq \operatorname{dist}(x_0, y_0)$ and there exists a sequence of elements in $\mathscr{A} \times \mathscr{A}$ that converge to (x_0, y_0). Thus,

$$\operatorname{diam}(\mathscr{A}) \geq \operatorname{diam}(\mathscr{A}') - \varepsilon.$$

Since this is true for every $\varepsilon > 0$ it must be that $\operatorname{diam}(\mathscr{A}) = \operatorname{diam}(\mathscr{A}')$. Therefore, the Lemma statement must also hold for $\mathscr{A}$.

$\square$

We will consider the *(uniform) random geometric graph* $\mathscr{G}(\boldsymbol{X}_n; r)$ to be the random geometric graph where we choose each X_i to be a point in $\mathcal{T}_d$ uniformly at random; we place n vertices in $\mathcal{T}_d$ by letting vertex $i \in V$ be at point X_i; and for any two vertices $i, j \in V$, $i \sim j$ if and only if $\operatorname{dist}(i, j) \leq r$. In other words, we are placing n vertices uniformly at random in $\mathcal{T}_d$ and connecting two vertices if they are within distance r of each other. Almost surely, the vertices are in general position, so no two vertices are at the same point, no $b+1$ vertices are on the same b-dimensional affine space, no $d+2$ vertices are on a single ball, and the distance between any two points is unique.

We consider the *random geometric graph process* to be the coupling $(\mathscr{G}(\boldsymbol{X}_n; r))_{r \geq 0}$, where $r \in [0, \infty)$ and each graph is on the same (random) vertex set V, with vertex positions given by $\boldsymbol{X}_n$. We may think of r as a time parameter, where we increase r from 0 so that we may add edges in increasing length, (a.a.s.) a single edge at a time. On the d-dimensional torus $[0, 1)^d$, which we call $\mathcal{T}_d$, $\mathscr{G}(\boldsymbol{X}_n; \operatorname{diam}(\mathcal{T}_d))$ is the complete graph (though the complete graph may be achieved with a smaller r).

We will be interested in the time in the process $(\mathscr{G}(\boldsymbol{X}_n; r))_{r \geq 0}$ in which the minimum degree of the graph is k:

$$r_{\delta \geq k} = \min\{r : \deg(i) \geq k \text{ for all } i \in V\}.$$

A graph is said to be *k-connected* if removing fewer than k vertices leaves the graph connected. We may then also define the time in the process $(\mathscr{G}(\boldsymbol{X}_n; r))_{r \geq 0}$ in which the graph becomes k-connected:

$$r_{k\text{-conn}} = \min\{r : \mathscr{G}(\boldsymbol{X}_n; r) \text{ is } k\text{-connected}\}.$$

In 1999, Penrose proved that a.a.s., the precise instant where a random geometric graph becomes k-connected is the same time that the minimum degree becomes k:

Theorem 2.1. [33] A.a.s., $r_{\delta \geq k} = r_{k\text{-conn}}$.

2.2 Bootstrap Percolation

Given a graph $\mathscr{G}(V, E)$ with vertex set V with $|V| = n$, edge set E, a subset of vertices $A_0 \subseteq V$, and a *bootstrap parameter* $k \in \mathbb{N}$, the *k-bootstrap percolation process*, $\mathfrak{B}_k(\mathscr{G}; A_0)$ is defined as follows: at each time step $t = 0, 1, 2, \ldots$, we define the set $A_t \subseteq V$, which we call *active vertices*. We call vertices in $V \setminus A_t$ *inactive* vertices. The vertices in A_0 are considered *initially active*. At time step $t + 1$, a vertex $v \in V \setminus A_t$ turns active (or *becomes activated*) if v has at least k neighbors in A_t, and we place v in A_{t+1}. Once a vertex is active, it remains active, so $A_t \subseteq A_{t+1}$ for every $t = 0, 1, 2, \ldots$. In other words, for $t \geq 0$,

$$A_{t+1} = A_t \cup \{v \in V \setminus A_t : v \text{ has at least } k \text{ neighbors in } A_t\}.$$

We set $A_\infty = \bigcup_{t=0}^{\infty} A_t$, so A_∞ is the set of all vertices that begin as initially active or eventually become activated in $\mathfrak{B}_k(\mathscr{G}; A_0)$.

We call a set of vertices $A \subseteq V$ *stable*[1] if each vertex in $V \setminus A$ has at most $k - 1$ neighbors in A. Once the k-bootstrap percolation process reaches a stable set of vertices, the process becomes stationary since no inactive vertex will become activated. Both $\emptyset$ and V are stable sets. We define $[A] := \bigcap_{\substack{A \subseteq A' \subseteq V \\ A' \text{ stable}}} A'$. Note that $[A]$ is stable and $[A]$ is the smallest stable set containing A. It is easy to check that in $\mathfrak{B}_k(\mathscr{G}; A_0)$, we have $A_\infty = [A_0]$.

[1]This definition of *stable* differs from the normal definition of a stable set of vertices in graph theory.

Given a graph $\mathscr{G}(V, E)$, $k \in \mathbb{N}$, and $A_0 \subseteq V$, we say that $\mathfrak{B}_k(\mathscr{G}; A_0)$ *percolates* (or achieves *percolation*) if $A_\infty = V$. If we consider a sequences of graphs $\mathscr{G}_n(V_n, E_n)$ where $|V_n| = n$, we say that $\mathfrak{B}_k(\mathscr{G}_n; A_{0,n})$ *nearly percolates* (or achieves *near percolation*) if $A_{\infty,n} = (1 - o(1))n$, or in other words, all but a vanishing fraction of vertices are eventually active.

Here, we will be choosing the vertices in A_0 randomly so that each vertex in V is included in A_0 independently with probability $p = p(n)$. We note that p will depend on n but we will hide this dependency in the notation. In order to emphasize that our A_0 is random, we will sometimes denote it with $A_0(p)$. Then $A_\infty(p) = [A_0(p)]$ is the final set of active vertices in $\mathfrak{B}_k(\mathscr{G}; A_0(p))$.

We may also think of this process in a slightly different way, where we add vertices to the set A_0 one at a time. In order to do this, we will consider the random vector $\boldsymbol{W}_n = (W_1, W_2, \ldots, W_n)$ where each W_i for $i \in [n]$ are i.i.d. uniform random variables taking values in $[0, 1]$. We then define the random set

$$A_0(\boldsymbol{W}_n; p) = \{v_i \in V : W_i \leq p\}.$$

Then for $0 \leq p \leq 1$, $A_0(\boldsymbol{W}_n; p)$ is distributed as $A_0(p)$. In addition, the process $(A_0(\boldsymbol{W}_n; p))_{0 \leq p \leq 1}$ adds one vertex at a time, since almost surely, $W_i \neq W_j$ for $i \neq j$. Then $A_\infty(\boldsymbol{W}_n; p) = [A_0(\boldsymbol{W}_n; p)]$ is distributed as $A_\infty(p)$.

2.3 Bootstrap Percolation on Random Geometric Graphs

We will now look at the k-bootstrap percolation process on a uniform random geometric graph, which is denoted by $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$ using the notation given in Sections 2.1 and 2.2 above. We will frequently hide arguments that do not differ from the normal definition of $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$, and write it as $\mathfrak{B}_k$. There are two sources of randomness here: (1) how we place the vertices (this is given by $\boldsymbol{X}_n$), and (2) which vertices begin as initially active (these are vertices in $A_0(p)$). We will be concerned with values of the parameters $r = r(n)$ and $p = p(n)$, both of which may depend on the number of vertices (n) in the random geometric graph. We will use the notation r and p with the assumption that both parameters depend on n. We will consider a sequence of processes $(\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p)))_{n\in\mathbb{N}}$ and analyze what happens as $n \to \infty$.

Little has been studied in the intersection of Bootstrap Percolation and Random Geometric Graphs. In the introduction of a 1997 paper [35], Penrose describes a situation similar to bootstrap percolation with $k = 1$ while analyzing random minimal spanning trees. In a 2012 paper [13], Bradonjić and Saniee consider $r = \sqrt{\frac{B\log n}{\pi n}}$ for $B > 1$, the supercritical regime of connectivity, and take k to depend on n, specifically for k to be logarithmic in n.

The following definition is found on page 18 of Janson, Łucsak, and Rucinski's book *Random graphs* [27], but is adapted to our situation:

Definition 2.2. Thresholds A function $p^*(n)$ is a *threshold* for a graph property $\mathcal{P}$ of $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$ if

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{P} \text{ holds for } \mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))) = \begin{cases} 0, & \text{if } \frac{p}{p^*} \to 0, \text{ and} \\ 1, & \text{if } \frac{p}{p^*} \to \infty, \end{cases}$$

or if the roles of 0 and 1 are reversed. A function $p^*(n)$ is a *sharp threshold* for a graph property $\mathcal{P}$ of $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{P} \text{ holds for } \mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))) = \begin{cases} 0, & \text{if } \frac{p}{p^*} \le 1 - \varepsilon, \text{ and} \\ 1, & \text{if } \frac{p}{p^*} \ge 1 + \varepsilon, \end{cases}$$

or if the roles of 0 and 1 are reversed. A function $p^*(n)$ that is a threshold, but not a sharp threshold, for property $\mathcal{P}$ of $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$ is said to be a *coarse threshold* for property $\mathcal{P}$ of $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$.

We will define the following:

$$\begin{aligned} p_{\text{nonstuck}} &= \min\{p \in [0,1] : A_1(p) \setminus A_0(p) \neq \emptyset\}, \\ p_{\text{perco}} &= \min\{p \in [0,1] : \mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p)) \text{ percolates}\}. \end{aligned}$$

In the proofs of the theorems from Chapter 3, as is common with Random Geometric Graphs, we will tesselate $\mathcal{T}_d$ into d-dimensional cubes called *cells* with side length Sr.

We say that a cell c is *active* if every vertex inside c is active. A cell c is called a *seed* if there are at least k initially active vertices within distance $(1 - \operatorname{diam}(\mathcal{T}_d)S)\, r$ from the center of c. A cell is called a *concentrated seed* if there are at least k initially active vertices inside of it. A vertex v is called *stuck* if it is not initially active and it does not have k or more initially active neighbors. Then a cell c is called *stuck* if all cells within distance $(1 + \operatorname{diam}(\mathcal{T}_d)S)\, r$ from the center of c have less than k initially active vertices total. This ball of radius $(1 + \operatorname{diam}(\mathcal{T}_d)S)\, r$ centered at the center of the cell c is called c's *ball of activation*. If there are at least k initially active vertices in c's ball of activation, then we say that c is *nonstuck*.

We will be finding the volume of d-dimensional balls frequently in this paper. For any norm, we will call the volume of the unit d-dimensional ball in that norm ξ. Then the volume of a d-dimensional ball of radius r is ξr^d.

For example, using the Euclidean norm the volume of the d-dimensional ball of radius r is $\dfrac{\sqrt{\pi^d}}{\Gamma\left(\frac{d}{2}+1\right)} r^d$, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ for $x > 0$. We would only be concerned with $\Gamma(x)$ where $2x \in \mathbb{N}$, and in this case, for $m \in \mathbb{N}$,

$$\Gamma(m) = (m-1)!, \quad \text{and}$$
$$\Gamma\left(m - \frac{1}{2}\right) = \left(m - \frac{3}{2}\right)\left(m - \frac{5}{2}\right)\cdots\frac{1}{2}\pi^{\frac{1}{2}}.$$

Then in this case, $\xi = \frac{\sqrt{\pi^d}}{\Gamma\left(\frac{d}{2}+1\right)}$.

2.4 Useful Probability Results

Lemma 2.3. Chernoff Bounds
Let $X = \sum_{i=1}^{n} X_i$, where $X_i \overset{d}{\sim}$ Bernoulli(p) with $\mu = \mathbb{E}X = pn$. Then for any $\varepsilon > 0$,

$$\mathbb{P}(X \geq (1+\varepsilon)\mu) \leq e^{-\varepsilon^2\mu/(2+\varepsilon)},$$

and for $0 < \varepsilon < 1$,

$$\mathbb{P}(X \leq (1-\varepsilon)\mu) \leq e^{-\varepsilon^2\mu/2}$$

and

$$\mathbb{P}(|X-\mu| \geq \varepsilon\mu) \leq 2e^{-\varepsilon^2\mu/3}.$$

These can be seen from Theorem 2.1, Corollary 2.2, and Corollary 2.3 (pages 27 and 28) in Janson, Łuczak, and Rucinski's book *Random Graphs* [27]. Chernoff Bounds will be used frequently in the proofs of the main results.

Lemma 2.4. Markov Inequality
Let $X \geq 0$ be a random variable and $t > 0$. Then $\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}X}{t}$. If in addition $\mathbb{E}X > 0$, then $\mathbb{P}(X > t) < \frac{\mathbb{E}X}{t}$.

This result can be found on page 8 of Janson, Łuczak, and Rucinski's book

Random Graphs [27]. The special case of this result that states $\mathbb{P}(X > 0) \leq \mathbb{E}X$ is referred to as the "First Moment Method". This Lemma and the next resulting Lemma will be used many times throughout this dissertation.

Lemma 2.5. Suppose $X \sim \text{Bin}(m, p)$ for $m \in \mathbb{N}$ and $p \in [0, 1]$. Then for any $k \in \mathbb{N}$ with $k \leq m$,

$$\sum_{j=k}^{m} \binom{m}{j} p^j (1-p)^{m-j} \leq \binom{m}{k} p^k.$$

Proof. Let $Y = \binom{X}{k}$. Then using Markov's Inequality (Lemma 2.4) for Y, we have

$$\sum_{j=k}^{m} \binom{m}{j} p^j (1-p)^{m-j} = \mathbb{P}(X \geq k) = \mathbb{P}(Y \geq 1) \leq \mathbb{E}Y = \mathbb{E}\binom{X}{k} = \binom{m}{k} p^k.$$

□

Lemma 2.6. Harris-Kleitman Inequality

Define $\mathscr{P}(n) = \{x : x \subseteq \{1, 2, \ldots, n\}\}$. If $A, B \subseteq \mathscr{P}(n)$ are increasing families and $C, D \subseteq \mathscr{P}(n)$ are decreasing families, then the following inequalities hold:

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(C \cap D) \geq \mathbb{P}(C)\mathbb{P}(D)$$

$$\mathbb{P}(A \cap C) \leq \mathbb{P}(A)\mathbb{P}(C).$$

The Harris-Kleitman Inequality is discussed in Chapter 6 of Alon and Spencer's book *The Probabilistic Method* [2]. This will be used in the proof of Theorem 3.5.

Chapter 3

Main Results

The first result states that for $\sqrt[d]{\dfrac{1}{n}} \ll r \ll 1$, a.a.s., there is a near percolation threshold for p at $p^* = \dfrac{1}{(n)^{1/k}(a)^{1-1/k}}$, where $a = \xi n r^d$, so a is the average degree.

Theorem 3.1. Suppose that each vertex inside the torus begins as activated independently with probability p. Consider $p^* = \frac{1}{(n)^{1/k}(a)^{1-1/k}}$ and $r = \sqrt[d]{\frac{a}{\xi n}}$. Then for $1 \ll a \leq n$ and $p/p^* \to 0$, a.a.s., no initially inactive vertex becomes activated and for $1 \ll a \ll n$ and $p/p^* \to \infty$, a.a.s., $\mathfrak{B}_k$ nearly percolates.

The second result shows that a.a.s., there is not a sharp near percolation threshold for p when $\sqrt[d]{\frac{1}{n}} \ll r \ll 1$. Theorems 3.1 and 3.2 imply that a.a.s., there is a coarse near percolation threshold for p.

Theorem 3.2. Suppose that each vertex inside the torus begins as activated with probability p independently. Consider $p^* = \frac{1}{(n)^{1/k}(a)^{1-1/k}}$ and $r = \sqrt[d]{\frac{a}{\xi n}}$. Then for $1 \ll a \ll n$ and $p = \gamma p^*$ with constant γ, a.a.s.

$$\alpha \leq \mathbb{P}(\mathfrak{B}_k \text{ is nonstuck}) \leq 1 - \alpha$$

where $\alpha = \alpha(\gamma)$ is a constant and $\alpha \in (0, 1)$.

The third result shows that for r past the threshold for k-connectivity, a.a.s., as soon as $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$ becomes nonstuck, it must percolate. When r is a bit less than the threshold for k-connectivity, then a.a.s., $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p_{\text{nonstuck}}))$ must percolate completely except for vertices of degree less than k that were not initially activated.

Theorem 3.3. Consider $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$.

(i) For $\sqrt[d]{\frac{1}{n}} \ll r \ll 1$, a.a.s., $\mathfrak{B}_k(A_0(p_{\text{nonstuck}}))$ nearly percolates.

(ii) Let $r_{\delta \geq k} \leq r \ll 1$. Then a.a.s. $p_{\text{nonstuck}}(r) = p_{\text{perco}}(r)$.

(iii) For $r' = \sqrt[d]{\frac{\log n + (k-1)\log\log n - \omega}{\xi n}}$, where $\omega \to \infty$ and $\omega = o(\log\log n)$, whenever $r' \leq r \ll 1$, a.a.s., $\mathfrak{B}_k(A_0(p_{\text{nonstuck}}))$ percolates completely except possibly for some subset of vertices with degree less than k.

In order to prove the above results, we will prove similar, more general results. Instead of activating every vertex in $\mathscr{G}(\boldsymbol{X}_n; r)$ independently with probability p, we will only activate vertices of $\mathscr{G}(\boldsymbol{X}_n; r)$ that are inside some d-dimensional cube $L \subseteq \mathcal{T}_d$ of side length $\ell = \ell(n)$, a restricted area of infection, independently with probability p. As with parameters r and p that depend on n, when we write ℓ, it is assumed that ℓ depends on n. We also note that the following results still hold when L is a ball of radius $\ell = \ell(n)$ or other nice shapes of volume on the order of ℓ^d. Since we are now only initially activating vertices that fall in L, we will denote the set of initially active vertices by $A_0(p; L)$.

Theorem 3.4 is the same as Theorem 3.1, but with the restricted area of infection L. Here, we consider $a = O(n\ell^d)$, which is equivalent to $r = O(\ell)$; and $a \ll n\ell^d$, which is equivalent to $r \ll \ell$.

Theorem 3.4. Let L (the restricted area of infection) denote a d-dimensional cube in the torus so that $\operatorname{diam}_\infty(L) = \ell(n)$. Suppose that each vertex inside L begins as activated independently with probability p. Consider $p^* = \frac{1}{(n\ell^d)^{1/k}(a)^{1-1/k}}$ and $r = \sqrt[d]{\frac{a}{\xi n}}$. Then for $1 \ll a \leq O(n\ell^d)$ and $p/p^* \to 0$, a.a.s., no initially inactive vertex becomes activated and for $1 \ll a \ll n\ell^d$ and $p/p^* \to \infty$, a.a.s., $\mathfrak{B}_k(A_0(p; L))$ nearly percolates.

Theorems 3.5 and 3.6 are the same as Theorems 3.2 and 3.3 respectively, but with the restricted area of infection L.

Theorem 3.5. Suppose that each vertex inside L (a cube so that $\operatorname{diam}_\infty(L) = \ell(n)$ inside the torus, the restricted area of infection) begins as activated with probability p independently. Consider $p^* = \frac{1}{(\ell^d n)^{1/k}(a)^{1-1/k}}$ and $r = \sqrt[d]{\frac{a}{\xi n}}$. Then for $1 \ll a \ll \ell^d n$ and $p = \gamma p^*$ with constant γ, a.a.s.

$$\alpha \leq \mathbb{P}(\mathfrak{B}_k(A_0(p; L)) \text{ is nonstuck}) \leq 1 - \alpha$$

where $\alpha = \alpha(\gamma)$ is a constant and $\alpha \in (0, 1)$.

Theorem 3.6. For the following two parts, consider a restricted area of infection L, a cube so that $\operatorname{diam}_\infty(L) = \ell(n)$ in the torus.

(i) For $\sqrt[d]{\frac{1}{n}} \ll r \ll \ell$, a.a.s., $\mathfrak{B}_k(A_0(p_{\text{nonstuck}}))$ nearly percolates.

(ii) Let $r_{\delta \geq k} < r \ll \ell$. Then a.a.s. $p_{\text{nonstuck}}(r) = p_{\text{perco}}(r)$.

(iii) For $r' = \sqrt[d]{\frac{\log n + (k-1)\log\log n - \omega}{\xi n}}$, where $\omega \to \infty$ and $\omega = o(\log\log n)$, whenever $r' \leq r \ll \ell$, a.a.s., $\mathfrak{B}_k(A_0(p_{\text{nonstuck}}; L))$ percolates completely except possibly for some subset of vertices with degree less than k.

Once we have proved Theorems 3.4, 3.5, and 3.6, then Theorems 3.1, 3.2, and 3.3 are immediate since we may take $\ell = 1$ (so $L = \mathcal{T}_d$).

Chapter 4

Proof of Theorem 3.4

Theorem 3.4. Let L (the restricted area of infection) denote a d-dimensional cube in the torus so that $\operatorname{diam}_\infty(L) = \ell(n)$. Suppose that each vertex inside L begins as activated independently with probability p. Consider $p^* = \frac{1}{(n\ell^d)^{1/k}(a)^{1-1/k}}$ and $r = \sqrt[d]{\frac{a}{\xi n}}$. Then for $1 \ll a \leq O(n\ell^d)$ and $p/p^* \to 0$, a.a.s., no initially inactive vertex becomes activated and for $1 \ll a \ll n\ell^d$ and $p/p^* \to \infty$, a.a.s., $\mathfrak{B}_k(A_0(p;L))$ nearly percolates.

