On the development of cut-generating functions

by

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Abstract

Cut-generating functions are tools for producing cutting planes for generic mixed-integer sets. Historically, cutting planes have advanced the progress of algorithms for solving mixed-integer programs. When used alone, cutting-planes provide a finite time algorithm for solving a large family of integer programs [12, 70]. Used in tandem with other algorithmic techniques, cutting planes play a large role in popular commercial solvers for mixed-integer programs [9, 34, 35].

Considering the benefit that cutting planes bring, it becomes important to understand how to construct good cutting planes. Sometimes information about the motivating problem can be used to construct problem-specific cutting planes. One prominent example is the history of the Traveling Salesman Problem [43]. However, it is unclear how much insight into the particular problem is required for these types of cutting-planes. In contrast, cut-generating functions (a term coined by Cornuéjols et al. [40]) provide a way to construct cutting planes without using inherent structure that a problem may have. Some of the earliest examples of cut-generating functions are due to Gomory [70] and these have been very successful in practice [34]. Moreover, cut-generating functions produce the strongest cutting planes for some commonly used mixed-integer sets such as Gomory's corner polyhedron [66, 95].

In this thesis, we examine the theory of cut-generating functions. Due to the success of the cut-generating function created by Gomory, there has been a proliferation of research in this direction with one end goal being the further advancement of algorithms for mixed-integer programs [78, 40, 28]. We contribute to the theory by assessing the usefulness of certain cut-generating functions and developing methods for constructing new ones.

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Secondary Reader: Daniel Robinson

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This thesis is dedicated to my parents, Ann and John.

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Chapter 1

Introduction

Mixed-integer programming is a powerful tool for modeling a wide array of real-world problems. These problems stem from fields including (but not limited to) airline and railway logistics [38, 73], astronomy [82], medical treatment and operations [47, 83, 89], chip design [93], finance [32], forestry and mining [63, 39], national defense [71, 77], and supply chain management [3]. For a given problem, this modeling technique considers the set of all outcomes that satisfy the problem's constraints. In turn, optimization algorithms can be run over these 'feasible regions' to search for a solution that best meets a given criterion. It stands to reason that understanding these feasible regions plays an important role in applying mixed-integer programming techniques.

So what do feasible regions look like? Feasible regions are represented with mixed-integer sets, that is sets of vectors where some coordinates are constrained to be integral. We define the *mixed-integer set with respect to* $R, P,$ *and* S to be

$$M_S(R,P) = \left\{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^l_+ : Rs + Py \in S\right\},$$

for a positive integer n, some set $S \subseteq \mathbb{R}^n$, a $n \times k$ matrix R, and a $n \times l$ matrix P. Figure 1.1 shows an example of a mixed-integer set.

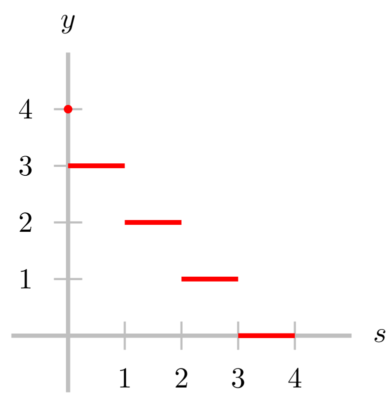
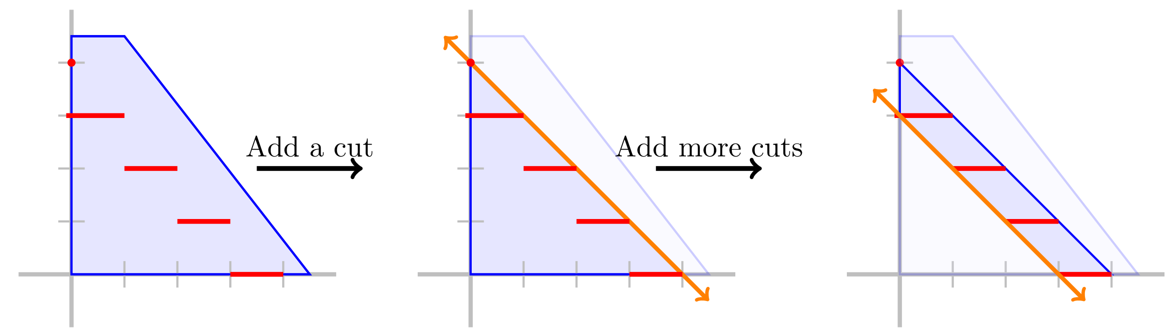


Figure 1.1: The mixed-integer set $\{(s, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ : s + y \in [3, 4]\}$ is in red.

At a high-level, one can think of the values R and P as representing data coming from the real-world problem, and the set S as representing the types of constraints placed on the feasible region. With this interpretation, the collection of mixed-integer sets sharing the same S follow similar modeling constraints. Thus for a fixed S, we say the *mixed-integer model* M_S is the collection of all mixed-integer sets defined by R, P, and S, over all R and P. Understanding that mixed-integer models and sets depend on S, we will often call them just *mixed-integer models* and *mixed-integer sets*. This particular definition of a mixed-integer model is quite inclusive, as it can be used as a framework to cast many commonly used optimization paradigms as special cases (see [40]). Examples include mixed-integer linear programming, complementarity problems, and semidefinite programming. We give more details in Chapter 2.

Understanding the structure of the mixed-integer sets in a model is very beneficial when it comes to subsequently implementing optimization algorithms. In particular, being able to describe the *convex hull* of a mixed-integer set is advantageous. For instance, Meyer showed in [86] that for a mixed-integer linear program with rational data, the convex hull of the mixed-integer set is a polyhedron. This result, commonly referred to as the 'Fundamental Theorem of Integer Programming' (see Chapter 4 of [43]), enables one to apply fast linear programming algorithms to more complicated mixed-integer linear programs, provided the convex hull of the mixed-integer set can be found.

Unfortunately, it is usually the case that convex hulls of mixed-integer sets are difficult to compute. Some notable examples include the feasible regions for the traveling salesman problem, the matching problem, and the spanning tree problem, all of which require at least an exponential number of linear constraints to describe (see Chapter 4 in [43] as well as [61, 62, 80]). It is often much simpler to describe a larger set, called a *relaxation*, that contains the mixed-integer set. From a relaxation, one can then 'cut away' parts of the relaxation in an attempt to carve out the convex hull of the mixed-integer set. This cutting can be accomplished using linear inequalities called *cutting planes* or just *cuts*. Figure 1.2 shows the idea of using cutting planes to describe the convex hull of the mixed-integer set from Figure 1.1.



A relaxation of the mixed-integer set. from Figure 1.1.

Cuts remove part of the relaxation, making a 'tighter' relaxation.

Adding enough cuts isolates the convex hull of the mixed-integer set.

Figure 1.2: Cutting planes (drawn in orange) carve out the convex hull of the mixed-integer set from a relaxation (in blue).

Used alone, cutting planes provide algorithms for solving certain mixed-integer programming problems. Gomory created a cutting plane algorithm for solving pure integer linear programs [70], and Balas et al. created a cutting plane algorithm for mixed-integer 0/1 problems [10, 11, 12]. Used alongside other techniques, cutting planes have been successful in mixed-integer solvers such as IBM's CPLEX [33, 34]. The results provided by Bixby et al. [34] show that cutting planes can reduce computation time significantly more than other tools such as heuristics or presolving techniques.

Recognizing the usefulness of cutting planes, it becomes important to know how to generate them. As a mixed-integer set $M_S(R, P)$ is derived from an underlying problem, there is occasionally structure from the problem that can be exploited to create cuts (see Chapter 7 of [43] for an extensive list of examples). However, such problem-specific cutting planes may be difficult to construct for arbitrary mixed-integer sets. This leads to the following question: for a fixed S, is it possible to just use the values R and P to construct a cutting plane? Put differently, can we develop a function for M_S that takes matrices R and P as input and outputs a cutting plane for $M_S(R, P)$? We call such a function a *cut-generating function*. The term cut-generating function was first introduced in [40], although cut-generating functions predate the name [10, 66, 67, 70].

By definition, cut-generating functions must satisfy the strong requirement that they can be used on any mixed-integer set in a particular model. So it is natural to ask if such functions even exist, and if they do, are the resulting cutting planes useful? It is not clear that the answer to either of these questions is yes. However, cut-generating functions have been constructed for a variety of mixed-integer models; for a comprehensive overview, see the surveys [21, 28] as well as [40, 94], the references therein, and Chapter 6 of [43]. Not only do they exist, but cut-generating functions have been created that produce very good cutting planes ('good' will be explicitly defined in Chapter 2). For example, the popular 'corner polyhedron' introduced by Gomory in [66, 67] is the convex hull of a particular type of mixed-integer set. It was shown in [95] that only cutting planes generated by cut-generating functions are required to describe the corner polyhedron. Cornuéjols, Wolsey, and Yıldız were able to extend this to an even larger collection of models [45].

Cut-generating functions provide a powerful tool for solving mixed-integer programs and have the potential to greatly enhance current mixed-integer solvers. The purpose of this thesis is to explore cut-generating functions within particular models, and connect functions coming from different models.

1.1 Contributions

1. **Approximation guarantees for intersection cuts**

Fix $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Consider the mixed-integer model $C_{b+\mathbb{Z}^n}$ with mixed-integer sets

$$C_{b+\mathbb{Z}^n}(R) = \{s \in \mathbb{R}^k_+ : Rs \in b + \mathbb{Z}^n\}. \tag{1.1}$$

Note that (1.1) is a mixed-integer set where the matrix P is always set to zero, i.e. $C_{b+\mathbb{Z}^n}(R) = M_{b+\mathbb{Z}^n}(R, 0)$. This model, which has received much attention in the literature [1, 16, 18, 19, 26, 36, 46, 49], has the nice property that good cut-generating functions can be characterized using lattice-free polyhedra (that is polyhedra that do not have lattice-points in their interiors). Cutting planes generated using lattice-free polyhedra are called intersection cuts. Provided the matrix R and vector b are rational, the convex hull of the mixed-integer sets in (1.1) are obtained by intersecting all intersection cuts. However, generating all intersection cuts is computationally impractical. Therefore a question of interest is if any subfamilies of intersection cuts can be used to closely approximate the mixed-integer sets. We show that such an approximation is possible and that it is determined by how many facets the underlying lattice-free polyhedra have. For polyhedra with only a small number of facets, the resulting cuts do not provide a close approximation of the mixed-integer sets. These results provide a theoretical guarantee for which families of cuts to be implemented in practice. This work was done jointly with Gennadiy Averkov and Amitabh Basu.

2. **Cut-generating functions for the infinite group problem**

Fix $b \in \mathbb{R} \setminus \mathbb{Z}$. Consider the model $I_{b+\mathbb{Z}}$ with mixed-integer sets

$$I_{b+\mathbb{Z}}(P) = \{y \in \mathbb{Z}^l_+ : Py \in b + \mathbb{Z}\}. \tag{1.2}$$

Note that (1.2) is indeed a mixed-integer set where the matrix R is always set to zero, i.e. $I_{b+\mathbb{Z}}(P) = M_{b+\mathbb{Z}}(0, P)$. The model $I_{b+\mathbb{Z}}$ was introduced by Gomory [66, 67] and is commonly referred to as the infinite group problem. Much research has been done on the infinite group problem; see the manuscripts of Basu et al. [28] and Dey and Richard [90], and the references therein, for a thorough review.

Gomory and Johnson showed that a cut-generating function for the infinite group problem is strong provided it is piecewise linear with exactly two slopes [66]. However, these sufficient conditions are not necessary as strong cut-generating functions with up to 28 slopes had since been discovered [81]. We extend this 'slope bound' by providing an explicit construction of strong cut-generating functions with k slopes, for any $k \in \mathbb{N}$ and $k \geq 2$. We also extend this result to the model known as the n-row infinite group problem $I_{b+\mathbb{Z}^n}$, which is defined similarly to $I_{b+\mathbb{Z}}$. While the practical importance of such complicated cut-generating functions has not yet been established, they do provide a better understanding of the infinite group problem. This work was done jointly with Amitabh Basu, Michele Conforti, and Marco Di Summa, and appears in [22].

3. **Building cut-generating pairs for a mixed-integer model**

   Fix $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Consider the model $M_{b+\mathbb{Z}^n}(R, P)$ with mixed-integer sets

$$M_{b+\mathbb{Z}^n}(R, P) = \{(s, y) \in \mathbb{R}^k_+ \times \mathbb{Z}^l_+ : Rs + Py \in b + \mathbb{Z}^n\}. \tag{1.3}$$

   Mixed-integer set in the models $C_{b+\mathbb{Z}^n}$ and $I_{b+\mathbb{Z}}$ are both instances of the mixed-integer sets (1.3). The connection between these models runs deeper, as strong cut-generating functions for both $C_{b+\mathbb{Z}^n}$ and $I_{b+\mathbb{Z}}$ can be used to construct cut-generating functions for $M_{b+\mathbb{Z}^n}$; see for example [17, 56, 79]. Moreover, the geometry of the lattice-free polyhedra in model (1.1) can be leveraged. One such geometric property is called the 'covering property' [5, 17, 23, 24, 30, 53, 54, 56]. The covering property was first introduced by Dey and Wolsey in [53]. If a lattice-free set has the covering property, then the corresponding cut-generating function can be quickly extended to (1.3) [13, 43].

   We consider both sets with and without the covering property, and examine how they can be used to construct cut-generating pairs.

   (a) **Sets with the covering property**

      It is not always simple to identify if a lattice-free set has the covering property. However, there are set operations that preserve the covering property and produce new lattice-free sets with the covering property (and therefore new good cut-generating functions for 1.3) [5, 23]. We show that these operations can also be used in more generalized mixed-integer models where $S = (b + \Lambda) \cap C$ is a polyhedrally-truncated affine lattice (these choices of S will be defined formally in Chapter 2, and a special instance is $(b + \mathbb{Z}^n) \cap P$ for a rational polyhedron P). This creates a large family of previously unknown cut-generating functions for M_S. We also apply a new topological proof technique, which can be used in proving similar results. This work was done jointly with Amitabh Basu and appears in [30].

(b) **Sets without the covering property**

Less work has been done in creating strong cut-generating functions for $M_{b+\mathbb{Z}^n}$ using sets without the covering property. However, Dey and Wolsey [54, 56] were able to identify certain situations in which such sets can be recycled to find strong cut-generating functions. Motivated by their work, we develop the notion of a fixing region which creates a geometric description for identifying when this recycling can occur. We rederive some of their results using a more geometric approach, in comparison to their algebraic viewpoint, as well as establish a foundation for further work. This research provides a framework for further study into cut-generating functions for M_S derived from sets without the covering property. This work was done jointly with Amitabh Basu and Santanu Dey, and appears in [25].

1.2 Outline of the thesis

Our contributions are organized into Chapters 3-6 of the thesis. Chapter 2 discusses most of the background material used throughout the thesis; each result has some specific background that is introduced in its corresponding section. We examine approximation guarantees for intersection cuts in Chapter 3. In Chapter 4 we provide the construction of strong cut-generating functions with an arbitrary number of slopes. Chapter 5 discusses our results on preserving the covering property, while Chapter 6 looks closer at the situation of creating cut-generating functions from sets without the covering property. We conclude with future research directions in Chapter 7.

Chapter 2

Background

This chapter provides a starting point for understanding the mathematical content in this thesis. Here we present notation and most of the background material required. Additional background will be introduced in later chapters as needed. We provide references for any results given without proofs.

2.1 Preliminaries and notation

Section 2.1 is notation heavy and is intended to be used as a reference. For a summary of this notation, as well as notation defined throughout the thesis, see the **List of notation** following the appendices.

2.1.1 Basic notation

For a positive integer n, let $[n] = \{1, \ldots, n\}$. We use the notation $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for each } i \in [n]\}$ and $\mathbb{Z}^n_+ := \{x \in \mathbb{Z}^n : x_i \geq 0 \text{ for each } i \in [n]\}$. When $n = 1$, we write $\mathbb{R}_+$ and $\mathbb{Z}_+$ for $\mathbb{R}^1_+$ and $\mathbb{Z}^1_+$, respectively. For $n, k \in \mathbb{N}$, the set $\mathbb{R}^{n\times k}$ denotes all $n \times k$ real-valued matrices. For a matrix $R \in \mathbb{R}^{n\times k}$, we let r_i denote the i-th column of R, for $i \in [k]$. If $\{r_1, \ldots, r_k\} \subseteq \mathbb{R}^n$ then we set $[r_1, ..., r_k]$ to be the matrix in $\mathbb{R}^{n\times k}$ with columns defined by $\{r_1, \ldots, r_k\}$. For $\epsilon > 0$ and $x \in \mathbb{R}^n$, the *open ball of radius ϵ centered at x* is $D(x; \epsilon) := \{y \in \mathbb{R}^n : \|x - y\|_2 < \epsilon\}$. If $x \in \text{int}(A)$ then we call A a *x-neighborhood.*

For sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}^n$, let $A^B := \{f : B \to A\}$. For $f \in A^B$, let the *support of*

f be denoted by $\mathrm{supp}(f) := \{b \in B : f(b) \neq 0\}$. A function $f \in A^B$ is said to have *finite support* if $|\mathrm{supp}(f)|$ is finite. Let $(A)^{(B)}$ denote the set of functions $f \in A^B$ with finite support.

Let $A, B \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^m$. We use $\mathrm{conv}(A)$, $\mathrm{int}(A)$, $\mathrm{relint}(A)$, $\mathrm{bd}(A)$, $\mathrm{cl}(A)$ and $\mathrm{aff}(A)$ to denote the *convex hull*, the *interior*, the *relative interior*, the *boundary*, the *closure*, and the *affine hull* of A, respectively. The *dimension of* A is the dimension of $\mathrm{aff}(A)$. The set $A \subseteq \mathbb{R}^n$ is said to be *full-dimensional* if $\dim(A) = n$. The *Minkowski sum of* A *and* B is $A + B := \{a + b : a \in A \quad b \in B\}$; when B is a singleton $\{b\}$, we will use $A + b$ to denote $A + \{b\}$. For $\mu \in \mathbb{R}$, $\mu A := \{\mu a : A \in A\}$. The *Cartesian product of* A *and* C is $A \times C = \{(a, c) \in \mathbb{R}^{n+m} : a \in A, c \in C\}$. The *polar of* A is defined to be $A^* := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in A\}$. A *relaxation of* A is any superset of A.

2.1.2 Convexity and lattices

For more on convexity and lattices, see [14, 43].

A set A is *convex* if $\lambda x + (1-\lambda)y \in A$ for all $x, y \in A$ and $\lambda \in [0, 1]$. Suppose A is closed and convex. The *recession cone of* A is $\mathrm{rec}(A) := \{x \in \mathbb{R}^n : a + \lambda x \in A \text{ for all } \lambda \geq 0 \text{ and } a \in A\}$, and the *lineality space of* A is $\mathrm{lin}(A) := \{x \in \mathbb{R}^n : a + \lambda x \in A \text{ for all } \lambda \in \mathbb{R} \text{ and } a \in A\}$.

Suppose that $A, B \subseteq \mathbb{R}^n$ are convex sets. An inequality $a \cdot x \leq b$ is *valid for* A if $a \cdot x \leq b$ for all $x \in A$. A *hyperplane* is a set of the form $\{x \in \mathbb{R}^n : a \cdot x = b\}$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane defining a valid inequality is called a *valid cutting plane for* A (or just a cutting plane). A hyperplane $\{x \in \mathbb{R}^n : a \cdot x = b\}$ *separates* A and B if $a \cdot x \leq b$ is valid for A and $-a \cdot x \leq -b$ is valid for B. The hyperplane *strictly separates* A and B if $a \cdot x < b$ for all $x \in A$ and $-a \cdot y < -b$ for all $y \in B$. The following 'separating hyperplane' theorem can be found in many sources; see for example Theorem 1.3 in Chapter III of [14].

Theorem 1 (Separating hyperplane theorem)**.** *Let $A \subseteq \mathbb{R}^n$ be a closed convex set and $x \notin A$. Then there is a hyperplane that strictly separates A and $\{x\}$.*

A convex set C is a *cone* if $\lambda c \in C$ for all $c \in C$ and $\lambda \geq 0$. A *polyhedron* $P \subseteq \mathbb{R}^n$ is a convex set of the form $P = \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i,\ i \in [k]\}$, for $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$, and $k \in \mathbb{N}$. A *rational polyhedron* is one where a_i and b_i have rational entries, for each

$i \in [k]$. A *polyhedral cone* is a set that is both a polyhedron and a cone. A set $Q \subseteq \mathbb{R}^n$ is a *polytope* if there is a finite set $\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$ so that $Q = \text{conv}\{v_1, ..., v_k\}$. The famous result of Minkowski and Weyl ties together polyhedra, cones, and polytopes; see for example Theorem 3.14 in [43].

Theorem 2 (Minkowski-Weyl)**.** *$P \subseteq \mathbb{R}^n$ is a polyhedron if and only if it is of the form $P = C + Q$ for a polyhedral cone C and a polytope Q. It follows that a polytope is a bounded polyhedron.*

Suppose $P \subseteq \mathbb{R}^n$ is a polyhedron and $a \cdot x \leq b$ is a valid inequality for P. We say that the hyperplane defined by a and b is a *supporting hyperplane for* P if $P \cap \{x \in \mathbb{R}^n : a \cdot x = b\} \neq \emptyset$. When analyzing a polyhedron, it is common to look at its faces. For $k \in [n]$, a *k-face of* P is a subset $F \subseteq P$ such that $\dim(F) = k$ and there exists a supporting hyperplane $\{x \in \mathbb{R}^n : a \cdot x = b\}$ of P such that $F = P \cap \{x \in \mathbb{R}^n : a \cdot x = b\}$. A *vertex of* P is a 0-face. A *facet of* P is a $(\dim(P) - 1)$-face of P.

A *lattice* $\Lambda \subseteq \mathbb{R}^n$ is of the form $\Lambda = \{\lambda_1 v_1 + \ldots + \lambda_n v_n : \lambda_i \in \mathbb{Z}\}$, where $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent vectors. When these generating vectors are the standard unit vectors in $\mathbb{R}^n$, we get the standard integer lattice $\mathbb{Z}^n$. A *lattice subspace* of a lattice Λ is a linear subspace which has a basis composed of vectors from Λ. We say a set S is a *truncated affine lattice* if $S = (b + \Lambda) \cap C$ for some lattice Λ in $\mathbb{R}^n$, some $b \in \mathbb{R}^n \setminus \Lambda$, and some convex set $C \subseteq \mathbb{R}^n$; if $C = \mathbb{R}^n$ we call S an *affine lattice* or a *translated lattice*. In general, for a truncated affine lattice S, $\text{conv}(S)$ is not a polyhedron; it may not even be closed [57]. If $\text{conv}(S)$ is a polyhedron, we specify further by saying S is a *polyhedrally-truncated affine lattice*. In this case, $S = (b + \Lambda) \cap \text{conv}(S)$.

Fact 1. *If* $\text{conv}(S)$ *is a polyhedron for a truncated affine lattice S, the* $\text{lin}(\text{conv}(S))$ *is a lattice subspace. See for example [30].*

2.2 Cut-generating functions

In mathematical programming, it is often the case that the feasible region over which optimization algorithms run can be represented by a mixed-integer set. Many of these mixed-integer sets share the same structure, and so they are organized into different models. The idea of a cut-generating function is to exploit this shared structure and generate cuts for all mixed-integer sets in the model. In this section, we begin to review these ideas. We start by providing a background on mixed-integer sets, the corresponding models, and cut-generating function pairs.

2.2.1 Mixed-integer sets

Definition 1 (Mixed-integer set $M_S(R,P)$ and mixed-integer model M_S)**.**

Let S be a nonempty, closed subset of $\mathbb{R}^n$ with $0 \notin S$. The set

$$M_S(R,P) := \left\{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^l_+ \,:\, Rs + Py \in S\right\}, \tag{2.1}$$

is called a mixed-integer set, where $k, l \in \mathbb{Z}_+, n \in \mathbb{N}$, $R \in \mathbb{R}^{n\times k}$ and $P \in \mathbb{R}^{n\times l}$ are matrices. We allow $k = 0$ or $l = 0$, but not both. The mixed-integer model M_S is the collection of all $M_S(R,P)$, where $R \in \mathbb{R}^{n\times k}$, $P \in \mathbb{R}^{n\times l}$, $l, n \in \mathbb{Z}_+$, and at most one of k and l is 0.

We provide a few examples showing how mixed-integer sets can model feasible regions in mathematical programming; for more see [40].

Example 1 (Mixed-integer linear programming)**.** *A mixed-integer linear program's feasible region is defined by $\left\{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ \,:\, Rs + Py = b\right\}$, where R, P are matrices and $b \in \mathbb{R}^n$ a vector. This is seen to be a mixed-integer set by letting $S = \{b\}$. An interesting, and useful, way to model MILPs is setting $S = b - \mathbb{Z}^n_+$. This comes from the tableaux form of general MILPs*

$$\left\{(x,s,y) \in \mathbb{Z}^n_+ \times \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ \,:\, Rs + Py + x = b\right\}, \tag{2.2}$$

where x are the basic variables and s, y as the nonbasic variables in a simplex tableaux. $\square$

Example 2 (Complementarity problems)**.** *In complementarity problems, the feasible region is given by all integer points in a polyhedron that satisfy complementarity constraints like $y_i y_j = 0$ (y_i, y_j are variables of the problem). Let $k = 0$ and $n = \ell$. Let E be a subset of $\{1,\ldots,n\} \times \{1,\ldots,n\}$. Let $C = \{y \in \mathbb{R}^n : y_i y_j = 0 \ \ \forall (i,j) \in E\}$ and $S = X \cap C$ where $X \subseteq \mathbb{R}^n$ is a polyhedron. Letting the matrix $P = I_n$ (the identity matrix of dimension n),* *we then obtain a method for representing complementarity problems with integer constraints by a mixed-integer set.* □

Example 3 (Integer semidefinite programming)**.**

The feasible region of a semidefinite program can be written as

$$\begin{aligned} \langle A^t, X\rangle &= b_t, \qquad t \in [k] \\ X &\succcurlyeq 0 \\ X &\in \mathbb{Z}^{n\times n} \end{aligned}$$

for an $n \times n$ *matrix* A^t *and* $b_t \in \mathbb{R}$ *for each* $t \in [k]$*. Suppose that* $b_t \neq 0$ *for some* $t \in [k]$*. Here* $\langle A^t, X\rangle = \sum_{i,j=1}^{n} A^t_{i,j} X_{i,j}$*. After reordering the* n^2 *indices of each* A^t *and* X*, we may treat* A^t *and* X *as vectors in* $\mathbb{R}^{n^2}$ *and write* $\langle A^t, X\rangle = \sum_i^{n^2} A^t_i X_i = b_t$*. With this vector notation, define* $A \in \mathbb{R}^{k\times n^2}$ *to be the matrix where the i-th row is* A^i*. Let* $I \in \mathbb{R}^{n^2\times n^2}$ *be the identity matrix. The condition* $X \succcurlyeq 0$ *enforces* X *to be a positive-semidefinite matrix. The collection of all* $n \times n$ *positive semidefinite matrices is a cone* $C \subseteq \mathbb{R}^{n\times n}$*. As with the matrices* A^t *and* X*, the set* C *can be thought of as a set in* $\mathbb{R}^{n^2}$*. Then, the feasible region of the integer semidefinite program can be written as a mixed-integer set:*

$$\left\{ x \in \mathbb{R}^{n^2}_+ : \begin{bmatrix} A \\ I \\ I \end{bmatrix} x \in [b_1, \ldots, b_k] \times \mathbb{Z}^{n^2} \times C \right\}.$$

□

Although the mixed-integer sets defined by (2.1) seem very general, it is necessary to place some structure on S to achieve practical utility. However, one must be cautious. Placing too much structure on S reduces the amount of modeling power of M_S, while too little structure limits how much analysis can be done that eventually leads to concrete methods and algorithms. This brings us to consider the necessity of the assumptions that we make on S: S is nonempty, S is closed, and $0 \notin S$.

Note that if S is empty, then mixed-integer sets in M_S are also empty and there is nothing for us to describe. So the interesting case occurs when S is nonempty. Using the assumptions that S is closed and $0 \notin S$, we can prove the following.

Proposition 1 (Conforti et al. [40])**.** *Suppose that S is closed and $0 \notin S$. Then $0 \notin \mathrm{cl}(\mathrm{conv}(M_S(R,P))$.*

Conforti et al. [40] actually prove the case when $P = 0$, but their result implies Proposition 1.

Observe that for a given mixed-integer set $M_S(R,P) \subseteq \mathbb{R}^k \times \mathbb{R}^l$, the nonnegative orthant $\mathbb{R}^k_+ \times \mathbb{R}^l_+$ is a relaxation. Proposition 1 shows that 0 is contained in this relaxation but not in $\mathrm{cl}(\mathrm{conv}(M_S(R,P))$. Together with Theorem 1, this implies that there exists a cutting plane strictly separating 0 from $\mathrm{cl}(\mathrm{conv}(M_S(R,P))$. Therefore, with the goal of cutting $\mathrm{cl}(\mathrm{conv}(M_S(R,P))$ from the relaxation $\mathbb{R}^k_+ \times \mathbb{R}^l_+$, Proposition 1 guarantees that we can at least separate 0.

The ability to separate 0 (as opposed to some other point) turns out to be quite useful so long as one is careful in fitting a feasible region to a model. Indeed, in handling the feasible region A for a mathematical program by using a relaxation B, it is often the case that we want to separate A from a particular point $x^* \in B$ (as an example, consider the cutting step of a branch-and-cut algorithm, see Chapter 9 of [43]). If we carefully transform A to look like a mixed-integer set $M_S(R,P)$, then we can also transform B into the nonnegative orthant $\mathbb{R}^k_+ \times \mathbb{R}^l_+$ and x^* to 0. Therefore, cutting 0 from $\mathrm{conv}(M_S(R,P))$ is equivalent to cutting x^* from A. The following example demonstrates this.

Example 4. *Consider the general tableaux form of a MILP described by Equation* (2.2) *in Example 1. Setting (s,y) to $(0,0)$ produces the basic feasible solution $(s,y,x) = (0,0,b)$ for the linear relaxation of the mixed-integer set. If $x = b \notin \mathbb{Z}^n_+$ then the solution (s,y,x) is not in the feasible region* (2.2)*. Therefore, we would like to separate the point $(s,y) = (0,0)$ from the mixed-integer set*

$$M_{b-\mathbb{Z}^n_+}(R,S) = \left\{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ \ : \ Rs + Py \in b - \mathbb{Z}^n_+\right\}.$$

□

2.2.2 Cut-generating function pairs

Following Proposition 1 and Example 4, we would like to separate 0 from a mixed-integer set $M_S(R,P)$ in some model M_S. Cut-generating functions give us a tool for constructing valid cuts for *all* mixed-integer sets in M_S.

Definition 2 (Cut-generating function pair for M_S)**.** *Let M_S be a mixed-integer model. A* cut-generating function pair (ψ,π) *for M_S is a pair of functions $\psi,\pi:\mathbb{R}^n\to\mathbb{R}$ such that for every mixed-integer set $M_S(R,P)$ in M_S and each $(s,y)\in M_S(R,P)$, it follows that*

$$\sum_{i=1}^{k}\psi(r_i)s_i+\sum_{j=1}^{l}\pi(p_j)y_j\geq 1. \tag{2.3}$$

That is, (ψ,π) produces a valid inequality for every mixed-integer set in M_S. We also call a cut-generating function pair just a cut-generating pair.

We emphasize that cut-generating pairs depend on n and S, but not on k,l, R nor P. This justifies the name 'cut-generating'.

Given a mixed-integer set $M_S(R,P)$, we would like to strictly separate $(0,0)\in\mathbb{R}^k_+\times\mathbb{R}^l_+$ from $\mathrm{cl}(\mathrm{conv}(M_S(R,P)))$ (see Proposition 1). Since the right hand side of inequality (2.3) is 1, it follows that $(0,0)$ does not satisfy (2.3), and so a cut-generating pair achieves this goal.

Note 1. *The choice of '1' for the right hand side of inequality* (2.3) *is not very restrictive. Indeed, a more general inequality generated by a cut-generating pair is of the form*

$$\sum_{i=1}^{k}\psi(r_i)s_i+\sum_{j=1}^{l}\pi(p_j)y_j\geq \alpha \tag{2.4}$$

for some $\alpha\in\mathbb{R}_+\setminus\{0\}$. Dividing (2.4) *through by α gives an inequality equivalent to* (2.3)*. In order to ensure that* 0 *is cut off, we require '$\geq$' instead of '$\leq$' as well as $\alpha\neq 0$.*

Since our goal is to create an inequality that is violated by 0, it is also possible to replace '$\geq$' in (2.4) *by a '$\leq$', provided that $\alpha<0$. However, we do not consider such cut-generating pairs in this thesis. For more on these see [28].*

□

Since cut-generating pairs seem like powerful tools, it is often helpful to provide an example showing one exists. Here we give one of the most famous cut-generating pairs, Gomory's mixed-integer cuts (see [64], also Chapter 6 in [43]).

Example 5 (Gomory mixed-integer cuts [64])**.** *Suppose that $n = 1$ and $S = b + \mathbb{Z}$ for $b \in (0, 1)$. Consider the functions*

$$\psi(r) = \max\left\{-\frac{r}{1-b}, \frac{r}{b}\right\}$$

and

$$\pi(r) = \begin{cases} \frac{r - \lfloor r \rfloor}{b}, & \text{if } r - \lfloor r \rfloor \le b \\ \frac{\lceil r \rceil - r}{1-b}, & \text{otherwise.} \end{cases}$$

Then (ψ, π) are a cut-generating pair for $M_{b+\mathbb{Z}}$. For a mixed-integer set in $M_{b+\mathbb{Z}}$, the cut created by (ψ, π) is called the Gomory mixed-integer cut.

□

Some cut-generating pairs produce stronger cuts than others. For instance, suppose that (ψ, π) and (ψ', π') are cut-generating pairs for M_S such that $\psi \le \psi'$ and $\pi \le \pi'$. Let $M_S(R, P)$ be a mixed-integer set in the model and take $(s, y) \in \mathbb{R}^k_+ \times \mathbb{R}^l_+$ (not necessarily in $M_S(R, P)$). Since (s, y) is nonnegative, it follows that

$$\sum_{i=1}^{k} \psi'(r_i) s_i + \sum_{j=1}^{l} \pi'(p_j) y_j \ge \sum_{i=1}^{k} \psi(r_i) s_i + \sum_{j=1}^{l} \pi(p_j) y_j.$$

Therefore if (ψ', π') cuts off the point (s, y) (that is, (s, y) does not satisfy (2.3) for (ψ', π')) then (s, y) is also cut off by (ψ, π). Thus (ψ', π') is redundant for describing $M_S(R, P)$ and is implied by (ψ, π). This leads to the notion of dominating and minimal cut-generating pairs.

Definition 3 (Dominate and minimal cut-generating pairs)**.** *Let $(\psi, \pi), (\psi', \pi')$ be cut-generating pairs for a model M_S. The pair (ψ, π) dominates (ψ', π') if $\psi \le \psi'$ and $\pi \le \pi'$. A minimal cut-generating pair for M_S is one not dominated by another cut-generating pair.*

The Gomory mixed-integer cuts in Example 5 are minimal [43]. Using Zorn's Lemma (see Theorem 1.1 in [29]), it can be shown that every cut-generating pair is dominated by a minimal one. We prove the following result in [30]; see also [94].

Proposition 2. *Every cut-generating pair for M_S is dominated by a minimal cut-generating pair for M_S.*

Proof. Fix $s^* \in S$ which is nonempty. Note that any cut-generating pair (ψ, π) satisfies $\psi(r) + \psi(s^* - r) \geq 1$ and $\pi(r) + \pi(s^* - r) \geq 1$ for every $r \in \mathbb{R}^n$.

Let $(\bar{\psi}, \bar{\pi})$ be a cut-generating pair. Define two new functions $\phi_1(r) = 1 - \bar{\psi}(s^* - r)$ and $\phi_2(r) = 1 - \bar{\pi}(s^* - r)$. Let $\mathcal{I}$ be the set of cut generating functions (ψ, π) such that $\psi \leq \bar{\psi}$ and $\pi \leq \bar{\pi}$. Note that any element $(\psi, \pi) \in \mathcal{I}$ satisfies $\psi(r) \geq 1 - \psi(s^* - r) \geq 1 - \bar{\psi}(s^* - r) = \phi_1(r)$ and similarly, $\pi(r) \geq \phi_2(r)$.

We show that every chain in $\mathcal{I}$ has a lower bound in $\mathcal{I}$. Then by Zorn's lemma, $\mathcal{I}$ will contain a minimal element, proving the result.

Consider any chain $\mathcal{C}$ in $\mathcal{I}$. For any element $(\psi, \pi) \in \mathcal{C}$, we know that $\psi \geq \phi_1$ and $\pi \geq \phi_2$. Therefore, $\tilde{\psi}(r) := \inf\{\psi(r) : (\psi, \pi) \in \mathcal{C}\}$ and $\tilde{\pi}(r) := \inf\{\pi(r) : (\psi, \pi) \in \mathcal{C}\}$ are well-defined real-valued functions. It is easy to verify that $(\tilde{\psi}, \tilde{\pi})$ are cut-generating functions, and are therefore in $\mathcal{I}$. This completes the proof that each chain has a lower bound in $\mathcal{I}$. $\square$

There are other notions of strength for cut-generating pairs (see the survey [28] or the manuscript [94] for a detailed discussion on the hierarchy of strength), and there are levels of strength that exceed minimal. However, we only require these in Chapter 4 and postpone discussion of them until that time.

2.2.3 An infinite dimensional interpretation

Let M_S be a mixed-integer model. Since cut-generating pairs produce cuts for all mixed-integer sets in M_S, one needs to analyze the model itself when developing these pairs. This leads to an alternative, albeit infinite dimensional, interpretation of mixed-integer sets and cut-generating functions.

Consider a mixed-integer set $M_S(R, P)$ and take $(s, y) \in M_S(R, P)$. The vector $s \in \mathbb{R}^k_+$ can be thought of as a function $s : \mathbb{R}^n \to \mathbb{R}_+$ where $s(r_i) = s_i$ for any $r_i \in \mathbb{R}^n$ that is a column of R, and $s(r) = 0$ for any $r \in \mathbb{R}^n$ that is not a column of R. With this interpretation,

s is a finitely-supported function from $\mathbb{R}^n$ to $\mathbb{R}_+$, that is $s \in (\mathbb{R}_+)^{(\mathbb{R}^n)}$. Similarly, the vector y can be thought of as a function in $(\mathbb{Z}_+)^{(\mathbb{R}^n)}$. Conversely, any pair of functions $(\overline{s}, \overline{y})$ in $(\mathbb{R})^{(\mathbb{R}^n)} \times (\mathbb{R})^{(\mathbb{R}^n)}$ such that $\overline{s} \in (\mathbb{R}_+)^{(\mathbb{R}^n)}$, $\overline{y} \in (\mathbb{Z}_+)^{(\mathbb{R}^n)}$ and

$$\sum_{r \in \mathbb{R}^n} r\overline{s}(r) + \sum_{p \in \mathbb{R}^n} p\overline{y}(p) \in S,$$

corresponds to a point in the mixed integer set $M_S(\text{supp}(s), \text{supp}(y))$ by restricting $(\overline{s}, \overline{y})$ to the finite set $(\text{supp}(s), \text{supp}(y))$. Note that these summations converge since $\overline{s}$ and $\overline{y}$ are assumed to have finite support. In this way, the infinite dimensional set

$$\left\{(s, y) \in (\mathbb{R})^{(\mathbb{R}^n)} \times (\mathbb{R})^{(\mathbb{R}^n)} : \sum_{r \in \mathbb{R}^n} rs_r + \sum_{p \in \mathbb{R}^n} py_p \in S,\ s \in (\mathbb{R}_+)^{(\mathbb{R}^n)},\ y \in (\mathbb{Z}_+)^{(\mathbb{R}^n)}\right\} \quad (2.5)$$

contains all mixed-integer sets in M_S as faces. More specifically, for $R \in \mathbb{R}^{n\times k}$ and $P \in \mathbb{R}^{n\times l}$, intersecting (2.5) and the hyperplane $\left\{(s, y) \in (\mathbb{R})^{(\mathbb{R}^n)} \times (\mathbb{R})^{(\mathbb{R}^n)} : \text{supp}(s) = R, \text{supp}(y) = P\right\}$ yields the mixed-integer set $M_S(R, P)$. Therefore (2.5) represents M_S. Figure 2.1 illustrates the idea of this infinite interpretation, typically called the *infinite relaxation*.

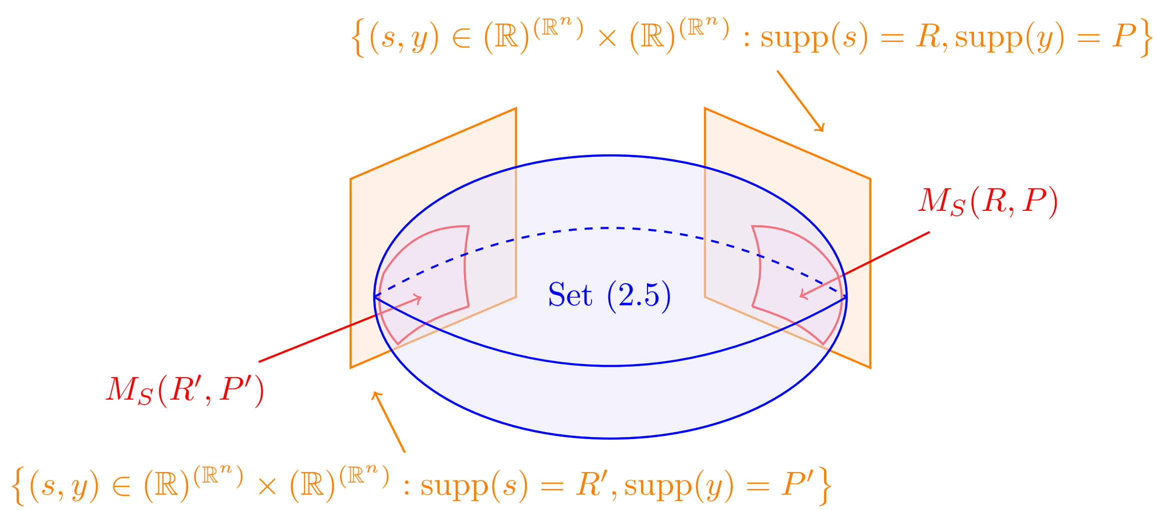


Figure 2.1: The faces of the infinite dimensional set are mixed-integer sets.

In the infinite space $(\mathbb{R})^{(\mathbb{R}^n)} \times (\mathbb{R})^{(\mathbb{R}^n)}$, a cut-generating pair (ψ, π) for S defines the hyperplane

$$\left\{(s,y)\in(\mathbb{R})^{(\mathbb{R}^n)}\times(\mathbb{R})^{(\mathbb{R}^n)}:\sum_{r\in\mathbb{R}^n}\psi(r)s_r+\sum_{p\in\mathbb{R}^n}\pi(p)y_p=1\right\} \tag{2.6}$$

that strictly separates the set (2.5) from the zero function in $(\mathbb{R}^n)^{(\mathbb{R})}\times(\mathbb{R}^n)^{(\mathbb{R})}$.

A benefit of this infinite dimensional interpretation is that it allows one to study all mixed-integer sets for a model M_S at the same time. Gomory and Johnson's initial work [66, 67, 70] uses this infinite dimensional interpretation, and it has inspired a long line of work in this direction (see [28] or [90] for a detailed survey). In this thesis, we will transition between the original notion of mixed-integer sets and this infinite dimensional interpretation. Our choice of interpretation will depend on the model under consideration and will remain consistent with current work in the literature.

2.3 The submodel C_S

A mixed-integer set $M_S(R,P)$ in the model M_S consists of tuples (s,y), where s are continuous variables and y are integral. Work has been done in studying submodels of M_S that consist of only continuous or integral variables, but not both. In this section, we review the model C_S which considers the continuous case. In Section 2.3, we review the integer setting.

Definition 4 (Mixed-integer set $C_S(R)$ and mixed-integer model C_S)**.** *Let S be a nonempty, closed subset of $\mathbb{R}^n$ with $0\notin S$. The mixed-integer model C_S is the collection of all mixed-integer sets of the form*

$$C_S(R):=\left\{s\in\mathbb{R}^k_+\ :\ Rs\in S\right\}, \tag{2.7}$$

where $k\in\mathbb{Z}_+, n\in\mathbb{N}$, and $R\in\mathbb{R}^{n\times k}$.

The article by Conforti et. al [40] provides a well-written account of the model C_S in this generality. For more structured S, see also [1, 18, 19, 36, 44, 46, 55].

Each mixed-integer set $C_S(R)$ can be thought of as a mixed-integer set in M_S of the form $C_S(R)=M_S(R,0)$. In this way, C_S is a submodel of M_S. Since there are only continuous variables 's' in the mixed-integer sets $C_S(R)$, cut-generating pairs for C_S reduce to just a single cut-generating function.

Definition 5 (Cut-generating function for C_S)**.** *A cut-generating function for C_S is a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that for every mixed-integer set $C_S(R)$ in C_S and each $s \in C_S(R)$, it follows that*

$$\sum_{i=1}^{k} \psi(r_i)s_i \geq 1. \tag{2.8}$$

That is, ψ produces a valid inequality for each mixed-integer set in C_S.

Just like with cut-generating pairs, there exists a partial ordering on the strength of cut-generating functions.

Definition 6 (Dominate and minimal cut-generating functions for C_S)**.** *Let ψ and ψ' be cut-generating functions for C_S. We say that ψ dominates ψ' if $\psi \leq \psi'$. If ψ is not dominated by any other cut-generating function, then it is minimal.*

Additionally, the results of Propositions 1 and 2 carry over to cut-generating functions. In particular, every cut-generating function is dominated by a minimal one. When restricting attention to C_S (as opposed to M_S), these minimal functions are also sublinear.

Proposition 3 (Basu et al. [19], Conforti et. al [40])**.** *Every cut-generating function for C_S is dominated by a minimal, sublinear cut-generating function.*

Cut-generating functions for C_S have been thoroughly studied, partly because of their relation with S-free sets. It turns out that for many specially structured S, we obtain closed-form formulas for minimal cut-generating functions using representations of S-free convex sets [15, 19, 40, 55].

2.3.1 S-free sets

This section quickly reviews S-free sets and their geometric structure.

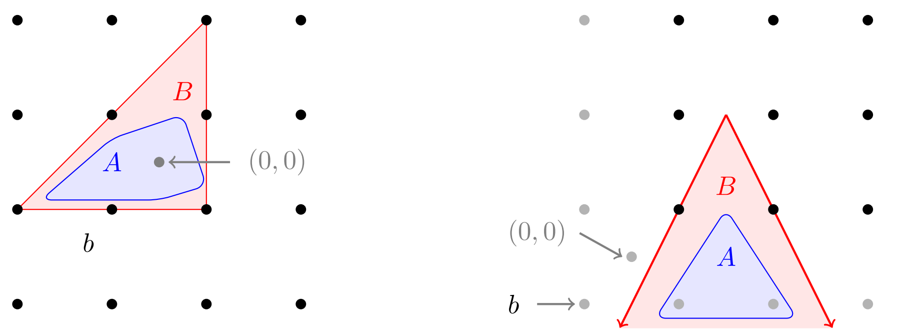
Definition 7 (S-free sets, Maximal S-free sets, Lattice-free sets)**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed set with $0 \notin S$. A convex set $B \subseteq \mathbb{R}^n$ is called S-free if $\operatorname{int}(B) \cap S = \emptyset$. A S-free set*

B *is* maximal *if there are no other S-free sets B' such that $B \subsetneq B'$. If S is a translated lattice, then a S-free set is called lattice-free.*

The following provides a few examples of S-free sets. Recall a 0-neighborhood $A \subseteq \mathbb{R}^n$ has $0 \in \text{int}(A)$.

Example 6 (Example of S-free sets)**.**

Let $b = \left(-\frac{1}{2}, -\frac{1}{2}\right) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$. The following figure exhibits S-free sets when (a) $S = b + \mathbb{Z}^2$ and (b) $S = (b + \mathbb{Z}^2) \cap \mathbb{R}^2_+$. In both (a) and (b), A and B are S-free sets with B maximal. In (a), both A and B are also 0*-neighborhoods, while in (b), neither A nor B are* 0*-neighborhoods.*



(a) Example of $b + \mathbb{Z}^2$-free set

(b) Example of $(b + \mathbb{Z}^2) \cap \mathbb{R}^2_+$-free set

Figure 2.2: Examples of S-free sets.

$\square$

From Example 6, one may conjecture that S-free sets are always contained within maximal S-free sets. An application of Zorn's Lemma shows this is indeed the case. The following is in [19] for the specialized case of $S = b + \mathbb{Z}^n$, and [40] for more general S.

Proposition 4 (Basu et al. [19], Conforti et al. [40])**.** *Every S-free set is contained in a maximal S-free set.*

In later chapters of this thesis, we typically focus on the situation when S is a polyhedrally-truncated affine lattice, i.e. $S = (b + \Lambda) \cap C$ with $b \in \mathbb{R}^n \setminus \Lambda$, $C \subseteq \mathbb{R}^n$ a convex set, and $\text{conv}(S)$ a polyhedron. For such an S, maximal S-free convex sets have a lot of structure [19, 55, 85].

Theorem 3 (Basu et. al [19], Dey and Wolsey [55], Lovasz [85]). *Suppose that* $S = (b + \Lambda) \cap C$ *for* $\Lambda \subseteq \mathbb{R}^n$ *a lattice,* $b \in \mathbb{R}^n \setminus \Lambda$*, and* $C \subseteq \mathbb{R}^n$ *a convex set so that* $\dim(S) = n$ *and* $\operatorname{conv}(S)$ *is a polyhedron. A set* $B \subseteq \mathbb{R}^n$ *is a maximal* S*-free set if and only if one of the following holds*

(a) B *is a polyhedron so that* $\operatorname{int}(B \cap \operatorname{conv}(S)) \neq \emptyset$, $\operatorname{int}(B) \cap S = \emptyset$*, and there is a point of* S *in the relative interior of each facet of* B*.*

(b) B *is a half-space of* $\mathbb{R}^n$ *so that* $\operatorname{int}(B \cap \operatorname{conv}(S)) = \emptyset$ *and* $\operatorname{bd}(B)$ *is a supporting hyperplane of* $\operatorname{conv}(S)$*.*

(c) B *is a hyperplane of* $\mathbb{R}^n$ *so that* $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ *is irrational.*

For the setting of $C = \mathbb{R}^n$ and $b+\mathbb{Z}^n$, we can say more. The following was shown in [18].

Proposition 5 (Basu et al. [18]). *Let* $B \subseteq \mathbb{R}^n$ *be a maximal* $(b + \mathbb{Z}^n)$*-free set. Then* $\operatorname{int}(\operatorname{rec}(B)) = \emptyset$*.*

Doignon, Bell, and Scarf showed that maximal $(b + \mathbb{Z}^n)$-free sets in $\mathbb{R}^n$ have at most 2^n facets [31, 58, 91]. Using Theorem 3, their proof can be extended to general S-free sets when $S = (b + \mathbb{Z}^n) \cap P$ for a polyhedron P. The following was first proved by Dey and Moran [87] and then extended to more general S by Averkov [4].

Proposition 6 (Averkov [4], Dey and Moran [87]). *Let* S *be as in Theorem 3. Suppose that* $B \subseteq \mathbb{R}^n$ *is a maximal* S*-free set and* $\dim(B) = n$*. Then* B *is a polyhedron with at most* 2^n *facets.*

If we further assume that $S = (b+\mathbb{Z}^n) \cap P$ contains the translated unit cube $b+w+\{0,1\}^n$, where $w \in \mathbb{Z}^n$, then the bound 2^n in Proposition 6 is tight. Moreover, there is a maximal S-free set with i facets for every i between 2 and 2^n .

Lemma 1. *Let* S *be as in Theorem 3. Assume that* S *contains the set* $b + w + \{0, 1\}^n$ *for* $w \in \mathbb{Z}^n$*. Then there exists a maximal* S*-free set with exactly* i *facets for each* $2 \leq i \leq 2^n$*.*

Proof. We begin with the following claim.

Claim 1. *Let $i, n \in \mathbb{N}$ so that $2 \leq i \leq 2^n$. There exists a family $\mathcal{G}$ of i disjoint faces of the unit cube $b + w + [0,1]^n$ that cover all of its vertices $b + w + \{0,1\}^n$.*

Proof of Claim 1. We provide a constructive argument for the existence of $\mathcal{G}$. Begin with $\mathcal{G} = \{F_0, F_1\}$, where $F_0 = \{x \in b + w + [0,1]^n : x_n = b_n + w_n\}$ and $F_1 = \{x \in [0,1]^n : x_n = b_n + w_n + 1\}$. Iteratively update $\mathcal{G}$ as follows.

If $|\mathcal{G}| = i$, then we are done. If $|\mathcal{G}| < i \leq 2^n$ then choose some $H \in \mathcal{G}$ such that H is not 0-dimensional. Since H is not a vertex, there exists some $k \in [n]$ so that neither $H_0 = \{x \in H : x_k = b_k + w_k\}$ nor $H_1 = \{x \in H : x_k = b_k + w_k + 1\}$ are empty. Recursively update $\mathcal{G}$ to be $\mathcal{G} \setminus \{H\} \cup \{H_0, H_1\}$.

After $i - 2$ steps of this recursion, $|\mathcal{G}| = i$. □

From Claim 1, there exist disjoint faces $F_1, \dots, F_i$ of the unit cube $b + w + [0,1]^n$ that cover the points $b + w + \{0,1\}^n$. For $j \in [i]$, let $u_j \in \mathbb{R}^n$ and $c_j \in \mathbb{R}$ be such that the inequality $u_j \cdot x \leq c_j$ is valid for $b + w + [0,1]^n$ and defines the face F_j. Then we claim that $B = \{x \in \mathbb{R}^n : u_j \cdot x \leq c_j,\ j \in [i]\}$ is S-free with i facets. Moreover, by construction, $v \in b + w + \{0,1\}^n$ satisfies $u_j \cdot v = c_j$ if and only if $v \in F_j$. Because the F_j were chosen to be disjoint, the vertices of F_j are contained in the relative interior of the facet of B corresponding to the inequality $u_j \cdot x \leq c_j$. It remains to show that B is lattice-free — indeed, the result then follows from Theorem 3.

Suppose $\operatorname{int}(B) \cap S = \operatorname{int}(B) \cap ((b + \mathbb{Z}^n) \cap P) \neq \emptyset$. Let $z \in \operatorname{int}(B) \cap S$ be such that it is the closest to $b + w + [0,1]^n$ amongst all points in $\operatorname{int}(B) \cap S$. This implies that $\{z - b - w\} \cup \{0,1\}^n$ are in $\mathbb{Z}^n$-convex position, i.e., $\operatorname{conv}(\{z - b - w\} \cup \{0,1\}^n) \cap \mathbb{Z}^n = \operatorname{vert}(\operatorname{conv}(\{z - b - w\} \cup \{0,1\}^n) \cap \mathbb{Z}^n) = \{z - b - w\} \cup \{0,1\}^n$. This contradicts a classical theorem due to Doignon-Bell-Scarf which states that one can have at most 2^n integer points in $\mathbb{Z}^n$-convex position [75].

□

2.3.2 Representations of S-free sets

By Proposition 3, looking for cut-generating functions for C_S only requires searching through sublinear functions. In this search, a certain class of sublinear functions that will prove useful are gauge functions and representations.

Let $A \subseteq \mathbb{R}^n$ be a closed and convex 0-neighborhood, that is $0 \in \operatorname{int}(A)$. Then the *gauge function* of A is the function

$$\psi_A(x) := \inf \{\lambda > 0 \,:\, x \in \lambda A\}\,. \tag{2.9}$$

The following is a list of properties satisfied by gauge functions. For more, see Hiriart-Urruty and Lemarechal [84].

Proposition 7 (Properties of gauge functions)**.** *Let $A \subseteq \mathbb{R}^n$ be a convex 0-neighborhood. The gauge function ψ_A of A satisfies the following properties:*

(a) $\psi_A(x) \geq 0$ for all $x \in \mathbb{R}^n$,

(b) $\psi_A(x_1 + x_2) \leq \psi_A(x_1) + \psi_A(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$,

(c) $\lambda\psi_A(x) = \psi_A(\lambda x)$ for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$,

(d) $\lambda\psi_A(x) = \psi_{\frac{1}{\lambda}A}(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$,

(e) for $x \in \mathbb{R}^n$, $\psi_A(x) = 0$ if and only if $x \in \operatorname{rec}(A)$,

(f) $A = \{x \in \mathbb{R}^n : \psi_A(x) \leq 1\}$.

Property (b) in Proposition 7 is called *subadditivity* and Property (c) is called *positive homogeneity.* A function is *sublinear* if it is both subadditive and positively homogeneous. Proposition 7 states that gauge functions are sublinear.

A generalized version of a gauge function is a representation. A function $\gamma_A : \mathbb{R}^n \to \mathbb{R}$ is called a *representation of A* if it is sublinear and $A = \{r \in \mathbb{R}^n : \gamma_A(r) \leq 1\}$. A convex neighborhood may have several representations, and from Proposition 7, the gauge is one of them. Representations of convex 0-neighborhoods was the main topic of study in [15, 40], where it was shown that there is always a smallest representation.

Proposition 8 (Basu et al. [15], Conforti et al. [40])**.** *Let $A \subseteq \mathbb{R}^n$ be a convex 0-neighborhood. Then A has a pointwise smallest representation γ_A^*, that is $\gamma_A^* \leq \gamma_A$ for all representations γ_A of A.*

It was also shown that all representations of some S-free 0-neighborhood B agree outside the interior of the recession cone of B.

Proposition 9 (Conforti et al. [40])**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed set with $0 \notin S$. Let $B \subseteq \mathbb{R}^n$ be a S-free 0-neighborhood and γ, γ' be representations of B. Then $\gamma(x) = \gamma'(x)$ for all $x \in \mathbb{R}^n \setminus (\operatorname{int}(\operatorname{rec}(B)))$. In particular, if $S = b + \mathbb{Z}^n$ then the gauge function is the unique representation of B.*

Tying representations back to cut-generating functions, the following result shows that representations of S-free 0-neighborhoods are precisely the cut-generating functions for C_S. See also [15, 19, 55] for more structured choices of S.

Proposition 10 (Conforti et al. [40])**.** *Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be sublinear and define $B = \{x \in \mathbb{R}^n : \gamma(x) \leq 1\}$. Note that γ is a representation of B, and B is a 0-neighborhood. Then γ is a cut-generating function for C_S if and only if B is S-free.*

2.3.3 Minimal cut-generating functions for C_S

Proposition 10 shows that S-free sets produce cut-generating functions for C_S. However, which S-free sets produce minimal cut-generating functions? Conforti et al. show that one sufficient condition is to consider representations of maximal S-free sets. See also [15].

Proposition 11 (Conforti et al. [40])**.** *Let B be a maximal S-free 0-neighborhood. Then the smallest representation γ_B^* is a minimal cut-generating function for C_S.*

The following example provides a bit of intuition behind how maximal S-free sets create stronger cutting planes.

Example 7 (The idea of maximal S-free sets)**.** *Let $b = \mathbb{R}^n \setminus \mathbb{Z}^n$ and $S = b + \mathbb{Z}^n$. Suppose that A and B are S-free 0 neighborhoods so that A is strictly contained in B. From Proposition 9, the gauge functions ψ_A and ψ_B are the unique representations of A and B, respectively. How do these gauge functions compare?*

Let $x \in \mathbb{R}^n$ and $\lambda > 0$ such that $x \in \lambda A$. Since $A \subsetneq B$, it follows that $x \in \lambda B$. Therefore

$$\psi_B(x) = \inf\{\lambda > 0 : x \in \lambda B\} \leq \inf\{\lambda > 0 : x \in \lambda A\} = \psi_A(x)$$

for all $x \in \mathbb{R}^n$. Furthermore, since $A \subsetneq B$, there is some $x^* \in B \setminus A$. From Proposition 7(f), $\psi_B(x^*) \leq 1 < \psi_A(x^*)$. Hence $\psi_B \neq \psi_A$ and $\psi_B \leq \psi_A$. Therefore ψ_B dominates ψ_A.

If A, B are S-free 0-neighborhoods so that $A \not\subseteq B$ and $B \not\subseteq A$, then the previous discussion implies the corresponding gauge functions can not be ordered via pointwise comparison. This, together with the previous derivation, implies that the gauge functions of maximal S-free 0-neighborhoods are minimal cut-generating functions. □

Depending on the structure of S, the converse of Proposition 11 might not hold. That is, for certain choices of S, there are minimal cut-generating functions that do not correspond to representations of *maximal* S-free 0-neighborhoods (see [40]). However, the converse is true when S is of the form $S = b + \mathbb{Z}^n$ for $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. In this case, Proposition 5 states that $\text{int}(\text{rec}(B)) = \emptyset$ for a maximal S-free set B. Using Proposition 9, this implies that the gauge ψ_B of B is its only representation, and so ψ_B is a minimal cut-generating function for C_S.

Proposition 12 (Basu et al. [15], Borozan and Cornuéjols [36])**.** *A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function for $C_{b+\mathbb{Z}^n}$ if and only if it is the gauge function of some maximal $(b + \mathbb{Z}^n)$-free 0-neighborhood.*

Let B be a maximal $(b+\mathbb{Z}^n)$-free 0-neighborhood. From Theorem 3, B is a polyhedron that can be given by

$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq \alpha_i \;\; i \in [m]\} \tag{2.10}$$

for some $m \in \mathbb{N}$. Since B is a 0-neighborhood, $0 \in \text{int}(B)$. Therefore, each b_i must be strictly positive, and after a proper scaling we may assume $\alpha_i = 1$ for each $i \in [m]$. The gauge function of B can then be written as

$$\begin{aligned}
\psi_B(x) &= \inf\{\lambda > 0 : x \in \lambda B\} \\
&= \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in B\right\} \\
&= \inf\left\{\lambda > 0 : a_i \cdot \left(\frac{x}{\lambda}\right) \leq 1 \text{ for all } i \in [m]\right\} \\
&= \max_{i \in [m]} a_i \cdot x.
\end{aligned} \tag{2.11}$$

Thus in order to construct minimal cut-generating functions for $C_{b+\mathbb{Z}^n}$, it suffices to identify a maximal lattice-free polyhedron and evaluate (2.11). It should be noted that while (2.11) is a simple expression, computing the gauge in this form might require 2^n evaluations (see Lemma 1).

When $S = (b+\mathbb{Z}^n) \cap P$ for $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ a rational polyhedron, it was shown in [19, 55] that maximal S-free 0-neighborhoods also have a unique minimal representation.

Proposition 13 (Basu et al. [19], Dey and Wolsey [55]). *Let $S = (b + \mathbb{Z}^n) \cap C$, where $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$, $C \subseteq \mathbb{R}^n$, and $\mathrm{conv}(S)$ is a polyhedron. Then a function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function for C_S if and only if it there is a maximal S-free 0-neighborhood B so that*

$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1 \;\; i \in [m]\}$$

and $\psi(r) = \max_{i \in [m]} a_i \cdot r$.

2.4 The submodel I_S

In this section, we review the submodel I_S of M_S which consists of mixed-integer sets with integer variables.

Definition 8 (Mixed-integer set $I_S(P)$ and mixed-integer model I_S). *Let S be a nonempty, closed subset of $\mathbb{R}^n$ with $0 \notin S$. The mixed-integer model I_S is defined by the mixed-integer sets*

$$I_S(P) := \left\{ y \in \mathbb{Z}^k_+ \; : \; Py \in S \right\}, \tag{2.12}$$

where $l \in \mathbb{Z}_+, n \in \mathbb{N}$, and $P \in \mathbb{R}^{n \times l}$.

Since only the integer variables 'y' appear in the sets $I_S(P)$, cut-generating pairs reduce to a single cut-generating function.

Definition 9 (Cut-generating function π for I_S). *A cut-generating function for I_S is a function $\pi : \mathbb{R}^n \to \mathbb{R}$ such that for every mixed-integer set $I_S(R)$ in I_S and each $y \in I_S(P)$, it follows that*

$$\sum_{i=1}^{l} \pi(p_i) y_i \geq 1. \tag{2.13}$$

That is, π produces a valid inequality for each mixed-integer set in I_S.

Just as with C_S, the model I_S only uses a single cut-generating function instead of a pair. We reserve ψ for the C_S model and π for I_S. Since both ψ and π serve the same purpose but in different models, we use the term cut-generating function to refer to both. The infinite dimensional representation from Section 2.2.3 can be used in this model. Also, the notions of dominating and minimal cut-generating functions are defined for π as they are in Definition 6.

Definition 10 (Dominate and minimal cut-generating functions for I_S)**.** *Let π and π' be cut-generating functions for I_S. We say that π dominates π' if $\pi \leq \pi'$. If π is not dominated by any other cut-generating function, then it is minimal.*

The model I_S has been thoroughly examined for various choices of S; see [28, 90] for an extensive overview of the case $S = b + \mathbb{Z}^n$ and [94] for the more general case. For the purpose of this thesis, we are only concerned with I_S when $S = b + \mathbb{Z}^n$ for $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Thus for the remainder of Section 2.4, we consider the model I_S when $S = b + \mathbb{Z}^n$. Many of the background results that we present have been generalized thanks to Cornuéjols and Yıldız [94].

