DESIGN AND ANALYSIS OF COMPETITIVE ELECTRICITY MARKETS

BY

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DISSERTATION

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ABSTRACT

This thesis focuses on the study of allocation mechanisms and pricing schemes for the design and analysis of competitive electricity markets. Motivated by the increasing demand-side participation in high- and low-voltage power grids, we consider two-sided competition models where a finite group of producers and consumers compete through scalar-parameterized supply offers and demand bids. Acting as a smooth approximation to supply offers used in practice, scalar-parameterized offers greatly facilitate mathematical analysis while preserving the primary determinants and mechanisms by which market power is exercised in electricity markets. In the framework of a pool-based market, characterized by a central dispatch and pricing mechanism, when strategic, capacity-constrained suppliers face strategic, price-responsive consumers, we show that market allocative efficiency loss and price markup at the Nash equilibrium are bounded. We demonstrate analogous efficiency bounds in the study of inter-area electricity markets where we exploit scalar-parameterized offers to model budget-constrained price arbitrageurs that compete against affine inter-area price spreads. Our analysis provides important insights on the type of behavior that may occur at the equilibrium including the pivotal role assumed by certain players, the impacts of aggregate liquidity and uncertainty as well financial positions in other electricity markets. Through the application of reinforcement learning algorithms we demonstrate that players can discover their equilibrium actions even when they know little to nothing about the game setting.

The simplicity of scalar-parameterized supply offers that grant market actors' one-dimensional action spaces while properly constraining their strategic flexibility, render such offer/bid structures an attractive candidate for the expansion of electricity markets to distribution grids. Motivated by the rapid proliferation of distributed energy resources that increasingly hold value for the grid either as power suppliers or flexible demand, we leverage scalar-parameterized supply offers together with appropriate pricing schemes to design a pool-based market for the retail sector. Our goal is complicated by the underlying physics of distribution grids that render the central dispatch problem, in its full generality, non-linear and non-convex. To get around this difficulty, we exploit semidefinite relaxations of the optimal power flow problem and leverage duality theory to define prices for electricity as the optimal Lagrange multipliers of nodal real and reactive power balance constraints. We demonstrate that such prices stand on sound economic principles that together with scalar-parameterized offers/bids, constitute a comprehensive mechanism for the expansion of markets to the low-voltage side of the electric power grid.

To all talented and hard-working women engineers who never give up.

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CHAPTER 1

INTRODUCTION

This chapter presents the motivation for the thesis, gives the outline of the chapters, and states the original contributions of the thesis. There are no dedicated chapters covering a literature review or to establish notation. Rather, the literature is reviewed and notation is established in each chapter and section where it is appropriate.

1.1 Electricity Markets: Where They Stand and Where They Are Progressing

Spot pricing of electricity [1, 2] laid the theoretical foundations for the restructuring of wholesale electricity markets. The underlying purpose for restructuring the power sector is, at least in theory, to unleash the forces of competition, improve efficiency and eventually reduce the costs for consumers. Decades of experience with restructuring has led to a bid-based, security constrained unit commitment and economic dispatch model as the reference paradigm for wholesale electricity market design, closely following the guidelines of the original theory.

Restructured electricity markets are, typically, administered by a central authority such as regional transmission organizations (RTOs) or independent system operators (ISOs) that are responsible for clearing the market while ensuring the stable and reliable operation of the power system. Figure 1.1 illustrates geographical areas in which organized electricity markets are operated by an ISO/RTO, henceforth referred to as system operator (SO). Each SO-operated electricity market reflects a two-settlement system with day-ahead markets (DAMs) and balancing real-time markets (RTMs). The DAM clears to meet the bid-in load for each hour of the day, one day in advance. Power schedules and locational marginal prices (LMPs) are cal-

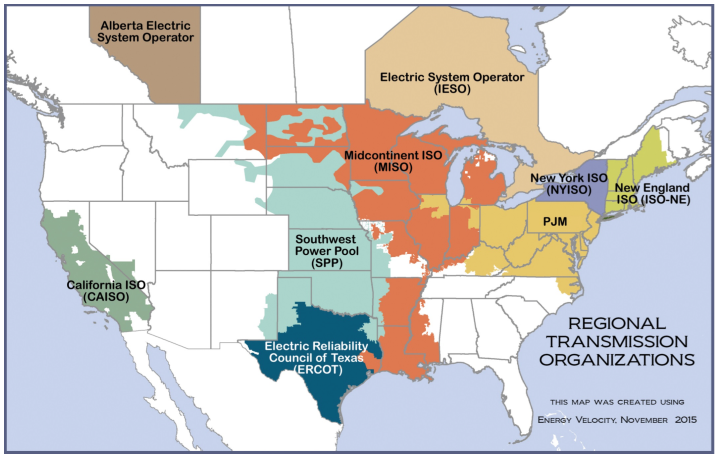
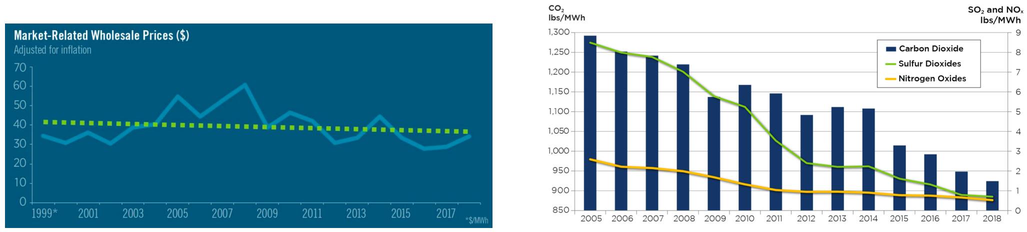


Figure 1.1: RTOs/ISOs in North America. Uncolored regions represent states that operate under the vertically integrated utility paradigm.

culated from the market-clearing process and these price-quantity pairs are settled for all market participants. To reflect changes that may occur between day-ahead and real-time, the RTM is used to re-dispatch resources to meet imbalances caused by variability and uncertainty, induced largely by demand fluctuations and intermittent renewable generation.

Organized wholesale electricity markets have, indeed, contributed to reduction of electricity costs and encouraged innovation through competition. For example, PJM reports that their market has saved consumers at least \$3.2 billion a year by integrating more efficient resources and ensuring the lowest production costs [3]. Figure 1.2 illustrates how wholesale electricity prices have remained largely flat since the introduction of markets in 1999, while the generation mix is about 30% less carbon-intensive than ten years ago. Even utilities located in fully regulated states benefit from organized markets as they can offer or purchase electricity in the market when it makes economic sense. However, there is growing concern that the ability of existing electricity market designs to continue to deliver such benefits will drastically diminish, as radical developments and transformations are under way.

This thesis focuses on three major forces propelling the overall transformation in the power industry and electricity markets: (1) massive deployment of large-scale renewable generation, (2) increased participation of the consumer-side, and (3) widespread adoption of distributed energy resources (DERs) in the low-voltage grid. Such developments—facilitated by the deployment of rapidly improving technologies that travel down the cost curve—raise nu-



(a) (b)

Figure 1.2: Benefits of organized electricity markets: wholesale prices have remained flat (left) and average emissions have steadily declined in PJM (right).

merous markets design questions and require a growing toolbox of solutions, a set of which is presented in this thesis.

Integration of grid-scale renewable resources such as wind and solar is accelerating (see Figure 1.3) and the trend is likely to continue. Such resources are uncertain, intermittent and largely uncontrollable, i.e., cannot be dispatched on demand. Such variability makes it challenging to balance demand and supply across a transmission-constrained power network at all times. Current electricity markets accommodate said uncertainty in a rather ad-hoc manner, e.g. by planning for the nominal scenario and choosing fixed reserve margins to deal with forecast errors. Unfortunately, deepening renewable penetrations ultimately lead to greater forecast errors and prohibitively large reserve margins [4, 5].

Unpredictability and variability are sometimes interpreted as “missing markets” problems. This has been largely addressed through the introduction of ancillary service markets. However, having separate markets may create opportunity for arbitrage or exercise of market power. Ideally, markets should optimize against uncertainty at the forward stage, i.e., DAM, with the explicit incorporation of uncertainty and produce price signals that internalize said uncertainty. Any such mechanism should yield efficient market allocations and guarantee revenue adequacy, at least in expectation. Moreover, it should ensure bounded efficiency losses in market allocations and societal welfare when participants exercise market power.

Besides suitable modeling and analytical tools, increasing coordination across larger geographic areas with diverse resources and weather patterns facilitates renewable resource integration. To harness the low-cost and clean energy from these resources we need an almost *seamless* operation of the interconnected grid; that is an efficient way to transfer power across different grids to move more renewable generation from the middle of the country to the coasts. As such, consumers can receive the benefits of these resources no matter how far away they are located. Moreover, a dynamic and efficient scheduling of power across different regions, allows for real-time reserve provision improving reliability, which is becoming increasingly critical due to the variable output of these renewable resources. Even systems that are not part of organized markets can benefit from regional coordination. For example, CAISO and PacifiCorp formed an Energy Imbalance Market (EIM) in 2014 to "manage resource deviations, smoothing out power flows so that renewable energy is effectively integrated into the grid" [6]. The EIM covers fourteen western states, comprising mostly of energy balancing authorities that are not organized into regional markets.

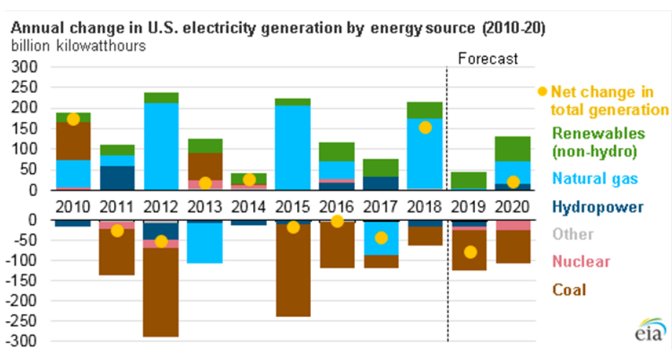


Figure 1.3: Annual change in U.S. electricity generation by source.

The gathered momentum with grid-scale renewable generation is spreading swiftly down the electric power value chain, as grids in many regions become increasingly decentralized and host a growing number of distributed energy resources (DERs). A DER is "any resource on the distribution system that produces electricity and is not otherwise included in the formal NERC definition of the Bulk Electric System (BES)" [7]. This encompasses many resource types and technologies such as solar photovoltaics (PV), combined heat and power (CHP), small-scale wind turbines, energy storage, demand response as well as electric vehicles (EV) and EV charges.

While once viewed primarily as a threat to utility business models, DERs are beginning to be considered as valuable tools to: reduce carbon emissions, access new revenue streams for utilities and other providers, defer investment in transmission and distribution infrastructure, improve grid efficiency and asset utilization for utilities and customers, and fulfill renewable portfolio standards (RPS). DERs could develop into a significant tool for managing integrated grid-scale renewable resources. For example, SOs could reach behind the meter to tap into customers' largely unused storage devices or demand response to smooth out fluctuations in renewable output. In the wholesale market, demand response at times of high renewable availability, helps mitigate the revenue slump caused by renewable suppliers with near-zero marginal costs [8]. However, the majority of demand flexibility is due to residential and commercial customers connected at the low-voltage side of the grid [9] and are currently largely excluded from wholesale markets. To harness said benefits of DERs, the industry needs first to develop the regulatory and market structures necessary to access, monitor and manage these resources.

The challenge involves efforts to simulate, quantify, and monetize the value created by DERs. Since these resources are connected to the distribution side of the grid, pricing energy and ancillary services for the distribution network is becoming an increasingly important aspect of electricity market design. The integration of DERs into a market clearing process raises numerous market design questions including bid/offer structures, products to be traded, network model, pricing and settlement schemes, interactions with wholesale markets and so forth. This thesis advocates that redesigning and expanding electricity markets is not only imperative but it constitutes a cost-effective solution to manage the challenges posed by these trends. Our goal is to provide market tools and solutions to appropriately model, analyze and redesign existing or future electricity markets.

1.2 Outline of Thesis

This thesis consists of six chapters (including introduction and conclusion), which concentrate on two main areas: first, the introduction of market mechanisms that are suitable for modeling and analytical purposes and second, the deployment of such mechanisms for the design of comprehensive market frameworks. To measure the performance of the aforementioned mechanisms we study the market outcome under two specific conditions: (i) market participants are pure price-takers, i.e., the market resembles conditions of perfect competition, and (ii) participants exercise market power and distort market outcomes. As such, we make extensive use of concepts and tools from welfare economics and game theory.

Chapter 2 introduces the market mechanism that is utilized for market design and analysis throughout this thesis. It consists of a particular family of scalar-parameterized supply offers and demand bids that allow suppliers and consumers to declare their private information in the market. We consider a two-sided market where both suppliers and consumers compete for a product with the allocations determined by a central manager or market operator. Our model incorporates production capacity constraints and minimum inelastic demand requirements. We explicitly show that under perfect competition, there exist prices such that the mechanism yields allocations that maximize social welfare. When market participants are strategic, we explicitly characterize the Nash equilibrium and the market allocation at the equilibrium. We prove that strategic interactions cannot cripple market performance and provide bounds on the welfare loss and price markup at the equilibrium. We conclude the chapter by showing that efficient allocations can be sustained for a market that operates under (convex) network constraints.

In Chapter 3 we demonstrate the analytical prowess of scalar-parameterized supply offers. To that end, we investigate regional, market-based, coordination mechanisms such as coordinated transaction scheduling (CTS). We discuss the mechanics of CTS and we set up a theoretical framework that we use as the proxy of CTS markets. We demonstrate how scalar-parameterized offers can be effectively utilized to model pure price-arbitrageurs facing interregional prices spreads, which we camouflage as demand functions. The effectiveness of said mechanism is explicitly illustrated by its capability to reveal the well-documented flaws of CTS: (1) lack of market liquidity, (2) transaction fees, and (3) SO's forecast errors. We explicitly quantify the impact of strategic participants by deriving the market allocation at the Nash equilibrium. Moreover, employing simple reinforcement learning algorithms we show that such outcomes can be learned through repeated interactions of players with CTS markets.

Moving in the direction of electricity market design, in Chapter 4 we focus on price formation and settlement schemes for electricity markets with full consideration of the power network. We start off with the non-convex, non-linear, market clearing model referred to as alternating current optimal power flow model (AC-OPF). Motivated by recent work on distribution LMPs (DLMPs), we explore economic properties of a pricing scheme that utilizes optimal Lagrange multipliers from a semidefinite programming (SDP) relaxation of AC-OPF. We call these prices RLMPs. We show explicitly how such prices possess a number of nice properties: RLMPs support efficient market equilibria, guarantee revenue adequacy and minimize a form of side-payments when the duality gap between AC and SDP is nonzero.

RLMPs together with the scalar-parameterized supply offers and demand bids are the primary constituents for the design of a comprehensive electricity market for distribution networks. In Chapter 5, we set up this market framework, which explicitly takes into consideration the characteristics of distribution networks. Here, we define appropriate DLMP price signals (a special case of RLMPs) that can be used to compensate resources connected at the distribution level, such as DERs. The said prices support efficient market allocations and ensure, under mild conditions, that the SO remains solvent after settling all transactions. The market framework presented in Chapter 5 constitutes a comprehensive effort to design appropriate mechanisms for low-voltage power customers to offer their services to the grid and receive compensation, while a central entity ensures the secure operation of the grid.

1.3 Original Contribution of the Thesis

First of all, to the best knowledge of the author, this thesis is the first piece of work that presents a comprehensive electricity market framework for distribution networks with explicitly defined offer/bid structures and clearing mechanisms. Moreover, the analysis presented in Chapters 2 and 6, is a generalization of earlier efforts that focused on properties of scalar-parameterized supply offers when the demand-side is perfectly inelastic. In this thesis, consumers are active participants in two ways: submit individual demand bids—the *dual* version of said offers—and in aggregate as a smooth, elastic demand function in CTS. In particular, said offer/bid structures can effectively model:

- Controllable power producers (e.g., natural gas, fossil fueled generation) with capacity constraints
- Renewable generation resources with uncertain supply and maximum production capacity
- Price-arbitrageurs / virtual bidders with budget constraints
- Inelastic demand/load
- Price-responsive demand
- Uncertain demand
- Generation assets and consumers connected in three-phase unbalanced distribution networks

Several results presented in this thesis are joint work with other research collaborators. Such results are merely stated for completeness. Whenever this is the case, we mention it explicitly and cite the relevant work for reference to the reader. In detail the original contributions of this thesis are:

- Section 2.3, Theorem 1: We prove that two-sided markets with scalar-parameterized offers/bids, support efficient market equilibria under pure price-taking participants. This result illustrates that when market actors are non-strategic and are restricted to use a particular family of bids/offers, then the market yields allocations that (i) maximize society's welfare, and (ii) are incentive compatible.
- Section 2.4, Theorem 2: For the two-sided market framework, we establish existence and uniqueness of the Nash equilibrium for strategic market participants. Furthermore, Theorem 2 provides a computationally efficient way to characterize the Nash equilibrium and the market allocation. This is achieved by solving a convex program.
- Section 2.5, Theorem 3: Under strategic interactions, the misrepresentation of private information has the potential to induce market allocations that are suboptimal to the efficient outcome. Theorem 3 explicitly characterizes the bounds on welfare loss and price markup

at the Nash equilibrium. This result demonstrates that the scalar-parameterized functions do not cripple market performance when participants are strategic.

- Section 3.3, Theorem 4: In the analysis of CTS markets, we establish sufficient conditions that guarantee existence of a Nash equilibrium in competition models with scalar-parameterized supply functions facing elastic demand functions.

- Section 3.4, Proposition 1: For affine demand models, we characterize the unique Nash equilibrium of the CTS game and quantify the impacts of market liquidity on scheduling efficiency. Specifically, CTS outcome yields efficient allocations when market liquidity is high. In the intermediate liquidity regime, efficiency loss of CTS is at most 25%. When market liquidity is prohibitively low, inefficient outcomes are a result of lack of liquidity and not the strategic interactions of players.

- Section 3.4.2: The Nash equilibrium of CTS can be discovered through the application of the upper confidence bound (UCB) algorithm. We apply UCB to a game with five players, where each player is given a discrete set of actions, including the Nash equilibrium strategy. Players' actions converge within a finite number of plays. To the best of the author's knowledge, this is the first application of UCB to supply function competition with scalar-parameterized models.

- Section 3.5, Proposition 2: This result demonstrates the coupling of CTS with virtual transactions in other energy markets. To the best of the author's knowledge this is the first analysis that highlights potential uneconomic bidding due to players holding combined positions in CTS and other energy markets.

- Section 3.6, Proposition 3: This result reflects a somewhat counter-intuitive outcome: the incentives of CTS bidders are aligned in a way that allows them to *correct* SO's forecasts or systematic bias. This result is counter-intuitive in the sense that, in electricity markets, the SO is, typically, viewed as the ultimate authority that has the best information on the system's state, and thus strategic participants are expected to drive markets to suboptimal outcomes. To the best of the author's knowledge, this is the first theoretical result that contradicts perceived suboptimality of CTS. Moreover, Proposition 3 demonstrates the impact of transaction fees on achieving the goal of CTS: to converge prices between neighboring markets. Our analysis demonstrates that transaction fees act as barrier to trade and prevents bidders from offering their entire budgets.

- Section 4.3, Theorem 5: We show that RLMPs, defined as the Lagrange multipliers of an optimal solution to the SDP dispatch problem, support efficient market equilibria and guarantee revenue adequacy under mild conditions. Moreover, we explicitly compute the duality gap between AC and SDP economic dispatch problems, and show that it comprises two terms: the lost opportunity cost (LOC) and the product revenue shortfall (PRS). Theorem 5 establishes that RLMPs minimize a form of side-payments whenever the duality gap is nonzero, a feature common to convex-hull pricing (CHP).

- Section 5.3: We define DLMPs for real and reactive power in multi-phase distribution networks. That is, there is a price for electricity for each phase at every node in the network. Moreover, in Theorem 6, we combine DLMPs together with supply function competition in scalar-parameterized offers/bids support efficient market allocations.

CHAPTER 2

A SCALAR-PARAMETERIZED MECHANISM FOR TWO-SIDED MARKETS

We introduce the market mechanism that serves as the main tool to analyze and design competitive electricity markets. We begin by considering a general market setting where both suppliers and consumers compete for a product. We restrict our attention to a particular family of supply offers and demand bids referred to as scalar-parameterized offers/bids. We study properties of said mechanism when market participants are pure price-takers, and when they are strategic. The goal of the analysis presented here is to demonstrate the strengths of scalar-parameterized offer/bid structures in the design and analysis of any market and electricity markets in particular.

2.1 Supply Function Competition in Electricity Markets

In wholesale electricity markets, the behavior of market participants has been largely modeled and analyzed through the well-known Bertrand and Cournot competition models, which are simple (degenerate) price/quantity offer strategies [10, 11, 12, 13, 14, 15, 16, 17]. However, the Bertrand model typically assumes that each participant is willing to supply the entire demand, which may not be satisfied in a number of cases. Variations of the Bertrand model with capacity constraints have been proposed, however, in such settings pure Nash equilibria may not exist [18]. The Cournot model has a number of appealing properties when studying oligopolies in markets with relatively high demand elasticity. However, when demand elasticity is low, Cournot competition may exhibit arbitrarily high welfare loss [19]. For day-ahead electricity markets in particular, pure quantity/price competition cannot adequately represent supply offers or demand bids of market participants. In such markets power producers submit varying quantities at successively higher prices and the demand-side specifies the quantity willing to purchase at successively lower prices.

One is then forced to consider offer/bid structures that allow participants more degrees of freedom to declare their preferences. The seminal work by Klemperer and Meyer [20] demonstrated that in the absence of uncertainty there exist an enormous multiplicity of equilibria in supply functions. Hence, we must restrict attention to a particular family of supply functions. Linear supply functions make up another candidate used to model electricity markets—although incorporating capacity constraints into linear supply offers is not straightforward [21]. Moreover, arbitrary high-efficiency loss at the Nash equilibrium is possible, particularly when suppliers have highly heterogeneous cost functions [22].

What is then a suitable mechanism for electricity markets? We answer this question in conjunction to the major transformations currently taking place in electric power systems. In particular, the emergence of a potential retail marketplace [23] incites greater demand-side participation all the while higher grid-scale renewable generation and DERs increase uncertainty in market outcomes. Hence, the ideal candidate for offer/bid structures: (i) must provide enough flexibility to participants to declare their preferences against a range of possible market outcomes, (ii) is simple enough to facilitate widespread participation in retail and wholesale sectors, (iii) facilitates modeling of the demand-side without severely complicating analysis, (iv) sustains competitive outcomes, and (v) does not cripple market performance when participants exercise market power.

In this thesis, we restrict our attention on a specific family of supply offers and demand bids, referred to as scalar-parameterized supply/demand functions, studied in [24, 25]. The specific family of supply functions allows market actors to have one-dimensional action spaces, when faced with a single market price. Such market mechanisms are simple to implement and are considered to be fair among market participants. In [26], the authors show that said mechanisms possess a number of attractive properties including bounded price of anarchy and price markup at the Nash equilibrium. The family of supply functions considered here is a capacitated version similar to those employed by [27, 28]. Such supply functions prohibit situations where firms can offer in the market beyond their means. Understanding the impact of capacity constraints is critical in many industries, including the electric power sector where the importance and irreversibility of investment on production capacities impose long-term decisions. In the sequel, we demonstrate why scalar-parameterized offer/bids is the best-possible mechanism available for certain market structures that exhibits a number of desired properties.

Notation: Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+$ the set of non-negative real numbers. Denote the transpose of a vector $\mathbf{x} \in \mathbb{R}^n$ by $\mathbf{x}^\intercal$. Let $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ be the vector including all but the ith element of $\mathbf{x}$. Finally, denote by $\mathbf{1}$ the vector of all ones with appropriate size.

2.2 A Two-Sided Market on Copperplate Power System

We consider a market that consists of a collection of consumers $\mathcal{I}$, a collection of suppliers $\mathcal{J}$, and a central entity or market manager. In particular,

- *Consumers:* Consumer $i \in \mathcal{I}$ demands amount d_i, which must be greater than some minimum inelastic demand requirement d_i^0. Each consumer derives utility $U_i(d_i)$ from consuming amount d_i. For each $i \in \mathcal{I}$, U_i is assumed smooth, concave and strictly increasing for $d_i \geq d_i^0$ with $U_i(d_i^0) = 0$.

- *Suppliers:* Supplier $j \in \mathcal{J}$ offers amount s_j, which must lie below some maximum (nameplate) production capacity denoted by κ_j^0. Each supplier incurs costs $C_j(s_j)$ for producing quantity s_j. For each $j \in \mathcal{J}$, $C_j(s_j)$ is assumed smooth, convex and strictly increasing with $C_j(s_j) \geq 0$ for $s_j \geq 0$. Over the domain $s_j \leq 0$, $C_j(s_j) = 0$.

- *Market manager:* The manager collects supply offers and demand bids from market participants and implements a centralized market mechanism that defines: (1) the amount each producer/consumer supplies/demands, and (2) the market price and payments for every producer/consumer.

The market manager would ideally like to compute a market allocation that maximizes society's welfare while operating within the constraints of the individual participants. Let $\mathbf{d} \in \mathbb{R}_+^{|\mathcal{I}|}$ and $\mathbf{s} \in \mathbb{R}_+^{|\mathcal{J}|}$ denote the collection of demand and supply quantities, respectively. Then, the market manager would like to solve the following program.

$$\begin{aligned} \text{maximize} \quad & \mathcal{W}(\mathbf{d},\mathbf{s}) := \sum_{i\in\mathcal{I}} U_i(d_i) - \sum_{j\in\mathcal{J}} C_j(s_j), && (2.1a)\\ \text{subject to} \quad & \sum_{i\in\mathcal{I}} d_i = \sum_{j\in\mathcal{J}} s_j, && (2.1b)\\ & 0 \le s_j \le \kappa_j^0, && (2.1c)\\ & d_i^0 \le d_i, && (2.1d)\\ & \text{for each } i\in\mathcal{I}, j\in\mathcal{J}, \end{aligned}$$

over the variables $\mathbf{d}, \mathbf{s}$. Any demand plan $\mathbf{d}$ and supply plan $\mathbf{s}$ constitutes an *efficient* market allocation if it solves (2.1). Such allocations can be determined if the market manager has perfect knowledge on the market and all participants. However, U_i and C_j are, typically, private information and thus, not available to the market manager. The first question we ask is: *Is there a mechanism that allows market participants to reveal their private information in a way that yields efficient market allocations?* In what follows, we present such a mechanism based on scalar-parameterized supply functions and demand bids.

Consider consumer $i \in \mathcal{I}$ that provides to the market manager demand bid θ_i^d such that, given a market price $\lambda > 0$, the consumer is willing to buy quantity

$$d_i = D(\theta_i^d, \lambda) := d_i^0 + \frac{\theta_i^d}{\lambda}, \quad \theta_i^d \ge 0. \tag{2.2}$$

The expression in (2.2) represents the quantity the consumer is willing to buy, given the inelastic component d_i^0, the market price and the parameter θ_i^d. The inelastic demand d_i^0 represents the minimum quantity the consumer must be supplied while θ_i^d/λ represents the price-responsive portion of their demand. Note that the demand bid is decreasing in price, i.e., it is downward sloping. Similarly, consider firm $j \in \mathcal{J}$ that submits to the market supply offer θ_j^s such that, given the market price $\lambda > 0$, they are willing to supply

$$s_j = S(\theta_j^s, \lambda) := \kappa_j^0 - \frac{\theta_j^s}{\lambda}, \quad \theta_j^s \geq 0. \tag{2.3}$$

The supply offer (2.3) represents the quantity supplier j is willing to offer as a function of price. The supply offer is further parameterized in the capacity κ_j^0, which represents supplier j's maximum production capacity. Observe that as demand approaches its inelastic portion d_i^0, consumer i's willingness to buy approaches infinity. Similarly, as the supply quantity approaches firm j's maximum capacity κ_j^0, the requested market price grows unbounded.

Remark 1. *A possible drawback of the class of supply functions in* (2.3) *is that it allows market participants to offer negative quantities. Nothing rules this out in the definition of the mechanism and suppliers may be rightly nervous to agree in a mechanism with such a property. However, we will show that such situations cannot arise at a market equilibrium, under both competitive and strategic behavior.*

Let $\boldsymbol{\theta}^d = (\theta_1^d, \ldots, \theta_{|\mathcal{I}|}^d)$ and $\boldsymbol{\theta}^s = (\theta_1^s, \ldots, \theta_{|\mathcal{J}|}^s)$ denote the collection of demand bids and supply offers, respectively. The market manager chooses price $\lambda(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s) > 0$ to clear the market such that supply equals demand, i.e.,

$$\sum_{i \in \mathcal{I}} D(\theta_i^d, \lambda) = \sum_{j \in \mathcal{J}} S(\theta_j^s, \lambda). \tag{2.4}$$

Such choice is only possible when $\mathbf{1}^\mathsf{T}\boldsymbol{\theta}^d + \mathbf{1}^\mathsf{T}\boldsymbol{\theta}^s > 0$ in which case the market price is given by

$$\lambda(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s) = \frac{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}^d + \mathbf{1}^\mathsf{T}\boldsymbol{\theta}^s}{\kappa_{|\mathcal{J}|}^0 - d_{|\mathcal{I}|}^0}, \tag{2.5}$$

where $d_{|\mathcal{I}|}^0 = \sum_{i \in \mathcal{I}} d_i^0$ and $\kappa_{|\mathcal{J}|}^0 = \sum_{j \in \mathcal{J}} \kappa_j^0$. We assume throughout that $\kappa_{|\mathcal{J}|}^0 > d_{|\mathcal{I}|}^0$ and the market price is well-defined. In the case where $\mathbf{1}^\mathsf{T}\boldsymbol{\theta}^d + \mathbf{1}^\mathsf{T}\boldsymbol{\theta}^s = 0$, i.e., every market participant submits zero parameter, we adopt the following conventions

$$D(0,0) = d_i^0, \ \forall i \in \mathcal{I} \text{ and } S(0,0) = \kappa_j^0, \ \forall j \in \mathcal{J}. \tag{2.6}$$

For markets with perfectly inelastic demand, the residual supply index (RSI) is often adopted as a suitable indicator of market power. Precisely, the RSI of firm j measures the capability of the aggregate market capacity—

excluding that of j—to meet total inelastic demand. In the market model considered here, the inelastic portion of demand is $d^0_{|\mathcal{I}|}$. Mathematically, if

$$\mathsf{RSI}_j := \frac{\kappa^0_{|\mathcal{J}|} - \kappa^0_j}{d^0_{|\mathcal{I}|}}$$

is strictly less than one, then firm j is said to be *pivotal*. See [29] and [30] for further details. As we show in Section 2.4, the presence of pivotal suppliers is critical in the analysis of the market outcome under strategic interactions.

2.3 Perfect Competition

In this section, we study the market outcome assuming all market participants are pure price-takers. We aim to establish the existence and characterization of the market equilibrium taking into account the profit-maximizing nature of suppliers and consumers.

Given market price $\lambda > 0$, each consumer maximizes the payoff

$$\pi_i(\theta^d_i, \lambda) = U_i\left(D\left(\theta^d_i, \lambda\right)\right) - \lambda D\left(\theta^d_i, \lambda\right), \; i \in \mathcal{I}. \tag{2.7}$$

Similarly, each supplier maximizes

$$\pi_j(\theta^s_j, \lambda) = \lambda S(\theta^s_j, \lambda) - C_j(S(\theta^s_j, \lambda)), \; j \in \mathcal{J}. \tag{2.8}$$

We now proceed with our first result which shows that when consumers bid in (2.2) and firms offer in (2.3) the mechanism supports an efficient market equilibrium.