We are going to tesselate the torus $\mathcal{T}_d$ into equally sized cells. If c is a cell, we will define the side length of the cell to be $s := \operatorname{diam}_\infty(c)$. We would like for all vertices in two topologically connected cells to be within distance r of each other (so they are adjacent in $\mathscr{G}(\boldsymbol{X}_n; r)$). For a cell c, let $\beta > 0$ be the value so that $\operatorname{diam}(c) \leq \beta \operatorname{diam}_\infty(c) = \beta s$ given by Lemma 2.2. Consider two topologically connected, equally sized cells c_1 and c_2. We would like

$$\max\{\operatorname{dist}(x,y) : x \in c_1, y \in c_2\} \leq 2\operatorname{diam}(c) \leq 2\beta \operatorname{diam}_\infty(c) = 2\beta s \leq r.$$

Thus, we may take[1] $s = \frac{r}{2\beta}$. So for $s = Sr$, we have $S = \frac{1}{2\beta}$. If m is the number

[1]We want $\frac{1}{s}$ to be an integer, so we really want to take $s = \frac{1}{\lceil 2\beta/r \rceil} \sim \frac{r}{2\beta}$, but we will omit roofs and ceilings for clarity.

of cells in $\mathcal{T}_d$, then $m = (2\beta)^d \xi \frac{n}{a}$. There are $m_L := \Theta(m\ell^d) = \Theta\left(\frac{(2\beta)^d \xi n \ell^d}{a}\right)$ cells intersecting L.

4.1 For $p \ll p^*$ and $1 \ll a \leq n\ell^d$

Consider $p \ll p^*$ and $1 \ll a \leq n\ell^d$. For any cell c, let Y_c be the number of vertices in c's ball of activation. Then $Y_c \sim \text{Bin}\left(n, \left(1 + \frac{\text{diam}(\mathcal{T}_d)}{2\beta}\right)^d \cdot \frac{a}{\xi n}\right)$. Call $\mu := \mathbb{E}Y_c = \left(1 + \frac{\text{diam}(\mathcal{T}_d)}{2\beta}\right)^d \cdot \frac{a}{\xi}$ the average number of vertices in a cell's ball of activation. Define a cell to be *typical* if it has at most 2μ vertices inside its ball of activation. If a cell is not typical, call it *atypical*. Let an atypical cell be called *atypical type i* if it has between $2^i\mu$ and $2^{i+1}\mu$ vertices inside its ball of activation. First note that the number of typical cells is at most m. Let $Y_{c,\text{act}}$ be the number of active vertices in cell c's ball of activation. Note that only cells whose balls of activation intersects L may be nonstuck. Then for a typical cell c and by Lemma 2.5, the probability that c is nonstuck is

$$\begin{aligned}
\mathbb{P}(Y_{c,\text{act}} \geq k | Y_c \leq 2\mu) &\leq \sum_{j=k}^{2\mu} \binom{2\mu}{j} p^j (1-p)^{2\mu - j} \leq \binom{2\mu}{k} p^k \sim \frac{(2\mu)^k}{k!} p^k \\
&= o\left(\frac{\left(\left(1 + \frac{\text{diam}(\mathcal{T}_d)}{2\beta}\right)^d \cdot \frac{2}{\xi}\right)^k (a)^k}{k!} \cdot \left(\frac{1}{(\ell^d n)^{1/k} (a)^{1-1/k}}\right)^k\right) \\
&= o\left(\frac{a}{\ell^d n}\right).
\end{aligned}$$

Thus, for Z_0 the number of typical cells that are nonstuck, we have that the expected number of typical cells that are nonstuck is

$$\mathbb{E}(Z_0) = \mathbb{P}(Y_{c,\text{act}} \geq k | Y_c \leq 2\mu) \cdot m_L \leq o\left(\frac{a}{n\ell^d}\right) \cdot \Theta\left(\frac{(2\beta)^d \xi n \ell^d}{a}\right) = o(1).$$

Thus, by the first moment method (Lemma 2.4), a.a.s., there are no typical cells that are nonstuck in L.

Define $T_i(c)$ to be the event that cell c is atypical type i and $T_{\text{ns}}(c)$ to be the event that cell c is nonstuck. By linearity of expectation, for $i \geq 1$, we have that if Z_i is the number of nonstuck atypical cells in L of type i, then for

$$Z_{i,c} = \begin{cases} 1, & \text{if cell } c \text{ is nonstuck and atypical type } i \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\mathbb{E}Z_i = \sum_{c \text{ cell in } L} \mathbb{E}Z_{i,c} = m_L \mathbb{P}(Z_{i,c} = 1) = m_L \mathbb{P}(T_i(c))\mathbb{P}(T_{\text{ns}}(c)|T_i(c)).$$

Note that

$$\mathbb{P}(T_i(c)) \leq \mathbb{P}(Y_c \geq 2^i \mu) \leq e^{\frac{-(2^i-1)^2 \mu}{3}}$$

by a Chernoff bound (Lemma 2.3), and by Lemma 2.5,

$$\mathbb{P}(T_{\text{ns}}(c)|T_i(c)) \leq \mathbb{P}(Y_{c,\text{act}} \geq k | Y_c = 2^{i+1}\mu) \leq \binom{2^{i+1}\mu}{k} p^k \leq (2^{i+1}\mu)^k p^k.$$

Thus, we have

$$\mathbb{P}(Z_{i,c} = 1) \leq e^{\frac{-(2^i-1)^2 \mu}{3}} (2^{i+1}\mu)^k p^k$$

and

$$\mathbb{E}Z_i \leq q_i := m_L e^{\frac{-(2^i-1)^2 \mu}{3}} (2^{i+1}\mu)^k p^k.$$

Then

$$\frac{q_{i+1}}{q_i} = e^{-\frac{\mu}{3}((2^{i+1}-1)^2 - (2^i-1)^2)} 2^k = o(1),$$

since $(2^{i+1}-1)^2-(2^i-1)^2>0$ for all $i\geq 1$ and $\mu\to\infty$. Define $Z_{i\geq 1}$ to be the number of nonstuck atypical cells. Then

$$\begin{aligned}
\mathbb{E}(Z_{i\geq 1}) &= \sum_{i=1}^{\infty}\mathbb{E}Z_i \leq \sum_{i=1}^{\infty} q_i \sim q_1\\
&= m_L e^{-\frac{\mu}{3}}(4\mu)^k p^k = \Theta\left(\frac{n}{\mu}\mu^k p^k \ell^d\right)e^{-\mu/3}\\
&= \Theta\left(\ell^d n\left(\left(1+\frac{\operatorname{diam}(\mathcal{T}_d)}{2\beta}\right)^d \frac{a}{\xi}\right)^{k-1} o((p^*)^k)\right)e^{-\mu/3}\\
&= \Theta\left(\ell^d n\,(a)^{k-1}\,o\left(\frac{1}{\ell^d n\,(a)^{k-1}}\right)\right)e^{-\mu/3}\\
&= o\left(e^{-\mu/3}\right) = o(1).
\end{aligned}$$

Then asymptotically almost surely (by Lemma 2.4) there are no nonstuck atypical cells in L. Thus, asymptotically almost surely, there are no nonstuck cells in L which implies all cells in L are stuck. So all cells must be stuck and thus, for $p\ll p^*$, a.a.s., $\mathfrak{B}_k$ does not almost percolate.

4.2 For $p\gg p^*$ and $1\ll a\ll n\ell^d$

Consider $p\gg p^*$ and $1\ll a\ll n\ell^d$. For any cell c, let W_c be the number of vertices in the cell. Then $W_c\sim \operatorname{Bin}\left(n,\frac{a}{(2\beta)^d\xi n}\right)$. Call $\nu:=\mathbb{E}W_c=\frac{a}{(2\beta)^d\xi}$, the average number of vertices in a cell. Define a cell to be *typical* if it has at least $\frac{\nu}{2}$ vertices inside the cell. If a cell has less than $\frac{\nu}{2}$ vertices, call it *atypical*. Then by a multiplicative Chernoff bound (Lemma 2.3), we have

$$\mathbb{P}\left(W_c\leq\frac{\nu}{2}\right)\leq e^{-\nu/8}=e^{\frac{-a}{8\cdot(2\beta)^d\xi}}.$$

Let Q_{atyp} be the number of atypical cells in $\mathcal{T}_d$. Then $\mathbb{E}(Q_{\text{atyp}}) \leq m \cdot e^{-a/(8\cdot(2\beta)^d\xi)}$. Then by Markov's Inequality (Lemma 2.4), we have that

$$\mathbb{P}(Q_{\text{atyp}} \geq m \cdot e^{-a/(16\cdot(2\beta)^d\xi)}) \leq e^{-a/(16\cdot(2\beta)^d\xi)} \to 0$$

as $n \to \infty$. Thus, a.a.s., the number of atypical cells is less than $m \cdot e^{-a/(16\cdot(2\beta)^d\xi)} = o(m)$.

Lemma 4.1. A.a.s. the largest topologically connected component of typical cells contains $(1 - o(1))m$ cells.

Proof. Note that the event that a cell is typical is not independent from other cells. To overcome that, we couple our model with a slightly different model involving a Poisson point process. More precisely, let $n_0 = n - n^{3/4}$ and N be a Poisson random variable with expectation n_0. In the Poisson model, we drop N points in the torus uniformly at random and independently. We define cells and typical cells as before. Note that in this model each cell receives a Poisson number of points (with expectation $n_0/m \sim \nu$) and each cell is typical independently of each other. Moreover the probability that a cell is typical is

$$\mathbb{P}(\text{Poisson}(n_0/m) \geq \nu/2) = 1 - o(1).$$

We will prove that the lemma holds for this Poisson model. We can extend it to the uniform model as follows. If $N \leq n$ (which happens with probability $1 - e^{-n^{1/2}} = 1 - o(1)$), then we can add $n - N$ points to the Poisson model chosen uniformly at random from the torus. This gives us the usual uniform model on n points. Since typical sets in the Poisson model are a subset of typical sets in this uniform model, if the event in the statement holds for the Poisson model then it also holds for the uniform model. Otherwise, if $N > n$, just resample n uniform points from the torus.

In that case the two models are not related, but this only happens with probability $o(1)$. To prove the result in the Poisson model, we can simply use some well-known percolation results. Let $m' = \sqrt[d]{m}$. We can identify the cells with points in the grid $[m']^d$ and declare a point open if the corresponding cell is typical and closed otherwise. We declare two points of $[m']^d$ adjacent if their ℓ_1 distance is 1 (i.e. the two points differ by 1 in one coordinate and are identical in all other coordinates). (Note that we are ignoring some adjacencies between cells and only considering adjacent those cells sharing a $d-1$-dimensional face. Also we are not considering adjacencies around the torus.) This is precisely the usual Bernoulli site percolation on $[m']^d$ with parameter $p = 1-o(1)$. By Thm 1.1 of [17] a.a.s. there is a unique giant component of open sites with all but a $o(1)$ fraction of the sites. This proves the statement for the Poisson model, since adding additional adjacencies between cells can only help. Also in view of the earlier coupling, the result also holds for the uniform model. $\square$

Let $M = \tau m$ for some $\tau \in (0,1)$. Then a.a.s., if X is the number of typical cells in the giant component, $X \geq M$. Let D be the giant component and let M_L be the number of cells in $D \cap L$ (so $M_L = \Theta(\tau m_L)$).

Let p be so that $p = \omega(p^*)$ but $p = o\left(\frac{1}{a}\right)$. Note that $a \ll n\ell^d$, so there are p's that satisfy this. For larger p, the result follows by monotonicity. Let $U_{c,\text{seed}}$ be the event that cell c is a concentrated seed. Then

$$\mathbb{P}\left(U_{c,\text{seed}} | W_c \geq \frac{\nu}{2}\right) \geq \binom{\nu/2}{k} p^k (1-p)^{\frac{\nu}{2}-k} \sim \frac{\nu^k}{2^k k!} p^k,$$

noting that since $p = o\left(\frac{1}{a}\right)$,

$$(1-p)^{\nu/2-k} = e^{-\nu p/2(1+O(p))} = e^{o(1)} \to 1.$$

Let $Q_{\text{typ,seed}}$ be the number of typical cells that are seeds in $D \cap L$ (the intersection of the giant component D and L). We have

$$\mathbb{P}(Q_{\text{typ,seed}} = 0) \leq \prod_{i=1}^{M_L} \left(1 - \mathbb{P}\left(U_{c,\text{seed}} | W_c \geq \frac{\nu}{2}\right)\right),$$

where we may bound the probability by restricting to M_L cells. We also note that for the set of cells in $D \cap L$, $\{c_i\}_{i=1,\ldots,M_L}$, the collection of events $\{U_{c_i,\text{seed}}\}_{i=1,\ldots,M_L}$ are independent of each other. Thus,

$$\begin{aligned}
\mathbb{P}(Q_{\text{typ,seed}} = 0) &\leq \prod_{i=1}^{M_L} \left(1 - \mathbb{P}\left(U_{c,\text{seed}} | W_c \geq \frac{\nu}{2}\right)\right) \\
&\leq \left(1 - (1 + o(1)) \frac{\nu^k}{2^k k!} p^k\right)^{M_L} \\
&\leq e^{-(1+o(1)) M_L \frac{\nu^k}{2^k k!} p^k}.
\end{aligned}$$

Note that

$$M_L \nu^k p^k = \Theta\left(\frac{(2\beta)^d \xi \tau \ell^d n}{a}\right) \cdot \left(\frac{a}{(2\beta)^d \xi}\right)^k p^k = \omega\left(\frac{\ell^d n a^k}{a} \cdot \frac{1}{\ell^d n a^{k-1}}\right) = \omega(1).$$

Therefore $\mathbb{P}(Q_{\text{typ,seed}}) \leq e^{-(1+o(1)) M_L \frac{\nu^k}{2^k k!} p^k} \to 0$. Hence, a.a.s., we will have a typical cell in $D \cap L$ that is a seed. Then, by the definition of the giant component, a.a.s., the giant component will become activated and thus, the graph nearly percolates. This completes the proof of Theorem 3.4.

Chapter 5

Proof of Theorem 3.5

Theorem 3.5. Suppose that each vertex inside L (a cube so that $\mathrm{diam}_\infty(L) = \ell(n)$ inside the torus, the restricted area of infection) begins as activated with probability p independently. Consider $p^* = \frac{1}{(\ell^d n)^{1/k}(a)^{1-1/k}}$ and $r = \sqrt[d]{\frac{a}{\xi n}}$. Then for $1 \ll a \ll \ell^d n$ and $p = \gamma p^*$ with constant γ, a.a.s.

$$\alpha \leq \mathbb{P}(\mathfrak{B}_k(A_0(p; L)) \text{ is nonstuck}) \leq 1 - \alpha$$

where $\alpha = \alpha(\gamma)$ is a constant and $\alpha \in (0, 1)$.

In the first part of this proof, we will show that there is a constant $\alpha_1 = \alpha_1(\gamma) \in (0, 1)$ so that $\mathbb{P}(\mathfrak{B}_k(A_0(p; L)) \text{ is nonstuck}) \geq \alpha_1$. In the second part, we will show that there is a constant $\alpha_2 = \alpha_2(\gamma) \in (0, 1)$ so that $\mathbb{P}(\mathfrak{B}_k(A_0(p; L)) \text{ is nonstuck}) \leq 1 - \alpha_2$. Then, if we let $\alpha = \min\{\alpha_1, \alpha_2\}$, the theorem is proved.

5.1 The Lower Bound

We are going to tesselate the torus $\mathcal{T}_d$ into equally sized cells. If c is a cell, we will define the side length of the cell to be $s := \mathrm{diam}_\infty(c)$. We would like for all vertices in two topologically connected cells to be within distance r of each other (so they are adjacent in $\mathscr{G}(\boldsymbol{X}_n; r)$). For a cell c, let $\beta > 0$ be the value so that $\operatorname{diam}(c) \le \beta \operatorname{diam}_\infty(c) = s$ given by Lemma 2.2. Consider two topologically connected, equally sized cells c_1 and c_2. We would like

$$\max\{\operatorname{dist}(x, y) : x \in c_1, y \in c_2\} \le 2 \operatorname{diam}(c) \le 2\beta \operatorname{diam}_\infty(c) = 2\beta s \le r.$$

Thus, we may take[1] $s = \frac{r}{2\beta}$. This gives us $S = \frac{1}{2\beta}$. If m_1 is the number of cells in $\mathcal{T}_d$, then $m_1 = (2\beta)^d \xi \frac{n}{a}$. We also have that there are $m_{1,L} := \Theta(m\ell^d) = \Theta\left(\frac{(2\beta)^d \xi n \ell^d}{a}\right)$ cells intersecting L. In addition, the volume of a cell is $\frac{a}{(2\beta)^d \xi n}$.

Let Y_c be the number of vertices in cell c. Then $Y_c \sim \operatorname{Bin}\left(n, \frac{a}{(2\beta)^2 \xi n}\right)$, so $\mathbb{E}Y_c = \frac{a}{(2\beta)^d \xi}$. Call a cell *typical* if it has between $\frac{1}{2}\mathbb{E}Y_c$ and $3\mathbb{E}Y_c$ vertices in it. Note that a multiplicative Chernoff bound (Lemma 2.3) gives us that

$$\begin{aligned}\Pr(\text{cell } c \text{ is atypical}) &= \Pr(Y_c \le \frac{1}{2}\mathbb{E}Y_c) + \Pr(Y_c \ge 3\mathbb{E}Y_c) \\ &\le e^{-\frac{a}{8(2\beta)^d \xi}} + e^{-\frac{a}{(2\beta)^d \xi}} \le 2e^{-\frac{a}{8(2\beta)^d \xi}}.\end{aligned}$$

Let Q_{atyp} be the number of atypical cells. Then

$$\mathbb{E}(Q_{\text{atyp}}) \le \frac{2(2\beta)^d \xi n}{a} e^{-\frac{a}{8(2\beta)^d \xi}} = o(m).$$

By Markov's Inequality, we have that

$$\mathbb{P}\left(Q_{\text{atyp}} \ge \frac{2(2\beta)^d \xi n}{a} e^{-\frac{a}{16(2\beta)^d \xi}}\right) \le \frac{\frac{2(2\beta)^d \xi n}{a} e^{-\frac{a}{8(2\beta)^d \xi}}}{\frac{2(2\beta)^d \xi n}{a} e^{-\frac{a}{16(2\beta)^d \xi}}} = e^{-\frac{a}{256\pi}} \to 0.$$

Thus, a.a.s., the number of atypical cells is at most $\frac{2(2\beta)^d \xi n}{a} e^{-\frac{a}{16(2\beta)^d \xi}}$, and therefore

[1] We again want $\frac{1}{s}$ to be an integer, so we really want to take $s = \frac{1}{\lceil 2\beta/r \rceil} \sim \frac{r}{2\beta}$, but we will omit roofs and ceilings for clarity.

the number of typical cells intersecting L is at least $m_{1,L}\left(1-2e^{-\frac{a}{16(2\beta)^d\xi}}\right)$, which is at least $0.9\frac{n\ell^d}{a}$ for sufficiently large n.

Let T_c denote the number of initially active vertices in cell c. Then for a cell c in L,

$$\begin{aligned}\mathbb{P}(T_c = k) &= \binom{\Theta(a)}{k} p^k (1-p)^{\Theta(a)-k} \\ &\sim \frac{\Theta(a^k)}{k!}\gamma^k (p^*)^k \\ &= \Theta\left(\frac{\gamma^k a}{k! n\ell^d}\right),\end{aligned}$$

since $(1-p)^{\Theta(a)-k} \to 1$ as $n \to \infty$. Let T_{seed} denote the number of concentrated seeds. Then since the probability that a cell is a concentrated seed is independent of whether other cells are concentrated seeds, we have

$$\mathbb{P}(T_{\text{seed}} = 0) \leq \prod_{c \text{ typical in L}} \mathbb{P}(T_c < k) \leq \left(1 - \Theta\left(\frac{\gamma^k a}{k! n\ell^d}\right)\right)^{0.9\frac{n\ell^d}{a}} \leq e^{-0.9\Theta\left(\frac{\gamma^k}{k!}\right)}.$$

By the definition of a typical cell, we know that the constants that appear in $\Theta\left(\frac{\gamma^k}{k!}\right)$ are bounded (both above and below). Note that if the graph has a concentrated seed, then $\mathfrak{B}_k$ is nonstuck. So

$$\mathbb{P}(\mathfrak{B}_k \text{ is nonstuck}) \geq 1 - \mathbb{P}(T_{\text{seed}} = 0) \geq 1 - e^{-.9\Theta\left(\frac{\gamma^k}{k!}\right)} =: \alpha_1(\gamma).$$

5.2 The Upper Bound

We are going to again tesselate the torus into cells of equal size. We would like for vertices in a cell c to only be adjacent to other vertices either in c or the cells topologically connected to c. In particular, we do not require two vertices in the same cell to be adjacent. We will take $s := \operatorname{diam}_\infty(c) = 2\beta r$, where $\beta > 0$ is the constant so that $\frac{1}{\beta}\operatorname{diam}_\infty(c) \le \operatorname{diam}(c) \le \beta \operatorname{diam}_\infty(c)$. Then $\operatorname{diam}(c) \ge 2r$ and $S = 2\beta$.

Define a *tile* to be a cube composed of 2^d cells (in a $\underbrace{2 \times 2 \times \cdot \times 2}_{d}$ arrangement). We will create 2^d different tilings created with these tiles where each will partition the torus in a different way. Note that each cell is in 2^d tiles and each of these 2^d tiles appears in a different tiling.

The volume of a tile is $2^d(2\beta r)^d = \frac{(4\beta)^d a}{\xi n}$ and there are $\frac{\xi n}{(4\beta)^d a}$ tiles in a single tiling. Let Y_t be the number of vertices in a tile t. Then $Y_t \sim \operatorname{Bin}\left(n, \frac{(4\beta)^d a}{\xi n}\right)$, so the average number of vertices in a tile is $\mu := \mathbb{E}Y_t = \frac{(4\beta)^d a}{\xi}$. Say a tile is of *type* i if $2^i\mu \le Y_t \le 2^{i+1}\mu$ for $i \ge 1$ and of *type* 0 if $Y_t \le 2\mu$. Call a tile *stuck* if it has less than k active vertices inside it. Call a tile *nonstuck* if it has at least k active vertices inside it.