2.4.1 The infinite group problem

The model I_S is typically analyzed using the infinite dimensional representation presented in Section 2.2.3. The analogue of the set (2.5) in the context of I_S is called the n-row infinite group problem.

Definition 11 (n-row infinite group problem and the infinite group problem)**.** *Let $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The n-row infinite group problem is*

$$R_b(\mathbb{R}^n, \mathbb{Z}^n) := \left\{ y \in (\mathbb{Z}_+)^{(\mathbb{R}^n)} : \sum_{p \in \mathbb{R}^n} p y(p) \in b + \mathbb{Z}^n \right\}. \tag{2.14}$$

For $n = 1$, $R_b(\mathbb{R}, \mathbb{Z})$ is referred to as the infinite group problem.

Note 2. *For a brief explanation of the term 'infinite group problem', we look back to the initial work by Gomory [65]. Note that $\mathbb{Z}^n$ is an additive subgroup of the additive group $\mathbb{R}^n$.*

Hence, we can rewrite (2.14) *in the more general form*

$$R_b(G,H) := \left\{ y \in (\mathbb{Z}_+)^{(G)} : \sum_{p \in G} p y(p) \in b + H \right\}, \tag{2.15}$$

for an additive group G, a subgroup $H \subseteq G$, and some $b \notin H$. Because of this 'group representation', (2.15) *is referred to as the group problem. The infinite group problem* (2.14) *is a group problem with the underlying group $G = \mathbb{R}^n$. The term infinite follows since $\mathbb{R}^n$ has infinite order. Although many groups G of infinite order can be placed in* (2.15)*, using $G = \mathbb{R}^n$ and $H = \mathbb{Z}^n$ is the most commonly studied, thus earning it the name of the infinite group problem. See Basu et al. [28] or Dey and Richard [90] for a comprehensive survey on the infinite group problem.* □

Just as with the model C_S, we aim to build minimal cut-generating functions π for the infinite group problem. Gomory and Johnson were able to identify necessary and sufficient conditions for nonnegative minimal cut-generating functions. A function $\theta \colon \mathbb{R}^n \to \mathbb{R}$ satisfies the *symmetry condition* if $\theta(x) + \theta(b - x) = 1$ for all $x \in \mathbb{R}^n$. The function θ is *periodic* with respect to $\mathbb{Z}^n$ if $\theta(x) = \theta(x + z)$ for all $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$.

Theorem 4 (Gomory and Johnson [66])**.** *A function $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is a minimal valid function for $R_b(\mathbb{R}^n, \mathbb{Z}^n)$ if and only if $\pi(z) = 0$, π is periodic with respect to $\mathbb{Z}^n$, π is subadditive, and π satisfies the symmetry condition.*

2.5 Connecting C_S and I_S to M_S

Stepping back to the mixed-integer model M_S, let $S \subseteq \mathbb{R}^n$ to be a nonempty, closed set with $0 \notin S$. Both models C_S and I_S are submodels of (2.1). As a consequence, any cut-generating pair (ψ, π) for M_S creates both a cut-generating function ψ for C_S and a cut-generating function π for I_S. A question of interest is if one can work in the other direction — can the ideas in Sections 2.3 and 2.4 for making cut-generating *functions* be used to make cut-generating *pairs*? Like with cut-generating functions, can we also focus on identifying minimal cut-generating pairs? This section reviews the work done in addressing these questions.

Fortunately, a characterization of minimal cut-generating pairs exists.

Theorem 5 (Cornuéjols and Yıldız [94])**.** *Let (ψ, π) be a cut-generating pair for M_S with $\psi : \mathbb{R}^n \to \mathbb{R}$ and $\pi : \mathbb{R}^n \to \mathbb{R}$. Then (ψ, π) is minimal if and only if π is minimal for I_S and ψ is of the form*

$$\limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon x)}{\epsilon}. \tag{2.16}$$

Theorem 5 was originally proved by Johnson [79] for $S = b + \mathbb{Z}^n$ under the assumption that π is nonnegative. This theorem outlines a procedure for creating minimal cut-generating pairs. First generate a minimal π for I_S and then use the construction in (2.16). While this approach has the benefit of producing all cut-generating pairs for M_S, it comes with its limitations. In particular, identifying minimal π for I_S is difficult and still generates a myriad of research (recently Basu et. al [28] surveyed these results); we confront this problem in Chapter 4.

For some choices of S, an alternative approach to creating minimal cut-generating pairs from functions exists. This alternative approach starts by considering minimal cut-generating functions for C_S, as opposed to I_S like in Theorem 5. In order to introduce this approach, which is referred to as 'lifting' from [56], consider the following observation.

Observation 1. *Let $S \subseteq \mathbb{R}^n$ be nonempty, closed with $0 \notin S$. Let ψ be a cut-generating function for C_S. Take a mixed-integer set $M_S(R, P)$ in the model M_S and $(s, y) \in M_S(R, P)$. Note that*

$$\sum_{i=1}^{k} \psi(r_i)s_i + \sum_{i=1}^{l} \psi(p_i)y_i = \sum_{r_i \notin P} \psi(r_i)s_i + \sum_{r_i = p_j} \psi(r_i)(s_i + y_j) + \sum_{p_i \notin R} \psi(p_i)y_i \geq 1,$$

where $r_i \notin P$ indicates that r_i is not a column of P. The inequality follows since $Rs + Py \in S$. Hence (ψ, ψ) is a cut-generating pair for M_S.

From Observation 1, (ψ, ψ) is a cut-generating pair for M_S when ψ is a cut-generating function. In general, any π that can be attached to ψ to create a cut-generating pair is called a lifting.

Definition 12 (Lifting)**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed set with $0 \notin S$. Let $\psi : \mathbb{R}^n \to \mathbb{R}$*

be a cut-generating function for C_S. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a lifting of ψ if (ψ, π) is a cut-generating pair for M_S.

Another lifting is called the *trivial lifting of ψ* defined by

$$\pi^*(x) = \inf_{w \in W_S} \psi(x + w), \tag{2.17}$$

where

$$W_S := \{w \in \mathbb{R}^n : s + \lambda w \in S \ , \forall s \in S, \forall \lambda \in \mathbb{Z}\}. \tag{2.18}$$

The trivial lifting is indeed a lifting of ψ [13, 66].

Proposition 14. *The trivial lifting* (2.17) *is a lifting of ψ.*

Proof. Let $M_S(R, P)$ be a mixed-integer set in M_S and take $(s, y) \in M_S(R, P)$. From the definition of a lifting, it is sufficient so show that

$$\sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{l} \pi^*(p_j)y_j \geq 1.$$

To this end, let $W \in \mathbb{R}^{n \times k}$ with columns w_i, $i \in [k]$, contained in W_S. Using $(s, y) \in M_S(R, P)$ and the definition of W, it follows that $Rs + Py + Wy \in S + Wy \subseteq S$. Hence $(s, y) \in M_S(R, P + W)$. As ψ is a cut-generating function for M_S we have

$$\sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{l} \pi^*(p_j + w_j)y_j \geq 1.$$

Therefore, as W was arbitrarily chosen, we see

$$\begin{aligned}
\sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{l} \pi^*(p_j)y_j &= \sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{l} \inf_{w_j \in W_S} \psi(p_j + w_j)y_j \\
&= \sum_{i=1}^{k} \psi(r_i)s_i + \inf_{W \subseteq W_S} \sum_{j=1}^{l} \psi(p_j + w_j)y_j \\
&= \inf_{W \subseteq W_S} \left\{ \sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{l} \psi(p_j + w_j)y_j \right\} \geq 1.
\end{aligned}$$

$\square$

Observe that $\pi^* \leq \psi$ and so as a cut-generating function for I_S, π^* dominates ψ. For a given cut-generating function ψ for C_S, the set of all liftings of ψ is partially ordered by pointwise dominance and one can define minimal liftings.

Definition 13 (Minimal lifting)**.** *Let ψ be a cut-generating function for C_S. A lifting π of ψ is minimal if there is no other lifting π' of ψ satisfying $\pi' \leq \pi$.*

Just as with minimal cut-generating functions and minimal cut-generating pairs, there always exists a minimal lifting.

Proposition 15. *Let ψ be a cut-generating function for C_S. Then every lifting of ψ is dominated by a minimal lifting of ψ.*

Proof. Fix $s^* \in S$ which is nonempty. For any lifting π of ψ, we must have $\psi(s^*-r)+\pi(r) \geq 1$ and therefore, if we define $\phi(r) = 1 - \psi(s^* - r)$, we have that $\pi(r) \geq \phi(r)$. The proof idea of Proposition 2 can again be used to show that every lifting is dominated by a minimal lifting.

□

Proposition 15 only gives a 'partial minimality' in that it does not mention minimality of ψ. If ψ is minimal for C_S and π is a minimal lifting of ψ, then (ψ, π) is minimal for M_S. However, ψ may have multiple minimal liftings, so how is one identified? This question leads to the notion of a lifting region.

Definition 14 (Lifting region)**.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a cut-generating function for C_S. The* lifting region *R_ψ of ψ is defined as:*

$$R_\psi := \{r \in \mathbb{R}^n \colon \psi(r) = \pi(r) \textit{ for all minimal liftings } \pi \textit{ of } \psi\}. \tag{2.19}$$

The lifting region of a function ψ [5, 19, 30, 56] is one of the main topics of focus in Chapters 5 and 6 of this thesis. By definition, all minimal lifting agree on R_ψ. The following result extends this 'uniqueness of minimal lifting' to a larger region.

Proposition 16. *Let $S \subseteq \mathbb{R}^n \setminus \{0\}$ and let ψ be a cut-generating function for S. Every minimal lifting of ψ is periodic along W_S.*

The proof of Proposition 16 is left for Appendix A.3. If $R_\psi + W_S = \mathbb{R}^n$, then we say that ψ has the *covering property.* Since every minimal lifting is periodic along W_S, if ψ has the covering property then it has a unique minimal lifting. Indeed, let π_1, π_2 be liftings of ψ and take $x \in \mathbb{R}^n$. If ψ has the covering property then $x = r + w$ for $r \in R_\psi$ and $w \in W_S$. Using the perioidicity of π_1 and π_2 along W_S, it follows that

$$\pi_1(x) = \pi_1(r + w) = \pi_1(r) = \pi_2(r) = \pi_2(r + w) = \pi_2(x).$$

Hence $\pi_1 = \pi_2$ and ψ has a unique minimal lifting.

It was shown in [17] that for the special case when S is a translated lattice, the covering property is a characterization for a unique minimal lifting, i.e., ψ has a unique minimal lifting if and only if $R_\psi + W_S = \mathbb{R}^n$. Note that when $S = b + \mathbb{Z}^n$, then $W_S = \mathbb{Z}^n$. In this situation, the question of whether ψ has a unique minimal lifting or not is equivalent to the geometric question of whether $R_\psi + \mathbb{Z}^n = \mathbb{R}^n$, i.e., whether R_ψ covers $\mathbb{R}^n$ by integer translates.

If ψ is a minimal cut-generating function with the covering property, can we identify its unique minimal lifting π? It turns out that this unique minimal lifting is just the trivial lifting.

Proposition 17. *Let $S \subseteq \mathbb{R}^n$ be nonempty, closed with $0 \notin S$. Let ψ be a minimal cut-generating function for C_S with the covering property. Then π^* is the unique minimal lifting of ψ.*

Proof. From Proposition 14, π^* is a lifting of ψ. Take any minimal lifting π of ψ. Consider any $r \in \mathbb{R}^n$ and let $w \in W_S$ such that $r + w \in R_\psi$. By Proposition 16, $\pi(r) = \pi(r + w) = \psi(r + w) \geq \pi^*(r)$. This implies that $\pi(r) \geq \pi^*(r)$ for all $r \in \mathbb{R}^n$. However, since π is minimal and π^* is a lifting, $\pi = \pi^*$. □

Now we have a second method for constructing minimal cut-generating pairs: identify a minimal ψ with the covering property and create its trivial lifting using (2.17). The trivial lifting has a simple formula in terms of ψ, making this approach appealing. Also, from Proposition 13, ψ has a nice formula in terms of S-free sets. However, this approach has limitations such as the fact that not every minimal cut-generating pair can be created this way [56] (we consider this in Chapter 6). Furthermore, classifying ψ with the covering property, let alone minimal ψ, is difficult [1, 6, 7, 53, 88]. We look at this classification in Chapter 5.

Chapter 3

Approximation guarantees

In this chapter we focus on the mixed-integer model $C_{b+\mathbb{Z}^n}$, where $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Proposition 12 states that for a mixed-integer set $C_{b+\mathbb{Z}^n}(R)$, maximal lattice-free sets generate strong valid cuts. Cuts generated in this way are called *intersection cuts*. If we assume that both b and R have rational entries, then a stronger result holds. In this setting, $\operatorname{conv}(C_{b+\mathbb{Z}^n}(R))$ is equal to the intersection of all intersection cuts [95]. Thus if we can obtain all intersection cuts, then we can perfectly describe $\operatorname{conv}(C_{b+\mathbb{Z}^n}(R))$.

However, obtaining and applying every intersection cut seems intractable. In light of this, perhaps one can approximate a mixed-integer set by using a subfamily of intersection cuts. This leads to the question considered in this chapter:

Can we identify a family of lattice-free sets $\mathcal{B}$ such that for any choice of R and b, the intersection cuts generated from $\mathcal{B}$ closely approximate $\operatorname{conv}(C_{b+\mathbb{Z}^n}(R))$*?*

We develop necessary and sufficient conditions for a family $\mathcal{B}$ to provide these approximations. Similar conditions hold when $\mathcal{B}$ is allowed to depend on b. These conditions help us find specific families that provide good approximations. In particular, the family of lattice-free sets with 'many' facets provides good approximations, while the family of sets with 'few' facets does not. This work was done in collaboration Gennadiy Averkov and Amitabh Basu.

3.1 Approximating corner polyhedra using intersection cuts

Let $b \in \mathbb{R}^n \backslash \mathbb{Z}^n$ and take ψ to be a minimal cut-generating function for $C_{b+\mathbb{Z}^n}$. Proposition 12 states that there is some maximal $(b + \mathbb{Z}^n)$-free 0-neighborhood $B' = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1, i \in [m]\}$ such that $\psi(x) = \max_{i \in [m]} a_i \cdot x$ for all $x \in \mathbb{R}^n$. Translating B' by $-b$, we obtain the maximal $\mathbb{Z}^n$-free set $B := B' - b$. Since B is a translate of B', we can write it as

$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq c_i, \ i \in [m]\}. \tag{3.1}$$

where $c_i = 1 + a_i \cdot b_i$ for $i \in [m]$.

The exciting part about B is that it defines a collection of cut-generating functions. Indeed, for any choice of $f \in \text{int}(B)$, $B - f$ is a maximal $(-f + \mathbb{Z}^n)$-free 0-neighborhood. Again using Proposition 12, the gauge function ψ_{B-f} of $B - f$ is a minimal cut-generating function for $C_{-f+\mathbb{Z}^n}$. Moreover, since $B - f$ is a translate of B, it follows that

$$\psi_{B-f}(x) = \max_{i \in [m]} \left(\frac{1}{c_i - a_i \cdot f} \right) a_i \cdot x. \tag{3.2}$$

This shows that in order to build cut-generating functions, we can consider $\mathbb{Z}^n$-free sets, also called *lattice-free* sets. In this chapter, we only consider lattice-free sets B with the condition $\dim(B) = n$, as the case $\dim(B) < n$ is not needed.

For $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$, let $\mathcal{C}_f$ denote all full-dimensional, convex sets containing f in the interior. For any $k \in \mathbb{N}$ and $R \in \mathbb{R}^{n \times k}$, a lattice-free set B in $\mathcal{C}_f$ generates valid cuts for the mixed-integer set $C_{-f+\mathbb{Z}^n}(R)$. We call these cuts the intersection cuts generated by B. Intersection cuts were first developed by Balas [10, 11].

Definition 15 (Intersection cuts generated by B)**.** *Let $R \in \mathbb{R}^{n \times k}$ and $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Given a lattice-free set B in $\mathcal{C}_f$, the* intersection cut for (R, f) generated by B *(or the B-*cut *of (R, f)) is*

$$C_B(R, f) := \left\{ s \in \mathbb{R}^k_+ \ : \ \sum_{i=1}^{k} s_i \psi_{B-f}(r_i) \geq 1 \right\}. \tag{3.3}$$

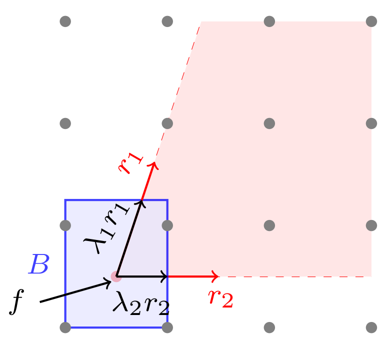
We extend this notation to include lattice-free sets B' such that $f \in \mathbb{R}^n \setminus \text{int}(B)$; in this case we define $C_B(R, f) := \mathbb{R}^k_+$.

The following example illustrates the geometry of intersection cuts.

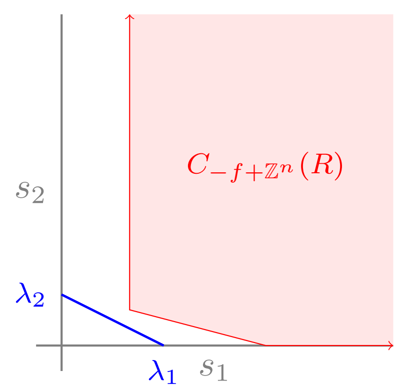
Example 8. *Suppose that $B \subseteq \mathbb{R}^2$ is a lattice-free set, $f \in \operatorname{int}(B)$ and $r_1, r_2 \in \mathbb{R}^2$; see Figure 3.1(a). Note that $f + r_1 \notin B$. However, for $\lambda_1 := \frac{1}{\psi_{B-f}(r_1)}$, the point $f + \lambda_1 r_1$ is on the boundary of B. Thus any $s_1 \in \mathbb{R}_+$ satisfying $f + s_1 r_1 \in \mathbb{Z}^2$ must have $s_1 \geq \lambda_1$; see Figure 3.1(b). Similarly, any $s_2 \in \mathbb{R}_+$ satisfying $f + s_2 r_2 \in \mathbb{Z}^2$ must have $s_2 \geq \lambda_2 =: \frac{1}{\psi_{B-f}(r_2)}$. Moreover, since B is lattice-free and convex (by definition of lattice-free), any choice of nonnegative scalars $s_1, s_2 \in \mathbb{R}_+$ satisfying $f + s_1 r_1 + s_2 r_2 \in \mathbb{Z}^2$ must satisfy*

$$1 \leq \frac{1}{\lambda_1} s_1 + \frac{1}{\lambda_2} s_2 = \psi_{B-f}(r_1) s_1 + \psi_{B-f}(r_2) s_2.$$

This is precisely the intersection cut $C_B(R, f)$ defined in Definition 15, for $R = [r_1, r_2]$. Figure 3.1(b) shows $C_B(R, f)$ and $C_{-f+\mathbb{Z}^n}(R)$.



(a) A lattice-free set $B \in \mathcal{C}_f$ and a choice of R.



(b) The intersection cut $C_B(R, f)$.

Figure 3.1: Illustrating an intersection cut.

□

For any $k \in \mathbb{N}$ and $R \in \mathbb{R}^{n \times k}$, since a single intersection cut is a relaxation of $C_{-f+\mathbb{Z}^n}(R)$, intersecting any number of intersection cuts also provides a relaxation of $C_{-f+\mathbb{Z}^n}(R)$. This leads to the notion of the closure of a family of intersection cuts.

Definition 16 ($\mathcal{B}$-closures)**.** *Given a family $\mathcal{B}$ of lattice-free subsets of $\mathbb{R}^n$ we call the set*

$$C_{\mathcal{B}}(R, f) := \bigcap_{B \in \mathcal{B}} C_B(R, f)$$

the $\mathcal{B}$-closure *of* (R, f). *We define* $C_\emptyset(R, f) = \mathbb{R}^k_+$.

The relative strength of intersection cuts can be partially ordered based on set inclusion of the underlying lattice-free sets. If $B_1 \subseteq B_2$ are lattice-free sets then $C_{B_2}(R, f) \subseteq C_{B_1}(R, f)$ for all (R, f). Hence maximal lattice-free sets produce the strongest cuts [18, 36]. Furthermore, Proposition 4 shows that all lattice-free sets are contained in maximal lattice-free sets, which are all polyhedra in $\mathbb{R}^n$ according to Theorem 3. Therefore intersection cuts from polyhedra are sufficient when it comes to applying intersection cuts.

It follows from Definition 15 that for any choice of $f \in \mathbb{R}^n$ and $R \subseteq \mathbb{R}^{n\times k}$, we have the containment $C_{-f+\mathbb{Z}^n}(R) \subseteq C_B(R, f)$. Since $C_B(R, f)$ is convex, we can also say that $\operatorname{conv}(C_{-f+\mathbb{Z}^n}(R)) \subseteq C_B(R, f)$. To reduce the use of cumbersome notation, we introduce *the convex hull of the mixed-integer set* $C_{-f+\mathbb{Z}^n}(R)$:

$$C(R, f) := \operatorname{conv}(C_{-f+\mathbb{Z}^n}(R)) = \operatorname{conv}\left\{ s \in \mathbb{R}^n_+ \;:\; f + \sum_{i=1}^{k} s_i r_i \in \mathbb{Z}^n \right\}. \tag{3.4}$$

When both R and f are rational, Meyer's theorem (see [43, 86]) yields that $C(R, f)$ is a rational polyhedron, called the corner polyhedron. The term corner polyhedron was first coined by Gomory [65] and initially was used for $\operatorname{conv}(I_{-f+\mathbb{Z}^n}(R))$ instead of $\operatorname{conv}(C_{-f+\mathbb{Z}^n}(R))$. However recent work considered different variants of the model, allowing for integer, continuous and mixed s-variables. It is difficult to give all relevant references to the large body of work in this direction; instead we refer the reader to the recent surveys and the references therein [21, 28, 41], as well as Chapter 6 of [43]. For the purposes of this chapter, the corner polyhedron will refer to $C(R, f)$ as defined in (3.4).

When considering the corner polyhedron $C(R, f)$, an important feature is that every valid cut for $C(R, f)$ is dominated by an intersection cut. Together with the validity of intersection cuts for the corner polyhedron, we get the following result [10, 42, 95].

Proposition 18. *Let* $C(R, f)$ *be a corner polyhedron. Then*

$$C(R, f) = \bigcap_{\substack{B \text{ is a} \\ \text{lattice-free} \\ f-\text{neighborhood}}} C_B(R, f).$$

As previously mentioned, it is enough to consider lattice-free *polyhedra*. In this chapter, we frequently distinguish families of polyhedra based upon how many facets they have.

Definition 17 (i-hedral closures $\mathcal{L}_i^n$, $\mathcal{L}_*^n$)**.** *Let $\mathcal{L}_i^n$ denote the family of all (not necessarily maximal) lattice-free polyhedra in $\mathbb{R}^n$ with at most i facets; we call $C_{\mathcal{L}_i^n}(R,f)$ the* i-hedral closure*. We use $\mathcal{L}_*^n$ to denote all lattice-free (not necessarily maximal) polyhedra in $\mathbb{R}^n$.*

When describing $C(R,f)$ using a $\mathcal{B}$-closure, $C(R,f) \subseteq C_{\mathcal{B}}(R,f)$ for every (R,f), and if $\mathcal{L}_*^n \subseteq \mathcal{B}$ then Proposition 18 gives equality $C_{\mathcal{B}}(R,f) = C_{\mathcal{L}_*^n}(R,f) = C(R,f)$ for every (R,f). So one approach to obtain $C(R,f)$ is to classify maximal lattice-free sets in $\mathcal{L}_*^n$ and compute the corresponding intersection cuts. Recent work has focused on this classification [1, 6, 7, 53, 88]. Maximal lattice-free sets in $\mathcal{L}_*^2$ have been classified [1, 36, 44, 53], but even for $n = 3$, a classification of is unknown. Moreover, even if such a classification was known, the resulting gauge functions may be expensive to compute. Indeed, for a lattice-free polyhedron with k facets, the resulting gauge function is a maximum over k inner products (see Proposition 23). Therefore for large values of k (for maximal lattice-free sets in $\mathbb{R}^n$ k is at most 2^n), the number of computations required to evaluate a single value of the gauge function is large.

In light of these difficulties, instead of completely describing $C(R,f)$ by classifying lattice-free sets, one can aim to find a small and simple family of intersection cuts whose closure closely approximates it [2, 8, 16]. In other words, one can search for a simple family $\mathcal{B} \subseteq \mathcal{L}_*^n$ and a constant $\alpha > 1$ so that the inclusion

$$C_{\mathcal{B}}(R,f) \subseteq \frac{1}{\alpha} C(R,f)$$

holds for all (R,f) (observe that both $C_{\mathcal{B}}(R,f)$ and $C(R,f)$ are convex sets of the blocking type, i.e., they have $\mathbb{R}_+^k$ as their recession cone). Since computation of the gauge function of a polyhedron depends on the number of facets, there is additional motivation to consider families made of polyhedra with few facets.

More generally, one can ask the following question.

(i) Let $\mathcal{B}$ and $\mathcal{L}$ be families of lattice-free sets in $\mathbb{R}^n$. Under what conditions does there exist some $\alpha > 1$ so that the inclusion

$$C_{\mathcal{B}}(R, f) \subseteq \frac{1}{\alpha} C_{\mathcal{L}}(R, f) \tag{3.5}$$

holds for all pairs (R, f)? Also, for a fixed $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$, when does (3.5) hold for all rational R?

If such an α exists, then the $\mathcal{B}$-closure approximates the $\mathcal{L}$-closure within a factor of α, that is the $\mathcal{B}$-closure provides a finite approximation of the $\mathcal{L}$-closure for all choices of (R, f) (or for a fixed f and all R).

Definition 18 (α-relaxations)**.** *For $\alpha \geq 1$, we call $\frac{1}{\alpha} C_B(R, f)$ the* α-relaxation *of the cut $C_B(R, f)$. Analogously, for a family of lattice-free sets $\mathcal{B}$, we call $\frac{1}{\alpha} C_{\mathcal{B}}(R, f)$ the* α-relaxation *of the $\mathcal{B}$-closure $C_{\mathcal{B}}(R, f)$.*

Using α-relaxations, the relative strength of cuts and closures can be quantified naturally as follows. For $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and lattice-free subsets B and L of $\mathbb{R}^n$, we define

$$\rho_f(B, L) := \inf \left\{ \alpha > 0 \,:\, C_B(R, f) \subseteq \frac{1}{\alpha} C_L(R, f) \;\forall R \right\}. \tag{3.6}$$

The value $\rho_f(B, L)$ quantifies up to what extent $C_B(R, f)$ can 'replace' $C_L(R, f)$. For $\alpha \geq 1$, the inclusion $C_B(R, f) \subseteq \frac{1}{\alpha} C_L(R, f)$ says that the cut $C_B(R, f)$ is at least as strong as the α-relaxation of the cut $C_L(R, f)$. For $\alpha < 1$, the previous inclusion says that not just $C_B(R, f)$ but also the $\frac{1}{\alpha}$-relaxation of the cut $C_B(R, f)$ is at least as strong as the cut $C_L(R, f)$. Thus, if $\rho_f(B, L) \leq 1$, the B-cuts of (R, f) are stronger than the L-cuts of (R, f) for every R, and the value $\rho_f(B, L)$ quantifies how much stronger they are. If $1 < \rho_f(B, L) < \infty$, then the B-cuts of (R, f) are not stronger than the L-cuts of (R, f) but stronger than α-relaxations of L-cuts for some $\alpha > 0$ independent of R, where the value $\rho_f(B, L)$ quantifies up to what extend the L-cuts should be relaxed. If $\rho_f(B, L) = \infty$, then $C_B(R, f)$ cannot 'replace' $C_L(R, f)$ because there is no $\alpha \geq 1$ independent of R such that $C_B(R, f)$ is stronger than the α-relaxation of $C_L(R, f)$.

In addition to comparing the cuts coming from two lattice-free sets, we want to compare the relative strength of a family $\mathcal{B}$ to one particular set L, and the relative strength of two families $\mathcal{B}$ and $\mathcal{L}$. We consider these comparisons both when f is fixed and when f is arbitrary. For the case of a fixed $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ we introduce the functional

$$\rho_f(\mathcal{B}, L) := \inf \left\{ \alpha > 0 \,:\, C_{\mathcal{B}}(R, f) \subseteq \frac{1}{\alpha} C_L(R, f) \ \forall R \right\}, \tag{3.7}$$

which compares $\mathcal{B}$-closures to L-cuts for a fixed f. We also introduce the functional

$$\rho_f(\mathcal{B}, \mathcal{L}) := \inf \left\{ \alpha > 0 \,:\, C_{\mathcal{B}}(R, f) \subseteq \frac{1}{\alpha} C_{\mathcal{L}}(R, f) \ \forall R \right\} \tag{3.8}$$

for comparing $\mathcal{B}$-closures to $\mathcal{L}$-closures for a fixed f. The analysis of $\rho_f(\mathcal{B}, \mathcal{L})$ can be reduced to the analysis of $\rho_f(\mathcal{B}, L)$ for $L \in \mathcal{L}$, since one obviously has

$$\rho_f(\mathcal{B}, \mathcal{L}) = \sup \left\{ \rho_f(\mathcal{B}, L) \,:\, L \in \mathcal{L} \right\}. \tag{3.9}$$

For the analysis in the case of a varying f, we introduce the following two functionals:

$$\begin{aligned} \rho(\mathcal{B}, L) &:= \sup \left\{ \rho_f(\mathcal{B}, L) \,:\, f \in \mathbb{R}^n \setminus \mathbb{Z}^n \right\}, \\ \rho(\mathcal{B}, \mathcal{L}) &:= \sup \left\{ \rho_f(\mathcal{B}, \mathcal{L}) \,:\, f \in \mathbb{R}^n \setminus \mathbb{Z}^n \right\}. \end{aligned}$$

Observe that

$$\rho(\mathcal{B}, L) = \sup \left\{ \rho_f(\mathcal{B}, L) \,:\, f \in \operatorname{int}(L) \right\}, \tag{3.10}$$

$$\rho(\mathcal{B}, \mathcal{L}) = \sup \left\{ \rho_f(\mathcal{B}, L) \,:\, f \in \operatorname{int}(L),\ L \in \mathcal{L} \right\}. \tag{3.11}$$

The functional $\rho(\mathcal{B}, \mathcal{L})$ was introduced in [8, § 1.2], where the authors initiated a systematic study for the case of $n = 2$. Since $C_{\mathcal{L}^n_*}(R, f) = C(R, f)$, the functional $\rho(\mathcal{B}, \mathcal{L}^n_*) \geq 1$ describes how well $C_{\mathcal{B}}(R, f)$ approximates $C(R, f)$.

3.1.1 Statement of results

In this chapter, we focus on answering (i). A consequence of this pursuit is that we study the trade-off between the complexity of a family $\mathcal{B}$ of lattice-free sets and the quality of approximation of $C(R,f)$ by $C_{\mathcal{B}}(R,f)$. In fact, our results will be about the quality of approximation of $C_{\mathcal{L}_*^n}(R,f)$ by $C_{\mathcal{B}}(R,f)$. Hence, we will be able to state many results without any rationality assumption on (R,f). Only when one wants to interpret these results for the corner polyhedron $C(R,f)$, one must keep the rationality assumption in mind, which is needed for the equivalence $C_{\mathcal{L}_*^n}(R,f) = C(R,f)$.

From (3.2), intersection cuts are defined by the facet structure of the underlying lattice-free set. So if two lattice-free sets have similar facet structure, the corresponding intersection cuts are 'close'. We formalize this by defining a metric called the f-metric on the collection of lattice-free f-neighborhoods, for each $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The f-metric is formally defined in Chapter 3.2. For some family $\mathcal{B}$ of lattice-free f-neighborhoods, we let $\mathrm{cl}_f(\mathcal{B})$ denote the closure of $\mathcal{B}$ under the f -metric. Since the f-metric is only defined on f-neighborhoods, for a family $\mathcal{B}$ of lattice-free sets, we define $\mathcal{B}_f := \mathcal{B} \cap \mathcal{C}_f$.

1. Our main tool is the following geometric result.

   Theorem 6. *Let $\mathcal{B}$ be a family of lattice-free sets of $\mathbb{R}^n$ and let $\mathcal{L}$ be a family of lattice-free polyhedra such that every $L \in \mathcal{L}$ is the direct sum $M \oplus U$ of a polytope M and a linear space U and one has $2(|\mathrm{vert}(M)|^2 + |\mathrm{vert}(M)|) \leq N$ for some $N \in \mathbb{N}$ independent of L. Then the following statements hold:*

   (a) For $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$, the condition $\rho_f(\mathcal{B}, \mathcal{L}) < \infty$ holds if and only if there exists $0 < \mu < 1$ such that for every $L \in \mathcal{L}$ with $f \in \mathrm{int}(L)$ some $B \in \mathrm{cl}_f(\mathcal{B}_f)$ satisfies $B \supseteq \mu L + (1-\mu)f$.

   (b) The condition $\rho(\mathcal{B}, \mathcal{L}) < \infty$ holds if and only if there exists $0 < \mu < 1$ such that for every $L \in \mathcal{L}$ and every $f \in \mathrm{int}(L)$ some $B \in \mathrm{cl}_f(\mathcal{B}_f)$ satisfies $B \supseteq \mu L + (1-\mu)f$.

   Theorem 6 establishes a relationship between the functionals $\rho_f(\mathcal{B}, \mathcal{L})$ and $\rho(\mathcal{B}, \mathcal{L})$, and individual lattice-free sets $B \in \mathcal{B}$ and $L \in \mathcal{L}$. Since an inclusion of lattice-free sets creates an inclusion of the corresponding intersection cuts, if B contains the homothetical copy $\mu L + (1-\mu)f$ then the cuts $C_B(R,f)$ are contained in the cuts $C_{\mu L + (1-\mu)f}(R,f)$. If such a μ can be found so that this homothetical inclusion holds for all L (where B is allowed to depend on L), then the inclusion of intersection cuts carries over to the inclusion $C_{\mathcal{B}}(R,f) \subseteq \mu C_{\mathcal{L}}(R,f)$. In turn, this translates into the finiteness of $\rho_f(\mathcal{B},\mathcal{L})$ and $\rho(\mathcal{B},\mathcal{L})$.

2. As previously mentioned, there is motivation to consider families of lattice-free polyhedra with few facets. Our first main result compares i-hedral closures with j-hedral closures using the functional (3.11).

   Theorem 7. *Let $i \in \{2,\dots,2^n\}$. Then $\rho(\mathcal{L}_i^n,\mathcal{L}_{i+1}^n) = \infty$ for every $i \le 2^{n-1}$ and $\rho(\mathcal{L}_i^n,\mathcal{L}_*^n) \le 4\mathit{Flt}(n)$ for every $i > 2^{n-1}$.*

   The value $\mathrm{Flt}(n)$ is a 'flatness constant' that is a number depending only on n (see Chapter VII [14]).

3. Another way to examine the relative strength of i-hedral closures is with the functional (3.9) for some fixed $f \in \mathbb{R}^n \backslash \mathbb{Z}^n$. Theorem 7 immediately implies that $\rho_f(\mathcal{L}_i^n,\mathcal{L}_*^n) < \infty$ for $i > 2^{n-1}$. However, this can be improved to $\rho_f(\mathcal{L}_i^n,\mathcal{L}_*^n) < \infty$ for $i > n$. This strengthening of Theorem 7 is our second main result.

   Theorem 8. *Fix $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$. Let $i \in \{2,\dots,2^n\}$ Then $\rho_f(\mathcal{L}_i^n,\mathcal{L}_{i+1}^n) = \infty$ for every $i \le n$ and $\rho_f(\mathcal{L}_i^n,\mathcal{L}_*^n) < \infty$ for every $i > n$.*

   The proofs of Theorems 7 and 8 are the focus of Section 3.6.

4. We apply Theorem 7 to identify other families of lattice-free sets that provide α-approximations of the corner polyhedron. When $n = 2$, we show that a special family of 'thin' Type 2 triangles provides a finite α-approximation (Section 3.7).

3.2 The f-metric

One nice property of $\rho_f(\mathcal{B},\mathcal{L})$ is that it can be approximated by the values $\rho_f(B,L)$ for individual sets $B \in \mathrm{cl}_f(\mathcal{B}_f)$ and $L \in L$ (in fact, this idea is one way to interpret Theorem 6). The following example shows that it is not enough to consider just sets $B \in \mathcal{B}_f$.

Example 9. *For each $n \in \mathbb{N}$, define the maximal lattice-free triangle $B_n = \{(x_1,x_2) : x_2 \ge 0, -\frac{2}{n}x_1 + x_2 \le 1, \frac{2}{n}x_1 + x_2 \le 1 + \frac{2}{n}\}$. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ and $L = \{(x_1,x_2) : 0 \le x_2 \le 1\}$.*

Let $f = \left(\frac{1}{2}, \frac{1}{2}\right)$, $r_1 = (1,0)^T, r_2 = (-1,0)^T$, and $R = (r_1, r_2)$. Then $C_L(R,f) = \emptyset$ while $C_{B_n}(R,f) \neq \emptyset$ for each $n \in N$. Hence $\rho_f(\mathcal{B}, L) = \infty$. However, it was shown in [16] that $\rho_f(\mathcal{B}, L) \leq 2$.

The issue in Example 9 is that the set L is a 'limit point' of the family $\mathcal{B}$, however $L \notin \mathcal{B}$. Examples such as this one motivates the use of a metric so that 'limit sets' may be considered.

For $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$, recall that $\mathcal{C}_f$ is all closed, full-dimensional convex sets in $\mathbb{R}^n$ that contain f in their interior. We define the *f-metric* $d_f : \mathcal{C}_f \times \mathcal{C}_f \to \mathbb{R}_+$ on $\mathcal{C}_f$ to be

$$d_f(B_1, B_2) := d_H\left((B_1 - f)^*, (B_2 - f)^*\right), \tag{3.12}$$

where $B_1, B_2 \in \mathcal{C}_f$, and d_H denotes the Hausdorff metric. Note that $d_f(\cdot,\cdot)$ is always finite. This follows since $f \in \text{int}(B)$ for each $B \in \mathcal{C}_f$, and the polar of a convex set is bounded if it contains the origin in its interior. Furthermore, the polar of a set B is independent of the representation of B and therefore the f-metric is well-defined.

In what follows, we use $B_t \xrightarrow{f} B$ to denote convergence in the f-metric and $C_t \xrightarrow{H} C$ to denote convergence in the Hausdorff metric. We will use $\text{cl}_f(\cdot)$ to denote the closure of a set under the f-metric.

3.2.1 Properties of the f-metric

Here we collect some notes on the f-metric, and many of these notes follow from the next result from [92].

Theorem 9 (Theorem 1.8.7 in [92]). *Let $(K_t)_{t=1}^{\infty}$ be a sequence of nonempty, convex, compact sets in $\mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ be nonempty, convex, and compact. Suppose that $K_t \xrightarrow{H} K$. This is equivalent to the following two conditions together:*

(a) each point $x \in K$ is a limit of points $(x_t)_{t=1}^{\infty}$, where $x_t \in K_t$ for $t \in \mathbb{N}$;

(b) the limit of any convergent sequence $(x_t)_{t=1}^{\infty}$, where $x_t \in K_t$ for $t \in \mathbb{N}$, is contained in K.

When dealing with intersection cuts, our interest in this metric will restrict to lattice-free polyhedra. The following proposition states that the collection of lattice-free sets is closed under this metric, allowing us to freely consider sequences of such sets.

Proposition 19. *Let $(B_t)_{t=1}^{\infty}$ be a sequence in $\mathcal{C}_f$ and $B \in \mathcal{C}_f$ so that $B_t \xrightarrow{f} B$. Then*

(a) if $x \in B_t$ for each $t \in \mathbb{N}$, then $x \in B$;

(b) if $x \notin \operatorname{int}(B_t)$ for each $t \in \mathbb{N}$, then $x \notin \operatorname{int}(B)$.

Proof.

(a) It is well known that $B - f = ((B - f)^*)^*$. Thus it is sufficient to show that $r \cdot (x - f) \leq 1$ for each $r \in (B - f)^*$. From Theorem 9 (a), each $r \in (B - f)^*$ is a limit of points $(r_t)_{t=1}^{\infty}$, where $r_t \in (B_t - f)^*$. Note that $r_t \cdot (x - f) \leq 1$ for all $t \in \mathbb{N}$. Therefore, by continuity of the inner product, $r \cdot (x - f) \leq 1$ for each $r \in (B - f)^*$.

(b) This follows using an argument similar to that of part (a).

□

The collection $\mathcal{B}$ in Example 9 is not a closed subset of $\mathcal{C}_f$, for $f = \left(\frac{1}{2}, \frac{1}{2}\right)$. In general, the collection $(\mathcal{L}_i^n \setminus \mathcal{L}_{i-1}^n) \cap \mathcal{C}_f$ of polyhedra in $\mathbb{R}^n$ with exactly i-facets is not closed in $\mathcal{C}_f$ under d_f. However, the set $\mathcal{L}_i^n \cap \mathcal{C}_f$ is closed under d_f.

Proposition 20. *Let $i, n \in \mathbb{N}$ with $i, n \geq 2$ and $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The set $\mathcal{L}_i^n \cap \mathcal{C}_f$ is a closed subset of $\mathcal{C}_f$ under the f-metric.*

Proof. Let $(B_t)_{t=1}^{\infty}$ be a sequence in $\mathcal{L}_i^n \cap \mathcal{C}_f$ and $B \in \mathcal{C}_f$ so that $B_t \xrightarrow{f} B$. For each $t \in \mathbb{N}$, let k_t denote the number of facets of B_t. Since $0 \in \operatorname{int}(B_t - f)$ there is some $A_t \in \mathbb{R}^{n \times k_t}$ so that $B_t = \{x \in \mathbb{R}^n : A_t x \leq 1\}$. Note that, as each $B_t - f$ is a polyhedron with at most i facets, $k_t \leq i$ for each $t \in \mathbb{N}$.

Writing each B_t using this system of inequalities, it follows that $(B_t - f)^* = \operatorname{conv}\{v : v \in \operatorname{row}(A_t)\}$. It follows from Theorem 9 (a) that any extreme point v of $(B - f)^*$ is a limit point of the form $v = \lim_{t \to \infty} v_t$, where $v_t \in \operatorname{row}(A_t)$. Hence there is some $A \subseteq \mathbb{R}^{n \times k}$ so that $k \leq i$ and $(B - f)^* = \operatorname{conv}\{v \in \operatorname{row}(A)\}$. This implies that B is a polyhedron with at most i facets. □

3.3 More on lattice-free sets and gauge functions

In this section we collect the additional tools on lattice-free sets and gauge functions that are necessary for discussing intersection cuts.

3.3.1 Lattice-free sets

We say that a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *affine unimodular* if $T(x) = Ux + v$ for a unimodular matrix $U \in \mathbb{R}^{n\times n}$ and an integral vector $v \in \mathbb{R}^n$. Theorem 3 and Proposition 6 can be combined into the following result [19, 55, 4, 87].

Theorem 10. *Let B be a lattice-free subset of $\mathbb{R}^n$. Then the following conditions hold*

(a) B is maximal lattice-free if and only if B is a lattice-free polyhedron and the relative interior of each facet of B contains a point of $\mathbb{Z}^n$.

(b) If B is maximal lattice-free, then B is a polyhedron with at most 2^n facets.

(c) If B is an unbounded maximal lattice-free set, then B coincides up to an affine unimodular transformation with $B' \times \mathbb{R}^k$, where $1 \le k \le n-1$ and B' is a bounded maximal lattice-free set.

Theorem 10 provides many useful corollaries. For instance, Theorem 10(c) implies the recession cone of a maximal lattice-free set is a linear space spanned by integer vectors. From Theorem 10(b), we can classify maximal lattice-free sets based upon number of facets (which is the classification used in Theorems 7 and 8) . Note that for $i = 2$, the family $\mathcal{L}_i^n$ contains all splits, and for $i > 2$, the family $\mathcal{L}_i^n \setminus \mathcal{L}_{i-1}^n$ contains all lattice-free sets with *exactly* i facets; for an account on the approximation factors obtained by $\mathcal{L}_2^2, \mathcal{L}_3^2 \setminus \mathcal{L}_2^2, \mathcal{L}_4^2 \setminus \mathcal{L}_3^2$ see [8, 16].

Lemma 2. *Let $P = \{x \in \mathbb{R}^n : a_i \cdot x \le b_i, i \in [m]\}$ be a lattice-free polyhedron in $\mathbb{R}^n$. Then there exists a nonempty subset $I' \subseteq [m]$ and some $k \in \mathbb{N}$ so that $\{x \in \mathbb{R}^n : a_i \cdot x \le b_i, i \in I'\} = S \oplus U$, where S is a k-simplex and U a linear space.*

Proof.

We may assume that $a_i \neq 0$ for each $i \in [m]$. Let $z \in P$. Since P is lattice-free, the linear subspace $L = \text{aff}(\text{rec}(P)) - z$ is not full-dimensional. Therefore $\dim(L^\perp) \geq 1$.

Let $u \in L^\perp$ be nonzero so that $u \cdot x \leq 1$ and $(-u) \cdot x \leq 1$ for each $x \in P$. By Farkas' Lemma, there exists nonnegative scalars $\{\lambda_i\}_{i=1}^m$ so that $\sum_{i=1}^m \lambda_i a_i = u$ and $0 < \sum_{i=1}^m \lambda_i \leq 1$. Set $M = \sum_{i=1}^m \lambda_i$ and note that $\frac{1}{M} u \in \text{conv}\{a_i : i \in I'\}$. A similar argument shows that $\alpha u \in \text{conv}\{a_i : i \in I'\}$ for some $\alpha < 0$. Hence $0 \in \text{conv}\{a_i : i \in [m]\}$. Let $I' \subseteq [m]$ be a minimal set so that $0 \in \text{conv}\{a_i : i \in I'\}$. Then $0 \in \text{relint}(\text{conv}\{a_i : i \in I'\})$. Since I' is minimal, Caratheodory's Theorem implies that $\text{conv}\{a_i : i \in I'\}$ is a k-simplex for $k = |I'| - 1$. Therefore, $\{x \in \mathbb{R}^n : a_i \cdot x \leq b_i, i \in I'\}$ is of the form $S \oplus U$ for U a $n-k$ dimensional linear space. □

Proposition 21. *Let $M \subseteq \mathbb{R}^n$ be a maximal lattice-free polyhedron with k facets. For each $f \in \text{int}(M)$, there is some $\alpha_M \in (0,1)$ so that for $\epsilon \geq \alpha_M$, any lattice-free polyhedron containing $\epsilon M + (1-\epsilon) f$ must have at least k facets.*

Proof. Let $F_1, ..., F_k$ denote the facets of M. From Theorem 10(a), for each $i \in [k]$ there is a $z_i \in \mathbb{Z}^n$ so that $z_i \in \text{relint}(F_i)$. We will show that for each pair of indices $\{i,j\} \subseteq [k]$, there is some $\alpha_{i,j} \in (0,1)$ so that no single valid inequality for $\epsilon_{i,j} M + (1-\epsilon_{i,j}) f$ separates both z_i and z_j for each $\epsilon_{i,j} \geq \alpha_{i,j}$. Then for $\alpha_M = \max\{\alpha_{i,j} : \{i,j\} \subseteq [k]\}$ and $\epsilon \geq \alpha_M$, any lattice-free polyhedron containing $\epsilon M + (1-\epsilon) f$ must contain a separate facet for each z_i. This will give the desired result.

Let $\{i,j\} \subseteq [k]$. Since $z_i \in \text{relint}(F_i)$ and $z_j \in \text{relint}(F_j)$, the point $m_{i,j} = \frac{1}{2}(z_i + z_j)$ is contained in $\text{int}(M)$. Therefore, there is an $\alpha_{i,j} > 0$ so that $m_{i,j} \in \text{int}(\alpha_{i,j} M + (1-\alpha_{i,j}) f)$. Assume to the contrary that some valid inequality $a \cdot x \leq b$ for $\alpha_{i,j} M + (1-\alpha_{i,j}) f$ separates both z_i and z_j. Then

$$b \leq \frac{1}{2}(a \cdot z_i + a \cdot z_j) = a \cdot m \leq b.$$

Hence $a \cdot m = b$ and so $m \notin \text{int}((\alpha_{i,j} M + (1-\alpha_{i,j}) f)$ which is a contradiction. Note that for each $\epsilon_{i,j} \geq \alpha_{i,j}$, we arrive at the same contradiction giving the desired result. □

The final result on lattice-free sets that we require relates to the width of a lattice-free set.

Definition 19 (Width function and lattice width)**.** *For every nonempty subset X of $\mathbb{R}^n$ we introduce the* width function $w(X,\,\cdot\,) : \mathbb{R}^n \to [0, \infty]$ *of X to be*

$$w(X, u) := \sup_{x \in X} x \cdot u - \inf_{x \in X} x \cdot u.$$

The value

$$w(X) := \inf_{u \in \mathbb{Z}^n \setminus \{0\}} w(X, u)$$

is called the lattice width *of X.*

Theorem 11 (Flatness Theorem)**.** *For every $n \in \mathbb{N}$, the value*

$$\mathit{Flt}(n) := \sup \{ w(B) \,:\, B \textit{ lattice free in } \mathbb{R}^n \}$$

is finite.

The literature contains a number of upper bounds on $\mathrm{Flt}(n)$. For example, it is known that $\mathrm{Flt}(n) \leq n^{5/2}$ (see Chapter VII [14]).

3.3.2 Gauge functions

The definition of an intersection cut is in terms of the gauge function of a lattice-free set, while the metric defined in Section 3.2 is on polyhedra and not functions. Proposition 22 states that the this metric behaves nicely when transitioning from lattice-free sets to their gauges (and consequently, to the intersection cuts).

Proposition 22. *Fix $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Let $B \in \mathcal{C}_f$, and take $(B_t)_{t \in \mathbb{N}} \subseteq \mathcal{C}_f$ so that $B_t \xrightarrow{f} B$. Then $\psi_{B_t - f} \to \psi_{B-f}$ pointwise.*

In order to prove Proposition 22, we require the following proposition which follows immediately from the definition of the gauge function.

Proposition 23. *Let B be closed convex set with $0 \in \mathrm{int}(B)$ given by a (possibly infinite) system of linear inequalities as*

$$B = \{ x \in \mathbb{R}^n \,:\, a_i \cdot x \leq b_i \ \forall i \in I \},$$

where I is an index set, $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$ and $(a_i, b_i) \neq (0,0)$ for every $i \in I$. Then $b_i > 0$ for each $i \in I$ and $\psi_B(r)$ can be given by

$$\psi_B(r) = \sup\left\{\frac{a_i \cdot r}{b_i} \;:\; i \in I\right\} \cup \{0\} \qquad \forall r \in \mathbb{R}^n.$$

Proof of Proposition 22. For each $t \in \mathbb{N}$, since $f \in \operatorname{int}(B_t)$ the set $B_t - f$ can be written as $B_t - f = \left\{x \in \mathbb{R}^n : a_i^t \cdot x \leq 1,\ i \in I_t\right\}$, where $I_t \in \mathbb{N}$ is an index set (possibly of infinite cardinality). Similarly, there exists some index set I so that $B - f = \{x \in \mathbb{R}^n : a_i \leq 1,\ i \in I\}$.

Fix $r \in \mathbb{R}^n$ and let $\epsilon > 0$. From Proposition 23, there is some a_i, $i \in I$ such that $\psi_{B-f}(r) - \epsilon \leq a_i \cdot r$. It is known that $(B-f)^* = \operatorname{conv}\{a_i : i \in I\}$. From Theorem 9, there is a sequence of points $(\alpha_t)_{t=1}^{\infty}$ so that $\alpha_t \in (B_t - f)^*$ and $\alpha_t \to a_i$. Note that, for each $t \in \mathbb{N}$, it follows that $(B_t - f)^* = \operatorname{conv}\{a_i^t : i \in I_t\}$. Therefore, each α_t is a convex combination of points in $\{a_i^t : i \in I_t\}$ and so $\alpha_t \cdot r \leq \psi_{B_t - f}(r)$ for each $t \in \mathbb{N}$. Again using Proposition 23, it follows that

$$\psi_{B-f}(r) - \epsilon \leq a_i \cdot r = \lim_{t\to\infty} \alpha_t \cdot r \leq \liminf_t \psi_{B_t - f}(r). \tag{3.13}$$

Letting ϵ go to 0, we see $\psi_{B-f}(r) \leq \liminf_t \psi_{B_t-f}(r)$.

From Equation (3.13), it is enough to show that $\limsup_t \psi_{B_t-f}(r) \leq \psi_{B-f}(r)$. Assume to the contrary that $\limsup \psi_{B_t-f}(r) > \psi_{B-f}(r)$. Then there exists some $\epsilon > 0$ and a sequence of points $(\alpha_{t_k})_{k=1}^{\infty}$ so that, for each $k \in \mathbb{N}$, $\alpha_{t_k} \in \{a_i^{t_k} : i \in I_{t_k}\}$ and $\alpha_{t_k} \cdot r > \psi_{B-f}(r) + \epsilon$. However, $(\alpha_{t_k})_{k=1}^{\infty}$ is a bounded sequence since $(B_t - f)^* \xrightarrow{H} (B-f)^*$, each of which is a polytope. Therefore, there is a convergent subsequence of $(\alpha_{t_k})_{k=1}^{\infty}$ with a limit $\alpha \in (B-f)^*$. Since $(B-f)^* = \operatorname{conv}\{a_i : i \in I\}$, this implies that $\psi_{B-f}(r) \geq \alpha \cdot r \geq \psi_{B-f}(r) + \epsilon$, which is a contradiction.

□

Proposition 22 implies that it is possible to restrict considerations to topologically closed families. For a family of lattice-free sets $\mathcal{B}$ and some $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$, let $\mathcal{B}_f := \mathcal{B} \cap \mathcal{C}_f$ be the subcollection (possibly empty) of $\mathcal{B}$ that contains f in their interiors.

Proposition 24. *Let $\mathcal{B}$ be a family lattice-free convex sets and $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ such that*

$\mathcal{B}_f \neq \emptyset$. *Then*

$$C_{\mathcal{B}}(R,f) = C_{\mathcal{B}_f}(R,f) = C_{\mathrm{cl}_f(\mathcal{B}_f)}(R,f) \tag{3.14}$$

for every R.

Proof. Choose a R with $R \in \mathbb{R}^{n\times k}$. The equality $C_{\mathcal{B}}(R,f) = C_{\mathcal{B}_f}(R,f)$ holds because $C_B(R,f) = \mathbb{R}^k_+$ for each $B \in \mathcal{B}$ with $f \notin \mathrm{int}(B)$. So we focus our attention on the second equality.

Since $\mathcal{B}_f \subseteq \mathrm{cl}_f(\mathcal{B}_f)$, it follows that $C_{\mathrm{cl}_f(\mathcal{B}_f)}(R,f) \subseteq C_{\mathcal{B}_f}(R,f)$. In order to show $C_{\mathcal{B}_f}(R,f) \subseteq C_{\mathrm{cl}_f(\mathcal{B}_f)}(R,f)$, we derive $C_{\mathcal{B}_f}(R,f) \subseteq C_B(R,f)$ for every $B \in \mathrm{cl}_f(\mathcal{B}_f)$. Consider a sequence $(B_t)_{t\in\mathbb{N}}$ of elements of $\mathcal{B}_f$ converging to B in the f-metric. Consider an arbitrary $s \in C_{\mathcal{B}_f}(R,f)$. For each $t \in \mathbb{N}$ we have $\sum_{i=1}^k \psi_{B_t - f}(r_i)s_i \geq 1$ since $B_t \in \mathcal{B}_f$. In view of Proposition 22, in the latter inequality we can pass to the limit as $t \to \infty$ obtaining $\sum_{i=1}^k \psi_{B-f}(r_i) \geq 1$. Thus, $s \in C_B(R,f)$. □

3.4 Properties of the functionals $\rho_f(B,L)$, $\rho_f(\mathcal{B},L)$, and $\rho_f(\mathcal{B},\mathcal{L})$

The following proposition shows that, for $\alpha \geq 1$, the α-relaxation $\frac{1}{\alpha}C_B(R,f)$ of the cut $C_B(R,f)$ is also an intersection cut (the result follows from the basic properties of the gauge functions).

Proposition 25. *Let* B *be a lattice-free subset of* $\mathbb{R}^n$. *Consider the intersection cut* $C_B(R,f)$ *of some* (R,f) *with* $f \in \mathrm{int}(B)$ *and let* $\alpha \geq 1$. *Then* $\frac{1}{\alpha}C_B(R,f)$ *is an intersection cut for* (R,f) *too, since*

$$\frac{1}{\alpha}C_B(R,f) = C_{B'}(R,f) \qquad \text{for} \qquad B' := \frac{1}{\alpha}B + \left(1 - \frac{1}{\alpha}\right)f,$$

where B' *is a lattice-free subset of* B.

We characterize the condition $C_B(R,f) \subseteq C_L(R,f)$, for all R and a fixed f, which can also be formulated as $\rho_f(B,L) \leq 1$.

Proposition 26. *Let* B *and* L *be lattice-free subsets of* $\mathbb{R}^n$ *and let* $f \in \mathrm{int}(B) \cap \mathrm{int}(L)$. *Take* $\alpha \geq 1$. *Then the following conditions are equivalent:*

(i) $C_B(R,f) \subseteq C_L(R,f)$ *holds for every* R,

(ii) $B \supseteq L$,

(iii) $\psi_{B-f} \leq \psi_{L-f}$

Proof.

$(i) \Rightarrow (ii)$: Take $f + r \in L$. This implies that $\psi_{L-f}(r) \leq 1$ and $1 \notin \operatorname{int}(C_L(r,f))$. By (i), this implies that $1 \notin \operatorname{int}(C_B(r,f))$, or equivalently, that $\psi_{B-f}(r) \leq 1$. Hence $r \in B - f$ and so $f + r \in B$.

$(ii) \Rightarrow (iii)$: Let $r \in \mathbb{R}^n$ and take $\lambda > 0$ such that $r \in \lambda(L - f)$. By (ii), $r \in \lambda(B - f)$ and so $\psi_{B-f}(r) \leq \lambda$. Taking the infimum over $\lambda > 0$ gives $\psi_{B-f}(r) \leq \psi_{L-f}(r)$.

$(iii) \Rightarrow (i)$: Choose $R \in \mathbb{R}^{n\times k}$ and take $s \in C_B(R,f)$. Note that

$$\sum_{i=1}^{k} \psi_{L-f}(r_i)s_i \geq \sum_{i=1}^{k} \psi_{B-f}(r_i)s_i \geq 1,$$

where the first inequality follows from (iii) and the second since $s \in C_B(R,f)$. Hence $s \in C_L(R,f)$. □

Using Propositions 25 and 26, we conclude that $\rho_f(B,L)$ can be described as follows.

Proposition 27. *Let* B *and* L *be maximal lattice-free subsets of* $\mathbb{R}^n$ *and let* $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. *Then one has*

$$\begin{aligned} \rho_f(B,L) &= \inf\left\{\alpha > 0 \,:\, B \supseteq \frac{1}{\alpha}L + \left(1 - \frac{1}{\alpha}\right)f\right\} \\ &= \inf\left\{\alpha > 0 \,:\, \alpha B + (1-\alpha)f \supseteq L\right\} \\ &= \inf\left\{\alpha > 0 \,:\, \psi_{B-f} \leq \alpha\psi_{L-f}\right\} \end{aligned} \tag{3.15}$$

if $f \in \operatorname{int}(L)$ *and* $\rho_f(B,L) = 0$, *otherwise.*

Observe that whenever the infimum in the above representation of $\rho_f(B,L)$ is finite, this infimum is attained for some α. We end this section by studying topological properties of the functionals (3.7) and (3.8).

Proposition 28. *Let $\mathcal{B}, \mathcal{M}, \mathcal{L} \subseteq \mathcal{L}_*^n$ be such that $\rho(\mathcal{B}, \mathcal{M}) \leq \alpha$ and $\rho(\mathcal{M}, \mathcal{L}) \leq \beta$. Then $\rho(\mathcal{B}, \mathcal{L}) \leq \alpha\beta$.*

Proof. Let $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Since $\rho(\mathcal{B}, \mathcal{M}) \leq \alpha$ and $\rho(\mathcal{M}, \mathcal{L}) \leq \beta$, the inclusions

$$C_{\mathcal{B}}(R, f) \subseteq \frac{1}{\alpha} C_{\mathcal{M}}(R, f) \quad \text{and} \quad C_{\mathcal{M}}(R, f) \subseteq \frac{1}{\beta} C_{\mathcal{L}}(R, f)$$

hold for all R. Combining these we see that $C_{\mathcal{B}}(R, f) \subseteq \frac{1}{\alpha\beta} C_{\mathcal{L}}(R, f)$ for all R. Hence $\rho_f(\mathcal{B}, \mathcal{L}) \leq \alpha\beta$ and so $\rho(\mathcal{B}, \mathcal{L}) \leq \alpha\beta$.

□

3.5 One-for-all theorems and finiteness of the relative strength

The section is dedicated to the proof of Theorem 6. We will see that Theorem 6 follows from Theorem 12, which states that if a $\mathcal{B}$-closure provides a finite approximation for a L-cut, then for any choice of f, there is some $B \in \mathrm{cl}_f(\mathcal{B})$ so that the B-cut finitely approximates the B-cut.

Theorem 12 (One-for-all Theorem for a family $\mathcal{B}$ and a set L)**.** *Let $\mathcal{B}$ be a family of lattice-free sets in $\mathbb{R}^n$. Let $L \in \mathcal{L}_*^n$ which can be represented as the direct sum $L = M \oplus U$, where M is a polytope and U is a linear subspace of $\mathbb{R}^n$. Fix $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Then*

$$\frac{1}{N} \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \leq \rho_f(\mathcal{B}, L) \leq \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \tag{3.16}$$

holds for $2(|\operatorname{vert}(M)|^2 + |\operatorname{vert}(M)|) \leq N$. Furthermore, one has

$$\frac{1}{N} \sup_{f \in \mathrm{int}(L)} \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \leq \rho(\mathcal{B}, L) \leq \sup_{f \in \mathrm{int}(L)} \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \tag{3.17}$$

Proof of Theorem 12.

Note (3.17) is a direct consequence of (3.16), so it is enough to prove (3.16). First we consider the upper bound in (3.16). If $\inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) = \infty$, then we are done (note that one such instance where this case occurs is when $\mathcal{B}_f = \emptyset$). So assume that $\inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) < \infty$. It is sufficient to show that $\rho_f(\mathcal{B}, L) < \alpha$ for each α satisfying

$\inf_{B\in\mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B,L) < \alpha$. For such an α, there exists a $B \in \mathrm{cl}_f(\mathcal{B}_f)$ so that $\rho_f(B,L) < \alpha$. Using Equation (3.6) and Proposition 24, it follows that

$$C_{\mathcal{B}}(R,f) = C_{\mathrm{cl}_f(\mathcal{B}_f)}(R,f) \subseteq C_B(R,f) \subseteq \frac{1}{\alpha}C_L(R,f)$$

for all choices of R. Hence $\rho_f(\mathcal{B},L) < \alpha$, as desired.

Now we consider the lower bound in (3.16).

Claim 2. *In order to prove the lower bound, it is enough to consider the situation when all of the following hold:*

(A1) $\rho_f(\mathcal{B},L) < \infty$*;*

(A2) $f \in \mathrm{int}(L)$*;*

(A3) $\mathcal{B}_f \neq \emptyset$*;*

(A4) $0 < \rho_f(\mathcal{B},L)$.

Proof.

We show that each assumption can be made by proving the lower bound in (3.16) holds when the assumption is violated or that a contradiction is reached.

(A1) If $\rho_f(\mathcal{B},L) = \infty$ and so the lower bound holds.

(A2) Assume that (A1) holds but $f \notin \mathrm{int}(L)$. Then we have $\rho_f(B,L) = 0$ for every $B \in \mathrm{cl}_f(\mathcal{B}_f)$ and $\rho_f(\mathcal{B},L) = 0$, so the lower bound holds.

(A3) Suppose $(A1)$, $(A2)$ hold and $\mathcal{B}_f = \emptyset$. Then we have $C_B(R,f) = \mathbb{R}^k_{\geq 0}$ for every $B \in \mathcal{B}$ and every choice of R, which implies that $\rho_f(\mathcal{B},L) = \infty$ and yields the lower bound.

(A4) Suppose $(A1)-(A3)$ hold and $\rho_f(\mathcal{B},L) = 0$; we claim that this creates a contradiction. Choose $R = [-f]$. Then the corner polyhedron $C(R,f)$ (and thus $C_{\mathcal{B}}(R,f)$) is not empty. Also, $0 \notin C_L(R,f)$ since $f \in \mathrm{int}(L)$, so for any $x > 0$, there is some $\alpha \in \mathbb{R}_{>0}$ so that $x \notin \frac{1}{\alpha}C_L(R,f)$. However, since $C_{\mathcal{B}}(R,f) \neq \emptyset$, this implies that $C_{\mathcal{B}}(R,f) \not\subseteq \frac{1}{\alpha}C_L(R,f)$ for some $\alpha > 0$, contradicting that $\rho_f(\mathcal{B},L) = 0$.