Theorem 1. *There exists a market equilibrium* $(\boldsymbol{\theta}^{d,\star}, \boldsymbol{\theta}^{s,\star}, \lambda)$ *satisfying*

$$\pi_i(\theta^{d,\star}_i, \lambda) \geq \pi_i(\theta^d_i, \lambda), \; \forall \; \theta^d_i \geq 0 \;\; \text{and } i \in \mathcal{I}, \tag{2.9a}$$

$$\pi_j(\theta^{s,\star}_j, \lambda) \geq \pi_j(\theta^s_j, \lambda), \; \forall \; \theta^s_j \geq 0 \;\; \text{and } j \in \mathcal{J}, \tag{2.9b}$$

$$\lambda \text{ is given by } (2.5). \tag{2.9c}$$

Moreover, the supply plan $s^\star_j = S(\theta^{s,\star}_j, \lambda)$ *for every* $j \in \mathcal{J}$ *and the demand plan* $d^\star_i = D(\theta^{d,\star}_i, \lambda)$ *for every* $i \in \mathcal{I}$*, constitute an efficient allocation.*

The proof of Theorem 1 is provided in Section 2.7. According to Theorem 1, under perfect competition, suppliers and consumers maximize their payoffs and the resulting market allocation is efficient. This implies that given price λ, the firms have no incentive to deviate from supplying $\mathbf{s}^\star$ and consumers have no incentive to deviate from buying $\mathbf{d}^\star$. Thus, the competitive market allocation is efficient and the market clearing price is the shadow price (or Lagrange multiplier) of the constraint $\sum_{i\in\mathcal{I}} d_i = \sum_{j\in\mathcal{J}} s_j$. Therefore, at price λ the marginal social benefit of additional output equals the marginal social cost, which establishes the *first fundamental theorem of welfare economics*.

2.4 Strategic Consumers and Suppliers

In contrast to the price-taking model, we now consider a model where market participants are price-anticipating. Price-anticipating suppliers and consumers realize that the market price is a function of their actions and adjust their bids/offers accordingly. In particular, the payoff for the price-anticipating consumer $i \in \mathcal{I}$ is

$$\pi_i(\theta_i^d, \boldsymbol{\theta}_{-i}^d, \boldsymbol{\theta}^s) = U_i\left(d_i^0 + \frac{\theta_i^d}{\lambda\left(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s\right)}\right) - \lambda\left(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s\right) d_i^0 - \theta_i^d. \tag{2.10}$$

Note that the payoff of each consumer depends on the actions of all other market participants that are collectively incorporated in the market price. Similarly, firm j's payoff depends on action θ_j^s and the actions of all other market participants. Specifically, for each $j \in \mathcal{J}$ we have

$$\pi_j(\theta_j^s, \boldsymbol{\theta}_{-j}^s, \boldsymbol{\theta}^d) = \lambda\left(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s\right)\kappa_j^0 - \theta_j^s - C_j\left(\kappa_j^0 - \frac{\theta_j^s}{\lambda\left(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s\right)}\right). \tag{2.11}$$

We define the game $\mathcal{G}$ with $\mathcal{I} \cup \mathcal{J}$ denoting the set of *players* with strategy spaces $\boldsymbol{\Theta}_\mathbf{i} = \mathbb{R}_+$ and payoffs given by (2.10) and (2.11). Our goal is to study the existence (and uniqueness) of the Nash equilibrium of $\mathcal{G}$ and provide an efficient way to compute the equilibrium allocation. A bid/offer profile

$\left(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s\right)$ constitutes a Nash equilibrium if

$$\pi_i(\widehat{\theta}_i^d, \widehat{\boldsymbol{\theta}}_{-i}^d, \widehat{\boldsymbol{\theta}}^s) \geq \pi_i(\theta_i^d, \widehat{\boldsymbol{\theta}}_{-i}^d, \widehat{\boldsymbol{\theta}}^s), \ \forall \ \theta_i^d \geq 0 \text{ and } i \in \mathcal{I}$$
$$\pi_j(\widehat{\theta}_j^s, \widehat{\boldsymbol{\theta}}_{-j}^s, \widehat{\boldsymbol{\theta}}^d) \geq \pi_j(\theta_j^s, \widehat{\boldsymbol{\theta}}_{-j}^s, \widehat{\boldsymbol{\theta}}^d), \ \forall \ \theta_j^s \geq 0 \text{ and } j \in \mathcal{J}.$$

We begin with the following result that illustrates how certain problem parameters influence the existence of a Nash equilibrium for $\mathcal{G}$.

Lemma 1. *$\mathcal{G}$ does not admit a Nash equilibrium if a pivotal supplier exists in the market.*

The proof of Lemma 1 is provided in Section 2.7. In effect, Lemma 1 implies that when any $|\mathcal{J}| - 1$ firms cannot supply the entire inelastic demand in the market, then there exists a pivotal supplier that faces a non-zero inflexible demand that has infinite willingness to pay. This makes the suppliers' payoff grow unbounded with respect their action $\boldsymbol{\theta}^s$. Hence, a Nash equilibrium cannot exist in this case. As a consequence of Lemma 1, there cannot exist a Nash equilibrium with $|\mathcal{J}| = 1$ since, by definition, the single supplier is pivotal. In view of Lemma 1, we impose the following assumption.

Assumption 1. $\mathsf{RSI}_j > 1$ *for each firm $j \in \mathcal{J}$.*

Equipped with the previous observations, we present our main result that explicitly characterizes the unique Nash equilibrium of $\mathcal{G}$.

Theorem 2. *Suppose Assumption 1 holds. $\mathcal{G}$ admits a unique Nash equilibrium $\left(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s\right)$. Moreover, the supply profile*

$$\widehat{s}_j = S_j\left(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right), \ j \in \mathcal{J}$$

and the demand profile

$$\widehat{d}_i = D_i\left(\widehat{\theta}_i^d, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right), \ i \in \mathcal{I}$$

are given by the unique solution of the following convex program

$$\text{maximize} \quad \widehat{\mathcal{W}}(\mathbf{d},\mathbf{s}) := \sum_{i\in\mathcal{I}} \widehat{U}_i(d_i) - \sum_{j\in\mathcal{J}} \widehat{C}_j(s_j), \tag{2.12a}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{I}} d_i = \sum_{j\in\mathcal{J}} s_j, \tag{2.12b}$$

$$0 \le s_j \le \kappa_j^0, \tag{2.12c}$$

$$d_i^0 \le d_i, \tag{2.12d}$$

$$\text{for each } i \in \mathcal{I}, j \in \mathcal{J},$$

where

$$\widehat{U}_i(d_i) := \left(1 - \frac{d_i}{\kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|} + d_i^0}\right) U_i(d_i) + \frac{1}{\kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|} + d_i^0} \int_{d_i^0}^{d_i} U_i(z)dz, \tag{2.13}$$

$$\widehat{C}_j(s_j) := \left(1 + \frac{s_i}{\kappa^0_{|\mathcal{J}|} - \kappa_j^0 - d^0_{|\mathcal{I}|}}\right) C_j(s_j) - \frac{1}{\kappa^0_{|\mathcal{J}|} - \kappa_j^0 - d^0_{|\mathcal{I}|}} \int_0^{s_j} C_j(z)dz. \tag{2.14}$$

The proof of Theorem 2 is provided in Section 2.7. Computing Nash equilibria is, in general, hard as shown in [31]. Theorem 2 establishes the computation of the market allocation at the Nash equilibrium—and the Nash equilibrium itself—through the solution of a convex program in $(\mathbf{d}, \mathbf{s})$ instead of solving $|\mathcal{J}|+|\mathcal{I}|$ problems in the actions $(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s)$, which can be cumbersome depending on the structure of the utility and cost functions. The crux of Theorem 2 is the construction of an appropriate convex program that yields the market allocation at the Nash equilibrium—a technique closely related to the use of potential functions in characterizing Nash equilibria ([32]). However, the functions (2.13) and (2.14) are not potentials for $\mathcal{G}$, since they depend on the allocations and not on the players' decisions. Hence, we cannot use these functions to conclude anything about convergence of best response dynamics to the Nash equilibrium. However, in Section 2.5, we exploit the structure of $\widehat{U}_i$ and $\widehat{C}_j$ to compute bounds on the efficiency loss and the price markup observed at the Nash equilibrium.

2.5 Efficiency Loss and Price Markup

The structure of the modified utility and cost functions allows us to make a number of interesting observations about the behavior of strategic market actors. First, note that since $C_j(s_j)$ are assumed convex and increasing, it follows that $\widehat{C}_j(s_j) \geq C_j(s_j),\ \forall\, s_j \geq 0$. Similarly, since $U_i(d_i)$ are concave and increasing, for each consumer we have $\widehat{U}_i(d_i) \leq U_i(d_i),\ \forall\, d_i \geq d_i^0$. In effect, strategic suppliers misrepresent their costs functions through $\widehat{C}_j(s_j)$, which are greater than the true cost $C_j(s_j)$ at every s_j. On the other hand, strategic consumers misrepresent their utilities through $\widehat{U}_i(d_i)$, which are smaller than the true utility $U_i(d_i)$ at every d_i.

Furthermore, $\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}}) \leq \mathcal{W}(\mathbf{d}^\star, \mathbf{s}^\star)$ since the maximum value of $\mathcal{W}$ occurs at $(\mathbf{d}^\star, \mathbf{s}^\star)$. However, in our next result, we show that the social welfare at the Nash is bounded below and can be relatively close to the optimal value provided some minimum flexible production capacity. In order to compute bounds on price markups at the Nash equilibrium we utilize the Lerner index [33], which we define as

$$\mathsf{LI}(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s) := 1 - \frac{1}{\lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)} \max_j \left\{ \frac{\partial}{\partial s_j} C_j \left(S(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)) \right) \right\}. \tag{2.15}$$

The Lerner index measures a firm's market power and it varies from zero to one, with higher values indicating greater market power. The following result summarizes the efficiency loss at the Nash equilibrium and the price markups.

Theorem 3. *Suppose Assumption 1 holds and let* $\kappa_m^0 = \max\{\kappa_1^0, \ldots, \kappa_{|\mathcal{J}|}^0\}$. *Let* $(\mathbf{d}^\star, \mathbf{s}^\star)$ *be the optimal allocation from* (2.1) *and* $(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ *be the market allocation at the Nash equilibrium of* $\mathcal{G}$. *It follows that*

$$\sum_{i\in\mathcal{I}} U_i(\widehat{d}_i) - \sum_{j\in\mathcal{J}} C_j(\widehat{s}_j) \geq \frac{3}{4} \sum_{i\in\mathcal{I}} U_i(d_i^\star) - \left(1 - \frac{\kappa_m^0}{\zeta}\right)^{-1} \sum_{j\in\mathcal{J}} C_j(s_j^\star), \tag{2.16}$$

where $\zeta := \kappa_{|\mathcal{J}|}^0 - d_{|\mathcal{I}|}^0$ *and* $\zeta \in (\kappa_m^0, \infty)$. *Moreover, when* $\zeta \in [4\kappa_m^0, \infty)$ *we have*

$$\sum_{i\in\mathcal{I}} U_i(\widehat{d}_i) - \sum_{j\in\mathcal{J}} C_j(\widehat{s}_j) \geq \frac{3}{4} \sum_{i\in\mathcal{I}} U_i(d_i^\star) - \frac{4}{3} \sum_{j\in\mathcal{J}} C_j(s_j^\star). \tag{2.17}$$

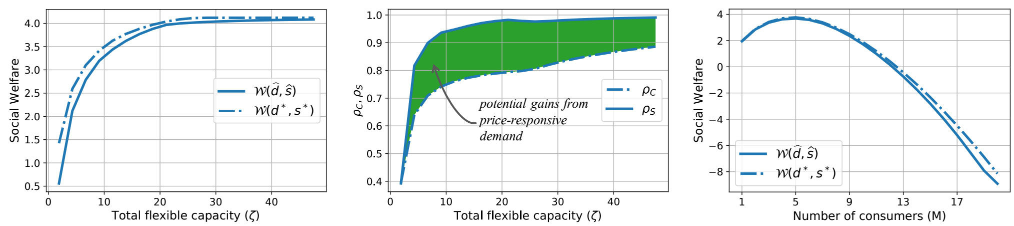
Finally, the Lerner index at the Nash equilibrium satisfies

$$\mathsf{LI}(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s) \leq \frac{\kappa_m^0}{\zeta} < 1. \tag{2.18}$$

The proof of Theorem 3 is provided in Section 2.7. In effect, Theorem 3 provides a lower bound on the social welfare at the Nash equilibrium and an upper bound on the market price with respect to the true marginal cost of suppliers. Notice that $\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ is in the worst case 3/4 of the aggregate utility less $\zeta/(\zeta - \kappa_m^0)$ of the aggregate costs at the efficient allocation. We do not claim this bound is tight; there may exist an even tighter bound on the social welfare the computation of which we relegate to future work. Higher values of ζ yield values of the social welfare at the Nash equilibrium closer to $\mathcal{W}(\mathbf{d}^\star, \mathbf{s}^\star)$. The worst-case values for $\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ arise when $\zeta \to \kappa_m^0$, although it never reaches it. Intuitively, when the aggregate production capacity of supply is relatively close to the total inelastic demand, then firms' market power increases over consumers, gradually inducing *pivotalness* for the supplier with the maximum production capacity. Specifically, for $\zeta \in (\kappa_m^0, 2\kappa_m^0)$ the efficiency loss can be arbitrarily high, similar to that derived by [27] for a market with capacity-constrained suppliers. When $\zeta \in [2\kappa_m^0, \infty)$ the worst-case aggregate cost coefficient in (2.16) is equal to two and we recover the worst-case bound of [26] derived for uncapacitated supply function competition. Moreover, (2.17) shows that provided some minimum flexible production capacity, the social welfare at the Nash equilibrium is no lower than 3/4 of the aggregate utility less 4/3 of the aggregate cost at the efficient allocation—not much lower than $\mathcal{W}(\mathbf{d}^\star, \mathbf{s}^\star)$. From (2.18) note that the Lerner index is strictly less than one due to the non-pivotal supplier assumption. As ζ grows large, $\mathsf{LI}(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ goes to zero, indicating less market power on the supply side. As $d^0_{|\mathcal{I}|}$ approaches $\kappa^0_{|\mathcal{J}|}$, $\mathsf{LI}(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ grows large implying high market power since there is little available capacity to supply anything more than the total inelastic demand.

2.6 Illustrative Examples

In this section we provide numerical experiments to illustrate the behavior of the social welfare under perfect competition and strategic interactions



(a) (b) (c)

Figure 2.1: Plot (a) shows values of the social welfare with respect to ζ at the efficient and Nash equilibrium allocations. In (b) we plot social welfare bounds for (strategic) price-responsive and perfectly inelastic demand. Plot (c) shows how the social welfare varies with respect to the number of consumers.

with respect to specific problem parameters. As shown in Section 2.5, the key parameter that affects social welfare is the total flexible capacity in the market ζ.

Consider a market with $|\mathcal{J}| = 6$ and $|\mathcal{I}| = 5$. Let each consumer $i \in \mathcal{I}$ have utility

$$U_i(d_i) := \beta_i \log(d_i), \qquad d_i^0 = 1.$$

Note that the above utility function is strictly concave and increasing and attains a minimum value $U_i(d_i^0) = 0$ for every $i \in \mathcal{I}$. Moreover, every supplier $j \in \mathcal{J}$ incurs costs given by

$$C_j(s_j) := \frac{1}{2}\alpha_j s_j^2.$$

The modified utility for each $i \in \mathcal{I}$ is

$$\widehat{U}_i(d_i) = \left(1 - \frac{d_i}{\zeta + d_i^0}\right)\beta_i \log(d_i) + \frac{\beta_i}{\zeta + d_i^0}\left(d_i \log(d_i) - d_i + 1\right).$$

Similarly, for each $j \in \mathcal{J}$ the modified cost is given by

$$\widehat{C}_j(s_j) = \left(1 + \frac{s_j}{\zeta - \kappa_j^0}\right)\frac{1}{2}\alpha_j s_j^2 - \frac{1}{6}\left(\frac{\alpha_j}{\zeta - \kappa_j^0}\right)s_j^3.$$

Figures 2.1a and 2.1b illustrate how social welfare at the optimal and Nash allocations varies with respect to ζ. For the experiments we assumed that the vector of utility coefficients β_i is $[1, 1, 1.5, 2, 2]$ and of the cost coefficients α_j is $[0.1, 0.2, 0.3, 0.4, 0.5, 0.5]$. More specifically, we start with a value of $\kappa_j^0 = 1.1$

for every $j \in \mathcal{J}$—just slightly higher than d_i^0 to avoid pivotal suppliers—and increase it gradually. Observe that the higher the value of ζ the closer $\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ is to $\mathcal{W}(\mathbf{d}^\star, \mathbf{s}^\star)$. On the other hand, the smaller ζ is, the higher the efficiency loss at the Nash equilibrium. To gain additional insights, define the following ratio

$$\rho_S := \frac{\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})}{\mathcal{W}(\mathbf{d}^\star, \mathbf{s}^\star)}.$$

For the special case in which the market has perfectly inelastic, non-strategic demand, we utilize the worst-case market performance metric ρ_C, which is adjusted from [27] and is given by

$$\rho_C = \left(1 + \frac{1}{\zeta - \kappa_j^0} \min\left\{\kappa_j^0, d_{|\mathcal{I}|}^0\right\}\right)^{-1}. \tag{2.19}$$

Figure 2.1b demonstrates that the worst-case value of $\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ occurs when $\zeta = 1.6 \in (\kappa_m^0, 2\kappa_m^0)$ where the ratio $\rho_S = 0.4$. Immediately after $\zeta \in [2\kappa_m^0, \infty)$, the ratio ρ_S jumps to 0.8 and stays above 0.9 after $\zeta \geq 4\kappa_m^0$. Note that ρ_S lies everywhere above ρ_C except when $\zeta \in (\kappa_m^0, 2\kappa_m^0)$ where $\rho_S = \rho_C$. This implies that although consumers are strategic, the market efficiency loss is lower-bounded by the worst-case performance of a market with perfectly inelastic demand. It remains to be shown whether this outcome holds more broadly, for any choice of cost and utility functions. Finally, increasing the number of consumers, while keeping the production capacity constant, widens the disparity between $\mathcal{W}(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ and $\mathcal{W}(\mathbf{d}^\star, \mathbf{s}^\star)$ as shown in Figure 2.1c. This illustrates the effect of increasing the inelastic portion of demand and as such inducing higher market power on the existing set of firms, which is also captured by the Lerner index in (2.18).

2.7 Proofs

2.7.1 Proof of Theorem 1

The crux of our derivations relies on Lagrangian duality to establish that the equilibrium conditions of (2.7) and (2.8) together with (2.5) are equivalent to the first-order optimality conditions of (2.1). We begin with the consumer's problem. The payoff in (2.7) is concave in each player's action θ_i^d. Hence, the Karush-Kuhn-Tucker (KKT) optimality conditions are both necessary and sufficient. For every $i \in \mathcal{I}$, an optimal strategy $\theta_i^{d,\star} \geq 0$ must satisfy

$$\frac{\partial}{\partial d_i} U_i \left(D(\theta_i^{d,\star}, \lambda) \right) = \lambda, \quad \text{if } \theta_i^{d,\star} > 0 \tag{2.20a}$$

$$\frac{\partial}{\partial d_i} U_i \left(D(\theta_i^{d,\star}, \lambda) \right) \leq \lambda, \quad \text{if } \theta_i^{d,\star} = 0. \tag{2.20b}$$

Each supplier's payoff is concave in the action θ_j^s. Moreover, an optimal strategy $\theta_j^{s,\star}$ must lie in the closed interval $\left[0, \lambda\kappa_j^0\right]$. If not, then it is easy to show that $S_j(\theta_j^s, \lambda) < 0$ for $\theta_j^s > \lambda\kappa_j^0$. Therefore, such strategies cannot occur at the equilibrium since they yield negative payoff. Therefore, an optimal strategy $\theta_j^{s,\star}$ must satisfy

$$\frac{\partial}{\partial s_j} C_j \left(S(\theta_j^{s,\star}, \lambda) \right) \leq \lambda, \quad \text{if } 0 \leq \theta_j^{s,\star} < \lambda\kappa_j^0, \tag{2.21a}$$

$$\frac{\partial}{\partial s_j} C_j \left(S(\theta_j^{s,\star}, \lambda) \right) \geq \lambda, \quad \text{if } 0 < \theta_j^{s,\star} \leq \lambda\kappa_j^0. \tag{2.21b}$$

We now turn to problem (2.1) solved by the market manager. Associate the Lagrange multiplier λ with the equality constraint (2.1b). The objective function is continuous and concave over a compact set. Therefore, there exists at least one optimal solution $(\mathbf{d}^\star, \mathbf{s}^\star)$ and $\lambda \geq 0$ that satisfy

$$\frac{\partial U_i(d_i^\star)}{\partial d_i} = \lambda, \quad \text{if } d_i^\star > d_i^0 \tag{2.22a}$$

$$\frac{\partial U_i(d_i^\star)}{\partial d_i} \leq \lambda, \quad \text{if } d_i^\star = d_i^0. \tag{2.22b}$$

Similarly, the supply vector $\mathbf{s}^\star$ must satisfy

$$\frac{\partial C_j(s_j^\star)}{\partial s_j} \geq \lambda, \quad \text{if } 0 \leq s_j^\star < \kappa_j^0 \tag{2.23a}$$

$$\frac{\partial C_j(s_j^\star)}{\partial s_j} \leq \lambda, \quad \text{if } 0 < s_j^\star \leq \kappa_j^0. \tag{2.23b}$$

Primal feasibility requires

$$\sum_{i\in\mathcal{I}} d_i^\star = \sum_{j\in\mathcal{J}} s_j^\star. \tag{2.24}$$

Note that $\lambda > 0$ since U_i and C_j are strictly increasing and there exists at least one $s_j^\star > 0$. If the pair $(\mathbf{s}^\star, \lambda)$ satisfies (2.23) and we let $\theta_j^s = \lambda\left(\kappa_j^0 - s_j^\star\right)$ for every $j \in \mathcal{J}$, then $(\boldsymbol{\theta}^s, \lambda)$ satisfy (2.21) and $\boldsymbol{\theta}^s \geq 0$. In effect (2.23) become equivalent to (2.21).

Similarly, if the pair $(\mathbf{d}^\star, \lambda)$ satisfies (2.22) and we let $\theta_i^d = \lambda(d_i^\star - d_i^0)$ then $(\boldsymbol{\theta}^d, \lambda)$ satisfy (2.20) and $\boldsymbol{\theta}^d \geq 0$. In this case, (2.22) become equivalent to (2.20). Finally, the market clearing condition in (2.24) yields λ is given by (2.5). Hence, $(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s, \lambda)$ is a market equilibrium. Now suppose that $(\boldsymbol{\theta}^{d,\star}, \boldsymbol{\theta}^{s,\star}, \lambda)$ satisfy (2.20),(2.21) and λ is given by (2.5). Let $s_j = S(\theta_j^{s,\star}, \lambda)$ for $j \in \mathcal{J}$ and $d_i = D(\theta_i^{s,\star}, \lambda)$ for $i \in \mathcal{I}$. Then, it is easy to verify that the vector $(\mathbf{d}, \mathbf{s})$ satisfies (2.22)-(2.24). Therefore, $(\mathbf{d}, \mathbf{s})$ is an efficient allocation.

2.7.2 Proof of Lemma 1

Let supplier j be pivotal. Then it must hold

$$\mathsf{RSI}_j = \frac{\kappa_{|\mathcal{J}|}^0 - \kappa_j^0}{d_{|\mathcal{I}|}^0} < 1, \tag{2.25}$$

i.e., the total production capacity less that of j's is less than the total inelastic demand in the market. In this case, the first derivative of the supplier's payoff becomes

$$\frac{\partial}{\partial \theta_j^s}\pi_j(\theta_j^s, \boldsymbol{\theta}_{-j}^s, \boldsymbol{\theta}^d) = \frac{\kappa_j^0 + d_{|\mathcal{I}|}^0 - \kappa_{|\mathcal{J}|}^0}{\kappa_{|\mathcal{J}|}^0 - d_{|\mathcal{I}|}^0} + (\kappa_{|\mathcal{J}|}^0 - d_{|\mathcal{I}|}^0)\frac{\partial C_j}{\partial s_j}\left(\frac{\mathbf{1}^\intercal \boldsymbol{\theta}^d + \mathbf{1}^\intercal \boldsymbol{\theta}_{-j}^s}{(\mathbf{1}^\intercal \boldsymbol{\theta}^d + \mathbf{1}^\intercal \boldsymbol{\theta}^s)^2}\right). \tag{2.26}$$

From (2.25) it follows that (2.26) is strictly positive. Therefore, the payoff is strictly increasing in the action θ_j^s and grows unbounded. A Nash equilibrium does not exist.

2.7.3 Proof of Theorem 2

We break the proof into five steps. First, we show that any Nash equilibrium has at least two positive components and we derive the necessary and sufficient conditions for such equilibrium. Next we establish the existence and uniqueness of the market allocation at the Nash equilibrium and derive the first-order necessary conditions for (2.12). We show that for the bids/offers given by (2.2) and (2.3), the equilibrium conditions of all players become equivalent to the first-order conditions of (2.12). Finally, we establish uniqueness of the Nash equilibrium.

Step 1. (Any Nash Equilibrium Has at Least Two Positive Components) First, it is straightforward to see that $\mathbf{1}^\intercal \boldsymbol{\theta}^d + \mathbf{1}^\intercal \boldsymbol{\theta}^s = 0$ cannot occur at the Nash equilibrium since $\kappa^0_{|\mathcal{J}|} > d^0_{|\mathcal{I}|}$ and therefore the market does not clear. Next, we consider two cases. First, assume that $\mathbf{1}^\intercal \boldsymbol{\theta}^d = 0$. Fix supplier j and let $\mathbf{1}^\intercal \boldsymbol{\theta}^s_{-j} = 0$. Note that, in this case, $\theta^s_j > 0$ is not possible by the non-pivotal supplier assumption. A Nash equilibrium cannot exist with all consumers bidding zero and all but one supplier offering a strictly positive θ^s_j. Second, assume $\mathbf{1}^\intercal \boldsymbol{\theta}^s = 0$. Fix consumer i and let $\mathbf{1}^\intercal \boldsymbol{\theta}^d_{-i} = 0$. Then, $\theta^d_i > 0$ implies $d_i > d^0_i$. In this case, the payoff of consumer i is given by

$$U_i\left(d^0_i + \kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|}\right) - \frac{d^0_i}{\kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|}}\theta^d_i - \theta^d_i, \tag{2.27}$$

which is strictly increasing as θ^d_i becomes small and attains its maximum when $\theta^d_i = 0$. Thus for any $\theta^d_i > 0$ there exists an infinitesimally smaller and positive θ^d_i that yields higher payoff. Moreover, by definition of $U_i(0,0) = U(d^0_i) = 0$. A Nash equilibrium does not exist in this case. Hence, at the Nash equilibrium, the vector $\boldsymbol{\theta} = \left(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s\right)$ has at least two positive components.

Step 2. (Necessary and Sufficient Nash Equilibrium Conditions) Having shown that any Nash equilibrium must have at least two positive components, we only focus in the region where $\mathbf{1}^\intercal \boldsymbol{\theta}^d + \mathbf{1}^\intercal \boldsymbol{\theta}^s > 0$. Note that, for each consumer (supplier), their payoff is strictly concave in the action θ^d_i (θ^s_j). Hence, the KKT conditions are both necessary and sufficient. Moreover, we must have

$$0 \le \widehat{\theta}^s_j \le \theta^{\max}_j := \frac{\kappa^0_j}{\kappa^0_{|\mathcal{J}|} - \kappa^0_j - d^0_{|\mathcal{I}|}}\left(\sum_{i\in\mathcal{I}} \theta^d_i + \sum_{k\neq j\in\mathcal{J}} \theta^s_k\right),$$

in order for $S\left(\widehat{\theta}_j^s, \lambda(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s)\right) \geq 0$. We have the following equilibrium conditions.

A demand profile $\widehat{\boldsymbol{\theta}}^d = \left(\widehat{\theta}_1^d, \ldots, \widehat{\theta}_{|\mathcal{I}|}^d\right)$ is a Nash profile if and only if

$$\left(1 - \frac{D(\widehat{\theta}_i^d, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s))}{\kappa_{|\mathcal{J}|}^0 - d_{|\mathcal{I}|}^0 + d_i^0}\right) \frac{\partial}{\partial d_i} U_i \left(D(\widehat{\theta}_i^d, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s))\right) = \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s), \text{ if } \widehat{\theta}_i^d > 0 \tag{2.28a}$$

$$\left(1 - \frac{D(\widehat{\theta}_i^d, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s))}{\kappa_{|\mathcal{J}|}^0 - d_{|\mathcal{I}|}^0 + d_i^0}\right) \frac{\partial}{\partial d_i} U_i \left(D(\widehat{\theta}_i^d, p(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s))\right) \leq \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s), \text{ if } \widehat{\theta}_i^d = 0. \tag{2.28b}$$

A supply profile $\widehat{\boldsymbol{\theta}}^s = \left(\widehat{\theta}_1^s, \ldots, \widehat{\theta}_{|\mathcal{J}|}^s\right)$ is a Nash equilibrium if and only if

$$\left(1 + \frac{S\left(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right)}{\kappa_{|\mathcal{J}|}^0 - \kappa_j^0 - d_{|\mathcal{I}|}^0}\right) \frac{\partial}{\partial s_j} C_j \left(S(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s))\right) \leq \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s), \text{ if } 0 \leq \widehat{\theta}_j^s < \theta_j^{\max} \tag{2.29a}$$

$$\left(1 + \frac{S\left(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s))\right)}{\kappa_{|\mathcal{J}|}^0 - \kappa_j^0 - d_{|\mathcal{I}|}^0}\right) \frac{\partial}{\partial s_j} C_j \left(S(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)))\right) \geq \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s), \text{ if } 0 < \widehat{\theta}_j^s \leq \theta_j^{\max}. \tag{2.29b}$$

The equilibrium conditions (2.28) and (2.29) are derived from the KKT conditions of each player's payoff maximization problem, where the payoff of each consumer and supplier is given by expressions (2.10) and (2.11), respectively.

Step 3. (Existence and Uniqueness of a Market Allocation) Equipped with the above relations we now proceed to the market manager's problem. Note that $\widehat{U}_i(d_i)$ is strictly concave and $\widehat{C}_j(s_j)$ is strictly convex. Hence, $\widehat{\mathcal{W}}$ is continuous and strictly concave over a compact set. Specifically, the Hessian matrix $\mathbf{H}$ of $\widehat{\mathcal{W}}$ has diagonal elements

$$h_{kk} = \begin{cases} \dfrac{\partial^2 \widehat{U}_k(d_k)}{\partial d_k^2} < 0, \text{ for } k = 1, \ldots, |\mathcal{I}| \\ \\ -\dfrac{\partial^2 \widehat{C}_k(s_k)}{\partial s_k^2} < 0, \text{ for } k = |\mathcal{I}| + 1, \ldots, |\mathcal{I}| + |\mathcal{J}|, \end{cases} \tag{2.30}$$

and $h_{km} = 0$ for $k \neq m$. Hence, $\mathbf{H}$ is negative definite and there exists a unique solution to (2.12).