Let E_j be the event that all tiles intersecting L are stuck for tiling j with $j = 1, 2, \ldots, 2^d$. We will focus on one tiling. Let L_j denote the tiles intersecting L in tiling j. Note that $|L_j| \le (\ell + 4\beta r)^d \frac{\xi n}{(4\beta)^d a}$. Let $X_{i,j}$ be the number of tiles in L_j of type i. Note that $X_{i,j}$ is at most the number of tiles in tiling j with $2^i\mu$ or more vertices. Define

$$X_{i,j,t} = \begin{cases} 1, & \text{tile } t \text{ in } L_j \text{ is type } i \\ 0, & \text{otherwise} \end{cases}.$$

Then using a multiplicative Chernoff bound, for $i \ge 1$,

$$\begin{aligned}
\mathbb{E}X_{i,j} &= \mathbb{E}\sum_{\text{tile } t} X_{i,j,t} = \sum_{\text{tile } t\in L_j} \mathbb{E}X_{i,j,t} = \sum_{\text{tile } t\in L_j} \mathbb{P}(2^i\mu \le Y_t \le 2^{i+1}\mu) \\
&\le \sum_{\text{tile } t\in L_j} \mathbb{P}(Y_t \ge 2^i\mu) \le \sum_{\text{tile } t\in L_j} e^{-\frac{(2^i-1)^2\mu}{2^i+1}} \le (\ell + 4\beta r)^d \frac{\xi n}{(4\beta)^d a} e^{-\frac{4^d(2^i-1)^2 a}{\xi(2^i+1)}}.
\end{aligned}$$

Set

$$b_i = \begin{cases} (\ell + 4\beta r)^d \frac{\xi n}{(4\beta)^d a}, & i = 0, \\ (\ell + 4\beta r)^d \frac{\xi n}{(4\beta)^d a} e^{-\frac{(4\beta)^d (2^i-1)^2 a}{2\xi(2^i+1)}}, & i \geq 1. \end{cases}$$

Then by a Markov inequality, for $i \geq 1$

$$\mathbb{P}(X_{i,j} > b_i) < \frac{\mathbb{E}X_{i,j}}{b_i} \leq \frac{(\ell + 4\beta r)^d \frac{\xi n}{(4\beta)^d a} e^{-\frac{(4\beta)^d (2^i-1)^2 a}{\xi(2^i+1)}}}{(\ell + 4\beta r)^d \frac{\xi n}{(4\beta)^d a} e^{-\frac{(4\beta)^d (2^i-1)^2 a}{2\xi(2^i+1)}}} = e^{-\frac{(4\beta)^d (2^i-1)^2 a}{2\xi(2^i+1)}}.$$

Note that $\mathbb{P}(X_{0,j} > b_0) = 0$ since b_0 is the number of tiles in L_j. Set

$$Z_{i,j} = \begin{cases} 1, & X_{i,j} > b_i \\ 0, & X_{i,j} \leq b_i \end{cases}$$

and let $Z = \sum_{j=1}^{2^d} \sum_{i=0}^{\infty} Z_{i,j}$. Note that

$$\mathbb{E}Z = \mathbb{E}\sum_{j=1}^{2^d}\sum_{i=0}^{\infty} Z_{i,j} = \sum_{j=1}^{2^d}\sum_{i=0}^{\infty} \mathbb{E}Z_{i,j} = \sum_{j=1}^{2^d}\sum_{i=0}^{\infty} \mathbb{P}(X_{i,j} > b_i) \leq \sum_{j=1}^{2^d}\left(0 + \sum_{i=1}^{\infty} e^{-\frac{(4\beta)^d (2^i-1)^2 a}{2\xi(2^i+1)}}\right).$$

Let $q_i := e^{-\frac{(4\beta)^d (2^i-1)^2 a}{2\xi(2^i+1)}}$ for $i \geq 1$. Note that $\frac{(2^{i+1}-1)^2}{(2^{i+1}+1)} - \frac{(2^i-1)^2}{(2^i+1)} > 0$, so

$$\frac{q_{i+1}}{q_i} = e^{-\frac{(4\beta)^d a}{2\xi}\left(\frac{(2^{i+1}-1)^2}{(2^{i+1}+1)} - \frac{(2^i-1)^2}{(2^i+1)}\right)} \to 0$$

as $n \to \infty$ since $a \to \infty$. So

$$\mathbb{E}Z \leq \sum_{j=1}^{2^d}\sum_{i=1}^{\infty} e^{-\frac{(4\beta)^d (2^i-1)^2 a}{2\xi(2^i+1)}} \sim \sum_{j=1}^{2^d} e^{-\frac{(4\beta)^d (2-1)^2 a}{2\xi(2+1)}} = \sum_{j=1}^{2^d} e^{-\frac{(4\beta)^d a}{6\xi}} = 2^d e^{-\frac{(4\beta)^d a}{6\xi}}.$$

Then

$$\mathbb{P}(Z \geq 1) \leq \mathbb{E}Z < 2^d e^{-\frac{(4\beta)^d a}{6\xi}} \to 0$$

as $n \to \infty$ since $a \to \infty$. Thus with high probability, for all $j = 1, 2, \ldots, 2^d$, for all $i \geq 0$,

$$X_{i,j} \leq b_i.$$

Consider $T_i(t)$ to be the event that tile t is of type i and T_{ns} to be the event that tile t is nonstuck. Using Lemma 2.5 and the upper bound on the number of vertices in a tile of type i, we see that

$$\begin{aligned}
\mathbb{P}(T_{\mathrm{ns}}(t)|T_i(t)) &\leq \sum_{q=k}^{2^{i+1}\mu} \binom{2^{i+1}\mu}{q} p^q (1-p)^{2^{i+1}\mu - q} \\
&\leq \binom{2^{i+1}\mu}{k} p^k \sim \frac{(2^{i+1}\mu)^k}{k!} (\gamma p^*)^k \\
&= \frac{1}{k!} \left(\frac{2^{i+1}(4\beta)^d \gamma}{\xi} \right)^k \frac{a}{n\ell^d}.
\end{aligned}$$

Note that the probability that a tile on the boundary of L has k or more active vertices is bounded above by the probability that a tile lying inside L has k or more active vertices, so we can treat all tiles as if they are inner tiles.

Define a *configuration* to be an assignment of vertices in the tiles of the tilings. Set

$$\mathscr{C} := \{C \text{ a configuration} : 0 \leq X_{i,j} \leq b_i\}.$$

Fix $C \in \mathscr{C}$. Note that $\frac{(\ell + 4\beta r)^d}{\ell^d} = 1 + o(1)$ since $a \ll n\ell^d$. Let $T_{\mathrm{s}}(t)$ be the event that tile t is stuck. Recall that E_j is the event that all tiles intersecting L are stuck for tiling j with $j = 1, 2, \ldots, 2^d$. Then a.a.s.,

$$
\begin{aligned}
\mathbb{P}(E_j|C) &= \prod_{i=0}^{\infty} \prod_{\substack{\text{tile } t \in L_j \\ \text{of type } i}} \mathbb{P}(T_{\mathrm{s}}(t)|T_i(t)) \\
&= \prod_{i=0}^{\infty} \prod_{\substack{\text{tile } t \in L_j \\ \text{of type } i}} (1 - \mathbb{P}(T_{\mathrm{ns}}(t)|T_i(t)) \\
&\geq \prod_{i=0}^{\infty} \prod_{\substack{\text{tile } t \in L_j \\ \text{of type } i}} \left(1 - \frac{1}{k!}\left(\frac{2^{i+1}(4\beta)^d\gamma}{\xi}\right)^k \frac{a}{n\ell^d}\right) \\
&\geq \prod_{i=0}^{\infty} \left(1 - \frac{1}{k!}\left(\frac{2^{i+1}(4\beta)^d\gamma}{\xi}\right)^k \frac{a}{n\ell^d}\right)^{b_i} \\
&= \prod_{i=0}^{\infty} \exp\left(-b_i\left(\frac{1}{k!}\left(\frac{2^{i+1}(4\beta)^d\gamma}{\xi}\right)^k \frac{a}{n\ell^d}\right)\left(1 + O\left((2^i)^k \frac{a}{n\ell^d}\right)\right)\right) \\
&= \exp\left(-\sum_{i=0}^{\infty} b_i\left(\frac{1}{k!}\left(\frac{2^{i+1}(4\beta)^d\gamma}{\xi}\right)^k \frac{a}{n\ell^d}\right)\left(1 + o(2^{ik})\right)\right) \\
&= \exp\left(-\sum_{i=0}^{\infty} \frac{\xi}{(4\beta)^d k!} \cdot \frac{(\ell + 4\beta r)^d}{\ell^d}\left(\frac{2^{i+1}(4\beta)^d\gamma}{\xi}\right)^k e^{-\frac{(4\beta)^d(2^i-1)^2 a}{2\xi(2^i+1)}}\left(1 + o(2^{ik})\right)\right) \\
&= \exp\left(-\frac{\xi}{(4\beta)^d k!} \cdot \frac{(\ell + 4\beta r)^d}{\ell^d}\left(\frac{2 \cdot (4\beta)^d\gamma}{\xi}\right)^k \left(\sum_{i=0}^{\infty} 2^{ik} e^{-\frac{(4\beta)^d(2^i-1)^2 a}{2\xi(2^i+1)}}\left(1 + o(2^{ik})\right)\right)\right) \\
&\sim \exp\left(-\frac{\xi}{(4\beta)^d k!}\left(\frac{2 \cdot (4\beta)^d\gamma}{\xi}\right)^k (1 + o(1))\right) \\
&\geq \exp\left(-\frac{2\xi}{(4\beta)^d k!}\left(\frac{2 \cdot (4\beta)^d\gamma}{\xi}\right)^k\right) =: (\alpha_2(\gamma))^{1/2^d}.
\end{aligned}
$$

Activate each vertex in L independently with probability p. Let

$$
x_i = \begin{cases} 1, & \text{vertex } i \text{ is initially active} \\ 0, & \text{vertex } i \text{ is initially inactive} \end{cases}
$$

and say that $\boldsymbol{x} := (x_i)_{i=1}^n \in E_j$ for each $j = 1, 2, \ldots, 2^d$ if E_j holds for the given vertex activation. Note that conditional on the configuration C, each E_j is decreasing in p since if $\boldsymbol{y} \subseteq \boldsymbol{x}$ (or in other words, the set of $\boldsymbol{y}$'s activated vertices are contained in the set of $\boldsymbol{x}$'s activated vertices), then $\boldsymbol{y} \in E_j$ as well. Then, by the Harris-Kleitman inequality (Lemma 2.6), we have that

$$\mathbb{P}\left(\bigcap_{j=1}^{2^d} E_j \middle| C\right) \geq \prod_{j=1}^{2^d} \mathbb{P}(E_j|C) = \left((\alpha_2(c))^{1/2^d}\right)^{2^d} = \alpha_2(c).$$

Let $E := \bigcap_{j=1}^{2^d} E_j$. Earlier we proved $\mathbb{P}(\text{having a configuration in } \mathscr{C}) = 1 - o(1)$. Then for $C \in \mathscr{C}$,

$$\begin{aligned} \mathbb{P}(E|C) &\geq \alpha_2(c) \\ \mathbb{P}(E \cap C) &\geq \alpha_2(c)\mathbb{P}(C) \\ \sum_{C \in \mathscr{C}} \mathbb{P}(E \cap C) &\geq \alpha_2(c) \sum_{C \in \mathscr{C}} \mathbb{P}(C) \\ \mathbb{P}\left(\bigcup_{C \in \mathscr{C}} (E \cap C)\right) = \mathbb{P}(E \cap \mathscr{C}) &\geq \alpha_2(c)\mathbb{P}(\mathscr{C}) \end{aligned}$$

So,

$$\mathbb{P}(E) \geq \mathbb{P}(E \cap \mathscr{C}) \geq \alpha_2(c)\mathbb{P}(\mathscr{C})$$

and hence $\mathbb{P}(E) \geq \alpha_2(c)$. Then since a.a.s.,

$$\mathbb{P}(\mathfrak{B}_k \text{ is stuck}) \geq \mathbb{P}(E),$$

we have that

$$\mathbb{P}(\mathfrak{B}_k \text{ is nonstuck}) \leq 1 - \alpha_2(c).$$

This completes the proof of Theorem 3.5.

Chapter 6

Proof of Theorem 3.6

Theorem 3.6. For the following two parts, consider a restricted area of infection L, a cube so that $\operatorname{diam}_\infty(L) = \ell(n)$ in the torus.

(i) For $\sqrt[d]{\frac{1}{n}} \ll r \ll \ell$, a.a.s., $\mathfrak{B}_k(A_0(p_{\text{nonstuck}}))$ nearly percolates.

(ii) Let $r_{\delta \geq k} < r \ll \ell$. Then a.a.s. $p_{\text{nonstuck}}(r) = p_{\text{perco}}(r)$.

(iii) For $r' = \sqrt[d]{\frac{\log n + (k-1)\log\log n - \omega}{\xi n}}$, where $\omega \to \infty$ and $\omega = o(\log\log n)$, whenever $r' \leq r \ll \ell$, a.a.s., $\mathfrak{B}_k(A_0(p_{\text{nonstuck}}; L))$ percolates completely except possibly for some subset of vertices with degree less than k.

In the following proof parts, we will be tessellating the torus $\mathcal{T}_d$ into cubic cells of equal size where for c a cell, $\operatorname{diam}_\infty(c) = Sr$ for a given $S > 0$, which will be taken to be small. We then define the following terms:

A cell is called *dense* if it has at least k vertices inside it. Otherwise, it will be called *sparse*. We will then create a *graph of cells* denoted $\mathscr{G}_{\text{cells}}(\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p)))$ (or simply $\mathscr{G}_{\text{cells}}$), where the vertices

$$V_{\text{cells}} = \{c : c \text{ is a cell of } \mathscr{G}(\boldsymbol{X}_n; r) \text{ in the tessellation of } \mathcal{T}_d\}$$

and cells $c_1, c_2 \in V_{\text{cells}}$ are adjacent if the distance between the center of c_1 and the center of c_2 is at most $\left(1 - \operatorname{diam}([0,1]^d)S\right) r$. Then we also create a *graph of dense cells*, denoted $\mathscr{G}_{\text{dense cells}}(\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p)))$ (or simply $\mathscr{G}_{\text{dense cells}}$), the subgraph of $\mathscr{G}_{\text{cells}}$ induced by the dense cells. More precisely, we have the vertices are

$$V_{\text{dense cells}} = \{c : c \text{ is a dense cell of } \mathscr{G}(\boldsymbol{X}_n; r) \text{ in the tessellation of } \mathcal{T}_d\}$$

and for $c_1, c_2 \in V_{\text{dense cells}} \subseteq V_{\text{cells}}$, c_1 and c_2 are adjacent if $c_1 \sim_{\mathscr{G}_{\text{cells}}} c_2$.

We will take $S = \varepsilon$, where $\varepsilon > 0$ is a sufficiently small constant. Then $\operatorname{diam}_\infty(\text{cell}) = \varepsilon r$ and thus by Remark 2.2, we know that there exists a $\beta > 0$ so that $\frac{\varepsilon r}{\beta} \leq \operatorname{diam}(\text{cell}) \leq \beta \varepsilon r$.

We will use the following definition mostly in parts (b) and (c):

Definition 6.1. Let D_1 be the largest component of dense cells in the graph of dense cells. Define cells in the graph of cells whose corresponding cell in the graph of dense cells is in D_1 to be *good*. If a cell is not good, but is sparse and adjacent to a good cell in the graph of cells, then that cell is called *bad*. If a cell is not adjacent to a good cell in the graph of cells, it is called *ugly*. Note that an ugly cell can be sparse or dense.

Lemma 6.1.

(i) Suppose that $r \gg \sqrt[d]{\frac{1}{n}}$. Then a.a.s. the largest component of the subgraph of the graph of cells induced by dense cells contains $(1 - o(1))m$ cells (in other words, all but a vanishing fraction of the cells are good).

(ii) Suppose that $r \geq r'$. Then a.a.s. all the components of the graph of cells induced by ugly cells must have diameter at most $Q\epsilon r$ (where $Q > 0$ is a constant depending only on the dimension d and the norm $\|\cdot\|$), and any two such components must be at distance at least Ar from each other (where constant $A > 0$ can be chosen arbitrarily large).

(iii) Suppose that $r \geq B\sqrt[d]{\frac{\log n}{\xi n}}$ for some constant $B > 1$. Then, for sufficiently small $\epsilon > 0$, a.a.s. the dense cells of the graph of cells induce a connected component, and every sparse cell is adjacent to some dense cell (that is, all dense cells are good and all sparse cells are bad, but there are no ugly cells).

Proof. The proof of part (i) is identical to that of Lemma 4.1, but replacing the role of typical cells by that of dense cells. Note that the only facts that we used about typical cells were that a cell is typical with probability $1 - o(1)$, and also that typicality cannot be destroyed by adding additional points. Dense cells satisfy these two properties, and thus the argument follows with virtually no changes.

Parts (ii) and (iii) are proved in [31] for ℓ_p norms, but the proof for general norms is the same. A proof of part (iii) for the 2-dimensional case appeared in [18]. $\square$

We also note that we prove this theorem in three parts: (a), (b), and (c), but these parts do NOT correspond to parts (i), (ii), and (iii) in Theorem 3.6. In part (a), we prove that if $B\sqrt[d]{\frac{\log n}{\xi n}} \leq r \ll \ell$, with $B > 1$, then a.a.s., $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}}$. In part (b), we show that if $\sqrt[d]{\frac{1}{n}} \ll r \leq \sqrt[d]{\frac{1.1\log n}{\xi n}}$, then a.a.s., $\mathfrak{B}_k(A_0(\hat{p}_{\text{nonstuck}}))$ nearly percolates. In part (c), we use results from (b) to prove that if $r' \leq r \leq \sqrt[d]{\frac{1.1\log n}{\xi n}}$, then a.a.s., the only surviving inactive vertices are vertices of degree less than k that are initially inactive. Then parts (a) and (b) together give us (i), (a) and (c) together give us (ii) and (iii).

We finally mention that parts (a) and (b) are broken into subsections. While the definitions given in (a) and (b) are contained within each section, they will span throughout their subsections.

6.1 Proof of Theorem 3.6(a)

Suppose $r \geq B\sqrt[d]{\frac{\log n}{\xi n}}$ for $B > 1$ and set $r = \sqrt[d]{\frac{a}{\xi n}}$, and thus the average degree $a = \xi n r^d$. Then $a \geq B^d \log n > \log n$. Note that we also have $a \leq o(n)$ and $a \ll n\ell^d$.

Let $p^* = \frac{1}{(n\ell^d)^{1/k}(a)^{1-1/k}}$, the threshold from Theorem 3.4. Let $\varepsilon > 0$ be sufficiently small, and in particular, small enough so that $\varepsilon < \frac{B-1}{B\,\mathrm{diam}([0,1]^d)}$. Suppose $p = \gamma p^*$ for $\gamma = \varepsilon^{-1/2k}$. This value of p will "typically" be past $\hat{p}_{\text{nonstuck}}$, since γ will be very large. We will tesselate the unit torus into cubic cells of equal size so that for a cell c, $s := \mathrm{diam}_\infty(c) = \varepsilon r$. So $S = \varepsilon$. Note that the volume of each cell c is $\varepsilon^d r^d = \frac{\varepsilon^d a}{\xi n}$. Thus, there are $m := \frac{1}{\varepsilon^d r^d} = \frac{\xi n}{\varepsilon^d a}$ cells in $\mathcal{T}_d$. As a reminder, we will use Θ_ε or O_ε to tell us the constants in Θ or O depend on ε.

We will define "bad" events $E_0, E_{1,p} \ldots, E_{5,p}$ and show that $\mathbb{P}\left(E_0 \cup (\cup_{i=1}^5 E_{i,p})\right)$ is small and that if none of $E_0, E_{1,p}, \ldots, E_{5,p}$ happen, then $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}}$. We define $E_0, E_{1,p}, \ldots, E_{5,p}$ as:

$E_0 =$ event that some cell is not adjacent to a dense cell or $\mathscr{G}_{\text{dense cells}}$ is not connected

$E_{1,p} =$ event that there is no seed box intersecting L at time p

$E_{2,p} =$ event that there is some wishy-washy box intersecting L at time p

$E_{3,p} =$ event that some superbox intersecting L has more than k active vertices at time p

$E_{4,p} =$ event that some superbox intersecting L with an atypical cell has exactly k active vertices at time p

$E_{5,p} =$ event that some superbox with no atypical cell intersecting the boundary of L has at least k active vertices at time p

where typical, box, seed box, wishy-washy box, and superbox are defined in their first respective subsections below.

6.1.1 $\mathbb{P}(E_0) = o(1)$

Note that E_0 is the event that some cell is not adjacent to a dense cell or $\mathscr{G}_{\text{dense cells}}$ is not connected. If every cell is adjacent to a dense cell and the graph of dense cells is connected, then if one cell's vertices all become active, then percolation will be achieved. Since $\varepsilon < \frac{B-1}{B\,\mathrm{diam}([0,1]^d)}$, we have that $\left(B - \varepsilon B\,\mathrm{diam}([0,1]^d)\right)^d - 1 > 0$. Let $\zeta > 0$ be so that $\zeta \leq \left(B - \varepsilon B\,\mathrm{diam}([0,1]^d)\right)^d - 1$. Define event T_1 to be the event that there exists a topologically connected set of at least $(1+\zeta)\frac{\xi}{\varepsilon^d B^d}$ sparse cells. Consider $b \geq (1+\zeta)\frac{\xi}{\varepsilon^d B^d}$ with $b \leq \left(\frac{1}{\varepsilon^d} - \mathrm{diam}([0,1]^d)\right)^d \xi$. Let J be the number of topologically connected sets of b cells (call them $c_1, c_2, \ldots, c_b$) that are all sparse. Further, define S_i to be the event that cell c_i is sparse. Then

$$
\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{b} S_i\right) &= \prod_{i=1}^{b} \mathbb{P}\left(S_i \middle| \bigcap_{j=1}^{i-1} S_j\right) \leq \prod_{i=1}^{b} \mathbb{P}(S_i) = \prod_{i=1}^{b} \sum_{j=0}^{k-1} \binom{n}{j} (\varepsilon^d r^d)^j (1-\varepsilon^d r^d)^{n-j} \\
&= \prod_{i=1}^{b} \sum_{j=0}^{k-1} O\left(n^j \varepsilon^{dj} \frac{a^j}{n^j}\right)(1-\varepsilon^d r^d)^{n-j} \leq \prod_{i=1}^{b} \sum_{j=0}^{k-1} O\left(\varepsilon^{dj} a^j\right) e^{-\varepsilon^d r^d n} \\
&\leq \prod_{i=1}^{b} \sum_{j=0}^{k-1} O\left(\varepsilon^{dj} a^j\right) e^{-\varepsilon^d B^d \frac{\log n}{\xi}} = \prod_{i=1}^{b} \sum_{j=0}^{k-1} O\left(\varepsilon^{dj} a^j\right) n^{-\frac{\varepsilon^d B^d}{\xi}} \\
&\leq \prod_{i=1}^{b} O\left(a^k n^{-\frac{\varepsilon^d B^d}{\xi}}\right) \leq O\left(\left(\frac{a^k}{n^{\frac{\varepsilon^d B^d}{\xi}}}\right)^{(1+\zeta)\frac{\xi}{\varepsilon^d B^d}}\right) \\
&= O\left(\frac{a^{\frac{k(1+\zeta)\xi}{\varepsilon^d B^d}}}{n^{(1+\zeta)}}\right).
\end{aligned}
$$

We would like to bound the number of topologically connected sets of b cells. For each of these connected sets, we wish to designate a specific cell c'. We will choose c' to be the cell with a point whose 1^{st} coordinate is closest to 0. If such a unique cell does not exist, call the set of these cells Ξ_1. Then for each $i > 2$, define Ξ_i to be the cells in Ξ_{i-1} that contain a point whose i^{th} coordinate is closest to 0. If $|\Xi_i| = 1$, then let that cell be c', otherwise continue the process. Note that there will have to be such a designated cell c' at or before step $i = d$.