$\square$

From Claim 2, it is left to consider the case when $0 < \rho_f(\mathcal{B}, L) < \infty$, $\mathcal{B}_f \neq \emptyset$, and $f \in \operatorname{int}(L)$. To this end, suppose $k := |\operatorname{vert}(M)|$ and let $\{r_1, \ldots, r_k\} \in \mathbb{R}^n$ be so that $\{f + r_i\}_{i=1}^k = \operatorname{vert}(M)$. Let $d := \dim(U)$ and choose $U_{bas} := \{u_1, \ldots, u_d\}$ to be an orthonormal basis of U (if $d = 0$ define $U_{bas} := \emptyset$). Note that $d < n$ because L is lattice-free. Also, $\psi_{L-f}(u_i) = 0$ for each $u_i \in U_{bas}$.

Define the set

$$\mathcal{B}_U := \{B \in \operatorname{cl}_f(\mathcal{B}_f) : U \subseteq \operatorname{lin}(B)\}.$$

Claim 3. *There exists some $B \in \mathcal{B}_U$ so that $\epsilon L + (1-\epsilon)f \subseteq B$, where $\epsilon = \frac{1}{2\rho_f(\mathcal{B},L)(k+1)}$.*

Proof of Claim. For $t \in \mathbb{N}$, define the vector $v^t \in \mathbb{R}^{d+k}$ coordinate-wise by

$$v_i^t = \begin{cases} t, & \text{if } i \leq d \\ \epsilon, & \text{if } i > d \end{cases}.$$

Set $R = [u_1, ..., u_d, r_1, ..., r_k]$. Observe that $v^t \notin \frac{1}{2\rho_f(\mathcal{B},L)} C_L(R, f)$ since

$$\sum_{i=1}^{d} \psi_{L-f}(u_i) v_i^t + \sum_{j=d+1}^{d+k} \psi_{L-f}(r_{j-d}) v_j^t = \sum_{j=d+1}^{d+k} \epsilon < \frac{1}{2\rho_f(\mathcal{B}, L)}.$$

Since $\rho_f(\mathcal{B}, L) < 2\rho_f(\mathcal{B}, L)$, Equation (3.7) implies $C_{\mathcal{B}}(R, f) \subseteq \frac{1}{2\rho_f(\mathcal{B},L)} C_L(R, f)$. Hence, as $v^t \notin \frac{1}{2\rho_f(\mathcal{B},L)} C_L(R, f)$, there is some $B_t \in \mathcal{B}$ so that $v^t \notin B_t$. From this it follows that

$$1 > \sum_{i=1}^{d} \psi_{B_t-f}(u_i) t + \sum_{j=d+1}^{d+k} \psi_{B_t-f}(r_{j-d}) \epsilon = \sum_{i=1}^{d} \psi_{B_t-f}(t u_i) + \sum_{j=d+1}^{d+k} \psi_{B_t-f}(\epsilon r_{j-d}) \quad (3.18)$$

where the equality follows from the positive homogeneity of ψ_{B_t-f}. Two implications of Equation (3.18) are

(i) $\psi_{B_t-f}(t u_i) < 1$ for each $u_i \in U_{bas}$, and so $t U_{bas} \subseteq B_t - f$;

(ii) $\epsilon M + (1-\epsilon) f \subseteq B_t$.

Statements (i) and (ii) imply that the ball of radius $\min\{1, \epsilon\}$ centered at 0 is contained in $B_t - f$ for every $t \in \mathbb{N}$. Hence the sequence of polars $((B_t - f)^\circ)_{t=1}^\infty$ is uniformly bounded, and so Blashke's Selection Theorem [92] yields a convergent subsequence $((B_{t_k} - f)^\circ)_{k=1}^\infty$ in the Hausdorff metric. A limit of such a subsequence is of the form $(B - f)^\circ$, for some $B \in \mathrm{cl}_f(\mathcal{B}_f)$. Moreover, from (i), $B_t - f \supseteq tU_{bas}$ for each $t \in \mathbb{N}$ and so $U \subseteq \mathrm{lin}(B)$. Therefore $B \in \mathcal{B}_U$. $\diamond$

Set $R = (r_1, \ldots, r_k, u_1, \ldots, u_d)$. By construction, $\psi_{L-f}(r_i) = 1$ for every $i \in [k]$ and $\psi_{L-f}(u_i) = 0$ for each $i \in [d]$. Thus

$$C_L(R, f) = \left\{ s \in \mathbb{R}^{k+d}_{\geq 0} \ : \ \sum_{i=1}^{k} s_i \geq 1 \right\}. \tag{3.19}$$

Let

$$\mu := \inf \left\{ \max\{\psi_{B-f}(r_1), \ldots, \psi_{B-f}(r_k)\} \ : \ B \in \mathcal{B}_U \right\}.$$

Claim 3 implies that μ is an infimum over a nonempty set.

Claim 4.

$$\inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \leq \inf_{B \in \mathcal{B}_U} \rho_f(B, L) \leq \mu. \tag{3.20}$$

Proof of Claim. The first inequality follows since $\mathcal{B}_U \subseteq \mathrm{cl}_f(\mathcal{B}_f)$, so it is left to show the second inequality.

Let $\delta > 0$ and choose $B \in \mathcal{B}_U$ so that for each $i \in [k]$, $\psi_{B-f}(r_i) < \mu + \delta$. Thus $\frac{1}{\mu+\delta}\psi_{B-f}(r_i) < 1$ and $f + r_i \in \mathrm{int}(B)$. Therefore, since the set $\{f + r_i \ : \ i \in [k]\}$ denotes the vertices of M, $\frac{1}{\mu+\delta}M + (1 - \frac{1}{\mu+\delta})f \subseteq B$. Moreover, since $B \in \mathcal{B}_U$, $\frac{1}{\mu+\delta}L + (1 - \frac{1}{\mu+\delta})f \subseteq B$. From (3.15), we see that $\rho_f(B, L) \leq \mu + \delta$. Taking the infimum on the left over $B \in \mathcal{B}_U$ and then sending δ to 0, we get (3.20). $\diamond$

Using (3.20) in the case of $\mu = 0$, $\inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L)$ is also 0 showing the lower bound in (3.16). So consider the case $\mu > 0$. We can verify that the point $\frac{1}{\mu}w \in \mathbb{R}^{k+d}_{\geq 0}$ belongs to $C_{\mathcal{B}_U}(R, f)$, where $w_i = 1$ for $i \leq k$ and $w_i = 0$ for $i > k$. Indeed, $\frac{1}{\mu}w \in C_{\mathcal{B}_U}(R, f)$ since from the definition of μ one gets $\sum_{i=1}^{k} \psi_{B-f}(r_i) \geq \mu$ for every $B \in \mathcal{B}_U$. In particular, Claim 3 implies there is some $B \in \mathcal{B}_U$ that contains $\epsilon L + (1 - \epsilon)f$. From Propositions 25 and 26, it follows that $C_B(R, f) \subseteq \epsilon C_L(R, f)$. Hence $\frac{1}{\mu\epsilon}(1, \ldots, 1) \in C_L(R, f)$, and in light of Equation (3.19) this implies

$$\mu \leq \frac{k}{\epsilon} = 2\rho_f(\mathcal{B}, L)(k+1)k.$$

It follows from (3.20) that

$$\left(\frac{1}{2(k+1)k}\right) \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \leq \left(\frac{1}{2(k+1)k}\right) \mu \leq \rho_f(\mathcal{B}, L),$$

which proves the lower bound in (3.16).

□

When deriving necessary and sufficient conditions for one family of cuts to approximate another, the One-for-all Theorem allows us to focus on comparing individual cuts coming from two sets. In particular, we obtain the following characterizations of $\rho_f(\mathcal{B}, L) < \infty$ and $\rho(\mathcal{B}, L) < \infty$.

Theorem 13. *Let $\mathcal{B}$ and L be as in Theorem 12. Then the following statements hold:*

(a) For $f \in \mathrm{int}(L)$, the condition $\rho_f(\mathcal{B}, L) < \infty$ holds if and only if there exist $0 < \mu < 1$ and $B \in \mathrm{cl}_f(\mathcal{B}_f)$ satisfying $B \supseteq \mu L + (1-\mu)f$.

(b) The condition $\rho(\mathcal{B}, L) < \infty$ holds if and only if there exists $0 < \mu < 1$ such that for every $f \in \mathrm{int}(L)$ some $B \in \mathrm{cl}_f(\mathcal{B}_f)$ satisfies $B \supseteq \mu L + (1-\mu)f$.

Another corollary of Theorem 12 is a One-for-all type result for comparing two closures. That is, if the $\mathcal{B}$-closure approximates some α relaxation of the $\mathcal{L}$-closure, one can compare individual cuts coming from some $B \in \mathcal{B}$ and $L \in \mathcal{L}$.

Theorem 14 (One-for-all Theorem for two families)**.** *Let $\mathcal{B}$ be a family of lattice-free subsets of $\mathbb{R}^n$ and let $\mathcal{L}$ be a family of lattice-free polyhedra such that every $L \in \mathcal{L}$ is the direct sum $M \oplus U$ of a polytope M and a linear space U and one has $2(|\mathrm{vert}(M)|^2 + |\mathrm{vert}(M)|) \leq N$ for some $N \in \mathbb{N}$ independent of L. Then*

$$\frac{1}{N} \sup_{L \in \mathcal{L}} \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L) \leq \rho_f(\mathcal{B}, \mathcal{L}) \leq \sup_{L \in \mathcal{L}} \inf_{B \in \mathrm{cl}_f(\mathcal{B}_f)} \rho_f(B, L), \tag{3.21}$$

holds for every $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and one has

$$\frac{1}{N} \sup_{L\in\mathcal{L}, f\in\text{int}(L)} \inf_{B\in\text{cl}_f(\mathcal{B}_f)} \rho_f(B,L) \leq \rho(\mathcal{B},\mathcal{L}) \leq \sup_{L\in\mathcal{L}, f\in\text{int}(L)} \inf_{B\in\text{cl}_f(\mathcal{B}_f)} \rho_f(B,L). \tag{3.22}$$

Similar to Theorem 13, the One-for-all Theorem for two families provides characterizations of conditions $\rho_f(\mathcal{B},\mathcal{L}) < \infty$ and $\rho(\mathcal{B},\mathcal{L}) < \infty$, and these characterizations form the content of Theorem 6.

3.6 The relative strength of i-hedral closures

Theorems 7 and 8 provide characterizations for when a $\mathcal{B}$-closure finitely approximates some $\mathcal{L}$-closure. However, these results do not immediately provide us with any specific (and interesting) families for which this happens. In this section, we characterize the relative strength of i-hedral closures. Section 3.6.2 develops results for an arbitrary f, while Section 3.6.3 considers a fixed f and develops stronger results in that situation. In Section 3.6.1 we introduce some geometric tools useful in applying Theorems 7 and 8.

3.6.1 Truncated pyramids

The One-for-all Theorems provide a way of proving Theorems 7 and 8 that requires scaling of lattice-free polyhedra. One set that is particularly useful in this scaling is a truncated pyramid.

Definition 20 (Truncated pyramid)**.** *Let $M \subseteq \mathbb{R}^n$ be a closed, convex set, and let $M' = (1+\alpha)M + p$, for $\alpha \geq 0$ and $p \in \mathbb{R}^n$. Suppose that $\text{aff}(M) \neq \text{aff}(M')$. Then $P := \text{conv}(M \cup M')$ is a truncated pyramid with bases M and M'.*

In this section, we collect useful results on truncated pyramids. The proofs are relegated to Appendix A.2.

Lemma 3. *Let P be a truncated pyramid with bases M and $M' = (1+\alpha)M + p$. Then*

(a) P can be given by

$$P = \bigcup_{0\leq\lambda\leq 1} \Big((1+\lambda\alpha)M + \lambda p\Big). \tag{3.23}$$

In particular every point $t \in P$ can be given as

$$t = (1 + \lambda\alpha)x + \lambda p$$

with $0 \leq \lambda \leq 1$ and $x \in M$.

(b) *If $f \in P$ can be given as $f = (1 + \mu\alpha)x + \mu p$ with $x \in M$ and $1/3 \leq \mu \leq 1$, then*

$$\frac{1}{4}P + \frac{3}{4}f \subseteq \mathrm{conv}(\{x\} \cup M') \subseteq P. \tag{3.24}$$

Proposition 29. *Suppose $P \subseteq \mathbb{R}^n$ is a truncated pyramid with bases M and M'. Let $f \in \mathrm{int}(P)$ and assume*

(i) *$M \subseteq (\mathbb{R}^{n-1} \times \{\lambda\})$ and $M' \subseteq (\mathbb{R}^{n-1} \times \{\gamma\})$, for $\lambda \in (0, 1)$ and $\gamma \in (-1, 0)$;*

(ii) *$\lambda - \gamma \leq 1$;*

(iii) *for some $k \in \mathbb{N}$, $M = S \oplus U$ with S a k-simplex and U a linear space;*

(iv) *$\frac{1}{4}(P - f) + f$ is not contained in a split.*

Then there exists a polyhedron B with at most $k + 2$ facets such that

$$\frac{1}{4}P + \frac{3}{4}f \subseteq B \subseteq (\mathbb{R}^{n-1} \times [-1, 1]), \tag{3.25}$$

and

$$B \cap (\mathbb{R}^{n-1} \times \{0\}) \subseteq P \cap (\mathbb{R}^{n-1} \times \{0\}). \tag{3.26}$$

3.6.2 Arbitrary f

This section is dedicated to the proof of Theorem 7.

Proposition 30. *Let $L \subseteq \mathcal{L}^n_*$ be such that $L \subseteq (\mathbb{R}^{n-1} \times [\gamma, \lambda])$ for $\gamma \in [-1, 0], \lambda \in [0, 1]$, and $\lambda - \gamma \leq 1$. Define $L_0 := L \cap (\mathbb{R}^{n-1} \times \{0\})$ and suppose there is some $k \in \mathbb{N}$ and $M_0 \in \mathcal{L}^{n-1}_k$ so that $L_0 \subseteq M_0 \times \{0\}$. Then for each $f \in \mathrm{int}(L)$, there is a lattice-free set B with at most $k + 1$ facets such that $\frac{1}{4}L + \frac{3}{4}f \subseteq B$.*

Proof. Fix $f \in \text{int}(L)$. In the case that $\frac{1}{4}L + \frac{3}{4}f$ is contained in a split, we may choose B to be such a split. So we can focus on the case when $\frac{1}{4}L + \frac{3}{4}f$ is not contained in a split.

We may write $M_0 \times \{0\}$ using the inequality description

$$M_0 \times \{0\} = \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : (a_i, 0) \cdot (x, x_n) \leq b_i, i \in [k]\} \cap (\mathbb{R}^{n-1} \times \{0\}).$$

Since $L_0 \subseteq M_0 \times \{0\}$ and L is not contained in a split, the facets of M_0 can be extended in $\mathbb{R}^n$ to create a polyhedron M containing L. Extending those facets and truncating at the hyperplanes $\mathbb{R}^{n-1} \times \{\lambda\}$ and $\mathbb{R}^{n-1} \times \{\gamma\}$, we obtain the polyhedron

$$M := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : (a_i, \alpha_i) \cdot (x, x_n) \leq b_i, i \in [k]\} \cap (\mathbb{R}^{n-1} \times [\gamma, \lambda]).$$

From Lemma 2, the cross-section $M \cap (\mathbb{R}^{n-1} \times \{0\})$ is contained in the set

$$\{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : (a_i, \alpha_i) \cdot (x, x_n) \leq b_i, i \in I'\} \cap (\mathbb{R}^{n-1} \times \{0\}) = S \oplus U,$$

for S a k-simplex and U a linear space, and $I' \subseteq [k]$. Define

$$M' := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : (a_i, \alpha_i) \cdot (x, x_n) \leq b_i, i \in I'\} \cap (\mathbb{R}^{n-1} \times [\gamma, \lambda]).$$

Note $B \subseteq M'$ and $M'_0 := M' \cap (\mathbb{R}^{n-1} \times \{0\}) = M_0$.

Proposition 29 gives a set $B' = \{x \in \mathbb{R}^n : c_j \cdot x \leq d_j, j \in J\}$, where $|J| \leq |I'| + 1$, so that

$$\frac{1}{4}L + \frac{3}{4}f \subseteq B' \subseteq (\mathbb{R}^{n-1} \times [-1, 1]), \tag{3.27}$$

and

$$B' \cap (\mathbb{R}^{n-1} \times \{0\}) \subseteq M_0. \tag{3.28}$$

Finally, define the set $B \subseteq \mathbb{R}^n$ to be

$$B := \{x \in \mathbb{R}^n : c_j \cdot x \leq d_j, j \in J\} \cap \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : (a_i, \alpha_i) \cdot (x, x_n) \leq b_i, i \in [k] \setminus I'\}.$$

Both sets defining B contain $\frac{1}{4}L + \frac{3}{4}f$, so B does as well. Also, B has at most $k+1$ facets. Finally, from Equations (3.27) and (3.28), B is lattice-free.

$\square$

Lemma 4. *Let $L \subseteq \mathbb{R}^n$ be a lattice-free set such that $w(L) \leq 1$. Then for each $f \in \text{int}(L)$, there exists a lattice-free set B with at most $2^{n-1}+1$ facets such that $\frac{1}{4}L + \frac{3}{4}f \subseteq B$.*

Proof. Since $w(L) \leq 1$, by performing a unimodular transformation we may assume that generality that

$$L \subseteq (\mathbb{R}^{n-1} \times [\gamma, \lambda]), \tag{3.29}$$

where $\gamma \in [-1, 0], \lambda \in [0, 1]$, and $\lambda - \gamma \leq 1$. Define $L_0 := L \cap \mathbb{R}^{n-1} \times \{0\}$. Note that L_0 is lattice-free in $\mathbb{R}^{n-1}$. Therefore there exists a maximal lattice free set $M_0 \subseteq \mathbb{R}^{n-1}$ containing L_0. Theorem 10(b) implies that M_0 is a polyhedron with at most 2^{n-1} facets. The result then follows from Proposition 30.

$\square$

Proposition 31. $\rho(\mathcal{L}_i^n, \mathcal{L}_*^n) \leq 4Flt(n)$ *for* $i > 2^{n-1}$.

Proof. From Theorem 6 and Proposition 20, it is enough to show that for each $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and each $L \in \mathcal{L}_*^n \cap \mathcal{C}_f$, there is some set $B \in \mathcal{L}_i^n$ so that $\frac{1}{4\text{Flt}(n)}L + \left(1 - \frac{1}{4\text{Flt}(n)}\right) f \subseteq B$. So let $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Note that $w\left(\frac{1}{\text{Flt}(n)}L + \left(1 - \frac{1}{\text{Flt}(n)}\right) f\right) \leq 1$. Lemma 4 gives the desired result.

$\square$

Proposition 32. $\rho(\mathcal{L}_i^n, \mathcal{L}_{i+1}^n) = \infty$ *for* $i \leq 2^{n-1}$.

Proof. Assuming $\rho(\mathcal{L}_i^n, \mathcal{L}_{i+1}^n) < \infty$, we derive a contradiction. Since $i \leq 2^{n-1}$, Lemma 1 yields that there is some $M \in \mathcal{L}_i^{n-1} \backslash \mathcal{L}_{i-1}^{n-1}$ that is maximal lattice-free for $\mathbb{R}^{n-1}$. We imagine M as embedded in $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. For any $z \in \text{relint}(M)$, let $\alpha_M(z)$ denote the value obtained from Proposition 21 by setting $c = z$.

Fix $z \in \text{relint}(M)$. For each $\varepsilon > 0$, we consider the pyramid

$$L^\varepsilon := \text{conv}\left(\{z + \varepsilon e_n\} \cup \left(\left(1 + \frac{1}{\varepsilon}\right)(M - z) + z - e_n\right)\right)$$

with apex $z+\varepsilon e_n$ and base $\left(1+\frac{1}{\varepsilon}\right)(M-z)+z-e_n$. The pyramid L^ε is a maximal lattice-free polyhedron with exactly $i+1$ facets.

Theorem 6(ii) and Proposition 20 imply the existence of $0<\mu<1$ independent of $\varepsilon>0$ such that for every $f \in L^\varepsilon$ some $B_\epsilon \in \mathcal{L}_i^n$ satisfies $\mu L^\varepsilon + (1-\mu)f \subseteq B_\epsilon$. Let $\epsilon > 0$ and $\gamma \in (0,\epsilon)$ be chosen so that

(a) $\varepsilon(1-\mu)-\mu<0$;

(b) $\mu\left(1+\frac{\gamma}{\varepsilon}\left(\frac{1-\mu}{\mu}\right)\right) > \alpha_M(z)$.

For example, one can choose $\epsilon = \frac{1}{2}(\frac{\mu}{1-\mu})$ and $\gamma = \max\{\frac{\epsilon}{2}, \frac{\epsilon}{2}(1+\frac{\alpha_M(z)-\mu}{1-\mu})\}$.

Now choose $f = z+\gamma e_n$. With this choice of f, the polyhedron

$$\begin{aligned}
L' &:= \mu L^\varepsilon + (1-\mu)f \\
&= \operatorname{conv}\Big(\left\{z+(\mu\varepsilon+(1-\mu)\gamma)e_n\right\} \cup \Big(\mu\left(\left(1+\frac{1}{\varepsilon}\right)(M-z)+z\right)+(\gamma(1-\mu)-\mu)e_n\Big)\Big)
\end{aligned}$$

is a pyramid with apex $z+(\mu\varepsilon+(1-\mu)\gamma)e_n$ and base $\mu\left(\left(1+\frac{1}{\varepsilon}\right)(M-z)+z\right)+(\gamma(1-\mu)-\mu)e_n$. From (a) and the fact that $\gamma<\epsilon$, we obtain that $\gamma(1-\mu)<\mu$, i.e., $\gamma(1-\mu)-\mu<0$. Thus, the base of L' is below the hyperplane $\mathbb{R}^{n-1}\times\{0\}$.

For $\lambda \in \mathbb{R}$, define $L^\varepsilon_\lambda := L_\varepsilon \cap (\mathbb{R}^{n-1}\times\{\lambda\})$ and $L'_\lambda := L' \cap (\mathbb{R}^{n-1}\times\{\lambda\})$.

Claim 5. *Let* $\lambda \in [-1,0]$. *Then* $L^\varepsilon_\lambda = \left(1-\frac{\lambda}{\varepsilon}\right)(M-z)+z+\lambda e_n$.

Proof of Claim. Using the definitions of L^ε_λ and L_ε it follows that

$$\begin{aligned}
x \in L^\varepsilon_\lambda \quad &\text{iff} \quad x_n = \lambda \text{ and } x \in L_\varepsilon \\
&\text{iff} \quad x = \left(\tfrac{1+\lambda}{1+\varepsilon}\right)(z+\varepsilon e_n) + \left(1-\tfrac{1+\lambda}{1+\varepsilon}\right)\left(\left(1+\tfrac{1}{\varepsilon}\right)(y-z)+z-e_n\right), \text{ for } y \in M \\
&\text{iff} \quad x = \left(1-\tfrac{\lambda}{\varepsilon}\right)(y-z)+z+\lambda e_n, \text{ for } y \in M \\
&\text{iff} \quad x \in \left(1-\tfrac{\lambda}{\varepsilon}\right)(M-z)+z+\lambda e_n.
\end{aligned}$$

$\diamond$

Define $\beta := \frac{-\gamma(1-\mu)}{\mu}$. From (a) and the fact that $\gamma<\epsilon$, $\beta \in (-1,0)$.

Claim 6. $L'_0 = \mu L^\epsilon_\beta + (1-\mu)f$.

Proof of Claim. L'_0 is the set of all points in L' such that the n-coordinate is 0. L' is exactly the set of points that can be written as $\mu x + (1-\mu)f$ for some $x \in L^\epsilon$. All points of this form that have 0 in the last coordinate must satisfy $x_n = -\frac{\gamma(1-\mu)}{\mu} = \beta$. Thus, L' is the set of all points that can be written as $\mu x + (1-\mu)f$ for some $x \in L^\epsilon_\beta$. $\diamond$

We now follow the following sequence of equalities.

$$
\begin{aligned}
L'_0 &= \mu L^\varepsilon_\beta + (1-\mu)f && \text{from Claim 6} \\
&= \mu\left(\left(1-\frac{\beta}{\varepsilon}\right)(M-z) + z + \beta e_n\right) + (1-\mu)(z+\gamma e_n) && \text{from Claim 5} \\
&= \mu\left(1-\frac{\beta}{\varepsilon}\right)M + \left(1-\mu\left(1-\frac{\beta}{\varepsilon}\right)\right)z.
\end{aligned}
$$

From (c) and the definition of $\alpha_M(z)$, any lattice-free polyhedron containing L'_0 requires i facets. In particular, B_ε must have at least i facets since the cross-section of B_ε by the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ contains C'_0. Since $B_\varepsilon \in \mathcal{L}^n_i$, B_ε must have exactly i facets. However, for small enough ε, the base of L' can be made to have lattice-width bigger than the flatness constant $\mathrm{Flt}(n-1)$. This would imply that B_ϵ is not a cylinder, because it must contain the base of L'. Therefore B_ε must have a full-dimensional recession cone. However, this contradicts that B_ε is lattice-free. $\square$

Proof of Theorem 7. The proof is the union of Propositions 31 and 32. $\square$

3.6.3 Fixed f

Theorem 7 implies that $\rho_f(\mathcal{L}^n_i, \mathcal{L}^n_*) < \infty$ for any $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$ as long as $i > 2^{n-1}$. Theorem 8, which is the main result in this section, strengthens this for the case of fixed f. Just as with Theorem 7, we organize the proof of Theorem 8 into a chain smaller results. Recall that for the results involving a fixed f, we restrict f to be rational.

Lemma 5. *Let $n \in \mathbb{N}$, $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and $\mu \in (0,1)$. Then*

(a) There exists a maximal lattice-free simplex $L \in \mathcal{L}^n_{n+1} \cap \mathcal{C}_f$ such that, for some choice of $n+1$ integer points $z_1, \dots, z_{n+1}$ in the relative interior of the $n+1$ distinct facets

of L, the following is fulfilled: every closed halfspace disjoint from $\operatorname{int}(\mu L + (1-\mu)f)$ *contains at most one point of the set* $\{z_1, \dots, z_{n+1}\}$.

(b) *For every simplex L in (a), $B \not\supseteq \mu L + (1-\mu)f$ for every $B \in \mathcal{L}_n^{n+1} \cap \mathcal{C}_f$.*

Proof. (a): We argue by induction on n. In the case $n = 1$, we can choose L, z_1 and z_2 such that $L = [z_1, z_2]$ and $z_1 < f < z_2 = z_1 + 1$ for some $z_1 \in \mathbb{Z}$. Assume the assertion is true for some dimension $n \in \mathbb{N}$ and let us verify the assertion in dimension $n+1$. Let $f \in \mathbb{Q}^{n+1} \setminus \mathbb{Z}^{n+1}$ and $\mu \in (0,1)$. Applying a unimodular transformation, we can assume that f has the form $f = (f', 0)$ with $f' \in \mathbb{Q}^n \setminus \mathbb{Z}^n$. By the induction assumption there exists a simplex $L' \in \mathcal{L}_{n+1}^n$ such that, for some choice of $n+1$ integer points $z'_1, \dots, z'_{n+1}$ in the relative interior of the $n+1$ distinct facets of L', every closed halfspace disjoint with $L'_\mu := \mu L' + (1-\mu)f$ contains at most one point of the set $\{z'_1, \dots, z'_{n+1}\}$.

For an $\alpha > 1$ that will be fixed in what follows, we choose $L \in \mathcal{L}_{n+2}^{n+1}$ to be the pyramid $L := \operatorname{conv}(\{a\} \cup F)$ with the apex

$$a := \left(f', \frac{1}{\alpha - 1}\right)$$

and the base

$$F := (\alpha L' + (1-\alpha) f') \times \{-1\}.$$

The cross-section $L \cap (\mathbb{R}^n \times \{0\})$ of L coincides with $L' \times \{0\}$. The apex a lies above the points $(f', 0) \in L' \times \{0\}$ and $(f', -1) \in F$. We now fix $z_i = (z'_i, 0)$ for $i \in [n+1]$. Fix $z_{n+2} := (z'_1, -1)$. Since $\alpha > 1$, one has $L' \times \{-1\} \subseteq \operatorname{relint}(F)$. Thus, $z_{n+2} \in \operatorname{relint}(F)$ and the chosen L is indeed a maximal lattice-free simplex with the integer points $z_1, \dots, z_{n+2}$ lying in the relative interior of the $n+2$ distinct facets of L.

Set $L_\mu := \mu L + (1-\mu) f$ and consider the facet

$$F_\mu := \mu F + (1-\mu)\, f = \big(\alpha\mu L' + (1 - \alpha\mu)\, f'\big) \times \{-\mu\} \tag{3.30}$$

of L_μ. We also consider the polytope $\operatorname{conv}(z_1, \dots, z_{n+2})$ with $n+1$ vertices $z_1, \dots, z_{n+1}$ above $\operatorname{aff}(F_\mu)$ and one vertex z_{n+2} below $\operatorname{aff}(F_\mu)$. The cross-section of the latter polytope by aff(F_μ) is the following polytope

$$C := \Big((1-\mu)\,\mathrm{conv}(z'_1,\dots,z'_{n+1}) + \mu z'_1\Big) \times \{-\mu\}\,.$$

We now observe that $\alpha\mu L' + (1-\alpha\mu)f'$ in (3.30) is a homothetical copy of L' of size $\alpha\mu$, the homothety center f' is in the interior of L', and $(1-\mu)\,\mathrm{conv}(z'_1,\dots,z'_{n+1}) + \mu z'_1$ is a bounded set independent of α. Consequently, the inclusion

$$(1-\mu)\,\mathrm{conv}(z'_1,\dots,z'_{n+1}) + \mu z'_1 \subseteq \mathrm{relint}(\alpha\mu L' + (1-\alpha\mu)\,f')$$

holds for all sufficiently large α. In terms of C and F_μ, the latter means that $C \subseteq \mathrm{relint}(F_\mu)$ holds if α is sufficiently large. It remains to show that with the choice of α established above, the desired property is fulfilled.

So consider an arbitrary closed half-space $H \subseteq \mathbb{R}^{n+1}$ with $H \cap \mathrm{int}(L_\mu) = \emptyset$. We show that H cannot contain more than one point of $\{z_1,\dots,z_{n+2}\}$. In the case of H containing point z_{n+2} we argue that H cannot contain any other point. In fact, if some z_i with $i \in [n+1]$ is also in H, then the whole segment $[z_i, z_{n+2}]$ is in H. This segment contains a point of $x \in C$, while by construction $C \subseteq \mathrm{relint}(F_\mu)$. From this, there is some $y \in [z_i, x] \cap \mathrm{int}(L_\mu)$. Thus, we have found a point of H belonging to $\mathrm{int}(L_\mu)$ which is a contradiction. So, if H contains z_{n+2}, then H does not contain any other point in $\{z_1,\dots,z_{n+2}\}$. Let us switch to the case that H contains a point z_i with $i \in [n+1]$. Note H does not cover $\mathbb{R}^n \times \{0\}$, since $f = (f', 0) \in (\mathbb{R}^n \times \{0\}) \cap \mathrm{int}(L_\mu)$. Thus, $H_0 := H \cap (\mathbb{R}^n \times \{0\})$ is a closed half-space of the vector space $\mathbb{R}^n \times \{0\}$. The induction assumption yields that, apart from z_i, the halfspace H_0 does not contain any other point z_j with $j \in [n+1]$ and $j \neq 0$.

(b): Assume to the contrary that $B \supseteq \mu L + (1-\mu) f$ holds for some $B \in \mathcal{L}_n^{n+1} \cap \mathcal{C}_f$. Since $B \in \mathcal{L}_n^{n+1}$, the interior of B is the intersection of n open halfspaces which have no common points in $\mathbb{Z}^n$. Consequently, $\mathbb{R}^n \setminus \mathrm{int}(B)$ is a union of closed halfspaces $H_1,\dots,H_n$ which cover $\mathbb{Z}^n$. Every half-space H_j with $j \in [n]$ is disjoint with $\mathrm{int}(L_\mu)$. Thus, by (a), every H_j covers at most one of the points $z_1,\dots,z_{n+1}$. On the other hand since the number of halspaces $H_1,\dots,H_n$ is smaller than the number of points $z_1,\dots,z_{n+1}$, there is a half-space

H_j that covers more than one of the points $z_1, \dots, z_{n+1}$, which is a contradiction. $\square$

Proposition 33. *$\rho_f(\mathcal{L}_i^n, \mathcal{L}_{i+1}^n) = \infty$ for each $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and every $i \in \{2, \dots, n\}$.*

Proof. From Theorem 6, it suffices to show that for all $i \in \{1, \dots, n\}$, $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and $\mu \in (0,1)$, there exists $L \in \mathcal{L}_{i+1}^n \cap \mathcal{C}_f$ satisfying $B \not\supseteq \mu L + (1-\mu)f$ for all $B \in \mathcal{L}_i^n \cap \mathcal{C}_f$. For $i = n$, the assertion follows by choosing L as in Lemma 5(b).

Consider the case $i < n$. After applying an appropriate unimodular transformation we may assume that $f = (f', 0, \dots, 0) \in \mathbb{R}^n$ for some $f' \in \mathbb{Q}^i \setminus \mathbb{Z}^i$. Application of Lemma 5 in dimension i yields the existence of a maximal lattice-free simplex $L' \in \mathcal{L}_{i+1}^i$ such that $B' \not\supseteq \mu L' + (1-\mu)f'$ holds for every $B' \in \mathcal{L}_i^i$. We choose $L = L' \times \mathbb{R}^{n-i}$ and show that $B \not\supseteq \mu L + (1-\mu)f$ for every $B \in \mathcal{L}_i^n$. The homothetical copy $\mu L + (1-\mu)f$ of L contains the affine space $A := \{f'\} \times \mathbb{R}^{n-i}$. If $B \not\supseteq A$, we get $B \not\supseteq \mu L + (1-\mu)f$. Otherwise, $B \supseteq A$ and thus B can be represented as $B = B' \times \mathbb{R}^{n-i}$ with $B' \in \mathcal{L}_i^i$. In this case, $B \not\supseteq \mu L + (1-\mu)f$ since $(B \cap \mathbb{R}_i) \not\supseteq \mu L' + (1-\mu)f'$. $\square$

Proposition 34. *$\rho_f(\mathcal{L}_i^n, \mathcal{L}_*^n) < \infty$ for each $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and every $i \in \{n+1, \dots, 2^n\}$.*

Proof. We use induction on n. As a base case, take $n = 1$. Note that $i \in \{n+1, \dots, 2^n\}$ implies $i = 2^n = 2$. Using Theorem 6(a), the result holds since $\mathcal{L}_2^1$ contains all maximal lattice-free sets for $\mathbb{R}^1$. So suppose that $\rho_{f'}(\mathcal{L}_i^{n-1}, \mathcal{L}_*^{n-1}) < \infty$ for each $f' \in \mathbb{Q}^{n-1} \setminus \mathbb{Z}^{n-1}$ and every $i \in \{n, \dots, 2^{n-1}\}$.

Fix $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and take $i \in \{n+1, \dots, 2^n\}$. From Theorem 6(a), it is enough to find some $\mu \in (0,1)$ so that for each $L \in \mathcal{L}_*^n \cap \mathcal{C}_f$, there exists a $B \in \mathcal{L}_i^n$ so that $\mu L + (1-\mu)f \subseteq B$. From here, we consider cases on f. In what follows, let $L \in \mathcal{L}_*^n \cap \mathcal{C}_f$. Since L is lattice-free, the set $L' := \frac{1}{\mathrm{Flt}(n)}(L - f) + f$ satisfies $w(L') \leq 1$. After a unimodular transformation, we may assume that $L' \subseteq (\mathbb{R}^{n-1} \times [-1, 1]) =: C$.

Case 1: Assume that $|f_n| > 0$. We claim $\mu = \frac{|f_n|}{(1+|f_n|)\mathrm{Flt}(n)} \in (0,1)$ gives the desired result. Note that $x \in L'$ implies $|x_n| \leq 1$ and so $|x_n - f_n| \leq 1 + |f_n|$. For each $x \in L'$ it follows that

$$\left| \left(\frac{|f_n|}{1+|f_n|}(x_n - f_n) + f_n \right) - f_n \right| = \frac{|f_n|}{1+|f_n|} |x_n - f_n| \leq |f_n|.$$

Hence $\frac{|f_n|}{1+|f_n|}(L' - f) + f$ is contained in the split $D := \mathbb{R}^{n-1} \times [f_n - |f_n|, f_n + |f_n|]$. Since $f \in \operatorname{int}(L')$ and L' is convex, it is also the case that $\frac{|f_n|}{1+|f_n|}(L' - f) + f \subseteq C$. Hence $\frac{|f_n|}{1+|f_n|}(L' - f) + f = \mu L + (1-\mu) f$ is contained in the split $B := C \cap D$, which is lattice-free.

Case 2: Assume that $f_n = 0$. Define $f' = (f_1, \ldots, f_{n-1}) \in \mathbb{Q}^{n-1} \setminus \mathbb{Z}^{n-1}$. From the induction hypothesis, $\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*) < \infty$. Note that $\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*) = \rho_{f'}(\mathcal{L}^{n-1}_{2^{n-1}}, \mathcal{L}^{n-1}_*)$ for $i - 1 \geq 2^{n-1}$. Set $\mu = \frac{1}{4\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*)} \in (0,1)$. To see that μ satisfies the conclusion, define $L'_0 := L' \cap (\mathbb{R}^{n-1} \times \{0\})$. Observe that, when restricted to $\mathbb{R}^{n-1}$, L'_0 is a lattice-free polyhedron in $\mathcal{L}^{n-1}_* \cap \mathcal{C}_{f'}$. Hence there exists some $B'_0 \in \mathcal{L}^{n-1}_{i-1}$ so that

$$\left(\frac{1}{\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*)}\right) L'_0 + \left(1 - \frac{1}{\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*)}\right) f' \subseteq B'_0.$$

From Proposition 30, there is some $B \in \mathcal{L}^n_i$ so that

$$\mu L + (1-\mu) f = \left(\frac{1}{4\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*)}\right) L + \left(1 - \frac{1}{4\rho_{f'}(\mathcal{L}^{n-1}_{i-1}, \mathcal{L}^{n-1}_*)}\right) f \subseteq B.$$

□

Proof of Theorem 8.

The proof is a union of Propositions 33 and 34.

□

3.7 The case of $n = 2$

When $n = 2$, much is known about approximation guarantees of cut closures [8, 16]. Let $\mathcal{Q} \subseteq \mathcal{L}^2_*$ denote the collection of all maximal lattice-free quadrilaterals. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \subseteq \mathcal{L}^2_*$ denote the collection of all maximal lattice-free Type-1, Type-2, and Type-3 triangles, respectively. The details of these triangle 'types' will not be necessary for our results; refer to [53, 8, 16] for an explanation. It was shown there that [8, 16] $\rho(\mathcal{Q}, \mathcal{L}^2_*) < \infty$ and $\rho(\mathcal{T}_i, \mathcal{L}^2_*) < \infty$ for each $i \in [3]$. In this section we add another family to this list, the family of δ-thin triangles.

Let $\mathcal{T}_2$ denote the set of all Type 2 triangles. A *Type 2 triangle* is a maximal lattice-free triangle that contains a single lattice point on two of its sides, and multiple lattice points on the third.

Definition 21 (δ-thin triangles and $\mathcal{T}_\delta$)**.** *Let $T \in \mathcal{T}_2$ be a Type 2 triangle. Suppose p, v, v' are the vertices of T so that the line segments $[p, v]$ and $[p, v']$ have one lattice point and $[v, v']$ has multiple lattice points (see Figure 3.2). Let $\delta \geq 2$ be a constant. T is δ-thin if $\frac{\|v-p\|}{\|z-p\|} \geq \delta$. Let $\mathcal{T}_\delta$ denote the set of δ-thin triangles.*

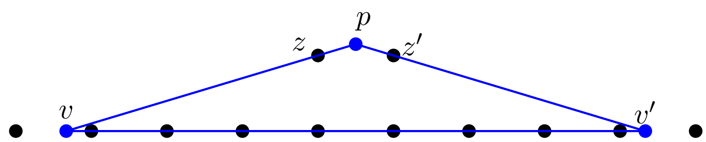


Figure 3.2: A δ-thin triangle.

Theorem 15. *Let $\delta \geq 2$. Then $\rho(\mathcal{T}_\delta, \mathcal{L}^2_*) \leq \frac{3}{2}\delta^3$.*

Proof. Awate et al. [8] showed that $\rho(\mathcal{T}_2, \mathcal{L}^2_*) \leq \frac{3}{2}$. If we can show that $\rho(\mathcal{T}_\delta, \mathcal{T}_2) \leq \delta^3$ then from Proposition 28

$$\rho(\mathcal{T}_\delta, \mathcal{L}^2_*) \leq \rho(\mathcal{T}_\delta, \mathcal{T}_2)\rho(\mathcal{T}_2, \mathcal{L}^2_*) \leq \frac{3}{2}\delta^3.$$

In order to argue that $\rho(\mathcal{T}_\delta, \mathcal{T}_2) \leq \delta^3$, Theorem 14 implies that it is enough to show that for every $T' \in \mathcal{T}_2$ and $f \in \text{int}(T')$, there exists some $T \in \mathcal{T}_\delta$ so that $\rho_f(T, T') \leq \delta^3$. From Proposition 26, $\rho_f(T, T') \leq \delta^3$ is equivalent to

$$\frac{1}{\delta^3}T' + \left(1 - \frac{1}{\delta^3}\right) f \subseteq T.$$

Moreover, if T' has vertices v, w and a, then it is sufficient to show

$$\left\{\frac{1}{\delta^3}v + \left(1 - \frac{1}{\delta^3}\right) f,\ \ \frac{1}{\delta^3}w + \left(1 - \frac{1}{\delta^3}\right) f,\ \ \frac{1}{\delta^3}a + \left(1 - \frac{1}{\delta^3}\right) f\right\} \subseteq \text{int}(T). \tag{3.31}$$

We will proceed by showing (3.31). We use the notation

$$\gamma := \frac{1}{\delta^3} \quad \text{and} \quad \beta := \frac{1}{(\delta-1)^2}$$

to make the computations more presentable.

Let $T' \in \mathcal{T}_2$. If T' is δ-thin, then the result follows by setting $T = T'$. So assume that T' is not δ-thin. Let a, v, and w be the vertices of T' such that the line segment $[v, w]$ contains multiple lattice points. After a unimodular transformation, we may assume that T' has the following form

$$T' = \Big\{(x_1, x_2) \in \mathbb{R}^2 : \quad x_2 \geq 0, \quad x_1 + (v_1 - 1)x_2 \leq v_1, \quad x_1 + w_1 x_2 \geq w_1\Big\}. \tag{3.32}$$

See Figure 3.3.

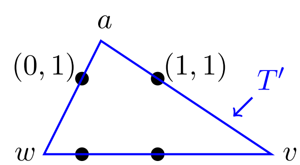


Figure 3.3: T' is not an δ-thin triangle.

The following properties can be proved using (3.32) and the assumption $T' \notin \mathcal{T}_\delta$.

A1. $1 < v_1 < \delta$.

A2. $1 - \delta < v_1 - \delta < w_1 < 0$.

A3. $a_1 \in (0, 1), \quad a = \left(a_1, \frac{v_1 - a_1}{v_1 - 1}\right), \quad \frac{v_1 - a_1}{v_1 - 1} > 1$.

Take $f \in \mathrm{int}(T')$. Without loss of generality, $f_1 \geq \frac{1}{2}$ (if $f_1 < \frac{1}{2}$ then swap the roles of v and w in the following argument). We consider cases dependent on the location of f and v:

Case 1. Suppose $v_1 \leq \frac{\delta}{\delta - 1}$.

Case 2. Suppose $v_1 > \frac{\delta}{\delta - 1}$ and $f_1 \leq 1 - \frac{1}{2}\beta$.

Case 3. Suppose $v_1 > \frac{\delta}{\delta - 1}$ and $f_1 > 1 - \frac{1}{2}\beta$.

In each case, we construct a type-2 triangle T and show that it satisfies five properties.

P1. $T \in \mathcal{T}_\delta$.

P2. $f \in \mathrm{int}(T)$.

P3. $\gamma v + (1 - \gamma) f \in \mathrm{int}(T)$.

P4. $\gamma w + (1 - \gamma) f \in \mathrm{int}(T)$.

P5. $\gamma a + (1-\gamma) f \in \text{int}(T)$.

Note that proving $\mathbf{P1} - \mathbf{P5}$ for each case gives the desired result. The proofs for each case reduces down to inequality manipulation, and so they are relegated to Appendix A.1.

$\square$

Chapter 4

Extreme functions with an arbitrary number of slopes

In this chapter, we focus on the mixed-integer model $I_{b+\mathbb{Z}^n}$, where $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The infinite dimensional interpretation (see Chapter 2.2.3) of $I_{b+\mathbb{Z}^n}$ is called n-row infinite group problem and is denoted by

$$R_b(\mathbb{R}^n, \mathbb{Z}^n) := \left\{ y \in (\mathbb{R}_+)^{(\mathbb{R}^n)} : \sum_{p \in \mathbb{R}^n} p y(p) \in b + \mathbb{Z}^n,\ y \in (\mathbb{Z}_+)^{(\mathbb{R}^n)} \right\}. \tag{4.1}$$

See Chapter 2.4.1 for an introduction to the n-row infinite group problem. We focus mainly on the 1-row problem, also called *the* infinite group problem. Our main contribution is showing the existence of extreme cut-generating functions (extreme is a stronger notion than minimal and will be defined in Section 4.1) that take the form of piecewise linear functions with k slopes, for $k \in \mathbb{N}$. We then extend this idea to the more general n-dimensional case. This work was done in collaboration with Amitabh Basu, Michele Conforti, and Marco Di Summa, and has been published in [22].

4.1 Piecewise linear extreme cut-generating functions

Cut-generating functions for the infinite group problem were first introduced by Gomory and Johnson [66, 67]. The most well known function is the Gomory mixed-integer function, defined as follows:

$$\phi(x) = \begin{cases} \frac{1}{b}x, & 0 \le x < b \\ \frac{1}{1-b} - \left(\frac{1}{1-b}\right)x, & b \le x < 1 \\ \phi(x-j), & x \in [j, j+1),\ j \in \mathbb{Z} \setminus \{0\}\ . \end{cases} \tag{4.2}$$

Figure 4.1 shows ϕ for $b = \frac{1}{2}$, restricted to the interval $[0, 1]$. Note that the highlighted slopes in Figure 4.1 stress that ϕ has two slopes. This property will be revisited later.

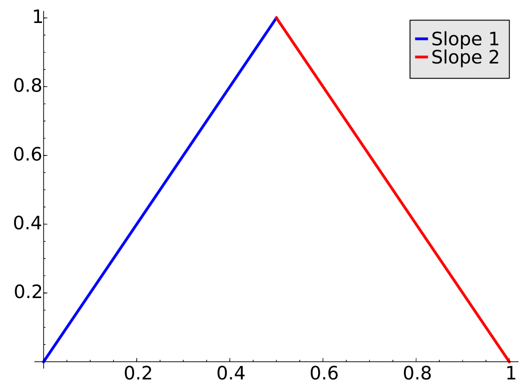


Figure 4.1: The Gomory mixed-integer function ϕ for $b = \frac{1}{2}$.

Theorem 4 in Chapter 2.4.1 states that a nonnegative function $\pi : \mathbb{R} \to \mathbb{R}_+$ is minimal for the 1-row infinite group problem if and only if $\pi(w) = 0$, π is periodic modulo $\mathbb{Z}$, π is subadditive, and π satisfies the symmetry condition. It is easy to check that the Gomory mixed-integer function is subadditive and satisfies the symmetry condition. Therefore, by Theorem 4, it is a minimal function.

Note 3. *When working with the infinite group problem, cut-generating functions are often referred to simply as* valid functions *in the literature [28, 90]. We adopt this convention for the rest of this chapter to stay consistent with current literature.*

Recall from Definition 10 that minimal valid functions are the ones that are not dominated by any other function. However, minimal functions may be implied by convex combinations of other valid functions. This leads to the notion of extreme functions. We define extreme functions only for the infinite group problem, but they can be extended to more general models I_S, and even models of the form C_S and M_S.

Definition 22 (Extreme)**.** *A valid function π for $R_b(\mathbb{R}^n, \mathbb{Z}^n)$ is extreme if it is not the convex combination of any other valid functions for $R_b(\mathbb{R}^n, \mathbb{Z}^n)$.*

If π is an extreme valid function, then π is easily seen to be minimal [66, 67]. We produce the the proof here.

Proposition 35 (Extreme functions are minimal, Gomory and Johnson [66, 67])**.** *Let π be an extreme valid function for $R_b(\mathbb{R}^n, \mathbb{Z}^n)$. Then π is minimal.*

Proof. Assume π is not minimal. From Proposition 2, there is a valid function π' that dominates π. Define the function π'' to be $\pi'' := 2\pi - \pi'$. Let $y \in R_b(\mathbb{R}^n, \mathbb{Z}^n)$ and note that

$$\sum_{p \in \mathbb{R}^n} \pi''(p)y(p) = \sum_{p \in \mathbb{R}^n} (2\pi(p) - \pi'(p))y(p) \geq \sum_{p \in \mathbb{R}^n} \pi(p)y(p) \geq 1,$$

where the first inequality is from $\pi' \leq \pi$ and the second since π is a valid function. Hence π'' satisfies Definition 9 and is therefore a valid function. Since π is equal to the convex combination $\frac{1}{2}\pi' + \frac{1}{2}\pi''$, it is not extreme. □

An even more stringent definition is that of a facet. For any valid function π, define

$$P(\pi) := \left\{ y \in R_b(\mathbb{R}^n, \mathbb{Z}^n) : \sum_{r \in \mathbb{R}^n} \pi(r)y(r) = 1 \right\}. \tag{4.3}$$

Definition 23 (Facet)**.** *A valid function π is a* facet *if $P(\pi) \subseteq P(\pi')$ implies $\pi = \pi'$ for all valid functions π'.*

It can be verified that a facet is extreme [29, Lemma 1.3]. It is not known whether every extreme function is a facet.

Gomory and Johnson [66] provide sufficient conditions for a nonnegative valid function to be a facet. We say a function $\theta : \mathbb{R} \to \mathbb{R}$ is *piecewise linear* if there is a set of closed, non-degenerate intervals I_j, $j \in J$ such that $\mathbb{R} = \cup_{j \in J} I_j$, any bounded subset of $\mathbb{R}$ intersects only finitely many intervals, and θ is affine linear over each interval I_j. Note that in this definition, a piecewise linear function is continuous.

Theorem 16 (2-Slope Theorem [66])**.** *Let $\pi \colon \mathbb{R} \to \mathbb{R}_+$ be a minimal valid function which is piecewise linear and has only 2 slopes. Then π is a facet (and therefore extreme).*

In particular, the above theorem implies that the Gomory mixed-integer function is a facet.

For the one-dimensional infinite group problem, extreme valid functions or facets that are piecewise linear and have few slopes received the largest number of hits in the shooting experiments of Gomory and Johnson [69] and seem to be the most useful in practice. Indeed Gomory and Johnson [68] conjectured that every valid function that is extreme is piecewise linear. This has been disproved by Basu et al. [20].

Note 4. *The shooting experiments [69] provide empirical evidence to determine effective valid functions. Here we provide some intuition for how the experiments work.*

Let $q \in \mathbb{N}$ and choose $b \in \frac{1}{q}\mathbb{Z} \setminus \mathbb{Z}$. One can then define the finite group problem (see (2.15)*)*

$$R_b\left(\frac{1}{q}\mathbb{Z}, \mathbb{Z}\right) := \left\{ y \in (\mathbb{Z}_+)^{\left(\frac{1}{q}\mathbb{Z}\right)} : \sum_{p \in \frac{1}{q}\mathbb{Z}} p y(p) \in b + \mathbb{Z}^n \right\}.$$

Gomory and Johnson showed that $\operatorname{conv}\left(R_b\left(\frac{1}{q}\mathbb{Z}, \mathbb{Z}\right)\right)$ *is isomorphic to a finite-dimensional polyhedron contained in the nonnegative orthant. Moreover, for every facet of* $\operatorname{conv}\left(R_b\left(\frac{1}{q}\mathbb{Z}, \mathbb{Z}\right)\right)$, *one can identify a valid function whose output is a cutting plane describing that particular facet. The shooting experiments 'shot' random directions into the nonnegative orthant and identified which facet of* $\operatorname{conv}\left(R_b\left(\frac{1}{q}\mathbb{Z}, \mathbb{Z}\right)\right)$ *was hit. Intuitively, if a facet is hit frequently then the corresponding valid function for $R_b(\mathbb{R}, \mathbb{Z})$ is more effective. See [69] for more on the shooting experiments.* $\square$

Minimal valid functions with 3 slopes are not always extreme. However, Gomory and Johnson constructed an extreme function that is piecewise linear with 3 slopes. It appears to be hard to construct extreme functions that are piecewise linear with many slopes. Indeed, until 2013, all known families of piecewise linear extreme functions had at most 4 slopes. This had led Dey and Richard to pose the question of constructing extreme functions with more than 4 slopes at a 2010 Aussois meeting [48]. In 2013, Hildebrand, in an unpublished result, constructed an extreme function that is piecewise linear with 5 slopes and very recently Köppe and Zhou [81] constructed an extreme function that is piecewise linear with 28 slopes. These functions were found through a clever computer search.

Köppe and Zhou [81] expressed the belief that there exist extreme functions that are piecewise linear and have an arbitrary number of slopes (this is also stated as an open question in the survey by Basu, Hildebrand and Köppe [28].) We focus on proving this in the current chapter.

4.1.1 Statement of Results

In this chapter, we focus on identifying extreme functions and facets for the n-row infinite group problem that have multiple slopes. Here we summarize the main results in this chapter.

1. We begin by addressing the situation when $n = 1$, i.e. the 1-row infinite group problem or just the infinite group problem.

   Theorem 17. *Let $b \in \mathbb{R} \setminus \mathbb{Z}$. For $k \geq 2$, there exists a facet (and therefore an extreme valid function) for $R_b(\mathbb{R}, \mathbb{Z})$ that is piecewise linear with k slopes.*

   The proof of Theorem 17 provided here is constructive. We define a sequence of functions $\{\pi_k\}_{k=2}^{\infty}$, where π_2 is the Gomory mixed-integer function, and π_3 is an instantiation of a construction of extreme functions that are piecewise linear and have 3 slopes provided by Gomory and Johnson. We first prove some properties about each function π_k. In Chapter 4.3 we use these properties to show that these functions are subadditive and satisfy the symmetry condition. Therefore each function π_k is a minimal valid function, as it satisfies the conditions of Theorem 4. Section 4.4 is devoted to the proof that each function π_k is a facet.

2. Our next result states that the function which is the pointwise limit of this sequence is an extreme function that is continuous and has an infinite number of slopes. The proof appears in Section 4.5.

   Theorem 18. *Let $b \in \mathbb{R} \setminus \mathbb{Z}$. There exists a continuous function π_∞ that is a facet (and therefore extreme) for $R_b(\mathbb{R}, \mathbb{Z})$ with an infinite number of slopes (i.e., values for the derivative of π_∞).*

Note that in Theorems 17 and 18, we may assume $b \in (0,1)$ since extreme functions are periodic with respect to $\mathbb{Z}$. We give constructions to establish Theorems 17 and 18 with b in the interval $(0, \frac{1}{2}]$. One may obtain extreme functions for values of $b \in [\frac{1}{2}, 1)$ by reflecting the constructions about 0. Indeed, this follows from the following result. The proof of Theorem 19 is in Appendix A.4.

Theorem 19. *π is minimal/extreme/facet for $R_b(\mathbb{R}, \mathbb{Z})$ when $b \in (0, 1/2]$ if and only if $\tilde{\pi} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{\pi}(x) := \pi(-x)$ is minimal/extreme/facet for $R_{1-b}(\mathbb{R}, \mathbb{Z})$, respectively.*

3. Our final result uses the *sequential-merge* operation invented by Dey and Richard [51] to construct facets for the n-dimensional infinite group relaxation (for any $n \geq 1$) with an arbitrary number of slopes. The idea is to use the sequential merge operation iteratively on the facets constructed for Theorem 17 and the GMI function from (4.2). See Theorem 21 for a detailed statement.

4.2 A construction of k-slope functions π_k

Let $b \in (0, \frac{1}{2}]$. Let π_2 be the mixed-integer Gomory function defined by (4.2).

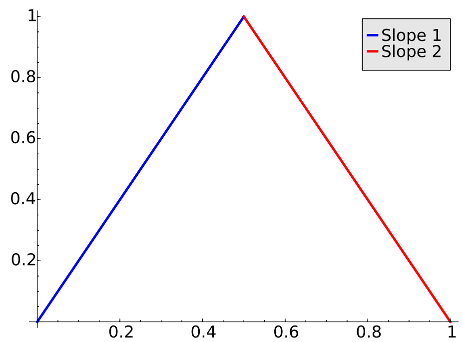
In constructing π_k for $k \geq 3$, we use the following intervals:

$$
\begin{aligned}
I_1^k &:= [0, b(\tfrac{1}{8})^{k-2}], & I_2^k &:= [b(\tfrac{1}{8})^{k-2}, 2b(\tfrac{1}{8})^{k-2}], \\
I_3^k &:= [2b(\tfrac{1}{8})^{k-2}, b - 2b(\tfrac{1}{8})^{k-2}], & I_4^k &:= [b - 2b(\tfrac{1}{8})^{k-2}, b - b(\tfrac{1}{8})^{k-2}], \\
I_5^k &:= [b - b(\tfrac{1}{8})^{k-2}, b], & I_6^k &:= [b, 1).
\end{aligned}
$$

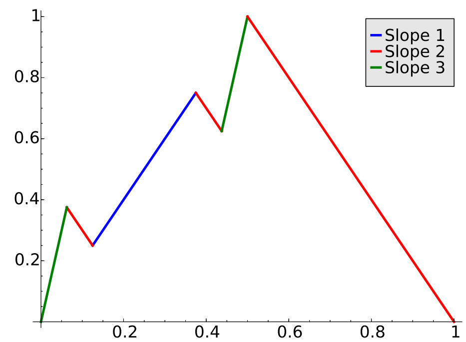
Given π_{k-1}, where $k-1 \geq 2$, define π_k to be

$$\pi_k(x) = \begin{cases} \left(\frac{2^{k-2}-b}{b-b^2}\right)x, & x \in I_1^k \\ \frac{4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right)x, & x \in I_2^k \\ \frac{1-4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right)x, & x \in I_4^k \\ \frac{1-2^{k-2}}{1-b} + \left(\frac{2^{k-2}-b}{b-b^2}\right)x, & x \in I_5^k \\ \pi_{k-1}(x), & x \in I_3^k \cup I_6^k \\ \pi_k(x-j), & x \in [j, j+1),\ j \in \mathbb{Z} \setminus \{0\}\ . \end{cases}$$

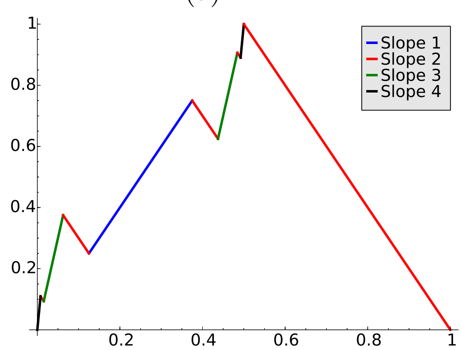
Figure 4.2 shows π_k for various values of k when $b = \frac{1}{2}$. The plots were generated using the help of a software package created by Hong, Köppe, and Zhou [76].



(a) k=2



(b) k=3



(c) k = 4

Figure 4.2: Plots of π_k for $b = \frac{1}{2}$.

Observe that π_k is built recursively with the Gomory mixed-integer function as the base case. Intuitively, π_k is created by adding to π_{k-1} a perturbation on a small interval to the right of 0 and applying a symmetric perturbation on an interval to the left of b; the interval

$[b, 1)$ is kept intact. These small perturbations allow π_k to maintain much of the structure of π_{k-1}, but the number of distinct slopes is increased by one.

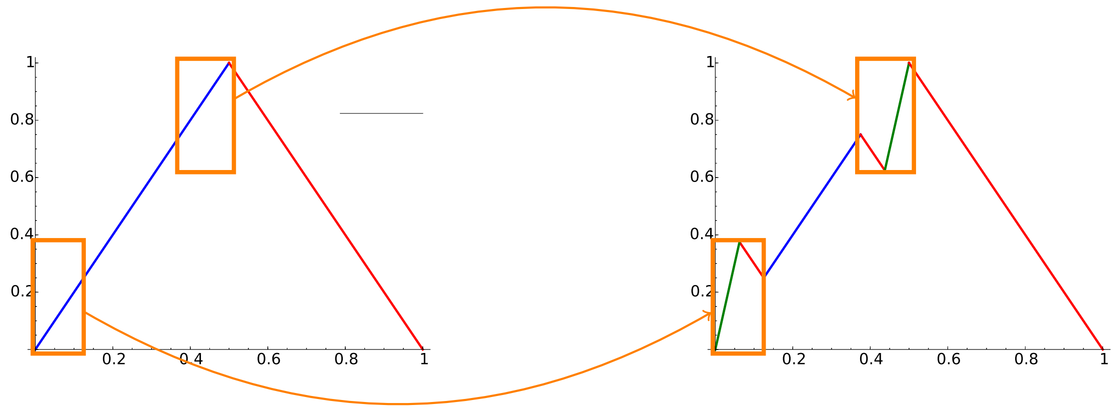


Figure 4.3: Perturbing π_2 to obtain π_3.

We collect some useful properties of π_k in Propositions 36 and 37.

Proposition 36. *Let $k \geq 3$. Then*

(i) $I_1^k \cup I_2^k \subsetneq I_1^{k-1}$ *and* $I_4^k \cup I_5^k \subsetneq I_5^{k-1}$

(ii) If $x \in I_3^k \cup I_6^k$, then $\pi_k(x) = \pi_{k-1}(x)$. If $x \in I_1^k \cup I_2^k$, then $\pi_k(x) \geq \pi_{k-1}(x)$. If $x \in I_4^k \cup I_5^k$, then $\pi_k(x) \leq \pi_{k-1}(x)$.

(iii) $-\pi_k$ is convex on $I_1^k \cup I_2^k$ and π_k is convex on $I_4^k \cup I_5^k$.

(iv) Let $y \in I_4^k \cup I_5^k$ such that $y \neq b$ and take $x \in [0, b-y]$. Then $\pi_k(x+y) \leq \pi_k(y) + \left(\frac{1-\pi_k(y)}{b-y}\right) x$. Also, $\left(\frac{\pi_k(b-y)}{b-y}\right) x \leq \pi_k(x)$.

(v) For any $x \in (0,1) \setminus \{b\}$, there exists some natural number N_x such that $x \in I_3^{N_x} \cup I_6^{N_x}$ and $\pi_{k_1}(x) = \pi_{k_2}(x)$ whenever $k_1, k_2 \geq N_x$.

Proof.

Proof of (i) Observe that

$$b\left(\frac{1}{8}\right)^{k-3} = 8b\left(\frac{1}{8}\right)^{k-2} > 2b\left(\frac{1}{8}\right)^{k-2}.$$

By the definitions of I_1^k, I_2^k and I_1^{k-1}, it follows that $I_1^k \cup I_2^k \subsetneq I_1^{k-1}$. A similar argument shows that $I_4^k \cup I_5^k \subsetneq I_5^{k-1}$.

Proof of (ii) Let $x \in [0,1)$. If $x \in I_3^k \cup I_6^k$, then $\pi_k(x) = \pi_{k-1}(x)$ by definition. If $x \in I_1^k$, then from (i) it follows that $x \in I_1^{k-1}$. Note that

$$\left(\frac{2^{k-2}-b}{b-b^2}\right) x \geq \left(\frac{2^{k-3}-b}{b-b^2}\right) x \ ,$$

and so $\pi_k(x) \geq \pi_{k-1}(x)$. If $x \in I_2^k$, then again from (i), $x \in I_1^{k-1}$ and it follows that

$$\begin{aligned}
\tfrac{4^{2-k}}{1-b} - \left(\tfrac{1}{1-b}\right) x &= \left(\tfrac{1}{1-b}\right)\left(4^{2-k} - x\right) \\
&\geq \left(\tfrac{1}{1-b}\right)\left(4^{2-k} - 2b\left(\tfrac{1}{8}\right)^{k-2}\right) && \text{since } x \in I_2^k \\
&= \left(\tfrac{1}{b-b^2}\right)\left(2^{k-3}\left(2b\left(\tfrac{1}{8}\right)^{k-2}\right) - b\left(2b\left(\tfrac{1}{8}\right)^{k-2}\right)\right) \\
&\geq \left(\tfrac{2^{k-3}-b}{b-b^2}\right) x && \text{since } x \in I_2^k.
\end{aligned}$$

Hence $\pi_k(x) \geq \pi_{k-1}(x)$ on $I_1^k \cup I_2^k$. A similar argument shows that $\pi_k(x) \leq \pi_{k-1}(x)$ on $I_4^k \cup I_5^k$

Proof of (iii) By definition, π_k is affine linear over I_1^k with positive slope and affine linear over I_2^k with negative slope. Since π_k is continuous, it is therefore concave. So $-\pi_k$ is a convex function over $I_1^k \cup I_2^k$. The same argument shows that π_k is convex over $I_4^k \cup I_5^k$.