Step 4. (Necessary and Sufficient Conditions for the Market Allocation) Let $(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ be the unique optimal solution to (2.12). There exists Lagrange multiplier λ such that

$$\left(1 - \frac{\widehat{d}_i}{\kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|} + d^0_i}\right) \frac{\partial U_i(\widehat{d}_i)}{\partial d_i} = \lambda, \text{ if } \widehat{d}_i > d_0, \tag{2.31a}$$

$$\left(1 - \frac{\widehat{d}_i}{\kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|} + d^0_i}\right) \frac{\partial U_i(\widehat{d}_i)}{\partial d_i} \leq \lambda, \text{ if } \widehat{d}_i = d_0, \tag{2.31b}$$

$$\left(1 + \frac{\widehat{s}_j}{\kappa^0_{|\mathcal{J}|} - \kappa^0_j - d^0_{|\mathcal{I}|}}\right) \frac{\partial C_j(\widehat{s}_j)}{\partial s_j} \geq \lambda, \text{ if } 0 \leq \widehat{s}_j < \kappa^0_j, \tag{2.31c}$$

$$\left(1 + \frac{\widehat{s}_j}{\kappa^0_{|\mathcal{J}|} - \kappa^0_j - d^0_{|\mathcal{I}|}}\right) \frac{\partial C_j(\widehat{s}_j)}{\partial s_j} \leq \lambda, \text{ if } 0 < \widehat{s}_j \leq \kappa^0_j. \tag{2.31d}$$

Moreover, primal feasibility requires

$$\sum_{i \in \mathcal{I}} \widehat{d}_i = \sum_{j \in \mathcal{J}} \widehat{s}_j. \tag{2.32}$$

Note that since there is at least one $\widehat{s}_j > 0$ and U_i and C_i are strictly increasing, then $\lambda > 0$. Consider bids $\widehat{\theta}^d_i = \lambda(\widehat{d}_i - d^0_i)$ for $i \in \mathcal{I}$ and offers $\widehat{\theta}^s_j = \lambda(\kappa^0_j - \widehat{s}_j)$ for $j \in \mathcal{J}$. Then, $\theta^d_i \geq 0$ and $\theta^s_j \geq 0$ for every consumer and every supplier, respectively. Suppose now that $d_i > d^0_i$ and $d_k = d^0_k$ for all $k \neq i$ and let $s_j = \kappa^0_j$ for all $j \in \mathcal{J}$. This implies that $d_i = d^0_i + \kappa^0_{|\mathcal{J}|} - d^0_{|\mathcal{I}|}$. Then from (2.31a) we have $\lambda = 0$. However, we have $\partial U_k(d^0_k)/\partial d_k > 0$ for each $k \neq i \in \mathcal{I}$, which violates (2.31b). Thus, the vector $(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ cannot have all components zero except one $\theta^d_i > 0$.

Similarly, $(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ cannot have all components zero except one $\theta^s_j > 0$ for some firm $j \in \mathcal{J}$. This is obvious by Assumption 1 since it holds $\kappa^0_{|\mathcal{J}|} - \kappa^0_j > d^0_{|\mathcal{I}|}$ for every supplier. Hence, at least two components of $(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ are positive. Moreover, since $\widehat{s}_j = \kappa^0_j$ if and only if $\widehat{\theta}^s_j = 0$, $\widehat{s}_j = 0$ if and only if $\widehat{\theta}^s_j = \theta^{\max}_j$, then it is not hard to see that (2.31) become equivalent to (2.28)-(2.29). Hence, the action vector $(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ is a Nash equilibrium. This also establishes existence of the Nash equilibrium.

We now reverse the argument. Let $(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ be a Nash equilibrium profile satisfying (2.28)-(2.29). Therefore, it has at least two positive components and $\lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s) > 0$. Define the demand allocation $\widehat{d}_i = d_i^0 + \widehat{\theta}_i^d/\lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ for $i \in \mathcal{I}$ and the supply allocation $\widehat{s}_j = \kappa_j^0 - \widehat{\theta}_j^s/\lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ for $j \in \mathcal{J}$. It follows that $(\widehat{\mathbf{d}}, \widehat{\mathbf{s}})$ satisfy (2.31) with $\lambda = \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$.

Step 5. (Uniqueness of the Nash Equilibrium) We have shown that all Nash equilibria yield a unique market allocation. Uniqueness of the Nash equilibrium follows from the fact that the transformation from $\left(\boldsymbol{\theta}^d, \boldsymbol{\theta}^s\right)$ to $(\mathbf{d}, \mathbf{s}, \lambda)$ is one-to-one.

2.7.4 Proof of Theorem 3

Step 1. (Bounding the Price Markup) To derive the upper bound on the Lerner index we note that at the Nash equilibrium there exists at least one firm such that $S_j\left(p(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right) < \kappa_j^0$ or $\widehat{\theta}_j^s > 0$. Therefore,

$$\begin{aligned}
\lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s) &\leq \left(1 + \frac{S\left(\widehat{\theta}_i^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right)}{\zeta - \kappa_j^0}\right) \frac{\partial}{\partial s_j} C_j\left(S_j\left(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right)\right) \\
&\leq \left(1 + \frac{\kappa_j^0}{\zeta - \kappa_j^0}\right) \frac{\partial}{\partial s_j} C_j\left(S_j\left(\widehat{\theta}_j^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right)\right) \\
&\leq \frac{\zeta}{\zeta - \kappa_j^0} \max_j \left\{\frac{\partial}{\partial s_j} C_j\left(S_j\left(\widehat{\theta}_i^s, \lambda(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)\right)\right)\right\}.
\end{aligned} \tag{2.33}$$

Utilizing (2.33) and substituting in the expression of $\mathsf{LI}(\widehat{\boldsymbol{\theta}}^d, \widehat{\boldsymbol{\theta}}^s)$ yields the bound in (2.18).

Step 2. (Bounding the Social Welfare) To simplify exposition let $\mathbf{x} = (\mathbf{d}, \mathbf{s})$. In this step we aim to bound the social welfare at the Nash equilibrium, i.e.,

$\mathcal{W}(\widehat{\mathbf{x}})$. Specifically,

$$
\begin{aligned}
\mathcal{W}(\widehat{\mathbf{x}}) &\geq \mathcal{W}(\widehat{\mathbf{x}}) + \sum_{k=1}^{|\mathcal{J}|+|\mathcal{I}|} \frac{\partial \widehat{\mathcal{W}}(\widehat{x}_k)}{\partial x_k}(x_k^\star - \widehat{x}_k) && \text{(2.34a)}\\
&= \mathcal{W}(\widehat{\mathbf{x}}) + \left\{ \sum_{k\in\mathcal{I}} \frac{\partial \widehat{U}_k(\widehat{d}_k)}{\partial d_k}(d_k^\star - \widehat{d}_k) - \sum_{k\in\mathcal{J}} \frac{\partial \widehat{C}_k(\widehat{s}_k)}{\partial s_k}(s_k^\star - \widehat{s}_k) \right\} && \text{(2.34b)}\\
&= \mathcal{W}(\widehat{\mathbf{x}}) + \sum_{k\in\mathcal{I}} \left(1 - \frac{\widehat{d}_k}{\zeta + d_k^0}\right) \frac{\partial U_k(\widehat{d}_k)}{\partial d_k}(d_k^\star - \tilde{d}_k)\\
&\quad - \sum_{k\in\mathcal{J}} \left(1 + \frac{\widehat{s}_k}{\zeta - \kappa_k^0}\right) \frac{\partial C_k(\widehat{s}_k)}{\partial s_k}(s_k^\star - \widehat{s}_k) && \text{(2.34c)}\\
&\geq \mathcal{W}(\widehat{\mathbf{x}}) + \sum_{k\in\mathcal{I}} \left(1 - \frac{\widehat{d}_k}{\zeta + d_k^0}\right) \left(U_k(d_k^\star) - U_k(\widehat{d}_k)\right)\\
&\quad - \sum_{k\in\mathcal{J}} \left(1 + \frac{\widehat{s}_k}{\zeta - \kappa_k^0}\right) (C_k(s_k^\star) - C_k(\widehat{s}_k)) && \text{(2.34d)}\\
&\geq \sum_{k\in\mathcal{I}} U_k(\widehat{d}_k) - \sum_{k\in\mathcal{J}} C_k(\widehat{s}_k) + \sum_{k\in\mathcal{I}} \left(1 - \frac{\widehat{d}_k}{d_k^\star}\right) \left(U_k(d_k^\star) - U_k(\widehat{d}_k)\right)\\
&\quad - \left(\frac{\zeta}{\zeta - \max_k \kappa_k^0}\right) \sum_{k\in\mathcal{J}} (C_k(s_k^\star) - C_k(\widehat{s}_k)) && \text{(2.34e)}\\
&\geq \sum_{k\in\mathcal{I}} \left(\left(\frac{\widehat{d}_k}{d_k^\star}\right)^2 + 1 - \frac{\widehat{d}_k}{d_k^\star} \right) U_k(d_k^\star) - \left(\frac{\zeta}{\zeta - \max_k \kappa_k^0}\right) \sum_{k\in\mathcal{J}} C_k(s_k^\star) && \text{(2.34f)}\\
&\geq \frac{3}{4} \sum_{k\in\mathcal{I}} U_k(d_k^\star) - \left(\frac{\zeta}{\zeta - \max \kappa_k^0}\right) \sum_{k\in\mathcal{J}} C_k(s_k^\star). && \text{(2.34g)}
\end{aligned}
$$

Inequality (2.34a) follows from optimality conditions of (2.12) while (2.34c) from the definitions of $\widehat{U}_k$ and $\widehat{C}_k$. Inequality (2.34d) follows from concavity of U_k and convexity of C_k. Step (2.34e) follows from the fact that $d_k^\star < \zeta + d_k^0$ for every $k \in \mathcal{I}$ and $\widehat{s}_k \leq \kappa_k^0$ for every $k \in \mathcal{J}$. Inequality (2.34f) follows from concavity of U_k and that

$$
U_k\left(\left(1 - \frac{\widehat{d}_k - d_k^0}{d_k^\star - d_k^0}\right) d_k^0 + \frac{\widehat{d}_k - d_k^0}{d_k^\star - d_k^0} d_k^\star\right) \geq \left(1 - \frac{\widehat{d}_k - d_k^0}{d_k^\star - d_k^0}\right) U_k(d_k^0) + \frac{\widehat{d}_k - d_k^0}{d_k^\star - d_k^0} U_k(d_k^\star), \tag{2.35}
$$

which implies that $U_k(\widehat{d}_k) \gtrsim \widehat{d}_k U_k(d_k^\star)/d_k^\star$. The last inequality follows from minimizing the expression $y^2 - y + 1$, which is minimized for $y^\star = 1/2$, where $y = \widehat{d}_k/d_k^\star$. Finally, note that $\zeta/(\zeta - \max_k \kappa_k^0)$ is a decreasing function of ζ.

Requiring $\zeta \in [4 \max_k \kappa_k^0, \infty)$ its highest value is 4/3.

2.8 Summary

We studied a market with $|\mathcal{J}|$ suppliers and $|\mathcal{I}|$ consumers that compete in supply offers and demand bids for a product. Our analysis showed that with a specific family of scalar-parameterized offers/bids, the market supports an efficient competitive equilibrium. Under strategic interactions, we showed there exists a unique Nash equilibrium and propose an efficient way of computing the induced market allocation. Moreover, the welfare loss and the price markups at the Nash equilibrium are bounded. We extended the two-sided market mechanism in wholesale electricity market models and established that it supports efficient market equilibria. The market mechanism has multiple interesting applications. For example, owing to their simplicity, scalar-parameterized offers/bids can be effectively utilized to model competition among retail electricity customers that are becoming both consumers and producers, due to the proliferation of distributed energy resources. We explore the properties of the mechanism in such applications in Chapter 5.

CHAPTER 3

ANALYSIS OF INTER-REGIONAL MARKETS VIA SCALAR-PARAMETERIZED SUPPLY FUNCTIONS

The importance of inter-regional markets and coordination among different SOs is underlined by the wave of geographically disperse renewable generation. In practice, inter-regional markets involve complex rules and mechanisms, often resulting in inefficient power schedules. In this chapter, we effectively analyze such markets utilizing scalar-parameterized supply offers and reveal the main factors that drive efficiency in inter-regional markets.

3.1 The Importance of Tie-Lines

Different parts of an interconnected power grid are controlled and managed by different system operators (SOs). We call the geographical footprint within each SO's jurisdiction an area, and transmission lines that interconnect two different areas as tie-lines. Efficient scheduling of power flows over tie-lines is paramount to improve market efficiency and exploit geographically diverse renewable resources. Tie-lines are capable of supplying a significant portion of each area's electricity demand. For example, the New York ISO (NYISO) and ISO New England (ISO-NE) share nine tie-lines with approximately 1800 MW capacity, capable of supplying 12% of New England's and 10% of New York's demand as of 2009 [34]. Even though tie-lines are important assets, they have been historically under-utilized or scheduled in the counter-economic direction [34]. The economic loss from inefficient tie-line schedules has been estimated at $784 million between NYISO and ISO-NE in 2006-10 [34], the burden of which has been ultimately borne by end-use customers. What causes such inefficiencies? There are a number of factors that include the inherent uncertainty about power requirements when tie-lines are scheduled prior to delivery time points, the lack of coordination and appropriate information exchange among the SOs, ad hoc use of proxy buses in deciding the schedule and transaction fees.

Conceptually, power flows over tie-lines should be determined through a joint economic dispatch problem geared toward maximizing the efficiency of the interconnected power grid as a whole. However, historical and legal reasons render such an aggregation of market information from different areas at a central location untenable. Naturally, a considerable effort has been made to solve the joint dispatch problem in a distributed fashion, focusing on primal [35, 36] and dual decomposition methods [37, 38, 39]. In such methods, SOs exchange information among themselves to compute the optimal tie-line schedule. This theoretical coordination mechanism, referred to as Tie Optimization (TO) in [34], proved challenging to implement in practice. It was perceived as requiring the SOs to trade directly with each other, violating their financial neutrality, in lieu of the earlier market-based, albeit inefficient, process for scheduling tie-line flows. Instead, many SOs adopted variants of Coordinated Transaction Scheduling (CTS), e.g., see [40, 41], that sought to blend the earlier market-based tie-line scheduling with the theoretically optimal TO, after receiving approval from FERC.

CTS is a market mechanism in which external market participants submit bids and offers to import or export from one area to the other. CTS market design is predicated on the simple premise that arbitrage opportunity will attract more participants, whose profit motivation will ultimately shrink that opportunity, pushing the schedule closer to the theoretically optimum. CTS has certainly improved tie-line scheduling as per [42, 43], but significant inefficiencies remain. Motivated by these inefficiencies, we present a theoretical model to analyze CTS and investigate the repercussions of strategic behavior on overall market performance. We provide palpable insights on the consequences of an illiquid market, errors in SOs' price forecasts and transaction fees on market efficiency, all of which have been named in [43] as crucial factors affecting CTS market efficiency.[1] In Section 3.2 we introduce the mechanics of CTS.

[1] We remark that the use of proxy buses as CTS trading locations results in the so-called "loop flow" problem (see [44]) that negatively impacts CTS market performance. We refer the reader to [45] for mechanisms to tackle this problem.

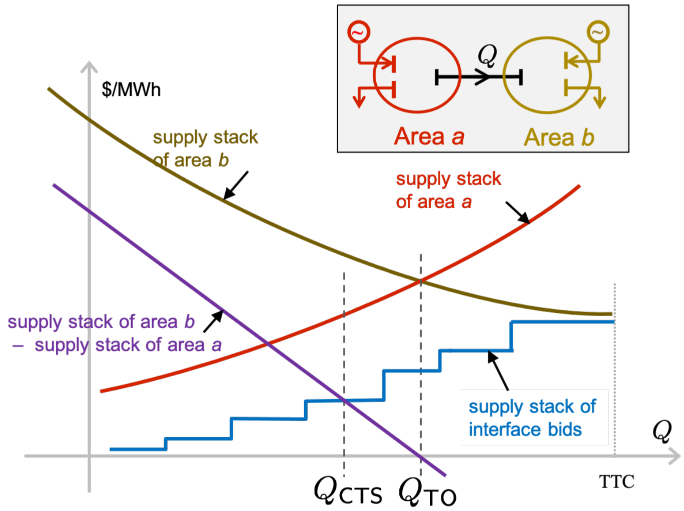


Figure 3.1: Illustration of the TO and CTS scheduling mechanisms.

3.2 The CTS Mechanism

CTS is a real-time, market-based mechanism for tie-line scheduling that replaced an earlier market-based structure in an effort to streamline the bidding and scheduling process. Among the important changes, CTS unified the bid submission and clearing process among the neighboring SOs, reduced the tie-line schedule duration from one hour to 15-minute intervals, and decreased time delays among bidding, scheduling, and power delivery. To illustrate the mechanics and economic rationale of CTS, consider a two-area power system, shown in Figure 3.1, that share a common interface, with the power flowing through the interface denoted by Q. Assume the SOs want to determine the tie-line schedule for an upcoming interval $[t, t+15]$. At $t-15$, both SOs compute their supply stacks by solving an area-wise parametric economic dispatch by varying the amount of power Q flowing on the tie-line. Notice that there is approximately a 30-minute time delay between when the tie-line is scheduled and when power delivery takes place.

An example of supply stack is shown in Figure 3.1. The stack of area a represents the expected incremental dispatch cost of delivering power at its side of the interface. Similarly, the stack of area b represents the expected decremental dispatch cost of reduced supply, shown in descending order. In this example, the direction of power should be from area a to b since at zero schedule, area b operates at higher costs than area a. At the level where dispatch costs at the border become equal or where the supply stacks intersect, the tie-line schedule minimizes the aggregate dispatch costs across the two areas. This schedule, denoted by Q_{TO}, corresponds to the outcome of the theoretical tie optimization (TO) scheme that minimizes the aggregate dispatch costs across the two areas. While CTS remains our focus in this thesis, TO serves as our theoretical benchmark to compare CTS against. Contrary to TO, CTS relies on virtual traders whose offers/bids are utilized together with the supply stacks to arrive at the tie-line schedule, as we describe next.

A CTS participant is a virtual bidder that can offer to transport power across areas without physically consuming or producing it. They only participate in the tie-line scheduling process, bearing no obligation for physical power delivery; the transaction is purely financial. In particular, CTS participants submit "interface" bids that consist of three elements: the minimum price difference the bidder is willing to accept, the maximum quantity to be transferred, and the direction of trade, i.e., the exporting and importing area. All the bids indicating a direction from a to b are stacked from lowest to highest price, to create their own interface supply stack as shown in Figure 3.1. Bids that indicate direction from b to a are rejected at the outset since they would widen the SO-predicted price spread. The price spread curve is derived by subtracting the supply stack of area a from that of area b. The CTS schedule, denoted by Q_{CTS}, is set at the intersection of the interface supply stack and the price spread.[2] An interface bid is accepted if its offer price is less than the price spread at the tie-line schedule. Therefore, all interface bids to the left of the CTS schedule are accepted; all bids to the right are not.

CTS bids can be submitted up to $t - 75$, are cleared at $t - 15$ and are settled at the ex-post LMPs calculated for the time period $[t, t + 15]$. Hence, there is approximately a 30-minute latency time for the SOs and 90 minutes for CTS participants. This latency problem exposes participants to financial risk since there is uncertainty at which LMPs CTS bids will settle. LMPs are highly volatile (see Figure II-7 in [34]) and bids that appeared economic at $t - 15$ may be uneconomic at $t + 15$, impacting overall efficiency of CTS

[2]The intersection of the supply stacks can occur to the right of the total transfer capability (TTC) of the interface. In such cases, Q_{CTS} is equal to TTC, preventing price convergence. However, according to [34], the primary interface between NYISO–ISO-NE was congested 0.3% and 1.2% of the hours eastbound and westbound, respectively, in 2009. In this work, we focus on the factors that cause price separation under CTS, other than TTC.

schedules. We emphasize that it may not be possible to eliminate latency time, irrespective of the scheduling mechanism in place, as this would require improvements in communication technology and SO commitment systems that often require look-ahead information. Thus, scheduling inefficiencies due to time delays will continue to occur under CTS or any other mechanism, unless bid submittal and market clearing come significantly close to delivery time.

In the sequel, we extract a theoretical model to study CTS. The purpose of our model is to serve as a useful tool for analysis and reveal fundamental design flaws of CTS and similar markets, and not to precisely describe reality. Despite being a theoretical abstraction, our model reveals a number of factors that drive CTS efficiency such as market liquidity, behavior of CTS bidders, transaction fees and SO's forecast errors.

3.3 Modeling the CTS Market as a Game

The first question we answer is whether the incentives of CTS bidders are aligned with those of the SOs and CTS design. Given that latency time will always influence efficiency, let us assume, for the time being, that bid submittal and scheduling times happen *near* real-time. We relegate the discussion on latency in Section 3.6. To reveal the impacts of bidding behavior on CTS, we model CTS as a game among virtual bidders who compete to transport power over the tie-lines against an elastic inter-area price spread that varies with Q. For areas a and b, denote by $\mathcal{P}_a(Q)$ and $\mathcal{P}_b(Q)$, the LMPs at CTS trading locations, respectively. Without loss of generality, let area a export and area b import power, and define

$$\mathcal{P}(Q) := \mathcal{P}_b(Q) - \mathcal{P}_a(Q) \tag{3.1}$$

as the price spread between the areas.

Assumption 2. $\mathcal{P} : \mathbb{R} \to \mathbb{R}$ *is differentiable, concave and strictly decreasing in* $Q \geq 0$ *with* $\mathcal{P}(0) > 0$.

Concavity, differentiability and decreasing nature of $\mathcal{P}$ are standard assumptions in prior literature on supply function and Cournot competition models, e.g., see [46, 21, 47, 20]. In CTS, the decreasing nature of $\mathcal{P}$ results organically from the fact that price differences decline as the lower-cost region exports to the higher-cost, displacing expensive generation from the dispatch solution. However, smoothness of $\mathcal{P}$ does not occur in practice. In the sequel, we argue that $\mathcal{P}_b(Q) - \mathcal{P}_a(Q)$ does exhibit some affine dependence on Q based on available data. Furthermore, without some theoretical assumptions, the game-theoretical analysis becomes challenging.

With the previous discussion in mind, consider N virtual bidders in the CTS market. Let bidder i provide two parameters θ_i, B_i to the SOs with the understanding that they are willing to transport up to

$$x_i(p) := B_i - \frac{\theta_i}{p}, \quad \theta_i \geq 0 \tag{3.2}$$

amount of power from area a to b at a price spread of $p > 0$. Figure 3.2 reveals how the parameters θ_i, B_i affect the shape of the transport offer. Bidder i is willing to transport a maximum quantity of B_i, but at a minimum price spread of θ_i/B_i. The required price difference increases with the power transport and grows unbounded as the latter approaches B_i. In effect, transporting power above B_i requires an infinite price difference. The parameterized "hockey-stick" shaped transport offer in (3.2) is a smooth approximation to the one in practice where a player is willing to transport up to B_i at a specified price difference. Therefore, bidder i expresses their total budget or their liquidity in B_i. In what follows, we assume that the bidder acts strategically in θ_i, given B_i that models their budget constraints. The transport offer considered in (3.2) allows market participants to submit negative quantities. Hence, we restrict θ_i to satisfy $\theta_i \leq \mathcal{P}(0)B_i$.

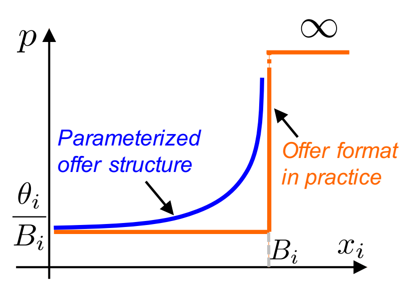


Figure 3.2: Parameterized interface bid of CTS market participant.

Using the family of transport offers in (3.2), the bidders participate in a capacitated scalar-parameterized supply function competition against an elastic demand in (3.1). Given the liquidities $\mathbf{B} = (B_1, \ldots, B_N)$, the choice of bids $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$ from the CTS bidders describes their willingness to transport power across the interface according to (3.2). The SOs calculate $\mathbf{x}^\star := (x_1^\star, \ldots, x_N^\star)$ as the allocations of the tie-line flow to the participants by solving

$$\mathbf{x}^\star(\boldsymbol{\theta}; \mathbf{B}) \in \underset{\mathbf{x} \leq \mathbf{B}}{\operatorname{argmax}} \int_0^{\mathbf{1}^\intercal \mathbf{x}} \mathcal{P}(z)dz - \sum_{i=1}^{N} \int_0^{x_i} \frac{\theta_i}{B_i - s} ds, \tag{3.3}$$

where $\mathbf{1}$ denotes a vector of ones of appropriate size. Notice that the transport offer enters the SOs' problem as the "bid-in cost" of each CTS bidder to transport quantity x_i. With this interpretation, the SOs' flow allocation problem in (3.3) seeks to maximize the social welfare of an economy that is composed of the wholesale markets in areas a and b together with the CTS bidders (see [45] for a similar interpretation of the CTS market objective). Observe that the reported transport offer of each participant resembles a logarithmic barrier function which encodes each participant's budget constraint. Thus, one can drop this constraint for any participant whose bid satisfies $\theta_i > 0$.

The CTS schedule occurs where the offer stack for inter-area power transport offers intersects the SOs' price spread function. Formally,

$$\mathcal{P}(Q_{\mathsf{CTS}}) = \frac{\mathbf{1}^\intercal \boldsymbol{\theta}}{\mathbf{1}^\intercal \mathbf{B} - Q_{\mathsf{CTS}}}. \tag{3.4}$$

Denote the solution of (3.4) by $Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})$. Then, the market clearing price is given by

$$p(\boldsymbol{\theta}; \mathbf{B}) = \mathcal{P}(Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})). \tag{3.5}$$

Let us now define a useful benchmark: the maximum inter-area demand or Q_{TO}. This schedule corresponds to the quantity for which the inter-area price spread vanishes or formally

$$Q_{\mathsf{TO}} \in \underset{Q \geq 0}{\operatorname{argmax}}\ \mathcal{W}(Q) := \int_0^Q \mathcal{P}(z)dz. \tag{3.6}$$

At Q_{TO} there is no more opportunity for arbitrage as $\mathcal{P}(Q_{\mathsf{TO}}) = 0$. With this definition in mind, we can now define the CTS flow allocation to every participant as

$$x_i^\star(\boldsymbol{\theta}; \mathbf{B}) = B_i - \frac{\theta_i}{p(\boldsymbol{\theta}; \mathbf{B})}, \text{ for } \mathbf{1}^\intercal \boldsymbol{\theta} > 0. \tag{3.7}$$

When $\mathbf{1}^\intercal \boldsymbol{\theta} = 0$, from (3.3) it follows that $Q_{\mathsf{CTS}} = \min\{\mathbf{1}^\intercal \mathbf{B}, Q_{\mathsf{TO}}\}$. When $\mathbf{1}^\intercal \mathbf{B} < Q_{\mathsf{TO}}$, $x_i^\star(\mathbf{0}; \mathbf{B}) = B_i$ irrespective of p. On the other hand, if $\mathbf{1}^\intercal \mathbf{B} \geq Q_{\mathsf{TO}}$, then any feasible solution of (3.3) is optimal. In this case, we specify $x_i^\star$ as the allocation of Q_{TO} proportional to each participant's budget, i.e., $x_i^\star(\mathbf{0}; \mathbf{B}) = (B_i/\mathbf{1}^\intercal \mathbf{B})Q_{\mathsf{TO}}$. With these additional conventions, $x_i^\star$ is well-defined for any $\boldsymbol{\theta}$ and $\mathbf{B}$.

While virtual bidders do not incur any costs to physically transport power, many pairs of SOs levy transaction fees on a per-MWh basis, e.g., in CTS between NYISO and PJM, NYISO charges physical exports to PJM at a rate ranging from \$4-\$8 per MWh, while PJM charges physical imports and exports rates that average less than \$3 per MWh. See [43] for details. For a willingness to transport x_i MW of power from area a to b, assume that transaction cost equals $c \cdot x_i$, where c is measured in \$/MWh. Then, each bidder's payoff equals the total revenue garnered less the transaction costs, formally given in

$$\begin{aligned}\pi_i(\theta_i, \boldsymbol{\theta}_{-i}) &= \mathcal{P}\left(Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})\right) x_i^\star(\boldsymbol{\theta}; \mathbf{B}) - c x_i^\star(\boldsymbol{\theta}; \mathbf{B}) \\ &= \mathcal{P}\left(Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})\right) B_i - \theta_i - c x_i^\star(\boldsymbol{\theta}; \mathbf{B}),\end{aligned} \tag{3.8}$$

where θ_{-i} denotes a vector with all but the ith component of $\boldsymbol{\theta}$. With this discussion in mind, we now proceed to define the CTS game. The set of players consists of N CTS participants. When players incur costs $c \geq 0$, any player bidding $\theta_i/B_i \leq c$ would incur loss. Thus, it makes sense to restrict each player's action space to the compact set $[cB_i, \mathcal{P}(0)B_i]$. Therefore, $\boldsymbol{\Theta} = \prod_{i=1}^n [cB_i, \mathcal{P}(0)B_i]$ is the strategy space of the game. Define $\mathcal{G}(\mathbf{B}, c)$ as the CTS game among N virtual bidders who bid $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, given $\mathbf{B}$, and receive a payoff described by (3.8). Bidders selfishly seek to maximize their own payoffs, given their liquidities. A bid profile $\boldsymbol{\theta}^{\mathsf{NE}}$ constitutes a *Nash equilibrium* of $\mathcal{G}(\mathbf{B}, c)$, if

$$\pi_i\left(\theta_i^{\mathsf{NE}}, \boldsymbol{\theta}_{-i}^{\mathsf{NE}}\right) \geq \pi_i\left(\theta_i, \boldsymbol{\theta}_{-i}^{\mathsf{NE}}\right)$$

for all $\theta_i \in [cB_i, \mathcal{P}(0)B_i]$. That is, no player has an incentive for a unilateral deviation from the equilibrium offer. We establish the existence of such an equilibrium profile in our first result.

Theorem 4 (Existence of Nash Equilibrium)**.** *Let Assumption 2 hold. Then, the CTS game* $\mathcal{G}(\mathbf{B}, c)$ *admits a Nash equilibrium if* $\mathcal{P}$ *satisfies*

$$\mathcal{P}''(Q)(\mathbf{1}^\intercal \mathbf{B} - Q) \geq 2\mathcal{P}'(Q). \tag{3.9}$$

The proof of Theorem 4 is provided in Section 3.7. The proof relies on Rosen's result in [48] after we establish that $\mathcal{G}(\mathbf{B}, c)$ is a concave game. Existence of an equilibrium requires the additional condition on $\mathcal{P}$ given by (3.9), that is satisfied by many commonly used demand function families including affine models. To explicitly characterize the Nash equilibrium we restrict our attention to affine price spreads

$$\mathcal{P}(Q) := \alpha - \beta Q \tag{3.10}$$

with $\alpha, \beta > 0$. Therefore, from Theorem 4 we conclude that an equilibrium always exists for $\mathcal{G}(\mathbf{B}, c, \alpha, \beta)$. The price spread can be shown to be affine in Q, when each area is represented as a copperplate power system, having a generator with quadratic generation costs and a fixed demand. This follows from properties of multiparametric quadratic programs in [49, Theorem 7.6]. To further justify our modeling choice, we perform a linear regression of New England's LMP at the CTS node (Roseton) as $\mathcal{P}_{\text{NE}} = w_1 \mathcal{P}_{\text{NY}} + w_2 Q + w_3$, where $\mathcal{P}_{\text{NY}}$ is the LMP at New York's CTS trading node. We obtain $w_1 \approx 1.0$ with an adjusted R^2 coefficient of 0.95, revealing an affine dependency of $\mathcal{P}_{\text{NY}} - \mathcal{P}_{\text{NE}}$ in Q. We obtain similar results when $\mathcal{P}_{\text{NY}}$ is the dependent variable and $\mathcal{P}_{\text{NE}}, Q$ are used as predictors. However, Q alone is not sufficient to accurately predict price differences between SOs. Spreads are, typically, noisy data that is influenced by multiple factors such as renewable generation [50], fuel prices [51], seasonality [52, 53], and so forth. The goal of this thesis is not to provide an accurate model to forecast inter-area price spreads; our focus is to reveal market design flaws particularly when participants are strategic. To this end, an affine model is satisfactory to perform the game-theoretic analysis. Finally, we remark that while it is challenging to establish uniqueness of equilibria in the setting of Theorem 4, the same does not hold with affine demand functions under additional assumptions, as we demonstrate next.

3.4 Impact of Liquidity in CTS Markets

Our first goal is to investigate the impacts of liquidity on the CTS scheduling efficiency. To isolate the effects of liquidity, neglect transaction fees and set $c \approx 0$. We define the efficiency of CTS as the ratio

$$\eta_{\mathsf{CTS}}(\mathbf{B}) := \frac{\mathcal{W}\left(Q_{\mathsf{CTS}}(\boldsymbol{\theta}^{\mathsf{NE}}, \mathbf{B})\right)}{\mathcal{W}\left(Q_{\mathsf{TO}}\right)},$$

where recall that $\mathcal{W}$ measures the aggregate welfare of the wholesale markets in the two areas attained at a particular tie-line schedule. TO seeks to maximize this welfare with $Q_{\mathsf{TO}} = \alpha/\beta$, while the outcome of CTS arises from the strategic interaction of the market participants.