There are $\Theta\left(\frac{\xi n}{\varepsilon^d a}\right)$ ways we can choose c' and there are $\Theta\left(\frac{1}{\varepsilon^d}\right)$ ways to choose each of the other cells. Thus, there are $\Theta\left(\frac{\xi n}{\varepsilon^d a} \cdot \left(\frac{1}{\varepsilon^d}\right)^b\right) = \Theta_\varepsilon\left(\frac{n}{a}\right)$ of these topologically connected sets of b cells. Then

$$\mathbb{E}J = O\left(\frac{a^{\frac{k(1+\zeta)\xi}{\varepsilon^d B^d}}}{n^{(1+\zeta)}}\right) \cdot \Theta_\varepsilon\left(\frac{n}{a}\right) = O_\varepsilon\left(\frac{a^{\frac{k(1+\zeta)\xi}{\varepsilon^d} B^d - 1}}{n^\zeta}\right) = o(1).$$

Then by Lemma 2.4 we have $\mathbb{P}(T_1) = \mathbb{P}(J \geq 1) = o(1)$. Thus, a.a.s., there are no topologically connected sets of at least $(1+\zeta)\frac{\xi}{\varepsilon^d B^d}$ cells that are all sparse.

Next, let T_2 be the event that there is a cell that is not adjacent to a dense cell. Let c be a cell. The number of cells in a ball of radius $(1 - \mathrm{diam}([0,1]^d)\varepsilon)r$ is

$$\frac{(1 - \varepsilon\,\mathrm{diam}([0,1]^d))^d r^d \xi}{\varepsilon^d r^d} = \left(\frac{1}{\varepsilon} - \mathrm{diam}([0,1]^d)\right)^d \xi \geq (1+\zeta)\frac{\xi}{\varepsilon^d B^d}.$$

Note that the number of cells c is adjacent to in $\mathscr{G}_{\text{cells}}$ is at least b, so $\mathbb{P}\left(T_2|\overline{T_1}\right) = 0$. Then

$$\mathbb{P}(T_2) = \mathbb{P}(T_2|T_1)\mathbb{P}(T_1) + \mathbb{P}\left(T_2|\overline{T_1}\right)\mathbb{P}\left(\overline{T_1}\right) \leq \mathbb{P}(T_1) = o(1).$$

Thus, a.a.s., for every cell c, there must be a dense cell that is adjacent to c.

Let T_3 be the event that $\mathscr{G}_{\text{dense cells}}$ is not connected. Recall:

Lemma 6.1. (iii) Suppose that $r \geq B\sqrt[d]{\frac{\log n}{\xi n}}$ for some constant $B > 1$. Then, for sufficiently small $\epsilon > 0$, a.a.s. the dense cells of the graph of cells induce a connected component, and every sparse cell is adjacent to some dense cell (that is, all dense cells are good and all sparse cells are bad, but there are no ugly cells).

Then by Lemma 6.1 (iii), $\mathbb{P}(T_3) = o(1)$.

Then

$$\mathbb{P}(E_0) = \mathbb{P}(T_2 \cup T_3) \leq \mathbb{P}(T_2) + \mathbb{P}(T_3) = o(1).$$

Therefore, a.a.s., all cells are adjacent to a dense cell in $\mathscr{G}_{\text{cells}}$ and $\mathscr{G}_{\text{dense cells}}$ is connected.

6.1.2 $\mathbb{P}(E_{1,p}) \leq e^{\Theta(1/\sqrt{\varepsilon})} + o(1)$

Event $E_{1,p}$ is the event that there is no seed box intersecting L at time p. Let Y_c be the number of vertices in a cell c. Note that $\mathbb{E}Y_c = \frac{\varepsilon^d}{\xi}a$. Define a cell to be *typical* if it has between $\frac{\varepsilon^d}{2\xi}a$ and $\frac{2\varepsilon^d}{\xi}a$ vertices. Otherwise, call the cell *atypical*. Then by Lemma 2.3, the probability that a cell c is atypical is

$$\begin{aligned}\mathbb{P}\left(Y_c \leq \frac{\varepsilon^d}{2\xi}a\right) + \mathbb{P}\left(Y_c \geq \frac{2\varepsilon^d}{\xi}a\right) &\leq e^{-\frac{\varepsilon^d a}{8\xi}} + e^{-\frac{\varepsilon^d a}{3\xi}} \\ &\leq 2e^{-\frac{\varepsilon^d a}{8\xi}} \leq 2e^{-\frac{\varepsilon^d \log n}{8\xi}} = 2n^{-\frac{\varepsilon^d}{8\xi}}.\end{aligned}$$

Then for Y_{atyp} the number of atypical cells,

$$\mathbb{E}Y_{\text{atyp}} \leq \frac{1}{\varepsilon^d r^d}\left(2n^{-\frac{\varepsilon^d}{8\xi}}\right) = 2\frac{n^{1-\frac{\varepsilon^d}{8\xi}}}{\frac{\varepsilon^d}{\xi}a} \leq 2\frac{n^{1-\frac{\varepsilon^d}{8\xi}}}{\frac{\varepsilon^d}{\xi}\log n}.$$

Define an *α-box* to be a cube with $\text{diam}_\infty(\alpha\text{-box}) = \alpha r$ so that each cell is either in the box or not in the box and the α-boxes partition the cells. We will consider $\text{diam}(\mathcal{T}_d)$-boxes. We will call a $\text{diam}(\mathcal{T}_d)$-box a *seed box* if it contains exactly k initially active points and all its cells are typical. Thus, if there exists a seed box intersecting L, all of the vertices in that box will become active.

Define event F to be the event that all except at most a $o(1)$ fraction of $\text{diam}(\mathcal{T}_d)$-

boxes have all its cells typical. Let Y be the number of diam($\mathcal{T}_d$)-boxes that do not have all its cells typical. Then

$$\mathbb{E}Y \leq \mathbb{E}Y_{\text{atyp}} \leq 2\frac{n^{1-\frac{\varepsilon^d}{8\xi}}}{\frac{\varepsilon^d}{\xi}a}.$$

Using Markov's Inequality (Lemma 2.4), we see that

$$\mathbb{P}\left(Y > \sqrt{\frac{2}{n^{\frac{\varepsilon^d}{8\xi}}}} \cdot \frac{n}{\frac{\varepsilon^d}{\xi}a}\right) \leq \sqrt{\frac{2}{n^{\frac{\varepsilon^d}{8\xi}}}} \to 0$$

as $n \to \infty$. So a.a.s.,

$$Y \leq \sqrt{\frac{2}{n^{\frac{\varepsilon^d}{8\xi}}}} \cdot \frac{n}{\frac{\varepsilon^d}{\xi}a} = o\left(\frac{n}{a}\right) = o\left(\#\text{ of } \operatorname{diam}(\mathcal{T}_d)\text{-boxes}\right).$$

Thus, a.a.s., F holds. So there are $\Theta\left(\frac{n\ell^d}{a}\right)$ diam($\mathcal{T}_d$)-boxes with all its cells typical. Let $Y_{\mathscr{b},\text{seed}}$ be the event that a diam($\mathcal{T}_d$)-box $\mathscr{b}$ with all its cells typical is a seed. Then $\mathbb{P}(Y_{\mathscr{b},\text{seed}}) \leq \Theta(p^k a^k)$.

Then since $pa \to 0$,

$$\begin{aligned}\mathbb{P}(E_{1,p}|F) = \mathbb{P}(\text{no seed box}|F) &\leq (1-\Theta((pa)^k))^{\Theta(n\ell^d/a)} = \left(1-\Theta\left(\gamma^k\frac{a}{n\ell^d}\right)\right)^{\Theta(n\ell^d/a)} \\ &\leq e^{-\Theta(\gamma^k)} = e^{-\Theta(1/\sqrt{\varepsilon})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(E_{1,p}) &= \mathbb{P}(E_{1,p}|F)\mathbb{P}(F) + \mathbb{P}(E_{1,p}|\overline{F}) + \mathbb{P}(\overline{F}) \\ &\leq \mathbb{P}(E_{1,p}|F) + \mathbb{P}(\overline{F}) \\ &\leq e^{-\Theta(1/\sqrt{\varepsilon})} + o(1).\end{aligned}$$

6.1.3 $\mathbb{P}(E_{2,p}) = O(\sqrt{\varepsilon})$

Event $E_{2,p}$ is the event that there is some wishy-washy box intersecting L at time p. Almost surely, the points are in general position. Define the *circumcenter* of k points to be the center of the smallest ball containing the k points. Note that the circumcenter is a point, not necessarily a vertex, and need not be unique. The radius of this ball is called the *circumradius*. We will call a cell containing the circumcenter of the k points the *circumcell*. Define a *superbox* to be 3^d 1-boxes arranged in a $\underbrace{3 \times 3 \times \cdots \times 3}_{d}$ formation. Define a 1-box (a cube whose ℓ_∞ diameter is $1r$) to be a *wishy-washy box* if the following properties hold:

- all cells in the superbox centered on the 1-box are typical,
- there exist exactly k active vertices in the superbox so that
  - the circumcell for those k points is in the 1-box,
  - the center of the circumcell sees the k active vertices inside the superbox within distance $(1 + \mathrm{diam}(\mathcal{T}_d)\varepsilon)\, r$ (the circumcell is nonstuck),
  - at least one active vertex in the superbox is **not** within distance $(1 - \mathrm{diam}(\mathcal{T}_d)\varepsilon)\, r$ of the center of the circumcell (the circumcell is not a seed)

Wishy-washy boxes are situations where there are k active vertices near each other, but there may not be a vertex that sees all of them, so it would be possible for those k initially active vertices to not infect any other vertices.

Let $\mathfrak{s}$ be a superbox centered on 1-box $\mathfrak{b}$. Define $m_{\mathfrak{b}}$ to be the number of 1-boxes in $\mathcal{T}_d$. Then $m_{\mathfrak{b}} = \frac{n\xi}{a}$. Note that there are $\Theta\left(\frac{n\ell^d}{a}\right)$ superboxes completely contained in L.

Define $Z(\mathfrak{b})$ to be the event that all cells in $\mathfrak{b}$'s associated superbox $\mathfrak{s}$ are typical. Note that the number of 1-boxes with an atypical cell in its associated superbox is at most 3^d times the number of atypical cells since there are 3^d 1-boxes in each superbox. For Y_{atyp} the number of atypical cells, we have that $\mathbb{E}Y_{\text{atyp}} = o\left(\frac{n}{\varepsilon^d a}\right) = o(m)$ from above. Let $Z_1(\mathscr{b})$ be the event that there is an atypical cell in $\mathscr{b}$'s associated superbox $\mathscr{s}$. Then take Z_1 to be the number of 1-boxes with an atypical cell in its associated superbox. Then

$$\mathbb{E}Z_1 = m_{\mathscr{b}} \cdot \mathbb{P}(Z_1(\mathscr{b})).$$

Then since $\mathbb{E}Z_1 \leq 3^d \mathbb{E}Y_{\text{atyp}} = o(m)$, we have that

$$\mathbb{P}(Z_1(\mathscr{b})) \leq \frac{o(m)}{m_{\mathscr{b}}} = \frac{o\left(\frac{n}{\varepsilon^d a}\right)}{\Theta\left(\frac{n}{a}\right)} = o\left(\frac{1}{\varepsilon^d}\right) = o(1).$$

Thus, $\mathbb{P}(Z(\mathscr{b})) = 1 - o(1)$.

In order for a 1-box to be wishy-washy, there must be k points in the associated superbox so that the circumcell is in the 1-box. Note that the circumradius $\rho \leq r$, else no point will be within range of all k initially active vertices, and hence the circumcell is nonstuck. Note that for any two points x, y in the circumcell c, by Lemma 2.2 there exists a $\beta > 0$ so that $\operatorname{dist}(x, y) \leq \beta \operatorname{dist}_\infty(x, y) \leq \beta \operatorname{diam}_\infty(c) = \varepsilon\beta r$. We must also have that the circumradius $\rho > (1 - \varepsilon\beta)r$, otherwise the circumcell c is a seed. Therefore a 1-box is wishy-washy when the circumradius of the k initially active points in the associated superbox satisfies $(1 - \beta\varepsilon)r \leq \rho \leq r$.

Let $\mathcal{W}_{\mathscr{b}}$ be the event that box $\mathscr{b}$ is wishy-washy and let $A_{\mathscr{b}}$ be the event that $\mathscr{b}$'s associated superbox $\mathscr{s}$ contains exactly k active vertices. Then

$$\mathbb{P}(\mathcal{W}_{\mathscr{b}}|A_{\mathscr{b}}) = \mathbb{P}\left((1 - \varepsilon\beta)r < \rho \leq r|A_{\mathscr{b}}\right) \leq \mathbb{P}(\rho \leq r|A_{\mathscr{b}}) - \mathbb{P}(\rho \leq (1 - \beta\varepsilon)r|A_{\mathscr{b}}).$$

Let $q \in \mathcal{T}_d$ be the corner of $\mathscr{s}$ with the lowest value in each coordinate. Then consider a slightly smaller cube $\mathscr{t} \subseteq \mathscr{s}$ so that $q \in \mathscr{t}$ and $\operatorname{diam}_\infty(\mathscr{t}) = (1 - \varepsilon\beta)3r$.

Consider the event $A_{\boldsymbol{t}}$ to be the event that all of the k initially active vertices in $\boldsymbol{s}$ fall in $\boldsymbol{t}$. Then[1]

$$\begin{aligned}
\mathbb{P}(\rho \leq (1-\beta\varepsilon)r|A_{\boldsymbol{b}}) &= \mathbb{P}(\rho \leq (1-\varepsilon\beta)r|A_{\boldsymbol{t}})\mathbb{P}(A_{\boldsymbol{t}}) + \mathbb{P}(\rho \leq (1-\varepsilon\beta)r|\overline{A_{\boldsymbol{t}}})\mathbb{P}(\overline{A_{\boldsymbol{t}}}) \\
&= \mathbb{P}(\rho \leq r|A_{\boldsymbol{b}})(1-\mathbb{P}(\overline{A_{\boldsymbol{t}}}) + \mathbb{P}(\rho \leq (1-\varepsilon\beta)r|\overline{A_{\boldsymbol{t}}})\mathbb{P}(\overline{A_{\boldsymbol{t}}}).
\end{aligned}$$

Note that $\mathbb{P}(\overline{A_{\boldsymbol{t}}}) \leq kd\varepsilon\beta = \Theta(\varepsilon)$. Then

$$\begin{aligned}
\mathbb{P}(\mathcal{W}_{\boldsymbol{b}}|A_{\boldsymbol{b}}) &= \mathbb{P}(\rho \leq r|A_{\boldsymbol{b}}) - \mathbb{P}(\rho \leq (1-\beta\varepsilon)r|A_{\boldsymbol{b}}) \\
&= \mathbb{P}(\rho \leq r|A_{\boldsymbol{b}})\mathbb{P}(\overline{A_{\boldsymbol{t}}}) - \mathbb{P}(\rho \leq (1-\beta\varepsilon)r|\overline{A_{\boldsymbol{t}}})\mathbb{P}(\overline{A_{\boldsymbol{t}}}) \\
&\leq \mathbb{P}(\overline{A_{\boldsymbol{t}}}) \leq \Theta(\varepsilon).
\end{aligned}$$

Note that there are at most $3^d \frac{1}{\varepsilon^d} \cdot \frac{2\varepsilon^d}{\xi} a = \frac{2\cdot 3^d a}{\xi}$ vertices in a superbox that has all its cells typical. Then

$$\begin{aligned}
\mathbb{P}(A_{\boldsymbol{b}}|Z(\boldsymbol{b})) &\leq \binom{\frac{2\cdot 3^d a}{\xi}}{k} p^k (1-p)^{\frac{2\cdot 3^d a}{xi}-k} \leq \Theta(a^k)p^k \\
&= \Theta(a^k)\gamma^k(p^*)^k = \Theta\left(\frac{a^k}{\sqrt{\varepsilon}}\right)\frac{1}{n\ell^d a^{k-1}} = \Theta\left(\frac{a}{\sqrt{\varepsilon}n}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{P}(\mathcal{W}_{\boldsymbol{b}}) &= \mathbb{P}(\mathcal{W}_{\boldsymbol{b}}|A_{\boldsymbol{b}})\mathbb{P}(A_{\boldsymbol{b}}|Z(\boldsymbol{b}))\mathbb{P}(Z(\boldsymbol{b})) \\
&\leq \Theta(\varepsilon)\Theta\left(\frac{a}{\sqrt{\varepsilon}n}\right)(1-o(1)) \sim \Theta\left(\frac{\sqrt{\varepsilon}a}{n}\right).
\end{aligned}$$

Let $\mathcal{W}$ be the number of wishy-washy boxes. Then

[1] Here we are noting that $\mathbb{P}(\rho \leq (1-\varepsilon\beta)r|A_{\boldsymbol{t}}) = \mathbb{P}(\rho \leq r|A_{\boldsymbol{b}})$ since they are equivalent if we apply a stretch by a factor of $\frac{1}{1-\varepsilon\beta}$.

$$\mathbb{E}\mathcal{W} \leq m_{\mathscr{b}}\mathcal{W}_{\mathscr{b}} \leq \frac{n\xi}{a} \cdot \Theta\left(\frac{\sqrt{\varepsilon}a}{n}\right) = O(\sqrt{\varepsilon}).$$

Therefore, by Lemma 2.4, $\mathbb{P}(E_{2,p}) = O(\sqrt{\varepsilon})$.

6.1.4 $\mathbb{P}(E_{3,p}) = o(1)$

Event $E_{3,p}$ is the event that some superbox intersecting L has more than k active vertices at time p. Note that a superbox has area $3^d r^d = \frac{3^d a}{\xi n}$. Let $\mathcal{Y}_{\mathscr{s}}$ be the number of vertices in a superbox $\mathscr{s}$. Then define $\mu := \mathbb{E}(\mathcal{Y}_{\mathscr{s}}) = \frac{3^d a}{\xi}$. Let $A(\mathscr{b})$ be the number of initially active vertices in $\mathscr{b}$'s associated superbox $\mathscr{s}$. We will split the following parts of the proof into two cases, each having a different definition of what it means for a superbox to be *typical* or *atypical*. Define $\mathcal{Y}_{\text{atyp}}$ to be the number of atypical superboxes, let $\mathcal{Y}_{\text{typ}}(\mathscr{b})$ be the event that $\mathscr{b}$'s associated superbox is typical, and call $\mathcal{Y}_{\text{typ}}$ the number of typical superboxes. Let $m_{\mathscr{s},L}$ be the number of superboxes intersecting L and $\mathcal{Y}_{\text{typ},L}$ be the number of typical superboxes intersecting L. Then $m_{\mathscr{s},L} = \Theta\left(\frac{n\ell^d}{a}\right)$. Finally, let $Z_{3,L}$ be the number of typical superboxes intersecting L with more than k active vertices. The cases we consider are: **Case (1)** where $\xi \leq 3^d$ and **Case (2)** where $\xi > 3^d$.