Proof of (iv) Fix $y \in I_4^k \cup I_5^k \setminus \{b\}$. It follows by assumption that $x+y \in [y,b]$. Therefore $\lambda = \frac{b-x-y}{b-y} \in [0,1]$. Using the facts that π_k is convex over $[y,b]$ from (iii) and $\pi_k(b) = 1$, we obtain

$$\pi_k(x+y) = \pi_k(\lambda y + (1-\lambda)b) \leq \lambda \pi_k(y) + (1-\lambda)\pi_k(b) = \pi_k(y) + \left(\frac{1-\pi_k(y)}{b-y}\right) x.$$

The other inequality follows from the fact that $-\pi_k$ is convex over $I_1^k \cup I_2^k$ by (iii).

Proof of (v) Notice that as $k \to \infty$, I_3^k converges to $(0,b)$ and thus, there exists N_x such that $x \in I_3^{N_x} \cup I_6^{N_x}$. Moreover, by definition on π_k, for any natural number N, $\pi_k(x) = \pi_N(x)$ $\forall x \in I_3^N \cap I_6^N$ for every $k \geq N$. $\square$

Proposition 37. *For each value of k, the function π_k is piecewise linear and has k slopes taking values $-\frac{1}{1-b}$ and $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k}$.*

Proof. We proceed by induction. For π_2, the result follows by definition, so assume that for $k-1 \geq 2$, π_{k-1} is piecewise linear and has slopes taking values $-\frac{1}{1-b}$ and $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-1}$.

Observe that for each value of j, π_j has a slope of $-\frac{1}{1-b}$ on the interval $(b,1)$. Therefore on the interval $[0,b)$, the function π_{k-1} must take on slope values $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-1}$ (π_{k-1} also admits a slope of $-\frac{1}{1-b}$ on subintervals contained in $[0,b)$). By Proposition 36 (ii), $\pi_k = \pi_{k-1}$ everywhere except $I_1^k \cup I_2^k$ and $I_4^k \cup I_5^k$, on which π_k takes on slope values $\frac{2^{k-2}-b}{b-b^2}$ and $-\frac{1}{1-b}$ by definition. Since $I_1^k \cup I_2^k \subsetneq I_1^{k-1}$ and $I_4^k \cup I_5^k \subsetneq I_5^{k-1}$ by Proposition 36 (i), it follows that π_k has slopes taking values $-\frac{1}{1-b}$ and $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k}$.

It is left to show that π_k is piecewise linear. By Proposition 36 (ii) and the induction hypothesis, it is sufficient to show that π_k is piecewise linear on $I_1^k \cup I_2^k$ and $I_4^k \cup I_5^k$, and that $\pi_k\left(2b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_{k-1}\left(2b\left(\frac{1}{8}\right)^{k-2}\right)$ and $\pi_k\left(b - 2f(\frac{1}{8})^{k-2}\right) = \pi_{k-1}\left(b - 2b\left(\frac{1}{8}\right)^{k-2}\right)$. Note that π_k is piecewise linear on $I_1^k \cup I_2^k$ and $I_4^k \cup I_5^k$ by definition. It is straightforward to check that $\pi_k\left(2b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_{k-1}\left(2b\left(\frac{1}{8}\right)^{k-2}\right)$ and $\pi_k\left(b - 2b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_{k-1}\left(b - 2b\left(\frac{1}{8}\right)^{k-2}\right)$. Thus π_k is piecewise linear, as desired. $\square$

4.3 Proof of minimality of π_k

In the proof of Theorem 17, it is required to show that π_k is a minimal valid function for $R_b(\mathbb{R},\mathbb{Z})$. Since by definition $\pi_k(0) = 0$, and π_k is periodic, by Theorem 4, it is sufficient to show that (a) $\pi_k(x) = \pi_k(b-x)$ for all $x \in [0,1)$, i.e. that π_k satisfies the symmetry condition, and (b) π_k is subadditive. We show (a) and (b) in Propositions 38 and 39, respectively.

Proposition 38. *π_k satisfies the symmetry condition for all $k \geq 2$.*

Proof. We proceed by induction on k. The Gomory mixed-integer function is known to be minimal and hence π_2 is symmetric. Assume π_{k-1} satisfies the symmetry condition for $k-1 \geq 2$ and consider $x \in [0,1)$. Observe that $x \in I_1^k$ if and only if $b - x \in I_5^k$. Therefore, if $x \in I_1^k$ then

$$\pi_k(x) + \pi_k(b-x) = \left(\frac{2^{k-2}-b}{b-b^2}\right)x + \frac{1-2^{k-2}}{1-b} + \left(\frac{2^{k-2}-b}{b-b^2}\right)(b-x) = 1 \ .$$

A similar argument can be used to show that π_k satisfies the symmetry condition on the intervals I_2^k and I_4^k. If $x \notin I_1^k \cup I_2^k \cup I_k^4 \cup I_k^5$ then $b - x \notin I_1^k \cup I_2^k \cup I_k^4 \cup I_k^5$, and so symmetry holds by induction. □

Proposition 39. *π_k is subadditive for all $k \geq 2$.*

Proof. We proceed by induction on k. Note that π_2 is subadditive, so assume π_{k-1} is subadditive for $k-1 \geq 2$. By periodicity of π_k, it suffices to check $\pi_k(x)+\pi_k(y) \geq \pi_k(x+y)$ for all $x, y \in [0,1)$ and $x \leq y$.

Claim. *If $y \in I_6^k = [b, 1)$, then $\pi_k(x+y) \leq \pi_k(x) + \pi_k(y)$.*

Proof of Claim. Since π_k is piecewise linear, we may integrate it over any bounded domain. Let π_k' denote the derivative of π_k (where defined). A direct calculation shows

$$\begin{aligned}
\pi_k(x+y) = &\ \pi_k(x + (y-1)) && \text{by periodicity of } \pi_k \\
= &\ \pi_k(x) + \textstyle\int_x^{x-(1-y)} \pi_k'(t)dt \\
= &\ \pi_k(x) + \textstyle\int_{x-(1-y)}^{x} -\pi_k'(t)dt \\
\leq &\ \pi_k(x) + \textstyle\int_y^1 -\pi_k'(t)dt \\
= &\ \pi_k(x) - \pi_k(1) + \pi_k(y) \\
= &\ \pi_k(x) + \pi_k(y) && \text{since } \pi_k(1) = 0.
\end{aligned}$$

The inequality follows from Proposition 37, as the minimum value of the slope for π_k is $-\frac{1}{1-b}$ and this is the slope over the interval $[b,1] \supseteq [y,1]$. This concludes the proof of the claim. ◇

By the above claim, it suffices to consider the case $y < b$. Since $b \leq \frac{1}{2}$, this implies that $x \leq y \leq x+y < 1$.

Case 1: $x+y \in I_1^k \cup I_2^k$. By Proposition 36 (iii), the function $-\pi_k$ is convex over $I_1^k \cup I_2^k$. Therefore $\pi_k(x) + \pi_k(y) \geq \pi_k(x+y)$.

Case 2: $x + y \in I_3^k$. Since $x, y \in I_1^k \cup I_2^k \cup I_3^k$ we have that

$$\pi_k(x) + \pi_k(y) \geq \pi_{k-1}(x) + \pi_{k-1}(y) \geq \pi_{k-1}(x + y) = \pi_k(x + y) \ ,$$

where the first inequality comes from Proposition 36 (ii), the second inequality comes from the induction hypothesis, and the final inequality comes again from Proposition 36 (ii).

Case 3: $x + y \in I_4^k \cup I_5^k$. If $y \in I_1^k \cup I_2^k \cup I_3^k$ then using the induction hypothesis and Proposition 36 (ii), it follows that

$$\pi_k(x) + \pi_k(y) \geq \pi_{k-1}(x) + \pi_{k-1}(y) \geq \pi_{k-1}(x + y) \geq \pi_k(x + y).$$

If $y \in I_4^k \cup I_5^k$ then $x \in [0, b - y]$ and $b - y \in I_1^k \cup I_2^k$. Thus, $x \in I_1^k \cup I_2^k$. Note that

$$\begin{aligned} \pi_k(x + y) \leq & \quad \pi_k(y) + \left(\tfrac{1-\pi_k(y)}{b-y}\right) x && \text{by Proposition 36 (iv)} \\ = & \quad \pi_k(y) + \left(\tfrac{\pi_k(b-y)}{b-y}\right) x && \text{by the symmetry property} \\ \leq & \quad \pi_k(y) + \pi_k(x) && \text{by Proposition 36 (iv).} \end{aligned}$$

Case 4: $x + y \in I_6^k$. π_k has a slope of $-\frac{1}{1-b}$ on the interval $[b, x + y]$. Moreover, by Proposition 37, this is the minimum slope that π_k admits. Therefore,

$$\begin{aligned} \pi_k(x + y) &= \pi(b) + \textstyle\int_b^{x+y} \pi'(t)dt \\ &\leq 1 + \textstyle\int_{b-x}^{y} \pi'(t)dt \\ &= 1 + (\pi_k(y) - \pi_k(b - x)) \\ &= \pi_k(x) + \pi_k(y) \ , \end{aligned}$$

where the last equality follows by the symmetry of π_k. □

4.4 π_k is a facet

By Proposition 37, in order to prove Theorem 17 it suffices to show the following result.

Proposition 40. *π_k is a facet for each $k \geq 2$.*

We dedicate the remainder of the section to proving Proposition 40. To this end, given a function $\theta : \mathbb{R}^n \to \mathbb{R}$, define

$$E(\theta) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \theta(x) + \theta(y) = \theta(x + y)\}. \tag{4.4}$$

Our proof of Proposition 40 is based on the *Facet Theorem*, which gives a sufficient condition for a function to be a facet [29, 66], and the *Interval Lemma*, which first appeared in [68], and was subsequently elaborated upon in [52, 51, 50, 27]; see also the survey [28].

Theorem 20 (Facet Theorem [29, 66])**.** *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a minimal valid function for $R_b(\mathbb{R}^n, \mathbb{Z}^n)$ for some $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Suppose that for every minimal function π' satisfying $E(\pi) \subseteq E(\pi')$, it follows that $\pi' = \pi$. Then π is a facet.*

Lemma 6 (Interval Lemma [68])**.** *Let U, V be non-degenerate closed intervals in $\mathbb{R}$. If $\theta : \mathbb{R} \to \mathbb{R}$ is bounded over U and V, and $U \times V \subseteq E(\theta)$, then θ is affine over U, V and $U + V$ with the same slope.*

We will often use the above lemma when θ is a minimal valid function. In this case θ is bounded, as $0 \leq \theta \leq 1$. We also say a function $\theta : \mathbb{R} \to \mathbb{R}$ is *locally bounded* if it is bounded on every compact interval.

Observation 2. *Let $\theta : \mathbb{R} \to \mathbb{R}_+$ be such that $\theta(0) = 0$ and $\theta(x + z) = \theta(x) + \theta(z)$ for all $x \in \mathbb{R}$ and $z \in \mathbb{Z}$. Then θ is periodic, i.e., $\theta(x + z) = \theta(x)$ for all $x \in \mathbb{R}$ and $z \in \mathbb{Z}$.*

Proof. It suffices to show that $\theta(z) = 0$ for all $z \in \mathbb{Z}$. This is true since $0 = \theta(0) = \theta(-z) + \theta(z)$ for all $z \in \mathbb{Z}$ and θ is nonnegative. □

In the following Claims 7, 8, 9, 10, we develop some tools towards proving facetness.

Claim 7. *Let $k \geq 3$ and let π be a minimal valid function such that $\pi = \pi_k$ on I_6^k. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ satisfying $\theta(0) = 0, \theta(b) = 1$, we must have $\theta = \pi = \pi_k$ on $I_6^k \cup \{1\}$.*

Proof of Claim. Note that $I_6^k \cup \{1\} \equiv [\frac{1+b}{2}, 1] + [\frac{1+b}{2}, 1]$ (modulo 1). Since π is minimal, Theorem 4 implies π is periodic. Since $E(\pi) \subseteq E(\theta)$, Observation 2 shows that θ is periodic. In particular, $\theta(1) = 0 = \pi(1)$ and $\theta(b) = 1 = \pi(b)$. Hence $\pi = \theta$ on the endpoints of $I_6^k \cup \{1\}$. Moreover, $x, y \in [\frac{1+b}{2}, 1]$ implies that

$$\begin{aligned} \pi(x) + \pi(y) = & \left(\tfrac{1}{1-b} - \left(\tfrac{1}{1-b}\right) x\right) + \left(\tfrac{1}{1-b} - \left(\tfrac{1}{1-b}\right) y\right) && \text{since } \pi = \pi_k \text{ on } I_6^k \\ = & \tfrac{1}{1-b} - \left(\tfrac{1}{1-b}\right)(x + y - 1) \\ = & \pi(x + y - 1) \\ = & \pi(x + y) && \text{by periodicity.} \end{aligned}$$

Hence $[\frac{1+b}{2}, 1] \times [\frac{1+b}{2}, 1] \subseteq E(\pi) \subseteq E(\theta)$. Lemma 6 then implies that θ is affine over $I_6^k \cup \{1\}$. Since π is also affine over $I_6^k \cup \{1\}$ and $\pi = \theta$ on the endpoints of $I_6^k \cup \{1\}$, we must have $\pi = \theta$ on $I_6^k \cup \{1\}$. $\diamond$

Claim 8. *Let $k \geq 3$ and let π be a minimal valid function such that $\pi = \pi_k$ on I_3^3. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ satisfying $\theta(\frac{b}{2}) = \frac{1}{2}$, we must have $\theta = \pi = \pi_k$ on I_3^3.*

Proof of Claim. Let $U = \left[\frac{b}{4}, \frac{3b}{8}\right] \subseteq I_3^3$ and note that $U + U = \left[\frac{b}{2}, \frac{3b}{4}\right] \subseteq I_3^3$. For $x, y \in U$, since $\pi = \pi_k$ on I_3^3 we see that

$$\pi(x) + \pi(y) = \frac{1}{b}x + \frac{1}{b}y = \frac{1}{b}(x + y) = \pi(x + y).$$

Hence $U \times U \subseteq E(\pi) \subseteq E(\theta)$. Using Lemma 6, θ is affine over $[\frac{b}{2}, \frac{3b}{4}]$. By assumption, $\theta(\frac{b}{2}) = \pi(\frac{b}{2}) = \frac{1}{2}$. Using this and $(\frac{b}{4}, \frac{b}{4}) \in E(\pi) \subseteq E(\theta)$, it follows that $\theta(\frac{b}{4}) = \pi(\frac{b}{4}) = \frac{1}{4}$. Since π satisfies the symmetry condition and $E(\pi) \subseteq E(\theta)$, θ also satisfies the symmetry condition. This implies $\theta(\frac{3b}{4}) = \pi(\frac{3b}{4}) = \frac{3}{4}$. Therefore, by the affine structure of θ and π over $[\frac{b}{2}, \frac{3b}{4}]$, it follows that $\theta = \pi$ on $[\frac{b}{2}, \frac{3b}{4}]$. The symmetric property of θ and π then yields $\theta = \pi$ on $[\frac{b}{4}, \frac{b}{2}]$ and thus on I_3^3. $\diamond$

Claim 9. *Let $k \geq 3$ and $j \in \{3, \ldots, k\}$. Let π be a minimal valid function such that $\pi = \pi_k$ on $I_2^j \cup I_4^j \cup I_6^j$. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ and $\theta = \pi$ on $I_3^j \cup I_6^j \cup \{1\}$, we must have $\theta = \pi = \pi_k$ on $I_2^j \cup I_4^j$.*

Proof of Claim. Let $U = \left[\frac{3}{2}b\left(\frac{1}{8}\right)^{j-2}, 2b\left(\frac{1}{8}\right)^{j-2}\right] \subseteq I_2^j$ and $V = \left[1 - \frac{1}{2}b\left(\frac{1}{8}\right)^{j-2}, 1\right] \subseteq I_6^j$. Observe that $U + V \equiv I_2^j$ (modulo 1). Moreover, $x \in U$ and $y \in V$ implies

$$\begin{aligned}
\pi(x) + \pi(y) &= \left(\tfrac{4^{2-j}}{1-b} - \left(\tfrac{1}{1-b}\right)x\right) + \left(\tfrac{1}{1-b} - \left(\tfrac{1}{1-b}\right)y\right) && \text{since } \pi = \pi_k \text{ on } I_2^j \cup I_6^j \\
&= \tfrac{4^{2-j}}{1-b} - \left(\tfrac{1}{1-b}\right)(x+y-1) \\
&= \pi(x+y-1) = \pi(x+y) && \text{by periodicity.}
\end{aligned}$$

Thus $U \times V \subseteq E(\pi) \subseteq E(\theta)$, and by Lemma 6, π and θ are affine over I_2^j. Since $\theta = \pi$ on I_3^j by assumption, it must be that $\theta\left(2b\left(\frac{1}{8}\right)^{j-2}\right) = \pi\left(2b\left(\frac{1}{8}\right)^{j-2}\right)$. Since π is periodic, $E(\pi) \subseteq E(\theta)$ and $\theta(1) = \pi(1) = 0$, Observation 2 says that θ is also periodic. Moreover,

$$\begin{aligned}
\theta\left(b\left(\frac{1}{8}\right)^{j-2}\right) &= \theta\left(b\left(\frac{1}{8}\right)^{j-2} + 1\right) && \text{by peridocity} \\
&= \theta\left(\frac{3}{2}b\left(\frac{1}{8}\right)^{j-2} + \left(1 - \frac{1}{2}b\left(\frac{1}{8}\right)^{j-2}\right)\right) \\
&= \theta\left(\frac{3}{2}b\left(\frac{1}{8}\right)^{j-2}\right) + \theta\left(1 - \frac{1}{2}b\left(\frac{1}{8}\right)^{j-2}\right) && \text{since } U \times V \subseteq E(\theta) \\
&= \pi\left(\frac{3}{2}b\left(\frac{1}{8}\right)^{j-2}\right) + \pi\left(1 - \frac{1}{2}b\left(\frac{1}{8}\right)^{j-2}\right) && \text{since } \theta = \pi \text{ on } I_3^j \cup I_6^j \\
&= \pi\left(\frac{3}{2}b\left(\frac{1}{8}\right)^{j-2} + \left(1 - \frac{1}{2}b\left(\frac{1}{8}\right)^{j-2}\right)\right) && \text{since } U \times V \subseteq E(\pi) \\
&= \pi\left(b\left(\frac{1}{8}\right)^{j-2}\right) && \text{by periodicity.}
\end{aligned}$$

This indicates that $\theta = \pi$ on the endpoints of I_2^j. Since both functions are affine over I_2^j, $\theta = \pi$ on I_2^j. Since π satisfies the symmetry condition and $E(\pi) \subseteq E(\theta)$, θ also satisfies the symmetry condition. Using symmetry, we see that $\theta = \pi$ over I_4^j. ◇

Claim 10. *Let $k \geq 3$ and let $j \in \{3, \ldots, k-1\}$. Let π be a minimal valid function such that $\pi = \pi_k$ on $I_1^j \setminus \mathrm{int}(I_1^{j+1} \cup I_2^{j+1})$ and $I_5^j \setminus \mathrm{int}(I_4^{j+1} \cup I_5^{j+1})$. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ and $\theta = \pi$ on $I_2^j \cup I_3^j \cup I_4^j \cup I_6^j$, we must have $\theta = \pi = \pi_k$ on $I_1^j \setminus \mathrm{int}(I_1^{j+1} \cup I_2^{j+1})$ and $I_5^j \setminus \mathrm{int}(I_4^{j+1} \cup I_5^{j+1})$.*

Proof of Claim. Set $I^* := I_1^j \setminus (\mathrm{int}(I_1^{j+1} \cup I_2^{j+1}) \cup \{0\}) = \left[2b\left(\frac{1}{8}\right)^{j-1}, b\left(\frac{1}{8}\right)^{j-2}\right]$. Let

$$U = \left[2b\left(\frac{1}{8}\right)^{j-1}, 4b\left(\frac{1}{8}\right)^{j-1}\right] \subseteq I^*.$$

Note that

$$U + U = \left[4b\left(\frac{1}{8}\right)^{j-1}, b\left(\frac{1}{8}\right)^{j-2}\right] \subseteq I^*$$

and $U \cup (U+U) = I^*$. Since $\pi = \pi_k$ over I^*, a direct calculation shows that $\pi(x) + \pi(y) = \pi_k(x) + \pi_k(x) = \pi_k(x+y) = \pi(x+y)$ for $x, y \in U$, and so $U \times U \subseteq E(\pi) \subseteq E(\theta)$. By Lemma 6, θ is affine over $U+U$ and U with the same slope, and thus affine over I^*. Similarly, π is affine over I^*.

Since $\theta = \pi$ on I_2^j, we have $\theta\left(b\left(\frac{1}{8}\right)^{j-2}\right) = \pi\left(b\left(\frac{1}{8}\right)^{j-2}\right)$. Also, since $2b\left(\frac{1}{8}\right)^{j-1}, 4b\left(\frac{1}{8}\right)^{j-1} \in U$ and $U \times U \subseteq E(\pi) \subseteq E(\theta)$, we see that

$$\begin{aligned}
4\theta\left(2b\left(\frac{1}{8}\right)^{j-1}\right) &= 2\theta\left(2b\left(\frac{1}{8}\right)^{j-1} + 2b\left(\frac{1}{8}\right)^{j-1}\right) \\
&= 2\theta\left(4b\left(\frac{1}{8}\right)^{j-1}\right) = \theta\left(4b\left(\frac{1}{8}\right)^{j-1} + 4b\left(\frac{1}{8}\right)^{j-1}\right) \\
&= \theta\left(b\left(\frac{1}{8}\right)^{j-2}\right) \\
&= \pi\left(b\left(\frac{1}{8}\right)^{j-2}\right) && \text{since } \theta = \pi \text{ on } I_2^j \\
&= 4\pi\left(2b\left(\frac{1}{8}\right)^{j-1}\right).
\end{aligned}$$

Thus $\theta\left(2b\left(\frac{1}{8}\right)^{j-1}\right) = \pi\left(2b\left(\frac{1}{8}\right)^{j-1}\right)$, and so $\theta = \pi$ on the endpoints of I^*. Since both functions are affine on I^*, it follows that $\theta = \pi$ on I^*. Since π satisfies the symmetry condition and $E(\pi) \subseteq E(\theta)$, θ also satisfies the symmetry condition. Symmetry of θ and π yields that $\theta = \pi$ over $I_5^j \setminus \mathrm{int}(I_4^{j+1} \cup I_5^{j+1})$. $\diamond$

Lemma 7. *Let $k \geq 3$ and $j \in \{3, \ldots, k\}$. Let π be a minimal valid function such that $\pi = \pi_k$ on $I_3^j \cup I_6^j$. Then for all locally bounded functions θ such that $E(\pi) \subseteq E(\theta)$ satisfying*

$\theta(0) = 0, \theta(b) = 1$, *we must have* $\theta = \pi = \pi_k$ *on* $I_3^j \cup I_6^j$.

Proof. By Claim 7, we obtain $\theta = \pi$ on $I_6^k = I_6^j = I_6^3$. We prove $\theta = \pi$ on I_3^j by induction on j. For $j = 3$, the result follows from Claim 8 (observe that $E(\pi) \subseteq E(\theta)$ implies that θ is symmetric and therefore $\theta(\frac{b}{2}) = \frac{1}{2}$). We assume the result holds for some j such that $3 \leq j \leq k-1$ and show that it holds for $j+1$. Note that $I_3^{j+1} \cup \{0, b\} = (I_1^j \setminus \operatorname{int}(I_1^{j+1} \cup I_2^{j+1})) \cup I_2^j \cup I_3^j \cup I_4^j \cup (I_5^j \setminus \operatorname{int}(I_4^{j+1} \cup I_5^{j+1}))$. By the induction hypothesis, $\theta = \pi$ on I_3^j. Then from Claims 9 and 10, it follows that $\theta = \pi$ on the rest of I_3^{j+1}. □

Proof of Proposition 40. Let θ be a minimal valid function for $R_b(\mathbb{R}, \mathbb{Z})$ such that $E(\pi_k) \subseteq E(\theta)$. Using $\pi = \pi_k$ in Lemma 7, it follows that $\theta = \pi_k$ on $I_3^k \cup I_6^k$. From Claim 9 and again setting $\pi = \pi_k$, we obtain that $\theta = \pi_k$ on $I_2^k \cup I_4^k$. It is left to show that $\theta = \pi_k$ on I_1^k and I_5^k.

Let $U = \left[0, \frac{b}{2}\left(\frac{1}{8}\right)^{k-2}\right]$ and observe that $U + U = \left[0, b\left(\frac{1}{8}\right)^{k-2}\right] = I_1^k$. Since π_k is additive on I_1^k by definition, $U \times U \subseteq E(\pi_k) \subseteq E(\theta)$. Since θ and π_k are minimal, $\theta(0) = \pi_k(0) = 0$. Also, since $\theta = \pi_k$ on I_2^k, $\theta\left(b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_k\left(b\left(\frac{1}{8}\right)^{k-2}\right)$. Thus $\theta = \pi_k$ on the endpoints of I_1^k. Moreover, Lemma 6 implies that θ is affine over I_1^k. Since π_k is also affine over I_1^k and $\theta = \pi_k$ at the endpoints, we have $\theta = \pi_k$ on I_1^k. The fact that $\theta = \pi_k$ on I_5^k follows by symmetry. Therefore, $\theta = \pi_k$ everywhere. By Theorem 20, π_k is a facet. □

4.5 Proof of Theorem 18

Proof of Theorem 18. Define $\pi_\infty : \mathbb{R} \to \mathbb{R}$ to be the pointwise limit of $\{\pi_i\}_{i=2}^{\infty}$. Since each π_k is minimal, by a standard limit argument, π_∞ is minimal (Proposition 4 in [52], Lemma 6.1 in [28]).

Using Proposition 36 (v), π_∞ is continuous over $(0, b)$ and $(b, 1)$. For $x = 0$ or $x = b$, note that, by definition of π_k, the maximum value of π_k on $I_1^k \cup I_2^k$ is $\frac{2^{4-3k}(2^k - 4b)}{1-b}$, which tends to 0 as $k \to \infty$. By symmetry, the smallest value of π_k on the interval $I_4^k \cup I_5^k$ tends to 1 as $k \to \infty$. Hence, the convergence $\pi_k \to \pi_\infty$ is actually uniform. Therefore π_∞ is continuous everywhere.

We next show that π_∞ is a facet. Let θ be any minimal function such that $E(\pi_\infty) \subseteq E(\theta)$.

If $x = 0$ or $x = b$, then $\pi_\infty(x) = \theta(x)$ by the minimality of π_∞ and θ. So assume that $x \notin \{0, b\}$. By Proposition 36 (v), $x \in I_3^{N_x} \cup I_6^{N_x}$. Observe that $\pi_\infty = \pi_{N_x}$ on $I_3^{N_x} \cup I_6^{N_x}$. Hence, by applying Lemma 7 with $k = j = N_x$ and $\pi = \pi_\infty$, we obtain that $\theta(x) = \pi_\infty(x)$. Therefore, $\theta = \pi_\infty$ everywhere. By Theorem 20, π_∞ is a facet.

We finally verify that π_∞ has infinitely many slopes. Note that for any $k \geq 3$, $\pi_\infty = \pi_k$ on $I_3^k \cup I_6^k$ and recall that π_k has $k-1$ different slopes on $I_3^k \cup I_6^k$. $\square$

4.6 Facets for higher dimensional group relaxations

One can ask if it is possible to find extreme functions with arbitrary number of slopes for the higher-dimensional infinite group relaxation. A trivial way to generalize to higher dimensions is to simply define $\pi_k^n : \mathbb{R}^n \to \mathbb{R}_+$ as $\pi_k^n(x_1, x_2, \ldots, x_n) = \pi_k(x_1)$ and $\pi_\infty^n : \mathbb{R}^n \to \mathbb{R}_+$ by defining $\pi_\infty^n(x_1, x_2, \ldots, x_n) = \pi_\infty(x_1)$. However, one can ask whether there are more "non-trivial" examples. In particular, one can ask whether there exist *genuinely n-dimensional* extreme functions with arbitrary number of slopes for all $n \geq 1$. A function $\theta : \mathbb{R}^n \to \mathbb{R}$ is genuinely n-dimensional if there does not exist a linear map $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and a function $\theta' : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\theta = \theta' \circ T$. The construction of such a "non-trivial" facet is the main result in this section. We use the notation 1_m to denote the vector of all ones in $\mathbb{R}^m$.

Theorem 21. *Let $n, k \in \mathbb{N}$ such that $k \geq n+1$. For any $b \in \mathbb{R}$, there exists a function $\Pi_k : \mathbb{R}^n \to \mathbb{R}_+$ such that Π_k has at least k slopes, is genuinely n dimensional, and is a facet (and thus extreme) for the n-dimensional infinite group relaxation $R_{b\mathbf{1}_n}(\mathbb{R}^n, \mathbb{Z}^n)$.*

We provide a constructive argument for the proof of Theorem 21 using the *sequential-merge* operation developed by Dey and Richard [51]. In particular, we employ Theorem 5 in [51], the assumptions of which will be proved throughout this section. The proof of Theorem 21 is the collection of these results and is presented at the end of the section. We begin with some definitions relating to sequential-merge.

Fix $b \in [0,1)^n \setminus \{0\}$. The *lifting-space representation* of any function $\theta : \mathbb{R}^n \to \mathbb{R}$ is

$[\theta]_b : \mathbb{R}^n \to \mathbb{R}$ defined by

$$[\theta]_b(x) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} b_i \theta\left(x - \lfloor x \rfloor\right).$$

The *group-space representation* of any function $\psi : \mathbb{R}^n \to \mathbb{R}$ is $[\psi]_b^{-1} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$[\psi]_b^{-1}(x) = \frac{\sum_{i=1}^{n} x_i - \psi(x)}{\sum_{i=1}^{n} b_i}.$$

A function $\theta : \mathbb{R}^n \to \mathbb{R}$ is called *superadditive* if $-\theta$ is subadditive. θ is called *pseudo-periodic* if $\theta(x + e_i) = \theta(x) + 1$ for all standard unit vectors $e_i \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The next result follows from Proposition 3 in [51].

Observation 3.

1. *If π is a minimal valid function for $R_b(\mathbb{R}^n, \mathbb{Z}^n)$, then $[\pi]_b$ is superadditive and pseudo-periodic.*

2. *If ψ is a superadditive and pseudo-periodic function then $[[\psi]_b^{-1}]_b = \psi$.*

If $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R}^m \to \mathbb{R}_+$ are valid functions for $R_{b_1}(\mathbb{R}, \mathbb{Z})$ and $R_{b_2}(\mathbb{R}^m, \mathbb{Z}^m)$, respectively, then the *sequential merge* $f \diamond g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is defined as

$$f \diamond g := [\psi]_{(b_1, b_2)}^{-1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is the function $\psi(x_1, x_2) = [f]_{b_1}(x_1 - \lfloor x_1 \rfloor + [g]_{b_2}(x_2 - \lfloor x_2 \rfloor))$.

For this section, we restrict $b \in [1/2, 1)$. Although the specific construction of π_k provided in Section 4.2 uses $b \in (0, 1/2]$, creating π_k for $b \in [1/2, 1)$ can be done by defining $\pi_k(x) := \tilde{\pi}(1-x)$ for $x \in [0, 1]$ (and then enforcing periodicity by $\mathbb{Z}$), where $\tilde{\pi}$ is the function for $R_{1-b}(\mathbb{R}, \mathbb{Z})$ constructed in Section 4.2 (see also Theorem 19). Let ϕ denote the GMI function for $R_b(\mathbb{R}, \mathbb{Z})$ (defined in (4.2)). For $n \in \mathbb{N}$, $n \geq 2$, let $\Pi_k : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\Pi_k(x_1, \ldots, x_n) := \pi_k \diamond (\phi \diamond (\phi \diamond (\ldots \diamond \phi) \ldots)) (x_1, \ldots, x_n),$$

where the sequential merge contains one copy of π_k and $n - 1$ copies of ϕ. Let Φ_m denote

$\phi \diamond (\phi \diamond (... \diamond \phi) ...)$, where there are m copies of ϕ in the sequential merge. One can show using induction on m that $\Phi_m(x) = 0$ if and only if $x \in \mathbb{Z}^m$.

A nice formula for the sequential merge procedure was stated in Proposition 5 of [51] and is provided below.

Observation 4. *For any* $v = (v_1, \dots, v_n) \in \mathbb{R}^n$*,*

$$\Pi_k(v_1, v_2, \dots, v_n) = \frac{(n-1)\Phi_{n-1}(v_2, \dots, v_n) + \pi_k\left(\sum_{i=1}^n v_i - b\Phi_{n-1}(v_2, \dots, v_n)\right)}{n}$$

We require a couple of definitions, before we proceed with the proof of Theorem 21.

1. A function $\theta : \mathbb{R}^d \to \mathbb{R}$ is *non-decreasing* if for all $x, y \in \mathbb{R}^d$, $x \leq y$ implies $\theta(x) \leq \theta(y)$.

2. For a valid function $\pi : \mathbb{R}^d \to \mathbb{R}_+$, the set $E(\pi)$ defined in (4.4) is said to be *unique up to scaling* if for any continuous nonnegative function $\theta : \mathbb{R}^d \to \mathbb{R}_+$ satisfying $E(\pi) \subseteq E(\theta)$, θ is a scaling of π, i.e, $\theta = \alpha\pi$ where $\alpha \in \mathbb{R}$.

We now state the main theorem about the sequential merge operation, due to Dey and Richard [51].

Theorem 22. *[Dey and Richard [51]] Let* $f : \mathbb{R} \to \mathbb{R}$ *be a valid function for* $R_{b_1}(\mathbb{R}, \mathbb{Z})$ *and* $g : \mathbb{R}^d \to \mathbb{R}$ *be a valid function for* $R_{b_2}(\mathbb{R}^d, \mathbb{Z}^d)$ *such that the following hold:*

1. $[f]_{b_1}$ *and* $[g]_{b_2}$ *are both non-decreasing,*

2. $E(f)$ *and* $E(g)$ *are unique up to scaling, and*

3. f *and* g *are facets for their respective infinite group relaxations.*

Then $f \diamond g$ *is a facet for* $R_{(b_1,b_2)}(\mathbb{R}^{d+1}, \mathbb{Z}^{d+1})$*.*[1]

We will prove that Π_k is a facet for $R_{b1_n}(\mathbb{R}^n, \mathbb{Z}^n)$ by verifying that the above hypotheses hold for π_k and Φ_{n-1} in the following propositions.

[1]The definition of facet used in [51] is slightly different from our definition, and corresponds to what the authors in [28] refer to as *weak facet*. However, the proof in [51] works for the definition of facet used in thesis.

Proposition 41. *For each value of $m \in \mathbb{N}$, the functions $[\pi_k]_b$ and $[\Phi_m]_{b\mathbf{1}_m}$ are nondecreasing.*

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Note that the periodicity of π_k implies

$$[\pi_k]_b(y) - [\pi_k]_b(x) = (y - b\pi_k(y)) - (x - b\pi_k(x)) = (y - x) - b(\pi_k(y) - \pi_k(x)).$$

If $[\pi_k]_b(y) < [\pi_k]_b(x)$ then $\frac{1}{b} < \frac{\pi_k(y)-\pi_k(x))}{y-x}$. However, this contradicts that the largest slope (and the only positive slope) in π_k is $\frac{1}{b}$ (this crucially uses the fact that we are using π_k with $b \in [1/2, 1)$). Thus $[\pi_k]_b$ is nondecreasing.

Since $\phi = \pi_2$, it follows that $[\phi]_b$ is nondecreasing. By induction, assume that $[\Phi_{m-1}]_{b\mathbf{1}_{m-1}}$ is nondecreasing and consider $[\Phi_m]_{b\mathbf{1}_m}$. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$ be such that $(x_1, x_2) \leq (y_1, y_2)$. Since $\Phi_m = \phi \diamond \Phi_{m-1}$, Observation 3 (ii) implies

$$\begin{aligned}
[\Phi_m]_{b\mathbf{1}_m}(x_1, x_2) &= [\phi]_b(x_1 + [\Phi_{m-1}]_{b\mathbf{1}_{m-1}}(x_2)) \\
&\leq [\phi]_b(y_1 + [\Phi_{m-1}]_{b\mathbf{1}_{m-1}}(y_2)) \qquad \text{since } [\phi], [\Phi_{n-1}] \text{ are nondecreasing} \\
&= [\Phi_m]_{b\mathbf{1}_m}(y_1, y_2).
\end{aligned}$$

Thus $[\Phi_m]_{b\mathbf{1}_m}$ is nondecreasing. □

Proposition 42. *For each value of $m \in \mathbb{N}$, the sets $E(\pi_k)$ and $E(\Phi_m)$ are unique up to scaling.*

Proof. First, consider π_k and let $\xi : \mathbb{R} \to \mathbb{R}_+$ be a continuous function such that $E(\xi) \supseteq E(\pi_k)$. We claim that $\xi = \xi(b)\pi_k$.

If $\xi(b) = 0$, then $\xi(x) + \xi(b - x) = 0$ for each $x \in \mathbb{R}$ since $E(\xi) \supseteq E(\pi_k)$. As ξ is nonnegative, this implies that $\xi(x) = 0$ for each $x \in \mathbb{R}$ and so $\xi = 0\pi_k$. Now suppose that $\xi(b) \neq 0$. It is sufficient to show that $\tilde{\xi} := \frac{1}{\xi(b)}\xi$ is equal to π_k. Note that $E(\tilde{\xi}) = E(\xi)$. Since $\tilde{\xi}(0) + \tilde{\xi}(b) = \tilde{\xi}(b)$, it follows that $\tilde{\xi}(0) = 0$. Since π_k is periodic and $E(\pi_k) \subseteq E(\tilde{\xi})$, Observation 2 implies that $\tilde{\xi}$ is periodic.

Using $\pi = \pi_k$ and $\theta = \tilde{\xi}$ in Lemma 7, it follows that $\tilde{\xi} = \pi_k$ on $I_3^k \cup I_6^k$. From Claim 9 and again setting $\pi = \pi_k$ and $\theta = \tilde{\xi}$, we obtain that $\tilde{\xi} = \pi_k$ on $I_2^k \cup I_4^k$. It is left to show that $\tilde{\xi} = \pi_k$ on I_1^k and I_5^k.

Let $U = \left[0, \frac{b}{2}\left(\frac{1}{8}\right)^{k-2}\right]$ and observe that $U + U = \left[0, b\left(\frac{1}{8}\right)^{k-2}\right] = I_1^k$. Since π_k is additive on I_1^k by definition, $U \times U \subseteq E(\pi_k) \subseteq E(\tilde{\xi})$. Recall that $\tilde{\xi}(0) = \pi_k(0) = 0$. Also, since $\tilde{\xi} = \pi_k$ on I_2^k, $\tilde{\xi}\left(b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_k\left(b\left(\frac{1}{8}\right)^{k-2}\right)$. Thus $\tilde{\xi} = \pi_k$ on the endpoints of I_1^k. Moreover, Lemma 6 implies that $\tilde{\xi}$ is affine over I_1^k. Since π_k is also affine over I_1^k and $\tilde{\xi} = \pi_k$ at the endpoints, we have $\tilde{\xi} = \pi_k$ on I_1^k. The fact that $\tilde{\xi} = \pi_k$ on I_5^k follows by symmetry (note that $\tilde{\xi}$ is also symmetric because $E(\pi_k) \subseteq E(\tilde{\xi})$). Therefore, $\tilde{\xi} = \pi_k$ everywhere.

Now consider Φ_m for $m \in \mathbb{N}$. Dey and Richard's proof of Theorem 22 shows that if $E(b)$ and $E(g)$ are unique up to scaling, then $E(f \diamond g)$ is also unique up to scaling. If $m = 1$, then $\Phi_m = \phi$. Since $\phi = \pi_2$, then $E(\phi)$ is unique up to scaling. Now an induction argument shows that $E(\Phi_m)$ is unique up to scaling. $\square$

Proposition 43. *For each value of $m \in \mathbb{N}$, the function Φ_m is a facet for $R_{b1_m}(\mathbb{R}^m, \mathbb{Z}^m)$.*

Proof. Using induction, the result is a consequence of Theorem 22; the assumptions of Theorem 22 are the results of Propositions 41 and 42. $\square$

The next two propositions argue that Π_k is genuinely n dimensional with at least k slopes. Note that, unlike the one dimensional setting in which exactly k slopes is attained, we are unsure of exactly how many slopes Π_k attains.

Proposition 44. *The function Π_k is genuinely n dimensional.*

Proof. Assume to the contrary that Π_k is not genuinely n dimensional. Then there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and a function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Pi_k = \Psi \circ T$. Since T is linear with non trivial kernel, there must exist $v = (v_1, v_2, \ldots, v_n) \in \ker(T)$ such that $v \notin \mathbb{Z}^n$. It follows that

$$\Pi_k(v) = \Psi \circ T(v) = \Psi(0) = \Psi \circ T(0) = \Pi_k(0) = 0.$$

Observation 4 states that

$$0 = \Pi_k(v_1, v_2, \ldots, v_n) = \frac{(n-1)\Phi_{n-1}(v_2, \ldots, v_n) + \pi_k\Big(\sum_{i=1}^n v_i - (n-1)b\Phi_{n-1}(v_2, \ldots, v_n)\Big)}{n}$$

which implies that

$$(n-1)\Phi_{n-1}(v_2,\ldots,v_n) = -\pi_k\left(\sum_{i=1}^{n} v_i - (n-1)b\Phi_{n-1}(v_2,\ldots,v_n)\right).$$

The left hand side is non-negative and the right hand side is non-positive, indicating that both expressions are 0. Since Φ_{n-1} is only 0 at $\mathbb{Z}^{n-1}$, $(v_2,\ldots,v_n)\in\mathbb{Z}^{n-1}$. Substituting this into the right hand side, we see that $\pi_k(v_1+z)=0$ where $z=v_2+\ldots+v_n\in\mathbb{Z}$, implying that $v_1\in\mathbb{Z}$. Hence $v\in\mathbb{Z}^n$, which is a contradiction. So Π_k is genuinely n-dimensional. $\square$

Lemma 8. *For each $m\in\mathbb{N}$, $[\Phi_m]_{b\mathbf{1_m}}(x)=0$ for all $x\in\mathbb{R}^m_+$ such that $\|x\|_\infty<b$.*

Proof. We proceed using induction. For $m=1$, $[\Phi_m]_{b1_m}(x)=[\phi]_b(x)=x-b\phi(x)$. Since $x\in(0,b)$, $\phi(x)=\frac{x}{b}$ and so $[\phi]_b(x)=0$, as desired.

Assume $[\Phi_{m-1}]_{b1_{m-1}}(x)=0$ for all $x\in\mathbb{R}^{m-1}_+$ such that $\|x\|_\infty<b$. Consider $[\Phi_m]_{b1_m}$ and take $x=(x_1,\ldots,x_m)\in\mathbb{R}^m$ such that each $x_i\in(0,b)$ of all i. Below, we will use the shorthand $x_{-1}:=(x_2,\ldots,x_m)$. Notice that

$$\begin{aligned}
&[\Phi_m]_{b1_m}(x_1,\ldots,x_m)\\
=&\sum_{i=1}^{m} x_i - mb\Phi_m(x_1,\ldots,x_m) && \text{by definition}\\
=&\sum_{i=1}^{m} x_i - mb\left(\frac{(m-1)\Phi_{m-1}(x_{-1})+\phi\left(x_1+\sum_{i=2}^{m}x_i-(m-1)b\Phi_{m-1}(x_{-1})\right)}{m}\right)\\
& && \text{by Observation 4}\\
=&\sum_{i=1}^{m} x_i - mb\left(\frac{(m-1)\Phi_{m-1}(x_{-1})+\phi\left(x_1\right)}{m}\right) && \text{by induction hypothesis}\\
=&(x_1-b\phi(x_1))+\left(\sum_{i=2}^{m}x_i-b(m-1)\Phi_{m-1}(x_{-1})\right)\\
=&(x_1-b\phi(x_1)) && \text{by induction hypothesis}\\
=&0 && \text{since } x_1\in(0,b).
\end{aligned}$$

$\square$

Lemma 9. *For each $m \in \mathbb{N}$, Φ_m is affine over $B_m := \{x \in \mathbb{R}^m_+ : \|x\|_\infty < b\}$.*

Proof. We proceed by induction. For $m = 1$, $\Phi_m = \phi$, which is affine over $[0, b)$. Suppose that Φ_{m-1} is affine over B_{m-1}. Thus, there exists some $d_{m-1} \in \mathbb{R}^{m-1}$ such that

$$\Phi_{m-1}(x) - \Phi_{m-1}(y) = d_{m-1} \cdot (x - y)$$

for all $x, y \in B_{m-1}$.

Consider Φ_m and let $(x_1, \dots, x_m), (y_1, \dots, y_m) \in B_m$. By Observation 4, it follows that $\Phi_m(x_1, \dots, x_m) - \Phi_m(y_1, \dots, y_m)$ can be written as

$$\begin{aligned} &\frac{(m-1)(\Phi_{m-1}(x_2, \dots, x_m) - \Phi_{m-1}(y_2, \dots, y_m))}{m} + \\ &\frac{(\phi\left(x_1 + \sum_{i=2}^m x_i - (m-1)b\Phi_{m-1}(x_2, \dots, x_m)\right))}{m} - \\ &\frac{(\phi\left(y_1 + \sum_{i=2}^m y_i - (m-1)b\Phi_{m-1}(y_2, \dots, y_m)\right))}{m}. \end{aligned}$$

Applying the induction hypothesis to the first fraction and Lemma 8 to the last two fractions, this can be reduced to

$$\frac{m-1}{m}(d_{m-1} \cdot ((x_2, \dots, x_m) - (y_2, \dots, y_m))) + \frac{1}{m}\left(\phi(x_1) - \phi(y_1)\right).$$

As $x_1, y_1 < b$, $\phi(x_1) = \frac{x_1}{b}$ and $\phi(y_1) = \frac{y_1}{b}$. Applying this identity and rearranging, we see that

$$\Phi_m(x_1, \dots, x_m) - \Phi_m(y_1, \dots, y_m) = \left(\frac{1}{bm}, \left(\frac{m-1}{m}\right) d_{m-1}\right) \cdot \left((x_1, \dots, x_m) - (y_1, \dots, y_m)\right).$$

Hence Φ_m is affine over B_m with gradient $\left(\frac{1}{bm}, \left(\frac{m-1}{m}\right) d_{m-1}\right)$. □

Proposition 45. *The function Π_k has at least k slopes.*

Proof. By Theorem 17, π_k has k intervals $J_1, \dots, J_k \subseteq \mathbb{R}$ such that π_k is affine over each J_i with slope σ_i. Moreover, $\sigma_i \neq \sigma_j$ for $i \neq j$. For each $i = 1, \dots, k$, let $R_i \subseteq \mathbb{R}^n$ be defined by

$$R_i := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \in J_i,\ (x_2, \dots, x_n) \in B_{n-1}\},$$

where $B_{n-1} = \{x \in \mathbb{R}_+^{n-1} : \|x\|_\infty < b\}$. We claim that Π_k is affine over each R_i, and attains a different slope on each R_i.

In order to see that Π_k is affine over R_i, let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in R_i$. We can expand $\Pi_k(x_1, \ldots, x_n) - \Pi_k(y_1, \ldots, y_n)$ using the definition of Π_k and Observation 4 to obtain

$$\frac{(n-1)\left(\Phi_{n-1}(x_2, \ldots, x_n) - \Phi_{n-1}(y_2, \ldots y_n)\right)}{n} + \\ \frac{\pi_k\left(x_1 + \sum_{i=2}^n x_i - (n-1)b\Phi_{n-1}(x_2, \ldots, x_n)\right)}{n} - \\ \frac{\pi_k\left(x_1 + \sum_{i=2}^n y_i - (n-1)b\Phi_{n-1}(y_2, \ldots, y_n)\right)}{n}.$$

Using Lemmas 8 and 9, we can reduce this to

$$\left(\frac{n-1}{n}\right) d_{n-1}\left((x_2, \ldots, x_n) - (y_2, \ldots, y_n)\right) + \frac{\pi_k(x_1) - \pi_k(y_1)}{n}$$

where $d_{n-1} \in \mathbb{R}^{n-1}$ is the gradient associated to Φ_{n-1} over B_{n-1}. Since $x_1, x_2 \in J_i$, the value $\pi_k(x_1) - \pi_k(y_1) = \sigma_i(x_1 - y_1)$. Substituting this in above and rearranging, we see that

$$\Pi_k(x_1, \ldots, x_n) - \Pi_k(y_1, \ldots, y_n) = \left(\frac{\sigma_i}{n}, \left(\frac{n-1}{n}\right) d_{n-1}\right) \cdot \left((x_1, \ldots, x_n) - (y_1, \ldots, y_n)\right).$$

Since each σ_i is distinct, for $i = 1, \ldots, n$, each gradient $\left(\frac{\sigma_i}{n}, \left(\frac{n-1}{n}\right) d_{n-1}\right)$ is distinct. Note that as R_i is full dimensional, this vector is indeed a gradient.

Hence, Π_k has at least k slopes, as desired. □

Proof of Theorem 21. Propositions 41, 42 and 43 show that π_k and Φ_{n-1} satisfy the assumptions for Theorem 5 in [51]. Thus Π_k is a facet for $R_{b1_n}(\mathbb{R}^n, \mathbb{Z}^n)$. Proposition 44 shows that Π_k is genuinely n dimensional, and Proposition 45 shows that Π_k has at least k slopes. □

Chapter 5

Operations that preserve the covering property

This chapter focuses on the mixed-integer model M_S, where $S = (b+\Lambda)\cap C$ is a polyhedrally-truncated affine lattice. In Section 2.5, we introduced the *lifting region*

$$R_\psi = \{r \in \mathbb{R}^n : \psi(r) = \pi(r) \text{ for all minimal liftings } \pi \text{ of } \psi\}$$

of a cut-generating function ψ for C_S. The lifting region provides a method for building a minimal cut-generating pair for M_S by using minimal a cut-generating function ψ for C_S. In particular, if ψ has the *covering property* then there is a unique minimal lifting π of ψ. If ψ is also minimal, then (ψ, π) is minimal for M_S. When $\Lambda = \mathbb{Z}^n$, minimal ψ for C_S correspond with maximal S-free sets 0-neighborhoods (see Proposition 13). This result extends to arbitrary lattices Λ. So what S-free sets correspond to functions with the covering property? In this chapter, we address this by developing set operations that preserve the covering property. This work was done in collaboration with Amitabh Basu and has been published in [30].

5.1 Identifying functions with the covering property

Let S be a polyhedrally-truncated affine lattice, that is $S = (b + \Lambda) \cap C$, where $\Lambda \subseteq \mathbb{R}^n$ is a lattice, $b \in \mathbb{R}^n \setminus \Lambda$, $C \subseteq \mathbb{R}^n$ convex, and $\operatorname{conv}(S)$ is a polyhedron. In the case of $\Lambda = \mathbb{Z}^n$, Proposition 13 states that minimal cut-generating functions for C_S have a nice correspondence with maximal S-free 0-neighborhoods. This result extends to general polyhedrally-truncated affine lattices.

Proposition 46. *Let $S = (b+\Lambda) \cap C$ be a polyhedrally-truncated affine lattice. A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function for C_S if and only if it is of the from*

$$\psi(x) = \max_{i \in [m]} a_i \cdot x, \tag{5.1}$$

where $m \in \mathbb{N}$ and B is a maximal S-free 0-neighborhood

$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1 \;\; i \in [m]\}. \tag{5.2}$$

The proof of Proposition 46 is given in Appendix A.5. The exciting observation is that we can compute the values $\psi(r)$ of the cut-generating function very quickly using the formula (5.1). Can we find similar formulas for cut-generating *pairs* ?

This led Dey and Wolsey [56] to import the idea of *monoidal strengthening* into this context. Monoidal strengthening was a method introduced by Balas and Jeroslow [13] to strengthen cutting planes by using integrality information. This inspired Dey and Wolsey to define the notion of a lifting of ψ (see Definition 12). As discussed in Chapter 2.5, liftings provide an approach to obtain formulas for minimal cut-generating pairs: start with a minimal cut-generating function ψ for C_S that has an easily computable formula like (5.1) and find minimal liftings π for ψ. Hopefully, a formula for π can also be derived easily from the formula for ψ. This was explicitly proved to be the case under certain conditions in [5]. This provides evidence to support Dey and Wolsey's method for finding efficient procedures to compute cut-generating pairs.

In Chapter 2.5, it was discussed that there is some regularity in the structure of minimal liftings. For instance, recall the set

$$W_S = \{w \in \mathbb{R}^n : s + \lambda w \in S \ , \forall s \in S, \forall \lambda \in \mathbb{Z}\} \tag{5.3}$$

and the lifting region

$$R_\psi = \{r \in \mathbb{R}^n : \psi(r) = \pi(r) \text{ for every minimal lifting } \pi \text{ of } \psi\}. \tag{5.4}$$

Proposition 16 shows that every minimal lifting π of ψ is periodic along W_S. This implies that every minimal lifting of ψ agrees on the set $R_\psi + W_S$.

Observation 5. *Suppose π_1 and π_2 are minimal liftings of ψ. Then $\pi_1(x) = \pi_2(x)$ for all $x \in R_\psi + W_S$.*

Proof of Observation. Let $x = r + w \in R_\psi + W_S$ for $r \in R_\psi$ and $w \in W_S$. Using Proposition 16 and the definition of R_ψ, It follows that

$$\pi_1(x) = \pi_1(r + w) = \pi_1(r) = \pi_2(r) = \pi_2(r + w) = \pi_2(x).$$

Since $x \in R_\psi + W_S$ was arbitrarily chosen, π_1 and π_2 agree on $R_\psi + W_S$. ◇

For a general S and a cut-generating function ψ for C_S, if $R_\psi + W_S = \mathbb{R}^n$ then ψ is said to have the *covering property.* From Observation 5, if ψ has the covering property then there is a unique minimal lifting of ψ. Moreover, from Proposition 17, the trivial lifting expresses this unique minimal lifting compactly in terms of ψ

$$\pi^*(r) = \inf_{w \in W_s} \psi(r + w). \tag{5.5}$$

In fact, Proposition 17 shows something stronger: π^* is a minimal lifting if $R_\psi + W_S = \mathbb{R}^n$ (and thus must be the unique minimal lifting) and the infimum in (5.5) is attained by any w such that $r + w \in R_\psi$. Therefore, if an explicit description for R_ψ can be obtained, then the coefficient $\pi^*(p_j)$ for the unique lifting can be computed by finding the w such that $p_j + w \in R_\psi$, and then using the formula for $\psi(p_j + w)$[1]. A central result in [17] was to show

[1] For the special case when S is the intersection of a translated lattice and a polyhedron, a proof similar

that when S is a polyhedrally-truncated affine lattice with $\Lambda = \mathbb{Z}^n$, R_ψ can be described as the finite union of full dimensional polyhedra, each of which has an explicit inequality description. This result can be extended to general Λ.

Therefore, in this approach of using liftings of minimal cut-generating functions to obtain computational efficiency with cut-generating pairs, two questions become important:

(i) For which kinds of sets S can we find explicit descriptions of R_ψ for any minimal cut-generating function ψ for C_S? The most general S that we know the answer to is when S is the intersection of a translated lattice with a rational polyhedron [17].

(ii) For which pairs S, ψ, where ψ is a minimal cut-generating function for C_S, is $R_\psi + W_S = \mathbb{R}^n$?

5.1.1 Statement of Results

In this chapter, we make some progress towards the covering question (ii) stated above for the special case when $S = (b+\Lambda) \cap C$ is a polyhedrally-truncated affine lattice. From Proposition 13, the minimal cut-generating functions for such S are in one-to-one correspondence with maximal S-free 0-neighborhoods. For any such maximal S-free 0-neighborhood B, we refer to the lifting region R_ψ for the minimal cut-generating function ψ corresponding to B by $R(S; B)$, to emphasize the dependence on S and B. We say $R(S; B)$ has the *covering property* if $R(S; B) + W_S = \mathbb{R}^n$. When S is clear from the context, we will also say B has the *covering property* if $R(S; B)$ has the covering property.

1. Let S be a translated lattice intersected with a rational polyhedron and let B be a maximal S-free 0-neighborhood. Then $R(S; B) + W_S = \mathbb{R}^n$ if and only if $R(T(S), T(B)) + W_{T(S)} = \mathbb{R}^n$ for all invertible affine transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(B)$ is also a 0-neighborhood. In other words, the covering property is preserved under invertible affine transformations. This is the content of Theorem 25. This result was first proved for the special case when S is a translated lattice, B is a maximal S-free

to Proposition 1.1 in [5] can be used to show that $\pi^*(p)$ can be computed in polynomial time when the dimension n is considered fixed, assuming the data is rational.

simplicial polytope and T is a simple translation [24]. In [5], the result was generalized to all maximal S-free sets when S is a translated lattice and T is a simple translation. Here we generalize the result to all maximal S-free sets where S is the intersection of a translated lattice and a rational polyhedron, and allow for T to be any general invertible affine transformation (which, of course, includes simple translations as a special case). Moreover, the proofs in [24] and [5] are based on volume arguments, whereas our proofs are based on a completely different topological argument. It makes the proof much cleaner, albeit at the expense of using more sophisticated topological tools like the "Invariance of Domain" theorem. The volume arguments are difficult to extend to tackle more general S sets and general affine transformations T, and hence we feel that our approach has a better chance of success for attacking the general covering question (ii) above.

2. In Section 5.4, we define a binary operation on polyhedra that preserves the covering property. Namely, given two polyhedra X_1 and X_2, we define the *coproduct* $X_1 \lozenge X_2$ which is a new polyhedron that has nice properties in terms of the lifting region. More precisely, let $n = n_1 + n_2$. For $i \in [2]$, let $S_i = (b_i \cap \Lambda_i) \cap C_i$ be a polyhedrally-truncated affine lattice. Theorem 26 shows that if B_i is maximal S_i-free such that $R(S_i, B_i)$ has the covering property for $i \in [2]$, then $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ is maximal $S_1 \times S_2$-free and $R(S_1 \times S_2, \frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu})$ has the covering property for every $\mu \in (0,1)$. This is an extremely useful operation to create higher dimensional maximal S-free sets with the covering property by "gluing" together lower dimensional such sets. This result is a generalization of a result from [5], where this was shown when S is a translated lattice, and only lattice-free *polytopes* were considered. Here we give the result for more general S sets, and perhaps more interestingly, extend the operation to unbounded S-free sets. It is worth noting that a trivial extension of the operation defined in [5] does not work in the more general setting. The operation defined in this manuscript utilizes prepolars which seems to be the right way to generalize and also leads to simpler proofs compared to [5]; see Section 5.4 for a discussion.

3. We show that if a sequence of maximal S-free sets all having the covering property,

converges to a maximal S-free set (in a precise mathematical sense), then the "limit" set also has the covering property; see Theorem 27. This result is a generalization of a result from [5] where this was shown when S is a translated lattice, and only lattice-free polytopes were considered. Here we consider general S sets and allow unbounded S-free sets.

The importance of these results in terms of cutting planes is the following. Result 1. above has important practical consequences in generating cutting planes, even in the special case when the affine transformation T is a simple translation. The cutting planes from maximal S-free sets for mixed-integer linear programs are useful for cutting off a basic feasible solution of the LP relaxation. Different basic feasible solutions correspond to different S sets, translated by a vector. The translation theorem tells us that if a certain S-free set B has good formulas because it has the covering property at a particular basic feasible solution, then B will give rise to good formulas at other basic feasible solutions as well, even though the S set has changed because the basic feasible solution has changed. The situation at the new basic feasible solution can be modeled by translating S and B.

Work by Dey and Wolsey [55, 56] has established a "base set" of maximal S-free sets with the covering property in $\mathbb{R}^2$. By iteratively applying the three operations stated in results 1., 2. and 3. above, we can then build a vast (infinite) list of maximal S-free sets (in arbitrarily high dimensions) with the covering property, enlarging this "base set". Moreover, in [5], specific classes of maximal S-free polytopes in general dimensions were shown to have the covering property. This contributes to a larger "base set" from which we can build using the operations in results 1., 2. and 3. Not only does this recover all the previously known sets with the covering property, it vastly expands this list. Earlier, ad hoc families of S-free sets were proven to have the covering property - now we have generic operations to construct infinitely many families. See Section 5.6 for more discussion. From a broader perspective, we believe it makes a contribution in the modern thrust on obtaining efficiently computable formulas for computing cutting planes, by giving a much wider class of cut-generating functions whose lifting regions have the covering property. As discussed earlier, this property is central for obtaining computable formulas for minimal liftings.

5.2 Preliminaries

In order to discuss the material in this chapter, we require a bit more background.

Properties of the translation set W_S

Given any arbitrary set $S \subseteq \mathbb{R}^n$, we collect some simple observations about the set W_S defined in (6.6). Note that W_S is a subgroup of $\mathbb{R}^n$, i.e., $0 \in W_S$, $w_1 + w_2 \in W_S$ for every $w_1, w_2 \in W_S$ and $-w \in W_S$ for every $w \in W_S$. We observe below how W_S changes as certain operations are performed on S. The proofs are straightforward and are relegated to Appendix A.6.

Proposition 47. *The following are true:*

(i) $W_{M(S)+m} = MW_S$ *for all sets* $S \subseteq \mathbb{R}^n$*, translation vectors* $m \in \mathbb{R}^n$*, and invertible linear transformations* $M : \mathbb{R}^n \to \mathbb{R}^n$*. In particular,* $W_{\mu S} = \mu W_S$ *for all sets* $S \subseteq \mathbb{R}^n$ *and all* $\mu \in \mathbb{R} \setminus \{0\}$*.*

(ii) $W_{S_1 \times S_2} = W_{S_1} \times W_{S_2}$ *for all sets* $S_1 \subseteq \mathbb{R}^{n_1}, S_2 \subseteq \mathbb{R}^{n_2}$*. Note that* $S_1 \times S_2 \subseteq \mathbb{R}^{n_1+n_2}$*.*

When S is a nonempty truncated affine lattice, W_S is a lattice; in particular, we can rewrite W_S as the intersection of $\operatorname{lin}(\operatorname{conv}(S))$ and the lattice Λ.

Proposition 48. *Let* $S = (b + \Lambda) \cap C$ *be a nonempty truncated affine lattice. Then* $W_S = \operatorname{lin}(\operatorname{conv}(S)) \cap \Lambda$*.*

Proof. Let $w \in W_S$. For each $y \in \operatorname{conv}(S)$, we can write $y = \sum_{i=1}^n \lambda_i s_i$ for $\lambda_i \in [0, 1]$, $\sum_{i=1}^n \lambda_i = 1$, and $s_i \in S$. It follows that

$$y + w = \left(\sum_{i=1}^n \lambda_i s_i\right) + w = \sum_{i=1}^n \lambda_i (s_i + w) \in \operatorname{conv}(S),$$

where the inclusion follows from the definition of W_S. Since $-w$ is also in W_S, this shows that $w \in \operatorname{lin}(\operatorname{conv}(S))$. As S is nonempty, there exists a $s \in S$, and we can write $s = b + z_1$ and $s + w = b + z_2$ for $z_1, z_2 \in \Lambda$. Thus, $w = z_2 - z_1 \in \Lambda$. Hence, $W_S \subseteq \operatorname{lin}(\operatorname{conv}(S)) \cap \Lambda$.

Conversely, take $w \in \operatorname{lin}(\operatorname{conv}(S)) \cap \Lambda$. For $\lambda \in \mathbb{Z}$ and $s \in S$, it follows that $s + \lambda w \in \operatorname{conv}(S) \subseteq C$. Furthermore, $s = b + z_1$ for $z_1 \in \Lambda$, and so $s + \lambda w = (z_1 + \lambda w) + b \in b + \Lambda$. Therefore $s + \lambda w \in S$, indicating that $\operatorname{lin}(\operatorname{conv}(S)) \cap \Lambda \subseteq W_S$. $\square$

Polyhedrally-truncated affine lattices and an explicit description of the lifting region

Let S be a polyhedrally-truncated affine lattice. Let $B = \{r \in \mathbb{R}^n : a_i \cdot r \leq 1,\ i \in [m]\}$ be a maximal S-free 0-neighborhood. For each $s \in B \cap S$, define the *spindle* $R(s; B)$ in the following way. Let $k \in [m]$ such that $a_k \cdot s = 1$; such an index exists since B is S-free, and therefore, s is on the boundary of B. Then

$$R(s; B) := \{r \in \mathbb{R}^n : (a_i - a_k) \cdot r \leq 0,\ \ (a_i - a_k) \cdot (s - r) \leq 0\ ,\ \forall i \in [m]\}. \tag{5.6}$$

Using these spindles, consider the set

$$R(S; B) := \bigcup_{s \in B \cap S} R(s; B). \tag{5.7}$$

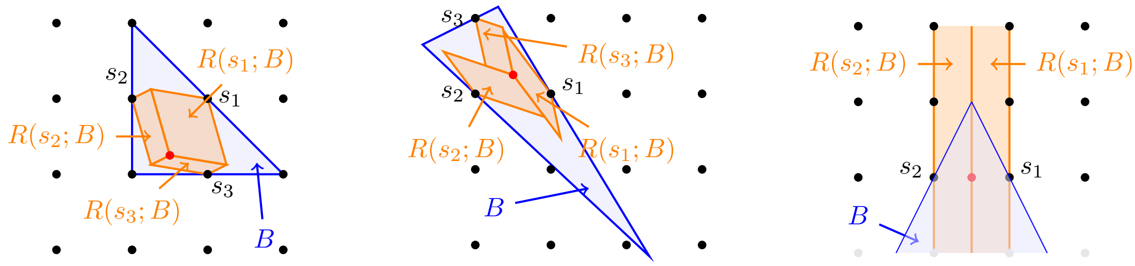
It was shown in [17] that when S is a polyhedrally-truncated affine lattice with $\Lambda = \mathbb{Z}^n$, the lifting region R_ψ for a cut-generating function ψ equals $R(S; B)$, where ψ is the minimal cut-generating function corresponding to B as defined by (5.1). Since every ψ is of this form when S is of this type, this gives an explicit description of the lifting region for any minimal cut-generating function in this situation.