Our next result characterizes the equilibrium and provides key insights into the behavior of $\eta_{\mathsf{CTS}} \leq 1$ in different liquidity regimes.

Proposition 1. *Consider the CTS game $\mathcal{G}(\mathbf{B}, 0, \alpha, \beta)$, where B_m is the unique maximal budget in $\{B_1, \ldots, B_N\}$. Then, $\mathcal{G}(\mathbf{B}, 0, \alpha, \beta)$ admits a unique Nash equilibrium $\boldsymbol{\theta}^{\mathsf{NE}}$ given by*

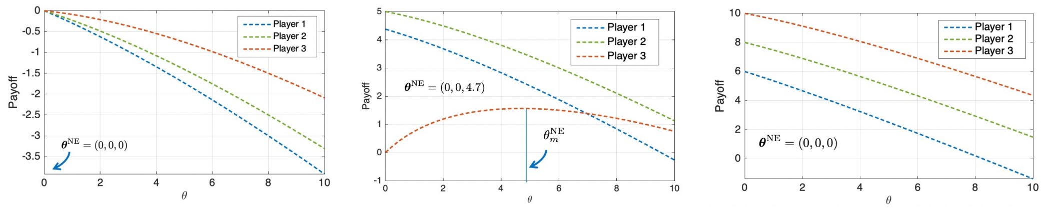
$$\theta_m^{\mathsf{NE}} = \begin{cases} \dfrac{1}{4\beta}\left(\beta^2 B_m - \mathcal{P}^2(\mathbf{1}^\intercal \mathbf{B})\right), & \text{if } |\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta| < B_m, \\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

and $\theta_i^{\mathsf{NE}} = 0$ for $i \neq m$. Furthermore, we have

$$\eta_{\mathsf{CTS}}(\mathbf{B}) \begin{cases} = 1, & \text{if } \mathbf{1}^\intercal \mathbf{B} - \alpha/\beta \geq B_m, \\ \geq \dfrac{3}{4}, & \text{if } |\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta| < B_m\ , \\ = 2z - z^2, & \text{otherwise} \end{cases} \tag{3.12}$$

where $z := \frac{\beta}{\alpha}\mathbf{1}^\intercal \mathbf{B}$.

The proof of Proposition 1 is provided in Section 3.7. The result highlights that allocation and efficiency vary widely with liquidity and the player with the maximal liquidity plays a rather central role in determining the outcome of the CTS market. To offer more insights, distinguish three different



(a) (b) (c)

Figure 3.3: Plots (a), (b) and (c) show payoffs of a 3-player CTS game $\mathcal{G}(\mathbf{B}, 0, \alpha, \beta)$ in the high, intermediate and low liquidity regimes, respectively. The liquidities satisfy $B_1 < B_2 < B_3$.

liquidity regimes. Identify the liquidity as high when $\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta \geq B_m$, where the aggregate liquidity of all players but m is sufficient to cover the efficient schedule $Q_{\mathsf{TO}} = \alpha/\beta$. The intermediate liquidity occurs where the aggregate liquidity is different from Q_{TO} by at most the liquidity of player m, i.e., $|\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta| < B_m$. Finally, the low liquidity regime is where $\mathbf{1}^\intercal \mathbf{B} + B_m < Q_{\mathsf{TO}}$. The outcome and the efficiency differ substantially across these regimes.

Using the equilibrium profile, it is easy to see that the flow allocation is given by

$$x_m^\star(\boldsymbol{\theta}^{\mathsf{NE}}; \mathbf{B}) = \begin{cases} \frac{1}{2}(\alpha/\beta - \mathbf{1}^\intercal \mathbf{B}_{-m}), & \text{if } |\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta| < B_m, \\ B_m, & \text{otherwise}, \end{cases},$$

$$x_i^\star(\boldsymbol{\theta}^{\mathsf{NE}}; \mathbf{B}) = B_i, \quad i \neq m,$$

where $\mathbf{B}_{-m}$ denotes the vector of liquidities of all players, except m. Thus, all but player m offer their maximum liquidity at equilibrium. These players benefit from being inframarginal, exploiting the bid of the marginal player m. This behavior is reminiscent of the so-called "free-rider problem" (see [54]). When the liquidity is too high or too low, player m does not have enough market power and does not benefit from bidding nonzero θ_m, implying that they do not withhold from their maximal budget B_m in their transport offer. In the intermediate liquidity case, player m enjoys market power and their flow allocation can be shown to be the Cournot best response to this residual price spread $\mathcal{P}(Q - \mathbf{1}^\intercal \mathbf{B}_{-m})$.

The tie-line schedule at the equilibrium of $\mathcal{G}(\mathbf{B}, 0, \alpha, \beta)$ is

$$Q_{\mathsf{CTS}} = \begin{cases} Q_{\mathsf{TO}}, & \text{if } \mathbf{1}^\intercal \mathbf{B} - B_m \geq \alpha/\beta, \\ \frac{1}{2}\left(Q_{\mathsf{TO}} + \mathbf{1}^\intercal \mathbf{B}_{-m}\right), & \text{if } |\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta| < B_m, \\ \mathbf{1}^\intercal \mathbf{B}, & \text{otherwise.} \end{cases}$$

When liquidity is high, Q_{CTS} coincides with Q_{TO}, implying that CTS yields the SOs' intended outcome. In other words, perfect competition arises as a result of strategic incentives. In the intermediate liquidity regime, CTS suffers welfare loss due to strategic interaction. The loss, however, is bounded; strategic behavior cannot cripple the welfare under perfect competition by more than 25%. When the liquidity is low, the lower bound on η_{CTS} can be arbitrarily small. However, in this case, lack of efficiency is not due to strategic interactions but rather due to the lack of market liquidity.

3.4.1 Strategic Selection of Budgets

To offer further insights in the previous discussion, consider an example of a CTS game with three players. The players' payoffs are shown in Figure 3.3 for each liquidity regime. When liquidity is high, all players garner zero payoffs by bidding $\boldsymbol{\theta}^{\mathsf{NE}}$. Any other action, induces negative reward and CTS yields the efficient schedule. Notice how the payoff of maximal player (B_3) changes in the intermediate regime, which leads to her choosing $\theta_m^{\mathsf{NE}} > 0$. This results in efficiency loss of CTS. Interestingly, the maximum payoff for all players (and highest efficiency loss for CTS) is attained at the low liquidity regime, which can result either from a small number of players or small budgets. This outcome is problematic from a market design perspective: it incentivizes players' to misrepresent B_i's.

Indeed, if players are aware that reporting lower B_i's increases their payoffs, strategic behavior would lead to even greater efficiency loss. To see why, consider the case where players, prior to choosing θ_i's, strategically select their budgets. The selection of budgets is such that no player would prefer to be in the high or intermediate liquidity regimes. To maximize their payoffs, players would have to select B_i's such that $\mathbf{1}^\intercal \mathbf{B} + B_m < Q_{\mathsf{TO}}$. In this regime, the outcome of game in $\boldsymbol{\theta}$ is fully characterized from 1. Hence, at the stage of selecting B_i's, player i faces payoff

$$\begin{aligned}\pi_i(B_i, \mathbf{B}_{-i}) &= \mathcal{P}(\boldsymbol{\theta}^{\mathsf{NE}}; \mathbf{B}) x_i^\star(\boldsymbol{\theta}^{\mathsf{NE}}; \mathbf{B}) - \gamma_i B_i \\ &= (\alpha - \beta \mathbf{1}^\mathsf{T} \mathbf{B}) B_i - \gamma_i B_i, \end{aligned} \tag{3.13}$$

where $\gamma_i > 0$ represents return on investment of a risk-free asset. Then, requiring each player to maximize (3.13) yields

$$B_i^\star(\mathbf{1}^\mathsf{T} \mathbf{B}_{-i}) = \frac{\alpha - \beta \mathbf{1}^\mathsf{T} \mathbf{B}_{-i}}{2\beta} - \frac{\gamma_i}{2\beta}, \quad i = 1, \dots, N, \tag{3.14}$$

which is the Cournot best response of player i to the budgets of all other players. Summing over i's in (3.14) yields budget

$$B_i^\star = \frac{\alpha + \mathbf{1}^\mathsf{T} \gamma}{\beta(N+1)} - \frac{\gamma_i}{\beta}, \tag{3.15}$$

for every CTS player i. It is straightforward to verify that the budgets computed by (3.15) satisfy the requirement $\mathbf{1}^\mathsf{T} \mathbf{B}^\star + B_m^\star < \alpha/\beta$, for any $\gamma_m > 0$. Then, the CTS schedule is given by

$$Q_{\mathsf{CTS}} = \frac{N\alpha}{(N+1)\beta} - \frac{\mathbf{1}^\mathsf{T} \gamma}{(N+1)\beta}, \tag{3.16}$$

which approaches Q_{TO} as $N \to \infty$. The previous discussion draws interesting parallels with earlier works that have established that quantity/capacity precommitment and Bertrand competition yield Cournot outcomes [55, 56]. Notice that if players are strategic only in $\boldsymbol{\theta}$, then $N = 2$ would suffice to restore competition, provided sufficient liquidity. However, if CTS bidders were to strategically select B_i's prior to bidding in the market, then $N = 2$ would no longer suffice to yield the efficient schedule. The previous discussion reveals that supply function competition in scalar-parameterized offers, behaves as Bertrand competition in $\boldsymbol{\theta}$ and as Cournot competition in $\mathbf{B}$. Constructing the more general framework to establish the conditions under which the previous equivalence holds, is an interesting direction for future research.

3.4.2 Learning Equilibria Through Repeated Play

Nash equilibria characterize how the incentives of market participants are oriented. However, the power of said equilibria to predict market outcomes may appear limited in that players are endowed with intelligence over their opponents' payoff and the system conditions to compute such an equilibrium. In practice, players interact repeatedly exploring the market environment while facing a noisy reward. Motivated to investigate if players can learn equilibria through repeated play, we study the game dynamics where bidders adopt action-value methods [57] to update their bids. More precisely, we implement an upper confidence bound (UCB) algorithm for each bidder. In such a setting, each player is agnostic to the presence of other players and the SOs' clearing process, i.e., they endogenize these as part of the environment that yields a random reward. UCB is a popular reinforcement learning algorithm that achieves logarithmic regret [58, 59] in static environments and balances between exploration and exploitation. In each round (an instance of a CTS market), each player selects the action that has the maximum observed payoff thus far plus some exploration bonus.

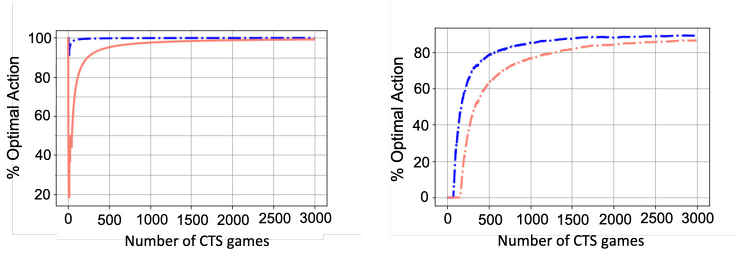
The game proceeds as follows: at each round, each bidder chooses θ from a finite set of actions $\Theta := \{\theta^1, \ldots, \theta^M\}$. Each bidder maintains a vector $\mathbf{R} \in \mathbb{R}^M$ of average rewards from each action and the number of times $\mathbf{T} \in \mathbb{N}^M$ each action is chosen, where $\mathbb{N}$ denotes the set of naturals. Here, the reward equals the revenue less the transaction cost from the CTS market. Bidders initialize $\mathbf{R}$ by selecting every action (possible bid from Θ) at least once. Upon bidding $\theta^k \in \Theta$ at a certain round, say she receives the reward r^k from the CTS market. Then, the bidder updates T^k and R^k as

$$T^k \leftarrow T^k + 1, \quad R^k \leftarrow R^k + \frac{1}{T^k}\left(r^k - R^k\right). \tag{3.17}$$

Then, the bidder bids the action θ^k, where

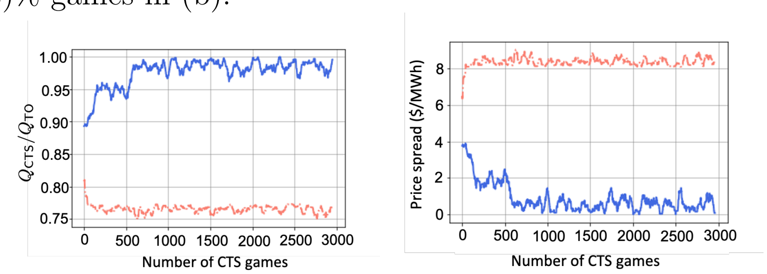
$$k = \underset{j \in \{1,\ldots,M\}}{\operatorname{argmax}} \left\{ R^j + \rho \sqrt{\ln\left(\mathbf{1}^\intercal \mathbf{T}\right)/T^j} \right\}. \tag{3.18}$$

The parameter $\rho > 0$ controls the degree of exploration. The larger the ρ, the player is eager to explore actions that have not been tried often enough. The smaller the ρ, the player tends to choose an action largely based on the



(a) Intermediate liquidity (b) High liquidity

Figure 3.4: Plot of cumulative percentage of times the Nash action is chosen across 3000 games for bidders 1 (——) and 5 (——). Bidder 5 is marginal for (a) and inframarginal for (b). After 3000 games, bidders 1-5 respectively select $\boldsymbol{\theta}^{\mathsf{NE}}$ in (99.9, 92.1, 99.9, 99.6, 99.2)% games in (a) and (90.1, 99.9, 86.4, 92.4, 88.2)% games in (b).



(a) Tie-line schedules (b) Price spread

Figure 3.5: Comparison of tie-line schedules and price spreads for a highly (——) and intermediately liquid (——) CTS market.

average reward seen thus far.

We utilize historical CTS data from the NYISO and ISONE markets to compute the affine price spread that yields $Q_{\mathsf{TO}} = 1493$ MW. We consider repeated play of the CTS game with five participants, first with $\mathbf{B} = (298, 223, 194, 149, 893)$ and then with $\mathbf{B} = (596, 522, 640, 373, 893)$. The first example corresponds to an intermediate liquidity regime with $\boldsymbol{\theta}^{\mathsf{NE}} = (0, 0, 0, 0, 4882)$. The second example belongs to the high liquidity category for which $\boldsymbol{\theta}^{\mathsf{NE}} = (0, 0, 0, 0, 0)$. In our simulations, we use $\rho = 2$ following [57, Chapter 2]. Each CTS bidder chooses from ten θ's in $\Theta = [0, 6000]$ that includes the optimal actions. Figure 3.4 shows percentages of optimal actions selected by bidders in a total of 3000 games for the high and intermediate liquidity regimes.

In the intermediate regime, the pivotal and inframarginal players act in a rather "greedy" fashion, exploiting their optimal action north of 99% of the games. This implies that the observed reward from playing the optimal action is large enough, even as the exploration bonus of other actions increases. Bidder 5 loses their role as the marginal player when the liquidity is high. In this regime, players are slower to discover their optimal actions, although selection percentages are north of 88% of the games. Our numerical experiments clearly demonstrate that even in a setting where players know little to nothing about the game setting, they are able to discover and play equilibrium actions (in majority of the games) through repeated play. This experiment lends credence to the conclusions from our equilibrium analysis. Indeed, $Q_{\mathsf{CTS}}/Q_{\mathsf{TO}}$ in Figure 3.5 remains close to unity and price spreads are below $2/MWh in most games for a highly liquid CTS market. A liquidity reduction of around 40% has palpable effects on market performance, although in aggregate, the players have the capacity to meet Q_{TO}. In particular, the price spread for intermediate liquidity is more than $6/MWh higher than the highly liquid case and $Q_{\mathsf{CTS}}/Q_{\mathsf{TO}}$ remains well below 80%. This experiment highlights how rise of pivotal players exercising market power exploiting the lack of liquidity can impact market performance.

3.5 Interactions with Virtual Trading in Energy Markets

CTS performance can be influenced by uneconomic bidding that aims to benefit financial positions of virtual transactions in energy markets. An example of said transactions are up-to-congestion (UTC) virtual bids [60]. A UTC is a bid in the day-ahead market to purchase congestion and losses between two nodes within each area. The UTC bid consists of a specified source and sink location together with a price spread that identifies how much the participant is willing to pay for congestion and losses between source and sink. The payoff of a UTC bid depends on the real-time and day-ahead prices at the specified locations.

Bidding behavior in CTS markets impacts CTS outcomes, that in turn affect price movements in both areas. Said price movements influence the return from UTC positions. Thus, bidders with existing UTC portfolios can engage in uneconomic bidding behavior. Here, we utilize our game model to illustrate one such case, where UTC positions negatively impact CTS performance. We remark that price manipulation via uneconomic virtual transactions has emerged as a central policy concern for FERC; several high-profile enforcement cases have ended in multi-million dollar settlements [61].

Denote by f_i^k, the UTC megawatt position of CTS bidder i from an internal node k inside area b to the CTS trading location. Let $\mathcal{P}_b^k$ denote the LMP at node k in area b. Denote by $\mathcal{P}_b^{k,\mathsf{DA}}$ and $\mathcal{P}_b^{\mathsf{DA}}$ the day-ahead prices at internal node k and CTS trading location, respectively. Then, the payoff of bidder i from their UTC positions is given by

$$\sum_k \left[(\mathcal{P}_b - \mathcal{P}_b^k) - (\mathcal{P}_b^{\mathsf{DA}} - \mathcal{P}_b^{k,\mathsf{DA}}) \right] f_i^k, \tag{3.19}$$

where the sum is taken with k ranging over buses within area b. The CTS outcome will not affect day-ahead prices, but it does influence real-time prices at other locations inside each area. We have assumed so far that $\mathcal{P}_b - \mathcal{P}_a$ has an affine dependence on Q, the amount that flows from bus a to bus b. Assume a similar affine dependence

$$\mathcal{P}_b(Q) - \mathcal{P}_b^k(Q) = \alpha_{\text{in}}^k - \beta_{\text{in}}^k Q$$

between the CTS trading location and an internal node k in area b. Albeit simplistic, this model is enough to reveal the impact of UTCs on CTS markets. To illustrate the coupling between UTC positions and CTS market, consider the joint payoff from them for bidder i in

$$\begin{aligned} \widetilde{\pi}_i(\theta_i, \boldsymbol{\theta}_{-i}) = &\underbrace{(\alpha - \beta Q)B_i - \theta_i}_{\text{from CTS}} \\ &+ \underbrace{\textstyle\sum_k (\alpha_{\text{in}}^k - \beta_{\text{in}}^k Q) f_i^k - (\mathcal{P}_b^{\mathsf{DA}} - \mathcal{P}_b^{k,\mathsf{DA}}) f_i^k}_{\text{from UTC}}, \end{aligned} \tag{3.20}$$

where Q depends on CTS market clearing with bids $\boldsymbol{\theta}$ and liquidities $\mathbf{B}$. Formally, call this game $\mathcal{G}_{\mathsf{UTC}}(\mathbf{B}, c, \alpha, \beta, \mathbf{f}, \alpha_{\text{in}}, \beta_{\text{in}})$ with payoffs in (3.20). Here, α_{in}, β_{in}, $\mathbf{f}$ collect the respective variables across all internal buses. Our next result characterizes the market outcome with UTC positions.

Proposition 2. *The game $\mathcal{G}_{\mathsf{UTC}}(\mathbf{B}, 0, \alpha, \beta, \mathbf{f}, \alpha_{in}, \beta_{in})$ admits a unique Nash equilibrium if $\mathbf{f}$ is elementwise non-negative, for which the tie-line schedule* *at the equilibrium is*

$$Q_{\mathsf{CTS}} = \begin{cases} Q_{\mathsf{TO}}, & \textit{if } \mathbf{1}^\mathsf{T}\mathbf{B} - \widetilde{B}_m \geq \alpha/\beta, \\ \frac{1}{2}\left(Q_{\mathsf{TO}} + \mathbf{1}^\mathsf{T}\mathbf{B} - \widetilde{B}_m\right), & \textit{if } |\mathbf{1}^\mathsf{T}\mathbf{B} - \alpha/\beta| < \widetilde{B}_m, \\ \mathbf{1}^\mathsf{T}\mathbf{B}, & \textit{otherwise}, \end{cases}$$

where $\widetilde{B}_i = B_i + \sum_k (\beta_{in}^k/\beta) f_i^k$ *for* $i = 1, \ldots, N$ *and* m *is the only player with maximal* $\widetilde{B}_m$.

The proof of Proposition 2 is provided in Section 3.7. The result reveals that the bidder with maximum combined CTS and UTC position emerges as the pivotal player in this market. Moreover, $\widetilde{B}_m \geq B_m$ dictates that less power is scheduled to flow in the tie-line when bidders have such positions. This results from the incentives of the pivotal player who benefits from higher prices at the importing region b's CTS bus as that yields a higher UTC payoff. In fact, the difference in the tie-line schedules with and without UTC, grows with $\widetilde{B}_m - B_m$ that is directly proportional to the UTC positions. Opposite conclusions can be drawn if we consider players with UTC positions that source at area b's proxy bus.

The following example illustrates the shift in market power and scheduling efficiency when participants hold UTCs. Consider the CTS market in Section 3.4.2 where the fifth bidder is pivotal in the intermediate liquidity regime. At the equilibrium, $Q_{\mathsf{CTS}} = 1176$ MW. Assume that the first bidder holds a UTC $f_1 = 800$ MW to an internal bus for which $\alpha_{\text{in}} = 35.7$ and $\beta_{\text{in}} = 0.02$. Then, $\widetilde{\mathbf{B}} = [1018, 463, 193, 149, 893]$. Notice that bidder one emerges as the new marginal bidder and has incentive to bid in a way that leads to less power being scheduled to flow into area b. Indeed, the new tie-line schedule is $Q_{\mathsf{CTS}} = 1113$ MW, 63 MW less than CTS without UTCs, falling even shorter of $Q_{\mathsf{TO}} = 1493$ MW.

3.6 Impact of Forecast Errors and Transaction Costs

Our analysis of the CTS game so far has assumed that players and the SOs have perfect forecasts into the price spread function. In practice, tie-line scheduling takes place with a lead time to power delivery, meaning that there is an inherent uncertainty in the price spread when these markets are convened. To model this uncertainty, assume that the SOs conjecture an affine price spread function

$$\mathcal{P}_{\mathsf{SO}}(Q) = \alpha_{\mathsf{SO}} - \beta_{\mathsf{SO}} Q$$

with $\alpha_{\mathsf{SO}}, \beta_{\mathsf{SO}} > 0$. The SOs use this spread to clear the CTS market as in (3.3). Let the realized price difference be

$$\mathcal{P}_{\star}(Q) = \alpha_{\star} - \beta_{\star} Q$$

with $\alpha_{\star}, \beta_{\star} > 0$. Then, the TO schedule and the optimal tie-line schedule, respectively, are given by

$$Q_{\mathsf{TO}} = \alpha_{\mathsf{SO}}/\beta_{\mathsf{SO}} \quad \text{and} \quad Q_{\star} = \alpha_{\star}/\beta_{\star}.$$

Modeling the uncertainty explicitly at the time of scheduling reveals that Q_{TO} may not equal $Q_{\star}$, the ex-post optimal tie-line schedule. Our interest lies in analyzing if strategic behavior of bidders in the CTS market can correct the errors in SOs' forecasts. Do bidders draw the outcome closer to $Q_{\star}$ than Q_{TO} or do they drive it further away as a result of their strategic interaction? We answer this question through a game-theoretic study. We also derive insights into how nonzero transaction fees ($c > 0$) affect these conclusions.

To isolate the impacts of uncertainty and transaction fees, we analyze the game under a simpler setting where the bidders are homogeneous, each with liquidity $B > 0$ and conjectured price spread $\mathcal{P}(Q) = \alpha - \beta Q$ with $\alpha, \beta > 0$. Notice that bidders' conjectured optimal schedule α/β may be different from both Q_{TO} and $Q_{\star}$. We assume here that players share a common belief that the market operates at an intermediate liquidity where the aggregate liquidity NB is close to their conjectured optimal tie-line schedule α/β, i.e.,

$$NB = \alpha/\beta + \mathcal{O}(1/N). \tag{3.21}$$

Under such an assumption, bidder i conjectures the market price from bidding $\boldsymbol{\theta}$ with liquidities $\mathbf{B} = B\mathbf{1}$ to be

$$\begin{aligned} p\left(\boldsymbol{\theta}, B\mathbf{1}\right) &= \frac{1}{2}\left(\mathcal{P}(NB) + \sqrt{\mathcal{P}^2(NB) + 4\beta\mathbf{1}^\intercal\boldsymbol{\theta}}\right) \\ &= \sqrt{\beta\mathbf{1}^\intercal\boldsymbol{\theta}} + \mathcal{O}(1/N), \end{aligned}$$

which yields the following perceived payoff for bidder i.

$$\begin{aligned} \pi_i(\theta_i, \boldsymbol{\theta}_{-i}) &= p(\boldsymbol{\theta}, \mathbf{B})B - \theta_i - c\left(B - \frac{\theta_i}{p\left(\boldsymbol{\theta}, B\mathbf{1}\right)}\right) \\ &\approx \sqrt{\beta\mathbf{1}^\intercal\boldsymbol{\theta}}B - \theta_i - c\left(B - \frac{\theta_i}{\sqrt{\beta\mathbf{1}^\intercal\boldsymbol{\theta}}}\right). \end{aligned} \tag{3.22}$$

Call the CTS game with conjectured price spreads $\mathcal{G}_{\text{conj}}(B, c, \alpha, \beta, \alpha_{\mathsf{SO}}, \beta_{\mathsf{SO}})$, where α, β satisfy (3.21) and the payoffs are given by (3.22). Assuming that all players offer based on an equilibrium profile for this game, the SOs then solve the CTS flow allocation problem in (3.3) with $\mathcal{P}_{\mathsf{SO}}$ to ultimately compute the tie-line schedule. Our next result characterizes both a (symmetric) equilibrium profile and the resulting tie-line schedule.

Proposition 3. *The CTS game $\mathcal{G}_{conj}(B, c, \alpha, \beta, \alpha_{\mathsf{SO}}, \beta_{\mathsf{SO}})$ admits a unique symmetric Nash equilibrium given by $\theta_i^{\mathsf{NE}} = \frac{\gamma^2}{4N\beta}$ for $i = 1, \ldots, N$, for which the tie-line schedule at equilibrium is*

$$Q_{\mathsf{CTS}} = \frac{1}{2}\left[Q_{\mathsf{TO}} + NB - \sqrt{\left(Q_{\mathsf{TO}} - NB\right)^2 + \frac{\gamma^2}{\beta\beta_{\mathsf{SO}}}}\right],$$

where $\gamma := c(2 - 1/N) + \beta B$.

The proof of Proposition 3 is provided in Section 3.7. Notice that players bid solely based on their own conjectures. The tie-line schedule, however, depends on the conjectures of both the bidders and the SOs. This result will allow us to study the effect of price spread forecasts and transaction costs on the scheduling efficiency in the sequel.

The lack of knowledge of $Q_\star$ by the SOs and market participants prompts us to investigate whether CTS can yield a more efficient schedule than the pure SO-driven TO. Proposition 3 implies $Q_{\mathsf{CTS}} \leq Q_{\mathsf{TO}}$, meaning that CTS cannot yield a more efficient schedule than TO if $Q_{\mathsf{TO}} < Q_\star$. Hence, CTS can only outperform TO when the SOs' forecast overestimates Q_{TO}. In this regime, Figure 3.1 yields that Q_{CTS} is always closer to $Q_\star$ when $Q_\star \leq Q_{\mathsf{TO}}/2$. Outside of this setting, the outcome of CTS depends on the liquidity and conjectures of players. Specifically, if $NB \in \mathcal{A}_1 \cup \mathcal{A}_2$, defined in Figure 3.6, Q_{CTS} is closer to $Q_\star$ than Q_{TO}, if

$$\frac{\gamma^2}{\beta\beta_{\mathsf{SO}}} \leq 8\left(Q_{\mathsf{TO}} - Q_\star\right)\left(Q_{\mathsf{TO}} - 2Q_\star + NB\right). \tag{3.23}$$

Such a premise appears to run counter to the intuition that TO is optimal. This situation can only arise under uncertainty where SOs make serious forecast errors in the expected price spread. Surprisingly, forecast errors are not that rare, according to [43], where the error in SOs' point forecast for the price spread between NYISO and ISO-NE averaged \$2.42/MWh. Notice how, in this liquidity regime, the presence of transaction fees makes it harder to satisfy (3.23). This is intuitively correct since transaction fees drive the tie-line schedule toward smaller values, as established in Proposition 3.

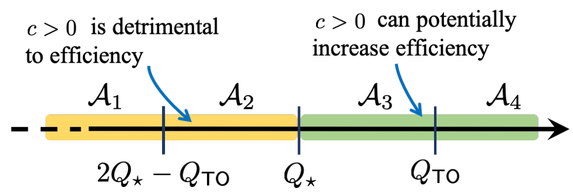


Figure 3.6: Ability of market participants to correct SO's forecast error depends on liquidity and transactions costs.

When $NB \in \mathcal{A}_3 \cup \mathcal{A}_4$, liquidity is sufficiently high and the presence of costs might improve scheduling efficiency since players bid higher prices to counter costs. Overall, players ability to correct SOs' forecast is somewhat limited and relies on many qualifications, indicating that the SOs forecasts and systematic bias plays a vital role in scheduling efficiency. Moving bid submittal and clearing timelines closer to power delivery should improve the efficiency of CTS.

Proposition 3 suggests that incentives of CTS bidders are aligned in a way that allows them to correct SOs' forecast errors in some settings. Can players learn such equilibria through repeated play? We employ the learning framework in Section 3.4.2, where players have their bids cleared against $(\alpha_{\mathsf{SO}}, \beta_{\mathsf{SO}})$ that are perturbed from $(\alpha_\star, \beta_\star)$ learned from historical data. That is, in every round, bidders receive reward from the ex-post price spread described by

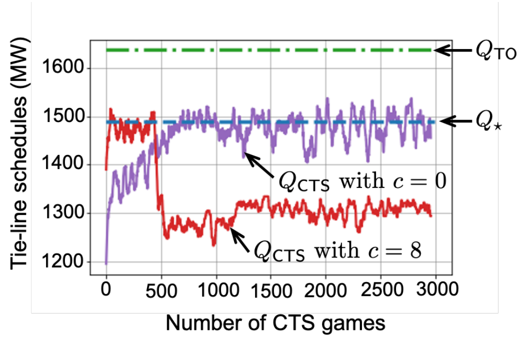
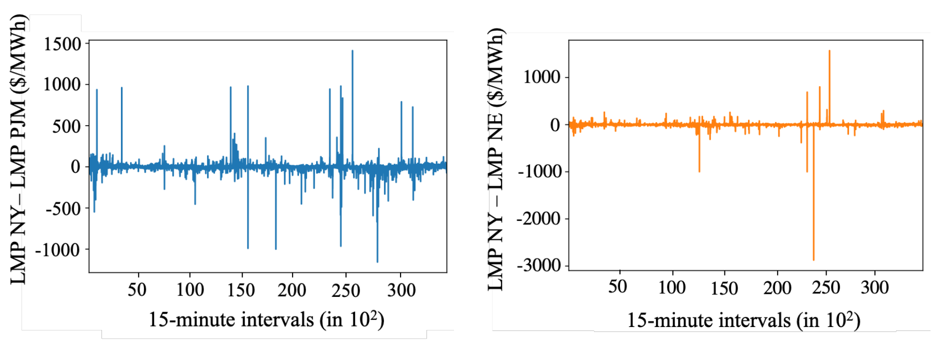


Figure 3.7: The trajectory of CTS schedules cleared against SO's forecasted prices with 10% error with $c = 0$ and $c = \$8$/MWh.

$\mathcal{P}_\star$. The trajectory of tie-line schedules in Figure 3.7 with $c = 0$ reveals that bidding behavior of players results in CTS schedules consistently closer to the ex-post optimal than TO. Despite the persistent forecast error, bidders correct the tie-line schedule to an extent by seeking actions that maximize their observed reward.