In **Case (1)**: Here, $\xi \leq 3^d$. We will call a superbox *typical* if it has at most 3μ vertices, and atypical otherwise. By Lemma 2.3,

$$\mathbb{P}(\mathcal{Y}_{\mathscr{s}} \geq 3\mu) \leq e^{-\frac{3^d a}{\xi}} \leq e^{\frac{-3^d \log n}{\xi}} = n^{-\frac{3^d}{\xi}}.$$

Then

$$\mathbb{E}(\mathcal{Y}_{\text{atyp}}) \leq n^{-\frac{3^d}{\xi}} \cdot \frac{n\xi}{a} = \frac{\xi}{n^{\frac{3^d}{\xi}-1}a} = o(1),$$

since $\xi \leq 3^d$. Thus, $\mathbb{P}(\mathcal{Y}_{\text{atyp}} > 0) \leq o(1)$, so a.a.s., all superboxes are typical and hence, a.a.s., $\mathcal{Y}_{\text{typ}} = \Theta(m_{\mathscr{b}})$, so $\mathcal{Y}_{\text{typ},L} = \Theta(m_{s,L}) = \Theta\left(\frac{n\ell^d}{a}\right)$. At time p and for a 1-box $\mathscr{b}$ whose associated superbox intersects L, using Lemma 2.5 we have

$$\begin{aligned}
\mathbb{P}(A(\mathscr{b}) > k | \mathcal{Y}_{\text{typ}}(\mathscr{b})) &= \sum_{j=k+1}^{3\mu} \binom{3\mu}{j} p^j (1-p)^{3\mu - j} \\
&\leq \binom{3\mu}{k+1} p^{k+1} \leq (3\mu)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \left(\frac{1}{(n\ell^d)^{k+1/k} (a)^{k-1/k}} \right) \\
&= \left(\frac{3^{d+1}}{\xi} \right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{k+1}}{(n\ell^d)^{k+1/k} (a)^{k-1/k}} \\
&= \left(\frac{3^{d+1}}{\xi} \right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1+1/k}}{(n\ell^d)^{k+1/k}}.
\end{aligned}$$

For **Case (2)**: Here, we have $\xi > 3^d$. We will call a superbox *typical* if it has at most $3\xi\mu$ vertices and *atypical* otherwise. By Lemma 2.3,

$$\mathbb{P}(\mathcal{Y}_s \geq 3\xi\mu) \leq e^{-\frac{(3\xi-1)^2 \mu}{3\xi+1}} = e^{-\frac{(3\xi-1)^2 3^d a}{3\xi^2+\xi}} \leq e^{-\frac{(3\xi-1)^2 3^d \log n}{3\xi^2+\xi}} = n^{-\frac{(3\xi-1)^2 3^d}{3\xi^2+\xi}}.$$

Then

$$\mathbb{E}(\mathcal{Y}_{\text{atyp}}) \leq n^{-\frac{(3\xi-1)^2 3^d}{3\xi^2+\xi}} \cdot \frac{n\xi}{a} = \frac{\xi}{n^{\frac{(3\xi-1)^2 3^d}{3\xi^2+\xi} - 1} a} = o(1),$$

since $\xi \geq 3^d$ and hence $\frac{(3\xi-1)^2}{3\xi^2+\xi} \geq 1$.

Thus, $\mathbb{P}(\mathcal{Y}_{\text{atyp}} > 0) \leq o(1)$, so a.a.s., all superboxes are typical hence, a.a.s., $\mathcal{Y}_{\text{typ}} = \Theta(m_{\mathscr{b}})$, so $\mathcal{Y}_{\text{typ},L} = \Theta(m_{s,L}) = \Theta\left(\frac{n\ell^d}{a}\right)$. At time p and for a 1-box $\mathscr{b}$ whose associated superbox intersects L, using Lemma 2.5, we have

$$\begin{aligned}
\mathbb{P}(A(\mathfrak{b}) > k|\mathcal{Y}_{\text{typ}}) &= \sum_{j=k+1}^{3\xi\mu} \binom{3\xi\mu}{j} p^j (1-p)^{3\xi\mu - j} \\
&\leq \binom{3\xi\mu}{k+1} p^{k+1} \leq (3\xi\mu)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \left(\frac{1}{(n\ell^d)^{k+1/k}(a)^{k-1/k}} \right) \\
&= \left(\frac{3^{d+1}\xi}{\xi} \right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{k+1}}{(n\ell^d)^{k+1/k}(a)^{k-1/k}} \\
&= \left(3^{d+1}\right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1+1/k}}{(n\ell^d)^{k+1/k}}.
\end{aligned}$$

So in both **Case (1)** and **Case (2)**, we have $\mathcal{Y}_{\text{typ}} = \Theta(m_{\mathfrak{b}})$, $\mathcal{Y}_{\text{typ},L} = \Theta\left(\frac{n\ell^d}{a}\right)$, and

$$\mathbb{P}(A(\mathfrak{b}) > k|\mathcal{Y}_{\text{typ}}) \leq \Theta\left(\varepsilon^{\frac{-(k+1)}{2k}} \frac{a^{1+1/k}}{(n\ell^d)^{k+1/k}} \right).$$

Then

$$\begin{aligned}
\mathbb{E}(Z_{3,L}) &= m_{s,L}\mathbb{P}(A(\mathfrak{b}) > k) = m_{s,L}\mathbb{P}(\mathcal{Y}_{\text{typ}}(\mathfrak{b}))\mathbb{P}(A(\mathfrak{b}) > k|\mathcal{Y}_{\text{typ}}(\mathfrak{b})) \\
&\leq \Theta\left(\frac{n\ell^d}{a}\right) \cdot \Theta\left(\varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1+1/k}}{(n\ell^d)^{k+1/k}} \right) \\
&= \Theta\left(\varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1/k}}{(n\ell^d)^{k-1+1/k}} \right) = o(1).
\end{aligned}$$

Then by Lemma 2.4,

$$\begin{aligned}
\mathbb{P}(E_{3,p}) &\leq \mathbb{P}(Z_{3,L} > 0)\mathbb{P}(\mathcal{Y}_{\text{typ}} = m_{\mathfrak{b}}) + \mathbb{P}(\mathcal{Y}_{\text{atyp}} > 0) \\
&\leq o(1)(1 - o(1)) + o(1) = o(1).
\end{aligned}$$

At time p, a.a.s., there is no superbox in L with more than k active vertices and so a.a.s., all superboxes in L have at most k active points.

6.1.5 $\mathbb{P}(E_{4,p}) = o(1)$

Event $E_{4,p}$ is the event that some superbox intersecting L with an atypical cell has exactly k active vertices at time p. Since $\mathbb{E}(Y_{\text{atyp}}) \leq 2\frac{n^{1-\frac{\varepsilon^d}{8\xi}}}{\frac{\varepsilon^d}{\xi}a}$ and each atypical cell is in 3^d different superboxes, we have that

$$\mathbb{E}(Z_1) \leq 2 \cdot 3^d \frac{n^{1-\frac{\varepsilon^d}{8\xi}}}{\frac{\varepsilon^d}{\xi}a} = o\left(\frac{n}{a}\right).$$

Take $Z_{1,L}$ to be the number of superboxes intersecting L that have an atypical cell. Then

$$\mathbb{E}(Z_{1,L}) = o\left(\frac{n\ell^d}{a}\right).$$

Let $Z_2(\mathscr{b})$ be the event that $\mathscr{b}$'s associated superbox is typical but contains an atypical cell and let Z_2 be the number of 1-boxes whose associated superbox is typical but contains an atypical cell. In **Case (1)**, we have

$$\begin{aligned}\mathbb{P}(A_{\mathscr{b}} | Z_2(\mathscr{b})) &\leq \binom{3\mu}{k} p^k (1-p)^{3\mu-k} \leq (3\mu)^k (\gamma p^*)^k \\ &= \Theta\left((a)^k \varepsilon^{-1/2} \frac{1}{n\ell^d (a)^{k-1}}\right) = \Theta\left(\varepsilon^{-1/2} \frac{a}{n\ell^d}\right),\end{aligned}$$

and in **Case (2)**, we have

$$\begin{aligned}\mathbb{P}(A_{\mathscr{b}} | Z_2(\mathscr{b})) &\leq \binom{3\xi\mu}{k} p^k (1-p)^{3\xi\mu-k} \leq (3\xi\mu)^k (\gamma p^*)^k \\ &= \Theta\left((a)^k \varepsilon^{-1/2} \frac{1}{n\ell^d (a)^{k-1}}\right) = \Theta\left(\varepsilon^{-1/2} \frac{a}{n\ell^d}\right).\end{aligned}$$

Note that $\mathbb{E}(Z_2) \le \mathbb{E}(Z_1)$. Define $Z_{4,L}$ to be the number of typical superboxes intersecting L with some atypical cell and k active vertices at time p. Then we have

$$\begin{aligned}\mathbb{E}(Z_{4,L}) &= m_{s,L}\mathbb{P}(Z_2(\mathfrak{b}) \cap A_{\mathfrak{b}}) = m_{s,L}\mathbb{P}(Z_2(\mathfrak{b})) \cdot \mathbb{P}(A_{\mathfrak{b}}|Z_2(\mathfrak{b}))\\ &\le m_{s,L}\mathbb{P}(Z_1(\mathfrak{b})) \cdot \Theta\left(\varepsilon^{-1/2}\frac{a}{n\ell^d}\right)\\ &= \mathbb{E}(Z_{1,L}) \cdot \Theta\left(\varepsilon^{-1/2}\frac{a}{n\ell^d}\right)\\ &= o\left(\frac{n\ell^d}{a}\right) \cdot \Theta\left(\varepsilon^{-1/2}\frac{a}{n\ell^d}\right) = o(1).\end{aligned}$$

Let Z_5 be the number of superboxes intersecting L with an atypical cell and k active vertices at time p. Thus,

$$\mathbb{E}(Z_5) \le \mathbb{E}(\mathcal{Y}_{\text{atyp}}) + \mathbb{E}(Z_{4,L}) \le o(1) + o(1) = o(1).$$

Therefore, $\mathbb{P}(E_{4,p}) = \mathbb{P}(Z_5 > 0) \le o(1)$ by Lemma 2.4 and a.a.s., there is no superbox intersecting L with an atypical cell that has k active vertices at time p.

6.1.6 $\mathbb{P}(E_{5,p}) = o(1)$

Event $E_{5,p}$ is the event that some superbox with no atypical cell intersecting the boundary of L has at least k active vertices at time p. If we can rule out this situation, then a superbox that has k active vertices in it must occur on the interior of L. There are $O\left(\frac{\ell}{r}\right) = O\left(\sqrt[d]{\frac{\ell^d n}{a}}\right)$ superboxes on the boundary of L. Then

$$\begin{aligned}
\mathbb{P}(A(\mathfrak{b}) \geq k | Z(\mathfrak{b})) &= \sum_{j=k}^{3\mu} \binom{3\mu}{j} p^j (1-p)^{3\mu-j} \\
&\leq \binom{3\mu}{k} p^k \leq (3\mu)^k \varepsilon^{-1/2} \left(\frac{1}{(n\ell^d)(a)^{k-1}}\right) \\
&= \left(\frac{3^{d+1}}{\xi}\right)^k \cdot \frac{1}{\sqrt{\varepsilon}} \cdot \frac{a}{n\ell^d}.
\end{aligned}$$

Let Z_6 be the number of superboxes on the boundary of L with no atypical cell and at least k initially active vertices. Then

$$\begin{aligned}
\mathbb{E}(Z_6) &= O\left(\sqrt[d]{\frac{\ell^d n}{a}}\right) \mathbb{P}(A(\mathfrak{b}) \geq k | Z(\mathfrak{b}))\mathbb{P}(Z(\mathfrak{b})) \\
&= O\left(\sqrt[d]{\frac{\ell^2 n}{a}}\right) \left(\frac{3^{d+1}}{\xi}\right)^k \cdot \frac{1}{\sqrt{\varepsilon}} \cdot \frac{a}{n\ell^d}(1-o(1)) \\
&= O_\varepsilon\left(\sqrt[d]{\frac{a}{n\ell^d}}\right) = o(1).
\end{aligned}$$

Therefore, $\mathbb{P}(E_{5,p}) = \mathbb{P}(Z_6 > 0) \leq o(1)$ by Lemma 2.4, and hence a.a.s., there are no superboxes on the boundary of L with no atypical cell that have k or more active vertices.

6.1.7 Showing a.a.s., $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}}$

Then

$$\begin{aligned}
\mathbb{P}\left(\overline{E_0} \cap \left(\bigcap_{i=1}^{5} \overline{E_{i,p}}\right)\right) &= 1 - \mathbb{P}\left(E_0 \cup \left(\bigcup_{i=1}^{5} E_{i,p}\right)\right) \\
&\geq 1 - \left(E_0 + \sum_{i=1}^{5} \mathbb{P}(E_{i,p})\right) \\
&\geq 1 - \left(o(1) + e^{-\Theta(1/\sqrt{\varepsilon})} + o(1) + \Theta(\sqrt{\varepsilon}) + o(1) + o(1) + o(1)\right) \\
&= 1 - o(1) - e^{-\Theta(1/\sqrt{\varepsilon})} - \Theta(\sqrt{\varepsilon}).
\end{aligned}$$

Note that the right hand side above can be made arbitrarily close to 1 by picking ε sufficiently small.

Claim: If $E_0, E_{1,p} \ldots, E_{5,p}$ do not hold, then $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}} \leq p$.

Once the claim is proved, then we have that $\mathbb{P}(\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}})$ can be made arbitrarily close to 1, so a.a.s., $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}}$, which finishes the proof of part (a).

Proof of Claim. Since $E_{1,p}$ fails, we have a seed box in L at time p. Thus, it must be $\hat{p}_{\text{perco}} \leq p$, since a seed box gives us percolation. We also know that $\hat{p}_{\text{nonstuck}} \leq \hat{p}_{\text{perco}}$. Since $E_{3,p}, E_{4,p}$, and $E_{5,p}$ fail, the events $E_{3,q}, E_{4,q}$, and $E_{5,p}$ fail for any $q \leq p$ as well since the active vertices at time q are a subset of the active vertices at time p, and the failure of these events are monotonically decreasing. At time $\hat{p}_{\text{nonstuck}}$, let v be one of the first vertices that becomes nonstuck. Since $E_{5,\hat{p}_{\text{nonstuck}}}$ fails, this vertex must not be in a superbox that intersects the boundary of L. Then since $E_{3,\hat{p}_{\text{nonstuck}}}$ fails, there must be exactly k active vertices in the ball of activation of the cell containing v. Consider the box that contains the center cell of this k-cloud. Since $E_{4,\hat{p}_{\text{nonstuck}}}$ fails, this center cell must be typical. We want to show that this box is not wishy-washy. Suppose $E_{2,q}$ holds for $q < p$, i.e., there is a wishy-washy box at time q. Since $E_{3,p}$

fails, we know there are not $k+1$ (or more) active vertices in a superbox, and thus, the wishy-washy box must remain wishy-washy. Thus, we have that $E_{2,p}$ holds. However, since $E_{2,p}$ fails, it must be that $E_{2,q}$ fails for any $q < p$ as well. Thus, the box must not be wishy-washy. Since $E_{4,q}$ fails for any $q < p$, this box must not have an atypical cell. The center cell is typical and becomes active. Then since E_0 fails, all dense cells become active, and then all sparse cells become active as well. Therefore, we have percolation and hence $\hat{p}_{\text{perco}} = \hat{p}_{\text{nonstuck}}$. $\square$

Thus, for $r \geq B\sqrt[d]{\frac{\log n}{\xi n}}$ with $B > 1$ and $\ell \gg r$, a.a.s., $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{perco}}$.

6.2 Proof of Theorem 3.6(b)

Let $r = \sqrt[d]{\frac{a}{\xi n}}$ with $a < 1.1 \log n$ and $p^* = \frac{1}{(n\ell^d)^{1/k}(a)^{1-1/k}}$. Note that $a = o(n)$ and $a \ll n\ell^d$. Suppose $p = \gamma p^*$ for $\gamma = \varepsilon^{-1/2k}$. Let $\varepsilon > 0$. We will tesselate the unit torus into cubic cells of equal size so that for a cell c, $s := \operatorname{diam}_\infty(c) = \varepsilon r$. So $S = \varepsilon$. Note that the volume of each cell c is $\varepsilon^d r^d = \frac{\varepsilon^d a}{\xi n}$, so there are $m := \frac{1}{\varepsilon^d r^d} = \frac{\xi n}{\varepsilon^d a}$ cells in $\mathcal{T}_d$.

Let D_1 be the largest component of dense cells in the graph of dense cells. If there are two "largest" components, we may pick one arbitrarily. We will define "bad" events $E_0, E_{1,p} \ldots, E_{5,p}$ and show that $\mathbb{P}\left(E_0 \cup (\cup_{i=1}^5 E_{i,p})\right)$ is small and that if none of

$E_0, E_{1,p} \ldots, E_{5,p}$ happen, then $\hat{p}_{\text{nonstuck}} = \hat{p}_{\text{near perco}}$. We define $E_0, E_{1,p} \ldots, E_{5,p}$ as:

$$\begin{aligned}
E_0 = &\ \text{event that } |D_1| \leq (1-f)m, \text{ for some } f = o(1), \\
E_{1,p} = &\ \text{event that there is no seed box intersecting } L \text{ at time } p, \\
E_{2,p} = &\ \text{event that there is some wishy-washy box intersecting } L \text{ at time } p, \\
E_{3,p} = &\ \text{event that some superbox intersecting } L \text{ has more than } k \text{ active vertices} \\
&\qquad \text{at time } p, \\
E_{4,p} = &\ \text{event that some superbox intersecting } L \text{ with an atypical cell has exactly} \\
&\qquad k \text{ active vertices at time } p, \\
E_{5,p} = &\ \text{event that some superbox with no atypical cell intersecting the boundary} \\
&\qquad \text{of } L \text{ has at least } k \text{ active vertices at time } p,
\end{aligned}$$

where typical, box, seed box, wishy-washy box, and superbox are defined in their first respective subsections below.

6.2.1 $\mathbb{P}(E_0) = o(1)$

Event E_0 is the event that $|D_1| \leq (1-f)m$ for some $f = o(1)$. Recall:

Lemma 6.1. (i) Suppose that $r \gg \sqrt[d]{\frac{1}{n}}$. Then a.a.s. the largest component of the subgraph of the graph of cells induced by dense cells contains $(1-o(1))m$ cells (in other words, all but a vanishing fraction of the cells are good).

Note that Lemma 6.1(i) tells us $\mathbb{P}(E_0) = o(1)$. Thus, a.a.s., the largest component of dense cells contains all except $o(m)$ cells.

6.2.2 $\mathbb{P}(E_{1,p}) \leq e^{-\Theta(1/\sqrt{\varepsilon})} + o(1)$

Event $E_{1,p}$ is the event that there is no seed box intersecting L at time p. Define a cell to be typical if it has between $\frac{\varepsilon^d}{2\xi}a$ and $\frac{2\varepsilon^d}{\xi}a$ vertices and the cell is in the largest component D_1. Let Y_c be the number of vertices in a cell c. Then by Lemma 2.3, we have

$$\begin{aligned}\mathbb{P}\left(Y_c \leq \frac{\varepsilon^d}{2\xi}a \text{ or } Y_c \geq \frac{2\varepsilon^d}{\xi}a\right) &= \mathbb{P}\left(Y_c \leq \frac{\varepsilon^d}{2\xi}a\right) + \mathbb{P}\left(X_c \geq \frac{2\varepsilon^d}{\xi}a\right) \\ &\leq e^{-\frac{\varepsilon^d a}{8\xi}} + e^{-\frac{\varepsilon^d a}{3\xi}} \leq 2e^{-\frac{\varepsilon^d a}{8\xi}}.\end{aligned}$$

Cells are atypical for one of two reasons: (1) the number of vertices in the cell is not in the correct range, or (2) the cell is not in the largest dense component. Then for Y_{atyp} the number of atypical cells, a.a.s.,

$$\begin{aligned}\mathbb{E}Y_{\text{atyp}} \leq \frac{1}{\varepsilon^2 r^2}\left(2e^{-\frac{\varepsilon^d a}{8\xi}}\right) + f\left(\frac{1}{\varepsilon^2 r^2}\right) &= 2\frac{n\xi}{\varepsilon^d a}e^{\frac{\varepsilon^d a}{8\xi}} + f\frac{n\xi}{\varepsilon^d a} \\ = \left(\frac{2}{e^{\varepsilon^d a}8\xi} + f\right)\frac{n\xi}{\varepsilon^d a} = o(m).\end{aligned}$$

Then by Markov's Theorem, we have that

$$\mathbb{P}\left(Y_{\text{atyp}} > \sqrt{\frac{2}{e^{\frac{\varepsilon^d a}{8\xi}}} + f} \cdot \frac{n\xi}{\varepsilon^d a}\right) \leq \sqrt{\frac{2}{e^{\frac{\varepsilon^d a}{8\xi}}} + f} \to 0$$

as $n \to \infty$. So a.a.s.,

$$Y_{\text{atyp}} \leq \sqrt{\frac{2}{e^{\frac{\varepsilon^d a}{8\xi}}} + f} \cdot \frac{n\xi}{\varepsilon^d a} = o\left(\frac{n\xi}{\varepsilon^d a}\right) = o(m).$$

Define an α-*box* to be a cube with $\text{diam}_\infty(\alpha\text{-box}) = \alpha r$ so that each cell is either in the box or not in the box and the α-boxes partition the cells. We will consider diam($\mathcal{T}_d$)-boxes. We will call a diam($\mathcal{T}_d$)-box a *seed box* if it contains exactly k initially active points and all its cells are typical. Thus, if there is a seed box, all of the vertices in the box will become activated since there are k initially active vertices inside it and any two vertices in the seed box are adjacent.

Define event F to be the event that all except at most a $o(1)$ fraction of diam($\mathcal{T}_d$)-boxes have all its cells typical. Let Y be the number of diam($\mathcal{T}_d$)-boxes that do not have all its cells typical. Then

$$\mathbb{E}Y \leq \mathbb{E}Y_{\text{atyp}} \leq \left(\frac{2}{e^{\varepsilon^d a} 8\xi} + f\right)\frac{n\xi}{\varepsilon^d a}.$$

Again, using Lemma 2.4, we see that

$$\mathbb{P}\left(Y > \sqrt{\frac{2}{e^{\frac{\varepsilon^d a}{8\xi}}} + f} \cdot \frac{n\xi}{\varepsilon^d a}\right) \leq \sqrt{\frac{2}{e^{\frac{\varepsilon^d a}{8\xi}}} + f} \to 0$$

as $n \to \infty$. So a.a.s.,

$$Y \leq \sqrt{\frac{2}{e^{\frac{\varepsilon^d a}{8\xi}}} + f} \cdot \frac{n\xi}{\varepsilon^d a} = o_\varepsilon\left(\frac{n}{a}\right) = o_\varepsilon\left(\#\text{ of diam}(\mathcal{T}_d)\text{-boxes}\right).$$

Thus, a.a.s., F holds. So there are $\Theta\left(\frac{n\ell^d}{a}\right)$ diam($\mathcal{T}_d$)-boxes with all its cells typical. Let $Y_{\mathscr{b},\text{seed}}$ be the event that a diam($\mathcal{T}_d$)-box $\mathscr{b}$ with all its cells typical is a seed. Then $\mathbb{P}(Y_{\mathscr{b},\text{seed}}) \leq \Theta(p^k a^k)$.