Example 10 provides some examples of spindles $R(s; B)$ and the set $R(S; B)$. Note that $R(S; B)$ itself may not be S-free, as seen in Figure 5.1(c).

Example 10.

Here are three examples of maximal S-free 0-neighborhoods B with their lifting regions $R(S; B)$ and spindles $R(s; B)$ highlighted. (a) Let $b = \left(-\frac{1}{2}, -\frac{1}{4}\right)$ and $S = b + \mathbb{Z}^2$. In Figure 5.1(a), S is drawn as black dots and the origin is drawn in red. Set $a_1 = \left(\frac{4}{5}, \frac{4}{5}\right)$, $a_2 = (-2, 0)$, and $a_3 = (0, -4)$. The set $B = \{x \in \mathbb{R}^2 : a_i \cdot x \leq 1, i \in [3]\}$ is a maximal S-free 0-neighborhood (in Figure 5.1(a), B is drawn in blue). Note that the points $s_1 = \left(\frac{1}{2}, \frac{3}{4}\right)$, $s_2 = \left(-\frac{1}{2}, \frac{3}{4}\right)$, and $s_3 = \left(\frac{1}{2}, -\frac{1}{4}\right)$ are in $S \cap B$. Using Equation (5.6), we can calculate the spindles $R(s_1; B), R(s_2; B)$, and $R(s_3; B)$ — these are drawn in orange in Figure 5.1(a). As can be seen from the diagram, there are other points in $S \cap B$, but the corresponding spindles are

lower dimensional. Therefore, the lifting region is $R(S;B) = R(s_1;B) \cup R(s_2;B) \cup R(s_3;B)$.
(b) Let $b = \left(-\frac{1}{2}, -\frac{1}{4}\right)$ *and* $S = b + \mathbb{Z}^2$. *Figure 5.1(b) shows another maximal* S*-free* 0*-neighborhood with corresponding spindles. (c) Let* $b = \left(-\frac{1}{2}, 0\right)$ *and* $S = b + \mathbb{Z}^2_+$. *Figure 5.1(c) shows a maximal* S*-free* 0*-neighborhood with corresponding spindles.*



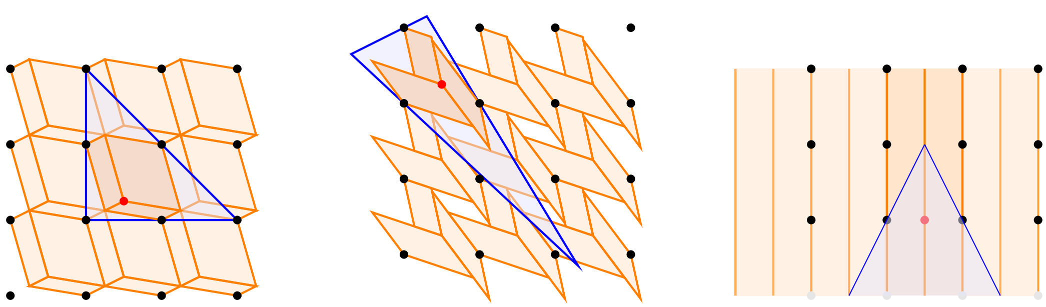
(a) (b) (c)

Figure 5.1: Examples of spindles $R(s;B)$ and the lifting region $R(S;B)$.

□

A maximal S-free 0-neighborhood is said to have the covering property if $R(S;B) + W_S = \mathbb{R}^n$. Example 11 shows that not every set has the covering property.

Example 11. *Consider again Example 10. For the sets in Figures 5.1(a) and (b),* $S = b + \mathbb{Z}^2$ *and* $W_S = \mathbb{Z}^2$. *For Figure 5.1(c),* $W_S = \mathbb{Z} \times \{0\}$. *Figure 5.2 shows that the sets in (a) and (c) have the covering property because* $R(S;B) + W_S = \mathbb{R}^n$. *However, the set in (b) does not have the covering property as seen by the 'holes' that arise. In this chapter, we consider sets with the covering property, like (a) and (c). In Chapter 6, we examine sets without the covering property, like (b).*



(a) (b) (c)

Figure 5.2: Not all S-free sets have the covering property.

$\square$

For the rest of the chapter, we will consider polyhedrally-truncated affine lattices S and analyze the properties of $R(S;B)$ as defined in (5.7) for maximal S-free convex sets B given by (5.2). It should be noted that the results described in Chapter 5.1.1 only concern the geometric object $R(S;B)$ defined in (5.7) and do not rely on any properties of cut-generating functions. We will also sometimes abbreviate $R(s;B)$ to $R(s)$ when the set B is clear from context.

Topological Facts

We collect here some basic tools from topology that will be used in our analysis.

Lemma 10. *[Theorem 9.4 in [60]] Let $P_\omega \subseteq \mathbb{R}^n, \omega \in \Omega$ be a (possibly infinite) family of polyhedra such that any bounded set intersects only finitely many polyhedra, and $\bigcup_{\omega\in\Omega} P_\omega = \mathbb{R}^n$. Suppose there is a family of functions $A_\omega : P_\omega \to \mathbb{R}^n, \omega \in \Omega$ such that A_ω is continuous over P_ω for each $\omega \in \Omega$, and for every pair $\omega_1, \omega_2 \in \Omega$, $A_{\omega_1}(x) = A_{\omega_2}(x)$ for all $x \in P_{\omega_1} \cap P_{\omega_2}$. Then there is a unique, continuous map $A : \mathbb{R}^n \to \mathbb{R}^n$ that equals A_ω when restricted to P_ω for each $\omega \in \Omega$.*

The following is a deep result in algebraic topology, first proved by Brouwer [37, 59].

Theorem 23. *[Invariance of Domain [37, 59]] If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective, continuous map, then $f(U)$ is open and f is a homeomorphism between U and $f(U)$.*

Structure of the lifting region $R(S;B)$

Let S be a polyhedrally-truncated affine lattice given as $S = (b + \Lambda) \cap C$ and let B be a maximal S-free polyhedron given by (5.2). We now collect some facts about the lifting region $R(S;B)$ as defined in (5.7).

Define $L_B = \{r \in \mathbb{R}^n : a_i \cdot r = a_j \cdot r, \forall i, j \in [m]\}$. The following is proved in [17] when $\Lambda = \mathbb{Z}^n$; the result can be seen to hold when Λ is a general lattice. This is an extension of Theorem 3.

Proposition 49. *[Theorem 1 and Proposition 6 in [17]] Let S be a polyhedrally-truncated affine lattice. B is a maximal S-free 0-neighborhood if and only if B is a polyhedron of the form (5.2) with a point from S in the relative interior of every facet. Further, either B is a halfspace or $\mathrm{int}(B \cap \mathrm{conv}(S)) \neq \emptyset$. When $\mathrm{int}(B \cap \mathrm{conv}(S)) \neq \emptyset$, the following are true:*

(i) $\mathrm{rec}(B \cap \mathrm{conv}(S)) = \mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) \subseteq \mathrm{lin}(B) \subseteq L_B$ *and* $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ *is a cone generated by vectors in* Λ.

(ii) $\mathrm{lin}(R(s)) = \mathrm{rec}(R(s)) = L_B$ *for every* $s \in B \cap S$.

(iii) $R(S; B)$ *is a union of finitely many polyhedra.*

Proposition 50. *Suppose* $\mathrm{int}(B \cap \mathrm{conv}(S) \neq \emptyset$. $L_B \cap \mathrm{lin}(\mathrm{conv}(S)) = \mathrm{lin}(B) \cap \mathrm{lin}(\mathrm{conv}(S))$ *and* $L_B \cap \mathrm{lin}(\mathrm{conv}(S))$ *is a lattice subspace of* Λ. *Consequently, if* $B \cap \mathrm{conv}(S)$ *is a polytope, then* $L_B \cap \mathrm{lin}(\mathrm{conv}(S)) = \{0\}$.

Proof. Consider $r \in L_B \cap \mathrm{lin}(\mathrm{conv}(S))$. It suffices to show that either r or $-r$ is in $\mathrm{lin}(B) \cap \mathrm{lin}(\mathrm{conv}(S))$. Since, $r \in L_B$, for all $i \in [m]$, $a_i \cdot r$ have the same sign. If $a_i \cdot r \leq 0$ for all $i \in [m]$, then $r \in \mathrm{rec}(B)$ and therefore, $r \in \mathrm{rec}(B) \cap \mathrm{lin}(\mathrm{conv}(S)) \subseteq \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S)) = \mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ (the equality follows from Proposition 49(i) – note that since B and $\mathrm{conv}(S)$ are both polyhedra, $\mathrm{rec}(B \cap \mathrm{conv}(S)) = \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$). Therefore $r \in \mathrm{lin}(B)$. Since $r \in \mathrm{lin}(\mathrm{conv}(S))$, we thus have $r \in \mathrm{lin}(B) \cap \mathrm{lin}(\mathrm{conv}(S))$. If $a_i \cdot r \geq 0$ for all $i \in [m]$, then $a_i \cdot (-r) \leq 0$ and so $-r \in \mathrm{rec}(B)$. Repeating the same argument, we obtain $-r \in \mathrm{lin}(B)$. Thus, $-r \in \mathrm{lin}(B) \cap \mathrm{lin}(\mathrm{conv}(S))$.

The assertion that $L_B \cap \mathrm{lin}(\mathrm{conv}(S))$ is a lattice subspace follows from Proposition 49 (i), the fact that $\mathrm{lin}(\mathrm{conv}(S))$ is a lattice subspace (Fact 1) and $\mathrm{lin}(B) \cap \mathrm{lin}(\mathrm{conv}(S)) = (\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))) \cap \mathrm{lin}(\mathrm{conv}(S))$. □

Theorem 24. *Suppose* $\mathrm{int}(B \cap \mathrm{conv}(S)) \neq \emptyset$. *A bounded set intersects only finitely many polyhedra from* $R(S; B) + W_S$.

Proof. Let $L = L_B \cap \mathrm{lin}(\mathrm{conv}(S))$; L is a lattice subspace by Proposition 50. Let V be a lattice subspace such that $V \cap L = \{0\}$ and $(V \cap \Lambda) + (L \cap \Lambda) = \Lambda$ (and so $V + L = \mathbb{R}^n$). Also define $L_1 := V \cap L_B$ and $L_2 := V \cap \mathrm{lin}(\mathrm{conv}(S))$.

Note that $L_2 \cap L_B = \{0\}$. Indeed,

$$L_2 \cap L_B = (V \cap \mathrm{lin}(\mathrm{conv}(S)) \cap L_B = V \cap (L_B \cap \mathrm{lin}(\mathrm{conv}(S))) = V \cap L = \{0\}.$$

Furthermore, $L_2 + L = \mathrm{lin}(\mathrm{conv}(S))$. In order to see this, observe that since $V + L = \mathbb{R}^n$, for every $x \in \mathrm{lin}(\mathrm{conv}(S))$ there exists $v \in V$ and $l \in L$ such that $x = v + l$. Since $v = x - l \in \mathrm{lin}(\mathrm{conv}(S))$, $x \in L_2 + L$. Thus $\mathrm{lin}(\mathrm{conv}(S)) \subseteq L_2 + L$. The other containment follows from the definitions of L and L_2.

We next show that $\mathrm{lin}(\mathrm{conv}(S)) \cap \Lambda = (L_2 \cap \Lambda) + (L \cap \Lambda)$. Consider some $x \in \mathrm{lin}(\mathrm{conv}(S)) \cap \Lambda$. Since $x \in \Lambda = (V \cap \Lambda) + (L \cap \Lambda)$ and $V \cap L = \{0\}$, there exists a unique $v \in V \cap \Lambda$ and $l \in L \cap \Lambda$ such that $x = v + l$. As $x \in \mathrm{lin}(\mathrm{conv}(S)) = L_2 + L$ and $L_2 \cap L \subseteq L_2 \cap L_B = \{0\}$, there exists a unique $l_2 \in L_2$ and $l' \in L$ such that $x = l_2 + l'$. By the uniqueness of v and l, it follows that $v = l_2$ and $l = l'$. Thus $v \in L_2 \cap \Lambda$ and $l \in L \cap \Lambda$. Hence, $\mathrm{lin}(\mathrm{conv}(S)) \cap \Lambda \subseteq (L_2 \cap \Lambda) + (L \cap \Lambda)$. The definitions of L_2 and L imply the $\supseteq$ containment.

Let L' be any linear subspace of $\mathbb{R}^n$ containing L_2 such that $L' \cap L_B = \{0\}$ and $L' + L_B = \mathbb{R}^n$; such a linear space exists since $L_2 \cap L_B = \{0\}$. Since L_B is the recession cone of each spindle in $R(S;B)$, $R(S;B) = (R(S;B) \cap L') + L_B$ and $R(S;B) \cap L'$ is a finite union of polytopes because $R(S;B)$ is a union of finitely many polyhedra. Moreover, by Proposition 48,

$$\begin{aligned}
R(S;B) + W_S &= R(S;B) + (\mathrm{lin}(\mathrm{conv}(S)) \cap \Lambda) \\
&= ((R(S;B) \cap L') + L_B) + ((L_2 \cap \Lambda) + (L \cap \Lambda)) \\
&= (R(S;B) \cap L') + (L_2 \cap \Lambda) + (L_B + (L \cap \Lambda)) \\
&= (R(S;B) \cap L') + (L_2 \cap \Lambda) + L_B,
\end{aligned}$$

where the last equality comes from $L \subseteq L_B$.

Observe that each bounded set D in $\mathbb{R}^n$ intersects at most as many polyhedra in $R(S;B) + W_S$ as $D + L_B$. Since $L' \cap L_B = \{0\}$, $D + L_B$ intersects the same number of polyhedra in $R(S;B) + W_S$ as $(D + L_B) \cap L'$ intersects polyhedra in $(R(S;B) \cap L') + (L_2 \cap \Lambda)$. The complementary assumption also implies that $(D + L_B) \cap L'$ is a bounded set. Since

$R(S;B) \cap L'$ is a finite union of polytopes and $L_2 \cap \Lambda$ is a lattice in $L_2 \subseteq L'$, the bounded set $(D + L_B) \cap L'$ intersects finitely many polytopes in $(R(S;B) \cap L') + (L_2 \cap \Lambda)$. $\square$

Lemma 11. *Suppose* $\operatorname{int}(B \cap \operatorname{conv}(S)) \neq \emptyset$. *$R(S;B) + W_S$ is a closed set.*

Proof. Let $x \notin R(S;B) + W_S$. Consider the closed ball $\operatorname{cl}(D(x;1))$ of radius one around x. By Theorem 24, $\operatorname{cl}(D(x;1))$ intersects only finitely many polyhedra from $R(S;B)+W_S$. The union of these finitely many polyhedra is a closed set and therefore, there exists an open ball $D(x;\epsilon)$, for some $\epsilon > 0$, around x that does not intersect any of these polyhedra. But since $D(x;\epsilon) \subseteq \operatorname{cl}(D(x;1))$, $D(x;\epsilon)$ does not intersect any other polyhedron from $R(S;B) + W_S$. Hence, the complement of $R(S;B) + W_S$ is open. $\square$

5.3 The covering property is preserved under affine transformations

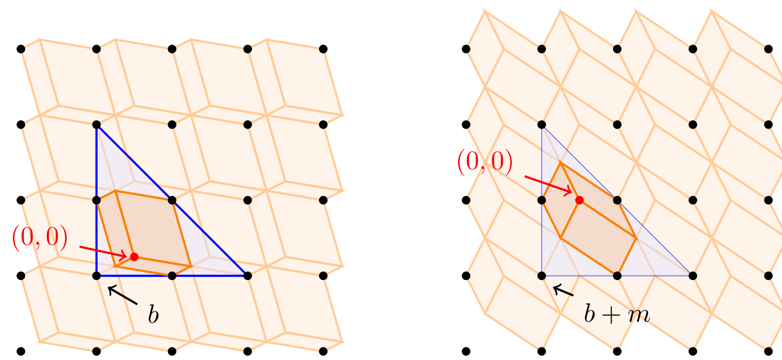
Let S be a polyhedrally-truncated affine lattice and let B be a maximal S-free polyhedron given by (5.2). We want to understand the covering properties of the lifting region when we transform S and B by the same invertible affine transformation. For any linear map $F : \mathbb{R}^n \to \mathbb{R}^n$, F^* will denote its adjoint, i.e., the unique linear map such that $x \cdot F(y) = F^*(x) \cdot y$ for all $x, y \in \mathbb{R}^n$; the adjoint corresponds to taking the transpose of the matrix form of the linear map F. To avoid an overuse of parentheses, we will often abbreviate $F(x)$ to Fx wherever this is possible without causing confusion.

Theorem 25. *[Affine Transformation Invariance Theorem] Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map and $m \in \mathbb{R}^n$. Let T denote the affine transformation $T(\cdot) := M(\cdot) + m$. Suppose that $T(B)$ also contains the origin in its interior (i.e., $a_i \cdot (-M^{-1}m) < 1$ for each $i \in [m]$). $R(S;B) + W_S = \mathbb{R}^n$ if and only if $R(S';B') + W_{S'} = \mathbb{R}^n$, where $S' = T(S)$ and $B' = T(B)$.*

Example 12 provides some geometric intuition behind Theorem 25 when the transformation T is a simple translation.

Example 12.

Let $b = \left(-\frac{1}{2}, -\frac{1}{4}\right)$ and $S = b + \mathbb{Z}^2$. Figure 5.3(a) shows the spindles computed in Figure 5.1(a). Let $m = \left(0, -\frac{3}{4}\right)$. Translating S and B by m yields a $S' = (b + m + \mathbb{Z}^2)$-free set $B' = B + m$. Figure 5.3(a) shows $R(S;B) + W_S$ in orange, with $R(S;B)$ highlighted in dark orange. Note that $R(S;B) + W_S$ is a tiling of the spindles in $R(S;B)$ (this tiling property will be proved in the Collision Lemma — see Lemma 12). Figure 5.3(b) shows $R(S';B') + W_{S'}$ Under this translation, the lifting region changes shape, but the covering property is maintained; see Figure 5.3. Note $W_S = W_{S'} = \mathbb{Z}^2$.



$S = b + \mathbb{Z}^2$

(a) $R(S;B) + W_S$

$S' = b + m + \mathbb{Z}^2$

(b) $R(S', B') + W_{S'}$

Figure 5.3: Some intuition for Theorem 25 when T is a translation. The lifting region $R(S;B)$ is in solid orange. The integer translates of $R(S;B)$ are in translucent orange.

$\square$

Observe that $B' = T(B) = M(B) + m$ is given by $\{r \in \mathbb{R}^n : a'_i \cdot r \leq 1 \;\; i \in [m]\}$, where

$$a'_i = \frac{(M^{-1})^*(a_i)}{1 + a_i \cdot M^{-1}(m)} \quad \text{for each} \;\; i \in [m].$$

Clearly, B' is a maximal S'-free polyhedron. For $s' \in B' \cap S'$, the spindle $R(s';B')$ is therefore given by

$$R(s';B') = \{r : (a'_i - a'_k) \cdot r \leq 0, \;\; (a'_i - a'_k) \cdot (s' - r) \leq 0 \quad , \forall i \in [m]\}.$$

The lifting region becomes $R(S';B') = \bigcup_{s' \in B' \cap S'} R(s';B')$.

Intersections modulo the lattice We show an interesting property of different spindles when they intersect after translations by vectors in W_S. In particular, two spindles from *different* facets cannot intersect in their interiors, and moreover, the "height" of the common intersection points from the different spindles is the same with respect to the respective facets.

Lemma 12. *[Collision Lemma] Let S be a polyhedrally-truncated affine lattice and let B be a maximal S-free polyhedron given by (5.2). Let $s_1, s_2 \in B \cap S$, and let $i_1, i_2 \in [m]$ be such that $a_{i_1} \cdot s_1 = 1$ and $a_{i_2} \cdot s_2 = 1$. If $x_1, x_2 \in R(S; B)$ are such that $x_1 - x_2 \in W_S$, $x_1 \in R(s_1)$, and $x_2 \in R(s_2)$, then $a_{i_1} \cdot x_1 = a_{i_2} \cdot x_2$. Moreover, if $x_1 \in \operatorname{int}(R(s_1))$ and $x_2 \in \operatorname{int}(R(s_2))$, then $a_{i_1} = a_{i_2}$.*

Proof. If $m = 1$, then the result is trivial. So suppose $m \geq 2$. Assume to the contrary that $a_{i_1} \cdot x_1 \neq a_{i_2} \cdot x_2$. Suppose that $a_{i_1} x_1 < a_{i_2} x_2$ (for the proof of the other case, switch the indices in the following argument). Since $x_1 - x_2 \in W_S$, the point $s_2 + (x_1 - x_2)$ is contained in S. In order to reach a contradiction, it is sufficient to show that $s_2 + (x_1 - x_2) \in \operatorname{int}(B)$. We will show this using the definition $B = \{r \in \mathbb{R}^n : a_i \cdot r \leq 1 \ \ , i \in [m]\}$.

Take $i \in [m]$. When $i = i_1$, it follows that

$$\begin{aligned}
a_{i_1}(s_2 + (x_1 - x_2)) &= a_{i_1}(s_2 - x_2) + a_{i_1} x_1 \\
&\leq a_{i_2}(s_2 - x_2) + a_{i_1} x_1 && \text{Since } x_2 \in R(s_2) \\
&= 1 - a_{i_2} x_2 + a_{i_1} x_1 \\
&< 1 - a_{i_1} x_1 + a_{i_1} x_1 \\
&= 1.
\end{aligned}$$

When $i = i_2$, it follows that

$$\begin{aligned}
a_{i_2}(s_2 + (x_1 - x_2)) &= 1 + a_{i_2} x_1 - a_{i_2} x_2 \\
&< 1 + a_{i_2} x_1 - a_{i_1} x_1 \\
&\leq 1 && \text{Since } x_1 \in R(s_1).
\end{aligned}$$

Finally, if $i \notin \{i_1, i_2\}$, then

$$\begin{aligned} a_i(s_2 + (x_1 - x_2)) &= a_i(s_2 - x_2) + a_i x_1 \\ &\leq a_{i_2}(s_2 - x_2) + a_i x_1 && \text{Since } x_2 \in R(s_2) \\ &= 1 - a_{i_2} x_2 + a_i x_1 \\ &< 1 - a_{i_1} x_1 + a_i x_1 \\ &\leq 1 && \text{Since } x_1 \in R(s_1). \end{aligned}$$

Hence $s_2 + (x_1 - x_2) \in \text{int}(B)$, giving a contradiction. Thus $a_{i_1} x_1 = a_{i_2} x_2$.

Now suppose that $x_1 \in \text{int}(R(s_1))$ and $x_2 \in \text{int}(R(s_2))$. Assume to the contrary that $a_{i_1} \neq a_{i_2}$. We will again show that $s_2 + (x_1 - x_2) \in \text{int}(B)$. Since $a_{i_1} \neq a_{i_2}$ and $x_2 \in \text{int}(R(s_2))$,

$$a_{i_1} \cdot x_2 < a_{i_2} \cdot x_2$$

and

$$a_{i_1} \cdot (s_2 - x_2) < a_{i_2} \cdot (s_2 - x_2).$$

Let $i \in [m]$. If $i = i_1$ then using $a_{i_1} \cdot x_1 = a_{i_2} \cdot x_2$, it follows that

$$a_{i_1} \cdot (s_2 + (x_1 - x_2)) = a_{i_1} \cdot (s_2 - x_2) + a_{i_1} \cdot x_1 < a_{i_2} \cdot (s_2 - x_2) + a_{i_1} \cdot x_1 = 1 - a_{i_2} \cdot x_2 + a_{i_1} \cdot x_1 = 1.$$

If $i = i_2$ then

$$a_{i_2} \cdot (s_2 + (x_1 - x_2)) = a_{i_2} \cdot s_2 + a_{i_2} \cdot x_1 - a_{i_2} \cdot x_2 = 1 + a_{i_2} \cdot x_1 - a_{i_1} \cdot x_1 = 1 + (a_{i_2} - a_{i_1}) \cdot x_1 < 1,$$

where the inequality comes from $x_1 \in \text{int}(R(s_1))$. Finally, if $i \notin \{i_1, i_2\}$ then

$$a_i \cdot (s_2 + (x_1 - x_2)) = a_i \cdot (s_2 - x_2) + a_i \cdot x_1 < a_{i_2} \cdot s_2 - a_{i_2} \cdot x_2 + a_i \cdot x_1 < 1 - a_{i_2} \cdot x_2 + a_{i_1} \cdot x_1 = 1,$$

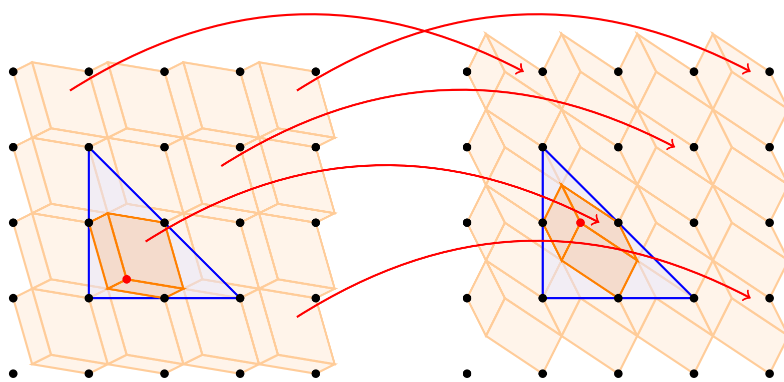
where the first inequality comes from $x_2 \in \text{int}(R(s_2))$ and the second from $x_1 \in \text{int}(R(s_1))$. Hence, $s_2 + (x_1 - x_2) \in \text{int}(B)$, yielding a contradiction. $\square$

Mapping $R(S;B)+W_S$ onto $R(S';B')+W_{S'}$

We now describe how one can bijectively map each spindle of $R(S;B)$ onto a spindle in $R(S';B')$ by a linear transformation. We will then be able to map $R(S;B)+W_S$ injectively onto $R(S';B')+W_{S'}$ by a piecewise affine map. The idea of this map will be to send the spindles in $R(S;B)$ to spindles in $R(S';B')$ in such a way that they 'align' properly. Example 13 illustrates this idea and gives intuition to what this mapping looks like.

Example 13.

Consider Example 12 with $b = \left(-\frac{1}{2}, -\frac{1}{4}\right)$, $S = b+\mathbb{Z}^2$, *and* $m = \left(0, -\frac{3}{4}\right)$. *For each* $s \in S \cap B$ *and* $z \in \mathbb{Z}^2$, *there is a somewhat natural correspondence with some* $s' \in S' \cap B'$ *and* $z \in \mathbb{Z}^2$. *This correspondence, which is essentially the function used to prove Theorem 25, is shown in Figure 5.4.*



$S = b + \mathbb{Z}^2$ $\qquad$ $S' = b + m + \mathbb{Z}^2$

Figure 5.4: The map used in Theorem 25 sends the translated spindles in $R(S;B)+W_S$ to the translated spindles in $R(S';B')+W_{S'}$.

□

Given a particular polyhedrally-truncated affine lattice S, a maximal S-free polyhedron B described as (2.10), and an invertible affine map $M(\cdot)+m$ such that $B' = M(B)+m$ contains the origin in its interior, we define linear transformations $T_i^{S,B,M,m}$ for each $i \in [m]$ given by

$$T_i^{S,B,M,m}(r) = Mr + (a_i \cdot r)m.$$

Lemma 13. *For every $i \in [m]$, $T_i^{S,B,M,m}(r)$ is an invertible linear transformation with inverse*

$$T_i^{S',B',M^{-1},-M^{-1}m}(r) = M^{-1}r - (a'_i \cdot r)M^{-1}m.$$

In the following two lemmas, we drop the superscripts in $T_i^{S,B,M,m}$ to save notational baggage; the lemmas are true for any tuple S, B, M, m such that S is a polyhedrally-truncated affine lattice, B is a maximal S-free 0-neighborhood and a polyhedron, and $M(\cdot)+m$ is an invertible affine transformation such that $M(B)+m$ is also a 0-neighborhood.

Lemma 14. *Let $s \in B \cap S$ and let $k \in [m]$ be such that $a_k \cdot s = 1$. Then $T_k(R(s;B)) = R(s';B')$, where $s' = Ms + m$.*

Proof. We first establish the following claim:

Claim 11. *For any $\bar{r} \in \mathbb{R}^n$ and $i \in [m]$ such that $(a_i - a_k) \cdot \bar{r} \le 0$, we have $(a'_i - a'_k) \cdot T_k(\bar{r}) \le 0$.*

Proof of Claim. Consider any such $i \in [m]$ and $\bar{r} \in \mathbb{R}^n$ such that $(a_i - a_k) \cdot \bar{r} \le 0$ (note that $i = k$ satisfies this hypothesis). We show that $a'_i \cdot T_k(\bar{r}) \le a_k \cdot \bar{r}$. Indeed,

$$\begin{aligned}
a'_i \cdot T_k(\bar{r}) &= \tfrac{(M^{-1})^*(a_i)\cdot(M\bar{r}+(a_k\cdot\bar{r})m)}{1+a_i\cdot M^{-1}m} \\
&= \tfrac{a_i\cdot M^{-1}((M\bar{r}+(a_k\cdot\bar{r})m))}{1+a_i\cdot M^{-1}m} \\
&= \tfrac{a_i\cdot(\bar{r}+(a_k\cdot\bar{r})M^{-1}m)}{1+a_i\cdot M^{-1}m} \\
&= \tfrac{a_i\cdot\bar{r}+(a_k\cdot\bar{r})(a_i\cdot M^{-1}m)}{1+a_i\cdot M^{-1}m} \\
&\le \tfrac{a_k\cdot\bar{r}+(a_k\cdot\bar{r})(a_i\cdot M^{-1}m)}{1+a_i\cdot M^{-1}m} \qquad \text{Using } (a_i - a_k)\cdot\bar{r} \le 0 \\
&= a_k \cdot \bar{r}.
\end{aligned}$$

Observe that the inequality above holds at equality when $i = k$. Therefore, $(a'_i - a'_k) \cdot T_k(\bar{r}) \le (a_k - a_k) \cdot \bar{r} = 0$.

◇

Now consider any $\hat{r} \in R(s;B)$. Therefore, for every $i \in [m]$ we have that $(a_i - a_k) \cdot \hat{r} \le 0$ and $(a_i - a_k) \cdot (s - \hat{r}) \le 0$. Observe that $T_k(s - \hat{r}) = T_k(s) - T_k(\hat{r}) = (Ms + m) - T_k(\hat{r}) = s' - T_k(\hat{r})$ where the second equality follows from the fact that $a_k \cdot s = 1$. By Claim 11, we therefore have $(a'_i - a'_k) \cdot T_k(\hat{r}) \le 0$ and $(a'_i - a'_k) \cdot (s' - T_k(\hat{r})) \le 0$. Hence, $T_k(\hat{r}) \in R(s';B')$.

This shows that $T_k(R(s;B)) \subseteq R(s';B')$. Using a similar reasoning with the transformation T_k^{-1}, one can show that $T_k^{-1}(R(s';B')) \subseteq R(s;B)$, i.e., $R(s';B') \subseteq T_k(R(s;B))$. This completes the proof. $\square$

Lemma 15. *Let $s_1, s_2 \in B \cap S$ and $w_1, w_2 \in W_S$ such that $(R(s_1)+w_1) \cap (R(s_2)+w_2) \neq \emptyset$ and let $x \in (R(s_1)+w_1) \cap (R(s_2)+w_2)$. Let $i_1, i_2 \in [m]$ be two indices such that $a_{i_1} \cdot s_1 = 1$ and $a_{i_2} \cdot s_2 = 1$. Then, $T_{i_1}(x - w_1) + Mw_1 = T_{i_2}(x - w_2) + Mw_2$.*

Proof. Observe that

$$\begin{aligned}
T_{i_1}(x-w_1) + Mw_1 &= M(x-w_1) + (a_{i_1} \cdot (x-w_1))m + Mw_1 \\
&= Mx + (a_{i_1} \cdot (x-w_1))m \\
&= Mx + (a_{i_2} \cdot (x-w_2))m \quad \text{using the Collision Lemma (Lemma 12)} \\
&= M(x-w_2) + (a_{i_2} \cdot (x-w_2))m + Mw_2 \\
&= T_{i_2}(x-w_2) + Mw_2.
\end{aligned}$$

$\square$

Proof of Theorem 25

Proof. Note that if B (and B') is a halfspace, then the lifting region is all of $\mathbb{R}^n$, and there is nothing to show. Thus, by Proposition 49, we assume $\operatorname{int}(B \cap \operatorname{conv}(S)) \neq \emptyset$. It suffices to show that $R(S;B) + W_S = \mathbb{R}^n$ implies $R(S';B') + W_{S'} = \mathbb{R}^n$ because the other direction follows by swapping the roles of S, B and S', B' and using the transformation $M^{-1}(\cdot) - M^{-1}m$ instead of $M(\cdot) + m$.

Assume $R(S;B) + W_S = \mathbb{R}^n$. For every $s \in B \cap S$ and $w \in W_S$, define the polyhedron $P_{s,w} = R(s;B) + w$ and define the map $A_{s,w} : P_{s,w} \to \mathbb{R}^n$ as $A_{s,w}(x) = T_k^{S,B,M,m}(x - w) + Mw$, where $k \in [m]$ is such that $a_k \cdot s = 1$. Since $R(S;B) + W_S = \mathbb{R}^n$, we have

$$\bigcup_{s \in B \cap S, w \in W_S} P_{s,w} = R(S;B) + W_S = \mathbb{R}^n.$$

By Theorem 24, any bounded set intersects only finitely many polyhedra from the family $\{P_{s,w} : s \in B \cap S, w \in W_S\}$. Moreover, by Lemma 15, we observe that for any two pairs

s_1, w_1 and s_2, w_2 we have that $A_{s_1,w_1}(x) = A_{s_2,w_2}(x)$ for all $x \in P_{s_1,w_1} \cap P_{s_2,w_2}$. Since each $A_{s,w}$ is an affine map on $P_{s,w}$, Lemma 10 shows that there exists a continuous map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that A restricted to $P_{s,w}$ is equal to $A_{s,w}$. Observe that

$$
\begin{aligned}
R(S'; B') + W_{S'} &= R(S'; B') + MW_S \qquad\qquad \text{by Proposition 47(i)} \\
&= \textstyle\bigcup_{s' \in B' \cap S', w \in W_S} (R(s'; B') + Mw) \\
&= \textstyle\bigcup_{s \in B \cap S, w \in W_S} (R(Ms + m, M(B) + m) + Mw) \\
&= \textstyle\bigcup_{s \in B \cap S, w \in W_S} A_{s,w}(R(s; B) + w) \\
&= A(\textstyle\bigcup_{s \in B \cap S, w \in W_S} (R(s; B) + w) \\
&= A(R(S; B) + W_S) \\
&= A(\mathbb{R}^n)
\end{aligned}
$$

where the fourth equality follows from the definition of $A_{s,w}$ and Lemma 14. If we can show that A is injective, then by Theorem 23, $A(\mathbb{R}^n) = R(S'; B') + W_{S'}$ is open. By Lemma 11, $R(S'; B') + W_{S'}$ is also closed (since $\operatorname{int}(B \cap \operatorname{conv}(S)) \neq \emptyset$ implies $\operatorname{int}(B' \cap \operatorname{conv}(S')) \neq \emptyset$). Since $\mathbb{R}^n$ is connected, the only nonempty closed and open subset of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. Thus, $R(S'; B') + W_{S'} = \mathbb{R}^n$.

Therefore, it is sufficient to show that A is an injective function. Choose $x, y \in \mathbb{R}^n$ such that $A(x) = A(y)$. Unfolding the definition, this implies that there exists $s_1, s_2 \in S \cap B$, $w_1, w_2 \in W_S$, and $k_1, k_2 \in [m]$ such that $x \in R(s_1) + w_1$, $y \in R(s_2) + w_2$, and $T^{S,B,M,m}_{k_1}(x - w_1) + Mw_1 = T^{S,B,M,m}_{k_2}(y - w_2) + Mw_2 =: z^*$. By Lemma 14, $z^* \in (R(s_1') + Mw_1) \cap (R(s_2') + Mw_2)$, where $s_1' = Ms_1 + m$ and $s_2' = Ms_2 + m$. Note that $R(s_1')$ and $R(s_2')$ are spindles corresponding to $R(S'; B')$, and by Proposition 47(i), $Mw_1, Mw_2 \in W_{S'}$. Therefore, by Lemma 15, $T^{S,B,M^{-1},-M^{-1}m}_{k_1}(z^* - Mw_1) + w_1 = T^{S,B,M^{-1},-M^{-1}m}_{k_2}(z^* - Mw_2) + w_2$. By Lemma 13, $T^{S',B',M^{-1},-M^{-1}m}_{i}$ is the inverse of $T^{S,B,M,m}_{i}$ for each $i \in [m]$, and so we have

$$
T^{S,B,M^{-1},-M^{-1}m}_{k_1}(z^* - Mw_1) + w_1 = T^{S,B,M^{-1},-M^{-1}m}_{k_1}\left((T^{S,B,M,m}_{k_1}(x - w_1)\right) + w_1 = x.
$$

Similarly, $T^{S',B',M^{-1},-M^{-1}m}_{k_2}(z^* - Mw_2) + w_2 = y$. Hence $x = y$ and A is injective. $\square$

5.4 Generation of S-free sets using coproducts

Here we display how the covering property is preserved under the so-called *coproduct* operation. Given a set $C \subseteq \mathbb{R}^n$ containing the origin in its interior, we say $X \subseteq \mathbb{R}^n$ is a *prepolar* of C if $X^* = C$, i.e., C is the polar of X. We use the notation $C^\bullet$ to denote the smallest prepolar of C with respect to set inclusion. To the best of our knowledge, this concept was first introduced in [40], where the authors establish that there is a *unique* smallest prepolar. Given closed sets $C_1 \subseteq \mathbb{R}^{n_1}, C_2 \subseteq \mathbb{R}^{n_2}$ (possibly unbounded) such that each contains the origin in its interior, define the *coproduct of* C_1, C_2 in $\mathbb{R}^{n_1+n_2}$ as

$$C_1 \lozenge C_2 := (C_1^\bullet \times C_2^\bullet)^*. \tag{5.8}$$

If the sets are polyhedra given using inequality descriptions, $P_1 = \{x \in \mathbb{R}^{n_1} : a_i^1 x \leq 1,\ \forall i \in [m_1]\}$ and $P_2 = \{x \in \mathbb{R}^{n_2} : a_j^2 x \leq 1,\ \forall j \in [m_2]\}$, then

$$P_1 \lozenge P_2 = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : (a_i^1, a_j^2) \cdot (x,y) \leq 1,\ \forall i \in [m_1],\ \forall j \in [m_2]\}. \tag{5.9}$$

See Example 14 at the end of the section for an example of coproducts.

The coproduct definition is motivated as a dual operation to Cartesian products: if P_1 and P_2 are polytopes containing the origin in their interiors, then $(P_1 \times P_2)^* = P_1^* \lozenge P_2^*$. In this case, our definition specializes to the operation known as the *free sum* in polytope theory [74, p. 250]: $P_1 \lozenge P_2 := \operatorname{conv}(P_1 \times \{o_2\} \cup \{o_1\} \times P_2)$. The free sum operation was utilized in Section 4 of [5], where the operation was also called the coproduct following a suggestion by Peter McMullen. Since our construction is a generalization to the case where P_1, P_2 are allowed to be unbounded polyhedra, we retain the terminology of coproduct. If we take closed hulls, then the free sum operation can be extended to unbounded sets. Using this extension for unbounded sets, the free sum operation is *different* from the coproduct operation defined in (5.8) – consider the coproduct and free sum of a ray in $\mathbb{R}$ containing the origin and an interval in $\mathbb{R}$ containing the origin. In fact, $\overline{\operatorname{conv}}(C_1 \times \{o_2\} \cup \{o_1\} \times C_2) = (C_1^* \times C_2^*)^*$ and the second term is different from $(C_1^\bullet \times C_2^\bullet)^*$ when C_1 or C_2 are unbounded. One can check that parts (ii) and (iii) of Theorem 26 below fail to hold if one uses $\overline{\text{conv}}(C_1 \times \{o_2\} \cup \{o_1\} \times C_2) = (C_1^* \times C_2^*)^*$ as the generalization of the operation defined in [5].

If each a_i^1, $i \in [m_1]$, gives a facet-defining inequality for P_1 and each a_j^2, $j \in [m_2]$, gives a facet-defining inequality for P_2, then each inequality in the description in (5.9) is facet-defining. This follows from the fact that each a_i^1, $i \in [m_1]$ is a vertex of P_1^*, and similarly, each a_j^2, $j \in [m_2]$ is a vertex of P_2^*, and so (a_i^1, a_j^2), $i \in [m_1], j \in [m_2]$ is a vertex of $P_1^* \times P_2^* = \overline{\text{conv}}(P_1^\bullet \times P_2^\bullet)$.

For $h \in [2]$, let $S_h = (b_h + \Lambda_h) \cap C_h$ be two polyhedrally-truncated affine lattices in $\mathbb{R}^{n_h}$ where $C_h = \text{conv}(S_h)$ is a polyhedron. Then $S_1 \times S_2 = ((b_1, b_2) + (\Lambda_1 \times \Lambda_2)) \cap (C_1 \times C_2)$ is also a polyhedrally-truncated affine lattice in $\mathbb{R}^{n_1+n_2}$. The following result creates $S_1 \times S_2$-free sets from S_1-free sets and S_2-free sets.

Theorem 26. *For $h \in [2]$, let $B_h \subseteq \mathbb{R}^{n_h}$ be given by facet defining inequalities $\{x \in \mathbb{R}^{n_h} : a_i^h x \leq 1,\ \forall i \in [m_h]\}$ and let S_h be polyhedrally-truncated affine lattices. Let $\mu \in (0, 1)$. Then*

(i) If B_h is S_h-free for $h \in [2]$, then $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ is $S_1 \times S_2$-free.

(ii) If B_h is maximal S_h-free for $h \in [2]$, then $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ is maximal $S_1 \times S_2$-free.

(iii) If B_h is maximal S_h-free with the covering property for $h \in [2]$, then $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ is maximal $S_1 \times S_2$-free with the covering property.

Proof.

(i) Note that

$$\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu} = \{(x_1, x_2) \in \mathbb{R}^{n_1+n_2} : (\mu a_i^1, (1-\mu) a_j^2) \cdot (x_1, x_2) \leq 1,\ \forall i \in [m_1],\ j \in [m_2]\}.$$

Let $(s_1, s_2) \in S_1 \times S_2$. As B_1 is S_1-free, there exists an $\bar{i} \in [m_1]$ such that $a_{\bar{i}}^1 \cdot s_1 \geq 1$. Similarly, there is a $\bar{j} \in [m_2]$ such that $a_{\bar{j}}^2 \cdot s_2 \geq 1$. This implies that $(\mu a_{\bar{i}}^1, (1-\mu) a_{\bar{j}}^2) \cdot (s_1, s_2) \geq 1$. Hence, $(s_1, s_2) \notin \text{int}(\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu})$.

(ii) From part (i) and Proposition 49, it is suffices to show that every facet of $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ contains an $S_1 \times S_2$ point in its relative interior. As noted earlier, each inequality in

$\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$, as given by (5.9), is facet defining. Consider the facet defined by $(\mu a^1_{\bar{i}}, (1-\mu)a^2_{\bar{j}})$. Since $a^1_{\bar{i}}$ defines a facet in B_1, there exists some $s_1 \in S_1$ such that $a^1_{\bar{i}} \cdot s_1 = 1$ and $a^1_i \cdot s_1 < 1$ for $i \in [m_1]$ with $i \neq \bar{i}$. Similarly, there exists a $s_2 \in S_2$ such that $a^2_{\bar{j}} \cdot s_2 = 1$ and $a^2_j \cdot s_2 < 1$ for $j \in [m_2]$ with $j \neq \bar{j}$. It follows that $(\mu a^1_{\bar{i}}, (1-\mu)a^2_{\bar{j}}) \cdot (s_1, s_2) = 1$ and $(\mu a^1_i, (1-\mu)a^2_j) \cdot (s_1, s_2) < 1$ for $(i, j) \neq (\bar{i}, \bar{j})$. Hence (s_1, s_2) is in the relative interior of the facet defined by $(\mu a^1_{\bar{i}}, (1-\mu)a^2_{\bar{j}})$.

(iii) In order to show that $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ has the covering property, it is sufficient to show that $\mathbb{R}^{n_1+n_2} \subseteq R + W_{S_1 \times S_2}$, where $R = R\left(S_1 \times S_2; \frac{B_1}{\mu} \lozenge \frac{B_s}{1-\mu}\right)$.

Consider $s_1 \in B_1 \cap S_1$ and $s_2 \in B_2 \cap S_2$. Let $\bar{i} \in [m_1]$ index the facet of B_1 containing s_1 and $\bar{j} \in [m_2]$ index the facet of B_2 containing s_2. Calculations similar to parts (i) and (ii) above show that (s_1, s_2) lies on the facet of $\frac{B_1}{\mu} \lozenge \frac{B_s}{1-\mu}$ indexed by $(\bar{i}, \bar{j})$. We claim that the spindle $R(s_1, s_2)$ corresponding to $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ contains the Cartesian product $R(s_1) \times R(s_2)$. Indeed, a vector $(x_1, x_2) \in R(s_1, s_2)$ if and only if

$$\left((\mu a^1_i, (1-\mu)a^2_j) - (\mu a^1_{\bar{i}}, (1-\mu)a^2_{\bar{j}})\right) \cdot (x_1, x_2) \leq 0, \ \forall i \in [m_1], \ j \in [m_2]$$

and

$$\left((\mu a^1_i, (1-\mu)a^2_j) - (\mu a^1_{\bar{i}}, (1-\mu)a^2_{\bar{j}})\right) \cdot ((s_1, s_2) - (x_1, x_2)) \leq 0, \ \forall i \in [m_1], \ j \in [m_2].$$

where $(\bar{i}, \bar{j})$ indexes the facet containing (s_1, s_2). Using the definition of $R(s_i)$, the latter condition follows since $x_1 \in R(s_1)$, $x_2 \in R(s_2)$, and $\mu \in (0, 1)$. Therefore, we get the containment $R(S_1; B_1) \times R(S_2; B_2) \subseteq R(S_1 \times S_2; \frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu})$.

From Proposition 47, we have that $W_{S_1 \times S_2} = W_{S_1} \times W_{S_2}$. Since B_1 and B_2 are assumed to each have the covering property, it follows that

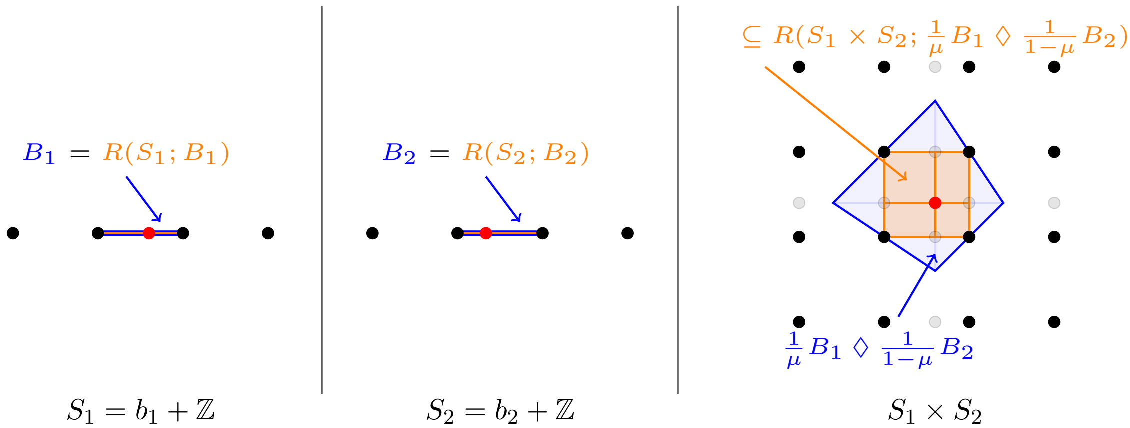
$$\begin{aligned}
\mathbb{R}^{n_1+n_2} &= \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \\
&= (R(S_1; B_1) + W_{S_1}) \times (R(S_2; B_2) + W_{S_2}) \\
&= (R(S_1; B_1) \times R(S_2; B_2)) + (W_{S_1} \times W_{S_2}) \\
&\subseteq R(S_1 \times S_2; \frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}) + (W_{S_1} \times W_{S_2}) \\
&= R(S_1 \times S_2; \frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}) + W_{S_1 \times S_2}.
\end{aligned}$$

Hence, $\frac{B_1}{\mu} \lozenge \frac{B_2}{1-\mu}$ has the covering property.

Note that (i) above holds for general closed sets S_h and S_h-free sets B_h. Example 14 provides an example of the coproduct operation and Theorem 26(c).

Example 14.

Let $n_1 = n_2 = 1$, $b_1 = \frac{1}{3}$, $b_2 = \frac{2}{3}$, $S_1 = b_1 + \mathbb{Z}$ *and* $S_2 = b_2 + \mathbb{Z}$. *Define the vectors* $a_1^1 = 3$, $a_2^1 = -\frac{3}{2}$, $a_1^2 = \frac{3}{2}$, *and* $a_2^2 = -3$. *Set* $B_1 = \{x \in \mathbb{R} : a_i^1 \cdot x \leq 1,\ i \in [2]\}$ *and* $B_2 = \{x \in \mathbb{R} : a_i^2 \cdot x \leq 1,\ i \in [2]\}$. *Note that* B_i *is a maximal* S_i*-free* 0*-neighborhood for* $i \in [2]$. *Figures 5.5(a) and (b) show* B_1 *and* B_2, *respectively. Using Equations* (5.7) *and* (5.6), *it can be seen that* $R(S_i; B_i) = B_i$ *for* $i \in [2]$. *For* $\mu \in (0,1)$, *Equation* (5.9) *gives an explicit formula for* $\frac{1}{\mu}B_1 \lozenge \frac{1}{1-\mu}B_2$*; see Figure 5.5(c). In the proof of Theorem 26(c), it is argued that the cross product of any two spindles* $R(s_j; B_1) \times R(s_k; B_2)$, *for* $j,k \in [2]$, *is contained in* $R(S_1 \times S_2; \frac{1}{\mu}B_1 \lozenge \frac{1}{1-\mu}B_2)$. *There are two spindles for* $R(S_1; B_1)$*:* $\left[-\frac{2}{3}, 0\right]$ *and* $\left[0, \frac{1}{3}\right]$. *Similarly, there are two spindles for* $R(S_2; B_2)$*:* $\left[-\frac{1}{3}, 0\right]$ *and* $\left[0, \frac{2}{3}\right]$. *Consider the four possible crossprodructs* $R(s_j; B_1) \times R(s_k; B_2)$, *we obtain the unit square. Furthermore,* $W_{S_1 \times S_2} = \mathbb{Z}^2$. *Hence* $\frac{1}{\mu}B_1 \lozenge \frac{1}{1-\mu}B_2$ *has the covering property.*



(a) S_1-free set with covering property

(b) S_2-free set with covering property

(c) $S_1 \times S_2$-free set with covering property

Figure 5.5: An example of the coproduct operation.

□

5.5 Limits of maximal S-free sets with the covering property

Let $m \in \mathbb{N}$ be fixed. For $t \in \mathbb{N}$, let $A^t \in \mathbb{R}^{m \times n}$ be a sequence of matrices and $b^t \in \mathbb{R}^m$ be a sequence of vectors such that $A^t \to A$ and $b^t \to b$ (both convergences are entrywise, i.e., convergence in the standard topology). Let $P_t = \{x \in \mathbb{R}^n : A^t \cdot x \leq b^t\}$ be the sequence of polyhedra defined A^t, b^t. We say that P_t converges to the polyhedron $P := \{x \in \mathbb{R}^n : A \cdot x \leq b\}$ and we write this as $P_t \to P$. We make some observations about this convergence.

Proposition 51. *Let $\{A^t\}_{t=1}^{\infty}$ be a sequence of matrices in $\mathbb{R}^{n \times m}$ converging entrywise to a matrix A. If the dimension of the nullspace of A^t is fixed for all t, say with value k, then the dimension of the nullspace of A is at least k.*

Proof. If $k = 0$, then the result is trivial. So assume that $k > 0$. For each value of t, there exists orthonormal vectors $\{v_1^t, v_2^t, \ldots, v_k^t\}$ that span the nullspace(A^t). Let $V^t \in \mathbb{R}^{n \times k}$ be the matrix with v_i^t as the i-th column. As each v_i^t is bounded in $\mathbb{R}^n$, V^t is bounded in $\mathbb{R}^{n \times k}$. Hence, we may extract a convergent subsequence converging to a matrix V. By continuity of the inner product of vectors, the columns of V are orthornormal and $AV = 0$. Hence, $\dim(\text{nullspace}(A)) \geq k$. □

Proposition 52. *Suppose that $\{P_t\}$ is a sequence of polyhedra defined by $P_t = \{x \in \mathbb{R}^n : A^t \cdot x \leq b^t\}$. If $P_t \to P$, where P is a polytope, and $P \cap P_t \neq \emptyset$ for each t, then there exists $M \in \mathbb{R}$ such that $P \subseteq [-M, M]^n$ and the sequence $\{P_t\}$ is eventually contained in $[-M, M]^n$. Consequently, the polyhedra in the sequence eventually become polytopes.*

Proof. It suffices to show that for every $\epsilon > 0$, there exists a sufficiently large t, $P_t \subseteq P + \epsilon D(0; 1)$, where $D(0; 1)$ is the unit ball around 0.

Assume to the contrary that this is not the case. This indicates that there exists a subsequence of points $\{x_{t_k}\}_{k=1}^{\infty}$ such that $x_{t_k} \in P_{t_k} \setminus (P + \epsilon D(0; 1))$. For each $k \in \mathbb{N}$, there exists some $z_k \in P_{t_k} \cap P$ since $P_{t_k} \cap P \neq \emptyset$. Since the distance function is continuous, there exists some point $y_k \in P_{t_k} \setminus P$ on the line segment $[x_k, z_k]$ such that $y_k \in Y := \{x \in \mathbb{R}^n : \epsilon/2 \leq d(P, x) \leq \epsilon\}$. Consider the sequence $\{y_k\}_{k=1}^{\infty}$. Note Y is compact since P is a polytope. Therefore, there exists a subsequence $\{y_{k_j}\}$ of $\{y_k\}$ such that $y_{k_j} \to y$ in Y. Let $A^t \to A$ and $b^t \to b$. Since $y \notin P$, there exists some $i^* \in [m]$ such that $a_{i^*} \cdot y > b_{i^*}$ where a_{i^*} is the row of A indexed by i^* and b_{i^*} is the i^*-th component of b. However, this implies that

$$a_{i^*} \cdot y > b_{i^*} = \lim_{j \to \infty} b_{i^*}^{t_{k_j}} \geq \lim_{j \to \infty} a_{i^*}^{t_{k_j}} \cdot y_{k_j} = a_{i^*} \cdot y,$$

where $a_{i^*}^{t_{k_j}}$ is the row of $A^{t_{k_j}}$ indexed by i^* and $b_{i^*}^{t_{k_j}}$ is the i^*-th component of $b^{t_{k_j}}$. Thus, we reach a contradiction. □

Proposition 53. *Suppose that $\{P_t\}$ is a sequence of polyhedra defined by $P_t = \{x \in \mathbb{R}^n : A^t \cdot x \leq b^t\}$. If $P_t \to P$ and $x \in \operatorname{int}(P)$, then there exists $t_0 \in \mathbb{N}$ such that $x \in \operatorname{int}(P_t)$ for all $t \geq t_0$.*

Proof. As $x \in \operatorname{int}(P)$, there exists $\delta > 0$ such that $\delta \mathbf{1}_m < b - Ax$, where $\mathbf{1}_m \in \mathbb{R}^m$ is the vector of all ones. Since $A^t \to A$ and $b^t \to b$, we have that $b^t - A^t x \to b - Ax$ and thus there exists $t_0 \in \mathbb{N}$ such that $b^t - A^t x \geq \delta \mathbf{1}_m$ for all $t \geq t_0$ and so $x \in \operatorname{int}(P_t)$ for all $t \geq t_0$. □

We next build some tools to prove our main result of this section, Theorem 27, which is about limits of maximal S-free sets that possess the covering property. For the rest of this section, we consider an arbitrary polyhedrally-truncated affine lattice S. Let B be a maximal S-free polyhedron given by (2.10), and recall the definition $L_B = \{r \in \mathbb{R}^n : a_i \cdot r = a_j \cdot r, \quad \forall i, j \in [m]\}$.

Proposition 54. *Let B be a maximal S-free set and assume that $B \cap \text{conv}(S)$ is a full-dimensional polytope. If B has the covering property, then $L_B + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$.*

Proof. Assume to the contrary that $L_B + \text{lin}(\text{conv}(S)) \neq \mathbb{R}^n$. We claim that $R(S; B) + W_S \neq \mathbb{R}^n$, yielding our contradiction.

Since $L_B + \text{lin}(\text{conv}(S)) \neq \mathbb{R}^n$, we may choose a subspace M of $\mathbb{R}^n$ such that $\text{lin}(\text{conv}(S)) \subsetneq M$ and $L_B + M = \mathbb{R}^n$. Furthermore, as a consequence of Proposition 50, we may choose M so that $M \cap L_B = \{0\}$. Define $M(S; B) := R(S; B) \cap M$. Note that $M(S; B)$ is compact as the recession cone of every spindle in $R(S; B)$ is L_B. Also, $R(S; B) = L_B + M(S; B)$. As $M(S; B)$ is compact, $\text{lin}(\text{conv}(S)) + M(S; B) \subsetneq M$. Therefore, using Proposition 48,

$$R(S; B) + W_S = L_B + M(S; B) + (\text{lin}(\text{conv}(S)) \cap \Lambda) \subsetneq L_B + M = \mathbb{R}^n.$$

$\square$

Proposition 55. *Suppose B is a maximal S-free set such that $L_B + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$ and $L_B \cap \text{lin}(\text{conv}(S)) = \{0\}$. Define $M := R(S; B) \cap \text{lin}(\text{conv}(S))$. Then the covering property $R(S; B) + W_S = \mathbb{R}^n$ is equivalent to $M + W_S = \text{lin}(\text{conv}(S))$.*

Proof. Suppose $R(S; B) + W_S = \mathbb{R}^n$. Intersecting both sides by $\text{lin}(\text{conv}(S))$, we see that $(R(S; B) + W_S) \cap \text{lin}(\text{conv}(S)) = \text{lin}(\text{conv}(S))$. It is sufficient to show that $(R(S; B) + W_S) \cap \text{lin}(\text{conv}(S)) = M + W_S$. Take $r + w \in (R(S; B) + W_S) \cap \text{lin}(\text{conv}(S))$ for $r \in R(S; B)$ and $w \in W_S \subseteq \text{lin}(\text{conv}(S))$ by Proposition 48. As $r + w \in \text{lin}(\text{conv}(S))$, $r \in \text{lin}(\text{conv}(S))$. Thus, $r \in R(S; B) \cap \text{lin}(\text{conv}(S))$. Hence, $r \in M$ and $(R(S; B) + W_S) \cap \text{lin}(\text{conv}(S)) \subseteq M + W_S$. The other inclusion follows immediately from $W_S \subseteq \text{lin}(\text{conv}(S))$.

Now suppose that $M + W_S = \text{lin}(\text{conv}(S))$ and take $x \in \mathbb{R}^n$. Since L_B and $\text{lin}(\text{conv}(S))$ are complementary spaces, there exists $l \in L_B$ and $s \in \text{lin}(\text{conv}(S))$ such that $x = l + s$. By our assumption, there is an $m \in M$ and $w \in W_S$ so that $x = l + s = l + (m + w) = (l + m) + w$. Since $m \in R(S; B)$, m is contained in some spindle belonging to $R(S; B)$. However, L_B is the lineality space of each spindle. Hence, $l+m \in R(S;B)$. This shows that $\mathbb{R}^n \subseteq R(S;B)+W_S$. The other inclusion follows as $\mathbb{R}^n$ is the ambient space. □

Proposition 56. *Suppose that* $\{B_t\}_{t=1}^{\infty}$ *is a sequence of maximal* S*-free sets such that* $L_{B_t} + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$*, where* $L_{B_t} = \{r : a_i^t \cdot r = a_j^t \cdot r, \quad \forall i,j \in [m]\}$*. If* $B_t \to B$*, and* $B \cap \text{conv}(S)$ *is a full dimensional polytope, then* $L_B + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$*, where* $L_B = \{r : a_i \cdot r = a_j \cdot r, \quad \forall i,j \in [m]\}$*.*

Proof. Suppose $\dim(\text{lin}(\text{conv}(S))) = k$. As $L_B \cap \text{lin}(\text{conv}(S)) = \{0\}$ from Proposition 50, it is sufficient to show that $\dim(L_B) \geq n-k$. Since $B_t \to B$, we have $B_t \cap \text{conv}(S) \to B \cap \text{conv}(S)$, and since $B \cap \text{conv}(S)$ is a full dimensional polytope, by Propositions 52 and 53 we eventually have that $B_t \cap \text{conv}(S)$ is a polytope. Thus, $L_{B_t} \cap \text{lin}(\text{conv}(S)) = \{0\}$ by Proposition 50. Since $L_{B_t} + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$ for each t, $\dim(L_{B_t}) = n-k$. For each $i \neq j \in [m]$, define the matrix A^t to have rows $a_i^t - a_j^t$ and A to have the rows $a_i - a_j$. As $L_{B_t} = \text{nullspace}(A^t)$, Proposition 51 implies that $\dim(\text{nullspace}(A)) \geq n-k$. Observing that $L_B = \text{nullspace}(A)$ yields the desired result. □

Theorem 27. *Suppose* $\{B_t\}_{t=1}^{\infty}$ *is a sequence of maximal* S*-free sets possessing the covering property. If* $B_t \to B$*, where* B *is a maximal* S*-free set and* $B \cap \text{conv}(S)$ *is a polytope, then* B *also possesses the covering property.*

Proof. If B is a halfspace, then it is easy to check that B has the covering property. Therefore, consider when B is not a halfspace and so $\text{int}(B \cap \text{conv}(S)) \neq \emptyset$ by Proposition 49.

From Proposition 52 and 53 we eventually have that $B_t \cap \text{conv}(S)$ is a full-dimensional polytope. By Proposition 54 we have $L_{B_t} + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$. By Proposition 56, $L_B + \text{lin}(\text{conv}(S)) = \mathbb{R}^n$. Moreover, since $B \cap \text{conv}(S)$ is a polytope, we have $L_B \cap \text{lin}(\text{conv}(S)) = \{0\}$ by Proposition 50. Define $M_t := R(S;B_t) \cap \text{lin}(\text{conv}(S))$ and $M := R(S;B) \cap \text{lin}(\text{conv}(S))$. From Proposition 55, it is sufficient to show that $\text{lin}(\text{conv}(S)) \subseteq M + W_S$.

Let $x \in \text{lin}(\text{conv}(S))$. Following Proposition 55, for each t there exists a spindle, $R(s_t;B_t)$, corresponding to B_t such that $x \in D(s_t;B_t) + w_t$, where $D(s_t;B_t) = R(s_t;B_t) \cap \text{lin}(\text{conv}(S))$ and $w_t \in W_S$.

Claim 12. *The values s_t and w_t can be chosen independently of t.*

Proof of Claim. From Proposition 52, there exists a bounded set, U, that contains $B \cap \operatorname{conv}(S)$ and $B_t \cap \operatorname{conv}(S)$ for sufficiently large t. Consider the tail subsequence $\{B_t\}$ that has the property $B_t \cap \operatorname{conv}(S) \subseteq U$ for all t. As U is bounded and S is discrete, there is a finite number of points in $U \cap S$. Note that each spindle in $R(S; B_t)$ is defined by a point in $B_t \cap S \subseteq U$. Therefore, there exists an $s \in S$ and a subsequence of $\{B_t\}$ such that $D(s_t; B_t) = D(s; B_t)$, for all t. Relabel such a subsequence by $\{B_t\}$.