The relation in (3.23) reveals that presence of nonzero transaction fees c make it more difficult for CTS market to drive the outcome closer to the ex-post optimal as γ increases with c. Bidders reacting to observed rewards with $c = \$8$/MWh in Figure 3.7 yield a CTS schedule farther from $Q_\star$, seeking actions that yield higher prices but smaller schedules. This result corroborates our theoretical finding that transaction fees impede bidders' ability to correct SOs' forecast errors.



(a) (b)

Figure 3.8: Plot (a) depicts the time series of spread between NYISO and PJM proxy buses in 2018 (absolute mean = 8.92 \$/MWh, std. deviation = 22.11 \$/MWh). Plot (b) shows the same between NYISO and ISO-NE for the same year (mean = 0.44, absolute mean = 5.59 \$/MWh, std. dev. = 18.14 \$/MWh).

Notice that equilibrium bid grows with c, per Proposition 3. With $c > 0$, bidders are reluctant to offer their entire liquidity. This prevents the price spread from converging to zero, even if the market is liquid. Moreover, transaction fees make it less attractive for CTS bidders overall, hurting long-term liquidity of the CTS market. Figure 3.8a indicates that the price spread in the CTS market between NYISO and PJM exhibits longer excursions from zero and higher volatility compared to that between NYISO and ISO-NE, depicted in Figure 3.8b. The average absolute spread between NYISO and PJM is approximately \$3.3/MWh higher than that between NYISO and ISO-NE. We surmise that transaction fees between NYISO and PJM and the lack thereof between NYISO and ISO-NE are largely responsible for this difference.

3.7 Proofs

3.7.1 Proof of Theorem 4

We break the proof into two parts—for $c > 0$ and $c = 0$. We argue that $\mathcal{G}(\mathbf{B}, c)$ is a concave game with a compact strategy set $\boldsymbol{\Theta}$ in each case. Then, the rest follows from Rosen's result in [48, Theorem 1].

<u>Case with $c > 0$:</u> Notice that $\mathbf{1}^\intercal \boldsymbol{\theta} > 0$ for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. The objective function of (3.3) can be shown to be strictly concave for all $\boldsymbol{\theta} \neq 0$ (the Hessian is negative definite), meaning that if a solution to the maximization problem exists, then it is unique. The first-order optimality conditions of (3.3) yield that such an optimal allocation $\mathbf{x}^\star$ must satisfy

$$\mathcal{P}(\mathbf{1}^\intercal \mathbf{x}^\star) - \frac{\theta_i}{B_i - x_i^\star} = 0, \ \forall i. \tag{3.24}$$

Summing the above relation over i, we get

$$\mathcal{P}(Q_{\mathsf{CTS}}) = \frac{\mathbf{1}^\intercal \boldsymbol{\theta}}{\mathbf{1}^\intercal \mathbf{B} - Q_{\mathsf{CTS}}}. \tag{3.25}$$

Since $\mathcal{P}$ is strictly decreasing with $\mathcal{P}(0) > 0$, the strictly increasing function of Q_{CTS} that grows to ∞ at $\mathbf{1}^\intercal \mathbf{B}$ in the RHS of (3.25) must intersect $\mathcal{P}$ at a unique point. Thus, Q_{CTS} is uniquely defined for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, and so is $\mathbf{x}^\star$

identified by

$$x_i^\star = B_i - \frac{\theta_i}{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}}(\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}). \tag{3.26}$$

To establish that $\mathcal{G}(\mathbf{B}, c)$ is a concave game, we now establish that the payoffs $\pi(\theta_i, \boldsymbol{\theta}_{-i})$ in (3.8) are continuous is $\boldsymbol{\theta}$ and concave in θ_i. Notice that the unique optimal allocation $\mathbf{x}^\star$ is continuous in $\boldsymbol{\theta}$, owing to Berge's maximum theorem [62], implying the same for Q_{CTS}. In turn, that proves the continuity of π_i in $\boldsymbol{\theta}$. Next, we prove that π_i is concave in θ_i, by showing that $\mathcal{P}(Q_{\mathsf{CTS}})$ is concave and $x_i^\star$ is convex in θ_i.

First, we show that $\mathcal{P}(Q_{\mathsf{CTS}})$ is concave. Notice that

$$\frac{\partial^2}{\partial\theta_i^2}\mathcal{P}(Q_{\mathsf{CTS}}) = \mathcal{P}''(Q_{\mathsf{CTS}})\left(\frac{\partial Q_{\mathsf{CTS}}}{\partial\theta_i}\right)^2 + \mathcal{P}'(Q_{\mathsf{CTS}})\frac{\partial^2 Q_{\mathsf{CTS}}}{\partial\theta_i^2}.$$

Since $\mathcal{P}$ is concave and strictly decreasing, it suffices to show that Q_{CTS} is convex in θ_i to conclude that $\frac{\partial^2}{\partial\theta_i^2}\mathcal{P}(Q_{\mathsf{CTS}}) \leq 0$ and hence, $\mathcal{P}(Q_{\mathsf{CTS}})$ is concave in θ_i.

To prove the convexity of Q_{CTS} in θ_i, rewrite (3.25) as $g(Q_{\mathsf{CTS}}) = \mathbf{1}^\mathsf{T}\boldsymbol{\theta}$, where

$$g(Q_{\mathsf{CTS}}) := (\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}})\mathcal{P}(Q_{\mathsf{CTS}}). \tag{3.27}$$

Now, $g(0) > 0$ and g is a continuous and strictly decreasing function of its argument. Also, g is convex because

$$g''(Q_{\mathsf{CTS}}) = \mathcal{P}''(Q_{\mathsf{CTS}})(\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}) - 2\mathcal{P}'(Q_{\mathsf{CTS}}) \geq 0, \tag{3.28}$$

where the inequality follows from (3.9), the strictly decreasing and concave nature of $\mathcal{P}$, and the non-negativity of $\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}$. These derivatives exist, owing to the implicit function theorem [63]. Then, Q_{CTS} is the inverse of a decreasing convex function, and is therefore decreasing convex itself in $\mathbf{1}^\mathsf{T}\boldsymbol{\theta}$, and therefore in θ_i. This completes the proof of the concavity of $\mathcal{P}(Q_{\mathsf{CTS}})$.

Next, we show that $x_i^\star$ is convex in θ_i. From (3.26), we get

$$\frac{\partial x_i^\star}{\partial \theta_i} = -\frac{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}_{-i}}{(\mathbf{1}^\mathsf{T}\boldsymbol{\theta})^2}(\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}) + \frac{\theta_i}{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}}\frac{\partial Q_{\mathsf{CTS}}}{\partial \theta_i}, \tag{3.29}$$

$$\frac{\partial^2 x_i^\star}{\partial \theta_i^2} = 2\frac{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}_{-i}}{(\mathbf{1}^\mathsf{T}\boldsymbol{\theta})^3}(\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}) + 2\frac{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}_{-i}}{(\mathbf{1}^\mathsf{T}\boldsymbol{\theta})^2}\frac{\partial Q_{\mathsf{CTS}}}{\partial \theta_i} + \frac{\theta_i}{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}}\frac{\partial^2 Q_{\mathsf{CTS}}}{\partial \theta_i^2}. \tag{3.30}$$

Again, the implicit function theorem guarantees that these derivatives exist for $\boldsymbol{\theta}$ away from the origin. The last term in (3.30) is non-negative by convexity of Q. Therefore, we require the sum of the remaining terms to be non-negative or

$$\frac{\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}}{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}} \geq -\frac{\partial Q_{\mathsf{CTS}}(\boldsymbol{\theta};\mathbf{B})}{\partial \theta_i}.$$

From (3.27) we have

$$\begin{aligned}\frac{\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}}{\mathbf{1}^\mathsf{T}\boldsymbol{\theta}} &= \frac{1}{\mathcal{P}(Q_{\mathsf{CTS}})}\\ &\geq \frac{1}{\mathcal{P}(Q_{\mathsf{CTS}}) - \mathcal{P}'(Q_{\mathsf{CTS}})(\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}})}\\ &= -\frac{\partial Q_{\mathsf{CTS}}(\boldsymbol{\theta},\mathbf{B})}{\partial \theta_i},\end{aligned}$$

where the inequality follows from the fact that $\mathcal{P}'(Q_{\mathsf{CTS}})(\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}}) < 0$. This finishes the proof of π_i being concave in θ_i.

Case with $c = 0$: The payoff π_i is continuous in $\boldsymbol{\theta}$ and concave in θ_i for all $\boldsymbol{\theta} > \mathbf{0}$. We extend the same to $\boldsymbol{\theta} = \mathbf{0}$ with $c = 0$. With zero costs, we have

$$\pi_i(\theta_i, \boldsymbol{\theta}_{-i}) = \mathcal{P}(Q_{\mathsf{CTS}}(\boldsymbol{\theta};\mathbf{B}))B_i - \theta_i. \tag{3.31}$$

It suffices to argue that Q_{CTS} is continuous at $\boldsymbol{\theta} = \mathbf{0}$. Recall that for $\mathbf{1}^\mathsf{T}\boldsymbol{\theta} > 0$, Q_{CTS} is given by the solution of

$$\mathbf{1}^\mathsf{T}\boldsymbol{\theta} = (\mathbf{1}^\mathsf{T}\mathbf{B} - Q_{\mathsf{CTS}})\mathcal{P}(Q_{\mathsf{CTS}}). \tag{3.32}$$

First, assume that $\mathbf{1}^\mathsf{T}\mathbf{B} < Q_{\mathsf{TO}}$. Then, $\mathcal{P}(Q_{\mathsf{CTS}}) \geq \mathcal{P}(\mathbf{1}^\mathsf{T}\mathbf{B}) > 0$ since $\mathcal{P}$ is strictly decreasing. Consider a sequence $\boldsymbol{\theta}_k \to \mathbf{0}$ as $k \to \infty$. Then, the LHS of (3.32) vanishes. Therefore, the RHS must vanish as well. Since $\mathcal{P}$ does not vanish, we must have $Q_{\mathsf{CTS}}(\boldsymbol{\theta}_k) \to \mathbf{1}^\intercal \mathbf{B} = Q_{\mathsf{CTS}}(\mathbf{0})$ as required. Now, consider the situation where $\mathbf{1}^\intercal \mathbf{B} > Q_{\mathsf{TO}}$. In this case, $Q_{\mathsf{CTS}} \leq Q_{TO} < \mathbf{1}^\intercal \mathbf{B}$. Consider the sequence $\boldsymbol{\theta}_k \to \mathbf{0}$ as $k \to \infty$. As the LHS of (3.32) vanishes, $\mathcal{P}$ must vanish in the RHS. Therefore, $\mathcal{P}(Q_{\mathsf{CTS}}(\boldsymbol{\theta}_k)) \to 0$ or $Q_{\mathsf{CTS}}(\boldsymbol{\theta}_k) \to \mathcal{P}^{-1}(0) = Q_{\mathsf{TO}} = Q_{\mathsf{CTS}}(\mathbf{0})$, as required for the case with $\mathbf{1}^\intercal \mathbf{B} > Q_{\mathsf{TO}}$. The case with $\mathbf{1}^\intercal \mathbf{B} = Q_{\mathsf{TO}}$ is trivially satisfied by the same line of arguments. Hence, $Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})$ is continuous in $\boldsymbol{\theta}$ at the origin. This completes the proof.

3.7.2 Proof of Proposition 1

Existence of the Nash equilibrium follows from Theorem 4. Solving (3.3) we find that $Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})$ is given by

$$Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B}) = \frac{\alpha + \beta \mathbf{1}^\intercal \mathbf{B}}{2\beta} - \frac{1}{2\beta} \left[\mathcal{P}^2\left(\mathbf{1}^\intercal \mathbf{B}\right) + 4\beta \mathbf{1}^\intercal \boldsymbol{\theta}\right]^{1/2}. \tag{3.33}$$

The payoff for player i is given by

$$\begin{aligned} \pi_i(\theta_i, \boldsymbol{\theta}_{-i}) &= \left(\alpha - \beta Q_{\mathsf{CTS}}(\boldsymbol{\theta}; \mathbf{B})\right) B_i - \theta_i \\ &= \frac{B_i}{2}\left(\mathcal{P}(\mathbf{1}^\intercal \mathbf{B}) + \left[\mathcal{P}^2(\mathbf{1}^\intercal \mathbf{B}) + 4\beta \mathbf{1}^\intercal \boldsymbol{\theta}\right]^{1/2}\right) - \theta_i. \end{aligned} \tag{3.34}$$

The payoff is continuous in $\boldsymbol{\theta}_{-i}$ and strictly concave in θ_i. The strategy space of each player is $[0, \alpha B_i]$. A bid profile $\boldsymbol{\theta}^{\mathsf{NE}} = \left(\theta_1^{\mathsf{NE}}, \ldots, \theta_N^{\mathsf{NE}}\right)$ is a Nash equilibrium if and only if

$$\left.\frac{\partial \pi_i(\theta_i, \boldsymbol{\theta}_{-i})}{\partial \theta_i}\right|_{\boldsymbol{\theta}^{\mathsf{NE}}} \leq 0, \quad \text{if } 0 \leq \theta_i^{\mathsf{NE}} < \alpha B_i \tag{3.35a}$$

$$\left.\frac{\partial \pi_i(\theta_i, \boldsymbol{\theta}_{-i})}{\partial \theta_i}\right|_{\boldsymbol{\theta}^{\mathsf{NE}}} \geq 0, \quad \text{if } 0 < \theta_i^{\mathsf{NE}} \leq \alpha B_i, \tag{3.35b}$$

where the above derivative is given by

$$\frac{\partial \pi_i(\theta_i, \boldsymbol{\theta}_{-i})}{\partial \theta_i} = \frac{\beta B_i}{\left[\mathcal{P}^2(\mathbf{1}^\intercal \mathbf{B}) + 4\beta \mathbf{1}^\intercal \boldsymbol{\theta}\right]^{1/2}} - 1. \tag{3.36}$$

From (3.36) we deduce that the payoff derivative cannot vanish for more than one player. Moreover, no player would bid $\theta_i^{\mathsf{NE}} = \alpha B_i$ since that yields negative payoff and each player profitably deviates by infinitesimally decreasing θ_i. From the previous discussion and the following observation

$$\frac{\partial \pi_m(\theta_m, \boldsymbol{\theta}_{-m})}{\partial \theta_m} > \frac{\partial \pi_i(\theta_i, \boldsymbol{\theta}_{-i})}{\partial \theta_i}, \ i \neq m \tag{3.37}$$

we conclude that $\boldsymbol{\theta}_{-m}^{\mathsf{NE}} = \mathbf{0}$. In search for positive $\theta_m > 0$ we find that

- If $|\mathbf{1}^\intercal \mathbf{B} - \alpha/\beta| < B_m$, then

$$\theta_m^{\mathsf{NE}} = \frac{\beta^2 B_m^2 - \mathcal{P}^2(\mathbf{1}^\intercal \mathbf{B})}{4\beta} > 0. \tag{3.38}$$

- Otherwise, $\theta_m^{\mathsf{NE}} = 0$ since (3.38) yields a negative value.

To prove the bounds on $\eta_{\mathsf{CTS}}(\mathbf{B})$ first note that the social welfare attains its maximum at $Q = Q_{\mathsf{TO}}$ with

$$\mathcal{W}(Q_{\mathsf{TO}}) = \frac{\alpha^2}{2\beta}. \tag{3.39}$$

Hence, in the high liquidity regime, i.e., $\mathbf{1}^\intercal \mathbf{B} - B_m \geq \alpha/\beta$, $Q_{\mathsf{CTS}} = Q_{\mathsf{TO}}$ and $\eta_{\mathsf{CTS}}(\mathbf{B}) = 1$. In the intermediate regime, the social welfare at Q_{CTS} is

$$\begin{aligned}\mathcal{W}(Q_{\mathsf{CTS}}) &= \frac{\alpha}{2}\left(\frac{\alpha}{\beta} + \mathbf{1}^\intercal \mathbf{B}_{-m}\right) - \frac{\beta}{8}\left(\frac{\alpha}{\beta} + \mathbf{1}^\intercal \mathbf{B}_{-m}\right)^2 \\ &= \frac{3}{4}\left(\frac{\alpha^2}{2\beta}\right) + \frac{\mathbf{1}^\intercal \mathbf{B}_{-m}}{4}\left(\alpha - \frac{1}{2}\beta \mathbf{1}^\intercal \mathbf{B}_{-m}\right) \\ &> \frac{3}{4}\mathcal{W}(Q_{\mathsf{TO}}).\end{aligned} \tag{3.40}$$

Finally, in the low liquidity regime, i.e., $\mathbf{1}^\intercal \mathbf{B} + B_m \leq \alpha/\beta$, we have

$$\begin{aligned}\frac{\mathcal{W}(Q_{\mathsf{CTS}})}{\mathcal{W}(Q_{\mathsf{TO}})} &= \frac{1}{\alpha^2}\left(2\beta(\mathbf{1}^\intercal \mathbf{B})\left(\alpha - \frac{\beta}{2}\mathbf{1}^\intercal \mathbf{B}\right)\right) \\ &= \frac{2\beta \mathbf{1}^\intercal \mathbf{B}}{\alpha} - \frac{\beta^2 (\mathbf{1}^\intercal \mathbf{B})^2}{\alpha^2} = 2z - z^2.\end{aligned} \tag{3.41}$$

This completes the proof.

3.7.3 Proof of Proposition 2

It is easy to verify that (3.20) is concave in θ_i for fixed $\boldsymbol{\theta}_{-i}$ and $\mathbf{f}$ non-negative. Moreover, Q is strictly decreasing in θ_i and as θ_i grows large the price spreads approach the limiting values α and α^k. Hence, in (3.20) the first two terms converge to constant values with the affine term approaching negative infinity as θ_i grows unbounded. Therefore, there exists $\theta_i^{\max}$ such that (3.20) becomes negative for $\theta_i \geq \theta_i^{\max}$. As such, we restrict our attention for a Nash equilibrium within the compact interval $[0, \theta_i^{\max}]$. Existence of a Nash equilibrium for $\mathcal{G}_{\mathsf{UTC}}\left(\widetilde{\mathbf{B}}, 0, \alpha, \beta, \alpha^k, \beta^k\right)$ is established by invoking [48, Theorem 1]. A bid profile $\boldsymbol{\theta}^{\mathsf{NE}} = \left(\theta_1^{\mathsf{NE}}, \ldots, \theta_N^{\mathsf{NE}}\right)$ is a Nash equilibrium if and only if (3.35) are satisfied where π_i is replaced with $\widetilde{\pi}_i$ and αB_i with $\theta_i^{\max}$. The payoff derivative is given by

$$\begin{aligned}\frac{\partial \widetilde{\pi}_i(\theta_i, \boldsymbol{\theta}_{-i})}{\partial \theta_i} &= \frac{\beta\left(B_i + \sum_k \frac{\beta^k_{\text{in}}}{\beta} f_i^k\right)}{\left[\mathcal{P}^2(\mathbf{1}^\intercal \mathbf{B}) + 4\beta \mathbf{1}^\intercal \boldsymbol{\theta}\right]^{1/2}} - 1 \\ &= \frac{\beta \widetilde{B}_i}{\left[\mathcal{P}^2(\mathbf{1}^\intercal \mathbf{B}) + 4\beta \mathbf{1}^\intercal \boldsymbol{\theta}\right]^{1/2}} - 1. \end{aligned} \tag{3.42}$$

The rest of proof is similar to that of Proposition 1.

3.7.4 Proof of Proposition 3

We are in search for a symmetric equilibrium for $\mathcal{G}_{\text{conj}}(B, c, \alpha, \beta, \alpha_{\mathsf{SO}}, \beta_{\mathsf{SO}})$. From first-order conditions we find that the payoff's derivative is given by

$$\frac{\partial \pi_i(\theta_i, \boldsymbol{\theta}_{-i})}{\partial \theta_i} = \frac{\beta B}{2p\left(\boldsymbol{\theta}, B\mathbf{1}\right)} - 1 + c\left[\frac{1}{p\left(\boldsymbol{\theta}, B\mathbf{1}\right)} - \frac{\theta_i}{2p\left(\boldsymbol{\theta}, B\mathbf{1}\right)\mathbf{1}^\intercal \boldsymbol{\theta}}\right], \tag{3.43}$$

where $p\left(\boldsymbol{\theta}, B\mathbf{1}\right) = \sqrt{\beta \mathbf{1}^\intercal \boldsymbol{\theta}}$. For $\theta_i^{\mathsf{NE}} > 0$ we require (3.43) to vanish, yielding the following

$$\frac{\theta_i^{\mathsf{NE}}}{\mathbf{1}^\intercal \boldsymbol{\theta}^{\mathsf{NE}}} = \frac{\beta B}{c} + 2 - \frac{2}{c}\sqrt{\beta \mathbf{1}^\intercal \boldsymbol{\theta}}. \tag{3.44}$$

Summing (3.44) over i's we find

$$\sqrt{\mathbf{1}^\intercal \boldsymbol{\theta}^{\mathsf{NE}}} = \frac{1}{N\sqrt{\beta}}\left(\frac{NB\beta}{2} + \frac{c}{2}(2N-1)\right) > 0. \tag{3.45}$$

From (3.45) and (3.44) we find that

$$\theta_i^{\mathsf{NE}} = \frac{1}{4N\beta}\left(\beta B + c(2 - \frac{1}{N})\right)^2, \tag{3.46}$$

which is strictly positive. The solution of (3.3) with $\mathcal{P}_{\mathsf{SO}}$ yields the CTS schedule

$$Q_{\mathsf{CTS}} = \frac{1}{2}\left(Q_{\mathsf{TO}} + NB\right) - \frac{1}{2\beta_{\mathsf{SO}}}\sqrt{\left(\alpha_{\mathsf{SO}} - \beta_{\mathsf{SO}} NB\right)^2 + 4\beta_{\mathsf{SO}}\mathbf{1}^{\mathsf{T}}\boldsymbol{\theta}}. \tag{3.47}$$

Substituting (3.46) in (3.47) we obtain the expression in Proposition 3.

3.8 Summary

We presented theoretical framework to model CTS as a game among arbitrage bidders who compete through scalar-parameterized transport offers. To the best of our knowledge, this is the first work that provides a concrete mathematical formulation to model CTS as a game. We established the existence of Nash equilibria for this game and study the impact of various factors on the nature of said equilibria to offer insights into the CTS market. We showed that when transaction costs (levied on a per-megawatt hour basis on bidders) are absent, then a highly liquid CTS market is efficient. Market efficiency degrades with liquidity shortfall, exhibiting bounded efficiency loss for intermediate liquidity and unbounded losses in low liquidity regimes. Second, with transaction costs, CTS fails to eradicate the price spread between adjacent markets even with a liquid market, implying that such costs undercut the vision behind the market design. Third, we showed that SOs' estimate of the price spread plays a central role in the efficiency of CTS markets in that bidders have limited ability to correct the effects of SOs' forecast errors. Fourth, portfolios of virtual trasactions such as up-to-congestion (UTC) bids held by CTS bidders can impact CTS market outcomes, revealing the dependency of efficiency of these inter-area markets on other energy markets. Our equilibrium analysis reveals how the strategic incentives in CTS markets are oriented but does not illustrate if bidders can learn equilibrium behavior through repeated participation in these markets. We simulate repeated play using historical data from the NYISO–ISO-NE market. In particular, we allow bidders to update their bids through a well-known upper confidence bound (UCB) algorithm that has been very well studied in the reinforcement learning literature. Our simulations confirm that our conclusions from equilibrium analysis continue to hold in a statistical sense in our numerical experiments.

CHAPTER 4

PRICING IN NON-CONVEX ELECTRICITY MARKETS

The design of participation mechanisms for electricity markets must take into consideration the aspects of the physical grid that determine the feasible power flows across the system. In Section **??** we adopted a convex model based on a lossless, linearized power flow model and derived prices for electricity based from the optimal dual multipliers of nodal power balance constraints. However, this model ignores key properties of the power system such as commitment decisions and non-convexities arising from the AC power flow model. In this chapter, we study price formation for electricity markets under a non-convex power flow model, which has received less consideration.

4.1 Sources of Non-Convexities

The core dispatch model in organized wholesale electricity markets relies on a bid-based, security-constrained problem with a linearized power flow model. LMPs stand on sound economic principles when the market-clearing problem is convex. Derived as the optimal dual multipliers from the popular DC approximation dispatch [64], LMPs exhibit several desirable properties. For example, they adequately incentivize market participants to follow the dispatch prescribed by the SO. Moreover, the SO never runs cash-negative after settling the payments with the market participants. See [65, 66] for details.

The core model for settlement design via LMPs ignores key properties of the power system. For example, using only the LMP, a generation unit may not be able to recover its as-bid cost including no-load and startup costs. Incorporation of unit commitment decisions render the market clearing problem non-convex. In this case, there may not exist a set of nodally uniform prices that support a market equilibrium, leading to revenue shortages for generation units that subsequently require out-of-market payments for them to follow the SO-prescribed dispatch. A long literature has emerged already to tackle non-convexities from unit commitment considerations, e.g., see [67, 68, 69, 70, 71, 72]. Unfortunately, such considerations do not define the only source of non-convexity in electricity pricing.

In this thesis, we focus on price formation in electricity markets with non-convexities in the market clearing problem that arise from an AC power flow model. Opposed to commitment considerations, this non-convexity is not a consequence of the cost structures of assets, but rather stems from the nature of the Kirchhoff's laws that govern the underlying power network. There is an increasing interest in the power industry to efficiently and optimally solve the non-convex market clearing problem with AC power flow, e.g., see recent efforts under the ongoing ARPA-E GO competition. As pricing under AC power flow models is gaining traction [73], we are motivated to design and analyze meaningful prices that can accompany such a dispatch. To that end, we formulate the economic dispatch problem with AC power flow equations and derive electricity prices from optimal dual multipliers of its convex semidefinite relaxation.

Linearized (real) power flow models such as DC approximations or local linearizations around an operating point have long been used to design prices in market environments. Some of these models ignore losses and reactive power considerations, making the settlement design somewhat divorced from the physics of the power grid. Ad hoc measures to incorporate losses are known to distort price signals, e.g., see [74]. The motivation behind explicitly incorporating reactive power in the pricing model is justified by two major trends. Declining natural gas prices and environmental regulations have caused baseload generation units that have historically provided reactive power support (e.g. coal plants) to run at economic loss or even plan retirement. Second, the deepening penetration of distributed generation has increased focus on ensuring reactive power capability exists given that high solar generation requires more reactive power. These trends—which are likely to continue—have resulted in substantial out-of-market payments in energy uplift, e.g. PJM paid \$199 million in 2018 according to [75]. As the Federal Energy Regulatory Commission (FERC) focuses on price formation and reactive power compensation in the era of increasing renewable generation [76], the need to incorporate reactive power as an explicit product with transparent price signals becomes compelling. By nature, our convex relaxation-based locational marginal prices (RLMPs) can accommodate AC power flow models in their full generality. As such, RLMPs assign prices to both real and reactive power, bringing reactive power compensation into the fold of competitive markets.

Semidefinite programming (SDP) based convex relaxation of economic dispatch problems has its origins in [77]. Popularized by [78], it has been analyzed in great detail in [79, 80, 81, 82, 83, 84], among others. In this thesis, we focus on RLMPs derived from this SDP relaxation. Moreover, we show that RLMPs exhibit properties of LMPs when the duality gap of the non-convex economic dispatch problem vanishes. Specifically, they incentivize market participants to follow the SO prescribed dispatch and the SO remains solvent after settling payments with the participants (under mild conditions). When the aforementioned duality gap is nonzero, the absence of a market equilibrium may provide incentives for certain generators to deviate from the SO prescribed dispatch signal. In such an event, side payments become necessary, which are undesirable for a number a reasons as highlighted in a recent order by the Federal Energy Regulatory Commission (FERC) [85]. We prove that RLMPs minimize a specific form of side payments necessary to provide dispatch-following incentives to market participants. These side payments are the sum of the lost opportunity cost for the generators and the product revenue shortfall, which arise due to network constraints. This result bears a striking resemblance to the properties of convex hull pricing (CHPs) that have been proposed and analyzed in [72, 86, 70] to tackle non-convexity due to commitment decisions. RLMPs, on the other hand, handle the non-convexity that arises due to power flow equations.

4.2 The Non-Convex Electricity Market Model

Consider an electric power network on n buses and m transmission lines. Let $\mathbf{V} \in \mathbb{C}^n$ denote the vector of nodal voltage phasors, where $\mathbb{C}$ is the set of complex numbers. Denote by $y_{k\ell}$, the admittance of the line joining buses k and ℓ.

The current flowing from bus k toward an adjacent bus ℓ is given by

$(V_k - V_\ell)y_{k\ell}$, yielding

$$p_{k\ell} + \boldsymbol{i}q_{k\ell} = V_k(V_k - V_\ell)^{\mathsf{H}} y_{k\ell}^{\mathsf{H}} \tag{4.1}$$

as the apparent power flow from bus k to bus ℓ. The notation $\mathbf{u}^{\mathsf{H}}$ stands for the conjugate transpose of $\mathbf{u}$ and $\boldsymbol{i} := \sqrt{-1}$. More succinctly, the above relation can be written as

$$p_{k\ell} + \boldsymbol{i}q_{k\ell} = \mathbf{V}^{\mathsf{H}}\boldsymbol{\Phi}_{k\ell}\mathbf{V} + \boldsymbol{i}\mathbf{V}^{\mathsf{H}}\boldsymbol{\Psi}_{k\ell}\mathbf{V}, \tag{4.2}$$

where $\boldsymbol{\Phi}_{k\ell}$ and $\boldsymbol{\Psi}_{k\ell}$ are $n \times n$ Hermitian matrices with all zeros except the following entries:

$$[\boldsymbol{\Phi}_{k\ell}]_{kk} := \frac{1}{2}(y_{k\ell} + y_{k\ell}^{\mathsf{H}}), \quad [\boldsymbol{\Phi}_{k\ell}]_{k\ell} = [\boldsymbol{\Phi}_{k\ell}]_{\ell k}^{\mathsf{H}} := -\frac{1}{2}y_{k\ell},$$
$$[\boldsymbol{\Psi}_{k\ell}]_{kk} := \frac{1}{2\boldsymbol{i}}(y_{k\ell}^{\mathsf{H}} - y_{k\ell}), \quad [\boldsymbol{\Psi}_{k\ell}]_{k\ell} := [\boldsymbol{\Psi}_{k\ell}]_{\ell k}^{\mathsf{H}} := \frac{1}{2\boldsymbol{i}}y_{k\ell}.$$

The two summands in the right-hand side (RHS) of (4.2) define the real and reactive power flows from bus k to bus ℓ, respectively. Assume that the real power flows on the lines are constrained as

$$p_{kl} \le f_{k\ell} \tag{4.3}$$

for a flow limit $f_{k\ell} > 0$. Such limits typically arise from thermal considerations, but may also serve as proxies for stability constraints.[1] Assume that y_{kk} is the shunt admittance at bus k. Then, the apparent power injection at bus k becomes

$$\begin{aligned} p_k + \boldsymbol{i}q_k &= V_k^{\mathsf{H}} V_k y_{kk}^{\mathsf{H}} + \sum_{\ell \sim k}(p_{k\ell} + \boldsymbol{i}q_{k\ell}) \\ &= \mathbf{V}^{\mathsf{H}}\boldsymbol{\Phi}_k\mathbf{V} + \boldsymbol{i}\mathbf{V}^{\mathsf{H}}\boldsymbol{\Psi}_k\mathbf{V}, \end{aligned}$$

[1]Line flow constraints are often formulated over the apparent power flow as $p_{k\ell}^2 + q_{k\ell}^2 \le f_{k\ell}^2$. These alternate formulations ultimately seek to constrain the magnitude of the current flowing over the transmission line. The formulation in (4.3) can alternately encode constraints on line current magnitudes.

where

$$\mathbf{\Phi}_k := \frac{1}{2}\left(y_{kk} + y_{kk}^{\mathsf{H}}\right)\mathbf{1}_k\mathbf{1}_k^{\mathsf{H}} + \sum_{\ell\sim k}\mathbf{\Phi}_{k\ell},$$
$$\mathbf{\Psi}_k := \frac{1}{2\boldsymbol{i}}\left(y_{kk}^{\mathsf{H}} - y_{kk}\right)\mathbf{1}_k\mathbf{1}_k^{\mathsf{H}} + \sum_{\ell\sim k}\mathbf{\Psi}_{k\ell}$$

and $\mathbf{1}_k \in \mathbb{C}^n$ is the vector of all zeros, except the k-th entry that is unity. The notation $\ell \sim k$ indicates that a transmission line connects buses ℓ and k in the power network. Voltage magnitudes across the network are deemed to remain close to rated voltage levels. We model such constraints at each bus k as $\underline{v}_k \leq |V_k| \leq \overline{v}_k$ that is equivalent to

$$\underline{v}_k^2 \leq \mathbf{V}^{\mathsf{H}}\mathbf{1}_k\mathbf{1}_k^{\mathsf{H}}\mathbf{V} \leq \overline{v}_k^2. \tag{4.4}$$

Consider two assets connected at each bus – an uncontrollable asset whose apparent power draw is fixed and known and a controllable asset whose power injection can vary within known capacity limits. Let p_k^D and q_k^D, respectively, denote the nominal real and reactive power draws at bus k from the uncontrollable asset. Similarly, let p_k^G and q_k^G denote the real and reactive power generation at bus k, respectively, that vary within known capacity limits as $p_k^G \in [\underline{p}_k, \overline{p}_k]$ and $q_k^G \in [\underline{q}_k, \overline{q}_k]$. Associated with that generation is a convex dispatch cost $c_k(p_k^G, q_k^G)$. Assume henceforth that c_k is jointly convex in its arguments. Such costs in wholesale markets are inferred from supply offers and demand bids. Uncontrollable assets represent the collective inelastic power demands at a bus. Generators and proxy demand resources comprise controllable assets. There may be one, more than one, or no controllable and uncontrollable assets at each bus, but we assume one asset of each kind to simplify notation.