Then since $pa \to 0$,

$$\begin{aligned}\mathbb{P}(E_{1,p}|F) &= \mathbb{P}(\text{no seed box}|F) = (1 - \Theta((pa)^k))^{\Theta(n\ell^d/a)}\\ &= \left(1 - \Theta\left(\gamma^k \frac{a}{n\ell^d}\right)\right)^{\Theta(n\ell^d/a)} = e^{-\Theta(\gamma^k)} \leq e^{-\Theta(1/\sqrt{\varepsilon})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(E_{1,p}) &= \mathbb{P}(E_{1,p}|F)\mathbb{P}(F) + \mathbb{P}(E_{1,p}|\overline{F})\mathbb{P}(\overline{F})\\ &\le \mathbb{P}(E_{1,p}|F) + \mathbb{P}(\overline{F})\\ &\le e^{-\Theta(1/\sqrt{\varepsilon})} + o_\varepsilon(1),\end{aligned}$$

which is small.

6.2.3 $\mathbb{P}(E_{2,p}) = O(\sqrt{\varepsilon})$

Event $E_{2,p}$ is the event that there is some wishy-washy box intersecting L at time p. Almost surely, the points are in general position. Define the *circumcenter* of k points to be the center of the smallest ball containing the k points. Note that the circumcenter is a point, not necessarily a vertex, and need not be unique. The radius of this ball is called the *circumradius.* We will call a cell containing the circumcenter of the k points the *circumcell.* Define a *superbox* to be 3^d 1-boxes arranged in a $\underbrace{3 \times 3 \times \cdots \times 3}_{d}$ formation. Define a 1-box to be a *wishy-washy box* if the following properties hold:

- all cells in the superbox centered on the 1-box are typical, and
- there exist exactly k active vertices in the superbox so that
  - the circumcell for those k points is in the 1-box,
  - the center of the circumcell sees the k active vertices in the superbox within distance $(1 + \mathrm{diam}(\mathcal{T}_d)\varepsilon)\, r$ (the circumcell is nonstuck),
  - at least one of the active vertices in the superbox is **not** within distance $(1 - \mathrm{diam}(\mathcal{T}_d)\varepsilon)\, r$ (the circumcell is not a seed).

Wishy-washy boxes are situations where there are k active vertices near each other, but there may not be a vertex that sees all of them, so it would be possible for those k initially active vertices to not infect any other vertices.

Let $\mathscr{s}$ be a superbox centered on 1-box $\mathscr{b}$. Define $m_{\mathscr{b}}$ to be the number of 1-boxes in $\mathcal{T}_d$. Then $m_{\mathscr{b}} = \frac{n\xi}{a}$. Note that there are $\Theta\left(\frac{n\ell^d}{a}\right)$ superboxes completely contained in L.

Define $Z(\mathscr{b})$ to be the event that all cells in $\mathscr{b}$'s associated superbox $\mathscr{s}$ are typical. Note that the number of 1-boxes with an atypical cell in its associated superbox is at most 3^d times the number of atypical cells since there are 3^d 1-boxes in each superbox. For Y_{atyp} the number of atypical cells, we have that $\mathbb{E}Y_{\text{atyp}} = o\left(\frac{n}{\varepsilon^d a}\right) = o(m)$ from above. Let $Z_1(\mathscr{b})$ be the event that there is an atypical cell in $\mathscr{b}$'s associated superbox $\mathscr{s}$. Then take Z_1 to be the number of 1-boxes with an atypical cell in its associated superbox. Then

$$\mathbb{E}Z_1 = m_{\mathscr{b}} \cdot \mathbb{P}(Z_1(\mathscr{b})).$$

Then since $\mathbb{E}Z_1 \leq 3^d \mathbb{E}Y_{\text{atyp}} = o(m)$, we have that

$$\mathbb{P}(Z_1(\mathscr{b})) \leq \frac{o(m)}{m_{\mathscr{b}}} = \frac{o\left(\frac{n}{\varepsilon^d a}\right)}{\Theta\left(\frac{n}{a}\right)} = o\left(\frac{1}{\varepsilon^d}\right) = o(1).$$

Thus, $\mathbb{P}(Z(\mathscr{b})) = 1 - o(1)$.

In order for a 1-box to be wishy-washy, there must be k points in the associated superbox so that the circumcell is in the 1-box. Note that the circumradius $\rho \leq r$, else no point will be within range of all k initially active vertices, and hence the circumcell is nonstuck. Note that for any two points x, y in the circumcell c, by Lemma 2.2 there exists a $\beta > 0$ so that $\operatorname{dist}(x,y) \leq \beta \operatorname{dist}_\infty(x,y) \leq \beta \operatorname{diam}_\infty(c) = \varepsilon\beta r$. We must also have that the circumradius $\rho > (1 - \varepsilon\beta)r$, otherwise the circumcell c is a seed. Therefore a 1-box is wishy-washy when the circumradius of the k initially active points in the associated superbox satisfies $(1-\beta\varepsilon)r \leq \rho \leq r$. Let $\mathcal{W}_{\mathscr{b}}$ be the event that box $\mathscr{b}$ is wishy-washy and let $A_{\mathscr{b}}$ be the event that $\mathscr{b}$'s associated superbox $\mathscr{s}$ contains exactly k active vertices. Then

$$\mathbb{P}(\mathcal{W}_{\mathscr{b}}|A_{\mathscr{b}}) = \mathbb{P}\left((1-\varepsilon\beta)r < \rho \leq r|A_{\mathscr{b}}\right) \leq \mathbb{P}(\rho \leq r|A_{\mathscr{b}}) - \mathbb{P}(\rho \leq (1-\beta\varepsilon)r|A_{\mathscr{b}}).$$

Let $q \in \mathcal{T}_d$ be the corner of $\mathscr{s}$ with the lowest value in each coordinate. Then consider a slightly smaller cube $\mathscr{t} \subseteq \mathscr{s}$ so that $q \in \mathscr{t}$ and $\operatorname{diam}_\infty(\mathscr{t}) = (1-\varepsilon\beta)3r$. Consider the event $A_{\mathscr{t}}$ to be the event that all of the k initially active vertices in $\mathscr{s}$ fall in $\mathscr{t}$. Then[2]

$$\begin{aligned}\mathbb{P}(\rho \leq (1-\beta\varepsilon)r|A_{\mathscr{b}}) &= \mathbb{P}(\rho \leq (1-\varepsilon\beta)r|A_{\mathscr{t}})\mathbb{P}(A_{\mathscr{t}}) + \mathbb{P}(\rho \leq (1-\varepsilon\beta)r|\overline{A_{\mathscr{t}}})\mathbb{P}(\overline{A_{\mathscr{t}}}) \\ &= \mathbb{P}(\rho \leq r|A_{\mathscr{b}})(1-\mathbb{P}(\overline{A_{\mathscr{t}}}) + \mathbb{P}(\rho \leq (1-\varepsilon\beta)r|\overline{A_{\mathscr{t}}})\mathbb{P}(\overline{A_{\mathscr{t}}}).\end{aligned}$$

Note that $\mathbb{P}(\overline{A_{\mathscr{t}}}) \leq kd\varepsilon\beta = \Theta(\varepsilon)$. Then

$$\begin{aligned}\mathbb{P}(\mathcal{W}_{\mathscr{b}}|A_{\mathscr{b}}) &= \mathbb{P}(\rho \leq r|A_{\mathscr{b}}) - \mathbb{P}(\rho \leq (1-\beta\varepsilon)r|A_{\mathscr{b}}) \\ &= \mathbb{P}(\rho \leq r|A_{\mathscr{b}})\mathbb{P}(\overline{A_{\mathscr{t}}}) - \mathbb{P}(\rho \leq (1-\beta\varepsilon)r|\overline{A_{\mathscr{t}}})\mathbb{P}(\overline{A_{\mathscr{t}}}) \\ &\leq \mathbb{P}(\overline{A_{\mathscr{t}}}) \leq \Theta(\varepsilon).\end{aligned}$$

Note that there are at most $3^d\frac{1}{\varepsilon^d} \cdot \frac{2\varepsilon^d}{\xi}a = \frac{2\cdot 3^d a}{\xi}$ vertices in a superbox that has all

[2] Here we are noting that $\mathbb{P}(\rho \leq (1-\varepsilon\beta)r|A_{\mathscr{t}}) = \mathbb{P}(\rho \leq r|A_{\mathscr{b}})$ since they are equivalent if we apply a stretch by a factor of $\frac{1}{1-\varepsilon\beta}$.

its cells typical. Then

$$\mathbb{P}(A_{\mathscr{b}}|Z(\mathscr{b})) \leq \binom{\frac{2\cdot 3^d a}{\xi}}{k} p^k (1-p)^{\frac{2\cdot 3^d a}{xi}-k} \leq \Theta(a^k)p^k$$
$$= \Theta(a^k)\gamma^k (p^*)^k = \Theta\left(\frac{a^k}{\sqrt{\varepsilon}}\right)\frac{1}{n\ell^d a^{k-1}} = \Theta\left(\frac{a}{\sqrt{\varepsilon}n}\right).$$

Thus,

$$\mathbb{P}(\mathcal{W}_{\mathscr{b}}) = \mathbb{P}(\mathcal{W}_{\mathscr{b}}|A_{\mathscr{b}})\mathbb{P}(A_{\mathscr{b}}|Z(\mathscr{b}))\mathbb{P}(Z(\mathscr{b}))$$
$$\leq \Theta(\varepsilon)\Theta\left(\frac{a}{\sqrt{\varepsilon}n}\right)(1-o(1)) \sim \Theta\left(\frac{\sqrt{\varepsilon}a}{n}\right).$$

Let $\mathcal{W}$ be the number of wishy-washy boxes. Then

$$\mathbb{E}\mathcal{W} \leq m_{\mathscr{b}}\mathcal{W}_{\mathscr{b}} \leq \frac{n\xi}{a}\cdot\Theta\left(\frac{\sqrt{\varepsilon}a}{n}\right) = O(\sqrt{\varepsilon}).$$

Therefore, by Lemma 2.4, $\mathbb{P}(E_{2,p}) = O(\sqrt{\varepsilon})$.

6.2.4 $\mathbb{P}(E_{3,p}) = o(1)$

Event $E_{3,p}$ is the event that some superbox intersecting L has more than k active vertices at time p. Note that a superbox has area $3^d r^d = \frac{3^d a}{\xi n}$. Let $\mathcal{Y}_{\mathscr{s}}$ be the number of vertices in a superbox $\mathscr{s}$. Then define $\mu := \mathbb{E}(\mathcal{Y}_{\mathscr{s}}) = \frac{3^d a}{\xi}$. Let $A(\mathscr{b})$ be the number of initially active vertices in $\mathscr{b}$'s associated superbox $\mathscr{s}$. We will split the following parts of the proof into two cases, each having a different definition of what it means for a superbox to be *typical* or *atypical*. Define $\mathcal{Y}_{\text{atyp}}$ to be the number of atypical superboxes, let $\mathcal{Y}_{\text{typ}}(\mathscr{b})$ be the event that $\mathscr{b}$'s associated superbox is typical, and call $\mathcal{Y}_{\text{typ}}$ the number of typical superboxes. Let $m_{\mathscr{s},L}$ be the number of superboxes intersecting L and $\mathcal{Y}_{\text{typ},L}$ be the number of typical superboxes intersecting L. Then

$m_{s,L} = \Theta\left(\frac{n\ell^d}{a}\right)$. Finally, let $Z_{3,L}$ be the number of typical superboxes intersecting L with more than k active vertices. The cases we consider are: **Case (1)** where $\xi \leq 3^d$ and **Case (2)** where $\xi > 3^d$.

In **Case (1)**: Here, $\xi \leq 3^d$. We will call a superbox *typical* if it has at most 3μ vertices. Otherwise, the superbox is *atypical*. Call an atypical superbox *atypical of type i* if it has between $3^i\mu$ and $3^{i+1}\mu$ vertices inside it. Then by Lemma 2.3, $\mathbb{P}(\mathcal{Y}_s \geq 3\mu) \leq e^{-\frac{3^d a}{\xi}}$.

Then using Lemma 2.5, we have

$$\begin{aligned}
\mathbb{P}(A(\mathscr{b}) > k|\mathcal{Y}_{\text{typ}}(\mathscr{b})) &= \sum_{j=k+1}^{3\mu} \binom{3\mu}{j} p^j(1-p)^{3\mu-j} \\
&\leq \binom{3\mu}{k+1} p^{k+1} \leq (3\mu)^{k+1}\varepsilon^{\frac{-(k+1)}{2k}} \left(\frac{1}{(n\ell^d)^{k+1/k}(a)^{k-1/k}}\right) \\
&= \left(\frac{3^{d+1}}{\xi}\right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{k+1}}{(n\ell^d)^{k+1/k}(a)^{k-1/k}} \\
&= \left(\frac{3^{d+1}}{\xi}\right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1+1/k}}{(n\ell^d)^{k+1/k}}.
\end{aligned}$$

For **Case 2**: Here $\xi > 3^d$. We will call a superbox *typical* if it has at most $3\xi\mu$ vertices and *atypical* otherwise. We will call an atypical superbox *atypical of type i* if the superbox has between $3^i\xi\mu$ and $3^{i+1}\xi\mu$ vertices. Then by Lemma 2.3, $\mathbb{P}(\mathcal{Y}_s \geq 3\xi\mu) \leq e^{-\frac{(3\xi-1)^2\mu}{3\xi+1}} = e^{-\frac{(3\xi-1)^2 3^d a}{3\xi^2+\xi}}$.

Then using Lemma 2.5, we have

$$\begin{aligned}
\mathbb{P}(A(\mathscr{b}) > k|\mathcal{Y}_{\text{typ}}(\mathscr{b})) &= \sum_{j=k+1}^{3\xi\mu} \binom{3\xi\mu}{j} p^j (1-p)^{3\xi\mu-j} \\
&\leq \binom{3\xi\mu}{k+1} p^{k+1} \leq (3\xi\mu)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \left(\frac{1}{(n\ell^d)^{k+1/k}(a)^{k-1/k}} \right) \\
&= \left(3^{d+1}\right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{k+1}}{(n\ell^d)^{k+1/k}(a)^{k-1/k}} \\
&= \left(3^{d+1}\right)^{k+1} \varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1+1/k}}{(n\ell^d)^{k+1/k}}.
\end{aligned}$$

Thus, in both **Case (1)** and **Case (2)**, we have

$$\mathbb{P}(A(\mathscr{b}) > k|\mathcal{Y}_{\text{typ}}(\mathscr{b})) \leq \Theta\left(\varepsilon^{-\frac{k+1}{2k}} \frac{a^{1+1/k}}{(n\ell^d)^{k+1/k}} \right).$$

So

$$\begin{aligned}
\mathbb{P}(\mathcal{Y}_{\text{typ}}(\mathscr{b}) \cap A(\mathscr{b}) > k) &= \mathbb{P}(A(\mathscr{b}) > k|\mathcal{Y}_{\text{typ}}(\mathscr{b})) \cdot \mathbb{P}(\mathcal{Y}_{\text{typ}}(\mathscr{b})) \\
&\leq \Theta\left(e^{-\frac{k+1}{2k}} \frac{a^{1+1/k}}{(n\ell^d)^{k+1/k}} \right).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}(Z_{3,L}) &\leq \Theta_\varepsilon\left(\frac{n\ell^d}{a}\right) \cdot \Theta\left(\varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1+1/k}}{(n\ell^d)^{k+1/k}} \right) \\
&= \Theta_\varepsilon\left(\varepsilon^{\frac{-(k+1)}{2k}} \frac{(a)^{1/k}}{(n\ell^d)^{k-1+1/k}} \right) = o(1).
\end{aligned}$$

Let $\mathcal{Z}_i(\mathscr{b})$ be the event that 1-box $\mathscr{b}$'s associated superbox is atypical of type i. Call $\mathcal{Z}_i$ the number of superboxes that are atypical of type i. Define $\mathcal{Z}_{1,i}$ to be the number of superboxes intersecting L that are atypical of type i that have more than

k active vertices inside of it. Then let

$$\mathcal{Z}_{1,i}(\mathscr{b}) = \begin{cases} 1, & \mathscr{b}\text{'s associated superbox is type } i \text{ and has more than } k \text{ active vertices,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{Z}_{1,i} = \sum_{\mathscr{b} \text{ in } L} \mathcal{Z}_{1,i}(\mathscr{b})$ and

$$\begin{aligned} \mathbb{E}\mathcal{Z}_{1,i} &= \sum_{\mathscr{b} \text{ in } L} \mathbb{E}\mathcal{Z}_{1,i}(\mathscr{b}) = \sum_{\mathscr{b} \text{ in } L} \mathbb{P}(\mathcal{Z}_{1,i}(\mathscr{b}) = 1) \\ &= \sum_{b \text{ in } L} \mathbb{P}(\mathcal{Z}_i(\mathscr{b}))\mathbb{P}(A(\mathscr{b}) > k | \mathcal{Z}_i(\mathscr{b})). \end{aligned}$$

Let $\mathcal{Z}_{2,L}$ be the number of atypical superboxes intersecting L with more than k active vertices and let $\mathscr{s}$ be $\mathscr{b}$'s associated superbox. We then break into cases again.

For **Case (1)**, by Lemma 2.3,

$$\mathbb{P}(\mathcal{Z}_i(\mathscr{b})) = \mathbb{P}(3^i\mu \leq \mathcal{Y}_{\mathscr{s}} < 3^{i+1}\mu) \leq \mathbb{P}(\mathcal{Y}_{\mathscr{s}} \geq 3^i\mu) \leq e^{-\frac{(3^i-1)^2\mu}{1+3^i}},$$

and using Lemma 2.5,

$$\begin{aligned} \mathbb{P}(A(\mathscr{b}) > k | \mathcal{Z}_i(\mathscr{b})) &\leq \mathbb{P}(A(\mathscr{b}) > k | \mathcal{Y}_{\mathscr{s}} = 3^{i+1}\mu) \\ &= \sum_{j=k+1}^{3^{i+1}\mu} \binom{3^{i+1}\mu}{j} p^j (1-p)^{3^{i+1}\mu - j} \\ &\sim \binom{3^{i+1}\mu}{k+1} p^{k+1} (1-p)^{3^{i+1}\mu - k - 1} \\ &\leq (3^{i+1}\mu p)^{k+1}. \end{aligned}$$

So

$$\mathbb{E}\mathcal{Z}_{1,i} \leq \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i-1)^2\mu}{1+3^i}} (3^{i+1}\mu p)^{k+1}.$$

Define $q_i = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i-1)^2\mu}{1+3^i}} (3^{i+1}\mu p)^{k+1}$. Then

$$\frac{q_{i+1}}{q_i} = \Theta(3^{k+1}) \exp\left(\mu\left(\frac{(3^i-1)^2}{1+3^i} - \frac{(3^{i+1}-1)^2}{1+3^{i+1}}\right)\right) = o(1)$$

since

$$\begin{aligned}
\frac{(3^i-1)^2}{1+3^i} - \frac{(3^{i+1}-1)^2}{1+3^{i+1}} &= \frac{3^{3i+1} - 3^{3i+2} + 3^{2i} - 3^{2i+2} + 2\cdot 3^{i+1}}{3^{2i+1} + 3^{i+1} + 3^i + 1} \\
&= \frac{-6\cdot 3^{3i} - 8\cdot 3^{2i} + 6\cdot 3^i}{3^{2i+1} + 3^{i+1} + 3^i + 1} \\
&\leq -6\cdot 27^i - 8\cdot 9^i + 6\cdot 3^i < 0
\end{aligned}$$

for all $i \geq 0$ and $\mu \to \infty$.

Then

$$\begin{aligned}
\mathbb{E}(\mathcal{Z}_{2,L}) &= \sum_{i=1}^{\infty} \mathbb{E}\mathcal{Z}_{1,i} \leq \sum_{i=1}^{\infty} q_i \sim q_1 \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\mu}(9\mu p)^{k+1} = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{3^d a}{\xi}} \left(\frac{3^d a}{\xi}\gamma p^*\right)^{k+1} \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\Theta(a)}\gamma^{k+1} \frac{a^{k+1}}{(n\ell^d)^{1+1/k} a^{k-1/k}} \\
&= \Theta(\gamma^{k+1}) e^{-\Theta(a)} \left(\frac{a}{n\ell^d}\right)^{1/k} = o(\varepsilon^{-1/2}) = o_\varepsilon(1).
\end{aligned}$$

For **Case (2)**, by Lemma 2.3,

$$\mathbb{P}(\mathcal{Z}_i(\mathscr{b})) = \mathbb{P}(3^i\xi\mu \leq \mathcal{Y}_s < 3^{i+1}\xi\mu) \leq \mathbb{P}(\mathcal{Y}_s \geq 3^i\xi\mu) \leq e^{-\frac{(3^i\xi-1)^2\mu}{1+3^i\xi}},$$

and using Lemma 2.5,

$$\begin{aligned}
\mathbb{P}(A(\mathscr{b}) > k | \mathcal{Z}_i(\mathscr{b})) &\leq \mathbb{P}(A(\mathscr{b}) > k | \mathcal{Y}_s = 3^{i+1}\xi\mu) \\
&= \sum_{j=k+1}^{3^{i+1}\xi\mu} \binom{3^{i+1}\xi\mu}{j} p^j (1-p)^{3^{i+1}\xi\mu - j} \\
&\sim \binom{3^{i+1}\xi\mu}{k+1} p^{k+1} (1-p)^{3^{i+1}\xi\mu - k - 1} \\
&\leq (3^{i+1}\xi\mu p)^{k+1}.
\end{aligned}$$

So

$$\mathbb{E}\mathcal{Z}_{1,i} \leq \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i\xi-1)^2\mu}{1+3^i\xi}} (3^{i+1}\xi\mu p)^{k+1}.$$

Define $q_i = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i\xi-1)^2\mu}{1+3^i\xi}} (3^{i+1}\xi\mu p)^{k+1}$. Then

$$\frac{q_{i+1}}{q_i} = \Theta(3^{k+1}) \exp\left(\mu\left(\frac{(3^i\xi-1)^2}{1+3^i\xi} - \frac{(3^{i+1}\xi-1)^2}{1+3^{i+1}\xi}\right)\right) = o(1)$$

since

$$\begin{aligned}
\frac{(3^i\xi-1)^2}{1+3^i\xi} - \frac{(3^{i+1}\xi-1)^2}{1+3^{i+1}\xi} &= \frac{6\cdot 3^i\xi - 6\cdot 3^{2i}\xi^2 - 6\cdot 3^{3i}\xi^3}{1+3^i\xi+3^{i+1}\xi+3^{2i+1}\xi^2} \\
&\leq 6\cdot 3^i - 6\cdot 9^i\xi^2 - 6\cdot 27^i\xi^3 \\
&\leq 6\xi^3(3^i - 9^i - 27^i) < 0
\end{aligned}$$

for all $i \geq 0$ and $\mu \to \infty$.