Since the inner product is a continuous function on $\mathbb{R}^n$, $s \in B_t$ implies $s \in B$. Since $B_t \to B$, for a fixed s it also follows that $D(s; B_t) \to D(s; B)$, where $D(s; B) := R(s) \cap \operatorname{lin}(\operatorname{conv}(S))$. As $L_{B_t} \cap \operatorname{lin}(\operatorname{conv}(S)) = \{0\}$ for each t, the set $D(s; B_t)$ is a polytope for each t. Similarly, $D(s; B)$ is a polytope. Again using Proposition 52, there exists a bounded set V such that $D(s; B) \subseteq V$ and $D(s; B_t) \subseteq V$ for large t (note that the origin is in each $D(s; B_t)$ and $D(s; B)$ and so the hypothesis of the Proposition 52 is satisfied). In the same manner as above, for large t, $w_t \in D(s; B_t) - x \subseteq V - x$, which is a bounded set. Since $W_S = \operatorname{lin}(\operatorname{conv}(S) \cap \Lambda$ by Proposition 48, W_S is discrete and there exists a $w \in W_S$ and a subsequence of $\{B_t\}$ (label this subsequence as $\{B_t\}$) such that $w_t = w$ for all t. Hence, $x \in D(s; B_t) + w$ for all t. $\diamond$

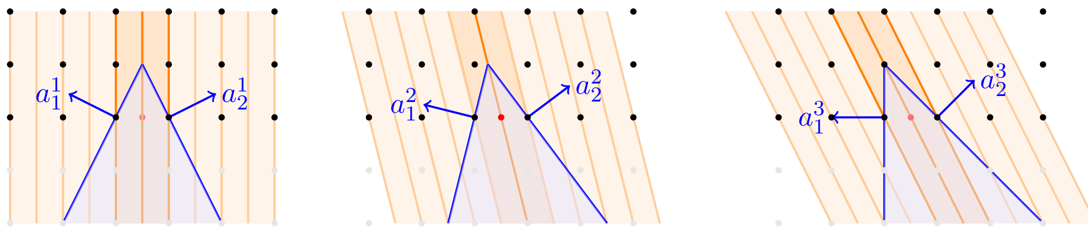
Since the inner product is a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$, $x \in D(s; B_t) + w$ implies $x \in D(s; B) + w$. As $D(s; B) \subseteq M$, it follows that $x \in M + W_S$. Hence, $\operatorname{lin}(\operatorname{conv}(S)) \subseteq M + W_S$, as desired. $\square$

Example 15 gives an example of how limits preserve the covering property.

Example 15.

Consider the situation in Figure 5.1(c)— $b = \left(-\frac{1}{2}, 0\right)$ and $S = f + \mathbb{Z}^2_+$. In this situation, note that $W_S = \mathbb{Z} \times \{0\}$. Let B_1 be the maximal S-free cylinder defined by $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{R}$. The region $R(S; B_1) + W_S$ is shown in Figure 5.6(a) in light orange, and the region $R(S; B_1)$ is highlighted in dark orange (the coloring scheme is the same for (b) and (c)). Tilting this cylinder while maintaining its base $\left[-\frac{1}{2}, \frac{1}{2}\right]$ creates a sequence of S-free sets. Figure 5.6(b) and (c) shows other sets B_2 and B_3 in this sequence, along with the corresponding $R(S; B_2)+$

W_S and $R(S;B_3)+W_S$. The normal vectors defining B_1, B_2, and B_3 are drawn in blue to emphasize convergence.



(a) $R(S,B_1)+W_S$ (b) $R(S,B_2)+W_S$ (c) $R(S,B_3)+W_S$

Figure 5.6: An example of how the covering property is preserved under limits.

□

The assumption that $B \cap \mathrm{conv}(S)$ is a polytope

We end this section with a short justification of the assumption that $B \cap \mathrm{conv}(S)$ is a polytope that was made in Theorem 27. Although it may seem restrictive at first, if $B \cap \mathrm{conv}(S)$ is not a polytope then one can reduce to that case in the following way. Let N be the linear space spanned by $\mathrm{rec}(B \cap \mathrm{conv}(S))$. By Proposition 49(i), N is a lattice subspace. Let $\bar{B}, \bar{S}, \bar{\Lambda}$ be the projection of B, S, Λ onto the orthogonal subspace $N^\perp$ of N. By a well-known property of lattices, $\bar{\Lambda}$ is a lattice. Also, since $\mathrm{conv}(\bar{S})$ is the projection of $\mathrm{conv}(S)$ and $S = \mathrm{conv}(S) \cap (b + \Lambda)$, we have $\bar{S} = \mathrm{conv}(\bar{S}) \cap (\bar{b} + \bar{\Lambda})$ where $\bar{b}$ is the projection of b. Hence, $\bar{S}$ is a polyhedrally-truncated affine lattice in $N^\perp$ and $\bar{B}$ is a maximal $\bar{S}$-free set. Moreover, $\bar{B} \cap \mathrm{conv}(\bar{S})$ is a polytope, since N is the linear space spanned by $\mathrm{rec}(B \cap \mathrm{conv}(S))$. Note that $N \subseteq L_B$ by Proposition 49(i), and by Proposition 49(ii), $R(S;B) = R(\bar{S};\bar{B}) + N$. Hence, B has the covering property with respect to S if and only if $\bar{B}$ has the covering property with respect to $\bar{S}$. Therefore, to check if B has the covering property with respect to S, one can check if $\bar{B}$ can be obtained as the limit of $\bar{S}$-free sets with the covering property.

5.6 Application: Iterative application of coproducts and limits

In this section, we show some examples demonstrating the versatility of the coproduct and limit operations to obtain new and interesting families of bodies with the covering property. We note that the coproduct operation is associative: $(C_1 \Diamond C_2) \Diamond C_3 = C_1 \Diamond (C_2 \Diamond C_3)$. Thus, we will use notation such as $C_1 \Diamond C_2 \Diamond \ldots \Diamond C_k$ without any ambiguity.

1. **Crosspolytopes.** Let $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that $a_j < 0 < b_j$ for all $j \in [n]$ and $\sum_{j=1}^n \frac{1}{b_j - a_j} = 1$. Consider the set of $2n$ points

$$X = \{(0, \ldots, a_j, \ldots, 0), (0, \ldots, b_j, \ldots, 0) : j \in [n]\},$$

   where the nonzero entry is in coordinate j. Define $S = (\frac{b_1}{b_1 - a_1}, \ldots, \frac{b_n}{b_n - a_n}) + \mathbb{Z}^n$. Then the crosspolytope $\text{conv}(X)$ is a maximal S-free set with the covering property.

   This follows from the fact that $\text{conv}(X) = (b_1 - a_1)I_1 \Diamond (b_2 - a_2)I_2 \Diamond \ldots \Diamond (b_n - a_n)I_n$ where I_j is the interval $[\frac{a_j}{b_j - a_j}, \frac{b_j}{b_j - a_j}]$; I_j is therefore a maximal S_j-free set with the covering property where $S_j = \frac{b_j}{b_j - a_j} + \mathbb{Z}$. Applying Theorem 26 shows that the crosspolytope $\text{conv}(X)$ has the covering property.

2. **Simplices.** Let $f_1, \ldots, f_n \in \mathbb{R}$ such that $0 < f_j$ for all $j \in [n]$ and $\sum_{j=1}^n \frac{1}{f_j} = 1$. Then the simplex $\text{conv}\{0, f_1 e^1, f_2 e^2, \ldots, f_n e^n\}$, where the e^i denotes the i-th unit vector in $\mathbb{R}^n$, is a maximal $\mathbb{Z}^n$-free set with the covering property. This follows from taking the limit of the crosspolytopes defined in 1. above as $a_i \to 0$, and applying Theorem 27. This generalizes the Type 1 triangle from the literature, as well as its higher dimensional analogue $\{0, ne^1, \ldots, ne^n\}$ that has been studied in [24, 42], where this special case was shown to have the covering property using completely different arguments.

3. **Further examples.** In three dimensions, one can show that there exist lattice-free sets with the covering property with 2,3,4,5,6, and 8 facets. By taking cylinders over the two-dimensional sets one can obtain 2,3, and 4 facets. The crosspolytope from 1.

above gives 8 facets. The coproduct of a triangle and an interval has 6 facets. Five facets can be obtained by taking the coproduct of a quadrilateral and an interval which gives a crosspolytope with 8 facets, and then taking a limit to reduce the number of facets from 8 to 5: four of the facets degenerate into a single facet. This can be iterated to generate bodies with the covering property in 4, 5, and any number of dimensions.

We give another example of the kind of results one can prove using coproducts and limit operations. In $\mathbb{R}^k$ ($k \geq 2$), one can explicitly construct a maximal $(b + \mathbb{Z}^k)$-free set with $2^{k-1} + 1$ facets with the covering property. This can be seen by taking the coproduct of k intervals (to get the crosspolytope with 2^k facets) and then taking the limit to reduce 2^{k-1} of the facets into a single facet.

Moreover, when considering S of the form $b + (\mathbb{Z}^n \times \mathbb{Z}^q_+)$ one can construct unbounded polyhedra, by taking the coproduct of a translated cone in $\mathbb{R}^2$ (which has been shown in the literature to be a maximal S-free set with the covering property when S is a translated lattice intersected by a halfspace) and quadrilaterals, triangles, and intervals (and iterating to get into arbitrarily high dimensions).

We feel establishing the covering property of the examples above, or even discovering that these bodies have the covering property, would have been challenging without the tools of the coproduct and the limit operation. We mention that the constructions for the crosspolytopes and simplices above were first given in [5]. These operations are constructive and therefore potentially useful beyond purely theoretical questions about the covering property.

Chapter 6

The Fixing Region

This chapter focuses on the mixed-integer model M_S for $S = (b+\Lambda) \cap C$, where Λ is a lattice, $b \in \mathbb{R}^n \setminus \Lambda$, $C \subseteq \mathbb{R}^n$ is convex, and $\mathrm{conv}(S)$ is a polyhedron (that is S is a polyhedrally-truncated affine lattice). In Chapter 5, we were able to create minimal cut-generating pairs for M_S as follows. First, identify a minimal cut-generating function ψ for C_S that has the covering property. Next, compute the trivial lifting π^* of ψ. Finally, pair (ψ, π^*) to obtain a minimal cut-generating pair. This process requires ψ to have the covering property, but, as seen in Example 10(b), not every minimal ψ does. What can be said about minimal ψ without the covering property? In this case, there is not a unique minimal lifting, so how can one find a minimal lifting? We address this question in this chapter. Using a lifting procedure, we identify a strict subfamily of minimal liftings of ψ. All liftings in this subfamily are equal on a subset called the fixing region. Moreover, we provide explicit formulas for the common values taken on the fixing region. Under certain conditions, the fixing region equals $\mathbb{R}^n$ in which case we identify a minimal lifting for ψ. This work was done in collaboration with Amitabh Basu and Santanu Dey, and has been submitted for publication [25].

6.1 Identifying functions without the covering property

Let $S = (b+\Lambda)\cap C$ be a polyhedrally-truncated affine lattice. For a cut-generating function ψ for C_S, recall the lifting region

$$R_\psi = \{r \in \mathbb{R}^n : \psi(r) = \pi(r) \text{ for every minimal lifting } \pi \text{ of } \psi\}, \tag{6.1}$$

and the set

$$W_S = \{w \in \mathbb{R}^n : s + \lambda w \in S \ , \forall s \in S, \forall \lambda \in \mathbb{Z}\}. \tag{6.2}$$

We made the following key observation.

Observation 6. *Suppose π_1 and π_2 are minimal liftings of ψ. Then $\pi_1(x) = \pi_2(x)$ for all $x \in R_\psi + W_S$.*

Thus if $R_\psi + W_S = \mathbb{R}^n$, then ψ has a unique minimal lifting and is said to have the *covering property*. Moreover, Proposition 46 states that $\psi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function for C_S precisely when there is some maximal S-free 0-neighborhood

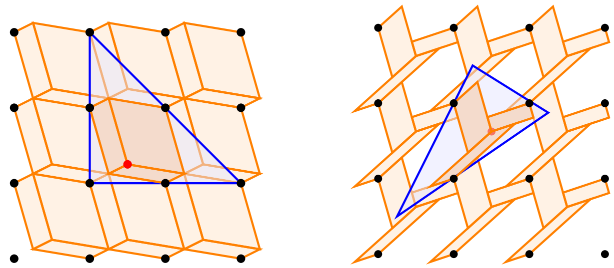
$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1 \ \ i \in [m]\} \tag{6.3}$$

such that

$$\psi(x) = \max_{i\in[m]} a_i \cdot x. \tag{6.4}$$

With this correspondence between minimal cut-generating functions ψ and maximal S-free 0-neighborhoods B, we can say B has the covering property if the cut-generating function ψ induced by (6.4) has the covering property. We also use $R(S;B)$ to refer to R_ψ, when B and ψ satisfy (6.3) and (6.4).

As seen in Chapter 5, there are a large class of maximal S-free 0-neighborhoods with the covering property that can be identified and these objects have been the focus of many recent papers on minimal liftings [5, 17, 23, 30, 56]. However, there are many sets that don't have the covering property. Figure 6.1 shows two $(b + \mathbb{Z}^2)$-free sets; (a) has the covering property while (b) does not.



(a) (b)

Figure 6.1: Examples of S-free sets with and without the covering property. The S-free sets are in blue and the region $R(S;B)+W_S$ is in orange.

The purpose of this chapter is to describe minimal cut-generating pairs that arise from maximal S-free sets *without* the covering property. Our work is very much inspired by Section 7 of [56], which initiated the study of this problem.

Since all minimal liftings agree of ψ agree on $R_\psi + W_S$, one may ask if the converse holds true. That is, for $p^* \in \mathbb{R}^n \setminus (R_\psi + W_S)$, are there two minimal liftings of ψ that disagree on p^*? For $S = \mathbb{Z}^n$, the existence of such a p^* was shown in [17]; the question is open for more general S. Dey and Wolsey observed that there is a largest lower bound $\pi(p^*)$ for all minimal lifting π of ψ [56]. Moreover, there exists a minimal lifting that satisfies this at equality (see Proposition 58).

Definition 24 ($V_\psi(p^*)$ and $\mathcal{L}_{\psi,p^*}$)**.**

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a minimal cut-generating function for C_S. For $p^* \in \mathbb{R}^n$, define

$$V_\psi(p^*) := \inf\{\pi(p^*) : \pi \text{ minimal lifting of } \psi\}. \tag{6.5}$$

The finiteness of $V_\psi(p^*)$ was shown in [56]. Let $\mathcal{L}_{\psi,p^*}$ to be the set of all minimal liftings π of ψ such that $\pi(p^*) = V_\psi(p^*)$.

By Definition 24 and Observation 6, all $\pi \in \mathcal{L}_{\psi,p^*}$ agree on both $V_\psi(p^*)$ and $R_\psi + W_S$. Are there more values on which these liftings agree? Analogous to the lifting region and its extension $R_\psi + W_S$, we define the *fixing region corresponding to* p^* to be the set of points on which all minimal liftings in $\mathcal{L}_{\psi,p^*}$ agree.

6.1.1 Statement of results

In this chapter, we explore questions such as: What is a good description of the fixing region? How does the fixing region depend on p^*? How much does the fixing region cover?

1. Our first main result is Theorem 31, which provides a partial, yet explicit, description of the fixing region.

2. Although we are not able to give a complete characterization of the fixing region, the partial subset we describe is used to show that for certain maximal S-free 0-neighborhoods B without the covering property, there exists a p^* such that the fixing region is all of $\mathbb{R}^n$. In other words, after finding the optimal lifting coefficient $V_\psi(p^*)$ for p^*, the lifting coefficients for all other rays get fixed. We say that such a set B is *one point fixable*. Proposition 66 shows that certain Type 3 triangles are one point fixable. As a corollary, in Proposition 67, we recover a result from [56] that Type 3 triangles resulting from the mixing set are one point fixable, using completely different and more geometric techniques. See [54, 72] for more on the mixing set.

3. Theorem 32 says if our partial description of the fixing region shows that an S-free 0-neighborhood B is one point fixable, then the $S+t$-free 0-neighborhood $B+t$ is also one point fixable for all vectors $t \in \mathbb{R}^n$ such that $B+t$ is still a 0-neighborhood. In other words, one point fixability is preserved under translations.

4. For our study, we develop a theory about so-called *partial cut-generating functions* – cut-generating functions that are only defined on a subset of $\mathbb{R}^n$. These results could be of general interest in the theory of cut-generating functions.

5. Our work is very much geometric in flavor, complementing the algebraic approach taken by Dey and Wolsey in [56] who studied the problem in $\mathbb{R}^2$. Proposition 60 gives some evidence that the geometric approach may yield stronger liftings compared to the algebraic approach.

6.2 Preliminaries and general facts about liftings

We denote the columns of a matrix A by $\operatorname{col}(A)$. For $S \subseteq \mathbb{R}^n \setminus \{0\}$, define:

$$\widehat{W}_S := \{w \in \mathbb{R}^n : s + \lambda w \in S \;, \forall s \in S, \forall \lambda \in \mathbb{Z}_+\}. \tag{6.6}$$

Note that $W_S = \widehat{W}_S \cap (-\widehat{W}_S)$.

6.2.1 Partial cut-generating functions

Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed subset with $0 \notin S$. The definition of a cut-generating pair for M_S takes as input any pair of matrices R and P with columns in $\mathbb{R}^n$. This can be generalized to only consider matrices R and P with columns coming from subsets of $\mathbb{R}^n$

Definition 25 (Partial cut-generating pairs, valid pairs)**.**

Let $\mathcal{R}, \mathcal{P}$ be subsets of $\mathbb{R}^n$ and let $\psi : \mathcal{R} \to \mathbb{R}$ and $\pi : \mathcal{P} \to \mathbb{R}$. The pair (ψ, π) is a partial cut-generating pair for $M_S, \mathcal{R}, \mathcal{P}$, if *for every choice of k, ℓ, R and P* where the columns of R and P come from $\mathcal{R}$ and $\mathcal{P}$ respectively,

$$\sum \psi(r) s_r + \sum \pi(p) y_p \geq 1 \tag{6.7}$$

is an inequality separating 0 from the mixed-integer set $M_S(R, P)$. We often call a partial cut-generating pair simply a *valid pair.*

Definition 26 (Partial cut-generating functions, valid functions)**.**

The function ψ is a partial cut-generating function $\psi : \mathcal{R} \to \mathbb{R}$ for $C_S, \mathcal{R}$ if for every choice of k and R, where the columns of R come from $\mathcal{R}$,

$$\sum \psi(r) s_r \geq 1 \tag{6.8}$$

is an inequality separating 0 from the mixed-integer set $C_S(R)$. We often call a partial cut-generating function simply a valid function.

For the setting of $\mathcal{R} = \mathbb{R}^n$, partial cut-generating functions are just cut-generating functions as defined in Definition 5. In this setting, a lifting π of a valid function ψ is a function $\pi : \mathbb{R}^n \to \mathbb{R}$ such that (ψ, π) forms a cut-generating pair for M_S. This concept of a lifting extends to partial cut-generating functions.

Definition 27 (Lifting of a valid function)**.**

Let $\mathcal{P} \subseteq \mathbb{R}^n$, then we say $\pi : \mathcal{P} \to \mathbb{R}$ is a lifting of a valid function ψ (valid for $C_S, \mathcal{R}$), if (ψ, π) is a valid pair for $M_S, \mathcal{R}, \mathcal{P}$.

When studying the 'standard' cut-generating pairs and functions, we looked for strong functions as defined by minimality. This concept extends to the setting of partial cut-generating pairs and partial cut-generating functions.

Note 5. *The concept of a minimal valid pair, minimal valid functions, and minimal liftings of ψ are defined analogously to the case $\mathcal{R} = \mathbb{R}^n, \mathcal{P} = \mathbb{R}^n$.*

The following proposition collects some results about partial cut-generating pairs. It is worth noting that setting $\mathcal{R} = \mathcal{P} = \mathbb{R}^n$ in the following recovers the setting of cut-generating pairs.

Proposition 57. *Let $S \subseteq \mathbb{R}^n \setminus \{0\}$, $\mathcal{R}, \mathcal{P}$ be subsets of $\mathbb{R}^n$ and $\psi : \mathcal{R} \to \mathbb{R}$ be a partial cut-generating function for $S, \mathcal{R}$. Then the following hold:*

(a) For every minimal lifting π of ψ, $\pi(p) \leq \pi(p+w)$ for all $p \in \mathcal{P}$ and $w \in \widehat{W}_S$ such that $p + w \in \mathcal{P}$. Thus, $\pi(p) = \pi(p + w)$ for all $p \in \mathcal{P}$ and $w \in W_S$ such that $p + w \in \mathcal{P}$.

(b) Define $\psi^* : \mathcal{R} \to \mathbb{R}$ as follows

$$\psi^*(r) = \inf\{\psi(r + w) : w \in \widehat{W}_S \text{ such that } r + w \in \mathcal{R}\}$$

Then (ψ, ψ^*) is a valid partial cut-generating pair for $S, \mathcal{R}, \mathcal{R}$.

(c) If $\mathcal{R} = \mathcal{P}$, then every minimal lifting π of ψ satisfies $\pi \leq \psi^*$.

Proof. We first establish that if π is any lifting of ψ, then $\hat{\pi}^* : \mathcal{P} \to \mathbb{R}$ defined as

$$\hat{\pi}^*(p) := \inf_{w \in \widehat{W}_S} \Big\{ \pi(p + w) : \; p + w \in \mathcal{P} \Big\}$$

is also a lifting of ψ. Consider any $R \in \mathcal{R}^k, P \in \mathcal{P}^\ell$ and $(s, y) \in M_S(R, P)$. Let $W \in \mathbb{R}^{n\times\ell}$ be any matrix whose columns are in $\widehat{W}_S$ such that $P+W \in \mathcal{P}^\ell$. Let $(\bar{s}, \bar{y})$ be constructed as follows: $\bar{s} = s$ and $\bar{y}_{p+w} = y_p$ for $p \in \operatorname{col}(P)$ and w the corresponding column of W. Then $R\bar{s}+(P+W)\bar{y} = Rs+Py+\bar{w}$ where $\bar{w} \in \widehat{W}_S$ by definition of W. Thus, $R\bar{s}+(P+W)\bar{y} \in S$. Since (ψ, π) is a valid pair, we obtain

$$\sum \psi(r)\bar{s}_r + \sum \pi(p+w)\bar{y}_{p+w} \geq 1.$$

The above holds for all matrices $W \in \mathbb{R}^{n\times\ell}$ whose columns are in $\widehat{W}_S$ and $P + W \in \mathcal{P}^\ell$. Taking an infimum over all such W gives

$$\begin{aligned}\sum \psi(r)s_r + \sum \hat{\pi}^*(p)y_p &= \sum \psi(r)s_r + \inf_W \left\{ \sum \pi(p+w)y_p \right\} \\ &= \inf_W \left\{ \sum \psi(r)\bar{s}_r + \sum \pi(p+w)\bar{y}_{p+w} \right\} \geq 1.\end{aligned}$$

From this we immediately obtain (a), since $\hat{\pi}^* \leq \pi$ for any minimal lifting π and also $(\psi, \hat{\pi}^*)$ is a valid pair. Thus by minimality of π, $\hat{\pi}^* = \pi$ and so $\pi(p) = \hat{\pi}^*(p) \leq \pi(p+w)$ for all $p \in \mathcal{P}$ and $w \in \widehat{W}_S$ such that $p + w \in \mathcal{P}$.

Part (b) follows from the fact that ψ is a valid function for S.

For any minimal lifting π, we have $\pi \leq \psi$. Using (a), $\pi(p) \leq \pi(p+w) \leq \psi(p+w)$ for all $p \in \mathcal{P}$ and $w \in \widehat{W}_S$ such that $p + w \in \mathcal{P}$. Taking an infimum over all such $w \in \widehat{W}_S$, we obtain (c). □

Theorem 28. *Let (ψ, π) be a minimal valid pair for $S, \mathcal{R}, \mathcal{P}$. Then ψ and π are both subadditive over $\mathcal{R}$ and $\mathcal{P}$ respectively, i.e., $\psi(r_1+r_2) \leq \psi(r_1)+\psi(r_2)$ for all $r_1, r_2 \in \mathcal{R}$ such that $r_1 + r_2 \in \mathcal{R}$, and $\pi(p_1 + p_2) \leq \pi(p_1) + \pi(p_2)$ for all $p_1, p_2 \in \mathcal{P}$ such that $p_1 + p_2 \in \mathcal{P}$. Moreover, ψ is positively homogeneous over $\mathcal{R}$, i.e., for all $r \in \mathcal{R}$ and $\lambda > 0$ such that $\lambda r \in \mathcal{R}$, we have $\psi(\lambda r) = \lambda\psi(r)$.*

When allowing partial cut-generating pairs to be defined on subsets of $\mathbb{R}^n$, it is natural to ask how such pairs behave when the domain is extended.

Question 1. *Given $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathbb{R}^n, \mathcal{P} \subseteq \mathcal{P}' \subseteq \mathbb{R}^n$, and a valid pair (ψ, π) valid for $M_S, \mathcal{R}, \mathcal{P}$,* *does there always exist functions ψ', π' such that (ψ', π') is valid for $M_S, \mathcal{R}', \mathcal{P}'$ and ψ', π' are extensions of ψ, π, i.e., they coincide on $\mathcal{R}$ and $\mathcal{P}$ respectively?*

The answer to Question 1 is NO in general. Indeed choosing $\mathcal{R} = \emptyset$ and $\mathcal{P} = \mathbb{R}^n$, we obtain Gomory and Johnson's pure integer model, and we know that the discontinuous valid functions π for this model cannot be appended with any ψ to give a valid pair for the full mixed-integer model (see [52]). However, under certain conditions, such extensions can be constructed.

Theorem 29. *Let $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathbb{R}^n, \mathcal{P} \subseteq \mathcal{P}' \subseteq \mathbb{R}^n$, and let (ψ, π) be a valid pair for $M_S, \mathcal{R}, \mathcal{P}$. Suppose $\mathcal{R}', \mathcal{P}' \subseteq \operatorname{cone}(\mathcal{R})$. Then there exist functions $\psi' : \mathcal{R}' \to \mathbb{R}$ and $\pi' : \mathcal{P}' \to \mathbb{R}$ such that (ψ', π') is a minimal valid pair for $M_S, \mathcal{R}', \mathcal{P}'$ and ψ', π' restricted to $\mathcal{R}, \mathcal{P}$ dominate ψ, π respectively.*

Proof. For $r' \in \mathcal{R}'$ define

$$v_\psi(r') := \inf \left\{ \sum_{r \in \mathcal{R}} \psi(r) s_r : \ r' = \sum_{r \in \mathcal{R}} r s_r, \ s \in \mathbb{R}^{\mathcal{R}}_+, s \text{ finite support} \right\}.$$

Similarly, for $p' \in \mathcal{P}'$ define

$$\begin{aligned} v_\pi(p') := \inf \Bigg\{ & \sum_{r \in \mathcal{R}} \psi(r) s_r + \sum_{p \in \mathcal{P}} \pi(p) y_p : \\ & p' = \sum_{r \in \mathcal{R}} r s_r + \sum_{p \in \mathcal{P}} p y_p, \ s \in \mathbb{R}^{\mathcal{R}}_+, y \in \mathbb{Z}^{\mathcal{R}}_+, s, y \text{ finite support} \Bigg\}. \end{aligned}$$

Since $\mathcal{R}', \mathcal{P}' \subseteq \operatorname{cone}(\mathcal{R})$, the infima defining $v_\psi(r')$ and $v_\pi(p')$ are over nonempty sets, so $v_\psi(r')$ and $v_\pi(p')$ are less than ∞.

Define functions $\tilde{\psi} : \mathcal{R}' \to \mathbb{R}$ and $\tilde{\pi} : \mathcal{P}' \to \mathbb{R}$ to be

$$\tilde{\psi}(r') := \begin{cases} v_\psi(r') & \text{if } v_\psi(r') > -\infty, \\ \psi(r') & \text{if } v_\psi(r') = -\infty \text{ and } r' \in \mathcal{R}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{\pi}(p') := \begin{cases} v_\pi(p') & \text{if } v_\pi(p') > -\infty, \\ \pi(p') & \text{if } v_\pi(p') = -\infty \text{ and } p' \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

From the comment above and the definition of $\tilde{\psi}, \tilde{\pi}$, both functions are well-defined. Moreover, $\tilde{\psi}(r) \le \psi(r)$ for $r \in \mathcal{R}$ and $\tilde{\pi}(p) \le \pi(p)$ for $p \in \mathcal{P}$. Therefore $(\tilde{\psi}, \tilde{\pi})$ dominates (ψ, π) on $\mathcal{R}, \mathcal{P}$. Since any valid pair is dominated by a minimal pair, it is sufficient to show that $(\tilde{\psi}, \tilde{\pi})$ is valid for $S, \mathcal{R}', \mathcal{P}'$.

Take (s', y') in $M_S(\mathcal{R}', \mathcal{P}')$ and let $\epsilon > 0$. Consider any $r' \in \mathcal{R}'$. By the definition of $\tilde{\psi}$, there exists $s(r') \in \mathbb{R}_+^{\mathcal{R}}$ so that $s(r')$ has finite support and

$$r' = \sum_{r \in \mathcal{R}} r s(r')_r \quad \text{and} \quad \tilde{\psi}(r') > \left(\sum_{r \in \mathcal{R}} \psi(r) s(r')_r \right) - \epsilon. \tag{6.9}$$

A similar argument shows that for each $p' \in \mathcal{P}'$, there exists $s(p') \in \mathbb{R}_+^{\mathcal{R}}$ and $y(p') \in \mathbb{Z}_+^{\mathcal{P}}$, both with finite support, satisfying

$$p' = \sum_{r \in \mathcal{R}} r s(p')_r + \sum_{p \in \mathcal{P}} p y(p')_p \quad \text{and} \quad \tilde{\pi}(r) > \left(\sum_{r \in \mathcal{R}} \psi(r) s(p')_r + \sum_{p \in \mathcal{P}} \psi(p) y(p')_p \right) - \epsilon. \tag{6.10}$$

Define $(\tilde{s}, \tilde{y})$ by

$$\tilde{s}_r := \sum_{r' \in \mathcal{R}'} s(r')_r s'_{r'} + \sum_{p' \in \mathcal{P}'} s(p')_r y'_{p'} \quad \textit{and} \quad \tilde{y}_p := \sum_{p' \in \mathcal{P}'} y(p')_p y'_{p'}$$

Notice $(\tilde{s}, \tilde{y}) \in M_S(\mathcal{R}, \mathcal{P})$ since $(s', y') \in M_S(\mathcal{R}', \mathcal{P}')$. Set $M = \sum_{r' \in \mathcal{R}'} s'_{r'} + \sum_{p' \in \mathcal{P}'} y'_{p'}$, which is a constant since s' and y' are fixed and have finite support. As (ψ, π) is valid for $S, \mathcal{R}, \mathcal{P}$, we see that

$$
\begin{aligned}
&\sum_{r'\in\mathcal{R}'} \tilde{\psi}(r')s'_{r'} + \sum_{p'\in\mathcal{P}'} \tilde{\pi}(p')y'_{p'} \\
\geq &\sum_{r'\in\mathcal{R}'} \left[\sum_{r\in\mathcal{R}} \psi(r)s(r')_r - \epsilon\right] s'_{r'} + \sum_{p'\in\mathcal{P}'} \left[\sum_{r\in\mathcal{R}} \psi(r)s(p')_r + \sum_{y\in\mathcal{P}} \pi(y)y(p')_p - \epsilon\right] y'_{p'} \\
= &\sum_{\substack{r\in\mathcal{R}\\ r'\in\mathcal{R}'}} \psi(r)s(r')_r s'_{r'} + \sum_{\substack{r\in\mathcal{R}\\ p'\in\mathcal{P}'}} \psi(r)s(p')_r y'_{p'} + \sum_{\substack{p\in\mathcal{P}\\ p'\in\mathcal{P}'}} \pi(p)y(p')_p y'_{p'} - \epsilon M \\
= &\sum_{r\in\mathcal{R}} \psi(r)\tilde{s}_r + \sum_{p\in\mathcal{P}} \pi(p)\tilde{y}_p - \epsilon M \\
\geq &1 - \epsilon M.
\end{aligned}
$$

Letting $\epsilon \to 0$ gives the desired result. □

Theorem 29 is useful in proving instances when the fixing region differs from the lifting region on a set of measure greater than 0.

6.3 Fixing Region

We now proceed to study minimal liftings of a valid function ψ (for a nonempty, closed set $S \subseteq \mathbb{R}^n \setminus \{0\}$). For any $p^* \in \mathbb{R}^n$, recall $V_\psi(p^*)$ from (6.5). Dey and Wolsey [56] provided an explicit formula:

$$V_\psi(p^*) = \sup_{w\in\mathbb{R}^n, N\in\mathbb{N}} \left\{\frac{1-\psi(w)}{N} : w + Np^* \in S\right\}. \tag{6.11}$$

Recall the set $\mathcal{L}_{\psi,p^*}$ from Definition 24. Define the *fixing region* as

$$\mathcal{F}_{\psi,p^*} := \{p \in \mathbb{R}^n : \pi_1(p) = \pi_2(p) \text{ for all } \pi_1, \pi_2 \in \mathcal{L}_{\psi,p^*}\}.$$

In other words, the fixing region is the set of all points where all minimal liftings from $\mathcal{L}_{\psi,p^*}$ take the same value.

Proposition 58. *$\mathcal{L}_{\psi,p^*}$ is nonempty.*

Proof. There are many ways to prove this; we do it via a particular construction from [56] that we will refer to later. Define

$$\phi(p) = \inf_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \psi(w) + N V_\psi(p^*) : w + Np^* \in p + W_S \right\}. \tag{6.12}$$

It was shown in [56] that ϕ is a valid lifting of ψ and $\phi(p^*) = V_\psi(p^*)$.[1] Any minimal lifting $\hat{\pi}$ dominating ϕ (which exists by an application of Zorn's lemma) is in $\mathcal{L}_{\psi,p^*}$. $\square$

For the rest of the paper, we will specialize to sets S that are polyhedrally truncated affine lattices, i.e. of the form $S = (b + \mathbb{Z}^n) \cap C$. From Proposition 13, maximal S-free 0-neighborhoods are all polyhedra that can be written in the form $B = \{r : a_i \cdot r \leq 1 \quad i \in [m]\}$, where $m \in \mathbb{N}$, and for such a B, the smallest representation is given by

$$\psi(r) = \max_{i \in [m]} a_i \cdot r. \tag{6.13}$$

The value of $V_\psi(p^*)$ can now be obtained geometrically in the following way. Define $B(\lambda, p^*)$ as the translated cone in $\mathbb{R}^n \times \mathbb{R}$ with $\frac{1}{\lambda}(p^*, 1)$ as the apex and $B \times \{0\}$ as the base, i.e.

$$B(\lambda, p^*) = \{(r, r_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : a_i \cdot r + (\lambda - a_i \cdot p^*) r_{n+1} \leq 1,\ i \in [m]\}. \tag{6.14}$$

Figure 6.2 shows an example of $B(\lambda, p^*)$ where B is the lattice-free set from Figure 6.1(b).

[1] Although [56] only deals with $\mathbb{R}^2$, the same proof works for general $\mathbb{R}^n$ with $n \geq 2$.

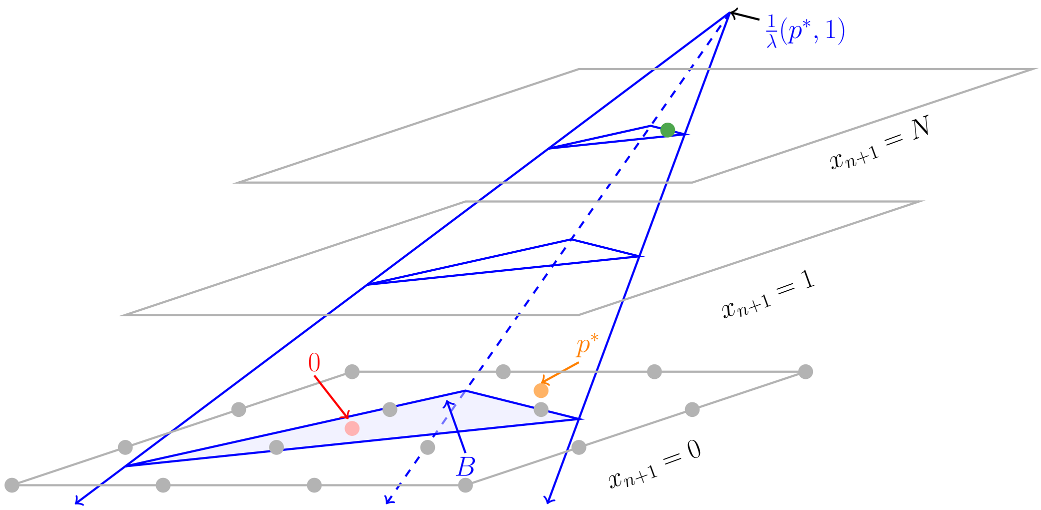


Figure 6.2: An example of $B(\lambda, p^*)$. Note that $B(\lambda, p^*)$ is a translated cone with base $B \times \{0\}$ and apex $\frac{1}{\lambda}(p^*, 1)$. The green point is a blocking point.

The following was observed in [17]:

Proposition 59. $V_\psi(p^*) = \inf\{\lambda > 0 : B(\lambda, p^*)$ *is* $S \times \mathbb{Z}_+$*-free*$\}$.

6.3.1 A geometric perspective on $\mathcal{L}_{\psi,p^*}$

The main tool for our geometric approach to understanding $\mathcal{L}_{\psi,p^*}$ will be the polyhedron $B(V_\psi(p^*), p^*)$ from (6.14).

Definition 28 (Blocking Point)**.**

Let ψ *be a cut-generating function for* C_S *obtained from a maximal* S*-free* 0*-neighborhood* B *as in* (6.13)*. Let* $p^* \in \mathbb{R}^n$*. A point* $(\bar{x}, \bar{x}_{n+1}) \in S \times \mathbb{Z}_+$ *with* $\bar{x}_{n+1} \geq 1$ *such that* $B(V_\psi(p^*), p^*)$ *contains* $(\bar{x}, \bar{x}_{n+1})$ *is called a* blocking point *for* $B(V_\psi(p^*), p^*)$*.*

In Figure 6.2, there is a point belonging to both $B(\lambda, p^*)$ and $S \times \mathbb{Z}_+$ lying on the hyperplane $\mathbb{R}^n \times \{N\}$. It was established in [17] that for every valid function ψ obtained from a maximal S-free 0-neighborhood B and every $p^* \in \mathbb{R}^n$, there exists a blocking point for $B(V_\psi(p^*), p^*)$. It is possible that there is more than one blocking point for a given B and p^*; this fact will be exploited in later sections. The following lemma relates the algebraic formula (6.11) for $V_\psi(p^*)$ and the important geometric notion of a blocking point for $B(V_\psi(p^*), p^*)$.

Lemma 16. *Suppose ψ is a valid function for S obtained from a maximal S-free 0-neighborhood B as in* (6.13)*. If $(\bar{x}, \bar{x}_{n+1}) \in S \times \mathbb{Z}_+$ is a blocking point for $B(V_\psi(p^*), p^*)$, then*

$$(\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1}) \in \operatorname*{argmax}_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \frac{1 - \psi(w)}{N} : w + Np^* \in S \right\}.$$

Conversely, if $(w, N) \in \mathbb{R}^n \times \mathbb{N}$ is a maximizer for (6.11)*, then $(w + Np^*, N)$ is a blocking point for $B(V_\psi(p^*), p^*)$.*

Proof. From Equation (6.14), $(\bar{x}, \bar{x}_{n+1})$ is a blocking point for $B(V_\psi(p^*), p^*)$ if and only if $a_i \cdot \bar{x} + (V_\psi(p^*) - a_i \cdot p^*)\bar{x}_{n+1} \leq 1$ for all $i \in [m]$, and there exists some $i^* \in [m]$ such that $a_{i^*} \cdot \bar{x} + (V_\psi(p^*) - a_{i^*} \cdot p^*)\bar{x}_{n+1} = 1$. Rearranging these inequalities and equality shows that $\bar{x}_{n+1}V_\psi(p^*) + \max_{i \in [m]}\{a_i \cdot (\bar{x} - \bar{x}_{n+1}p^*)\} = 1$. Thus $(\bar{x}, \bar{x}_{n+1})$ is a blocking point for $B(V_\psi(p^*), p^*)$ if and only if $V_\psi(p^*) = \frac{1 - \psi(\bar{x} - \bar{x}_{n+1}p^*)}{\bar{x}_{n+1}}$. We are then done because of formula (6.11). $\square$

Since blocking points always exist, Lemma 16 says that the supremum in (6.11) is actually a maximum.

Algebra v/s Geometry Although the function ϕ defined in (6.12) may not be a minimal lifting for ψ, it gives an explicit formula for computing the lifting values. Our geometric perspective provides an alternative function, which we show in Proposition 60 always dominates the algebraic construction of (6.12). Let $\psi_{B(V_\psi(p^*),p^*)} : (\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}$ be defined from $B(V_\psi(p^*), p^*)$ using (6.13). Then the restriction of

$$\psi^*_{B(V_\psi(p^*),p^*)}(r, r_{n+1}) := \inf_{(w,z) \in \widehat{W}_{S \times \mathbb{Z}_+}} \psi_{B(V_\psi(p^*),p^*)}\Big((r, r_{n+1}) + (w, z)\Big) \tag{6.15}$$

to $\mathbb{R}^n$ can be shown to be a valid lifting of ψ. Notice that $\psi^*_{B(V_\psi(p^*),p^*)}$ is the lifting defined in Proposition 57 for the function $\psi_{B(V_\psi(p^*),p^*)}$.

Proposition 60. *Consider ϕ and $\psi^*_{B(V_\psi(p^*),p^*)}$ defined in Equations* (6.12) *and* (6.15)*, respectively. Then $\psi^*_{B(V_\psi(p^*),p^*)}(p, 0) \leq \phi(p)$ for all $p \in \mathbb{R}^n$.*

Proof. For any $w \in \mathbb{R}^n$ and $N \in \mathbb{N}$, we have

$$
\begin{aligned}
\psi(w) + NV_\psi(p^*) = & \quad \psi_{B(V_\psi(p^*),p^*)}(w) + N\psi^*_{B(V_\psi(p^*),p^*)}(p^*) \\
\geq & \quad \psi^*_{B(V_\psi(p^*),p^*)}(w) + N\psi^*_{B(V_\psi(p^*),p^*)}(p^*) \\
\geq & \quad \psi^*_{B(V_\psi(p^*),p^*)}(w + Np^*) \\
\geq & \quad \psi^*_{B(V_\psi(p^*),p^*)}(p + x) && \text{for some } x \in W_S \\
= & \quad \psi^*_{B(V_\psi(p^*),p^*)}(p) && \text{since } x \in W_S
\end{aligned}
$$

Taking the infimum on the left gives the desired result. □

A universal upper bound In answering what vectors lie in $\mathcal{F}_{\psi,p^*}$, we first show an upper bound on the value of minimal liftings and then show this upper bound is tight. Theorem 30 gives such an upper bound, stating that the restriction of $\psi^*_{B(V_\psi(p^*),p^*)}$ to $\mathbb{R}^n$ is a universal upper bound for all minimal liftings $\pi \in \mathcal{L}_{\psi,p^*}$. The following technical lemma will be useful for this purpose.

Lemma 17. *Let $B = \{r : a_i \cdot r \leq 1 \quad i \in [m]\}$ be a polyhedron in $\mathbb{R}^n$ containing 0 and let $p^* \in \mathbb{R}^n$ and $\lambda > 0$. For $(\bar{r}, \bar{r}_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$ and $\mu \geq 0$, define $r' = (\bar{r}, \bar{r}_{n+1}) - \mu(p^*, 1)$. The following hold:*

(a)

$$\operatorname*{argmax}_{i\in[m]}\{a_i\cdot\bar{r}+(\lambda-a_i\cdot p^*)\bar{r}_{n+1}\} = \operatorname*{argmax}_{i\in[m]}\{a_i\cdot(\bar{r}-\bar{r}_{n+1}p^*)\} = \operatorname*{argmax}_{i\in[m]}\{(a_i,(\lambda-a_i\cdot p^*))\cdot r'\}$$

and

(b) $\psi_{B(\lambda,p^*)}(\bar{r}, \bar{r}_{n+1}) = \psi_{B(\lambda,p^*)}(r') + \mu\psi_{B(\lambda,p^*)}(p^*, 1)$.

Proof.

(a) The first and second terms are equal since $\lambda\bar{r}_{n+1}$ is a constant, while the first and the third terms are equal because for every $i \in [m]$,

$$a_i\cdot\bar{r}+(\lambda-a_i\cdot p^*)\bar{r}_{n+1} = a_i\cdot(\bar{r}-\mu p^*)+(\lambda-a_i\cdot p^*)(\bar{r}_{n+1}-\mu)+\lambda\mu = (a_i,(\lambda-a_i\cdot p^*))\cdot r'+\lambda\mu.$$

(b) From (a), let $i^* = \operatorname*{argmax}_{i \in [m]} \{a_i \cdot \bar{r} + (\lambda - a_i \cdot p^*)\bar{r}_{n+1}\}$, and so

$$\begin{aligned}\psi_{B(\lambda,p^*)}(\bar{r}, \bar{r}_{n+1}) &= a_{i^*} \cdot \bar{r} + (\lambda - a_{i^*} \cdot p^*)\bar{r}_{n+1} \\ &= (a_{i^*}, (\lambda - a_{i^*} \cdot p^*)) \cdot r' + (a_{i^*}, (\lambda - a_{i^*} \cdot p^*)) \cdot \mu(p^*, 1) \\ &= \psi_{B(\lambda,p^*)}(r') + \mu\psi_{B(\lambda,p^*)}(p^*, 1),\end{aligned}$$

where the last equality follows from $(a_{i^*}, (\lambda - a_{i^*} \cdot p^*)) \cdot (p^*, 1) = \lambda = \psi_{B(\lambda,p^*)}(p^*, 1)$.

□

Theorem 30. *Let ψ be a cut-generating function for C_S obtained from a maximal S-free 0-neighborhood B as in (6.13). Let $p^* \in \mathbb{R}^n$. Let $\psi^*_{B(V_\psi(p^*),p^*)}$ be obtained from $\psi_{B(V_\psi(p^*),p^*)}$ as in Proposition 57. Then for every $\pi \in \mathcal{L}_{\psi,p^*}$ and $p \in \mathbb{R}^n$, $\pi(p) \le \psi^*_{B(V_\psi(p^*),p^*)}(p, 0)$.*

Proof.

We would like to apply Theorem 29 by extending $\mathbb{R}^n \times \{0\}$ to $\mathbb{R}^n \times \mathbb{R}_+$, which are the domains of $\pi \in \mathcal{L}_{\psi,p^*}$ and $\psi^*_{B(V_\psi(p^*),p^*)}$, respectively. However, $\mathbb{R}^n \times \mathbb{R}_+ \not\subseteq \operatorname{cone}(\mathbb{R}^n \times \{0\})$. Instead, we will create related functions $\hat{\psi}$ and $\hat{\pi}$ in $n+1$ dimensions for which we can employ Theorem 29. This application of Theorem 29 will yield a minimal pair (ψ', π') in $n+1$ dimensions that matches (ψ, π) on the n-dimensional restricted space, but also dominates $\psi^*_{B(V_\psi(p^*),p^*)}$ in $n+1$ dimensions.

First, define $\mathcal{R} := (\mathbb{R}^n \times \{0\}) \cup \{(p^*, 1)\} \subseteq \mathbb{R}^n \times \mathbb{R}_+$ and $\mathcal{P} := \mathbb{R}^n \times \{0\}$. Define $\hat{\psi} : \mathcal{R} \to \mathbb{R}$ by $\hat{\psi}(r, 0) = \psi(r)$ for all $r \in \mathbb{R}^n$ and $\hat{\psi}(p^*, 1) = V_\psi(p^*)$. Define $\hat{\pi} : \mathcal{P} \to \mathbb{R}$ as $\hat{\pi}(p, 0) = \pi(p)$.

Claim 13. *$(\hat{\psi}, \hat{\pi})$ is valid for $(S \times \mathbb{Z}_+), \mathcal{R}, \mathcal{P}$.*

Proof of Claim. Consider any R, P with columns in $\mathcal{R}, \mathcal{P}$ respectively and let $(\bar{s}, \bar{y}) \in M_{S \times \mathbb{Z}_+}(R, P)$. If R does not contain $(p^*, 1)$ or $\bar{s}_{(p^*,1)} = 0$, we are done by the validity of (ψ, π). Otherwise, since $R\bar{s} + P\bar{y} \in S \times \mathbb{Z}_+$ and $\mathcal{P} \subseteq \mathbb{R}^n \times \{0\}$, we must have $\bar{s}_{(p^*,1)} \in \mathbb{Z}_+$. Define $\tilde{R}$ to be the matrix with columns in $\mathbb{R}^n$ that arise by truncating the columns of $R \setminus \{(p^*, 1)\}$ to the first n coordinates. Define $\tilde{P}$ to be the matrix with columns in $\mathbb{R}^n$ that arise by truncating the columns of $P \cup \{(p^*, 1)\}$ to the first n coordinates.

Consider the pair $(\tilde{s}, \tilde{y})$ defined by $\tilde{s}_r = \bar{s}_{(r,0)}$ and $\tilde{y}_p = \bar{y}_{(p,0)}$ if $p \neq p^*$ and $\tilde{y}_{p^*} = \bar{y}_{(p^*,0)} + \bar{s}_{(p^*,1)}$. Observe that $\tilde{R}\tilde{s} + \tilde{P}\tilde{y} \in S$ since $\bar{s}_{(p^*,1)} \in \mathbb{Z}_+$ and $R\bar{s} + P\bar{y} \in S \times \mathbb{Z}_+$. Thus $(\tilde{s}, \tilde{y}) \in M_S(\tilde{R}, \tilde{P})$ A direct calculation shows

$$
\begin{aligned}
& \sum_{r \in R} \hat{\psi}(r,0)\bar{s}_r + \sum_{p \in P} \hat{\pi}(p)\bar{y}_p \\
= \quad & \sum_{r \in R \setminus \{(p^*,1)\}} \hat{\psi}(r,0)\bar{s}_r + \hat{\psi}((p^*,1))\bar{s}_{(p^*,1)} + \sum_{p \in P} \hat{\pi}(p)\bar{y}_p \\
= \quad & \sum_{r \in R \setminus \{(p^*,1)\}} \hat{\psi}(r,0)\bar{s}_r + V_\psi(p^*)\bar{s}_{(p^*,1)} + \sum_{p \in P} \hat{\pi}(p)\bar{y}_p \\
= \quad & \sum_{r \in \tilde{R}} \psi(r)\tilde{s}_r + \sum_{p \in \tilde{P}} \pi(p)\tilde{y}_p \\
\geq \quad & 1,
\end{aligned}
$$

where the inequality comes from the validity of (ψ, π). $\diamond$

Theorem 29 states there exist functions $\psi' : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ and $\pi' : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that (ψ', π') is a minimal valid pair for $(S \times \mathbb{Z}_+), \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}_+$ whose restriction to $\mathbb{R}^n$ dominate (ψ, π) (because the restriction dominates $(\hat{\psi}, \hat{\pi})$). Since ψ is a minimal valid function for S, the restriction of ψ' to $\mathbb{R}^n$ must match ψ. Similarly, since π is a minimal lifting of ψ, π' restricted to $\mathbb{R}^n$ must coincide with π. This also implies that

$$\psi'(p^*, 1) = \hat{\psi}(p^*, 1) = V_\psi(p^*) \tag{6.16}$$

from the construction of $B(V_\psi(p^*), p^*)$. Now for any $r \in \mathbb{R}^n$, $\psi'(r, 0) \leq \psi_{B(V_\psi(p^*),p^*)}(r, 0)$. Indeed, by Lemma 17 there exists a $r' \in \mathbb{R}^n$ and $\mu \geq 0$ such that $(r, 0) = (r', 0) + \mu(p^*, 1)$

and $\psi_{B(V_\psi(p^*),p^*)}(r',0) + \mu\psi_{B(V_\psi(p^*),p^*)}(p^*,1) = \psi_{B(V_\psi(p^*),p^*)}(r,0)$. A direct calculation gives

$$\begin{aligned}
\psi'(r,0) \leq \quad & \psi'(r',0) + \mu\psi'(p^*,1) && \text{by Theorem 28} \\
= \quad & \hat{\psi}(r',0) + \mu\hat{\psi}(p^*,1) && \text{by Equation (6.16)} \\
= \quad & \psi_{B(V_\psi(p^*),p^*)}(r',0) + \mu\psi_{B(V_\psi(p^*),p^*)}(p^*,1) \\
= \quad & \psi_{B(V_\psi(p^*),p^*)}(r,0).
\end{aligned}$$

Using this inequality, observe that, for any $p \in \mathbb{R}^n \times \{0\}$,

$$\begin{aligned}
\pi(p) =& \pi'(p,0) && \text{from Equation (6.16)} \\
\leq& \inf\{\psi'((p,0)+(w,z)) : (w,z) \in \widehat{W}_{S\times\mathbb{Z}_+}, (p,0)+(w,z) \in \mathbb{R}^n \times \mathbb{R}_+\} \\
& && \text{from Proposition 57} \\
\leq& \inf\{\psi_{B(V_\psi(p^*),p^*)}((p,0)+(w,z)) : (w,z) \in \widehat{W}_{S\times\mathbb{Z}_+}, (p,0)+(w,z) \in \mathbb{R}^n \times \mathbb{R}_+\} \\
=& \psi^*_{B(V_\psi(p^*),p^*)}(p,0).
\end{aligned}$$

This is the desired result. □

6.3.2 A partial description of the fixing region

In this section, we will start with a maximal S-free 0-neighborhood B and a point p^*. We then define a collection of polyhedra (given by explicit inequalities) whose union will be shown to be a subset of $\mathcal{F}_{\psi,p^*}$, where ψ is defined from B using (6.13).

Let $\tilde{B} = \{r \in \mathbb{R}^d : a_i \cdot r \leq 1 \quad i \in [m]\}$ be a polyhedral 0-neighborhood. For $x \in \mathbb{R}^d$, the *spindle corresponding to* x is defined as

$$R(x;\tilde{B}) = \{r \in \mathbb{R}^d : (a_i - a_k)\cdot r \leq 1,\ (a_i - a_k)\cdot(x - r) \leq 1,\ \forall i \in [m]\},$$

where $\psi(x) = a_k \cdot x$. The original motivation for this definition was the following observation made in [17, 56]:

Observation 7. *Let ψ be a valid function for S obtained from a maximal S-free 0-neighborhood B as in (6.13), and let $\bar{x} \in B \cap S$. If $p^* \in R(\bar{x};B)$, then $V_\psi(p^*) = \psi(p^*)$ and $(\bar{x},1)$ is a* *blocking point for* $B(V_\psi(p^*), p^*)$.

The definition of a spindle is also valid when we consider the $n+1$ sets $B(V_\psi(p^*), p^*)$.

Definition 29 (Spindle)**.**

For a blocking point $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)$,

$$R\left((\bar{x}, \bar{x}_{n+1}); B(V_\psi(p^*), p^*)\right) \subseteq \mathbb{R}^n \times \mathbb{R}$$

denotes the $n+1$*-dimensional* spindle corresponding to $(\bar{x}, \bar{x}_{n+1})$.

The next result states that translating $(\mathbb{R}^n \times \{t\}) \cap R((\bar{x}, \bar{x}_{n+1}); B(V_\psi(p^*), p^*))$ by tp^* is equivalent to projecting the 'height-t slice' $(\mathbb{R}^n \times \{t\}) \cap R((\bar{x}, \bar{x}_{n+1}); B(V_\psi(p^*), p^*))$ onto the first n-coordinates.

Proposition 61. *Let* $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)$ *be a blocking point and* $t \in \mathbb{R}$. *Then*

$$(\mathbb{R}^n \times \{t\}) \cap R((\bar{x}, \bar{x}_{n+1}); B(V_\psi(p^*), p^*)) = \Big((\mathbb{R}^n \times \{0\}) \cap R((\bar{x}, \bar{x}_{n+1}); B(V_\psi(p^*), p^*))\Big) + t(p^*, 1).$$

Proof. This follows from a direct calculation. □

Our geometric, partial description of $\mathcal{F}_{\psi,p^*}$ is the content of the next theorem.

Theorem 31. *Let* ψ *be a valid function for* C_S *obtained from a maximal* S*-free* 0*-neighborhood* B *as in* (6.13)*. Let* $p^* \in \mathbb{R}^n$ *and* (w, N) *be a maximizer in* (6.11)*. Then*

$$(R(w; B) \cup (R(w; B) + p^*) \cup \ldots \cup (R(w; B) + Np^*)) + W_S \subseteq \mathcal{F}_{\psi,p^*}. \tag{6.17}$$

Equivalently, let $(\bar{x}, \bar{x}_{n+1})$ *be a blocking point for* $B(V_\psi(p^*), p^*)$*. Then* (6.17) *holds for* $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$.

An important result in previous literature is that for every $\bar{x} \in S \cap B$, $R(s; B) + W_S \subseteq R(S; B) + W_S$, where $R(S; B)$ is the lifting region corresponding to B (see the discussion following Equation (5.1)). Of course, for every $p^* \in \mathbb{R}^n$, $R(S; B) \subseteq \mathcal{F}_{\psi,p^*}$. Thus we obtain the following corollary of Theorem 31.

Corollary 1. *Let ψ be a valid function for S obtained from a maximal S-free 0-neighborhood B as in* (6.13)*. For any $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)$, let $(w, N) = (\bar{x} - \bar{x}_{n+1} p^*, \bar{x}_{n+1})$. Then*

$$\left(\bigcup_{i \in [N] \cup \{0\}} R(w; B) + ip^* \right) + W_S \subseteq \mathcal{F}_{\psi, p^*}. \tag{6.18}$$

Note that $(\bar{x}, \bar{x}_{n+1})$ in the above Corollary need not be a blocking point as $\bar{x}_{n+1}$ could be 0. Figure 6.3 gives some geometric intuition behind Theorem 31 and Corollary 1. Consider the $b + \mathbb{Z}^2$ triangle in Figures 6.1 and 6.2. Figure 6.3(a) shows the maximal $(S \times \mathbb{Z}_+)$-free set $B(V_\psi(p^*), p^*)$ along with a blocking point (w, N). The green cylinder is the spindle $R((w, N); B(V_\psi(p^*), p^*))$. Figure 6.3(a) shows the spindle $R(w; B)$, which is equal to $R((w, N); B(V_\psi(p^*), p^*)) \cap (\mathbb{R}^2 \times \{0\})$. Figure (b) also shows that the lifting region is supplemented by these new spindles and their translates by p^*.

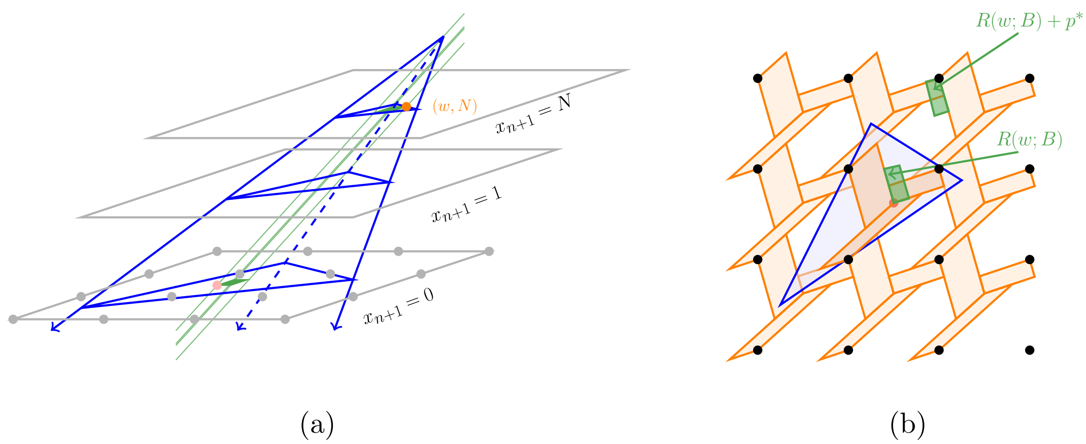


Figure 6.3: The intuition behind Theorem 31. Note that the fixing region is a way to 'extend' the lifting region.

The remainder of this section is dedicated to the proof of Theorem 31. Before proving the result, we build a few tools. For $p_1^*, p_2^* \in \mathbb{R}^n$, consider sequentially lifting p_1^* then p_2^*. Define

$$V_\psi(p_2^*;\ p_1^*) := \inf \left\{ \pi(p_2^*) :\ \pi \in \mathcal{L}_{\psi, p_1^*} \right\}. \tag{6.19}$$

The geometric construction used for $V_\psi(p_1^*)$ may be extended to calculate $V_\psi(p_2^*;\ p_1^*)$. Intuitively, $V_\psi(p_1^*)$ is found by constructing a translated cone in $\mathbb{R}^{n+1}$ with base B, and $V_\psi(p_2^*;\ p_1^*)$ is found similarly by constructing a translated cone in $\mathbb{R}^{n+2}$ with base $B(V_\psi(p^*), p^*)$.

For $\lambda > 0$, define

$$B\left(\lambda, p_2^*;\ V_\psi(p_1^*)\right) := \left\{(r, r_{n+1}, r_{n+2}) \in \mathbb{R}^{n+2} : a_i \cdot r + (V_\psi(p_1^*) - a_i \cdot p_1^*) r_{n+1} + (\lambda - a_i \cdot p_2^*) r_{n+2} \le 1\right\}. \tag{6.20}$$

The following result confirms the geometric approach is valid for finding $V_\psi(p_2^*;\ p_1^*)$.

Proposition 62.

$$V_\psi(p_2^*; p_1^*) = \inf\left\{\lambda > 0 :\ B\left(\lambda, p_2^*;\ V_\psi(p_1^*)\right) \text{ is } S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ free}\right\}. \tag{6.21}$$

The proof of Proposition 62 is similar to the reasoning in [42] that leads to Proposition 59 and is therefore relegated to Appendix A.7.

A consequence of the following proposition is that $V_\psi(p_2^*; p_1^*)$ is invariant under certain translations of q.

Proposition 63. *Let $S \subseteq \mathbb{R}^n \backslash \{0\}$ be a closed subset and let $\hat{B} \subseteq \mathbb{R}^{n+1}$ be a maximal $S \times \mathbb{Z}_+$-free 0-neighborhood. Let $\hat{\psi}$ the corresponding function derived using* (6.13)*. Consider any $\hat{p} \in \mathbb{R}^{n+1}$ and let $(\overline{x}, \overline{x}_{n+1}, 1) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ be a blocking point of $\hat{B}(V_{\hat{\psi}}(\hat{p}), \hat{p})$. Let $w \in \mathbb{Z}$ such that $(\overline{x}, \overline{x}_{n+1} + w, 1) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$. Then $V_{\hat{\psi}}\left(\hat{p} + (0_n, w)\right) = V_{\hat{\psi}}(\hat{p})$, where 0_n is the zero vector in $\mathbb{R}^n$.*

Proof. By Proposition 59, it is sufficient to show

$$\begin{aligned} &\inf\left\{\lambda > 0 :\ \hat{B}_{\hat{\psi}}(\lambda, \hat{p}) \text{ is } S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ free}\right\} \\ =&\inf\left\{\lambda > 0 :\ \hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w)) \text{ is } S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \text{ free}\right\}. \end{aligned}$$

Observe that the sets $\hat{B}_{\hat{\psi}}(\lambda, \hat{p})$ and $\hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w))$ are translated cones in $\mathbb{R}^{n+2}$. The geometric interpretation of the equality above is that the ratio of the 'lifting' vector to the apex is preserved between the two cones. The idea of the proof is to create a unimodular transformation between the two cones that preserves this ratio.

Define the linear transformation $U : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ by

$$U\left(y, y_{n+1}, y_{n+2}\right) = \left(y, y_{n+1} + y_{n+2}w, y_{n+2}\right),$$

where $y \in \mathbb{R}^n$ and $y_{n+1}, y_{n+2} \in \mathbb{R}$. Note that U is invertible with $U^{-1}\left(y, y_{n+1}, y_{n+2}\right) = (y, y_{n+1} - y_{n+2}w, y_{n+2})$. Furthermore, $w \in \mathbb{Z}$ by definition, and so U is unimodular. In the following arguments, it is useful to note $U(\hat{p}, 1) = (\hat{p} + (0_n, w), 1)$.

Since U is unimodular, it maps $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free sets to $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free sets. Therefore, since $(\hat{p}, 1)$ and $(\hat{p} + (0_n, w), 1)$ define $\hat{B}_{\hat{\psi}}(\lambda, \hat{p})$ and $\hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w))$, respectively, we have

$$U\left(\hat{B}_{\hat{\psi}}(\lambda, \hat{p})\right) \subseteq \hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p} + (0_n, w)), \hat{p} + (0_n, w))$$

for each value of λ. Similarly, as U^{-1} is unimodular, it follows that

$$U^{-1}\left(\hat{B}_{\hat{\psi}}(\lambda, \hat{p} + (0_n, w))\right) \subseteq \hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p}), \hat{p})$$

for each λ. Thus $U\left(\hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p}), \hat{p})\right) = \hat{B}_{\hat{\psi}}(V_{\hat{\psi}}(\hat{p} + (0_n, w)), \hat{p} + (0_n, w))$. Since U is unimodular, ratios of vector magnitudes are preserved and the result follows. $\square$

Proof of Theorem 31. The equivalence of the two statements follows from Lemma 16: if (w, N) is a maximizer in (6.11), then $(w + Np^*, N)$ is a blocking point for $B(V_\psi(p^*), p^*)$, and conversely, if $(\bar{x}, \bar{x}_{n+1})$ is a blocking point for $B(V_\psi(p^*), p^*)$, then $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$ is a maximizer in (6.11). Thus, it suffices to prove the result for a blocking point $(\bar{x}, \bar{x}_{n+1})$ and $(w, N) = (\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1})$.