The SO seeks to compute a dispatch that minimizes the aggregate dispatch costs from the collection of grid-connected controllable assets and meets the power requirements of the uncontrollable ones, meeting the engineering constraints of the power network as follows:

$$
\begin{aligned}
\mathcal{P}_{\mathsf{AC}} : \text{minimize} \quad & \sum_{k=1}^{n} c_k(p_k^G, q_k^G), \\
\text{subject to} \quad & p_k^G - p_k^D = \mathbf{V}^{\mathsf{H}} \mathbf{\Phi}_k \mathbf{V}, && \text{(4.5a)} \\
& q_k^G - q_k^D = \mathbf{V}^{\mathsf{H}} \mathbf{\Psi}_k \mathbf{V}, && \text{(4.5b)} \\
& \mathbf{V}^{\mathsf{H}} \mathbf{\Phi}_{k\ell} \mathbf{V} \le f_{k\ell}, && \text{(4.5c)} \\
& \underline{p}_k \le p_k^G \le \overline{p}_k, \quad \underline{q}_k \le q_k^G \le \overline{q}_k, && \text{(4.5d)} \\
& \underline{v}_k^2 \le \mathbf{V}^{\mathsf{H}} \mathbf{1}_k \mathbf{1}_k^{\mathsf{H}} \mathbf{V} \le \overline{v}_k^2 && \text{(4.5e)} \\
& \text{for } k = 1, \ldots, n, \ \ell \sim k
\end{aligned}
$$

over the variables $\mathbf{p}^G, \mathbf{q}^G$ and $\mathbf{V}$. The boldfaced symbols collect the corresponding variables across the network. In $\mathcal{P}_{\mathsf{AC}}$, (4.5a) and (4.5b) enforce the power balance at each bus, (4.5c) limits the power flows over each transmission line, (4.5d) defines capacity limits for the power production from dispatchable assets, and finally, (4.5e) defines bounds on voltage magnitudes. The above market clearing problem is an instance of an optimal power flow (OPF) problem with AC power flow. $\mathcal{P}_{\mathsf{AC}}$ is nonconvex, owing to quadratic equalities. In Section 4.3 we ask: *how should we price such a dispatch?*

4.3 Relaxation-Based Locational Marginal Prices

We associate nodal prices to real and reactive powers based on a semidefinite programming (SDP) based convex relaxation of $\mathcal{P}_{\mathsf{AC}}$ in $\mathcal{P}_{\mathsf{SDP}}$ in (4.6) that seeks to optimize the same objective function as $\mathcal{P}_{\mathsf{AC}}$, but over a convex superset of the feasible set of $\mathcal{P}_{\mathsf{AC}}$. To arrive at the relaxation, notice that

$$
\mathbf{V}^{\mathsf{H}} \mathbf{M} \mathbf{V} = \mathrm{Tr}\,(\mathbf{M} \mathbf{V} \mathbf{V}^{\mathsf{H}}) = \mathrm{Tr}\,(\mathbf{M} \mathbf{W})
$$

for any $\mathbf{M} \in \mathbb{C}^{n \times n}$ and $\mathbf{W} = \mathbf{V}\mathbf{V}^{\mathsf{H}}$. Here, Tr stands for the trace operator. The above representation reduces quadratic forms in $\mathbf{V}$ to linear forms in $\mathbf{W} \in \mathbb{C}^{n \times n}$. Also, any $\mathbf{W}$ that admits the representation $\mathbf{W} = \mathbf{V}\mathbf{V}^{\mathsf{H}}$ is a rank-1 positive semidefinite matrix (henceforth denoted $\mathbf{W} \succeq 0$). Therefore, one can reformulate $\mathcal{P}_{\mathsf{AC}}$ by replacing all quadratic forms in $\mathbf{V}$ by linear expressions in $\mathbf{W}$ and enforce $\mathbf{W}$ to be a rank-1 positive semidefinite matrix. This reformulation encodes the nonconvexity of $\mathcal{P}_{\mathsf{AC}}$ in the rank constraint. Drop this constraint to arrive at the following SDP relaxation of $\mathcal{P}_{\mathsf{AC}}$:

$$
\begin{aligned}
\mathcal{P}_{\mathsf{SDP}} : \text{minimize} \quad & \sum_{k=1}^{n} c_k(p_k^G, q_k^G), \\
\text{subject to} \quad & p_k^G - p_k^D = \text{Tr } (\mathbf{\Phi}_k \mathbf{W}), && \text{(4.6a)} \\
& q_k^G - q_k^D = \text{Tr } (\mathbf{\Psi}_k \mathbf{W}), && \text{(4.6b)} \\
& \text{Tr } (\mathbf{\Phi}_{k\ell} \mathbf{W}) \le f_{k\ell}, && \text{(4.6c)} \\
& \underline{p}_k \le p_k^G \le \overline{p}_k, \ \underline{q}_k \le q_k^G \le \overline{q}_k, && \text{(4.6d)} \\
& \underline{v}_k^2 \le \text{Tr } (\mathbf{1}_k \mathbf{1}_k^{\mathsf{H}} \mathbf{W}) \le \overline{v}_k^2, && \text{(4.6e)} \\
& \mathbf{W} \succeq 0, && \text{(4.6f)} \\
& \text{for } k = 1, \ldots, n, \ \ell \sim k
\end{aligned}
$$

over the variables $\mathbf{W}, \mathbf{p}^G, \mathbf{q}^G$. For any variable $\mathbf{z}$ in $\mathcal{P}_{\mathsf{SDP}}$, we use the notation $\mathbf{z}^\star$ to denote $\mathbf{z}$ at an optimum.

We now define prices using the optimal Lagrange multipliers from $\mathcal{P}_{\mathsf{SDP}}$. The prices we advocate are locational in nature, i.e., they vary based on location within the power network. However, the prices are uniform across assets connected at the same bus. Toward that goal, associate the multipliers λ_k^p and λ_k^q to the real and reactive power balance constraints in (4.6a) and (4.6b), respectively. Similarly, associate $\mu_{k\ell}$ to the one in (4.6c). Assign $\overline{\mu}_k^p, \underline{\mu}_k^p$ to the upper and lower limits, respectively, on the real power generation in (4.6d), and $\overline{\mu}_k^q, \underline{\mu}_k^q$ to the respective limits on the reactive power generation in (4.6d). Define $\overline{\mu}_k^v, \underline{\mu}_k^v$, respectively, as the multipliers for the upper and lower bounds on voltage magnitudes in (4.6e). Finally, associate the matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ as the multiplier for (4.6f).

Definition 1. *Define $\lambda_k^{p,\star}$ and $\lambda_k^{q,\star}$, the optimal Lagrange multipliers from $\mathcal{P}_{\mathsf{SDP}}$ for the real and reactive power balance constraints at bus k, respectively, as the prices for real and reactive power at bus k.*

The market proceeds as follows. The SO collects bids and offers from market participants and solves the market clearing problem $\mathcal{P}_{\mathsf{AC}}$ to compute the dispatch decisions $\mathbf{p}^{G,\star}$ and $\mathbf{q}^{G,\star}$. Then, the SO solves $\mathcal{P}_{\mathsf{SDP}}$ and computes the optimal Lagrange multipliers $\lambda^{p,\star}$ and $\lambda^{q,\star}$ as the RLMPs. Under our pricing scheme, the controllable asset at bus k produces $p_k^{G,\star}$ and $q_k^{G,\star}$ and collects the payment

$$\pi_k^G := \lambda_k^{p,\star} p_k^{G,\star} + \lambda_k^{q,\star} q_k^{G,\star}$$

from the SO. Further, the uncontrollable asset with its demand p_k^D and q_k^D pays

$$\pi_k^D := \lambda_k^{p,\star} p_k^D + \lambda_k^{q,\star} q_k^D$$

to the SO. What justifies RLMPs for price formation in electricity markets? We describe a wish-list of properties for market mechanisms and argue that RLMPs exhibit a number of these desirable properties, thus providing the rationale behind our proposed mechanism.

1. Efficient Market Equilibrium: The dispatch is said to be efficient and clears the market, if it optimally solves $\mathcal{P}_{\mathsf{AC}}$. It is individually rational, if the SO prescribed dispatch indeed maximizes the profit of a controllable asset, given the prices. Said mathematically, the dispatch $\left(p_k^{G,\star}, q_k^{G,\star}\right)$ must solve

$$\begin{aligned} \underset{p_k^G, q_k^G}{\text{maximize}} \quad & \lambda_k^{p,\star} p_k^G + \lambda_k^{q,\star} q_k^G - c_k(p_k^G, q_k^G), \\ \text{subject to} \quad & \underline{p}_k \leq p_k^G \leq \overline{p}_k, \quad \underline{q}_k \leq q_k^G \leq \overline{q}_k, \end{aligned} \tag{4.7}$$

   given $(\lambda_k^{p,\star}, \lambda_k^{q,\star})$. A controllable asset then has no incentive to deviate from its prescribed dispatch, given the prices. A pricing scheme is nodally uniform if all assets connected at a bus pay or are paid at the same price. Thus, co-located assets do not have incentives to trade among themselves. A market mechanism supports an efficient market equilibrium if the dispatch is efficient, clears the market, and is individually rational, given nodally uniform prices.

2. Side Payment Minimization: When the dispatch mechanism incorporates non-convexities, it is typically challenging to find a set of nodally uniform prices that adequately incentivize all assets to follow the SO-prescribed dispatch. The SOs then provide side-payments to controllable assets to deter possible deviations. Such payments are often socialized among end-use customers. Ideally, the market mechanism

should minimize such out-of-market settlements in aggregate to increase market transparency. Lost opportunity costs constitute a specific form of side-payment, defined as

$$\mathrm{LOC}(\lambda^p, \lambda^q) := \sum_{k=1}^{n} \pi_k^{\mathsf{opt}}(\lambda_k^p, \lambda_k^q) - \pi_k^{\mathsf{SO}}(\lambda_k^p, \lambda_k^q), \tag{4.8}$$

where $\pi_k^{\mathsf{opt}}(\lambda_k^p, \lambda_k^q)$ is the optimal cost of (4.7) with $(\lambda_k^{p,\star}, \lambda_k^{q,\star})$ replaced with $(\lambda_k^p, \lambda_k^q)$ and

$$\pi_k^{\mathsf{SO}}(\lambda_k^p, \lambda_k^q) := \lambda_k^p p_k^{G,\star} + \lambda_k^q q_k^{G,\star} - c_k(p_k^{G,\star}, q_k^{G,\star}). \tag{4.9}$$

Said differently, given the electricity prices, π_k^{SO} denotes the profit of the controllable asset at bus k from following the SO prescribed dispatch, while π_k^{opt} is the maximum profit that asset can garner.

3. Revenue Adequacy: A market mechanism is revenue adequate if the rents collected from power sales are enough to cover the rents payable to suppliers, i.e., the merchandising surplus defined as

$$\mathrm{MS} := \sum_{k=1}^{n} \left(\pi_k^G - \pi_k^D\right) \tag{4.10}$$

is non-negative. Non-negativity of MS ensures the solvency of the SO after each market clearing.

4.4 Properties of RLMPs

Having described the qualities we seek in a market mechanism, we now characterize the properties of RLMPs in Theorem 5, the proof of which relies on duality theory of semidefinite programming. Assume throughout that $\mathcal{P}_{\mathsf{SDP}}$ satisfies Slater's condition. To present the result, we need the following definition.

$$\begin{aligned}\text{PRS}(\mu, \overline{\mu}^v, \underline{\mu}^v, \mathbf{U}) &:= \mathbf{V}^{\mathsf{H},\star}\mathbf{U}\mathbf{V}^\star + \sum_{k\ell=1}^{m} \mu_{k\ell} \left(f_{k\ell} - \mathbf{V}^{\mathsf{H},\star}\mathbf{\Phi}_{k\ell}\mathbf{V}^\star\right) \\ &+ \sum_{k=1}^{n} \overline{\mu}_k^v \left(\overline{v}_k^2 - |V_k^\star|^2\right) + \sum_{k=1}^{n} \underline{\mu}_k^v \left(|V_k^\star|^2 - \underline{v}_k^2\right)\end{aligned} \tag{4.11}$$

for $\mu \geq 0, \overline{\mu}^v \geq 0, \underline{\mu}^v \geq 0, \mathbf{U} \succeq 0$ as the product revenue shortfall, where $\mathbf{V}^\star$ constitutes an optimal solution of $\mathcal{P}_{\mathsf{AC}}$.

Theorem 5. *$\mathcal{P}_{\mathsf{SDP}}$ is the dual of the dual problem of $\mathcal{P}_{\mathsf{AC}}$, and the duality gap of $\mathcal{P}_{\mathsf{AC}}$ is given by*

$$\begin{aligned}\underset{\substack{\lambda^p,\lambda^q,\mathbf{U},\\ \mu,\overline{\mu}^v,\underline{\mu}^v}}{\text{minimum}} \quad & LOC(\lambda^p, \lambda^q) + PRS(\mu, \overline{\mu}^v, \underline{\mu}^v, \mathbf{U}), \\ \text{subject to} \quad & \mathbf{U} = \sum_{k=1}^{n} \lambda_k^p \mathbf{\Phi}_k + \sum_{k=1}^{n} \lambda_k^q \mathbf{\Psi}_k + \sum_{k\ell=1}^{m} \mu_{k\ell} \mathbf{\Phi}_{k\ell} \\ & \quad + \sum_{k=1}^{n} \left(\overline{\mu}_k^v - \underline{\mu}_k^v\right) \mathbf{1}_k \mathbf{1}_k^{\mathsf{H}}, \\ & \mu \geq 0,\ \overline{\mu}^v \geq 0,\ \underline{\mu}^v \geq 0,\ \mathbf{U} \succeq 0.\end{aligned}$$

When the duality gap is zero (rank $\mathbf{W}^\star = 1$ in $\mathcal{P}_{\mathsf{SDP}}$), then the proposed market mechanism supports an efficient market equilibrium. Moreover, if the voltage lower limit constraint is non-binding at all buses, i.e., $\text{Tr}\,(\mathbf{1}_k \mathbf{1}_k^{\mathsf{T}} \mathbf{W}^\star) > \underline{v}_k^2$ *for $k = 1, \ldots, n$, then the mechanism is revenue adequate.*

The fact that $\mathcal{P}_{\mathsf{SDP}}$ is the double dual of $\mathcal{P}_{\mathsf{AC}}$ is well known, e.g., see [78, 87]. We include it in the result for completeness. The proof of the duality gap in Theorem 5 is provided in Section 4.7. For a proof on revenue adequacy and market equilibrium we refer the reader to [88]. Theorem 5 reveals that when the duality gap is nonzero, RLMPs seek to minimize the sum of two terms that are individually non-negative—the lost opportunity cost (LOC) and the product revenue shortfall (PRS), very similar in spirit to convex hull pricing (CHP). See [86, 72] for comparison. Having LOC as a component implies that RLMP in a way attempts to minimize side payments necessary to incentivize controllable assets to follow the SO's dispatch signals, thereby increasing market transparency. The economic interpretation of PRS remains challenging—a feature that is again common to both RLMP and CHP. A nonzero PRS can give rise to counter-intuitive situations where prices can be positive even with non-binding constraints. For example, one can end up with $\mu^{\star}_{k\ell} > 0$ from $\mathcal{P}_{\mathsf{SDP}}$ together with $f_{k\ell} > \mathbf{V}^{\mathsf{H},\star}\mathbf{\Phi}_{k\ell}\mathbf{V}^{\star}$ from $\mathcal{P}_{\mathsf{AC}}$. In such an event, the SO will garner congestion revenue, even when the line may not be congested at an optimal dispatch. This again is a property that CHP exhibits. Notice that PRS in (4.11) collects terms that appear in complementary slackness conditions for $\mathcal{P}_{\mathsf{AC}}$. However, the primal and the dual variables come from two different problems. Thus, when the duality gap is nonzero, one cannot expect the complementary slackness-like condition to hold. Theorem 5 indicates that RLMP tries to force PRS towards zero, similar in spirit to CHP. These parallels between CHP and RLMP are not surprising, given that both advocate pricing based on the dual (or the double dual) of the nonconvex market clearing problem, albeit to tackle two different kinds of nonconvexities. Understanding how RLMPs compare to the optimal dual multipliers of a local optimum of the dispatch problem is an interesting subject for future research.

When the duality gap is zero, Theorem 5 establishes that RLMPs have similar properties as LMPs. No controllable asset has incentive to deviate from the dispatch described by the optimal solution of $\mathcal{P}_{\mathsf{AC}}$. Under the additional condition of non-binding lower bounds for voltage constraints at each bus, the payments from uncontrollable assets cover the rents of those that are controllable. Given the strong coupling between reactive power injection and voltage magnitudes, one expects non-negative MS with adequate reactive power support. For illustrative examples on the fact that non-binding lower voltage limit is sufficient but not necessary, we refer the reader to [88].

Despite progress in wholesale electricity markets, the low-voltage distribution grid has been largely excluded from day-ahead and real-time markets. The role of distribution grids is largely passive with commercial and residential customers exposed to fixed or time-of-use rates that do not reflect real time conditions of the system. However, rapid proliferation of distributed energy resources (DERs) and the aim to harness demand flexibility of end-use customers have motivated research in defining appropriate price signals for compensating energy transactions in distribution networks (e.g., see [89], [90] and [91]). Suggested distribution LMPs (DLMPs) aim to reflect the locational value of DERs and physics of the network as discussed in [92] and [93]. We argue that RLMPs from $\mathcal{P}_{\mathsf{SDP}}$ become the second-order cone programming (SOCP) based DLMPs in [90] over radial (acyclic) distribution grids. Indeed, in [88, 94] we show that RLMPs restricted to radial networks coincides with DLMPs. Such prices are locational in nature and compensate market participants for both real and reactive power.

Our exposition in [94] focus on the mathematical foundations of DLMPs and sidestep a range of issues surrounding the adoption of such prices in practice. For example, what is the right trading platform that needs to be established and what products should be traded in such platforms that DERs can participate in? How should such platforms coordinate their operations with wholesale markets governed by transmission system operators? See [95] and [96] for insightful discussions on the same. We align with the view in [23] to consider a retail market operated by an independent distribution system operator (DSO) responsible for the dispatch and pricing of DERs, but leave the specifics of a coordinated wholesale-retail market design to a future effort.

4.5 Practical Considerations for Market Adoption

RLMPs associate prices for real and reactive power, thereby making reactive power compensation a part of competitive market processes. Creation of markets for reactive power has led to celebrated debates in the last two decades, e.g., see [97, 98, 99]. Reactive power is alleged to not "travel too far" and hence, a market is often deemed unnecessary. However, real and reactive power are intimately coupled with each other through the power flow equations. Therefore, pricing one and not the other ignores that coupling. Inadequate reactive power resources, especially under line/generator failure scenarios (contingencies), can lead to brown and blackouts (see [100]). To keep the notation simple, we have not modeled contingencies in formulating $\mathcal{P}_{\mathsf{AC}}/\mathcal{P}_{\mathsf{SDP}}$. That extension, however does not offer any conceptual difficulties. With such an extension, a competitive market for both real and reactive power will systematize the procurement process for both.

Pricing via RLMPs requires the SO to solve $\mathcal{P}_{\mathsf{SDP}}$. SDPs are known to scale poorly with problem dimension and pose serious algorithmic challenges to possible adoption of RLMPs. The difficulty typically arises from the need to solve large linear system of equations within interior-point methods to solve $\mathcal{P}_{\mathsf{SDP}}$ that require $\mathcal{O}(n^3)$ operations. Such scaling is prohibitive for practical power systems. In [80, 83], it has been shown that $\mathcal{P}_{\mathsf{SDP}}$ can be equivalently formulated in terms of semidefinite matrices corresponding to maximal cliques of chordal extensions of the sparse power network graph. The size of the largest maximal clique then determines the size of the semidefinite matrices involved in the computational step, and also the size of the linear systems solved at each iteration. In other words, the sparsity of the power network allows orders of magnitude speedups in algorithms for $\mathcal{P}_{\mathsf{SDP}}$. While algorithms to solve $\mathcal{P}_{\mathsf{SDP}}$ have come a long way (see [101, 84]), additional research is required to make it scalable for market adoption. Surprisingly enough, CHPs have faced the same difficulty in tractable computation, although for a completely different reason (see [70, 69]).

There are additional concerns in adopting RLMPs for day-ahead markets. Algorithms for market clearing with linearized power flows and unit commitment decisions lead to mixed-integer linear programs (MILPs). Software for MILP is much more mature than the nonlinear counterpart (see [102]). Thus, adoption of RLMP for day-ahead markets will impose a heavy computational burden on market clearing software. In addition, one needs a way to enhance RLMP to price commitment decisions—a topic we are eager to pursue in future work.

4.6 Illustrative Examples

In this section, we report results from numerical experiments on different power network examples to illustrate the behavior of RLMPs as well as to discuss main insights from our theoretical results. In our first experiment, we compute the RLMPs on the IEEE 30-bus test system adopted from Matpower, developed by [103]. Reactive power demands are computed from the real power demands, assuming a lagging power factor of 0.9. The resulting RLMPs for real and reactive power across the network are illustrated through heatmaps in Figures 4.1a and 4.1b respectively. An increase in real and reactive power demands at buses 29 and 30 demonstrate the locational nature of these prices. In particular, once the real power demand on bus 30 exceeds the flow limit on branch 29-30, the prices for both real and reactive power at buses 29 and 30 significantly exceed those at other locations in the network, as Figures 4.1c and 4.1d reveal. In effect, these prices reflect that in the presence of congestion it becomes more expensive to supply demand at buses 29 and 30. We then narrow the voltage magnitude limits on a subset of nodes; the effect on RLMPs is illustrated in Figures 4.1e and 4.1f for real and reactive power prices, respectively. The impact is significantly larger on reactive power prices than on real power prices. Intuitively, maintaining the voltage level within acceptable bounds across the network requires sufficient injections of reactive power in the appropriate locations on the network. Thus, enforcing stricter voltage limits, increases demand for reactive power injections and therefore their RLMPs. For further insights on RLMPs as well as their applications to distribution grids see [88, 94].

4.7 Proof of Theorem 5

The proof proceeds in four steps. The first step establishes that the dual program of $\mathcal{P}_{\mathsf{AC}}$ coincides with the dual of $\mathcal{P}_{\mathsf{SDP}}$. This part of the proof is provided in [88]. In the second step, we compute the duality gap of $\mathcal{P}_{\mathsf{AC}}$. We find that the duality gap constitutes of two terms: the LOC and PRS. The proof of this step proceeds as follows. Define the partial Lagrangian of $\mathcal{P}_{\mathsf{AC}}$ as

$$
\begin{aligned}
&\mathcal{L}_V(\mathbf{p}^G, \mathbf{q}^G, \mathbf{V}, \boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q, \boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v) := \sum_{k=1}^{n} c_k(p_k^G, q_k^G) \\
&- \sum_{k=1}^{n} \lambda_k^p \left(p_k^G - p_k^D - \mathbf{V}^{\mathsf{H}} \boldsymbol{\Phi}_k \mathbf{V}\right) - \sum_{k=1}^{n} \lambda_k^q \left(q_k^G - q_k^D - \mathbf{V}^{\mathsf{H}} \boldsymbol{\Psi}_k \mathbf{V}\right) \\
&+ \sum_{k\ell=1}^{m} \mu_{k\ell} \left(\mathbf{V}^{\mathsf{H}} \boldsymbol{\Phi}_{k\ell} \mathbf{V} \ - f_{k\ell}\right) + \sum_{k=1}^{n} \overline{\mu}_k^v \left(\mathbf{V}^{\mathsf{H}} \mathbf{1}_k \mathbf{1}_k^{\mathsf{H}} \mathbf{V} - \overline{v}_k^2\right) \\
&- \sum_{k=1}^{n} \underline{\mu}_k^v \left(\mathbf{V}^{\mathsf{H}} \mathbf{1}_k \mathbf{1}_k^{\mathsf{H}} \mathbf{V} - \underline{v}_k^2\right),
\end{aligned}
$$

and the set $\mathbb{S}$ as

$$
\mathbb{S} := \left\{ \left(\mathbf{p}^G, \mathbf{q}^G\right) \ | \ \underline{\mathbf{p}} \le \mathbf{p}^G \le \overline{\mathbf{p}}, \ \underline{\mathbf{q}} \le \mathbf{q}^G \le \overline{\mathbf{q}} \right\}. \tag{4.12}
$$

Then, the dual program of $\mathcal{P}_{\mathsf{AC}}$ is given by

$$\begin{aligned} &\underset{\substack{\boldsymbol{\lambda}^p,\boldsymbol{\lambda}^q,\\ \boldsymbol{\mu},\overline{\boldsymbol{\mu}}^v,\underline{\boldsymbol{\mu}}^v}}{\text{maximize}} \quad \underset{\substack{\mathbf{V},\\ (\mathbf{p}^G,\mathbf{q}^G)\in\mathbb{S}}}{\text{minimum}} \quad \mathcal{L}_V(\mathbf{p}^G,\mathbf{q}^G,\mathbf{V},\boldsymbol{\lambda}^p,\boldsymbol{\lambda}^q,\boldsymbol{\mu},\overline{\boldsymbol{\mu}}^v,\underline{\boldsymbol{\mu}}^v), \\ &\text{subject to} \quad \boldsymbol{\mu}\geq 0,\ \overline{\boldsymbol{\mu}}^v\geq 0,\ \underline{\boldsymbol{\mu}}^v\geq 0. \end{aligned} \tag{4.13}$$

In the above problem, the inner minimization with respect to $\mathbf{V}$ amounts to minimizing $\mathbf{V}^{\mathsf{H}}\mathbf{U}\mathbf{V}$, where

$$\mathbf{U} := \sum_{k=1}^{n}\lambda_k^p\boldsymbol{\Phi}_k + \sum_{k=1}^{n}\lambda_k^q\boldsymbol{\Psi}_k + \sum_{k\ell=1}^{m}\mu_{k\ell}\boldsymbol{\Phi}_{k\ell} + \sum_{k=1}^{n}\left(\overline{\mu}_k^v - \underline{\mu}_k^v\right)\mathbf{1}_k\mathbf{1}_k^{\mathsf{H}}. \tag{4.14}$$

It equals $-\infty$ unless $\mathbf{U}\succeq 0$. Thus, (4.13) can be written as

$$\begin{aligned} \underset{\substack{\boldsymbol{\lambda}^p,\boldsymbol{\lambda}^q,\mathbf{U},\\ \boldsymbol{\mu},\overline{\boldsymbol{\mu}}^v,\underline{\boldsymbol{\mu}}^v}}{\text{maximize}} \quad \underset{(\mathbf{p}^G,\mathbf{q}^G)\in\mathbb{S}}{\text{minimum}} \quad & \sum_{k=1}^{n}\left[c_k(p_k^G,q_k^G) - \lambda_k^p p_k^G - \lambda_k^q q_k^G\right] \\ & + \sum_{k=1}^{n}\left(\lambda_k^p p_k^D + \lambda_k^q q_k^D\right) - \sum_{k\ell=1}^{m}\mu_{k\ell}f_{k\ell} \\ & - \sum_{k=1}^{n}\left(\overline{\mu}_k^v\overline{v}_k^2 - \underline{\mu}_k^v\underline{v}_k^2\right), \\ \text{subject to} \quad \boldsymbol{\mu}\geq 0,\ \overline{\boldsymbol{\mu}}^v\geq 0,\ & \underline{\boldsymbol{\mu}}^v\geq 0,\ \mathbf{U}\succeq 0,\ (4.14). \end{aligned} \tag{4.15}$$

Therefore, (4.15) defines the common dual program of $\mathcal{P}_{\mathsf{AC}}$ and $\mathcal{P}_{\mathsf{SDP}}$. With Slater's condition, strong duality holds for $\mathcal{P}_{\mathsf{SDP}}$, and hence, the optimal cost of $\mathcal{P}_{\mathsf{SDP}}$ is the same as that of (4.15). Call this cost c^*_{SDP}.