Then

$$
\begin{aligned}
\mathbb{E}(\mathcal{Z}_{2,L}) &= \sum_{i=1}^{\infty} \mathbb{E}\mathcal{Z}_{1,i} \le \sum_{i=1}^{\infty} q_i \sim q_1 \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3\xi-1)^2\mu}{1+3\xi}} (9\xi\mu p)^{k+1} = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{3^d a}{\xi}} \left(\frac{3^d a}{\xi}\gamma p^*\right)^{k+1} \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\Theta(a)} \gamma^{k+1} \frac{a^{k+1}}{(n\ell^d)^{1+1/k} a^{k-1/k}} \\
&= \Theta(\gamma^{k+1}) e^{-\Theta(a)} \left(\frac{a}{n\ell^d}\right)^{1/k} = o(\varepsilon^{-1/2}) = o_\varepsilon(1).
\end{aligned}
$$

Thus, in both **Case (1)** and **Case (2)**, we have $\mathbb{E}(\mathcal{Z}_{2,L}) = o(1)$.

Let $\mathcal{Z}_{3,L}$ be the number of superboxes intersecting L with more than k active vertices. Therefore,

$$
\begin{aligned}
\mathbb{E}(\mathcal{Z}_{3,L}) &= \mathbb{E}(Z_{3,L}) + \mathbb{E}(\mathcal{Z}_{2,L}) \\
&\le o_\varepsilon(1) + o_\varepsilon(1) = o_\varepsilon(1).
\end{aligned}
$$

Then $\mathbb{P}(E_{3,p}) \le o(1)$. At time p, a.a.s., there is no superbox contained in L with more than k active vertices and so a.a.s., all superboxes contained in L have at most k active points.

6.2.5 $\mathbb{P}(E_{4,p}) = o(1)$

Event $E_{4,p}$ is the event that some superbox intersecting L with an atypical cell has exactly k active vertices at time p. Since $\mathbb{E}(Y_{\text{atyp}}) = o\,(m)$ and each atypical cell is in 3^d different superboxes, we have that

$$
\mathbb{E}(Z_1) \le 3^d o\,(m) = o_\varepsilon\left(\frac{n}{a}\right).
$$

Thus,

$$\mathbb{E}(Z_{1,L}) = o\left(\frac{n\ell^d}{a}\right) = o(m_{s,L}).$$

Let $Z_2(\mathscr{b})$ be the event that $\mathscr{b}$'s associated superbox is typical but contains an atypical cell and Z_2 be the number of 1-boxes whose associated superbox is typical but contains an atypical cell. In **Case (1)**, we have

$$\begin{aligned}\mathbb{P}(A_{\mathscr{b}}|Z_2(\mathscr{b})) &\leq \binom{3\mu}{k} p^k(1-p)^{3\mu-k} \leq (3\mu)^k(\gamma p^*)^k \\ &= \Theta\left((a\cdot\gamma)^k \cdot \frac{1}{n\ell^d a^{k-1}}\right) = \Theta\left(\frac{a}{\sqrt{\varepsilon} n\ell^d}\right),\end{aligned}$$

and in **Case (2)**, we have

$$\begin{aligned}\mathbb{P}(A_{\mathscr{b}}|Z_2(\mathscr{b})) &\leq \binom{3\xi\mu}{k} p^k(1-p)^{3\xi\mu-k} \leq (3\xi\mu)^k(\gamma p^*)^k \\ &= \Theta\left((a\cdot\gamma)^k \cdot \frac{1}{n\ell^d a^{k-1}}\right) = \Theta\left(\frac{a}{\sqrt{\varepsilon} n\ell^d}\right).\end{aligned}$$

Note that $\mathbb{E}(Z_{2,L}) \leq \mathbb{E}(Z_{1,L})$. Define $Z_{4,L}$ to be the number of typical superboxes intersecting L with some atypical cell and exactly k active vertices at time p. Then we have

$$\begin{aligned}\mathbb{E}(Z_{4,L}) &= m_{s,L}\mathbb{P}(Z_2(\mathscr{b}) \cap A_{\mathscr{b}}) = m_{s,L}\mathbb{P}(Z_2(\mathscr{b}))\mathbb{P}(A_{\mathscr{b}}|Z_2(\mathscr{b})) \\ &= \mathbb{E}(Z_{1,L}) \cdot \Theta\left(\frac{a}{\sqrt{\varepsilon} n\ell^d}\right) \\ &= o\left(\frac{n\ell^2}{a}\right) \cdot \Theta\left(\frac{a}{\sqrt{\varepsilon} n\ell^2}\right) = o(1).\end{aligned}$$

Define $\mathcal{Z}_{4,i}$ to be the number of superboxes intersecting L that are atypical of type i with exactly k active vertices and again let $\mathscr{s}$ be the superbox that is centered at box

$\mathscr{b}$. Then define

$$\mathcal{Z}_{4,i}(\mathscr{b}) = \begin{cases} 1, & \mathscr{b}\text{'s associated superbox is atypical of type } i \text{ and has } k \text{ active vertices,} \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let $\mathcal{Z}_{4,L}$ be the number of atypical superboxes intersecting L with exactly k active vertices and $\mathcal{Z}_{5,L}$ be the number of superboxes intersecting L with exactly k active vertices. Then we have

$$\begin{aligned} \mathbb{E}\mathcal{Z}_{4,i} &= \sum_{\mathscr{b} \text{ in } L} \mathbb{E}\mathcal{Z}_{4,i}(\mathscr{b}) = \sum_{\mathscr{b} \text{ in } L} \mathbb{P}(\mathcal{Z}_i(\mathscr{b}) \cap A_{\mathscr{b}}) \\ &= \sum_{\mathscr{b} \text{ in } L} \mathbb{P}(\mathcal{Z}_i(\mathscr{b})) \cdot \mathbb{P}(A_{\mathscr{b}} | \mathcal{Z}_i(\mathscr{b})). \end{aligned}$$

In **Case (1)**, from above we have

$$\mathbb{P}(s \text{ is type } i) \leq e^{-\frac{(3^i-1)^2\mu}{1+3^i}}$$

We also have

$$\mathbb{P}(A_{\mathscr{b}} | \mathcal{Z}_i(\mathscr{b})) \leq \binom{3^{i+1}\mu}{k} p^k (1-p)^{3^{i+1}\mu - k} \leq (3^{i+1}\mu p)^k.$$

So

$$\mathbb{E}\mathcal{Z}_{4,i} \leq \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i-1)^2\mu}{1+3^i}} (3^{i+1}\mu p)^k.$$

Define $q_i = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i-1)^2\mu}{1+3^i}} (3^{i+1}\mu p)^k$. Then

$$\frac{q_{i+1}}{q_i} = \Theta(3^k) \exp\left(\mu\left(\frac{(3^i-1)^2}{1+3^i} - \frac{(3^{i+1}-1)^2}{1+3^{i+1}}\right)\right) = o(1),$$

since $\frac{(3^i-1)^2}{1+3^i} - \frac{(3^{i+1}-1)^2}{1+3^{i+1}} < 0$ for all $i \geq 0$ and $\mu \to \infty$. So

$$\begin{aligned}
\mathbb{E}(\mathcal{Z}_{4,L}) &= \sum_{i=1}^{\infty} \mathbb{E}\mathcal{Z}_{4,i} = \sum_{i=1}^{\infty} q_i \sim q_i \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\mu}(9\mu p)^k = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{3^d}{\xi}a} \left(\frac{3^d}{\xi} a\gamma p^*\right)^k \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\Theta(a)}\varepsilon^{-1/2} \cdot \frac{a^k}{n\ell^d a^{k-1}} = \Theta(\varepsilon^{-1/2}) e^{-\Theta(a)} = o(\varepsilon^{-1/2}) = o_\varepsilon(1).
\end{aligned}$$

In **Case (2)**, from above we have

$$\mathbb{P}(s \text{ is type } i) \leq e^{-\frac{(3^i\xi-1)^2\mu}{1+3^i\xi}}$$

We also have

$$\mathbb{P}(A_{\not{b}}|\mathcal{Z}_i(\not{b})) \leq \binom{3^{i+1}\xi\mu}{k} p^k(1-p)^{3^{i+1}\xi\mu-k} \leq (3^{i+1}\xi\mu p)^k.$$

So

$$\mathbb{E}\mathcal{Z}_{4,i} \leq \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i\xi-1)^2\mu}{1+3^i\xi}} (3^{i+1}\xi\mu p)^k.$$

Define $q_i = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3^i\xi-1)^2\mu}{1+3^i}} (3^{i+1}\xi\mu p)^k$. Then

$$\frac{q_{i+1}}{q_i} = \Theta(3^k) \exp\left(\mu\left(\frac{(3^i\xi-1)^2}{1+3^i\xi} - \frac{(3^{i+1}\xi-1)^2}{1+3^{i+1}\xi}\right)\right) = o(1),$$

since $\frac{(3^i\xi-1)^2}{1+3^i\xi} - \frac{(3^{i+1}\xi-1)^2}{1+3^{i+1}\xi} < 0$ for all $i \geq 0$ and $\mu \to \infty$. So

$$\begin{aligned}
\mathbb{E}(\mathcal{Z}_{4,L}) &= \sum_{i=1}^{\infty} \mathbb{E}\mathcal{Z}_{4,i} = \sum_{i=1}^{\infty} q_i \sim q_i \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3\xi-1)^2\mu}{1+3\xi}} (9\xi\mu p)^k = \Theta\left(\frac{n\ell^d}{a}\right) e^{-\frac{(3\xi-1)^2 3^d}{\xi+3\xi^2}a} \left(3^d a\gamma p^*\right)^k \\
&= \Theta\left(\frac{n\ell^d}{a}\right) e^{-\Theta(a)}\varepsilon^{-1/2} \cdot \frac{a^k}{n\ell^d a^{k-1}} = \Theta(\varepsilon^{-1/2})e^{-\Theta(a)} = o(\varepsilon^{-1/2}) = o_\varepsilon(1).
\end{aligned}$$

So in both **Case (1)** and **Case (2)**, we have that $\mathbb{E}(\mathcal{Z}_{4,L}) = o(1)$. Let Z_5 be the number of superboxes intersecting L with an atypical cell and k active vertices at time p. Thus,

$$\mathbb{E}(Z_{5,L}) \leq \mathbb{E}(\mathcal{Z}_{4,L}) + \mathbb{E}(Z_{4,L}) = o_\varepsilon(1) + o(1) = o(1).$$

Therefore, $\mathbb{P}(E_{4,p}) = \mathbb{P}(Z_5 > 0) \leq o(1)$ by Lemma 2.4 and a.a.s., there is no superbox with an atypical cell that has k active vertices at time p.

6.2.6 $\mathbb{P}(E_{5,p}) = o(1)$

Event $E_{5,p}$ is the event that some superbox with no atypical cell intersecting the boundary of L has at least k active vertices at time p. If we can rule out this situation, then a superbox that has k active vertices in it must occur on the interior of L. There are $O\left(\frac{\ell}{r}\right) = O\left(\sqrt[d]{\frac{\ell^d n}{a}}\right)$ superboxes on the boundary of L. Note that

$$\begin{aligned}
\mathbb{P}(A(\mathfrak{b}) \geq k | Z(\mathfrak{b})) &= \sum_{j=k}^{3\mu} \binom{3\mu}{j} p^j (1-p)^{3\mu-j} \\
&\leq \binom{3\mu}{k} p^k \leq (3\mu)^k \varepsilon^{-1/2} \left(\frac{1}{n\ell^d a^{k-1}}\right) \\
&= \left(\frac{3^{d+1}}{\xi}\right)^k \cdot \frac{1}{\sqrt{\varepsilon}} \cdot \frac{a}{n\ell^d}
\end{aligned}$$

Let Z_6 be the number of superboxes on the boundary of L with no atypical cell and at least k initially active vertices. So

$$\begin{aligned}
\mathbb{E}(Z_6) &= O\left(\sqrt{\frac{\ell^d n}{a}}\right)\mathbb{P}(A(\mathfrak{b}) \geq k | Z(\mathfrak{b}))\mathbb{P}(Z(\mathfrak{b})) \\
&= O\left(\sqrt{\frac{\ell^d n}{a}}\right)\left(\frac{3^{d+1}}{\xi}\right)^k \cdot \frac{1}{\sqrt{\varepsilon}} \cdot \frac{a}{n\ell^d}(1-o(1)) \\
&= O_\varepsilon\left(\sqrt[d]{\frac{a}{n\ell^d}}\right) = o(1).
\end{aligned}$$

Therefore, $\mathbb{P}(E_{5,p}) = \mathbb{P}(Z_6 > 0) \leq o(1)$ by Lemma 2.4, and hence a.a.s., there are no superboxes on the boundary of L with no atypical cell that have k or more active vertices.

6.2.7 Showing a.a.s., $\mathfrak{B}_k(A_0(\hat{p}_{\textbf{nonstuck}}))$ nearly percolates

Then

$$\begin{aligned}
\mathbb{P}\left(\overline{E_0} \cap \left(\bigcap_{i=1}^{5} \overline{E_{i,p}}\right)\right) &= 1 - \mathbb{P}\left(E_0 \cup \left(\bigcup_{i=1}^{5} E_{i,p}\right)\right) \\
&\geq 1 - \left(E_0 + \sum_{i=1}^{5} \mathbb{P}(E_{i,p})\right) \\
&\geq 1 - \left(o(1) + e^{-\Theta(1/\sqrt{\varepsilon})} + o_\varepsilon(1) + \Theta(\sqrt{\varepsilon}) + o(1) + o(1) + o(1)\right) \\
&= 1 - o(1) - e^{-\Theta(1/\sqrt{\varepsilon})} - \Theta(\sqrt{\varepsilon}).
\end{aligned}$$

Claim: If $E_0, E_{1,p} \ldots, E_{5,p}$ do not hold, then $\mathfrak{B}_k(A_0(\hat{p}_{\text{nonstuck}}))$ nearly percolates.

Proof of Claim. Since $E_{1,p}$ fails, we have a seed box at time p. Thus, it must be that $\mathfrak{B}_k$ nearly percolates, since a seed box in L percolates to the vertices in the good and bad cells. Since $E_{3,p}, E_{4,p}$, and $E_{5,p}$ fail, then events $E_{3,q}, E_{4,q}$, and $E_{5,p}$ fail for any

$q \leq p$ as well since the active vertices at time q are a subset of the active vertices at time p. At time $\hat{p}_{\text{nonstuck}}$, let v be one of the first vertices that becomes nonstuck. Since $E_{5,p}$ fails, this vertex must not be in a superbox that intersects the boundary of L. Then since $E_{3,\hat{p}_{\text{nonstuck}}}$ fails, there must be exactly k active vertices in the ball of activation of the cell containing v. Consider the box that contains the center cell of this k-cloud. Since $E_{4,\hat{p}_{\text{nonstuck}}}$ fails, this center cell must be typical. We want to show that this box is not wishy-washy. Suppose $E_{2,q}$ holds for $q < p$, i.e., there is a wishy-washy box contained in L at time q. Since $E_{3,p}$ fails, we know there are not $k+1$ (or more) active vertices in a superbox contained in L, and thus, the wishy-washy box must remain wishy-washy. Thus, we have that $E_{2,p}$ holds. However, since $E_{2,p}$ fails, it must be that $E_{2,q}$ fails for any $q < p$ as well. Thus, the box must not be wishy-washy. Since $E_{4,q}$ fails for any $q < p$, this box must not have an atypical cell. The center cell is typical and becomes active. Then since E_0 fails, we have percolation of all but $o(1)$ cells, and hence all but $o(1)$ vertices. Thus $\mathfrak{B}_k(A_0(\hat{p}_{\text{near perco}}))$ nearly percolates.

$\square$

Therefore

$$\mathbb{P}(\mathfrak{B}_k(A_0(\hat{p}_{\text{nonstuck}}))\text{nearly percolates}) = 1 - o(1) - e^{-\Theta(1/\sqrt{\varepsilon})} - \Theta(\sqrt{\varepsilon}).$$

Thus taking ε sufficiently small, we have that

$$\mathbb{P}(\mathfrak{B}_k(A_0(\hat{p}_{\text{nonstuck}}))\text{nearly percolates}) \to 1 \quad \text{as } n \to \infty.$$

So a.a.s., $\mathfrak{B}_k(A_0(\hat{p}_{\text{nonstuck}}))$ nearly percolates.

6.3 Proof of Theorem 3.6(c)

Consider the same tessellation of $\mathcal{T}_d$ as in Section 6.2. Take

$$r' = \sqrt[d]{\frac{\log n + (k-1)\log\log n - \omega}{\xi n}}$$

where $\omega \to \infty$ and $\omega = o(\log\log n)$. Suppose $r' \leq r \leq \sqrt[d]{\frac{1.1\log n}{\xi n}}$, and consider the process on $\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n, r); A_0(\hat{p}_{\text{nonstuck}}))$. Define a *simple obstruction* to be a vertex of degree less than k. Consider the graph of cells restricted to only ugly cells, call it $G[U]$.

Recall:

Lemma 6.1. (ii) Suppose that $r \geq r'$. Then a.a.s. all the components of the graph of cells induced by ugly cells must have diameter at most $Q\epsilon r$ (where $Q > 0$ is a constant depending only on the dimension d and the norm $\|\cdot\|$), and any two such components must be at distance at least Ar from each other (where constant $A > 0$ can be chosen arbitrarily large).

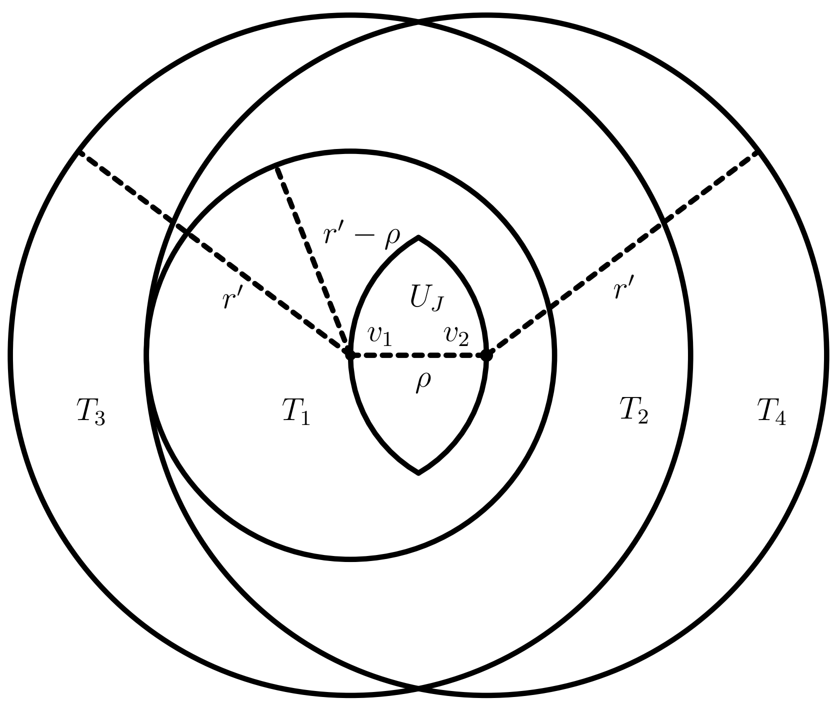
It is sufficient for us to take $A > 100$.

Consider one connected component of $G[U]$. Suppose there are $j \geq 2$ vertices in all of the ugly cells in this connected component of ugly cells that are not initially active, call the set of these vertices J. Without loss of generality, take the vertices v_1 and v_2 to be the two vertices in J that are the furthest apart from one another. Call $\rho = \text{dist}(v_1, v_2)$. Then by Lemma 6.1(ii), $\rho \leq Q\varepsilon r$. Let the ball of radius y centered at point x be denoted by $B(x, y)$. Set $U_J = B(v_1, \rho) \cap B(v_2, \rho)$. Then all vertices in J must fall in U_J, else v_1 and v_2 would not be the two vertices in J with maximum distance. We will consider the following parts of the torus:

$$
\begin{aligned}
T_1 &= B(v_1, r' - \rho) \setminus U_J, \\
T_2 &= (B(v_1, r') \cap B(v_2, r')) \setminus (U_J \cup T_1), \\
T_3 &= B(v_1, r') \setminus B(v_2, r'), \\
T_4 &= B(v_2, r') \setminus B(v_1, r'), \\
T_0 &= (\mathcal{T}_d \setminus (B(v_1, r') \cup B(v_2, r')) \cup U_J.
\end{aligned}
$$

Note that each of T_0, T_1, T_2, T_3, and T_4 are mutually disjoint and $\mathcal{T}_d = \bigcup_{i=0}^{4} T_i$.

Figure 6.1: A possible obstruction caused by connected components of ugly cells.



Suppose the number of vertices in T_i for $i = 1, \ldots, 4$ is t_i. We will denote the volume of T_i to be $|T_i|$ for $i = 0, \ldots, 4$, and the volume of U_J to be denoted by $|U_J|$.

Lemma 6.2. We have that $|T_1| = O((r')^d)$, $|T_2| = \Theta((r')^{d-1}\rho)$, $|T_3| = |T_4| = \Theta((r')^{d-1}\rho)$, and $|T_0| = 1 - (\xi(r')^d + C(r')^{d-1}\rho)$ for some $C > 0$.