To reduce notational baggage, we introduce $\hat{B} := B(V_\psi(p^*), p^*)$ and let $\hat{\psi} : \mathbb{R}^{n+1} \to \mathbb{R}$ denote the function obtained by applying (6.13) with $B = \hat{B}$. Consider any

$$q \in \bigcup_{i \in [N] \cup \{0\}} R(w; B) + ip^*$$

and any $\pi \in \mathcal{L}_{\psi, p^*}$. By definition, $V_\psi(q; p^*) \leq \pi(q)$, independent of π. Therefore it is sufficient to show that this inequality holds at equality.

Let $q \in jp^* + R(w; B)$ for some $j \in [N] \cup \{0\}$. Observe that

$$V_\psi(q; p^*) \leq \pi(q) \leq \inf_{n \in \mathbb{N} \cup \{0\}} \hat{\psi}(q, n) \leq \hat{\psi}(q, j),$$

where the second inequality follows from Theorem 30. By the definition of w and Proposition 61, $(q, j) \in R((\bar{x}, \bar{x}_{n+1}); \hat{B}) \cap (\mathbb{R}^n \times \{j\})$. Thus Observation 7 implies

$$\hat{\psi}(q, j) = V_{\hat{\psi}}(q, j),$$

and $(\bar{x}, \bar{x}_{n+1}, 1)$ is a blocking point for $\hat{B}\left(V_{\hat{\psi}}(q, j), (q, j)\right)$. Since $N - j \geq 0$ and $(\hat{x}, N) + (0_d, -j) \in S \times \mathbb{Z}_+$, we can apply Propositions 62 and 63 to conclude that

$$V_{\hat{\psi}}(q, j) = V_{\hat{\psi}}(q, 0) = V_\psi(q; p^*).$$

Combining inequalities and equalities, we get $V_\psi(q; p^*) = \pi(q)$. $\square$

6.3.3 Translation invariance of fixing region

Fix a maximal S-free 0-neighborhood B. For any $p^* \in \mathbb{R}^n$ and any point $z = (\bar{x}, \bar{x}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)$, we define $w(z) := \bar{x} - \bar{x}_{n+1} p^*$ and $N(z) := \bar{x}_{n+1}$. From Corollary 1, it is clear that if p^* is chosen such that

$$\left(\bigcup_{\substack{z = (\bar{x}, \bar{x}_{n+1}) \in \\ B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)}} \bigcup_{i \in [N(z)] \cup \{0\}} R(w(z); B) + ip^* \right) + W_S = \mathbb{R}^n. \tag{6.22}$$

then $\mathcal{L}_{\psi, p^*}$ is a singleton. In other words, after fixing the coefficient for p^*, all other lifting coefficients are fixed. Let us introduce a more compact notation:

$$\mathcal{X}(B, p^*) := \bigcup_{\substack{z = (\bar{x}, \bar{x}_{n+1}) \in \\ B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)}} \bigcup_{i \in [N(z)] \cup \{0\}} R(w(z); B) + ip^* \tag{6.23}$$

Let $m \in \mathbb{R}^n$ such that $0 \in \operatorname{int}(B + m)$. It has been shown that such a translation preserves the covering property of the extended lifting region (see [5, 23, 30]). Such a translation also preserves the equality in Equation (6.22).

Theorem 32. *Let B be a maximal S-free 0-neighborhood and let $f \in \mathbb{R}^n$ such that $0 \in \operatorname{int}(B+f)$; thus $B+f$ is a maximal $S+f$-free 0-neighborhood. For $p^* \in \mathbb{R}^n$,*

$$\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$$

if and only if

$$\mathcal{X}(B+f, p^* + V_\psi(p^*)f) + W_{S+f} = \mathbb{R}^n.$$

In other words, if for a given maximal S-free 0-neighborhood B, there exists a p^* that makes B one point fixable, then for any translation $B+f$, there exists a $\hat{p} := p^* + V_\psi(p^*)f$ that makes $B+f$ one point fixable.

The proof of Theorem 32 is very technical in nature and is similar to that of Theorem 3.1 in [30]. For this reason, we relegate the proof to Appendix A.8.

6.4 Application: Fixing Regions of Type 3 triangles

In this section, we let $S = b + \mathbb{Z}^2$ for $b = (b_1, b_2) \notin \mathbb{Z}^2$. Without loss of generality, we can assume $-1 \le b_1, b_2 \le 0$. Moreover, by relabeling the coordinates, we can further assume without loss of generality $-1 \le b_2 \le b_1 \le 0$. This means that the origin $(0,0)$ is contained in the triangle $\operatorname{conv}\{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$, where $\bar{s}_1 = (1+b_1, 1+b_2)$, $\bar{s}_2 = (b_1, 1+b_2)$, and $\bar{s}_3 = (b_1, b_2)$.

Let $\gamma_1, \gamma_2, \gamma_3 > 0$ with $\gamma_2, \gamma_3 < 1$, and define $T(\gamma_1, \gamma_2, \gamma_3) \subseteq \mathbb{R}^2$ by

$$T(\gamma_1,\gamma_2,\gamma_3) := \Bigg\{(x_1,x_2) \in \mathbb{R}^2 : \quad \begin{aligned} &\tfrac{1}{(1,\gamma_1)\cdot(b_1+1,b_2+1)}x_1 + \tfrac{\gamma_1}{(1,\gamma_1)\cdot(b_1+1,b_2+1)}x_2 \le 1, \\ -&\tfrac{1}{(-1,\gamma_2)\cdot(b_1,b_2+1)}x_1 + \tfrac{\gamma_2}{(-1,\gamma_2)\cdot(b_1,b_2+1)}x_2 \le 1, \\ &\tfrac{\gamma_3}{(\gamma_3,-1)\cdot(b_1,b_2)}x_1 - \tfrac{1}{(\gamma_3,-1)\cdot(b_1,b_2)}x_2 \le 1 \end{aligned} \Bigg\}. \tag{6.24}$$

The family of triangles $T(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1, \gamma_2, \gamma_3 > 0$ and $\gamma_2, \gamma_3 < 1$ are all maximal S-free 0-neighborhoods, and the three sides contain the points $\bar{s}_1, \bar{s}_2$ and $\bar{s}_3$ from S respectively in their relative interiors. They are known in the literature as the *Type 3* family of maximal S-free triangles. The lattice-free set in Figure 6.1(b) is an example of a Type-3 triangle.

6.4.1 Nonempty interior for the fixing region

The next result says that for any Type 3 triangle, there always exists a $p^* \in \mathbb{R}^2$ that fixes a set of measure greater than 0.

Proposition 64. *Let T be a Type 3 triangle as described above and let ψ be the valid function derived from T using* (6.13). *There exists $p^* \in \mathbb{R}^2$ and an $\epsilon > 0$ such $D(p^*; \epsilon) \subseteq \mathcal{F}_{\psi,p^*}$.*

Proposition 64 is a simple consequence of the following.

Proposition 65. *Let $T \subseteq \mathbb{R}^2$ be as above. There exists $P \subseteq \mathbb{R}^3$ such that*

(i) P is a translated cone with three facets and an apex $a = (a_1, a_2, a_3)$ satisfying $a_3 > 0$,

(ii) $P \cap \{x \in \mathbb{R}^3 : x_3 = 0\} = T \times \{0\}$,

(iii) P is maximal $S \times \mathbb{Z}_+$ free, and

(iv) each facet of P contains a point $(s_i, z_i) \in S \times \mathbb{Z}_+$, $i = 1, 2, 3$, with $z_i \geq 1$, in its relative interior.

Proposition 64 follows from Proposition 65 by observing that $(R(w_1; T)+p^*) \cup (R(w_2; T)+p^*) \cup (R(w_3; T)+p^*)$ contains p^* in its interior, where $w_i := s_i - z_i p^*$, and $(R(w_1; T)+p^*) \cup (R(w_2; T)+p^*) \cup (R(w_3; T)+p^*) \subseteq \mathcal{F}_{\psi,p^*}$ by Theorem 31 and the fact that $z_i \geq 1$.

The idea for constructing P in Proposition 65 is to first extend the three edges of T to hyperplanes in $\mathbb{R}^3$ and create a translated cone satisfying (i), (ii), and (iii). We then 'rotate' the hyperplanes one at a time until (iv) is satisfied. This rotation preserves (i), (ii), and (iii).

For $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$, define the vectors

$$n_{\alpha_1} = \left(\frac{1}{1+\gamma_1+(1,\gamma_1)\cdot(b_1,b_2)}, \frac{\gamma_1}{1+\gamma_1+(1,\gamma_1)\cdot(b_1,b_2)}, \left(\frac{-\gamma_1}{(1-\alpha_1)(1+\gamma_1+(1,\gamma_1)\cdot(b_1,b_2))} \right) \right)$$
$$n_{\alpha_2} = \left(\frac{-1}{\gamma_2+(-1,\gamma_2)\cdot(b_1,b_2)}, \frac{\gamma_2}{\gamma_2+(-1,\gamma_2)\cdot(b_1,b_2)}, \frac{\alpha_2}{(\alpha_2-1)(\gamma_2+(-1,\gamma_2)\cdot(b_1,b_2))} \right)$$
$$n_{\alpha_3} = \left(\frac{\gamma_3}{(\gamma_3,-1)\cdot(b_1,b_2)}, \frac{-1}{(\gamma_3,-1)\cdot(b_1,b_2)}, \frac{\beta(1-\alpha_3)}{(\gamma_3,-1)\cdot(b_1,b_2)} \right),$$

the hyperplanes

$$\begin{aligned}
H_1(\alpha_1) :=& \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \; n_{\alpha_1} \cdot (x_1, x_2, x_3) \leq 1\}, \\
H_2(\alpha_2) :=& \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \; n_{\alpha_2} \cdot (x_1, x_2, x_3) \leq 1\}, \\
H_3(\alpha_3) :=& \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \; n_{\alpha_3} \cdot (x_1, x_2, x_3) \leq 1\},
\end{aligned}$$

and the polyhedron

$$P(\alpha_1, \alpha_2, \alpha_3) := H_1(\alpha_1) \cap H_2(\alpha_2) \cap H_3(\alpha_3).$$

The scalar β in the definition of n_{α_3} is chosen so that $\beta > 0$, $P(0,0,0)$ is a translated cone with apex (a_1, a_2, a_3), and $a_3 \in (0,1)$ (a necessary and sufficient condition on β is $\beta > \frac{1+2\gamma_1+\gamma_2-\gamma_2\gamma_3-\gamma_1\gamma_2\gamma_3}{\gamma_1+\gamma_2}$). Whenever $P(\alpha_1, \alpha_2, \alpha_3)$ is a translated cone, we will use $(a_1, a_2, a_3) \in \mathbb{R}^3$ to denote the apex and $F_1(\alpha_1, \alpha_2, \alpha_3), F_2(\alpha_1, \alpha_2, \alpha_3)$, and $F_3(\alpha_1, \alpha_2, \alpha_3)$ to denote the facets defined by $H_1(\alpha_1), H_2(\alpha_2)$ and $H_3(\alpha_3)$, respectively.

Let $\alpha, \alpha^* \in [0,1)$ be such that $\alpha \leq \alpha^*$, $\alpha_2, \alpha_3 \in [0,1)$, and set $H_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \; x_3 \geq 0\}$. In this situation, observe that

Observation 8. $P(\alpha, \alpha_2, \alpha_3) \cap H_+ \subseteq P(\alpha^*, \alpha_2, \alpha_3) \cap H_+$.

Observation 8 is about $P(\alpha_1, \alpha_2, \alpha_3)$ when α_1 is allowed to vary in $[0,1)$ that follows from direct computation. Similar statements can be made when α_2 or α_3 is allowed to vary instead of α_1.

Here are two more properties about $P(\alpha_1, \alpha_2, \alpha_3)$ for $\alpha_3 \in [0,1)$.

Claim 14. *Suppose that $P(\alpha_1, \alpha_2, \alpha_3)$ is $S \times \mathbb{Z}_+$ free. If $(p_1, p_2, 1) \in \operatorname{rec}(P(\alpha_1, \alpha_2, \alpha_3))$ then $(p_1, p_2, 1) \in \mathbb{Z}^3$.*

Proof of Claim 14. Assume to the contrary that $(p_1, p_2, 1) \in \mathbb{R}^3 \backslash \mathbb{Z}^3$. Since $(b_1, b_2, 0), (b_1, b_2 + 1, 0), (b_1+1, b_2+1, 0) \in P(\alpha_1, \alpha_2, \alpha_3)$, either $(b_1+p_1-\lfloor p_1 \rfloor, b_2+p_2-\lfloor p_2 \rfloor, 0) \in \operatorname{int}(P(\alpha_1, \alpha_2, \alpha_3))$ or $(b_1+1, b_2+1, 0) - (p_1 - \lfloor p_1 \rfloor, p_2 - \lfloor p_2 \rfloor, 0) \in \operatorname{int}(P(\alpha_1, \alpha_2, \alpha_3))$. Suppose that $(b_1 + p_1 - \lfloor p_1 \rfloor, b_2 + p_2 - \lfloor p_2 \rfloor, 0) \in \operatorname{int}(P(\alpha_1, \alpha_2, \alpha_3))$ (a similar argument can be made in the other case). Therefore, there exists an open ball $B \subseteq \mathbb{R}^3$ centered at $(b_1+p_1-\lfloor p_1 \rfloor, b_2+p_2-\lfloor p_2 \rfloor, 0)$ and contained in $\mathrm{int}(P(\alpha_1, \alpha_2, \alpha_3))$.

Consider the cylinder $C := B+(p_1, p_2, 1)\mathbb{R}$. Note that $B+(p_1, p_2, 1)\mathbb{R}_+ \subseteq \mathrm{int}(P(\alpha_1, \alpha_2, \alpha_3))$, C is symmetric about $(b_1 - \lfloor p_1 \rfloor, b_2 - \lfloor p_2 \rfloor, -1)$, and $\mathrm{vol}(C) = \infty$. Therefore, by Minkowski's Convex Body Theorem, there exists a point $(z_1, z_2, z_3) \in (S \times \mathbb{Z}) \cap C$ with $z_3 \geq 0$. However, this implies that $(z_1, z_2, z_3) \in (S \times \mathbb{Z}_+) \cap \mathrm{int}(P(\alpha_1, \alpha_2, \alpha_3))$, contradicting that $P(\alpha_1, \alpha_2, \alpha_3)$ is $S \times \mathbb{Z}_+$ free. $\square$

Claim 15. *Assume that $P(0, 0, \alpha_3)$ is $S \times \mathbb{Z}_+$ free for $\alpha_3 \in [0, 1)$. Then $\alpha_3 \leq 1 - \frac{1-\gamma_3}{\beta}$. Furthermore, if there also exists $(z_1, z_2, 1) \in (S \times \mathbb{Z}_+) \cap F_3(0, 0, \alpha_3)$ then equality holds and $(z_1, z_2, 1) = (b_1 + 1, b_2 + 1, 1)$.*

Proof of Claim 15. If $\alpha_3 > 1 - \frac{1-\gamma_3}{\beta}$ then $(b_1 + 1, b_2 + 1, 1) \in \mathrm{int}(P(0, 0, \alpha_3))$, contradicting that $P(0, 0, \alpha_3)$ is $S \times \mathbb{Z}_+$ free (this can be seen since $n_{\alpha_i} \cdot (b_1 + 1, b_2 + 1, 1) < 1$ for each i).

Now suppose that there exists $(z_1, z_2, 1) \in (S \times \mathbb{Z}_+) \cap \mathrm{relint}(F_3(0, 0, \alpha_3))$. Suppose that $\alpha_3 = 1 - \frac{1-\gamma_3}{\beta}$. A direct calculation shows that $(b_1 + 1, b_2 + 1, 1)$ is contained in $\mathrm{relint}(F_3)$ and

$$F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\} \subseteq C_1 \cup C_2,$$

where

$$\begin{aligned} C_1 &:= \{(x_1, x_2, 1) : -b_1 + b_2 \leq -x_1 + x_2 \leq 1 - b_1 + b_2\} \\ C_2 &:= \{(x_1, x_2, 1) : 1 + b_2 \leq x_2 \leq 2 + b_2\}. \end{aligned}$$

Furthermore, it can be seen that $(F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\}) \setminus \{(b_1 + 1, b_2 + 1, 1)\}$ is contained in $\mathrm{relint}(C_1) \cup \mathrm{relint}(C_2)$ and so $(b_1 + 1, b_2 + 1, 1)$ is the only $S \times \mathbb{Z}_+$ point in $(F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\})$. Hence the result holds when equality holds.

If $\alpha_3 < 1 - \frac{1-\gamma_3}{\beta}$ then

$$\begin{aligned} &\mathrm{relint}\left(F_3(0, 0, \alpha_3) \cap \{(x_1, x_2, x_3) : x_3 = 1\}\right) \\ \subseteq\, &\mathrm{relint}\left(P_3\left(0, 0, 1 - \frac{1-\gamma_3}{\beta}\right) \cap \{(x_1, x_2, x_3) : x_3 = 1\}\right). \end{aligned}$$

Hence relint(F_3) contains no $\mathbb{Z}^2 \times \mathbb{Z}_+$ points, which is a contradiction. $\square$

Proof of Proposition 65. Let $P_\emptyset := P(0,0,0)$. Observe that $P_\emptyset$ satisfies (i), (ii), and (iii). Indeed, by the choice of β, $P_\emptyset$ is a translated cone such that (i) holds. Furthermore, the definitions of $P_\emptyset$ and T imply that $P_\emptyset$ has T as its base. Hence (ii) holds. Finally, in order to see that (iii) holds, let $(s,z) \in (S \times Z_+) \cap P_\emptyset$. By the choice of β, we have $a_3 \in (0,1)$ and so $z = 0$. However, since $P_\emptyset$ has T as its base and T is a maximal lattice-free triangle, s must be contained in the boundary of T, and therefore $(s,z) \notin \operatorname{int}(P_\emptyset)$. Hence $P_\emptyset$ is $S \times Z_+$-free. The maximality of $S \times Z_+$ free sets follows from the characterization provided in [17].

Let $P_{\{3\}} := P(0,0,\alpha_3^*)$, where α_3^* is defined by

$$\alpha_3^* := \sup\left\{\alpha \in [0,1) : \; P(0,0,\alpha) \text{ is } S \times \mathbb{Z}_+ \text{ free}\right\}.$$

We claim that $P_{\{3\}}$ satisfies (i), (ii), (iii), and (iv) for $F_3(0,0,\alpha_3^*)$. By definition, $P_{\{3\}}$ satisfies (ii) and (iii). Suppose that $P_{\{3\}}$ satisfies (i). Then $F_3(0,0,\alpha_3^*) \cap \{(x_1,x_2,x_3) : x_3 \geq 0\}$ is compact and there exists a point $(s,z) \in S \times \mathbb{Z}_+$ that is closest to $F_3(0,0,\alpha_3^*)$. Therefore, by the definition of α_3^*, $P_{\{3\}}$ satisfies (iv) for $F_3(0,0,\alpha_3^*)$. Hence, in order to prove the claim, it is sufficient to show $P_{\{3\}}$ satisfies (i). Assume to the contrary that there exists $(p_1,p_2,1) \in \operatorname{rec}(P_{\{3\}})$. From Claims 14 and 15, we have that $(p_1,p_2,1) = (1,1,1)$. However, this implies that $(b_1, b_2+1, 0) + (p_1,p_2,1) = (b_1+1, b_2+2, 1)$ is in $F_2(0,0,\alpha_3^*)$, and, through a direct calculation, it can be seen that $(b_1+1, b_2+2, 1) \notin F_2(0,0,\alpha_3^*)$. This is a contradiction, and so $P_{\{3\}}$ satisfies (i).

Suppose that (r_1,r_2,r_3) is a blocking point on $F_3(0,0,\alpha_3^*)$ arising in the construction of $P_{\{3\}}$. Define $P_{\{2,3\}} := P(0,\alpha_2^*,\alpha_3^*)$, where α_2^* is defined by

$$\alpha_2^* := \sup\left\{\alpha \in [0,1) : \; P(0,\alpha,\alpha_3^*) \text{ is } S \times \mathbb{Z}_+ \text{ free}\right\}.$$

We claim that $P_{\{2,3\}}$ satisfies (i), (ii), (iii), and (iv) for $F_2(0,\alpha_2^*,\alpha_3^*)$ and $F_3(0,\alpha_2^*,\alpha_3^*)$. By definition, $P_{\{2,3\}}$ satisfies (ii) and (iii), and similar to above, if $P_{\{2,3\}}$ satisfies (i) then it satisfies (iv) for $F_2(0,\alpha_2^*,\alpha_3^*)$. Finally, from Observation 8, $(r_1,r_2,r_3) \in \operatorname{relint}(F_3(0,\alpha_2^*,\alpha_3^*))$. Hence, it is sufficient to show that $P_{\{2,3\}}$ satisfies (i).

Suppose to the contrary that $P_{\{2,3\}}$ does not satisfy (i). Then, from Claims 14 and 15, $\operatorname{rec}(P_{\{2,3\}})$ equals the cone generated by some $(p_1, p_2, 1) \in \mathbb{Z}^3$. From (6.24) and Observation 8,

$$(b_1, b_2, 0) \in \operatorname{relint}(F_3(0, 0, \alpha_3^*)) \subseteq \operatorname{relint}(F_3(0, \alpha_2^*, \alpha_3^*)).$$

Moreover, from assumption (ii), $(b_1, b_2, 0)$ is the only point in $(S \times \{0\}) \cap \operatorname{relint}(F_3(0, \alpha_2^*, \alpha_3^*))$. Therefore, since $\operatorname{rec}(P_{\{2,3\}})$ is generated by $(p_1, p_2, 1)$, there exists exactly one point in $(S \times \{k\}) \cap \operatorname{relint}(F_3(0, \alpha_2^*, \alpha_3^*))$ for each $k \in \mathbb{Z}_+$, and such a point is of the form $(b_1, b_2, 0) + k(p_1, p_2, 1)$. In particular,

$$(r_1, r_2, r_3) = (b_1, b_2, 0) + r_3(p_1, p_2, 1). \tag{6.25}$$

Since (r_1, r_2, r_3) is a blocking point, $r_3 \geq 1$. However, as $(r_1, r_2, r_3), (b_1, b_2, 0) \in \operatorname{relint}(F_3(0, 0, \alpha_3^*))$, (6.25) implies that $(b_1, b_2, 0) + k(p_1, p_2, 1) \in \operatorname{relint}(F_3(0, 0, \alpha_3^*))$ for $1 \leq k \leq r_3$. In particular, $(b_1, b_2, 0) + (p_1, p_2, 1) \in \operatorname{relint}(F_3(0, 0, \alpha_3^*))$. Using Claim 15, we see $(p_1, p_2, 1) = (1, 1, 1)$, and again using (6.24), $(b_1, b_2 + 1, 0) \in \operatorname{relint}(F_2(0, \alpha_2^*, \alpha_3^*))$. Therefore

$$(b_1, b_2 + 1, 0) + (p_1, p_2, 1) = (b_1, b_2 + 1, 0) + (1, 1, 1) = (b_1 + 1, b_2 + 2, 1) \in \operatorname{relint}(F_2(0, \alpha_2^*, \alpha_3^*)).$$

However, a direct calculation shows that this is not the case. This is a contradiction and so $P_{\{2,3\}}$ satisfies (i).

The final facet to tilt is $F_1(0, \alpha_2^*, \alpha_3^*)$. Define $P_{\{1,2,3\}} := P(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, where α_1^* is defined by

$$\alpha_1^* := \sup\left\{\alpha \in [0, 1) : \ P(\alpha, \alpha_2^*, \alpha_3^*) \text{ is } S \times \mathbb{Z}_+ \text{ free}\right\}.$$

Using the same argument as for $P_{\{2,3\}}$, we have that $P_{\{1,2,3\}}$ satisfies (i), (ii), (iii). Furthermore, $P_{\{1,2,3\}}$ satisfies (iv) for $F_1(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, $F_2(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, and $F_3(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. $\square$

6.4.2 Sufficient condition for Type 3 triangles to be one point fixable

Let $T := T(\gamma_1, \gamma_2, \gamma_3)$ be a Type 3 triangle given by (6.24) with appropriate values for $\gamma_1, \gamma_2, \gamma_3$. Let $P \subseteq \mathbb{R}^3$ be the polyhedron defined by

$$P := \Bigg\{ \begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3 : \\ \frac{1}{(1,\gamma_1)\cdot(b_1+1,b_2+1)} x_1 + \frac{\gamma_1}{(1,\gamma_1)\cdot(b_1+1,b_2+1)} x_2 + \left(1 - \frac{(1,\gamma_1)\cdot(b_1+1,b_2+2)}{(1,\gamma_1)\cdot(b_1+1,b_2+1)}\right) x_3 \leq 1, \\ -\frac{1}{(-1,\gamma_2)\cdot(b_1,b_2+1)} x_1 + \frac{\gamma_2}{(-1,\gamma_2)\cdot(b_1,b_2+1)} x_2 \leq 1, \\ \frac{\gamma_3}{(\gamma_3,-1)\cdot(b_1,b_2)} x_1 - \frac{1}{(\gamma_3,-1)\cdot(b_1,b_2)} x_2 + \left(\frac{1}{2} - \frac{(\gamma_3,-1)\cdot(1+b_1,2+b_2)}{2(\gamma_3,-1)\cdot(b_1,b_2)}\right) x_3 \leq 1 \Bigg\}. \end{array} \tag{6.26}$$

Note that $T \times \{0\} = P \cap \{(x_1, x_2, x_3) : \; x_3 = 0\}$, and P contains the $S \times \mathbb{Z}_+$ points $z_1 = (\overline{x}^1, \overline{x}^1_{n+1}) = (1 + b_1, 1 + b_2, 0)$, $z_2 = (\overline{x}^2, \overline{x}^2_{n+1}) = (b_1, 1 + b_2, 0)$, $z_3 = (\overline{x}^3, \overline{x}^3_{n+1}) = (b_1, b_2, 0)$, $z_4 = (\overline{x}^4, \overline{x}^4_{n+1}) = (1 + b_1, 2 + b_2, 1)$, $z_5 = (\overline{x}^5, \overline{x}^5_{n+1}) = (b_1, 1 + b_2, 1)$, and $z_6 = (\overline{x}^6, \overline{x}^6_{n+1}) = (1 + b_1, 1 + b_2, 2)$. Furthermore P has three facets, F_1, F_2, and F_3, containing the points $\{z_1, z_4\}$, $\{z_2, z_5\}$, and $\{z_3, z_6\}$, respectively.

In the situation of Type 3 triangles, $S = b + \mathbb{Z}^2$ and $W_S = \mathbb{Z}^2$. Assuming a certain relation of $\gamma_1, \gamma_2, \gamma_3$, P is a translated cone with apex contained in $\mathbb{R}^2 \times \mathbb{R}_+$. If P is also $S \times \mathbb{Z}_+$ free then it is possible to find a p^* such that $\mathcal{X}(T, p^*) + \mathbb{Z}^2 = \mathbb{R}^2$ (recall Equation (6.23)). This implies that $\mathcal{L}_{\psi,p^*}$ is a singleton, and thus T is one point fixable. This is the content of Proposition 66.

Proposition 66. *Let T and P be described as above with $\gamma_1, \gamma_2, \gamma_3 > 0$ and $\gamma_2, \gamma_3 < 1$. Let ψ be the valid function for S obtained from T using* (6.13)*. Then the following hold.*

(i) P is a translated cone whose apex $a^* = (a^*_1, a^*_2, a^*_3)$ satisfies $a^*_3 > 0$ if and only if $\gamma_2(2 - \gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$.

(ii) If P is $S \times \mathbb{Z}_+$-free then setting $p^* = (a^*_1, a^*_2)$ implies $\mathcal{X}(T, p^*) + W_S = \mathbb{R}^n$ and consequently, $\mathcal{L}_{\psi,p^*}$ consists of a unique lifting function.

Proof. We can symbolically compute $a^* = (a_1^*, a_2^*, a_3^*) = F_1 \cap F_2 \cap F_3$:

$$a^* := \begin{pmatrix} b_1 + \frac{\gamma_2(2+2\gamma_1-\gamma_3)}{\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3}, \\ \\ b_2 + \frac{\gamma_1(2-\gamma_3+2\gamma_2\gamma_3)-(1+\gamma_2)(-2+\gamma_3)}{\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3}, \\ \\ \frac{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}{\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3} \end{pmatrix}.$$

In order for P to be translated cone with apex in the upper-half space, it is equivalent to show that $2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3) > 0$ and $\gamma_2(2-\gamma_3+2\gamma_1\gamma_3)-\gamma_1\gamma_3 > 0$. The first inequality holds since $\gamma_3 < 1$ while the second holds by hypothesis. Hence (i) is shown.

By Proposition 58, $\mathcal{L}_{\psi,p^*}$ is nonempty. According to Theorem 31, in order to see that $\mathcal{L}_{\psi,p^*}$ is indeed a unique lifting function, it is sufficient to show that $\mathcal{X}(T,p^*) + \mathbb{Z}^2 = \mathbb{R}^2$ (recall $W_S = \mathbb{Z}^2$). We draw inspiration from [56]. The crucial observation is that for the choice of p^* in the hypothesis, $P = B(V_\psi(p^*), p^*)$.

Figure 8 in [56] labels the vertices of the translates of the spindles $R(w(z_1);T)$, $R(w(z_2);T)$ and $R(w(z_3);T)$. For completeness, we reproduce the labels in Figure 6.4.

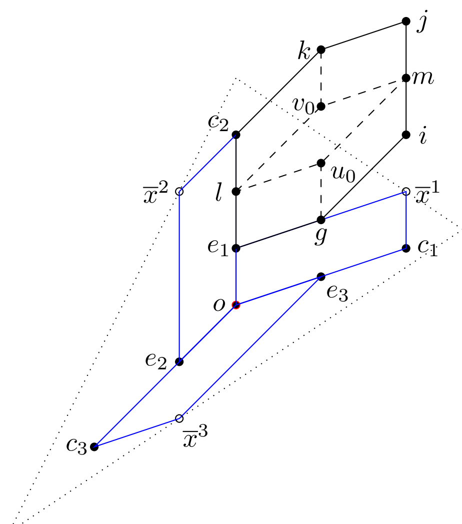


Figure 6.4: The point o denotes the origin. This diagram from [56] denotes the spindles of T.

The region $K = \text{conv}\{c_2, k, j, i, g, e_1\} \notin R(S;T) + W_S$, i.e. K is not contained in the translated lifting region of T. Moreover, up to W_S translations, K is the only region not contained in $R(S;T) + W_S$ [56]. Hence, to complete the proof it suffices to show that $K \subseteq \mathcal{X}(T,p^*) + \mathbb{Z}^2$. For this, partition K into $K = \cup_{i=1}^5 K_i$, where

$$K_1 = \text{conv}\{l, e_1, g, u_0\}$$

$$K_2 = \text{conv}\{u_0, m, i, g\}$$

$$K_3 = \text{conv}\{m, j, k, v_0\}$$

$$K_4 = \text{conv}\{c_2, k, v_0, l\}$$

$$K_5 = \text{conv}\{l, v_0, m, u_0\}.$$

From Theorem 31, $R(w(z_4);B), R(w(z_5);B)+p^*, R(w(z_5);B)+(1,1), R(w(z_5);B)+p^*$, and $R(w(z_6);B)+p^*$ are contained in $\mathcal{X}(T,p^*)+\mathbb{Z}^2$. We claim the following which will complete the proof.

Claim 16. *$K_1 \subseteq R(w(z_4);B)$, $K_2 \subseteq R(w(z_5);B) + (1,1)$, $K_3 \subseteq R(w(z_4);B) + p^*$, $K_4 \subseteq R(w(z_5);B) + p^*$, and $K_5 \subseteq R(w(z_6);B) + p^*$.*

The proof of Claim 16 appears in Appendix A.9. □

6.4.3 Type 3 triangles from the mixing set

Proposition 66 assumes that the pyramid P is $S \times \mathbb{Z}_+$ free. This is the situation for Type 3 triangles derived from the *mixing set* [54, 72]. The mixing set Type 3 triangles are defined for $S = b + \mathbb{Z}^2$ where $b \in \text{int}(\text{conv}\,((0,-1),(0,-1/2),(-1,-1)))$, which is a subset of our earlier restriction $-1 < b_2 < b_1 < 0$, with the additional constraint that $b_1 - 2b_2 > 1$. Define $\delta_b = -b_1^2 - b_2^2 + b_1 b_2 - b_2$. Observe $\delta_b = b_1(b_2 - b_1) - b_2(1 + b_2) > 0$.

Consider the set $T(b) \subseteq \mathbb{R}^2$ defined by

$$\begin{aligned} T(b) := \Bigg\{ (x_1,x_2) \in \mathbb{R}^2 :\ & \left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 \le 1, \\ & \left(\frac{-b_1 - 1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 \le 1, \\ & \left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2 - 1}{\delta_b}\right) x_2 \le 1 \Bigg\}. \end{aligned}$$

It can be checked directly that $T(b)$ is a Type 3 triangle by setting $\gamma_1 = \frac{b_2-b_1}{b_1}, \gamma_2 = \frac{b_1-b_2}{1+b_1}, \gamma_3 = \frac{b_1}{b_1-b_2-1}$. Note that the constraints on b imply that $\gamma_1, \gamma_2, \gamma_3 > 0$ and $\gamma_2, \gamma_3 < 1$, as required. By construction, $T(b) \cap S = \{(b_1, b_2), (b_1, 1+b_2), (1+b_1, 1+b_2)\}$. Plugging in these values of $\gamma_1, \gamma_2, \gamma_3$ in (6.26) we obtain

$$
\begin{aligned}
P(b) := \Bigg\{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ & \left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 - \left(\frac{b_1 - b_2}{\delta_b}\right) x_3 \leq 1, \\
& \left(\frac{-b_1 - 1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2 \leq 1, \\
& \left(\frac{-b_1}{\delta_b}\right) x_1 + \left(\frac{b_1 - b_2 - 1}{\delta_b}\right) x_2 + \left(\frac{2 - b_1 + 2b_2}{2\delta_b}\right) x_3 \leq 1\Bigg\}.
\end{aligned}
$$

We verify the two conditions in Proposition 66, concluding that there always exists a $p^* \in \mathbb{R}^2$ satisfying one point fixability for mixing set triangles. The condition $\gamma_2(2 - \gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$ can be checked by using the values $\gamma_1 = \frac{b_2-b_1}{b_1}, \gamma_2 = \frac{b_1-b_2}{1+b_1}, \gamma_3 = \frac{b_1}{b_1-b_2-1}$, and the constraints $-1 < b_2 < b_1 < 0$. We verify that $\operatorname{int}(P(b)) \cap (S \times \mathbb{Z}_+) = \emptyset$ in the next proposition.

Proposition 67. $\operatorname{int}(P(b)) \cap (S \times \mathbb{Z}_+) = \emptyset$ *for all* $b \in \operatorname{int}(\operatorname{conv}((0,0), (0, -1/2), (-1, -1)))$.

Proof. Recall that $P(b) \cap (\mathbb{R}^2 \times \{0\}) = T(b) \times \{0\}$, which is an S-free triangle. Thus we only need to show $\operatorname{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\})) \cap (S \times \{k\}) = \emptyset$ for $k \in \mathbb{N}$.

For a fixed $k \geq 1$, define the split sets

$$
\begin{aligned}
C_1 &:= \left\{(x_1, x_2, k) \in \mathbb{R}^3 :\ k \leq x_2 \leq k+1\right\} + (b_1, b_2, 0) \\
C_2 &:= \left\{(x_1, x_2, k) \in \mathbb{R}^3 :\ 0 \leq -2x_1 + x_2 \leq 1\right\} + (b_1, b_2, 0) \\
C_3 &:= \left\{(x_1, x_2, k) \in \mathbb{R}^3 :\ \frac{k}{2} \leq -x_1 + x_2 \leq \frac{k}{2} + \frac{1}{2}\right\} + (b_1, b_2, 0).
\end{aligned}
$$

Note that for each $k \geq 1$, the splits C_1, C_2 and C_3 have no $S \times \{k\}$ points in their relative interior. Hence if we can show that

$$
\operatorname{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\})) \subseteq \operatorname{relint}(C_1) \cup \operatorname{relint}(C_2) \cup \operatorname{relint}(C_3),
$$

then we will be done. To this end, suppose $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\}))$ but not in $\text{relint}(C_1) \cup \text{relint}(C_2)$. This implies that (x_1^*, x_2^*, k) does not strictly satisfy one of the inequalities defining C_1 and one of the inequalities defining C_2. This leads to four cases.

Case 1 Suppose $x_2^* - b_2 \leq k$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$. Observe that

$$\begin{aligned}
& \left(\tfrac{-b_1}{\delta_b}\right) x_1^* + \left(\tfrac{b_1 - b_2 - 1}{\delta_b}\right) x_2^* + \left(\tfrac{2 - b_1 + 2b_2}{2\delta_b}\right) \\
\geq\ & \left(\tfrac{-b_1}{\delta_b}\right)\left(\tfrac{2b_1 + x_2^* - b_2}{2}\right) + \left(\tfrac{b_1 - b_2 - 1}{\delta_b}\right) x_2^* + \left(\tfrac{2 - b_1 + 2b_2}{2\delta_b}\right) \\
=\ & \left(\tfrac{b_1 - 2b_2 - 2}{2\delta_b}\right) x_2^* + \left(\tfrac{2 - b_1 + 2b_2}{2\delta_b}\right) k + \left(\tfrac{-2b_1^2 + b_1 b_2}{2\delta_b}\right) \\
\geq\ & \left(\tfrac{b_1 - 2b_2 - 2}{2\delta_b}\right)(k + b_2) + \left(\tfrac{2 - b_1 + 2b_2}{2\delta_b}\right) k + \left(\tfrac{-2b_1^2 + b_1 b_2}{2\delta_b}\right) \\
=\ & 1.
\end{aligned}$$

The first inequality follows from the assumption $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$, and the second inequality follows from the assumption $x_2^* - b_2 \leq k$. This contradicts that $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\}))$ because the third inequality defining $P(b)$ is violated.

Case 2 Suppose $x_2^* - b_2 \leq k$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$. We claim that $(x_1^*, x_2^*, k) \in \text{relint}(C_3)$. For this, it is sufficient to show that $\frac{k}{2} < -(x_1^* - b_1) + (x_2^* - b_2) < \frac{k}{2} + \frac{1}{2}$. Note that since $(x_1^*, x_2^*, k) \in \text{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\}))$, the third inequality defining $P(b)$ gives the following bound on x_2^*

$$x_2^* > \frac{-b_1}{1 + b_2 - b_1} x_1^* + \frac{k}{2} + \frac{1 + b_2}{2(1 + b_2 - b_1)} k + \frac{-\delta_b}{1 + b_2 - b_1}.$$

Using this, we see that

$$
\begin{aligned}
-(x_1^*-b_1)+(x_2^*-b_2) &> -(x_1^*-b_1)+\left(\tfrac{-b_1}{1+b_2-b_1}x_1^*+\tfrac{k}{2}+\tfrac{1+b_2}{2(1+b_2-b_1)}k+\tfrac{-\delta_b}{1+b_2-b_1}\right)-b_2\\
&= \tfrac{k}{2}+\left(\tfrac{-1-b_2}{1+b_2-b_1}\right)x_1^*+\left(\tfrac{1+b_2}{2(1+b_2-b_1)}\right)k+\left(\tfrac{b_1+b_1b_2}{1+b_2-b_1}\right)\\
&\geq \tfrac{k}{2}+\left(\tfrac{-1-b_2}{1+b_2-b_1}\right)\left(\tfrac{x_2^*-b_2-1+2b_1}{2}\right)+\left(\tfrac{1+b_2}{2(1+b_2-b_1)}\right)k+\left(\tfrac{b_1+b_1b_2}{1+b_2-b_1}\right)\\
&= \tfrac{k}{2}+\left(\tfrac{-1-b_2}{2(1+b_2-b_1)}\right)x_2^*+\left(\tfrac{1+b_2}{2(1+b_2-b_1)}\right)k+\left(\tfrac{2b_2+b_2^2+1}{2(1+b_2-b_1)}\right)\\
&\geq \tfrac{k}{2}+\left(\tfrac{-1-b_2}{2(1+b_2-b_1)}\right)(k+b_2)+\left(\tfrac{1+b_2}{2(1+b_2-b_1)}\right)k+\left(\tfrac{2b_2+b_2^2+1}{2(1+b_2-b_1)}\right)\\
&= \tfrac{k}{2}+\tfrac{1+b_2}{2(1+b_2-b_1)}\\
&> \tfrac{k}{2}.
\end{aligned}
$$

The second inequality follows since $-2(x_1^*-b_1)+(x_2^*-b_2)\geq 1$ and $\frac{-1-b_2}{-b_1+b_2+1}<0$, the third inequality follows since $x_2^*\leq k+b_2$, and the fourth inequality follows since $\frac{1+b_2}{2(1+b_2-b_1)}>0$.

Since $(x_1^*,x_2^*,k)\in \operatorname{relint}(P(b)\cap(\mathbb{R}^2\times\{k\}))$, the second inequality defining $P(b)$ implies

$$
\begin{aligned}
-(x_1^*-b_1)+(x_2^*-b_2) &< -x_1^*+b_1+\left(\tfrac{\delta_b}{b_1-b_2}+\tfrac{1+b_1}{b_1-b_2}x_1^*\right)-b_2\\
&= \left(\tfrac{1+b_2}{b_1-b_2}\right)x_1^*+\tfrac{-b_2-b_1b_2}{b_1-b_2}\\
&\leq \left(\tfrac{1+b_2}{b_1-b_2}\right)\left(\tfrac{2b_1+x_2^*-b_2-1}{2}\right)+\tfrac{-b_2-b_1b_2}{b_1-b_2}\\
&= \left(\tfrac{1+b_2}{2(b_1-b_2)}\right)x_2^*+\left(\tfrac{2b_1-4b_2-b_2^2-1}{2(b_1-b_2)}\right)\\
&\leq \left(\tfrac{1+b_2}{2(b_1-b_2)}\right)(k+b_2)+\left(\tfrac{2b_1-4b_2-b_2^2-1}{2(b_1-b_2)}\right)\\
&= \tfrac{k}{2}+\left(\tfrac{1-b_1+2b_2}{b_1-b_2}\right)\tfrac{k}{2}+\left(\tfrac{2b_1-3b_2-1}{2(b_1-b_2)}\right)\\
&\leq \tfrac{k}{2}+\left(\tfrac{1-b_1+2b_2}{b_1-b_2}\right)\tfrac{1}{2}+\left(\tfrac{2b_1-3b_2-1}{2(b_1-b_2)}\right)\\
&= \tfrac{k}{2}+\tfrac{1}{2}.
\end{aligned}
$$

The second inequality follows since $-2(x_1^*-b_1)+(x_2^*-b_2)\geq 1$ and $\frac{1+b_2}{-b_1+b_2+1}>0$, the third inequality follows since $x_2^*\leq k+b_2$, and the fourth inequality follows since $k\geq 1$ and $1<b_1-2b_2$.

Case 3 Suppose $x_2^* - b_2 \geq k+1$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$. Observe

$$\begin{aligned}
\left(\tfrac{-b_1}{\delta_b}\right) x_1^* + \left(\tfrac{b_1-b_2}{\delta_b}\right) x_2^* - \left(\tfrac{b_1-b_2}{\delta_b}\right) k &\geq \left(\tfrac{-b_1}{\delta_b}\right)\left(\tfrac{2b_1+x_2^*-b_2}{2}\right) + \left(\tfrac{b_1-b_2}{\delta_b}\right) x_2^* - \left(\tfrac{b_1-b_2}{\delta_b}\right) k \\
&= \left(\tfrac{b_1-2b_2}{2\delta_b}\right) x_2^* - \left(\tfrac{b_1-b_2}{\delta_b}\right) k + \left(\tfrac{-2b_1^2+b_1b_2}{2\delta_b}\right) \\
&\geq \left(\tfrac{b_1-2b_2}{2\delta_b}\right)(k+1+b_2) - \left(\tfrac{b_1-b_2}{\delta_b}\right) k + \left(\tfrac{-2b_1^2+b_1b_2}{2\delta_b}\right) \\
&= \left(\tfrac{-b_1}{2\delta_b}\right) k + \left(\tfrac{b_1}{2\delta_b}\right) + 1 \\
&\geq \left(\tfrac{-b_1}{2\delta_b}\right) + \left(\tfrac{b_1}{2\delta_b}\right) + 1 \\
&= 1.
\end{aligned}$$

The first inequality follows since $\frac{-b_1}{\delta_b} > 0$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 0$, the second inequality follows since $b_1 - 2b_2 > 1$ and $x_2^* \geq k+1+b_2$, and the third inequality follows since $k \geq 1$. This contradicts that $(x_1^*, x_2^*, k) \in \operatorname{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\}))$ because the first inequality defining $P(b)$ is violated.

Case 4 Suppose $x_2^* - b_2 \geq k+1$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$. Observe that

$$\begin{aligned}
\left(\tfrac{-b_1-1}{\delta_b}\right) x_1^* + \left(\tfrac{b_1-b_2}{\delta_b}\right) x_2^* &\geq \left(\tfrac{-b_1-1}{\delta_b}\right)\left(\tfrac{x_2^*-1+2b_1-b_2}{2}\right) + \left(\tfrac{b_1-b_2}{\delta_b}\right) x_2^* \\
&= \left(\tfrac{b_1-2b_2-1}{2\delta_b}\right) x_2^* + \left(\tfrac{-b_1-1}{\delta_b}\right)\left(\tfrac{2b_1-b_2-1}{2}\right) \\
&\geq \left(\tfrac{b_1-2b_2-1}{2\delta_b}\right)(2+b_2) + \left(\tfrac{-b_1-1}{\delta_b}\right)\left(\tfrac{2b_1-b_2-1}{2}\right) \\
&= 1 + \tfrac{b_1-2b_2-1}{2\delta_b} \\
&> 1.
\end{aligned}$$

The first inequality comes from $\frac{-b_1-1}{\delta_b} < 0$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \geq 1$. The second inequality comes from the fact that $b_1 - 2b_2 > 1$ and $\delta_b > 0$ so the term $\left(\frac{b_1-2b_2-1}{2\delta_b}\right)$ is positive; furthermore, $x_2^* \geq k+1+b_2 \geq 2+b_2 > 0$ since $k \geq 1$ and $-1 < b_2$. The last inequality follows since $\delta_b > 0$ and $b_1 - 2b_2 > 1$. This contradicts that $(x_1^*, x_2^*, k) \in \operatorname{relint}(P(b) \cap (\mathbb{R}^2 \times \{k\}))$ because the second inequality defining $P(b)$ is violated.

$\square$

Chapter 7

Final Remarks

Here we discuss some future research directions that stem from the material in this thesis. We then conclude with a few final remarks.

7.1 Future research directions

7.1.1 Questions about the submodel C_S

In Chapter 3, we considered the question of approximating the corner polyhedron with a subfamily of intersection cuts. That is we searched for families $\mathcal{B}$ of lattice-free sets such that the approximation functional $\rho(\mathcal{B}, \mathcal{L}_*^n)$ is finite. Theorem 7 states that $\rho(\mathcal{L}_i^n, \mathcal{L}_*^n) \leq 4\mathrm{Flt}(n)$ provided that i is large enough. However, the proof of this result is independent of i and therefore does not stratify the values $\rho(\mathcal{L}_i^n, \mathcal{L}_*^n)$ for $i \geq 2^{n-1}$. Such a stratification could result from the further study of lower and upper bounds for $\rho(\mathcal{L}_i^n, \mathcal{L}_*^n)$. This leads to the following question, which was answered in the affirmative in [8].

Open Question 1. *Let $i, j, n \in \mathbb{N}$ such that $i > j \geq 2^{n-1}$. Are there values $L_i, L_j, U_i, U_j \in \mathbb{R}$ so that*

$$1 < L_i \leq \rho(\mathcal{L}_i^n, \mathcal{L}_*^n) \leq U_i < L_j \leq \rho(\mathcal{L}_j^n, \mathcal{L}_*^n) \leq U_j < 4Flt(n)?$$

Suppose one prescribes an approximation factor $\alpha \geq 1$ and desires $\rho(\mathcal{L}_i^n, \mathcal{L}_*^n) < \alpha$. An affirmative answer to Open Question 1 would imply that i needs to be large enough in order to obtain this approximation factor α. Furthermore, recall that for $L \in \mathcal{L}_i^n$ with i

facets, computing a coefficient for a L-intersection cut requires i inner product evaluations. Therefore Open Question 1 would indicate that in order to achieve an approximation factor of α, an additional computational expense is required.

One may also be interested in *any* family (not necessarily one of the form $\mathcal{L}_i^n$) that approximates the corner polyhedron within a specified factor. This leads to our second open question.

Open Question 2. *Let $n \in \mathbb{N}$ and $\alpha \geq 1$. Is there a minimal family of lattice-free sets $\mathcal{B} = \mathcal{B}(\alpha)$ such that $\rho(\mathcal{B}, \mathcal{L}_*^n) \leq \alpha$? Here 'minimal' refers to smallest with respect to set inclusion.*

Note the use of 'minimal' in Question 2. If we disregard minimal families, then setting $\mathcal{B} = \mathcal{L}_{2^n}^n$ would result in the best approximation value of $\rho(\mathcal{B}, \mathcal{L}_*^n) = 1$. However, this choice $\mathcal{B}$ returns us to the issue of intractability discussed in Chapter 3.1 and thus is not very helpful.

7.1.2 Questions about the submodel I_S

In Chapter 4, we inductively created extreme cut-generating functions for $R_b(\mathbb{R}, \mathbb{Z})$. While these functions are mostly untested when it comes to actual implementation, they do add to the ever-growing library of potential functions that may be used. The main idea of our induction was to apply a perturbation to an existing extreme function, using the Gomory function as the base case. Therefore, with the goal of expanding the library of extreme cut-generating functions, the following question may be of interest.

Open Question 3. *Let $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and suppose $\pi : \mathbb{R} \to \mathbb{R}_+$ is a piecewise linear extreme function for $R_b(\mathbb{R}, \mathbb{Z})$. Is there a 'perturbation' that can be applied to π that yields a different extreme cut-generating function π'?*

7.1.3 Questions about connecting C_S and I_S to M_S

Let $S = (b + \Lambda) \cap C$ be a polyhedrally-truncated affine lattice in $\mathbb{R}^n$. When developing minimal cut-generating pairs for M_S, one approach is to find a minimal cut-generating function ψ for C_S that has the covering property. Once this is done, the pair (ψ, π^*) provides a minimal cut-generating pair. In Chapter 5, the coproduct and limit operations were introduced as covering-property-preserving operations. One application of these operations is to use minimal cut-generating functions for M_S, where $S \subseteq \mathbb{R}^n$, and construct minimal cut-generating functions for $M_{S'}$, where $S' \subseteq \mathbb{R}^m$ for $m > n$. We believe that these two operations could be useful in attacking questions of the following flavor.

Open Question 4. *For a fixed $n \in \mathbb{N}$, for which natural numbers in the range $2 \leq k \leq 2^n$ do there exist maximal lattice-free sets in $\mathbb{R}^n$ with k facets that have the covering property?*

If Question 4 can be answered using the coproduct and limit operations, then one requires a base collection of sets with the covering property. This base collection is known for $n = 1$ and $n = 2$, but even for $n = 3$ the classification is unclear. However, can all lattice-free sets with the covering property in $\mathbb{R}^n$, where $n \geq 3$, be obtained via a sequence of limits and coproducts of sets in $\mathbb{R}^2$ and $\mathbb{R}$?

Open Question 5. *Let $n \geq 3$. Suppose $B \subseteq \mathbb{R}^n$ be a maximal lattice-free 0-neighborhood with the covering property. Using only the limit and coproduct operations, can B be expressed using lattice-free sets with the covering property in $\mathbb{R}^j$ for $j < n$?*

Open Question 5 was partially answered in the negative in [5], where the authors show that the coproduct operation alone is not sufficient for generating all lattice-free sets with the covering property.

Now consider the setting when $S = b + \mathbb{Z}^n$. The discussion above shows that cut-generating pairs can be built in higher dimensions using the coproduct operation $\Diamond$. Another approach is to use the sequential merge operation $\diamond$ discussed in 4. Note that the coproduct operation acts on cut-generating functions for C_S, while the sequential merge operation acts on functions for I_S. Is there a connection between the two operations when connected back to cut-generating pairs for M_S? We formalize this in the following (lengthy) question.

Open Question 6. *Suppose ψ_1 and ψ_2 are minimal cut-generating functions for C_S that have the covering property. Let B_1 and B_2 denote S-free sets corresponding to ψ_1 and ψ_2, respectively, and suppose ψ_3 is the minimal cut-generating function corresponding to $B_1 \Diamond B_2$. Then (ψ_1, π_1^*) and (ψ_2, π_2^*) are minimal cut-generating pairs for M_S, and (ψ_3, π_3^*)*

is minimal for $M_{S\times S}$. Now let π_4 be the sequential merge of π_1^* and π_2^*, that is $\pi_4 := \pi_1^* \diamond \pi_2^*$. Equation 2.16 gives an equation for some ψ_4 so that (ψ_4, π_4) is minimal for $M_{S\times S}$.

How does (ψ_3, π_3^*) compare to (ψ_4, π_4)?

7.2 Conclusion

The work presented in this thesis centers around creating cutting planes using cut-generating functions. We focus on the versatile models M_S, C_S, and I_S, which provide the framework for commonly used mathematical programming techniques. On one hand, examining a particular model allows us to construct minimal and extreme cut-generating functions for that model. On the other hand, understanding the relationship between models lets us construct cut-generating functions for one by using cut-generating functions for the other. This thesis contributes to the theory by addressing questions that pertain to the individual models as well as questions about their relationship.

In the author's opinion, there are still many interesting questions that are yet to be answered. Combining this with the fact that cutting planes have been extremely useful in practice, there seems to be an enormous benefit in future cut-generating function research. Hopefully this body of work can serve as a reference for future work and contribute to the development of cut-generating functions.

Appendix A

A.1 Case analysis for the proof of Theorem 15

Case 1. Suppose $v_1 \le \frac{\delta}{\delta-1}$. Define

$$T = \{(x_1, x_2) : x_1 \ge 0, \quad x_2 \ge 0, \quad x_1 + (v_1 - 1)x_2 \le v_1\}.$$

See Figure A.1. The vertices of T are $\left\{(v_1, 0), (0, 0), \left(0, \frac{v_1}{v_1-1}\right)\right\}$.

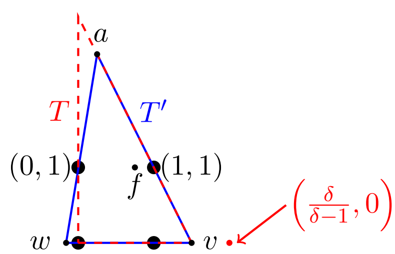


Figure A.1: Case 1 of Theorem 15.

P1. Note T is δ-thin because $v_1 \le \frac{\delta}{\delta-1}$.

P2. Since $f \in \text{int}(T')$ and $f_1 \ge \frac{1}{2}$, it follows that $f \in \text{int}(T)$.

P3. By construction $v \in T$. Thus $\gamma v + (1-\gamma)f$ is a convex combination of a point in T and a point $f \in \text{int}(T)$. Hence $\gamma v + (1-\gamma)f \in \text{int}(T)$.

P4. Since $f \in \text{int}(T)$ and $w \in T'$, the convex combination $\gamma w + (1-\gamma) f$ satisfies the inequalities $x_2 > 0$ and $x_1 + (v_1 - 1)x_2 < v_1$. It is left to show $\gamma w + (1-\gamma) f$ satisfies

$x_1 > 0$. Note

$$
\begin{aligned}
\gamma w_1 + (1-\gamma)\, f_1 &> \gamma(1-\delta) + (1-\gamma)\,\frac{1}{2}, && \text{from } \mathbf{A2.} \text{ and } f_1 \geq \frac{1}{2},\\
&= \frac{\delta^3 - 2\delta + 1}{2\delta^3}, && \text{by definition of } \gamma,\\
&> 0 && \text{since } \delta \geq 2.
\end{aligned}
$$

P5. The same argument from Case 1 **P3.** holds here.

Case 2. Suppose $v_1 > \frac{\delta}{\delta-1}$ and $f_1 \leq 1 - \frac{1}{2}\beta$. Define

$$T = \{(x_1, x_2) : x_1 \geq 0, \quad x_2 \geq 0, \quad (\delta-1)x_1 + x_2 \leq \delta\}.$$

See Figure A.2. The vertices of T are $\left\{(0,0), \left(\frac{\delta}{\delta-1}, 0\right), (0,\delta)\right\}$. Note that T is a type-2 triangle.

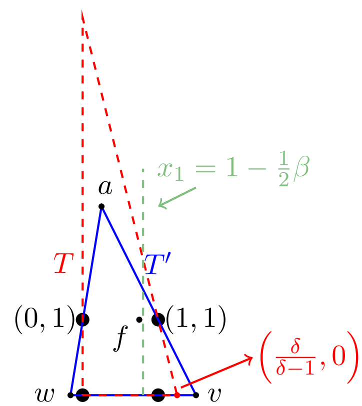


Figure A.2: Case 2 of Theorem 15.

P1. It follows from the vertex description of T that $T \in \mathcal{T}_\delta$.

P2. Since $f_1 \geq 1/2$ and $f \in \operatorname{int}(T')$, f satisfies the inequalities $x_1 > 0$ and $x_2 > 0$. It is left to show that f satisfies $(\delta-1)x_1 + x_2 < \delta$. Note

$$
\begin{aligned}
(\delta-1)f_1 + f_2 &< (\delta-1)f_1 + \frac{v_1 - f_1}{v_1 - 1}\\
&\leq (\delta-1)\left(\frac{2-\beta}{2}\right) + \left(\frac{2v_1 - 2 + \beta}{v_1 - 1}\right),
\end{aligned}
\tag{A.1}
$$

where the first inequality follows from $f_1 + (v_1 - 1)f_2 < v_1$ and the second since

$f_1 < 1 - \frac{1}{2}\beta$. Let $\tilde{f} = \left(\frac{2-\beta}{2}, \frac{2v_1-2+\beta}{v_1-1}\right)$. Note that $\tilde{f}$ is a strict convex combination of $b = (1,1)$ and v, where b and v lie in the relative interiors of the facets of T defined by $(\delta - 1)x_1 + x_2 \leq \delta$ and $x_1 \geq 0$, respectively. Hence $\tilde{f}$ satisfies the inequality $(\delta-1)x_1+x_2 \leq \delta$ strictly. Therefore, from (A.1), $(\delta-1)f_1+f_2 < \delta$ and so $f \in \text{int}(T)$.

P3. Observe that v satisfies $x_1 \geq 0$ and $x_2 \geq 0$, and $f \in \text{int}(T)$. Thus $\gamma v + (1-\gamma)f$ satisfies $x_1 > 0$ and $x_2 > 0$. Note

$$(\delta - 1)v_1 + 0 > \delta \tag{A.2}$$

since $v_1 > \frac{\delta}{\delta-1}$, and

$$(\delta - 1)\tilde{f}_1 + \tilde{f}_2 < \delta, \tag{A.3}$$

where $\tilde{f}$ is defined in Case 2 **P2**. Let $\lambda = \frac{\beta}{2v_1-2+\beta} \in (0,1)$. Note

$$\begin{aligned}
\lambda - \gamma &= \frac{\delta^3\beta - 2v_1 + 2 + \beta}{\delta^3(2v_1 - 2 + \beta)} \\
&\geq \frac{\delta^3 - 2v_1(\delta-1)^2}{\delta^3(\delta-1)^2(2v_1 - 2 + \beta)}, && \text{by reducing and dropping nonnegative terms} \\
&\geq \frac{\delta^3 - 2\delta(\delta-1)^2}{\delta^3(\delta-1)^2(2v_1 - 2 + \beta)}, && \text{since } v_1 < \delta \\
&> 0, && \text{since } \delta \geq 2.
\end{aligned}$$

Moreover, a computation shows that $\lambda v+(1-\lambda)\tilde{f}$ satisfies the equality $(\delta-1)x_1+x_2 = \delta$. This inequality along with $\lambda > \gamma$, (A.2), and (A.3) implies the point $\gamma v + (1-\gamma)f$ satisfies the inequality $(\delta - 1)x_1 + x_2 < \delta$.

P4. This follows from arguments similar to Case 1 **P4**.

P5. From **A3**, note that a is a convex combination of $(1,1)$ and $(0, \frac{v_1}{v_1-1})$. These two points lie in the relative interior different facets of T defined by $(\delta - 1)x_1 + x_2 \leq \delta$ and $x_1 \geq 0$, respectively. Hence a satisfies these inequalities strictly. Finally, it can be seen that a satisfies the inequality $x_2 > 0$ from **A3**.

Case 3. Suppose $v_1 > \frac{\delta}{\delta-1}$ and $f_1 > 1 - \frac{1}{2}\beta$. Define

$$T = \{(x_1, x_2) : x_2 \geq 0, \quad x_1 + (v_1 - 1)x_2 \leq v_1, \quad x_1 + (v_1 - \delta)x_2 \geq v_1 - \delta\}.$$

See Figure A.3. The vertices of T are $\left\{(v_1, 0), \left(\frac{\delta - v_1}{\delta-1}, \frac{\delta}{\delta-1}\right), (v_1 - \delta, 0)\right\}$.

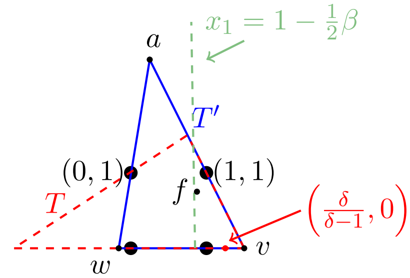


Figure A.3: Case 3 of Theorem 15.

P1. From the vertex description of T, it follows that $T \in \mathcal{T}_\delta$.

P2. Since $f \in \text{int}(T')$, f satisfies $x_2 > 0$ and $x_1 + (v_1 - 1)x_2 < v_1$. Note

$$\begin{aligned} f_1 + (v_1 - \delta)f_2 &> f_1 + (v_1 - \delta)\left(\frac{v_1 - f_1}{v_1 - 1}\right) \\ &> \frac{2 - \beta}{2} + (v_1 - \delta)\left(\frac{2v_1 - 2 + \beta}{v_1 - 1}\right) \end{aligned} \tag{A.4}$$

where the first inequality follows from **A1** and $f \in \text{int}(T')$, and the second inequality follows since $f_1 > \frac{2-\beta}{2}$. Define

$$\tilde{f} = \left(\frac{2 - \beta}{2}, \frac{2v_1 - 2 + \beta}{v_1 - 1}\right).$$

Note that $\tilde{f}$ is a strict convex combination of the points $\tilde{g} := \left(\frac{\delta - v_1}{\delta-1}, \frac{\delta}{\delta-1}\right)$ and v, both of which are in T. Since $\tilde{g}$ is on the facet defined by $x_1 + (v_1 - \delta)x_2 \geq v_1 - \delta$ and v is not, it follows that

$$\tilde{f}_1 + (v_1 - \delta)\tilde{f}_2 > v_1 - \delta. \tag{A.5}$$

From (A.4), it follows that $f_1 + (v_1 - \delta)f_2 > v_1 - \delta$.

P3. The same argument as Case 1 **P3** holds here.

P4. From **A2**, $w \in T$. Thus $\gamma w + (1-\gamma) f \in \text{int}(T)$.

P5. Observe that $\gamma a + (1-\gamma) f$ satisfies $x_2 > 0$. Note $a \in T'$ by construction and $f \in \text{int}(T')$ from **P2**. Thus $\gamma a + (1-\gamma) f \in \text{int}(T')$ and satisfies $x_1 + (v_1 - 1)x_2 < v_1$.

Note $a_1 < \frac{\delta - v_1}{\delta - 1}$. Indeed, assume to the contrary that $a_1 \geq \frac{\delta - v_1}{\delta - 1}$. Since T' is not δ-thin, $\frac{v_1 - a_1}{1 - a_1} < \delta$. Rearranging, we see that $v_1 < \delta + (1-\delta)a_1 \leq \delta + (1-\delta)\frac{\delta - v_1}{\delta - 1} = v_1$, which is a contradiction.