Next, consider an optimal solution $\left(\mathbf{p}^{\mathbf{G},\star},\mathbf{q}^{\mathbf{G},\star},\mathbf{V}^\star\right)$ of $\mathcal{P}_{\mathsf{AC}}$ with an optimal cost

$$c^*_{\mathsf{AC}} = \sum_{k=1}^{n} c_k\left(p_k^{G,\star},q_k^{G,\star}\right). \tag{4.16}$$

Then, the nodal demands satisfy

$$p_k^D = p_k^{G,\star} - \mathbf{V}^{\mathsf{H},\star}\boldsymbol{\Phi}_k\mathbf{V}^\star,\quad q_k^D = q_k^{G,\star} - \mathbf{V}^{\mathsf{H},\star}\boldsymbol{\Psi}_k\mathbf{V}^\star. \tag{4.17}$$

Utilizing (4.16) and (4.17), the objective function in (4.15) can be written as

$$c^*_{\mathsf{AC}} + \sum_{k=1}^{n} \left[c_k(p_k^G, q_k^G) - \lambda_k^p p_k^G - \lambda_k^q q_k^G\right] + \sum_{k=1}^{n} \left[\lambda_k^p p_k^{G,\star} + \lambda_k^q q_k^{G,\star} - c_k(p_k^{G,\star}, q_k^{G,\star})\right]$$
$$- \sum_{k\ell=1}^{m} \mu_{k\ell} f_{k\ell} + \sum_{k=1}^{n} \left(\underline{\mu}_k^v \underline{v}_k^2 - \overline{\mu}_k^v \overline{v}_k^2\right) - \sum_{k=1}^{n} \left[\lambda_k^p \mathbf{V}^{\mathsf{H},\star} \mathbf{\Phi}_k \mathbf{V}^\star + \lambda_k^q \mathbf{V}^{\mathsf{H},\star} \mathbf{\Psi}_k \mathbf{V}^\star\right],$$

that using the notation in (4.9) allows us to rearrange (4.15) as

$$\begin{aligned}
c^*_{\mathsf{AC}} - c^*_{\mathsf{SDP}} = -\underset{\substack{\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q, \mathbf{U} \\ \boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v}}{\text{maximum}} \quad & \sum_{k=1}^{n} \left[-\pi^{\mathsf{opt}}(\lambda_k^p, \lambda_k^q) + \pi_k^{\mathsf{SO}}(\lambda_k^p, \lambda_k^q)\right] \\
& - \sum_{k\ell=1}^{m} \mu_{k\ell} f_{k\ell} + \sum_{k=1}^{n} \left(\underline{\mu}_k^v \underline{v}_k^2 - \overline{\mu}_k^v \overline{v}_k^2\right) \\
& - \sum_{k=1}^{n} \left[\lambda_k^p \mathbf{V}^{\mathsf{H},\star} \mathbf{\Phi}_k \mathbf{V}^\star + \lambda_k^q \mathbf{V}^{\mathsf{H},\star} \mathbf{\Psi}_k \mathbf{V}^\star\right], \\
\text{subject to} \quad & \boldsymbol{\mu} \geq 0,\ \overline{\boldsymbol{\mu}}^v \geq 0,\ \underline{\boldsymbol{\mu}}^v \geq 0,\ \mathbf{U} \succeq 0,\ (4.14) \\
= \underset{\substack{\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q, \\ \boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v}}{\text{minimum}} \quad & \text{LOC}\,(\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q) + \eta(\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q, \boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v, \mathbf{U}) \\
\text{subject to} \quad & \boldsymbol{\mu} \geq 0,\ \overline{\boldsymbol{\mu}}^v \geq 0,\ \underline{\boldsymbol{\mu}}^v \geq 0,\ \mathbf{U} \succeq 0,\ (4.14),
\end{aligned}$$

where η is given by

$$\eta(\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q, \boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v, \mathbf{U}) := \sum_{k\ell=1}^{m} \mu_{k\ell} f_{k\ell} - \sum_{k=1}^{n} \left(\underline{\mu}_k^v \underline{v}_k^2 - \overline{\mu}_k^v \overline{v}_k^2\right)$$
$$+ \sum_{k=1}^{n} \lambda_k^p \mathbf{V}^{\mathsf{H},\star} \mathbf{\Phi}_k \mathbf{V}^\star + \sum_{k=1}^{n} \lambda_k^q \mathbf{V}^{\mathsf{H},\star} \mathbf{\Psi}_k \mathbf{V}^\star.$$

It remains to show that η indeed equals PRS. To that end, utilize the definition of $\mathbf{U}$ in (4.14) to get

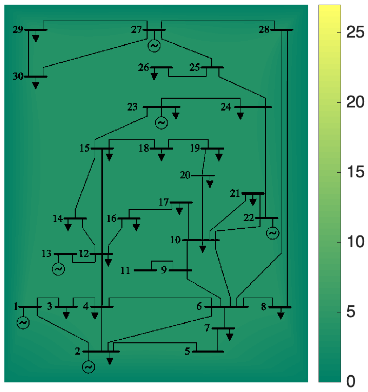
$$
\begin{aligned}
\eta(\boldsymbol{\lambda}^p, \boldsymbol{\lambda}^q, \boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v, \mathbf{U}) &= \sum_{k\ell=1}^{m} \mu_{k\ell} f_{k\ell} - \sum_{k=1}^{n} \left(\underline{\mu}_k^v \underline{v}_k^2 - \overline{\mu}_k^v \overline{v}_k^2 \right) \\
&+ \mathbf{V}^{\mathsf{H},\star} \mathbf{U} \mathbf{V}^\star - \sum_{k\ell=1}^{m} \mu_{k\ell} \mathbf{V}^{\mathsf{H},\star} \boldsymbol{\Phi}_{k\ell} \mathbf{V}^\star - \sum_{k=1}^{n} \left(\overline{\mu}_k^v - \underline{\mu}_k^v \right) \mathbf{V}^{\mathsf{H},\star} \mathbf{1}_k \mathbf{1}_k^{\mathsf{H}} \mathbf{V}^\star \\
&= \sum_{k\ell=1}^{m} \mu_{k\ell} f_{k\ell} - \sum_{k=1}^{n} \left(\underline{\mu}_k^v \underline{v}_k^2 - \overline{\mu}_k^v \overline{v}_k^2 \right) + \mathbf{V}^{\mathsf{H},\star} \mathbf{U} \mathbf{V}^\star \\
&- \sum_{k\ell=1}^{m} \mu_{k\ell} p_{k\ell}^\star - \sum_{k=1}^{n} \left(\overline{\mu}_k^v - \underline{\mu}_k^v \right) |V_k|^2 \\
&= \text{PRS} \left(\boldsymbol{\mu}, \overline{\boldsymbol{\mu}}^v, \underline{\boldsymbol{\mu}}^v, \mathbf{U} \right),
\end{aligned}
$$

where the last line follows from the definition of PRS in (4.11). This completes the derivation of the duality gap of $\mathcal{P}_{\mathsf{AC}}$.

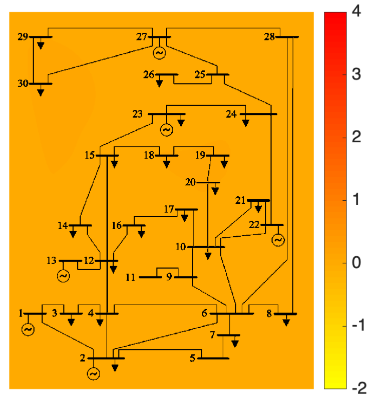
Steps 3 and 4 show that when the relaxation is exact or equivalently when rank $\mathbf{W}^\star = 1$, prices defined as the Langrange of $\mathcal{P}_{\mathsf{SDP}}$ support an efficient market equilibrium. Moreover, when the voltage lower bound is inactive, revenue adequacy is of the SO is guaranteed. The proof of these steps is provided in [88].

4.8 Summary

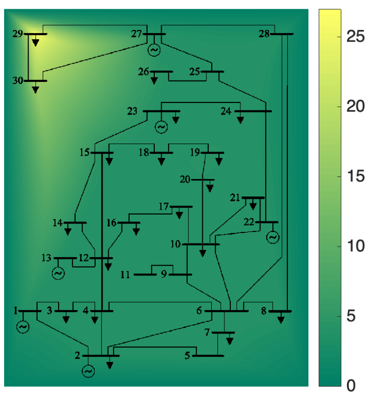
The non-convex nature of the market clearing problem $\mathcal{P}_{\mathsf{AC}}$ with AC power flow introduces challenges in defining appropriate price signals to compensate grid-connected assets in electricity markets. In this chapter, we addressed the question: *what price signals are deemed meaningful for market clearing with AC power flow?* We proposed and analyzed relaxation-based locational marginal prices (RLMPs) for real and reactive power, based on optimal dual multipliers of the SDP relaxation of the market clearing problem. Our market model relies on a central entity, the system operator, that determines the dispatch for all grid-connected assets from the solution of $\mathcal{P}_{\mathsf{AC}}$ while the compensation of each market participant is derived from the optimal Lagrange multipliers of $\mathcal{P}_{\mathsf{SDP}}$. We showed that when the duality gap of the market clearing problem is zero, RLMPs support an efficient market equilibrium and the mechanism is revenue adequate under mild conditions—properties that are reminiscent of LMPs defined with linear power flow models. With nonzero duality gap, we proved that RLMPs possess properties similar to convex hull prices. We also argued that RLMPs adapted to acyclic distribution networks define distribution LMPs (DLMPs) proposed in the literature.



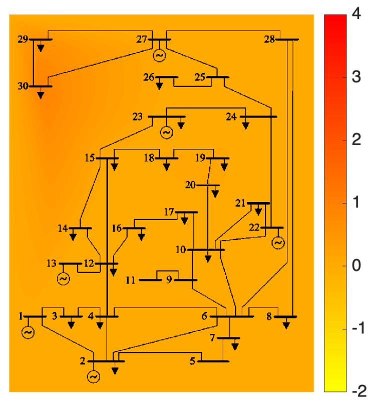
(a)



(b)



(c)



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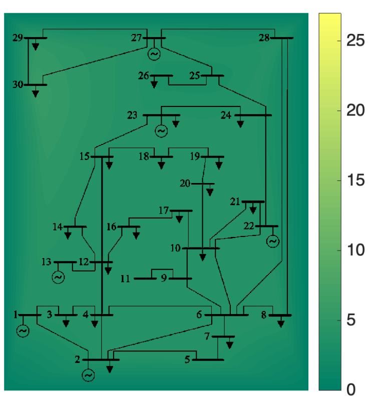
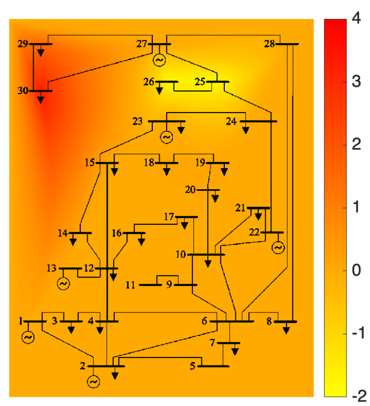


(e)



(f)

Figure 4.1: Plots (a), (b) show heatmaps of RLMP on the 30-bus IEEE network. Plots (c), (d) are derived with $p^D_{26} = 7.5$, $q^D_{26} = 3.3$, $p^D_{29} = 12.2$, $q^D_{29} = 4.9$, $p^D_{30} = 16.1$, and $q^D_{30} = 5.9$. Plots (e), (f) are derived with $\underline{v}_k = 0.99$, $\overline{v}_k = 1.01$ for $k = 24, \ldots, 30$. Prices are in \$/MWh.

CHAPTER 5

A COMPETITIVE ELECTRICITY MARKET FOR DISTRIBUTION GRIDS

In this chapter, we extend the electricity market model and pricing mechanism developed in Chapter 4 to a multi-phase, unbalanced distribution grid together with the demand bids and supply offers introduced in Chapter 2. RLMPs together with scalar-parameterized offers/bids serve as the vehicle to design a comprehensive framework for competitive electricity markets at the retail sector.

5.1 Why Markets for Electricity Retail?

The current state of distribution grids is passive. Low-voltage customers consume power and DERs inject power whenever it becomes available without any coordination with the rest of the system. It is then, the responsibility of the distribution system operator (DSO) to ensure network constraints while the SO at the transmission level ensures sufficient reserves to meet imbalances in supply and demand. Retail and commercial customers are largely excluded from any market process since they are exposed to fixed time-of-use rates that do not reflect the real-time conditions of the system. The paradigm shift envisioned in recent works [104, 90] entails a move toward active participation of low-voltage suppliers and consumers to potential market mechanisms designed for the retail side of the grid. This shift is propelled by the rapid proliferation of DERs in low and medium voltage distribution grids, which has generated considerable interest in designing appropriate price signals for distribution networks, e.g., see [89, 91, 23, 105].

The first fundamental question that emerges is what price signals are deemed appropriate and meaningful to compensate such resources and motivate them to offer their services to the grid. To this end, we utilize the concept of RLMPs developed in Chapter 4. Specifically, we consider a central dispatch problem for a local market in the low voltage grid administered by the DSO. The DSO clears the market for real and reactive power and determines the DLMPs at every node in the distribution grid. In this market, generation assets submit supply offers and consumers submit demand bids. The DSO determines the cleared bids and offers over a particular time horizon.

In contrast to the bulk power system, where linearized lossless power flow models are often deemed acceptable, distribution networks must explicitly account for reactive power flows and voltage considerations. In particular, distribution grids typically have lines with relatively high resistance to reactance ratios, and reactive power transactions play a crucial role in maintaining voltage magnitudes within tight bounds. Hence, the analysis of DLMPs becomes more complicated as we cannot ignore losses and reactive power—often a source of non-linearities in the dispatch model. However, the radial topology of distribution grids implies that loop flows, which have lead to heated debates surrounding LMPs, are less of a concern in DLMPs. The central dispatch model considered here, captures such characteristics of distribution grids. Specifically, in Section 5.2 we set up the network model that constitutes the backbone of the central dispatch problem. We define DLMPs as the optimal Lagrange multipliers of a semidefinite relaxation of the original dispatch problem. When market actors compete in scalar-parameterized offers/bids, we show that DLMPs support the efficient dispatch when the relaxation is exact.

5.2 Three-Phase, Unbalanced Distribution Grid Model

Distribution grids in practice are often multi-phase with a radial topology. In a distribution network, components such as capacitor banks and tap-changing transformers play a vital role in maintaining voltage magnitudes within specified limits. In this thesis, we ignore the tap-changing transformers and model the distribution grid as a three-phase network with shunt elements at each node. We also include controllable and uncontrollable assets operated by asset-owners at the various nodes of the network.

Throughout, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Let $\mathbb{H}^{n\times n}$ denote the space of all n-by-n Hermitian matrices. For

$y \in \mathbb{C}$, denote its real and imaginary parts by $\Re(x)$ and $\Im(y)$, respectively, and $\boldsymbol{i} := \sqrt{-1}$. For any scalar, vector, or matrix A, let A^{T} and A^{H} denote its transpose and conjugate transpose, respectively. Throughout Chapter 5 we slightly abuse notation and do not use boldface to denote vector quantities as we did in previous chapters. For a column vector A, let $\text{diag}(A)$ denote a diagonal matrix with entries of A on the diagonal. For a square matrix A, let $\text{diag}(A)$ denote a column vector consisted of its diagonal entries. Let $\text{Tr}\ (A)$ denote the trace of a square matrix A.

Let $\mathbb{N} = \{0, 1, ..., n\}$ denote the set of nodes in a multi-phase radial distribution network. Represent the network by a directed graph with $\mathbb{E}$ as the collections of directed edges. For $i, j \in \mathbb{N}$, the edge $i \to j \in \mathbb{E}$ represents a line joining nodes i and j. We assume that all the nodes $i \in \mathbb{N}$ and lines $i \to j \in \mathbb{E}$ have three phases: a, b, c collectively defined by $\Phi = \{a, b, c\}$. For $i \in \mathbb{N}$ and $\phi \in \Phi$, let V_i^ϕ denote the voltage phasor on phase ϕ at bus i. For $i \to j \in \mathbb{E}$, let I_{ij}^ϕ denote the phase ϕ current on the line from node i to j. Define vectors $V_i := [V_i^a, V_i^b, V_i^c]^{\mathsf{T}}$, $I_{ij} := [I_{ij}^a, I_{ij}^b, I_{ij}^c]^{\mathsf{T}}$. Let symmetric matrix $y_i := \boldsymbol{i} b_i \in \mathbb{C}^{3\times 3}$ denote the shunt admittance at node i, and symmetric matrix $z_{ij} := r_{ij} + \boldsymbol{i} x_{ij} \in \mathbb{C}^{3\times 3}$ denote the series impedance of line $i \to j$. If a particular phase is missing on certain node or line, the corresponding entries in current/power vectors and impedance/admittance matrices are set to zero.

We adopt a multi-phase, unbalanced distribution network model presented in [106] with some modifications. In this thesis, we ignore the delta-connected variables. Thermal considerations are included in this thesis as limits on both sending and receiving ends for real power on each line in $\mathbb{E}$. The network should satisfy the following constraints:

1. Ohm's law:

$$V_i - V_j = z_{ij} I_{ij}, \quad \forall i \to j \in \mathbb{E}. \tag{5.1}$$

2. Definition of auxiliary variables:

$$l_{ij} = I_{ij} I_{ij}^{\mathsf{H}}, \ S_{ij} = V_i I_{ij}^{\mathsf{H}}, \ \forall i \to j \in \mathbb{E}. \tag{5.2}$$

3. Power balance:

$$s_i^G - s_i^D = \sum_{j:i\to j} \operatorname{diag}(S_{ij}) - \sum_{k:k\to i} \operatorname{diag}(S_{ki} - z_{ki} l_{ki}) + \operatorname{diag}(V_i V_i^{\mathsf{H}} y_i^{\mathsf{H}}),\ \forall i \in \mathbb{N}. \tag{5.3}$$

4. Thermal constraints:

$$\operatorname{diag}(P_{ij}) \le f_{ij}, \quad \operatorname{diag}(\Re(z_{ij} l_{ij}) - P_{ij}) \le f_{ij}. \tag{5.4}$$

5. Voltage magnitude:

$$\underline{V}_i^{\phi} \le |V_i^{\phi}| \le \overline{V}_i^{\phi},\ \forall \phi \in \Phi. \tag{5.5}$$

We introduce the following auxiliary variable for the voltage at node i

$$w_i := V_i V_i^{\mathsf{H}} \in \mathbb{H}^{3\times 3},\ \forall i \in \mathbb{N}. \tag{5.6}$$

Notice that $\operatorname{diag}(w_i)$ denotes the squared magnitude of three phases of voltage V_i. The voltage constraint in (5.5) becomes

$$\underline{v}_i \le \operatorname{diag}(w_i) \le \overline{v}_i, \tag{5.7}$$

where the limits $\underline{v}_i$ and $\overline{v}_i$ are defined as:

$$\begin{aligned} \underline{v}_i &= \left[(\underline{V}_i^a)^2, (\underline{V}_i^b)^2, (\underline{V}_i^c)^2\right]^{\mathsf{T}},\ \forall i \in \mathbb{N} \\ \overline{v}_i &= \left[(\overline{V}_i^a)^2, (\overline{V}_i^b)^2, (\overline{V}_i^c)^2\right]^{\mathsf{T}},\ \forall i \in \mathbb{N}. \end{aligned} \tag{5.8}$$

Given the definition of w_i and utilizing (5.1) we obtain:

$$\begin{aligned} w_j &= (V_i - z_{ij} I_{ij})(V_i - z_{ij} I_{ij})^{\mathsf{H}} \\ &= V_i V_i^{\mathsf{H}} + z_{ij} I_{ij} I_{ij}^{\mathsf{H}} z_{ij}^{\mathsf{H}} - V_i I_{ij}^{\mathsf{H}} z_{ij}^{\mathsf{H}} - z_{ij} I_{ij} V_i^{\mathsf{H}} \\ &= w_i - (S_{ij} z_{ij}^{\mathsf{H}} + z_{ij} S_{ij}^{\mathsf{H}}) + z_{ij} l_{ij} z_{ij}^{\mathsf{H}}. \end{aligned} \tag{5.9}$$

We can rewrite (5.3) in terms of real and reactive power balance constraints. Define the real and imaginary parts of the following auxiliary variables:

$$S_{ij} := P_{ij} + \boldsymbol{i}Q_{ij}, \tag{5.10}$$

$$w_i := w_i^{\mathrm{R}} + \boldsymbol{i}w_i^{\mathrm{I}}, \tag{5.11}$$

$$l_{ij} := l_{ij}^{\mathrm{R}} + \boldsymbol{i}l_{ij}^{\mathrm{I}}. \tag{5.12}$$

Since w_i, l_{ij} are Hermitian, the following properties hold

$$\begin{aligned}(w_i^{\mathrm{R}})^{\intercal} = w_i^{\mathrm{R}}, \quad (w_i^{\mathrm{I}})^{\intercal} = -w_i^{\mathrm{I}},\\ (l_{ij}^{\mathrm{R}})^{\intercal} = l_{ij}^{\mathrm{R}}, \quad (l_{ij}^{\mathrm{I}})^{\intercal} = -l_{ij}^{\mathrm{I}}.\end{aligned} \tag{5.13}$$

Then, the nodal power balance constraints in (5.3) can be equivalently written as

$$p_i^G - p_i^D = \mathrm{diag}(w_i^{\mathrm{I}} b_i) + \sum_{j:i\to j} \mathrm{diag}(P_{ij}) - \sum_{k:k\to i} \mathrm{diag}\left(P_{ki} - r_{ki} l_{ki}^{\mathrm{R}} + x_{ki} l_{ki}^{\mathrm{I}}\right). \tag{5.14}$$

$$q_i^G - q_i^D = -\,\mathrm{diag}(w_i^{\mathrm{R}} b_i) + \sum_{j:i\to j} \mathrm{diag}(Q_{ij}) - \sum_{k:k\to i} \mathrm{diag}\left(Q_{ki} - x_{ki} l_{ki}^{\mathrm{R}} - r_{ki} l_{ki}^{\mathrm{I}}\right). \tag{5.15}$$

In (5.4), note that the real part of $z_{ij} l_{ij}$ can be explicitly written as

$$\Re(z_{ij} l_{ij}) = r_{ij} l_{ij}^{\mathrm{R}} - x_{ij} l_{ij}^{\mathrm{I}}. \tag{5.16}$$

Moreover, since $\mathrm{diag}(w_i)$ denotes squared magnitude of three phase voltages at node i, it is equivalent to $\mathrm{diag}(w_i^{\mathrm{R}})$.

We associate with each node i a set $\mathcal{J}(i)$ of controllable generation resources with three-phase apparent power injection $s_k^G := p_k^G + \boldsymbol{i}q_k^G \in \mathbb{C}^3$ for $k \in \mathcal{J}(i)$. Moreover, we assume that a set $\mathcal{I}(i)$ of consumers is connected at bus i drawing power demand $s_k^D := p_k^D + \boldsymbol{i}q_k^D \in \mathbb{C}^3$ for $k \in \mathcal{I}(i)$. Assume each generation resource can produce power within some capacity limits:

$$\begin{aligned}\underline{p}_k^G \le p_k^G \le \overline{p}_k^G,\\ \underline{q}_k^G \le q_k^G \le \overline{q}_k^G,\end{aligned} \tag{5.17}$$

for every $k \in \mathcal{J}(i)$. Without loss of generality, assume $\underline{p}_k^G = \underline{q}_k^G = 0$. We model both inelastic and price responsive loads. To this end, let $\underline{p}_k^D$ and $\underline{q}_k^D$ denote the minimum demand for real and reactive power that must be supplied to consumer $k \in \mathcal{I}(i)$. Finally, assume each generation asset incurs costs C_k associated with production of amount p_k^G and each consumer receives utility U_k from meeting load p_k^D.

5.3 Distribution Locational Marginal Prices

Assume the DSO has knowledge on the private costs and utilities of market participants. Then, the market allocation would be determined from the solution to the following dispatch problem:

$$
\begin{aligned}
\mathcal{P}_{\mathsf{AC}} : \text{maximize} \quad & \sum_{k\in\mathcal{I}} U_k(p_k^D) - \sum_{k\in\mathcal{J}} C_k(p_k^G), \\
\text{subject to} \quad & w_j = w_i - (S_{ij} z_{ij}^{\mathsf{H}} + z_{ij} S_{ij}^{\mathsf{H}}) + z_{ij} l_{ij} z_{ij}^{\mathsf{H}}, && (5.18a) \\
& \sum_{k\in\mathcal{J}(i)} p_k^G - \sum_{k\in\mathcal{I}(i)} p_k^D = \operatorname{diag}(w_i^{\mathrm{I}} b_i) + \sum_{j:i\to j} \operatorname{diag}(P_{ij}) \\
& - \sum_{k:k\to i} \operatorname{diag}\left(P_{ki} - r_{ki} l_{ki}^{\mathrm{R}} + x_{ki} l_{ki}^{\mathrm{I}}\right), && (5.18b) \\
& \sum_{k\in\mathcal{J}(i)} q_k^G - \sum_{k\in\mathcal{I}(i)} q_k^D = -\operatorname{diag}(w_i^{\mathrm{R}} b_i) + \sum_{j:i\to j} \operatorname{diag}(Q_{ij}) \\
& - \sum_{k:k\to i} \operatorname{diag}\left(Q_{ki} - x_{ki} l_{ki}^{\mathrm{R}} - r_{ki} l_{ki}^{\mathrm{I}}\right), && (5.18c) \\
& \operatorname{diag}(P_{ij}) \leq f_{ij}, && (5.18d) \\
& \operatorname{diag}\left(r_{ij} l_{ij}^{\mathrm{R}} - x_{ij} l_{ij}^{\mathrm{I}} - P_{ij}\right) \leq f_{ij}, && (5.18e) \\
& \underline{v}_i \leq \operatorname{diag}(w_i^{\mathrm{R}}) \leq \overline{v}_i, && (5.18f) \\
& 0 \leq p_k^G \leq \overline{p}_k^G, \ k \in \mathcal{J}(i) && (5.18g) \\
& 0 \leq q_k^G \leq \overline{q}_k^G, \ k \in \mathcal{J}(i) && (5.18h) \\
& p_k^D \geq \underline{p}_k^D, \ q_k^D \geq \underline{q}_k^D, \ k \in \mathcal{I}(i) && (5.18i) \\
& \begin{bmatrix} w_i & S_{ij} \\ S_{ij}^{\mathsf{H}} & l_{ij} \end{bmatrix} \succeq 0, && (5.18j) \\
& \operatorname{rank} \begin{bmatrix} w_i & S_{ij} \\ S_{ij}^{\mathsf{H}} & l_{ij} \end{bmatrix} = 1, && (5.18k) \\
& \text{for all } i \in \mathbb{N}, \ i \to j \in \mathbb{E},
\end{aligned}
$$

over decision variables $p_k^G, q_k^G, p_k^D, q_k^D \in \mathbb{R}^3$ and $P_{ij}, Q_{ij}, l_{ij}^{\mathrm{R}}, l_{ij}^{\mathrm{I}}, w_i^{\mathrm{R}}, w_i^{\mathrm{I}} \in \mathbb{R}^{3\times 3}$. Note that (5.18j)-(5.18k) ensure that the network constraints formulated in the auxiliary variables in $\mathcal{P}_{\mathsf{AC}}$ are equivalent to (5.1)-(5.5). This reformulation is useful in order to derive the convex surrogate of $\mathcal{P}_{\mathsf{AC}}$ via semidefinite relaxation. Notice that

$$\begin{bmatrix} w_i & S_{ij} \\ S_{ij}^{\mathsf{H}} & l_{ij} \end{bmatrix} = \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix} \begin{bmatrix} V_i \\ I_{ij} \end{bmatrix}^{\mathsf{H}}, \tag{5.19}$$

which is consistent with the definition of the auxiliary variables w_i, l_{ij} and explains why the semidefinite constraint (5.18j) and the rank-1 constraint (5.18k) must hold. The non-convexity of the dispatch problem in (5.18) lies in the rank-1 condition (5.18k). When the dispatch problem is non-convex, there may not exist a set of prices such that market participants are incentivized to follow the DSO-prescribed dispatch [74, 72, 86]. To get around this difficulty, we propose prices derived as the optimal dual multipliers of a convex relaxation of (5.18). In particular, dropping the non-convex, rank-1 constraint we arrive at the following relaxation of (5.18):

$$\mathcal{P}_{\mathsf{SDP}} : \text{maximize} \quad \sum_{k\in\mathcal{I}} U_k(p_k^D) - \sum_{k\in\mathcal{J}} C_k(p_k^G), \tag{5.20a}$$

$$\text{subject to} \quad (5.18\text{a}) - (5.18\text{j}) \tag{5.20b}$$

$$\text{for all } i \in \mathbb{N},\ i \to j \in \mathbb{E}.$$

When the optimal solution of $\mathcal{P}_{\mathsf{SDP}}$ satisfies the rank-1 condition in (5.18k) of $\mathcal{P}_{\mathsf{AC}}$, then we say the SDP relaxation is exact and an unique voltage and current vector $(V^\star, I^\star)$ can be recovered from the auxiliary variables $(w^\star, l^\star)$ [107]. Associate Lagrange multipliers $\lambda_i^p, \lambda_i^q \in \mathbb{R}^3$ with the real and reactive power balance constraints (5.18b)-(5.18c), respectively.

Definition 2. *The relaxation-based DLMPs for multi-phase, real and reactive power at node $i \in \mathbb{N}$ are defined as the optimal Lagrange multipliers $\lambda_i^{p,\star}$ and $\lambda_i^{q,\star}$ obtained from the solution of $\mathcal{P}_{\mathsf{SDP}}$.*

The problem with $\mathcal{P}_{\mathsf{SDP}}$ is that the DSO is agnostic to the the true utilities and cost functions of market participants. Hence, we require suppliers and consumers to submit offers and bids to reveal their preferences to the DSO. To this end, in Section 5.4 we exploit the family of scalar-parameterized supply offers/bids to arrive at an optimal solution of $\mathcal{P}_{\mathsf{SDP}}$.

5.4 Two-Sided Electricity Market for Distribution Grids

We consider a market mechanism based on scalar-parameterized supply offers and demand bids introduced in Chapter 2. Specifically, let generation asset owner k connected at node i submit to the DSO a multi-phase offer $\theta_k^G \in \mathbb{R}_+^3$ with the understanding that they are willing to supply up to

$$p_k^G := \overline{p}_k^G - \theta_k^G \oslash \lambda_k^p, \ \forall\ k \in \mathcal{J}(i), \tag{5.21}$$

when faced with the a positive, multi-phase price λ_i^p at node i. In (5.21), $a \oslash b$ denotes the Hadamard division of vectors a and b. When $\theta_k^G = 0$, power producer i submit their full generation capacity, and ever decreasing values as θ_k^G grows large. We assume that each power supplier submits only offers for real power; there are no offers for reactive power in the market. Similarly, let consumer k connected at node i submit to the DSO the multi-phase demand bid $\theta_k^D \in \mathbb{R}_+^3$ with the understanding that they are willing to consume up to

$$p_k^D := \underline{p}_k^D + \theta_k^D \oslash \lambda_k^p, \ \forall\ k \in \mathcal{I}(i), \tag{5.22}$$

given price vector λ_k^p. Notice that the p_k^D is a function of the price at bus i and consists of the inelastic demand $\underline{p}_k^D$ and the price-responsive part $\theta_k^D \oslash \lambda_k^p$. The higher the market price, the lower the desired quantity by power consumer k located at node i. Again, we assume that there are no demand bids for reactive power. However, consumers (producers) pay (are paid) the DSO for the reactive power they consume (supply). This is a standard assumption in the design of retail electricity markets [90]. Given the supply offers and demand bids, the DSO seeks to solve a central dispatch problem with the objective to maximize the induced social welfare while respecting various network and individual participant's constraints. Formally,

$$\mathcal{P}_\theta : \text{maximize} \quad \sum_{k\in\mathcal{I}} \int_{\underline{p}_k^D}^{p_k^D} \frac{\theta_k^D}{z_k - \underline{p}_k^D} dz_k - \sum_{k\in\mathcal{J}} \int_0^{p_k^G} \frac{\theta_k^G}{\overline{p}_k^G - z_k} dz_k, \tag{5.23a}$$

$$\text{subject to} \quad (5.18a) - (5.18j) \tag{5.23b}$$

$$\text{for all } i \in \mathbb{N},\ i \to j \in \mathbb{E}.$$

Our goal is to investigate whether, given (θ_k^D, θ_k^G) from each individual participant's profit maximization problem, there exist prices (λ^p, λ^q) such that the resulting allocations (p_k^G, q_k^G) and (p_k^D, q_k^D) are solutions to $\mathcal{P}_{\mathsf{SDP}}$. Moreover, if said dispatch satisfies the rank-1 condition, then the market mechanism in (5.23) yields efficient allocations, i.e., allocations that solve $\mathcal{P}_{\mathsf{AC}}$. We formalize the previous discussion in the following definition.

Definition 3. *The supply offer profile (θ^G, q^G), demand bid profile (θ^D, q^D) together with prices (λ^p, λ^q) constitute a market equilibrium if they satisfy the following conditions:*

- *Individual rationality for all controllable assets: At each node $i \in \mathbb{N}$, given prices λ_i^p, λ_i^q, controllable asset $k \in \mathcal{J}(i)$ maximize their payoff, i.e.,*

$$\theta_k^G, q_k^G \in \text{argmax} \left\{ \lambda_i^{p,\mathsf{T}} \left(\overline{p}_k^G - \theta_k^G \oslash \lambda_i^p\right) + \lambda_i^{q,\mathsf{T}} q_k^G - C_k(\overline{p}_k^G - \theta_k^G \oslash \lambda_i^p) \right.$$
$$\left. | 0 \le \theta_k^G \le \theta_k^{G,max}, 0 \le q_k^G \le \overline{q}_k^G \right\}. \tag{5.24}$$

  Similarly, demand $k \in \mathcal{I}(i)$ maximize their payoff

$$\theta_k^D, q_k^D \in \text{argmax} \left\{ U_k(\underline{p}_k^D + \theta_k^D \oslash \lambda_i^p) - (\lambda_i^p)^\mathsf{T} \left(\underline{p}_k^D + \theta_k^D \oslash \lambda_i^p \right) - (\lambda_i^q)^\mathsf{T} q_k^D \right.$$
$$\left. | \theta_k^D \ge 0, q_k^D \ge \underline{q}_k^D \right\}. \tag{5.25}$$

- *Market clearing condition: The dispatch $\sum_{k\in\mathcal{J}(i)} S(\theta_k^G, \lambda_i^p) + \boldsymbol{i} q_k^G$ meets the power demands $\sum_{k\in\mathcal{I}(i)} D(\theta_k^D, \lambda_i^p) + \boldsymbol{i} q_k^D$ at each node $i \in \mathbb{N}$ over the network and induce feasible power flows, i.e., there exist $w_i, P_{ij}, Q_{ij}, l_{ij}$ such that $\left(S(\theta_k^G, \lambda_i^p), q_k^G, D(\theta_k^D, \lambda_i^p), q_k^D, w_i, P_{ij}, Q_{ij}, l_{ij}\right)$ satisfy* (5.18a)-(5.18j) *for all $k \in \mathcal{J}(i), \mathcal{I}(i), i \in \mathbb{N},\ i \to j \in \mathbb{E}$.*

- *The DSO solves $\mathcal{P}_\theta$: For every $k \in \mathcal{J}(i), \mathcal{I}(i), i \in \mathbb{N},\ i \to j \in \mathbb{E}$, there* *exist* $w_i, P_{ij}, Q_{ij}, l_{ij}$ *such that* $\left(S(\theta_k^G, \lambda_i^p), q_k^G, D(\theta_k^D, \lambda_i^p), q_k^D, w_i, P_{ij}, Q_{ij}, l_{ij}\right)$ *optimizes* $\mathcal{P}_\theta$*.*

The second property that we seek in a pricing mechanism is revenue adequacy. We formally describe this in the following definition.