Proof. Note that $|T_1| = \xi(r' - \rho)^d = O((r')^d)$. We have that

$$B(v_1, r') \setminus B(v_1, r' - \rho) \subseteq T_3 \subseteq B(v_2, r' + \rho) \setminus B(v_2, r').$$

Then

$$|T_3| \leq \xi(r' + \rho)^d - \xi(r')^d = \Theta((r')^{d-1}\rho)$$

and

$$|T_3| \geq \xi(r')^d - \xi(r' - \rho)^d = \Theta((r')^{d-1}\rho).$$

Therefore $|T_3| = \Theta((r')^{d-1}\rho)$. By symmetry, we also have $|T_4| = \Theta((r')^{d-1}\rho)$.

We also have that

$$T_2 = B(v_1, r') \setminus (B(v_1, r' - \rho) \cup T_3),$$

so

$$\begin{aligned} |T_2| &= \xi(r')^d - (\xi(r' - \rho)^d + \Theta((r')^{d-1}\rho)) \\ &= \xi\left((r')^d - (r')^d + \Theta((r')^{d-1}\rho) - \Theta((r')^{d-1}\rho)\right) = \Theta((r')^{d-1}\rho). \end{aligned}$$

Note that $|U_J| = \Theta(\rho^d)$. Then there must be positive constants c_1 and c_2 so that $c_1\rho^d \leq |U_J| \leq c_2\rho^d$. Since $|T_4| = \Theta((r')^{d-1}\rho)$, there must be positive constants c_3 and c_4 so that $c_3(r')^{d-1}\rho \leq |T_4| \leq c_4(r')^{d-1}\rho$. Furthermore, there must exist nonnegative constants c_5 and c_6 so that $c_5\varepsilon r' \le \rho \le c_6\varepsilon r'$. Note that $c_3 > c_1(c_6\varepsilon)^{d-1}$ and $c_4 > c_2(c_5\varepsilon)^{d-1}$ since ε is small. Therefore, we have

$$\begin{aligned}
|T_0| &= 1 - |B(v_1, r')| - |T_4| + |U_J| \\
&\le 1 - \xi(r')^d - c_3(r')^{d-1}\rho + c_1\rho^d \le 1 - \xi(r')^d - c_3(r')^{d-1}\rho + c_1 \cdot (c_6\varepsilon)^{d-1}(r')^{d-1}\rho \\
&= 1 - \left(\xi(r')^d + \Theta((r')^{d-1}\rho)\right),
\end{aligned}$$

and similarly

$$\begin{aligned}
|T_0| &= 1 - |B(v_1, r')| - |T_4| + |U_J| \\
&\ge 1 - \xi(r')^d - c_4(r')^{d-1}\rho + c_2\rho^d \ge 1 - \xi(r')^d - c_4(r')^{d-1}\rho + c_2 \cdot (c_5\varepsilon)^{d-1}(r')^{d-1}\rho \\
&= 1 - \left(\xi(r')^d + \Theta((r')^{d-1}\rho)\right).
\end{aligned}$$

Thus, $|T_0| = 1 - \left(\xi(r')^d + \Theta((r')^{d-1}\rho)\right)$, so there exists a constant $C > 0$ so that $|T_0| = 1 - \left(\xi(r')^d + C(r')^{d-1}\rho\right)$. □

Let J be defined as before. Define J to be a *non-simple obstruction* with v_1 and v_2 defined as above so that $0 \le t_1, t_2, t_3, t_4 < k$. These bounds on t_1, t_2, t_3, and t_4 are necessary for J to remain active, since if any of the sets T_1, T_2, T_3, or T_4 had more than k vertices, these k vertices would become active, which would then activate either v_1 or v_2 giving us a contradiction.

Lemma 6.3. A.a.s., there are no non-simple obstructions.

Proof. Let $X =$ the number of non-simple obstructions and $X_{t_1,t_2,t_3,t_4} =$ the number of non-simple obstructions with parameters t_1, t_2, t_3, and t_4. Then

$$X = \sum_{0 \le t_1, t_2, t_3, t_4 < k} X_{t_1,t_2,t_3,t_4}.$$

Take $t = t_1 + t_2 + t_3 + t_4$. Then there are t vertices in $\bigcup_{i=1}^4 T_i$. Then there are $\Theta(n^{t+2})$ ways to choose each of these t vertices, v_1, and v_2. Further, let D be the distance between v_1 and v_2. Note that the cumulative distribution function $F_D(\rho) = \xi\rho^d$ and thus the probability density function is $\frac{d}{d\rho}F_D(\rho) = \xi d\rho^{d-1}$. Then by Lemma 6.2, we have[3]

$$\begin{aligned}
\mathbb{E}X_{t_1,t_2,t_3,t_4} &= O(1)n^{t+2}\int_0^{Q\varepsilon r} ((r')^d)^{t_1}((r')^{d-1}\rho)^{t_2+t_3+t_4}(1-(\xi(r')^d + C(r')^{d-1}\rho)^{n-t-2}\rho^{d-1}\, d\rho \\
&= O(1)n^{t+2}((r')^d)^t \int_0^{Q\varepsilon r}\left(\frac{\rho}{r'}\right)^{t_2+t_3+t_4} e^{-(\xi(r')^d+C(r')^{d-1}\rho)n+O(n(r')^{2d})}\rho^{d-1}\, d\rho \\
&= O(1)n^2(n(r')^d)^t \int_0^{Q\varepsilon r}\left(\frac{\rho}{r'}\right)^{t-t_1} e^{-(\xi(r')^d+C(r')^{d-1}\rho)n}\rho^{d-1}\, d\rho \\
&= O(1)n^2(\log n)^t \frac{e^\omega}{n(\log n)^{k-1}}\int_0^{Q\varepsilon r}\left(\frac{\rho}{r'}\right)^{t-t_1} e^{-C(r')^{d-1}\rho n}\rho^{d-1}\, d\rho,
\end{aligned}$$

where in the last line we use that $n(r')^d = \Theta(\log n)$ and $e^{-\pi(r')^d n} = \frac{e^\omega}{n(\log n)^{k-1}}$. Substituting $y = (r')^{d-1}\rho n$ and again using that $n(r')^d = \Theta(\log n)$, we have

$$\begin{aligned}
\mathbb{E}X_{t_1,t_2,t_3,t_4} &= O(1)e^\omega n(\log n)^{t-k+1}\int_0^{Q\varepsilon r(r')^{d-1}n}\left(\frac{y}{(r')^d n}\right)^{t-t_1} e^{-Cy}\frac{y}{(r')^{d-1}n}\cdot\frac{dy}{(r')^{d-1}n} \\
&= O(1)ne^\omega(\log n)^{t_1-k+1}\left(\frac{1}{(r')^{d-1}n}\right)^d \int_0^{Q\varepsilon r(r')^{d-1}n} y^{t-t_1+d-1}e^{-Cy}\, dy \\
&= O(1)ne^\omega(\log n)^{t_1-k+1}\left(\frac{r'}{\Theta(\log n)}\right)^d \int_0^{\Theta_\varepsilon(\log n)} y^{t-t_1+d-1}e^{-Cy}\, dy \\
&= O(1)ne^\omega(\log n)^{t_1-k+1-d}\frac{\Theta(\log n)}{n}\int_0^{\Theta_\varepsilon(\log n)} y^{t-t_1+d-1}e^{-Cy}\, dy \\
&= O(1)e^\omega(\log n)^{t_1-k+2-d}\int_0^{\Theta_\varepsilon(\log n)} y^{t-t_1+d-1}e^{-Cy}\, dy
\end{aligned}$$

[3] We wish to clarify that $d\rho$ in the integral below is the differential of ρ, not the dimension d multiplied by ρ.

Since $t - t_1 + d - 1$ is a constant and $C > 0$, $e^{-Cy} \ll y^{t-t_1+d-1}$, and thus

$$\int_0^\infty y^{t-t_1+d-1} e^{-Cy}\, dy \quad \text{and} \quad \int_0^\infty y^{t-t_1+d-1} e^{-Cy}\, dy - \int_0^{\Theta_\varepsilon(\log n)} y^{t-t_1+d-1} e^{-Cy}\, dy$$

are both constants. Thus,

$$\begin{aligned} \mathbb{E}X_{t_1,t_2,t_3,t_4} &= O(1) e^{\omega} (\log n)^{t_1-k+2-d} \int_0^\infty y^{t-t_1+1} e^{-Cy}\, dy \\ &= O(1) e^{\omega} (\log n)^{t_1-k+2-d}. \end{aligned}$$

Since $t_1 < k$, $2 \le d$, and $\omega = o(\log\log n)$, we have that $\mathbb{E}X_{t_1,t_2,t_3,t_4} = o(1)$. Thus,

$$\mathbb{E}X = \sum_{0 \le t_1,t_2,t_3,t_4 < k} \mathbb{E}X_{t_1,t_2,t_3,t_4} = \sum_{0 \le t_1,t_2,t_3,t_4 < k} o(1) = o(1).$$

Thus, a.a.s., no non-simple obstructions happen. $\square$

Since $G(n, r, p_{\text{nonstuck}}(r))$ is non-stuck, we know that all vertices in good and bad cells will eventually be active. Define a *survivor* to be a vertex that never becomes active. Then any survivors must happen in an ugly cell. By Lemma 6.1(ii), we know that clusters of ugly cells are far apart from one another (at least distance $100r$). If a cluster has one survivor, this must be because that survivor has degree less than k, so this is a simple obstruction. If a cluster has two or more survivors, find the two survivors that are furthest away from each other and call them v_1 and v_2. Call the set of survivors in this cluster J. Consider U_J, T_i, and t_i for $i = 0, \ldots, 4$ as defined above. Note that all survivors must fall in U_J. It must be that each of $t_2, t_3, t_4 < k$, else v_1 or v_2 would not be a survivor. If $t_1 \ge k$, then v_1 and v_2 are both not survivors. Thus, $t_1 < k$ and this gives a non-simple obstruction, which by Lemma 6.3, do not occur a.a.s. Therefore, the only survivors must be the vertices of degree less than k

that were not initially activated.

List of Global Symbols

r**:** In $\mathscr{G}(\boldsymbol{X}_n, r)$, the value so that for any two vertices $v_1, v_2 \in V$, if $\mathrm{dist}(v_1, v_2) \leq r$, then the edge $v_1 v_2 \in E$. 8

$\boldsymbol{X}_n$**:** $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_n)$; the set of (uniformly) random points in $\mathcal{T}_d$ that define the n vertices in $\mathscr{G}(\boldsymbol{X}_n, r)$ so that vertex $i \in V$ is at point X_i. 8

$\mathcal{T}_d$**:** The d-dimensional unit torus $[0,1)^d$. 8

$\mathscr{G}(\boldsymbol{X}_n; r)$**:** The random geometric graph defined on n vertices placed (uniformly) at random (where vertex i is given by the point X_i) in $\mathcal{T}_d$ so that for any two vertices $v_1, v_2 \in V$, if $\mathrm{dist}(v_1, v_2) \leq r$, then the edge $v_1 v_2 \in E$. 12

$r_{\delta \geq k}$**:** $\min\{r : \deg(i) \geq k \text{ for all } i \in V\}$. 13

$r_{k\textbf{-conn}}$**:** $\min\{r : \mathscr{G}(\boldsymbol{X}_n; r) \text{ is } k\text{-connected}\}$. 13

$\mathfrak{B}_k(\mathscr{G}; A_0)$**:** At each time step $t = 0, 1, 2, \ldots$, we define the set $A_t \subseteq V$, which we call *active vertices*. We call vertices in $V \setminus A_t$ *inactive* vertices. The vertices in A_0 are considered *initially active*. At time step $t+1$, a vertex $v \in V \setminus A_t$ turns active (or *becomes activated*) if v has at least k neighbors in A_t, and we place v in A_{t+1}. Once a vertex is active, it remains active, so $A_t \subseteq A_{t+1}$ for every $t = 0, 1, 2, \ldots$. In other words, for $t \geq 0$,

$$A_{t+1} = A_t \cup \{v \in V \setminus A_t : v \text{ has at least } k \text{ neighbors in } A_t\}.$$

We set $A_\infty = \bigcup_{t=0}^{\infty} A_t$, so A_∞ is the set of all vertices that begin as initially active or eventually become activated in $\mathfrak{B}(\mathscr{G}; A_0)$. 14

$A_0(p)$: The set of initially active vertices where each vertex is independently initially active with probability p. 15

$\boldsymbol{W}_n$: $(W_1, W_2, \ldots, W_n)$ where each W_i for $i \in [n]$ are i.i.d. uniform random variables taking values in $[0, 1]$. 15

$A_0(\boldsymbol{W}_n; p)$: $\{v_i \in V : W_i \leq p\}$. 15

$p_{\mathbf{nonstuck}}$: $\min\{p \in [0, 1] : A_1(p) \setminus A_0(p) \neq \emptyset\}$. 17

$p_{\mathbf{perco}}$: $\min\{p \in [0, 1] : \mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p))$ percolates$\}$. 17

ξ: The volume of the unit d-dimensional ball in the given norm $\|\cdot\|$. 17

$\mathscr{G}_{\mathbf{cells}}(\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p)))$: The graph of cells where the vertices

$$V_{\text{cells}} = \{c : c \text{ is a cell of } \mathscr{G}(\boldsymbol{X}_n; r) \text{ in the tessellation of } \mathcal{T}_d\}$$

and cells $c_1, c_2 \in V_{\text{cells}}$ are adjacent if the distance between the centers of c_1 and the center of c_2 is at most $\left(1 - \operatorname{diam}([0, 1]^d) S\right) r$. 39

$\mathscr{G}_{\mathbf{dense\ cells}}(\mathfrak{B}_k(\mathscr{G}(\boldsymbol{X}_n; r); A_0(p)))$: The graph of dense cells where the vertices

$$V_{\text{dense cells}} = \{c : c \text{ is a dense cell of } \mathscr{G}(\boldsymbol{X}_n; r) \text{ in the tessellation of } \mathcal{T}_d\}$$

and for $c_1, c_2 \in V_{\text{dense cells}} \subseteq V_{\text{cells}}$, c_1 and c_2 are adjacent if $c_1 \sim_{\mathscr{G}_{\text{cells}}} c_2$. 39

List of Theorem 3.4 Symbols

$Q_{\mathbf{atyp}}$**:** The number of atypical (with respect to ν) cells in $\mathcal{T}_d$. 27

$U_{c,\mathbf{seed}}$**:** The event that cell c is a concentrated seed. 29

$Q_{\mathbf{typ},\mathbf{seed}}$**:** The number of typical cells that are seeds in $D \cap L$. 29

List of Theorem 3.5 Symbols

List of Theorem 3.6(a) Symbols

S_i: The event that cell c_i is sparse. 43

T_2: The event that there is a cell that is not adjacent to a dense cell. 44

T_3: The event that $\mathscr{G}_{\text{dense cells}}$ is not connected. 44

Y_c: The number of vertices in a cell c. 45

$Y_{\mathbf{atyp}}$: The number of atypical cells in $\mathcal{T}_d$. 45

F: The event that all except at most a o fraction of diam$(\mathcal{T}_d)$-boxes have all its cells typical. 45

Y: The number of diam$(\mathcal{T}_d)$-boxes that do not have all its cells typical. . 45

$Y_{\mathscr{b},\mathbf{seed}}$: The event that a diam$(\mathcal{T}_d)$-box $\mathscr{b}$ with all its cells typical is a seed. 46

$Z(\mathscr{b})$: The event that all cells in 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$ are typical. 47

$Z_1(\mathscr{b})$: The event that there is an atypical cell in 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$. 47

Z_1: The number of 1-boxes with an atypical cell in its associated superbox. 47

$\mathcal{W}_{\mathscr{b}}$: The event that 1-box $\mathscr{b}$ is wishy-washy. 48

$A_{\mathscr{b}}$: The event that 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$ contains exactly k active vertices. 48

A_t: The event that all of the k initially active vertices in $\mathscr{s}$ fall in $\mathscr{t}$. 48

$\mathcal{W}$: The number of wishy-washy boxes. 49

$\mathcal{Y}_s$: The number of vertices in superbox $\mathscr{s}$. 50

μ: $\mathbb{E}\mathcal{Y}_s = \frac{3^d a}{\xi}$; the average number of vertices in a superbox. 50

$A(\mathscr{b})$: The number of initially active vertices in 1-box $\mathscr{b}$'s associated superbox. 50

$\mathcal{Y}_{\mathbf{atyp}}$: The number of atypical superboxes. 50

$\mathcal{Y}_{\mathbf{typ}}(\mathscr{b})$: The event that 1-box $\mathscr{b}$'s associated superbox is typical. 50

$\mathcal{Y}_{\mathbf{typ}}$: The number of typical superboxes. 50

$m_{s,L}$: The number of superboxes intersecting L. 50

$\mathcal{Y}_{\mathbf{typ},L}$: The number of typical superboxes intersecting L. 50

$Z_{3,L}$: The number of typical superboxes intersecting L with more than k active vertices. 50

$Z_{1,L}$: The number of superboxes intersecting L that have an atypical cell. 53

$Z_2(\mathscr{b})$: The event that 1-box $\mathscr{b}$'s associated superbox is typical but contains an atypical cell. 53

Z_2: The number of 1-boxes whose associated superbox is typical but contains an atypical cell. 53

$Z_{4,L}$: The number of typical superboxes intersecting L with some atypical cell and exactly k active vertices at time p. 53

Z_5: The number of superboxes intersecting L with an atypical cell and exactly k active vertices at time p. 54

Z_6: The number of superboxes on the boundary of L with no atypical cell and at least k initially active vertices. 55

List of Theorem 3.6(b) Symbols

p^*: $\frac{1}{(n\ell^d)^{1/k}(a)^{1-1/k}}$. 57

γ: $\varepsilon^{-1/2k}$. 57

D_1: The largest component of dense cells in the graph of dense cells. 57

E_0: The event that $|D_1| \leq (1-f)m$ for some $f = o(1)$. 57

$E_{1,p}$: The event that there is no seed box intersecting L at time p. 57

$E_{2,p}$: The event that there is some wishy-washy box intersecting L at time p. 57

$E_{3,p}$: The event that some superbox intersecting L has more than k active vertices at time p. 57

$E_{4,p}$: The event that some superbox intersecting L with an atypical cell has exactly k active vertices at time p. 57

$E_{5,p}$: The event that some superbox with no atypical cell intersecting the boundary of L has at least k active vertices at time p. 57

Y_c: The number of vertices in a cell c. 58

$Y_{\mathbf{atyp}}$: The number of atypical cells in $\mathcal{T}_d$. 59

F: The event that all except at most a $o(1)$ fraction of diam($\mathcal{T}_d$)-boxes have all its cells typical. 60

Y: The number of diam($\mathcal{T}_d$)-boxes that do not have all its cells typical. 60

$Y_{\mathscr{b},\mathbf{seed}}$: The event that a diam($\mathcal{T}_d$)-box $\mathscr{b}$ with all its cells typical is a seed. 60

$m_{\mathscr{b}}$: $\frac{n\xi}{a}$; the number of 1-boxes in $\mathcal{T}_d$. 62

$Z(\mathscr{b})$: The event that all cells in 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$ are typical. 62

$Z_1(\mathscr{b})$: The event that there is an atypical cell in 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$. 62

Z_1: The number of 1-boxes with an atypical cell in its associated superbox. 62

$\mathcal{W}_{\mathscr{b}}$: The event that box $\mathscr{b}$ is wishy-washy. 62

$A_{\mathscr{b}}$: The event that 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$ contains exactly k active vertices. 62

$\mathcal{W}$: The number of wishy-washy boxes. 64

$\mathcal{Y}_{\mathscr{s}}$: The number of vertices in a superbox $\mathscr{s}$. 64

μ: $\mathbb{E}\mathcal{Y}_{\mathscr{s}} = \frac{3^d a}{\xi}$; the average number of vertices in a superbox. 64

$A(\mathscr{b})$: The number of initially active vertices in 1-box $\mathscr{b}$'s associated superbox $\mathscr{s}$. 64

$\mathcal{Y}_{\mathbf{atyp}}$: The number of atypical superboxes. 64

$\mathcal{Y}_{\mathbf{typ}}(\mathscr{b})$: The event that 1-box $\mathscr{b}$'s associated superbox is typical. 64

$\mathcal{Y}_{\mathbf{typ}}$: The number of typical superboxes. 64

$m_{\mathscr{s},L}$: The number of superboxes intersecting L. 64

$\mathcal{Y}_{\mathbf{typ},L}$: The number of typical superboxes intersecting L. 64

$Z_{3,L}$: The number of typical superboxes intersecting L with more than k active vertices. 64

$\mathcal{Z}_i(\mathscr{b})$**:** The event that 1-box $\mathscr{b}$'s associated superbox is atypical of type i. 66

$\mathcal{Z}_i$**:** The number of superboxes that are atypical of type i. 66

$\mathcal{Z}_{1,i}$**:** The number of superboxes intersecting L that are atypical of type i and that have more than k active vertices. 66

$\mathcal{Z}_{1,i}(\mathscr{b})$**:** $\begin{cases} 1, & \mathscr{b}\text{'s associated superbox is type } i \text{ and has more than } k \text{ active vertices,} \\ 0, & \text{otherwise.} \end{cases}$. 66

$\mathcal{Z}_{2,L}$**:** The number of atypical superboxes intersecting L with more than k active vertices. 67

$\mathcal{Z}_{3,L}$**:** The number of superboxes intersecting L with more than k active vertices. 70

$Z_2(\mathscr{b})$**:** The event that 1-box $\mathscr{b}$'s associated superbox is typical but contains an atypical cell. 71

Z_2**:** The number of 1-boxes whose associated superbox is typical but contains an atypical cell. 71

$Z_{4,L}$**:** The number of typical superboxes intersecting L with some atypical cell and exactly k active vertices at time p. 71

$\mathcal{Z}_{4,i}$**:** The number of superboxes intersecting L that are atypical of type i with exactly k active vertices. 71

$\mathcal{Z}_{4,i}(\mathscr{b})$**:** $\begin{cases} 1, & \mathscr{b}\text{'s associated superbox is atypical of type } i \text{ and has } k \text{ active vertices,} \\ 0, & \text{otherwise.} \end{cases}$. 71

List of Theorem 3.6(c) Symbols

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