Using **A3** and then $a_1 < \frac{\delta - v_1}{\delta - 1}$, we see

$$\begin{aligned} a_1 + (v_1 - \delta)a_2 &= \frac{a_1(\delta - 1) - v_1(\delta - v_1)}{v_1 - 1} \\ &< \frac{(\delta - v_1) - v_1(\delta - v_1)}{v_1 - 1} \\ &= v_1 - \delta. \end{aligned} \tag{A.6}$$

Define $\lambda \in \mathbb{R}$ to be

$$\lambda = \frac{2(\delta-1)^2 - 2(\delta-1)(\delta-v_1) - 1}{2(\delta-1)^2 - 2a_1(\delta-1)^2 - 1}.$$

Note that

$$\begin{aligned} & 2(\delta-1)^2 - 2a_1(\delta-1)^2 - 1 \\ >& 2(\delta-1)^2 - 2(\delta-1)(\delta-v_1) - 1, && \text{since } a_1 < \frac{\delta - v_1}{\delta - 1} \\ \geq& 2(\delta-1)^2 - 2(\delta-1)\left(\delta - \frac{\delta}{\delta-1}\right) - 1, && \text{since } v_1 > \frac{\delta}{\delta - 1} \\ =& 1, \end{aligned}$$

and so $\lambda \in (0,1)$. Also $\lambda > \gamma$ because

$$
\begin{aligned}
\lambda - \gamma &= \frac{2(\delta-1)^2 - 2(\delta-1)(\delta-v_1) - 1}{2(\delta-1)^2 - 2a_1(\delta-1)^2 - 1} - \frac{1}{\delta^3} \\
&> \frac{2(\delta-1)^2 - 2(\delta-1)(\delta-v_1) - 1}{2(\delta-1)^2 - 1} - \frac{1}{\delta^3}, && \text{since } 2a_1(\delta-1)^2 > 0 \\
&> \frac{2(\delta-1)^2 - 2(\delta-1)(\delta-\frac{\delta}{\delta-1}) - 1}{2(\delta-1)^2 - 1} - \frac{1}{\delta^3}, && \text{since } v_1 > \frac{\delta}{\delta-1} \\
&= \frac{1}{2(\delta-1)^2 - 1} - \frac{1}{\delta^3} \\
&> 0 && \text{since } \delta \geq 2.
\end{aligned}
$$

A calculation shows $\lambda a + (1-\lambda)\tilde{f}$ satisfies the equality $x_1 + (v_1 - \delta)x_2 = v_1 - \delta$. Using this along with $\lambda > \gamma$, (A.5) and (A.6), we see that $\gamma a + (1-\gamma)f$ satisfies $x_1 + (v_1 - \delta)x_2 > v_1 - \delta$.

A.2 Proofs of Lemma 3 and Proposition 29

Proof of Lemma 3. (a): Equality (3.23) can be shown directly:

$$P = \operatorname{conv}(M \cup M') = \bigcup_{0 \leq \lambda \leq 1} ((1-\lambda)M + \lambda M') = \bigcup_{0 \leq \lambda \leq 1} \Big((1+\lambda\alpha)M + \lambda p\Big).$$

(b): $\operatorname{conv}(\{x\} \cup M') \subseteq P$ is clear, since $x \in M$. We show $\frac{1}{4}P + \frac{3}{4}f \subseteq \operatorname{conv}(\{x\} \cup M')$. Since $P = \operatorname{conv}(M \cup M')$, it suffices to check the two inclusions

$$
\begin{aligned}
\frac{1}{4}M + \frac{3}{4}f &\subseteq \operatorname{conv}(\{x\} \cup M') \\
\frac{1}{4}M' + \frac{3}{4}f &\subseteq \operatorname{conv}(\{x\} \cup M').
\end{aligned}
$$

This is equivalent to showing the following inclusions obtained by translating the right and the left hand sides by $-x$:

$$
\begin{aligned}
\frac{1}{4}(M - x) + \frac{3}{4}(f - x) &\subseteq \operatorname{conv}(\{0\} \cup (M' - x)) \\
\frac{1}{4}(M' - x) + \frac{3}{4}(f - x) &\subseteq \operatorname{conv}(\{0\} \cup (M' - x)),
\end{aligned}
$$

where

$$M' - x = (1+\alpha)(M-x) + \alpha x + p,$$
$$f - x = \mu(\alpha x + p).$$

This shows that for proving the two inclusions we can assume $x = 0$ (this corresponds to substitution of M for $M - x$ and p for $\alpha x + p$). So we assume $x = 0$ and need to verify

$$\frac{1}{4}M + \frac{3}{4}f \subseteq \operatorname{conv}(\{0\} \cup M')$$
$$\frac{1}{4}M' + \frac{3}{4}f \subseteq \operatorname{conv}(\{0\} \cup M'),$$

where $f = \mu p$ and $M' = (1+\alpha)M + p$. Since $\operatorname{conv}(\{0\} \cup M') = \bigcup_{0 \leq \lambda \leq 1} \lambda M'$ it suffices to verify to show that the left hand sides are subsets of $\lambda M'$ for some appropriate choices of λ. For the first inclusion we choose $\lambda = \frac{3}{4}\mu$. We have to check

$$\frac{1}{4}M + \frac{3}{4}f \subseteq \frac{3}{4}\mu M'$$

that is

$$\frac{1}{4}M + \frac{3}{4}\mu p \subseteq \frac{3}{4}\mu(1+\alpha)M + \frac{3}{4}\mu p.$$

The latter is fulfilled since $\mu \geq \frac{1}{3}$.

For the second inclusion we choose $\lambda = \frac{3}{4}\mu + \frac{1}{4}$. We have to check

$$\frac{1}{4}M' + \frac{3}{4}f \subseteq \left(\frac{3}{4}\mu + \frac{1}{4}\right) M'$$

that is

$$\frac{1}{4}(1+\alpha)M + \frac{1}{4}p + \frac{3}{4}\mu p \subseteq \left(\frac{3}{4}\mu + \frac{1}{4}\right)(1+\alpha)M + \left(\frac{3}{4}\mu + \frac{1}{4}\right)p.$$

The latter is obviously fulfilled.

$\square$

Proof of Proposition 29. Since $M = S \oplus U$ has $k+1$ facets, P can be written as

$$P = \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i, i \in [k+1]\} \cap (\mathbb{R}^{n-1} \times [\gamma, \lambda]). \tag{A.7}$$

We consider two cases on the facet structure of P.

Case 1: Suppose that neither $P \cap (\mathbb{R}^{n-1} \times \{\lambda\})$ nor $P \cap (\mathbb{R}^{n-1} \times \{\gamma\})$ is a facet of P. Set $B := P$. Equation (3.26) follows immediately. Also, B has at most $k+2$ facets as seen from the representation in Equation (A.7). Moreover, since $f \in \text{int}(P)$ and P is convex, Equation (3.25) follows.

Case 2: Suppose that $P \cap (\mathbb{R}^{n-1} \times \{\lambda\})$ and $P \cap (\mathbb{R}^{n-1} \times \{\gamma\})$ are facets of P. Define the polyhedron

$$P' := \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i, i \in [k+1]\} \cap (\mathbb{R}^{n-1} \times [-1, 1]), \tag{A.8}$$

and consider the faces $M_1 = P \cap (\mathbb{R}^{n-1} \times \{1\})$ and $M_{-1} = P \cap (\mathbb{R}^{n-1} \times \{-1\})$ of P'. If either M_1 or M_{-1} is not a facet of P', then set $B = P'$. Equation (3.26) holds immediately and (A.8) implies that B has at most $k+2$ facets. Also, since $f \in \text{int}(P) \subseteq P' \subseteq (\mathbb{R}^{n-1} \times [-1, 1])$, Equation (3.25) follows.

In Case 2, it is left to consider when both M_1 and M_{-1} are facets of P'. We move the proof of the following claim to after the main proof.

Claim 17. *P' is a truncated pyramid with bases M_1 and $M_{-1} = (1+\alpha')M_1 + p'$ for $\alpha' > 0$ and $p' \in \mathbb{R}^n$.*

Using Claim 17 and Lemma 3(a), we can write $f = (1+\mu\alpha')x + \mu p'$ for $x \in M_1$, $\mu \in [0, 1]$, $p' \in \mathbb{R}^n$, and $\alpha' > 0$. Observe that

$$f_n = (1+\mu\alpha')x_n + \mu p'_n = \mu[(1+\alpha')x_n + p'_n] + (1-\mu)x_n = -\mu + (1-\mu) = 1 - 2\mu. \tag{A.9}$$

From (i), we get the inclusion $\frac{1}{4}P + \frac{3}{4}f \subseteq (\mathbb{R}^{n-1} \times [-1, 1])$. From this inclusion and assumption (iv), there exist $x, y \in \frac{1}{4}P + \frac{3}{4}f$ so that $x_n \in (0, 1)$ and $y_n \in (-1, 0)$. This implies the inequalities $\frac{1}{4}\lambda + \frac{3}{4}f_n > 0$ and $\frac{1}{4}\gamma + \frac{3}{4}f_n < 0$. Using $\lambda \in [0, 1]$ and $\gamma \in [-1, 0]$ from (i), these inequalities can be rearranged to show $f_n \in \left(-\frac{1}{3}, \frac{1}{3}\right)$. Applying these bounds on of f_n to Equation (A.9), we get $\mu \in \left(\frac{1}{3}, \frac{2}{3}\right)$. From Lemma 3(b), the set $B := \text{conv}(\{x\} \cup ((1+\alpha')M_1 + p'))$ satisfies Equation (3.25). Note B has at most $k+2$ facets since $(1+\alpha')M_1 + p'$ has at most $k+1$ facets. Equation (3.26) holds as $P \subseteq P'$ is convex. □

Proof of Claim 17.

Note that truncated pyramids are preserved under invertible linear transformations and translations. Therefore we may apply an invertible linear transformation so that assume $M = S \oplus U \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k-1}$ and $p = -e_n$. The projection of P onto $\mathbb{R}^k$ is a translated cone, since S is a simplex. Furthermore, the projections of $P \cap (\mathbb{R}^{n-1} \times \{\lambda\})$ and $P \cap (\mathbb{R}^{n-1} \times \{\gamma\})$ are two cross-section of this cone, as are the projections of M_1 and M_{-1}. Since $-1 \le \gamma < \lambda \le 1$ and $\alpha > 0$, it follows that M_{-1} is of the form $(1+\alpha')M_1 + \beta p$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.

□

A.3 Proof of Proposition 16

Proof of Proposition 16. Let π be a minimal lifting of ψ. Assume to the contrary that π is not periodic along W_S. Therefore, there exists some $\hat{p} \in \mathbb{R}^n$ and $w \in W_S$ such that $\pi(\hat{p}) \neq \pi(\hat{p} + w)$. Since $-w \in W_S$, we may assume $\pi(\hat{p}) > \pi(\hat{p}+w)$. Define a function $\tilde{\pi} : \mathbb{R}^n \to \mathbb{R}$ by $\tilde{\pi}(p) = \pi(\hat{p}+w)$ if $p = \hat{p}$, and $\tilde{\pi}(p) = \pi(p)$ otherwise. If $\tilde{\pi}$ is a lifting of ψ, then we will have π is not minimal, yielding a contradiction. Hence, it is sufficient to show that $\tilde{\pi}$ is a lifting of ψ.

Take $k, l \in \mathbb{Z}_+, R \in \mathbb{R}^{n \times k}$, and $P \in \mathbb{R}^{n \times l}$. We must show that (6.8) holds for all $(s, y) \in M_S(R, P)$, so take $(s, y) \in M_S(R, P)$. Note that the columns of P may be taken to be distinct by adding the components of y that correspond to equal columns. Consider three cases.

Case 1: Suppose that P does not contain $\hat{p}$ as one of its columns. Then

$$\sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{\ell} \tilde{\pi}(p_j)y_j = \sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{\ell} \pi(p_j)y_j \ge 1,$$

where the inequality arises since π is a lifting of ψ.

Case 2: Suppose that P contains $\hat{p}$ as one of its columns, but not $\hat{p}+w$. Let P^o and y^o be the columns and values of P and y, respectively, that do not correspond to $\hat{p}$. Let $y_{\hat{j}}$ be the component of y corresponding to $\hat{p}$. Using the definition of W_S and the fact that $y_{\hat{j}} \in \mathbb{Z}_+$, it follows that

$$Rs + Py = Rs + P^o y^o + \hat{p} y_{\hat{j}} \in S \iff Rs + P^o y^o + \hat{p} y_{\hat{j}} + w y_{\hat{j}} = Rs + P^o y^o + (\hat{p}+w) y_{\hat{j}} \in S.$$

If we define $P' \in \mathbb{R}^{n \times k}$ to be the columns of P^o adjoined with $\hat{p}+w$, then the equivalence above implies

$$\sum_{i=1}^{k} \psi(r_i) s_i + \sum_{j=1}^{\ell} \tilde{\pi}(p_j) y_j = \sum_{i=1}^{k} \psi(r_i) s_i + \sum_{j=1, j \neq \hat{j}}^{\ell} \pi(p_j) y_j + \pi(\hat{p}+w) y_{\hat{j}} \geq 1,$$

where the inequality arises since π is a lifting of ψ and we can apply the cut-generating pair (ψ, π) to $(s, (y^o, y_{\hat{j}})) \in M_S(R, P')$.

Case 3: Suppose that P contains $\hat{p}$ and $\hat{p}+w$ as columns. Using a similar argument as above, define P' to be the columns of P without $\hat{p}$. This yields the same inequality as Case 2. $\square$

A.4 Proof of Theorem 19

Proof of Theorem 19. We show one direction as the other follows from swapping the roles of f and $1-f$. Suppose that π is minimal for $R_f(\mathbb{R}, \mathbb{Z})$ with $f \in (0, 1/2]$. Define $\tilde{\pi}(x) := \pi(-x)$. We check that $\tilde{\pi}$ is minimal using Theorem 4. Observe that $\tilde{\pi}$ is nonnegative since π is. If $w \in \mathbb{Z}$ then so is $-w$, and therefore $\tilde{\pi}(w) = \pi(-w) = 0$ since π is minimal. Let $x, y \in \mathbb{R}$ and note that

$$\tilde{\pi}(x+y) = \pi(-x-y) \leq \pi(-x) + \pi(-y) = \tilde{\pi}(x) + \tilde{\pi}(y),$$

where the inequality follows from the subadditivity of π. Hence $\tilde{\pi}$ is subadditive. Finally, let $r \in \mathbb{R}$ and note that

$$\tilde{\pi}(r) + \tilde{\pi}((1-f)-r) = \pi(-r) + \pi(f-(1-r)) = \pi(-r) + \pi(f-(-r)) = 1,$$

where the second equation follows from the periodicity of π and the third equation from the symmetry of π. Hence $\tilde{\pi}$ is symmetric about $1-f$. From Theorem 4, $\tilde{\pi}$ is minimal.

Now assume that π is extreme. Let θ_1, θ_2 be valid for $R_{1-f}(\mathbb{R},\mathbb{Z})$ such that $\tilde{\pi} = \frac{\theta_1+\theta_2}{2}$. We claim that $\tilde{\theta}_i(r) := \theta_i(-r)$, $i = 1,2$, is a valid function for $R_f(\mathbb{R},\mathbb{Z})$. This would imply $\tilde{\pi} = \theta_1 = \theta_2$ from the extremality of π. Let $y \in R_f(\mathbb{R},\mathbb{Z})$. Then $\tilde{y}(r) := y(-r) \in R_{1-f}(\mathbb{R},\mathbb{Z})$. Note that for $i = 1,2$,

$$\sum_{r\in\mathbb{R}} \tilde{\theta}_i(r)y(r) = \sum_{r\in\mathbb{R}} \theta_i(-r)y(r) = \sum_{r\in\mathbb{R}} \theta_i(-r)\tilde{y}(-r) \geq 1,$$

since θ_i is valid for $R_{1-f}(\mathbb{R},\mathbb{Z})$.

The proof that π is a facet if and only if $\tilde{\pi}$ is a facet is similar. □

A.5 Proof of Proposition 46

Let $S = (f+\Lambda) \cap C$ be a polyhedrally-truncated affine lattice, where $\Lambda \subseteq \mathbb{R}^n$ is a lattice, $f \in \mathbb{R}^n \setminus \Lambda$, $C \subseteq \mathbb{R}^n$ is convex, and $\operatorname{conv}(S)$ is a polyhedron. Let $\{v_1, \ldots, v_n\}$ be a basis of Λ and $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}^n$. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the invertible linear transformation defined by $Av_i = e_i$ for each $i \in [n]$. Set $f' := Af$, $C' := \{Ac : c \in C\}$, and $S' := (f' + \mathbb{Z}^n) \cap C'$. Note that $f' \in \mathbb{R}^n \setminus \mathbb{Z}^n$ because $f \in \mathbb{R}^n \setminus \Lambda$.

We prove Proposition 46 via a sequence of claims.

Claim 18. *$S' = \{As : s \in S\}$.*

Proof of Claim. Consider the vector As for $s \in S$. Since $s \in S = (f+\Lambda) \cap C$, there is some $\lambda \in \Lambda$ so that $s = f + \lambda$. Hence $As = Af + A\lambda = f' + A\lambda \in f' + \mathbb{Z}^n$. Further, $s \in C$ and so $As \in C'$. Thus $As \in S'$ and $S' \supseteq \{As : s \in S\}$. Showing the inclusion $S' \subseteq \{As : s \in S\}$ uses a similar argument, but swapping the roles of S and S'. ◇

Claim 19. conv(S') *is a polyhedron.*

Proof of Claim.

From Claim 18, $\text{conv}(S') = \text{conv}(\{As : s \in S\})$. Furthermore $\text{conv}(\{As : s \in S\}) = \{As : s \in \text{conv}(S)\}$ because A is linear. Since S is a polyhedrally-truncated affine lattice, conv(S) is a polyhedron. Theorem 2 implies that $\text{conv}(S) = Q + K$, where $Q \subseteq \mathbb{R}^n$ is a polytope with vertices in S and K is a polyhedral cone. From Claim 18, $\text{conv}(S') = Q' + K'$, where $Q' = \{Aq : q \in Q\}$ and $K' = \{Ak : k \in K\}$. Hence conv(S') is also a polyhedron. $\diamond$

Claim 20. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ and set $\psi' : \mathbb{R}^n \to \mathbb{R}$ to be $\psi'(x) := \psi(A^{-1}x)$. The function ψ is a (minimal) cut-generating function for C_S if and only if ψ' is a (minimal) cut-generating function for $C_{S'}$.*

Proof of Claim. Assume that ψ is a cut-generating function for C_S. Let $R \in \mathbb{R}^{n\times k}$, where $k \in \mathbb{N}$, and let $s \in C_{S'}(R)$. By definition of $C_{S'}(R)$, it holds that $Rs \in S'$ and so $A^{-1}Rs \in S$. Therefore,

$$\sum_{i=1}^{k} \psi'(r_i)s_i = \sum_{i=1}^{k} \psi(A^{-1}r_i)s_i \geq 1$$

showing that ψ' is a cut-generating function for $C_{S'}$. A similar argument shows that if ψ' is a cut-generating function for $C_{S'}$ then ψ is a cut-generating function for C_S.

The correspondence between cut-generating functions above shows that minimality is preserved between ψ and ψ'.

$\diamond$

Proof of Proposition 46.

Assume ψ be a minimal cut-generating function for C_S. From Claim 20, this happens if and only if $\psi' = \psi \circ A^{-1}$ is a minimal cut-generating function for $C_{S'}$. Using Proposition 13 and Claim 19, this is equivalent to the existence of a maximal S'-free 0-neighborhood $B' = \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i,\ i \in [m]\}$ so that $\psi'(x) = \max_{i\in[m]} a_i \cdot x$. Set $B := \{x \in \mathbb{R}^n : A^T a_i \cdot x \leq 1,\ i \in [m]\}$ is a maximal S-free 0-neighborhood and A^T is the transpose of A. Note that

$$\psi(x) = \psi'(Ax) = \max_{i\in[m]} a_i \cdot (Ax) = \max_{i\in[m]} A^T a_i \cdot x.$$

Hence the existence of a minimal ψ is equivalent to a maximal S-free 0-neighborhood satisfying Equation 5.1 such that ψ satisfies (5.2).

$\square$

A.6 Proof of Proposition 47

Proof of Proposition 47.

(i) Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation and $m \in \mathbb{R}^n$. Note that

$$\begin{aligned} W_{M(S)+m} &= \{w \in \mathbb{R}^n : (M(s)+m) + \lambda w \in M(S)+m, \forall\ s \in S,\ \lambda \in \mathbb{Z}\} \\ &= \{w \in \mathbb{R}^n : M(s) + \lambda w \in M(S), \forall\ s \in S,\ \lambda \in \mathbb{Z}\} \\ &= \left\{w \in \mathbb{R}^n : s + \lambda M^{-1}(w) \in S, \forall\ s \in S,\ \lambda \in \mathbb{Z}\right\} \\ &= \left\{w \in \mathbb{R}^n : M^{-1}(w) \in W_S\right\} \\ &= \{w \in \mathbb{R}^n : w \in M(W_S)\} \\ &= M(W_S). \end{aligned}$$

(ii) Note that

$$\begin{aligned} (x_1, x_2) \in W_{S_1 \times S_2} &\iff (s_1 + \lambda x_1, s_2 + \lambda x_2) \in S_1 \times S_2,\ \forall (s_1, s_2) \in S_1 \times S_2,\ \forall \lambda \in \mathbb{Z} \\ &\iff s_i + \lambda x_i \in S_i,\ \forall i \in \{1,2\},\ \forall s_i \in S_i,\ \forall \lambda \in \mathbb{Z} \\ &\iff (x_1, x_2) \in W_{S_1} \times W_{S_2}. \end{aligned}$$

$\square$

A.7 Proof of Proposition 62

Proof of Proposition 62. Observe that if $B\left(\lambda, p_2^*;\ V_\psi(p_1^*)\right)$ is $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free for some $\lambda > 0$ then it is also maximal $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free. This follows from the characterization of maximal S-free sets given in [19].

Consider the model

$$\left\{(s, y_1, y_2) \in \mathbb{R}^n_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ : \sum_{r \in \mathbb{R}^n} r s_r + p_1^* y_{p_1^*} + p_2^* y_{p_2^*} \in S\right\} \tag{A.10}$$

and note that $(s, y_1, y_2) \in$ (A.10) if and only if (s, y_1, y_2) is in

$$\left\{(s, y_1, y_2) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ : \sum_{r \in \mathbb{R}^n} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} s_r + \begin{pmatrix} p_1^* \\ 1 \\ 0 \end{pmatrix} y_{p_1^*} + \begin{pmatrix} p_2^* \\ 0 \\ 1 \end{pmatrix} y_{p_2^*} \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+ \right\}. \tag{A.11}$$

Claim 21. *Let $\lambda > 0$. If the inequality*

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r + V(p_1^*) y_{p_1^*} + \lambda p_2^* \geq 1 \tag{A.12}$$

is valid for (A.10) *then $B\left(\lambda, p_2^*;\ V_\psi(p_1^*)\right)$ is $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free.*

Proof of Claim 21. Take $(\overline{x}, \overline{x}_{n+1}, \overline{x}_{n+2}) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$. Let $\overline{r} = \overline{x} - \overline{x}_{n+1} p_1^* + \overline{x}_{n+2} p_2^*$, $\overline{z}_1 = \overline{x}_{n+1}$, $\overline{z}_2 = \overline{x}_{n+2}$ and $\overline{s}_r = 1$ if $r = \overline{r}$ and $\overline{s}_r = 0$ otherwise. Note that

$$\sum_{r \in \mathbb{R}^n} r \overline{s}_r + p_1^* \overline{z}_1 + p_2^* \overline{z}_2 = \overline{x} \in S.$$

Since λ is valid for (A.10), it follows that

$$\begin{aligned}
1 &\leq \sum_{r \in \mathbb{R}^n} \psi(r) \overline{s}_r + V_\psi(p_1^*) \overline{z}_1 + \lambda \overline{z}_2 \\
&= \psi(\overline{r}) + V(p_1^*) \overline{x}_{n+1} + \lambda \overline{x}_{n+2} \\
&= \max_{i \in [m]} \left\{ a_i(\overline{x} - \overline{x}_{n+1} p_1^* - \overline{x}_{n+2} p_2^*) + V_\psi(p_1^*) \overline{x}_{n+1} + \lambda \overline{x}_{n+2} \right\} \\
&= \max_{i \in [m]} \left\{ a_i \overline{x} + (V_\psi(p_1^*) - a_i p_1^*) \overline{x}_{n+1} + (\lambda - a_i \cdot p_2^*) \overline{x}_{n+2} \right\}.
\end{aligned}$$

Hence B is $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free. $\square$

The converse of the Claim 21 is also true.

Claim 22. *Let $\lambda > 0$. If $B\left(\lambda, p_2^*;\ V_\psi(p_1^*)\right)$ is $S \times \mathbb{Z}_+ \times \mathbb{Z}_+$ free then* (A.12) *is valid for* (A.10).

Proof of Claim 22. Consider the function

$$\Psi = \max_{i\in[m]} \left\{a_i \cdot r + (V_\psi(p_1^*) - a_i \cdot p_1^*)r_{n+1} + (\lambda - a_i \cdot p_2^*)r_{n+2}\right\}.$$

Take (s, y_1, y_2) satisfying (A.10). From the observation above, (s, y_1, y_2) also satisfies (A.11). Note that $\Psi(r, 0, 0) = \psi(r)$, $\Psi(p_1^*, 1, 0) = V_\psi(p_1^*)$, and $\Psi(p_2^*, 0, 1) = \lambda$. It follows that

$$\begin{aligned}
&\sum_{r\in\mathbb{R}^n} \psi(r)s_r + V_\psi(p_1^*)y_1 + \lambda y_2 \\
=&\sum_{r\in\mathbb{R}^n} \Psi\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} s_r + \Psi\begin{pmatrix} p_1^* \\ 1 \\ 0 \end{pmatrix} y_1 + \Psi\begin{pmatrix} p_2^* \\ 0 \\ 1 \end{pmatrix} y_2 \\
\geq& 1.
\end{aligned}$$

Hence (A.12) is valid for (A.10). $\square$

Suppose λ^* is the argmin of the infimum in (6.21). Note that $\lambda^* \leq V_\psi(p_2^*; p_1^*)$ from Claim 21 and $\lambda^* \geq V_\psi(p_2^*; p_1^*)$ from Claim 22. This gives the desired result. $\square$

A.8 Proof of Theorem 32

We state relevant results, before giving the final proof of Theorem 32 at the end of the section.

The first result is an extension of the so-called 'Collision Lemma' (see Lemma 12). Recall the definition of $B(V_\psi(p^*), p^*)$ in Equation 6.14.

Proposition 68. *Let $B \subseteq \mathbb{R}^n$ be any maximal S-free 0-neighborhood and let $p^* \in \mathbb{R}^n$. Let $(\overline{x}, \overline{x}_{n+1}), (\overline{y}, \overline{y}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)$, and $i_x, i_y \in [m]$ (the index set defining B) such that $(a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*) \cdot (\overline{x}, \overline{x}_{n+1}) = 1$ and $(a_{i_y}, V_\psi(p^*) - a_{i_y} \cdot p^*) \cdot (\overline{y}, \overline{y}_{n+1}) = 1$. Let $(x, k_x) \in R((\overline{x}, \overline{x}_{n+1}); B(V_\psi(p^*), p^*))$ and $(y, k_y) \in R((\overline{y}, \overline{y}_{n+1}); B(V_\psi(p^*), p^*))$ with $k_x, k_y \in$*

$\mathbb{Z}$, $0 \le k_x \le \overline{x}_{n+1}$, and $0 \le k_y \le \overline{y}_{n+1}$. If $x - y \in W_S$ then

$$(a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*) \cdot (x, k_x) = (a_{i_y}, V_\psi(p^*) - a_{i_y} \cdot p^*) \cdot (y, k_y).$$

Furthermore, if $(x, k_x) \in \operatorname{int}(R((\overline{x}, \overline{x}_{n+1}); B(V_\psi(p^*), p^*)))$ and $(y, k_y) \in \operatorname{int}\big(R((\overline{y}, \overline{y}_{n+1}); B(V_\psi(p^*), p^*))\big)$ then $(a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*) = (a_{i_y}, V_\psi(p^*) - a_{i_y} \cdot p^*)$.

Proof of Proposition 68.

Let $(x, k_x) \in R((\overline{x}, \overline{x}_{n+1}); B(V_\psi(p^*), p^*))$ and $(y, k_y) \in R((\overline{y}, \overline{y}_{n+1}); B(V_\psi(p^*), p^*))$. Assume to the contrary that $(a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*) \cdot (x, k_x) < (a_{i_y}, V_\psi(p^*) - a_{i_y} \cdot p^*) \cdot (y, k_y)$ and consider $(\overline{y}, \overline{y}_{n+1}) + (x - y, k_x - k_y)$ (if the inequality is reversed then consider $(\overline{x}, \overline{x}_{n+1}) + (y - x, k_y - k_x)$ instead). Since $x - y \in W_S$ and $k_y \le \overline{y}_{n+1}$, it follows that $(z, z_{n+1}) := (\overline{y}, \overline{y}_{n+1}) + (x - y, k_x - k_y) = (\overline{y} + (x - y), (\overline{y}_{n+1} - k_y) + k_x) \in S \times \mathbb{Z}_+$. We claim that $(z, z_{n+1}) \in \operatorname{int}(B(V_\psi(p^*), p^*))$, contradicting that $B(V_\psi(p^*), p^*)$ is $S \times \mathbb{Z}_+$ free.

The remainder of the proof is identical to that of the Collision Lemma (see Lemma 12).

□

Proposition 69. *Let B be a maximal S-free 0-neighborhood such that $\operatorname{int}(B \cap \operatorname{conv}(S)) \neq \emptyset$. Then any bounded set $U \subseteq \mathbb{R}^n$ intersects a finite number of polyhedra from $\mathcal{X}(B, p^*) + W_S$.*

Proof. Recall that $B \subseteq \mathbb{R}^n$ is a full dimensional set, and by construction, so is $B(V_\psi(p^*), p^*)$. Furthermore, $\operatorname{int}(\operatorname{conv}(S) \cap B) \neq \emptyset$ and therefore, $\operatorname{int}(\operatorname{conv}(S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*)) \neq \emptyset$. Define $\tilde{U} := U \times [0, 1] \subseteq \mathbb{R}^{n+1}$. Note that $\tilde{U}$ is bounded in $\mathbb{R}^{n+1}$, and using Theorem 2.8 in [30], $\tilde{U}$ intersects finitely many polyhedra from $R(B(V_\psi(p^*), p^*)) + W_{S \times \mathbb{Z}_+} = R(B(V_\psi(p^*), p^*)) + W_S \times \{0\}$. Say $\tilde{U}$ intersects $\tilde{P}_i + (w_i, 0)$, where $i \in [k]$, $(w_i, 0) \in W_S \times \{0\}$ and $\tilde{P}_i$ is a polyhedron in $R(B(V_\psi(p^*), p^*))$.

For any $t \in \mathbb{Z}$, Proposition 61 states that the projection of $(\mathbb{R}^n \times \{t\}) \cap (\tilde{P}_i + (w_i, 0))$ onto $\mathbb{R}^n$ is $((\mathbb{R}^n \times \{0\}) \cap \tilde{P}_i)|_{\mathbb{R}^n} + tp^* + w_i$, where $\cdot|_{\mathbb{R}^n}$ denotes the projection onto the first n coordinates. By definition of $\mathcal{X}(B, p^*) + W_S$, all polyhedra in $\mathcal{X}(b, p^*) + W_S$ are of the form $((\mathbb{R}^n \times \{0\}) \cap \tilde{P}_i)|_{\mathbb{R}^n} + tp^* + w_i$ for some t less than a blocking point corresponding to $B(V_\psi(p^*), p^*)$. Notice that, since $\tilde{U}$ is bounded, $(\mathbb{R}^n \times \{t\}) \cap \tilde{U} \cap (\tilde{P}_i + (w_i, 0)) \neq \emptyset$ for only a finite number of integral t, for each $i = 1, \ldots, k$. Hence U only intersects a finite number of polyhedra from $\mathcal{X}(B, p^*)$. $\square$

Proposition 70. *Let B be a maximal S-free 0-neighborhood. For $p^* \in \mathbb{R}^n$, the set $\mathcal{X}(B, p^*) + W_S$ is closed.*

Proof. Let $x \in \mathcal{X}(B, p^*) + W_S$ and consider the open ball $D(x; 1)$. From Proposition 69, $D(x; 1)$ intersects only finite many polyhedra $P_1, \ldots, P_k$ from $\mathcal{X}(B, p^*) + W_S$. Since each P_i is closed, so is the finite union $\cup_{i=1}^k P_i$. Since $x \notin \cup_{i=1}^k P_i$, there exists $\epsilon > 0$ such that the open ball $D(x; \epsilon) \subseteq D(x; 1)$ does not intersect P_i, for $i \in [k]$. Therefore, $D(x; \epsilon) \cap \mathcal{X}(B, p^*) + W_S = \emptyset$. This implies that $\mathbb{R}^n \setminus (\mathcal{X}(B, p^*) + W_S)$ is open, and thus $\mathcal{X}(B, p^*) + W_S$ is closed. $\square$

Let f be as in Theorem 32. For each $i \in [m]$, define $a_i^f := \frac{a^i}{1 + a_i \cdot f}$. Note that

$$B + f = \{r \in \mathbb{R}^n : \ a_i^f \cdot r \leq 1, \forall i \in [m]\}$$

and

$$B(V_\psi(p^*), p^*) + (f, 0) = \left\{(r, r_{n+1}) \in \mathbb{R}^{n+1} : \ a_i^f \cdot r + \left(\frac{V_\psi(p^*) - a_i \cdot p^*}{1 + a_i \cdot f}\right) r_{n+1} \leq 1, \forall i \in [m]\right\}$$

For each $k \in \mathbb{Z}$, $k \geq 0$, and $i \in [m]$ define $T_i^k : \mathbb{R}^n \to \mathbb{R}^n$ to be $T_i^k : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$T_i^k(x) := x + (a_i, V_\psi(p^*) - a_i p^*) \cdot (x, k) f.$$

Proposition 71. *The function $T^{s,k}$ is invertible with the inverse defined by*

$$(T_i^k)^{-1}(x) = x - \left(a_i^f, \frac{V_\psi(p^*) - a_i \cdot p^*}{1 + a_i \cdot f}\right) \cdot (x, k) f.$$

Lemma 18. *Let $(\overline{x}, \overline{x}_{n+1}), (\overline{y}, \overline{y}_{n+1}) \in B(V_\psi(p^*), p^*) \cap (S \times \mathbb{Z}_+)$ and suppose $i_x, i_y \in [m]$ such that $(a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*) \cdot (\overline{x}, \overline{x}_{n+1}) = 1$ and $(a_{i_y}, V_\psi(p^*) - a_{i_y} \cdot p^*) \cdot (\overline{y}, \overline{y}_{n+1}) = 1$. Assume $(z, k_x) \in R((\overline{x}, \overline{x}_{n+1}); B(V_\psi(p^*), p^*)) + (w_x, 0)$ and $(z, k_y) \in R((\overline{y}, \overline{y}_{n+1}); B(V_\psi(p^*), p^*)) + (w_y, 0)$, where $w_x, w_y \in W_S$, $k_i \in \mathbb{Z}_+$, $k_x \leq \overline{x}_{n+1}$, and $k_y \leq \overline{y}_{n+1}$. Then*

$$T_{i_x}^{k_x}(z - w_x, k_x) + w_x = T_{i_y}^{k_y}(z - w_y, k_y) + w_y.$$

Proof. A direct calculation shows that

$$\begin{aligned}
T_{i_x}^{k_x}(z-w_x,k_x)+w_x = &\ (z-w_x)+(a_{i_x},V_\psi(p^*)-a_{i_x}\cdot p^*)\cdot(z,k_x)f+w_x && \text{by definition}\\
= &\ z+(a_{i_x},V_\psi(p^*)-a_{i_x}\cdot p^*)\cdot(z,k_x)f && \\
= &\ z+(a_{i_y},V_\psi(p^*)-a_{i_y}\cdot p^*)\cdot(z,k_y)f && \text{by Proposition 68}\\
= &\ (z-w_y)+(a_{i_y},V_\psi(p^*)-a_{i_y}\cdot p^*)\cdot(z,k_y)f+w_y && \text{by definition}\\
= &\ T_{i_y}^{k_y}(z-w_y,k_y)+w_y &&
\end{aligned}$$

□

Proposition 72. *Let* $z=(\overline{x},\overline{x}_{n+1})\in B(V_\psi(p^*),p^*)$ *and* $z_f=z+(f,0)$. *Consider* $R(w(z);B)+kp^*$ *for* $k\in\mathbb{Z}_+,k\leq N(z)$. *Let* $i_x\in[m]$ *be such that* $(a_{i_x},V_\psi(p^*)-a_{i_x}\cdot p^*)\cdot(\overline{x},\overline{x}_{n+1})=1$. *Then*

$$T_{i_x}^{k}\left(R(w(z);B)+kp^*\right)=R(w(z_f);B+f)+k\left(p^*+V_\psi(p^*)f\right).$$

Proof. Let $y\in R(w(z);B)+kp^*$. By Proposition 61, $(y,k)\in R((\overline{x},\overline{x}_{n+1});B(V_\psi(p^*),p^*))$. Also,

$$T_{i_x}^{k}(y)\in R(w(z_f);B+f)+k\left(p^*+V_\psi(p^*)f\right)$$

if and only if

$$(T_{i_x}^{k}(y),k)\in R(z_f;B(V_\psi(p^*),p^*)+(f,0)).$$

Therefore, we will show that $(T^{s,k}(y),k)\in R(z_f;B(V_\psi(p^*),p^*)+(f,0))$.

We first show that for any $i\in[m]$ such that

$$[(a^i,V_\psi(p^*)-a_i\cdot p^*)-(a_{i_x},V_\psi(p^*)-a_{i_x}\cdot p^*)]\cdot(y,k)\leq 0,$$

it follows that

$$\left(a_i^f,\frac{V_\psi(p^*)-a_i\cdot p^*}{1+a_i\cdot f}\right)\cdot(T_{i_x}^{k}(y),k)\leq a_{i_x}\cdot y+k\left(V_\psi(p^*)-a_{i_x}\cdot p^*\right)$$

with equality for $i = i_x$. Indeed, direct calculation shows that

$$\begin{aligned}
\left(a_i^f, \tfrac{V_\psi(p^*)-a_i\cdot p^*}{1+a_i\cdot f}\right)\cdot(T_{i_x}^k(y),k) = &\quad \left(\tfrac{a^i}{1+a_i\cdot f}, \tfrac{V_\psi(p^*)-a_i\cdot p^*}{1+a_i\cdot f}\right)\cdot(y+(a_{i_x}\cdot y+(V_\psi(p^*)-a_{i_x}\cdot p^*)k)f, k) \\
= &\quad \tfrac{a_i\cdot y+k\left(V_\psi(p^*)-a_i\cdot p^*\right)+(a_{i_x}\cdot y)(a_i\cdot f)+(a_i\cdot f)k\left(V_\psi(p^*)-a_{i_x}\cdot p^*\right)}{1+a_i\cdot f} \\
\leq &\quad \tfrac{a_{i_x}\cdot y+k\left(V_\psi(p^*)-a_{i_x}\cdot p^*\right)+(a_{i_x}\cdot y)(a_i\cdot f)+(a_i\cdot f)k\left(V_\psi(p^*)+a_{i_x}\cdot p^*\right)}{1+a_i\cdot f} \\
= &\quad \tfrac{(1+a_i\cdot f)\left(a_{i_x}\cdot y+k\left(V_\psi(p^*)-a_{i_x}\cdot p^*\right)\right)}{1+a_i\cdot f} \\
= &\quad a_{i_x}\cdot y+k\left(V_\psi(p^*)-a_{i_x}\cdot p^*\right),
\end{aligned}$$

where the inequality arises since $[(a^i, V_\psi(p^*) - a_i \cdot p^*) - (a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*)] \cdot (y, k) \leq 0$. Note that equality holds when $i = i_x$.

Using a similar argument, it follows that for any $i \in [m]$ such that $[(a^i, V_\psi(p^*) - a_i \cdot p^*) - (a_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*)] \cdot (y, k) \leq 0$, it follows that $\left(a_i^f, \tfrac{V_\psi(p^*)-a_i\cdot p^*}{1+a_i\cdot f}\right)\cdot(\overline{x}+f-T_{i_x}^k(y), \overline{x}_{n+1}-k) \leq 1 - (a_{i_x}\cdot y + (V_\psi(p^*) - a_{i_x}\cdot p^*))\, k$ with equality for $i = i_x$.

Since $(y, k) \in R((\overline{x}, \overline{x}_{n+1}); B(V_\psi(p^*), p^*))$, it follows that $[(a^i, V_\psi(p^*) - a_i\cdot p^*) - (a_{i_x}, V_\psi(p^*) - a_{i_x}\cdot p^*)]\cdot(y,k) \leq 0$ for each $i \in [m]$. Applying the arguments to each $i \in [m]$, with equality for $i = i_x$, we see that

$$\left[\left(a_i^f, \frac{V_\psi(p^*) - a_i\cdot p^*}{1 + a_i\cdot f}\right) - \left(a_{i_x}^f, \frac{V_\psi(p^*) - a_{i_x}\cdot p^*}{1 + a_{i_x}\cdot f}\right)\right]\cdot(T_{i_x}^k(y), k) \leq 0$$

and

$$\left[\left(a_i^f, \frac{V_\psi(p^*) - a_i\cdot p^*}{1 + a_i\cdot f}\right) - \left(a_{i_x}^f, \frac{V_\psi(p^*) - a_{i_x}\cdot p^*}{1 + a_{i_x}\cdot f}\right)\right]\cdot(\overline{x}_{n+1} + f - T_{i_x}^k(y), \overline{x}_{n+1} - k) \leq 0$$

for each $i \in [m]$. Hence $(T_{i_x}^k(y), k) \in R(z_f; B(V_\psi(p^*), p^*) + (f, 0))$ and so

$$T_{i_x}^k\left(R(w(z); B) + kp^*\right) \subseteq R(w(z_f); B + f) + k\left(p^* + V_\psi(p^*) f\right).$$

Using similar reasoning applied to $(T^k_{i_x})^{-1}$, we get the reverse inclusion. $\square$

Proof of Theorem 32. We will first show that if $\mathcal{X}(B,p^*) + W_S = \mathbb{R}^n$ then $\mathcal{X}(B+f, p^* + V_\psi(p^*)f) + W_{S+f} = \mathbb{R}^n$. The converse holds by switching the roles of (B,p^*) and $(B + f, p^* + V_\psi(p^*)f)$.

Note that a direct calculation shows that $W_S = W_{S+f}$ (see Proposition 2.1 in [30]). If B is a half-space, then the lifting region is equal to $\mathbb{R}^n$. Note the lifting region is contained in $\mathcal{X}(B,p^*) + W_S$, and therefore $\mathcal{X}(B,p^*) + W_S = \mathcal{X}(B+f, p^* + V_\psi(p^*)f) + W_{S+f} = \mathbb{R}^n$. So assume that B is not a half-space.

Define the map $A : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$A(y) := T^k_{i_x}(y-u) + u, \quad \text{if } y \in R(w(z);B) + kp^* + u,$$

for a blocking point $z = (\overline{x}, \overline{x}_{n+1})$, $k \in \mathbb{Z}_+, k \leq \overline{x}_{n+1}, u \in W_S$, and $\left(a^f_{i_x}, V_\psi(p^*) - a_{i_x} \cdot p^*\right) \cdot (\overline{x}, \overline{x}_{n+1}) = 1$. Since $\mathcal{X}(B,p^*) + W_S = \mathbb{R}^n$, each y is in some $R(w(z);B) + kp^* + u$. Moreover, A is well defined from Lemma 18.

By assumption, $\mathbb{R}^n = \mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$. Therefore,

$$\begin{aligned}
A(\mathbb{R}^n) =& A\left(\mathcal{X}(B, p^*) + W_S\right) \\
=& A\left(\bigcup_{\substack{z \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*),\\ k \in \mathbb{Z}_+, k \leq N(z), u \in W_S}} R(w(z); B) + kp^* + u \right) \\
=& \bigcup_{\substack{z \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*),\\ k \in \mathbb{Z}_+, k \leq N(z), u \in W_S}} A\Big(R(w(z); B) + kp^* + u \Big) \\
=& \bigcup_{\substack{z \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*),\\ k \in \mathbb{Z}_+, k \leq N(z), u \in W_S}} \Big(R(w(z + (f, 0); B + f) + k(p^* + V_\psi(p^*)f) + u \Big) \\
=& \left(\bigcup_{\substack{z \in (S \times \mathbb{Z}_+) \cap B(V_\psi(p^*), p^*),\\ k \in \mathbb{Z}_+, k \leq N(z)}} R(w(z + (f, 0); B + f) + k(p^* + V_\psi(p^*)f) \right) + W_{S+f} \\
=& \mathcal{X}(B + f, p^* + V_\psi(p^*)f) + W_{S+f}.
\end{aligned}$$

The fourth equality follows from Proposition 72. Hence, A maps the translated fixing region to the translated fixing region.

For the time being, suppose that A is injective. From Lemma 10 and Proposition 69, A is continuous. Therefore, the Invariance of Domain Theorem (see [37, 59]) states that A is an open map. As A maps $\mathbb{R}^n$ to the translated fixing region, the translated fixing region is open. From Proposition 70, the translated fixing region is closed, and so the translated fixing region is both open and closed. As the translated fixing region is non-empty and there is only one non-empty open set, we see that the translated fixing region is equal to $\mathbb{R}^n$, and so B' has the fixing covering property. Thus it is sufficient to show that A is injective.

Suppose that $A(y_1) = A(y_2)$ for some $y_1, y_2 \in \mathbb{R}^n$. Let $\alpha := A(y_1) = A(y_2)$. By definition, for $j = 1, 2$, there exists a blocking point $z_j = (\overline{x}^j, \overline{x}^j_{n+1}) \in S \times \mathbb{Z}_+$, $k_j \in \mathbb{Z}_+$ with

$k_j \leq \overline{x}^j_{n+1}$, and $w_j \in W_S$ such that $y_j \in R(w(z_j); B) + k_j p^* + w_j$. Moreover

$$\alpha = A(y_1) = T^{k_1}_{i_{x_1}}(y_1 - w_1) + w_1 = T^{k_2}_{i_{x_2}}(y_2 - w_2) + w_2 = A(y_2).$$

From Proposition 72, we see that

$$\alpha \in \left(R(w(z + (f, 0)); B + f) + w_1\right) \cap \left(R(w(z + (f, 0)); B + f) + w_2\right).$$

According to Lemma 18 applied to $(T^{k_1}_{i_{x_1}})^{-1}$ and $(T^{k_2}_{i_{x_2}})^{-1}$, we see that

$$(T^{k_1}_{i_{x_1}})^{-1}\left(T^{k_1}_{i_{x_1}}(y_1 - w_1) + w_1 - w_1\right) + w_1 = (T^{k_2}_{i_{x_2}})^{-1}\left(T^{k_2}_{i_{x_2}}(y_2 - w_2) + w_2 - w_2\right) + w_2.$$

Applying the definition of each $\left(T^{k_j}_{i_{x_j}}\right)^{-1}$ for $j = 1, 2$ and simplifying the results, we see that $y_1 = y_2$. Hence A is injective. □

A.9 Case Analysis for K_i from Claim 16

We first obtain explicit formulas for the vertices of $K_1, \ldots, K_5$ in terms of $\gamma_1, \gamma_2, \gamma_3, b_1$, and b_2. For sake of exposition, we subtract b from each term.

$$l - b = \Bigg(\frac{-\gamma_3(-2+\gamma_3) + b_1(2+\gamma_1\gamma_3 - \gamma_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}, \frac{-(1+\gamma_1+\gamma_2)(-2+\gamma_3) + b_2(2+\gamma_1\gamma_3-\gamma_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)} \Bigg)$$

$$e_1 - b = \left(\frac{b_1+\gamma_2+b_1\gamma_1\gamma_3-\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, \frac{b_2-(1+\gamma_1+\gamma_2)(-1+\gamma_3)+b_2\gamma_1\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3} \right)$$

$$g - b = \left(1 + b_1 + \frac{(1+\gamma_1)(-\gamma_2+b_1(-1+\gamma_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, 1 + b_2 + \frac{(1+\gamma_1)(-\gamma_2\gamma_3+b_2(-1+\gamma_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3} \right)$$

$$u_0 - b = \Bigg(\frac{2 - b_1\gamma_2(-2+\gamma_3) - \gamma_2\gamma_3 + \gamma_1(2 - b_1\gamma_3 + 2\gamma_2(-1+b_1\gamma_3))}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}, \frac{2+\gamma_3+\gamma_2(2+2b_2-3\gamma_3-b_2\gamma_3)+\gamma_1(2+(-1+b_2)(-1+2\gamma_2)\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)} \Bigg)$$

$$m - b = \Bigg(\frac{2+\gamma_1(2-b_1(-2+\gamma_3)) - b_1\gamma_2(-2+\gamma_3) - \gamma_2\gamma_3}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}, \frac{2+\gamma_3+\gamma_1(2-b_2(-2+\gamma_3)+\gamma_3)+\gamma_2(2+2b_2-3\gamma_3-b_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)} \Bigg)$$

$$i - b = \left(\frac{1+\gamma_1+b_1\gamma_1+b_1\gamma_2-\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, \frac{1+\gamma_1+b_2\gamma_1+\gamma_2+b_2\gamma_2-2\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3} \right)$$

$$j - b = \Bigg(\frac{1+\gamma_1+b_1\gamma_1+b_1\gamma_2-b_1\gamma_1\gamma_3-b_1\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, \frac{1-(1+b_2)(-1+\gamma_3)\gamma_2+\gamma_3+\gamma_1(1+b_2+\gamma_3-b_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3} \Bigg)$$

$$k - b = \left(-\frac{(1+\gamma_1)(-\gamma_2+b_1(-1+\gamma_2\gamma_3))}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, \frac{1+\gamma_1+\gamma_2+\gamma_1\gamma_2\gamma_3-b_2(1+\gamma_1)(-1+\gamma_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3} \right)$$

$$v_0 - b = \Bigg(\frac{\gamma_2(2+2+\gamma_1-\gamma_3)+b_1(2-\gamma_2\gamma_3+\gamma_1(2+\gamma_3-2\gamma_2\gamma_3))}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}, \frac{-(1+\gamma_2)(-2+\gamma_3)+\gamma_1(2-\gamma_3+2\gamma_2\gamma_3)+b_2(2-\gamma_2\gamma_3+\gamma_1(2+\gamma_3-2\gamma_2\gamma_3))}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)} \Bigg)$$

$$c_2 - b = \left(\frac{b_1+\gamma_2-b_1\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, \frac{1+b_2+\gamma_1+\gamma_2-b_2\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3} \right).$$

Proof of Claim 16. To prove this claim, we use the half-space definition for spindles to compute the vertices $R(w(z_4);B)$, $R(w(z_4);B)+p^*$, $R(w(z_5);B)+(1,1)$, $R(w(z_5);B)+p^*$, and $R(w(z_6);B)+p^*$ directly. We then show that the vertices of each hole K_i is a convex combination of vertices in the corresponding translated spindle. For sake of presentation, we provide the vertices translated by $-b$.

$$R(w(z_4); B) - b = \operatorname{conv}\Bigg\{(b_1, b_2),$$
$$\left(\frac{-\gamma_2(-2+\gamma_3) + b_1(2+\gamma_1\gamma_3 - \gamma_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}, \frac{-(1+\gamma_1+\gamma_2)(-2+\gamma_3) + b_2(2+\gamma_1\gamma_3-\gamma_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}\right),$$
$$\left(\frac{1-\gamma_2(1+b_1(-2+\gamma_3)) + \gamma_1(1+b_1-\gamma_2-b_1\gamma_3+b_1\gamma_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3},\right.$$
$$\left.\frac{-(1+\gamma_1)(-1+\gamma_2)\gamma_3 + b_2(\gamma_1+2\gamma_2-\gamma_1\gamma_3-\gamma_2\gamma_3+\gamma_1\gamma_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}\right),$$
$$\left(1-\frac{b_1\gamma_1\gamma_3+\gamma_2(2+2\gamma_1-\gamma_3+b_1(-2+\gamma_3-2\gamma_1\gamma_3))}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)},\right.$$
$$\left.\frac{2+\gamma_3+\gamma_2(2+2b_2-3\gamma_3-b_2\gamma_3)+\gamma_1(2+(-1+b_2)(-1+2\gamma_2)\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}\right)\Bigg\}$$

$$R(w(z_5); B) - b + (1,1) = \operatorname{conv}\Bigg\{(1+b_1, 1+b_2),$$
$$\left(\frac{2+2\gamma_1+2\gamma_2-\gamma_2\gamma_3+b_1(2+2\gamma_2-\gamma_1(-2+\gamma_3)-3\gamma_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)},\right.$$
$$\left.\frac{2+2\gamma_2+\gamma_3-\gamma_2\gamma_3+\gamma_1(2+\gamma_3)+b_2(2+2\gamma_2-\gamma_1(-2+\gamma_3)-3\gamma_2\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}\right),$$
$$\left(\frac{1+\gamma_1+b_1\gamma_2-\gamma_1\gamma_2-\gamma_2\gamma_3+b_1\gamma_1\gamma_2\gamma_3}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}, \frac{1+\gamma_2(1+b_2-2\gamma_3)+\gamma_1(1+(-1+b_2)\gamma_2\gamma_3)}{1+\gamma_1+\gamma_2-\gamma_2\gamma_3}\right),$$
$$\left(1-\frac{b_1\gamma_1\gamma_3+\gamma_2(2+2\gamma_1-\gamma_3+b_1(-2+\gamma_3-2\gamma_1\gamma_3))}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)},\right.$$
$$\left.\frac{2+\gamma_3+\gamma_2(2+2b_2-3\gamma_3-b_2\gamma_3)+\gamma_1(2+(-1+b_2)(-1+2\gamma_2)\gamma_3)}{2(1+\gamma_1+\gamma_2-\gamma_2\gamma_3)}\right)\Bigg\}$$

$$
\begin{aligned}
&R(w(z_4);B) - b + p^* = \operatorname{conv}\Bigg\{(1,2),\\
&\Bigg(b_1 + \frac{b_1\gamma_1\gamma_3 + \gamma_2(2 + 2\gamma_1 - \gamma_3 + b_1(-2 + \gamma_3 - 2\gamma_1\gamma_3))}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&b_2 + \frac{(-1 + (-1 + b_2)\gamma_2)(-2 + \gamma_3) + \gamma_1(2 + (-1 + b_2 + 2\gamma_2 - 2b_2\gamma_2)\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg),\\
&\Bigg(\frac{\gamma_2(2 + \gamma_1 - \gamma_3) - b_1(-1 + \gamma_2)(1 + \gamma_1\gamma_3)}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3},\\
&\frac{-(1 + \gamma_2)(-2 + \gamma_3) - b_2(-1 + \gamma_2)(1 + \gamma_1\gamma_3) + \gamma_1(2 + (-1 + \gamma_2)\gamma_3)}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3}\Bigg),\\
&\Bigg(b_1 + \frac{2 + 2\gamma_1 - \gamma_2\gamma_3 + b_1(-2 - \gamma_1\gamma_3 + \gamma_2\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&b_2 + \frac{2 + 2\gamma_2 + \gamma_3 - 3\gamma_2\gamma_3 + \gamma_1(2 + \gamma_3) + b_2(-2 - \gamma_1\gamma_3 + \gamma_2\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&R(w(z_5);B) - b + p^* = \operatorname{conv}\Bigg\{(0,1)\\
&\Bigg(b_1 + \frac{b_1\gamma_1\gamma_3 + \gamma_2(2 + 2\gamma_1 - \gamma_3 + b_1(-2 + \gamma_3 - 2\gamma_1\gamma_3))}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&b_2 + \frac{(-1 + (-1 + b_2)\gamma_2)(-2 + \gamma_3) + \gamma_1(2 + (-1 + b_2 + 2\gamma_2 - 2b_2\gamma_2)\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg),\\
&\Bigg(b_1 + \frac{\gamma_2(1 + \gamma_1 - b_1(1 + \gamma_1\gamma_3))}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3},\\
&\frac{1 + \gamma_1 + \gamma_2 + \gamma_1\gamma_2\gamma_3 - b_2(1 + \gamma_1)(-1 + \gamma_2\gamma_3)}{1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3}\Bigg),\\
&\Bigg(\frac{(-\gamma_2 + b_1(\gamma_1 + \gamma_2))\gamma_3}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&\frac{2 - \gamma_3 + \gamma_2(2 + (-3 + b_2)\gamma_3) + \gamma_1(2 + (-1 + b_2)\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&R(w(z_6);B) - b + p^* = \operatorname{conv}\Bigg\{\\
&\Bigg(b_1 + \frac{b_1\gamma_1\gamma_3 + \gamma_2(2 + 2\gamma_1 - \gamma_3 + b_1(-2 + \gamma_3 - 2\gamma_1\gamma_3))}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&b_2 + \frac{(-1 + (-1 + b_2)\gamma_2)(-2 + \gamma_3) + \gamma_1(2 + (-1 + b_2 + 2\gamma_2 - 2b_2\gamma_2)\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg),\\
&\Bigg(\frac{2 + \gamma_1(2 - b_1(-2 + \gamma_3)) - b_1\gamma_2(-2 + \gamma_3) - \gamma_2\gamma_3}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&\frac{2 + \gamma_3 + \gamma_1(2 - b_2(-2 + \gamma_3) + \gamma_3) + \gamma_2(2 + 2b_2 - 3\gamma_3 - b_2\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg),\\
&\Bigg(\frac{-\gamma_2(-2 + \gamma_3) + b_1(2 + \gamma_1\gamma_3 - \gamma_2\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&\frac{-(1 + \gamma_1 + \gamma_2)(-2 + \gamma_3) + b_2(2 + \gamma_1\gamma_3 - \gamma_2\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg),\\
&\Bigg(1 - \frac{b_1\gamma_1\gamma_3 + \gamma_2(2 + 2\gamma_1 - \gamma_3 + b_1(-2 + \gamma_3 - 2\gamma_1\gamma_3))}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)},\\
&\frac{2 + \gamma_3 + \gamma_2(2 + 2b_2 - 3\gamma_3 - b_2\gamma_3) + \gamma_1(2 + (-1 + b_2)(-1 + 2\gamma_2)\gamma_3)}{2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3)}\Bigg)\Bigg\}
\end{aligned}
$$

One can verify that the labeled vertices in each $K_i \subseteq K$ are either (1) vertices of the corresponding translated spindle or (2) a convex combination of two such vertices. From both situations, the result follows. □

List of notation

A^*	......	the polar of A
A^B	......	the collection of functions $f : B \to A$
$(A)^{(B)}$	......	the collection of functions $f : B \to A$ with finite support
$A \Diamond B$	......	the coproduct of A and B
$A^\bullet$	......	the smallest prepolar of A
$A + B$	......	the Minkowski sum of A and B
$\mathrm{aff}(A)$	......	the affine hull of A
$\mathrm{bd}(A)$	......	the topological boundary of A
$\mathcal{B}_f$	......	all sets in $\mathcal{B}$ that contain f in the interior
$B(\lambda, p^*)$	......	the set $\{(r, r_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : a_i \cdot r + (\lambda - a_i \cdot p^*) r_{n+1} \leq 1,\ i \in [m]\}$ for some maximal S-free set $B = \{r \in \mathbb{R}^n : a_i \cdot r \leq 1\}$ and $\lambda > 0$
$\mathcal{C}_f$	......	the collection of closed, full-dimensional, convex sets in $\mathbb{R}^n$ with f in the interior
$C(R, f)$	......	the convex hull of the $C_{-f+\mathbb{Z}^n}(R)$
$C_B(R, f)$	......	the intersection cut for (R, f) corresponding to the lattice-free set B
$C_{\mathcal{B}}(R, f)$	......	the intersection cut closure for (R, f) corresponding to the family $\mathcal{B}$ of lattice-free sets
C_S	......	the mixed-integer model for S with continuous variables
$C_S(R)$	......	the mixed-integer set in C_S corresponding to $R \in \mathbb{R}^{n \times k}$
$A \times B$	......	the cartesian product of sets A and B
$\mathrm{cl}(A)$	......	the topological closure of A
$\mathrm{conv}(A)$	......	the convex hull of A

$D(x;\epsilon)$	......	the open ball of radius ϵ centered at x
$\dim(A)$	......	the dimension of A
$\mathcal{F}_{\psi,p^*}$	......	the fixing region of ψ with respect to p^*
I_S	......	the mixed-integer model for S with integer variables
$I_S(P)$	......	the mixed-integer set in I_S corresponding to $P \in \mathbb{R}^{n\times l}$
$\mathrm{int}(A)$	......	the interior of A
$\mathrm{lin}(A)$	......	the lineality space of A
$\mathcal{L}_{\psi,p^*}$	......	the set of all minimal liftings π of π such that $\pi(p^*) = V_\psi(p^*)$
M_S	......	the mixed-integer model for S with continuous and integer variables
$M_S(R,P)$	......	the mixed-integer set in M_S corresponding to $R \in \mathbb{R}^{n\times k}$, $P \in \mathbb{R}^{n\times l}$
$[n]$	......	the set of integers $\{1,\dots,n\}$
$\mathbb{R}_+$	......	the nonnegative real numbers
$\mathbb{R}^n_+$	......	the vectors in $\mathbb{R}^n$ with nonnegative entries
$\mathbb{R}^{n\times k}$	......	the collection of real-valued $n\times k$ matrices
R_ψ	......	the lifting region of a cut-generating function ψ for C_S
$R(s;B)$	......	the spindle corresponding to $s \in B\cap S$
$R(S;B)$	......	the lifting region R_ψ of a cut-generating function ψ that represents a S-free 0-neighborhood B
$R_f(\mathbb{R},\mathbb{Z})$	......	the infinite group problem
$R_f(\mathbb{R}^n,\mathbb{Z}^n)$	......	the n-row infinite group problem
$R_f(G,H)$	......	the group problem corresponding to G and H
$[r_1,...,r_k]$	......	the matrix with columns $r_1,...,r_k$
$\rho(B,L)$	......	the functional measuring approximation strength between B-cuts and L-cuts
$\rho(\mathcal{B},L)$	......	the functional measuring approximation strength between the $\mathcal{B}$-closure and L-cuts
$\rho(\mathcal{B},\mathcal{L})$	......	the functional measuring approximation strength between the $\mathcal{B}$-closure and the $\mathcal{L}$-closure

$\rho_f(B, L)$	......	the functional measuring approximation strength between B-cuts and L-cuts, for a fixed f
$\rho_f(\mathcal{B}, L)$	......	the functional measuring approximation strength between the $\mathcal{B}$-closure and L-cuts, for a fixed f
$\rho_f(\mathcal{B}, \mathcal{L})$	......	the functional measuring approximation strength between the $\mathcal{B}$-closure and the $\mathcal{L}$-closure, for a fixed f
$\mathrm{rec}(A)$	......	the recession cone of A
$\mathrm{relint}(A)$	......	the relative interior of A
ψ_B	......	the gauge function of a 0-neighborhood B
$\mathrm{supp}(f)$	......	the support of a function f
S	......	a nonempty, closed subset of $\mathbb{R}^n$ that does not contain 0
$\hat{W}_S$	......	the set $\{w \in \mathbb{R}^n : s + \lambda w \in S\ , \forall s \in S, \forall \lambda \in \mathbb{Z}_+\}$.
$V_\psi(p^*)$	......	the largest $\lambda \in \mathbb{R}$ that lower bounds $\pi(p^*)$ for all minimal liftings π of ψ
W_S	......	the set $\{w \in \mathbb{R}^n : s + \lambda w \in S\ , \forall s \in S, \forall \lambda \in \mathbb{Z}\}$.
x-neighborhood	......	a set A containing x in the interior
$\mathbb{Z}_+$	......	the set of nonnegative integers
$\mathbb{Z}^n_+$	......	the set of vectors in $\mathbb{R}^n$ with nonnegative, integer entries
$\mathbf{1}_m$	......	the vector of all 1's in $\mathbb{R}^m$

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Curriculum Vitae

Joseph Paat was born in Chicago, Illinois on April 11, 1989. In 2007, Joseph graduated from Mayo High School, in Rochester, MN. He then attended Denison University in Granville, OH, and studied math and computer science. While at Denison, he participated in various activities to foster his interest in math. During the spring of 2010, he spent a semester abroad in Budapest, Hungary at the Budapest Semester in Mathematics. During the summers of 2009 and 2010, he worked with Dr. Lew Ludwig on questions in mathematical knot theory. These experiences, in tandem with a point-set topology course taught by Dr. Ludwig, encouraged Joseph to apply to graduate school.

In the spring of 2011, Joseph graduated from Denison with a Bachelors of Science in Mathematics and a Bachelors of Science in Computer Science; he graduated with *Cum Laude* honors. He next enrolled in the Wake Forest University graduate program in Mathematics. Under the guidance of Dr. Kenneth Berenhaut, he examined aspects of unstable walking. Using statistical and topological techniques, the pair developed means for identifying unstable walking patterns in the human gait cycle. In 2013, Joseph graduated with a Masters of Arts from Wake Forest and enrolled in the Applied Mathematics and Statistics Department at Johns Hopkins University.

At Johns Hopkins University, Joseph quickly gained an interest in combinatorial optimization and integer programming. He was fortunate enough to work alongside Dr. Amitabh Basu, who nurtured his interest in topology and exhibited how it could be applied to questions in integer programming. Joseph's work was generously supported by various fellowships including the GAANN, the Harriet H. Cohen, and the Newman Family Fellowship. In addition to research, he taught Statistical Analysis and Discrete Mathematics, and served as a teaching assistant for half a dozen classes. In 2015, he won the Joel Dean Award for Excellence in Teaching. Starting in the spring of 2017, Joseph will be working as a postdoctoral researcher at the Institute for Operations Research at ETH Zürich in Zürich, Switzerland.