Definition 4. *The prescribed dispatch* $(p_k^G, q_k^G, p_k^D, q_k^D)$ *and prices* $(\lambda_i^p, \lambda_i^q)$ *for* $k \in \mathcal{I}(i), \mathcal{J}(i)$ *and* $i \in \mathbb{N}$ *define a revenue adequate market mechanism if the merchandizing surplus (MS) defined by*

$$MS := \sum_{i \in \mathbb{N}} \left[\lambda_i^{p,\intercal} \left(\sum_{k \in \mathcal{I}(i)} p_k^D - \sum_{k \in \mathcal{J}(i)} p_k^G \right) + \lambda_i^{q,\intercal} \left(\sum_{k \in \mathcal{I}(i)} q_k^D - \sum_{k \in \mathcal{J}(i)} q_k^G \right) \right] \tag{5.26}$$

is non-negative.

We now present our main result.

Theorem 6. *Let the multi-phase, unbalanced dispatch problem* $\mathcal{P}_{\mathsf{AC}}$ *be strictly feasible. There exist prices* (λ^p, λ^q) *such that* $\left(\theta^G, q^G, \theta^D, q^D, \lambda^p, \lambda^q\right)$ *constitute a market equilibrium. Moreover, the following assertions hold:*

- *For every* $i \in \mathbb{N},\ i \to j \in \mathbb{E}$*, the dispatch* $(p_k^G, q_k^G, p_k^D, q_k^D, w_i, P_{ij}, Q_{ij}, l_{ij})$*, where* $p_k^G = S(\theta_k^G, \lambda_i^p)$*, for* $k \in \mathcal{J}(i)$ *and* $p_k^D = D(\theta_k^D, \lambda_k^p)$*, for* $k \in \mathcal{I}(i)$*, is an optimal solution to* $\mathcal{P}_{\mathsf{SDP}}$*.*

- *If the optimal solution of* $\mathcal{P}_{\mathsf{SDP}}$ *satisfies the rank-1 condition in* (5.18k)*, then* $\left(\theta^G, q^G, \theta^D, q^D, \lambda^p, \lambda^q\right)$ *support an efficient market equilibrium.*

The proof of Theorem 6 is provided in Section 5.5. Theorem 6 establishes a fundamental property of DLMPs: they always support an optimal dispatch of $\mathcal{P}_{\mathsf{SDP}}$, and an efficient dispatch whenever the relaxation is exact. This implies that, given DLMPs, each market participant has no incentive to deviate from the DSO-prescribed dispatch. This result demonstrates that said DLMPs together with the bid/offer structures, constitute a promising mechanism for the design of retail electricity markets. However, in addition to support of efficient market equilibria, said mechanism must yield non-negative MS. We relegate the proof of revenue adequacy to future efforts.

In terms of physical intuition, numerical studies on distribution feeders are required to illustrate how voltage constraints, congestion and losses influence said DLMPs. In [106], the authors perform numerical simulations of $\mathcal{P}_{\mathsf{SDP}}$ (including delta-connected loads) and demonstrate numerical exactness of the relaxed model with respect to voltages and branch flows. This indicates that one can recover the values of $(V^\star, I^\star)$ that are optimal for $\mathcal{P}_{\mathsf{AC}}$, which provides validity to the central dispatch modeled considered in Section 5.3. Understanding conditions for which the semidefinite relaxation of $\mathcal{P}_{\mathsf{AC}}$ is exact, is beyond the scope of this thesis. Interested readers are referred to [106, 108, 109] and references therein for insightful discussions.

Our exposition focuses on the mathematical foundations of DLMPs and sidesteps a range of issues surrounding the adoption of such prices in practice. For example, what is the right trading platform that needs to be established and what products should be traded in such platforms that DERs can participate in? How should such platforms coordinate their operations with wholesale markets governed by transmission system operators? See [95] and [96] for insightful discussions on the same. We align with the view in [23] to consider a retail market operated by an independent distribution system operator (DSO) responsible for the dispatch and pricing of DERs, but leave the specifics of a coordinated wholesale-retail market design to a future effort.

5.5 Proof of Theorem 6

We begin by proving the first assertion. To show that at a market equilibrium, defined in Definition 3, the resulting allocation is an optimal dispatch of $\mathcal{P}_{\mathsf{SDP}}$, we utilize the KKT conditions of $\mathcal{P}_{\mathsf{SDP}}$ and show they are equivalent to those satisfied by a market equilibrium.

To motivate the proof, we first write the complex constraints in $\mathcal{P}_{\mathsf{SDP}}$ as real-valued constraints. Given (5.10)-(5.12), complex constraint (5.18a) can be written as two real-valued constraints, representing the real and imaginary parts respectively

$$\begin{aligned} w_j^{\mathrm{R}} =& w_i^{\mathrm{R}} - (P_{ij}r_{ij} + Q_{ij}x_{ij} + r_{ij}P_{ij}^{\intercal} + x_{ij}Q_{ij}^{\intercal}) \\ &+ (r_{ij}l_{ij}^{\mathrm{R}}r_{ij} + r_{ij}l_{ij}^{\mathrm{I}}x_{ij} + x_{ij}l_{ij}^{\mathrm{R}}x_{ij} - x_{ij}l_{ij}^{\mathrm{I}}r_{ij}), \end{aligned} \tag{5.27}$$

$$\begin{aligned} w_j^{\mathrm{I}} =& w_i^{\mathrm{I}} - (x_{ij}P_{ij}^{\intercal} - P_{ij}x_{ij} + Q_{ij}r_{ij} - r_{ij}Q_{ij}^{\intercal}) \\ &+ (r_{ij}l_{ij}^{\mathrm{I}}r_{ij} - r_{ij}l_{ij}^{\mathrm{R}}x_{ij} + x_{ij}l_{ij}^{\mathrm{R}}r_{ij} + x_{ij}l_{ij}^{\mathrm{I}}x_{ij}). \end{aligned} \tag{5.28}$$

For all $i \in \mathbb{N}$ and $i \to j \in \mathbb{E}$, associate Lagrange multiplier vectors λ_i^p, λ_i^q to the real and reactive power constraints (5.18b)-(5.18c) in $\mathcal{P}_{\mathsf{SDP}}$, respectively. Similarly, associate vectors $\alpha_{ij}, \alpha'_{ij}$ to the line thermal limits (5.18d)-(5.18e). Assign vectors $\overline{\mu}_k^p, \underline{\mu}_k^p, \overline{\mu}_k^q, \underline{\mu}_k^q$ to the upper and lower bounds on real and reactive power generation (5.18g)-(5.18h), respectively. Associate Lagrange multipliers $\underline{\gamma}_k^d, \underline{\delta}_k^d$ with demand lower limits in (5.18i). Define vectors $\overline{\mu}_i^w, \underline{\mu}_i^w$ as the multipliers for the squared voltage upper and lower constraints in (5.18f). For (5.27)-(5.28), let matrices $\nu_{ij}^{\mathrm{R}}, \nu_{ij}^{\mathrm{I}}$ be the multipliers, respectively. Finally, we assign matrix σ_{ij} as the multiplier for the positive semi-definite constraint (5.18j), where σ_{ij} is partitioned into four 3-by-3 matrices as:

$$\sigma_{ij} := \begin{bmatrix} \sigma_{ij}^w & \sigma_{ij}^S \\ \sigma_{ij}^{S,\mathsf{H}} & \sigma_{ij}^l \end{bmatrix}. \tag{5.29}$$

Also, we define the following:

$$\sigma_{ij}^w := \sigma_{ij}^{w,\mathrm{R}} + \boldsymbol{i}\sigma_{ij}^{w,\mathrm{I}}, \quad \sigma_{ij}^l := \sigma_{ij}^{l,\mathrm{R}} + \boldsymbol{i}\sigma_{ij}^{l,\mathrm{I}}, \tag{5.30}$$

$$\sigma_{ij}^S := \sigma_{ij}^{S,\mathrm{R}} + \boldsymbol{i}\sigma_{ij}^{S,\mathrm{I}}. \tag{5.31}$$

Since w_i, l_{ij} are Hermitian matrices, their corresponding multipliers σ_{ij}^w and σ_{ij}^l are also Hermitian. Then, $\sigma_{ij}^{w,\mathrm{R}}$, $\sigma_{ij}^{l,\mathrm{R}}$ are symmetric, and $\sigma_{ij}^{w,\mathrm{I}}$, $\sigma_{ij}^{l,\mathrm{I}}$ are skew-symmetric. Given (5.10)-(5.12), the term associated with the semi-definite constraint (5.18j) in the Lagrangian of problem (5.20) is essentially real-valued as shown:

$$\begin{aligned}
&\mathrm{Tr}\ \left(\sigma_{ij}^w w_i + \sigma_{ij}^S S_{ij}^{\mathsf{H}} + \sigma_{ij}^{S,\mathsf{H}} S_{ij} + \sigma_{ij} l_{ij}\right) \\
&\qquad = \mathrm{Tr}\ \left(\left(\sigma_{ij}^{w,\mathrm{R}} + \boldsymbol{i}\sigma_{ij}^{w,\mathrm{I}}\right)\left(w_i^{\mathrm{R}} + \boldsymbol{i}w_i^{\mathrm{I}}\right) + \left(\sigma_{ij}^{S,\mathrm{R}} + \boldsymbol{i}\sigma_{ij}^{S,\mathrm{I}}\right)\right. \\
&\qquad\qquad \left(P_{ij}^\top - \boldsymbol{i}Q_{ij}^\top\right) + \left(\sigma_{ij}^{S,\mathrm{R},\top} - \boldsymbol{i}\sigma_{ij}^{S,\mathrm{I},\top}\right)\left(P_{ij} + \boldsymbol{i}Q_{ij}\right) \\
&\qquad\qquad\qquad \left. + \left(\sigma_{ij}^{l,\mathrm{R}} + \boldsymbol{i}\sigma_{ij}^{l,\mathrm{I}}\right)\left(l_{ij}^{\mathrm{R}} + \boldsymbol{i}l_{ij}^{\mathrm{I}}\right)\right) \\
&\qquad = \mathrm{Tr}\ \left(\sigma_{ij}^{w,\mathrm{R}} w_i^{\mathrm{R}} - \sigma_{ij}^{w,\mathrm{I}} w_i^{\mathrm{I}} + \sigma_{ij}^{l,\mathrm{R}} l_{ij}^{\mathrm{R}} - \sigma_{ij}^{l,\mathrm{I}} l_{ij}^{\mathrm{I}} + 2\left(\sigma_{ij}^{S,\mathrm{R}} P_{ij}^\top\right.\right. \\
&\qquad\qquad \left. + \sigma_{ij}^{S,\mathrm{I}} Q_{ij}^\top\right) + \boldsymbol{i}\left(\sigma_{ij}^{w,\mathrm{I}} w_i^{\mathrm{R}} + \sigma_{ij}^{w,\mathrm{R}} w_i^{\mathrm{I}} + \sigma_{ij}^{l,\mathrm{I}} l_{ij}^{\mathrm{R}} + \sigma_{ij}^{l,\mathrm{R}} l_{ij}^{\mathrm{I}}\right)\Big) \\
&\qquad = \mathrm{Tr}\ \left(\sigma_{ij}^{w,\mathrm{R}} w_i^{\mathrm{R}} - \sigma_{ij}^{w,\mathrm{I}} w_i^{\mathrm{I}} + \sigma_{ij}^{l,\mathrm{R}} l_{ij}^{\mathrm{R}} - \sigma_{ij}^{l,\mathrm{I}} l_{ij}^{\mathrm{I}} + 2\left(\sigma_{ij}^{S,\mathrm{R}} P_{ij}^\top\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left. + \sigma_{ij}^{S,\mathrm{I}} Q_{ij}^\top\right)\right).
\end{aligned} \tag{5.32}$$

The last equality follows from

$$\begin{aligned}\text{Tr}\ (X) &:= \text{Tr}\ \left(\sigma_{ij}^{w,\text{I}} w_i^{\text{R}} + \sigma_{ij}^{w,\text{R}} w_i^{\text{I}} + \sigma_{ij}^{l,\text{I}} l_{ij}^{\text{R}} + \sigma_{ij}^{l,\text{R}} l_{ij}^{\text{I}}\right) \\ &= \text{Tr}\ \left(\left(\sigma_{ij}^{w,\text{I}} w_i^{\text{R}} + \sigma_{ij}^{w,\text{R}} w_i^{\text{I}} + \sigma_{ij}^{l,\text{I}} l_{ij}^{\text{R}} + \sigma_{ij}^{l,\text{R}} l_{ij}^{\text{I}}\right)^{\top}\right) \\ &= -\text{Tr}\ \left(\sigma_{ij}^{w,\text{I}} w_i^{\text{R}} + \sigma_{ij}^{w,\text{R}} w_i^{\text{I}} + \sigma_{ij}^{l,\text{I}} l_{ij}^{\text{R}} + \sigma_{ij}^{l,\text{R}} l_{ij}^{\text{I}}\right) \\ &= -\text{Tr}\ (X) = 0. \end{aligned} \tag{5.33}$$

Since $\mathcal{P}_{\mathsf{SDP}}$ is convex, the Slater's condition holds and the Karush-Kuhn-Tucker (KKT) optimality conditions in Figure 5.1 are necessary and sufficient. The KKT conditions are all real-valued. Notation $\mathbf{0}_{n\times m}$ denotes an n-by-m matrix with zero entries.

From Definition 3, in order for $(\theta^G, q^G, \theta^D, q^D, \lambda^p, \lambda^q)$ to be a market equilibrium, the following conditions must hold.

For each generation asset $k \in \mathcal{J}(i)$, the optimality conditions yield:

1. Primal feasibility:

$$0 \le \theta_k^G \le \theta_k^{G,\max},\ 0 \le q_k^G \le \overline{q}_k^G. \tag{5.36}$$

2. Dual feasibility:

$$\underline{\mu}_k^\theta,\ \overline{\mu}_k^\theta,\ \underline{\mu}_k^q,\ \overline{\mu}_k^q \ge 0. \tag{5.37}$$

3. Gradient conditions:

$$\nabla_{p_k^G} C_k \left(S(\theta_k^G, \lambda_i^p)\right) \begin{cases} \le \lambda_i^p, & 0 \le \theta_k^G < \theta_k^{G,\max}, \\ \ge \lambda_i^p, & 0 < \theta_k^G \le \theta_k^{G,\max}, \end{cases} \quad \forall\ k \in \mathcal{J}(i) \tag{5.38}$$

$$\lambda_i^q - \overline{\mu}_k^q + \underline{\mu}_k^q = 0, \quad \forall\ k \in \mathcal{J}(i). \tag{5.39}$$

4. Complementary slackness:

$$\overline{\mu}_k^\theta(\theta_k^{G,\max} - \theta_k^G) = 0,\ \underline{\mu}_k^\theta \theta_k^G = 0,\ \overline{\mu}_k^q(\overline{q}_k^G - q_k^G) = 0,\ \underline{\mu}_k^q q_k^G = 0. \tag{5.40}$$

For each consumer $k \in \mathcal{I}(i)$, the optimality conditions yield:

1. Primal feasibility:

$$\theta_k^D \ge 0,\ q_k^D \ge \underline{q}_k^D. \tag{5.41}$$

2. Dual feasibility:

$$\underline{\gamma}_k^{\theta} \geq 0,\ \underline{\delta}_k^{d} \geq 0. \tag{5.42}$$

3. Gradient conditions:

$$\nabla_{p_k^D} U_k\left(D(\theta_k^D, \lambda_i^p)\right) \begin{cases} \leq \lambda_i^p, & \theta_k^D \geq 0, \\ = \lambda_i^p, & \theta_k^D > 0, \end{cases} \quad \forall\ k \in \mathcal{I}(i) \tag{5.43}$$

$$\lambda_i^q - \underline{\delta}_k^d = 0,\ \ \forall\ k \in \mathcal{I}(i). \tag{5.44}$$

4. Complementary slackness:

$$\underline{\gamma}_k^{\theta} \theta_k^D = 0, \underline{\delta}_k^d (q_k^D - \underline{q}_k^D) = 0. \tag{5.45}$$

At a market equilibrium, the DSO solves $\mathcal{P}_\theta$. The KKT for $\mathcal{P}_\theta$ yield:

1. Primal feasibility: (5.18a)-(5.18j), for $i \in \mathbb{N}, i \to \in \mathbb{E}$.
2. Dual feasibility: $\alpha_{ij}, \alpha'_{ij}, \overline{\mu}_i^w, \underline{\mu}_i^w \geq \mathbf{0}, \sigma_{ij} \succeq \mathbf{0}$.
3. Gradient conditions: (5.34a), (5.34b), (5.34g), (5.34h), (5.34i), (5.34j) together with

$$p_k^G = \overline{p}_k^G - \theta_k^G \oslash \lambda_i^p \tag{5.46a}$$

$$p_k^D = \underline{p}_k^D + \theta_k^D \oslash \lambda_i^p. \tag{5.46b}$$

4. Complementary slackness: (5.35a), (5.35e), (5.35f).

Complementary slackness of $\mathcal{P}_\theta$ together with (5.40) and (5.45) are equivalent to (5.35a)-(5.35f) under the maps (5.46a) and (5.46b). Specifically, it is not hard to see that $\underline{\mu}_k^\theta \lambda_i^p = \overline{\mu}_k^p$, $\overline{\mu}_k^\theta \lambda_i^p = \underline{\mu}_k^p$, $\underline{\gamma}_k^\theta \lambda_i^p = \underline{\gamma}_k^d$. The rest of the dual variables are the same. Primal feasibility of $\mathcal{P}_\theta$ together with (5.36) and (5.41) are equivalent to primal feasibility conditions of $\mathcal{P}_{\mathsf{SDP}}$. Dual feasibility of $\mathcal{P}_\theta$ together with (5.37) and (5.42) are equivalent to dual feasibility conditions of $\mathcal{P}_{\mathsf{SDP}}$. Finally, (5.38),(5.39), (5.43), (5.44) together with the gradient conditions of $\mathcal{P}_\theta$, are equivalent (5.34c)-(5.34f). The previous discussion establishes that at a market equilibrium, the resulting allocation is an optimal solution to $\mathcal{P}_{\mathsf{SDP}}$. If at the given solution, the rank-1 condition is also satisfied, then the duality gap of $\mathcal{P}_{\mathsf{AC}}$ and $\mathcal{P}_{\mathsf{SDP}}$ is zero and the optimal allocation from $\mathcal{P}_{\mathsf{SDP}}$ is also an optimal solution of $\mathcal{P}_{\mathsf{AC}}$. Hence, $(\theta^g, q^G, \theta^D, q^D, \lambda^p, \lambda^q)$ supports an efficient allocation. This completes the proof.

5.6 Summary

We presented a central dispatch model for multi-phase, unbalanced distribution grids. The dispatch model incorporates multi-phase, supply offers and demand bids from market participants connected at the low-voltage end of the grid. A central market-maker, the DSO, determines the allocation for real and reactive power at each node of the grid together with the DLMPs, which are used to compensate resources at every location. We showed that when market actors compete in scalar-parameterized offers/bids and the relaxation is exact, DLMPs support efficient market equilibria. This result demonstrates the applicability of said offers/bids in a wide range of market settings and competition models. In future efforts, we aim to establish the revenue adequacy of proposed DLMPs and perform numerical simulations to reveal how structural network characteristics impact the behavior of DLMPs.

Primal feasibility conditions: (5.18a)-(5.18j), for $i \in \mathbb{N}, i \to\in \mathbb{E}$.
Dual feasibility conditions: $\alpha_{ij}^{\star}, \alpha_{ij}^{\prime,\star}, \overline{\mu}_i^{p,\star}, \overline{\mu}_k^{q,\star}, \underline{\mu}_k^{p,\star}, \underline{\mu}_k^{q,\star}, \underline{\gamma}_k^{d,\star}, \underline{\delta}_k^{d,\star}, \overline{\mu}_i^{w,\star}, \underline{\mu}_i^{w,\star} \geq \mathbf{0}, \sigma_{ij}^{\star} \succeq \mathbf{0}$.
Gradient conditions: For $i \in \mathbb{N},\ i \to j \in \mathbb{E}$,

$$\operatorname{diag}(\lambda_i^{p,\star} - \lambda_j^{p,\star}) + \operatorname{diag}(\alpha_{ij}^{\star} - \alpha_{ij}^{\prime,\star}) - (\nu_{ij}^{\mathrm{R},\star} + \nu_{ij}^{\mathrm{R},\star,\intercal}) r_{ij} + (\nu_{ij}^{\mathrm{I},\star} - \nu_{ij}^{\mathrm{I},\star,\intercal}) x_{ij} + 2\sigma_{ij}^{S,\mathrm{R},\star} = \mathbf{0}_{3\times3}, \tag{5.34a}$$

$$\operatorname{diag}(\lambda_i^{q,\star} - \lambda_j^{q,\star}) - (\nu_{ij}^{\mathrm{R},\star} + \nu_{ij}^{\mathrm{R},\star,\intercal}) x_{ij} + (\nu_{ij}^{\mathrm{I},\star,\intercal} - \nu_{ij}^{\mathrm{I},\star}) r_{ij} + 2\sigma_{ij}^{S,\mathrm{I},\star} = \mathbf{0}_{3\times3}, \tag{5.34b}$$

$$\nabla_{p_k^G} C_k(p_k^{G,\star}) \begin{cases} \leq \lambda_i^{p,\star}, & 0 < p_k^{G,\star} \leq \overline{p}_k^G \\ \geq \lambda_i^{p,\star}, & 0 \leq p_k^{G,\star} < \overline{p}_k^G \end{cases} \quad \forall\ k \in \mathcal{J}(i), \tag{5.34c}$$

$$\lambda_i^{q,\star} - \overline{\mu}_k^{q,\star} + \underline{\mu}_k^{q,\star} = 0, \quad \forall k \in \mathcal{J}(i), \tag{5.34d}$$

$$\nabla_{p_k^D} U_k(p_k^{D,\star}) \begin{cases} \leq \lambda_k^{p,\star}, & p_k^{D,\star} \geq \underline{p}_k^D \\ = \lambda_i^{p,\star}, & p_k^{D,\star} > \underline{p}_k^D \end{cases}, \quad \forall\ k \in \mathcal{I}(i), \tag{5.34e}$$

$$\lambda_i^{q,\star} - \delta_k^{d,\star} = 0, \quad \forall\ k \in \mathcal{I}(i), \tag{5.34f}$$

$$-\operatorname{diag}(\lambda_i^{q,\star}) b_i + \operatorname{diag}(\overline{\mu}_i^{w,\star} - \underline{\mu}_i^{w,\star}) + \sum_{j:i\to j} \nu_{ij}^{\mathrm{R},\star} + \sigma_{ij}^{w,\mathrm{R},\star,\intercal} - \sum_{k:k\to i} \nu_{ki}^{\mathrm{R},\star} = \mathbf{0}_{3\times3}, \tag{5.34g}$$

$$\operatorname{diag}(\lambda_i^{p,\star}) b_i + \sum_{j:i\to j} \nu_{ij}^{\mathrm{I},\star} - \sum_{k:k\to i} \nu_{ki}^{\mathrm{I},\star} - \sum_{j:i\to j} \sigma_{ij}^{w,\mathrm{I},\star,\intercal} = \mathbf{0}_{3\times3}, \tag{5.34h}$$

$$r_{ij} \operatorname{diag}(\lambda_j^{p,\star}) + x_{ij} \operatorname{diag}(\lambda_j^{q,\star}) + r_{ij} \operatorname{diag}(\alpha_{ij}^{\prime,\star}) + r_{ij} \nu_{ij}^{\mathrm{R},\star} r_{ij} + x_{ij} \nu_{ij}^{\mathrm{R},\star} x_{ij} - r_{ij} \nu_{ij}^{\mathrm{I},\star} x_{ij} + x_{ij} \nu_{ij}^{\mathrm{I},\star} r_{ij} + \sigma_{ij}^{l,\mathrm{R},\star,\intercal} = \mathbf{0}_{3\times3}, \tag{5.34i}$$

$$-x_{ij} \operatorname{diag}(\lambda_j^{p,\star}) + r_{ij} \operatorname{diag}(\lambda_j^{q,\star}) - x_{ij} \operatorname{diag}(\alpha_{ij}^{\prime,\star}) + r_{ij} \nu_{ij}^{\mathrm{R},\star} x_{ij} - x_{ij} \nu_{ij}^{\mathrm{R},\star} r_{ij} + r_{ij} \nu_{ij}^{\mathrm{I},\star} r_{ij} + x_{ij} \nu_{ij}^{\mathrm{I},\star} x_{ij} - \sigma_{ij}^{l,\mathrm{I},\star,\intercal} = \mathbf{0}_{3\times3}, \tag{5.34j}$$

Complementary slackness conditions: For $i \in \mathbb{N},\ i \to j \in \mathbb{E}$,

$$\operatorname{diag}(\alpha_{ij}^{\star})^{\intercal} \Big(\operatorname{diag}(P_{ij}^{\star}) - f_{ij} \Big) = \operatorname{diag}(\alpha_{ij}^{\prime,\star})^{\intercal} \Big(\operatorname{diag}(r_{ij} l_{ij}^{\mathrm{R},\star} - x_{ij} l_{ij}^{\mathrm{I},\star} - P_{ij}^{\star}) - f_{ij} \Big) = \mathbf{0}_{3\times1}, \tag{5.35a}$$

$$\operatorname{diag}(\overline{\mu}_k^{p,\star})^{\intercal} (p_k^{G,\star} - \overline{p}_k^G) = \operatorname{diag}(\underline{\mu}_k^{p,\star})^{\intercal} (\underline{p}_k^G - p_k^{G,\star}) = \mathbf{0}_{3\times1}, \tag{5.35b}$$

$$\operatorname{diag}(\overline{\mu}_k^{q,\star})^{\intercal} (q_k^{G,\star} - \overline{q}_k^G) = \operatorname{diag}(\underline{\mu}_k^{q,\star})^{\intercal} (\underline{q}_k^G - q_k^{G,\star}) = \mathbf{0}_{3\times1}, \tag{5.35c}$$

$$(\underline{\gamma}_k^{d,\star})^{\intercal} \Big(p_k^{D,\star} - \underline{p}_k^D \Big) = 0, \quad (\underline{\delta}_k^{d,\star})^{\intercal} \Big(q_k^{D,\star} - \underline{q}_k^D \Big) = 0, \tag{5.35d}$$

$$\operatorname{diag}(\overline{\mu}_i^{w,\star})^{\intercal} \Big(\operatorname{diag}(w_i^{\mathrm{R},\star}) - \overline{v}_i \Big) = \operatorname{diag}(\underline{\mu}_i^{w,\star})^{\intercal} \Big(\underline{v}_i - \operatorname{diag}(w_i^{\mathrm{R},\star}) \Big) = \mathbf{0}_{3\times1}, \tag{5.35e}$$

$$\operatorname{Tr} \Big\{ \sigma_{ij}^{w,\mathrm{R},\star} w_i^{\mathrm{R},\star} - \sigma_{ij}^{w,\mathrm{I},\star} w_i^{\mathrm{I},\star} + 2(\sigma_{ij}^{S,\mathrm{R},\star} P_{ij}^{\star,\intercal} + \sigma_{ij}^{S,\mathrm{I},\star} Q_{ij}^{\star,\intercal}) + \sigma_{ij}^{l,\mathrm{R},\star} l_{ij}^{\mathrm{R},\star} - \sigma_{ij}^{l,\mathrm{I},\star} l_{ij}^{\mathrm{I},\star} \Big\} = 0. \tag{5.35f}$$

Figure 5.1: The KKT optimality conditions for (5.20).

CHAPTER 6

CONCLUSION AND FUTURE DIRECTIONS

This thesis presented allocation and pricing mechanisms for competitive electricity markets. The market allocation mechanism is based on a particular family of scalar-parameterized supply offers and demand bids. Under these offer/bid structures, we analyzed a generic, two-sided market and demonstrated a number of useful properties including support for efficient market allocations, existence of a unique Nash equilibrium and its explicit characterization, and bounded efficiency loss and price markup at the equilibrium. We demonstrated how scalar-parameterized mechanisms adequately capture the primary means by which market power is exercised in electricity markets, e.g., through the economic withholding of generation capacity. This allowed us to extend the two-sided competition model over a power network with additional considerations on network security and reliability.

We demonstrated the analytical strengths of scalar-parameterized offers in the study of inter-regional electricity markets where these offers are utilized to model a game among pure price-arbitrageurs. Support for efficient outcomes together with efficiency bounds under strategic interactions are demonstrated when players face affine price spreads. In addition, through application of reinforcement learning algorithms we showed that computed Nash equilibria can be learned by the players in a setting of imperfect information.

A comprehensive competition framework for electricity markets cannot ignore the underlying physics of power grids and security considerations. Motivated by the increased customer participation at the low-voltage side of the grid, we explored pricing mechanisms when the central dispatch problem incorporates losses, reactive power and voltage constraints—sources of non-linearities and non-convexities. To this end, we exploited semidefinite relaxations of the optimal power flow problem to leverage the extensive literature on pricing based on duality theory. We proposed and analyzed relaxation-based LMPs (RLMPs) and illustrated a number of key properties that render RLMPs meaningful price signals to compensate electricity market actors. Moving toward the design of a retail electricity market, we utilized RLMPs together with scalar-parameterized offers/bids, and explicitly constructed the central market clearing problem solved by a distribution market maker. We defined prices for real and reactive power, referred to as DLMPs, and showed that such prices together with the offer-based participation mechanism, support efficient market allocations.

There are several interesting directions for future research. First, with regard to the two-sided competition model with scalar-parameterized offers/bids, a study of the market outcomes when only one group (suppliers or consumers) are strategic would provide further insights on the nature of competition. We analyzed one-sided competition under affine demand functions in Chapter 3. Perhaps exploration of strategic interactions under other families of demand functions would reveal further properties of said offer structures. We briefly explored how, under certain conditions, competition in scalar-parameterized supply functions sustains Cournot outcomes, drawing interesting parallels with existing literature on pure price competition models. Establishing the general framework and conditions under which scalar-parameterized supply function competition yields similar outcomes to Bertrand-Cournot models, is another interesting direction for future research. Moreover, understanding how uncertainty on the maximum production capacity and/or minimum inelastic demand affects dispatch solutions would provide useful insights in electricity market design, given the deepening penetration of stochastic renewable generation.

The development of a systematic framework for a retail market that leverages our RLMP-based DLMPs with a clearly defined role for the DSO and the information exchanged with the SO, is another direction for future research. Furthermore, we plan to combine our analysis on RLMPs with that of CHPs analyzed by [72] and [86] to account for non-convexity in market clearing problems that arise due to power flow equations and integer commitment decisions. Finally, we intend to establish a general theory of pricing in non-convex markets along the lines of [110] to include non-convexity due to physical constraints of an underlying network. Such an analysis has potential applications beyond electricity markets, e.g., for gas networks as in [111].

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