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# ON LOCAL AND GLOBAL PROPERTIES OF CONVEX SETS AND HYPERSURFACES

by

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Abstract

In the first chapters, there is obtained a generalization of a theorem of Tietze characterizing convex sets by local properties. This result is used in the second chapter to prove the main result: Let S be a (sufficiently smooth) isometric immersion of a complete n-dimensional Riemannian manifold $M^n$ in a Euclidean space $E^{n+1}$ with the intrinsic property that the second fundamental form of S is semi-definite at every point and definite at some point. Then S is the boundary of a convex body (although if $M^n$ is not complete, S need not even be locally convex).

The third chapter deals with a number of topics. Examples are given to show that theorems of Hilbert and Weyl, concerning extrema of principle curvatures or of mean and Gauss curvatures at non-umbilic points, are false without suitable smoothness assumptions.


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## INTRODUCTION

Each of the three chapters of this dissertation deals with a connection between local and global properties of a class of subsets of a topological vector space. In Chapters II and III the subsets are hypersurfaces in Euclidean space, but in Chapter I subsets of more general spaces are considered. While the results of the first chapter are of interest by themselves, they were developed primarily for use in the second chapter. The last chapter is logically independent of the other two.

The main results of Chapter I, *Local characterizations of convexity*, are two theorems which show that under certain conditions the global property of the convexity of a set can be deduced from a local property. These results generalize a theorem of Tietze [25]. One of the theorems of the first chapter is applied in Chapter II, *On hypersurfaces with a semi-definite second fundamental form*. Here, it is shown that if an n-dimensional manifold is immersed in Euclidean $(n + 1)$-space, the local property (which is independent of the immersion) that the second fundamental form be semi-definite suffices in certain situations to prove that the immersion is the boundary (or part of the boundary) of a convex set. The main theorem asserts roughly that if a hypersurface with this local property is complete, then it bounds an $(n + 1)$-dimensional convex body. This theorem is of the same type as a classical theorem of Hadamard [7] dealing with compact surfaces of positive curvature and its generalizations due to Stoker [23], Van Heijenoort [26], and Chern and Lashof [3].

Chapter III, *Maximum principles for partial differential operators and their applications in the theory of surfaces*, falls into two parts as the title suggests. The first part is concerned with theorems on the validity of weak maximum principles for a non-hyperbolic partial differential operator. These results generalize some of the theorems of an earlier work of Hartman and the author [8] and at the same time the proofs are simpler than the proofs of the corresponding theorems of [8]. Several counter-examples are given to show that in some cases a non-hyperbolic partial differential operator need not have a weak maximum principle.

The second part of Chapter III is concerned with the application of maximum principles to uniqueness theorems in the theory of surfaces. The possibility of sharpening some of the results of this type due to Aleksandrov [1] is discussed. Finally, some counter-examples are given which show that an older method based on the extrema of curvatures cannot be as effective in proving sharp results on the uniqueness of surfaces as are the methods employing maximum principles.

## Chapter I. Local characterization of convexity

1. Statement of the theorems. Tietze [25] proved that a local condition, "Konvexheit im kleinen" is sufficient for a closed, connected subset of the Euclidean space $E^n$ to be convex. Schoenberg [21] showed that Tietze's theorem remains valid if $E^n$ is replaced by any real or complex normed vector space. Moreover, by combining the arguments of Tietze and Schoenberg, it can be shown that the theorem is correct in even more general spaces, cf. [2], p. 56, ex. 22. In this chapter, theorems of a similar kind are proved in which "Konvexheit im kleinen" is replaced by a weaker condition called "almost convexity".

For the rest of this chapter the term topological vector space should be understood to mean a real, complete, locally convex, Hausdorff, topological vector space. The convex closure of a subset S of a topological vector space will mean the smallest closed convex set containing S. A subset S of a topological vector space will be called almost convex if every point x in S has a neighborhood $U_x$ such that each component of $U_x \cap S$ is convex. Almost convexity is a weaker condition than "Konvexheit im kleinen" which requires that $U_x \cap S$ itself be convex. A point x in S is called an extremal point of S if there do not exist points $x_1$ and $x_2$ in S, distinct from x, such that $x = \lambda x_1 + (1 - \lambda)x_2$ for some $\lambda$, $0 < \lambda < 1$.

The main results of this chapter are the two theorems below.

Theorem I.1. *Let S be a connected, compact, and almost convex subset of a topological vector space. Then S is convex.*

Theorem I.2. *Let C be an open convex subset of a locally compact* *topological vector space. Let S be a connected almost convex subset of C which is closed in C. Then S is convex.*

Since almost convexity is weaker than "Konvexheit im kleinen", Theorem I.1 generalizes the results which can be obtained by Tietze's and Schoenberg's arguments for the case of compact sets in the class of vector spaces considered. For non-compact sets it will remain open whether the restriction that the space be locally compact is essential in Theorem I.2. It would be desirable to remove this restriction, because it is not clear, for instance, that a closed, bounded and almost convex subset of Hilbert space need be convex. The set C is included in the statement of Theorem I.2 to facilitate its application in Chapter II. Usually, one would be interested in the case where C is the whole space.

2. *A lemma.* The proofs of the theorem below depend on the following lemma, which is probably known.

*Lemma I.1. Let W,X,Y be closed, non-void subsets of a Hausdorff space. Suppose $W \cup X \cup Y$ is connected, W is compact, and $X \cap Y$ is empty. Then there is a connected subset of W which intersects both X and Y.*

*Proof.* Let U denote the union of all of the components of W which intersect X and V the union of those which intersect Y. Then $W = U \cup V$ and the conclusion of the theorem is equivalent to the assertion that $U \cap V$ is non-void.

If $U \cap V$ is void, at least one of the sets U,V is not closed, for otherwise $W \cup X \cup Y$ is not connected. Suppose V is not closed and let $x_1$ be a point in $V' - V \subset U$. If N is a neighborhood of $x_1$, let Y(N) denote the intersection of Y with the union of all of the components of V which intersect N. Y(N) is a subset of $V \cap Y$. Let B denote the collection of sets $Y(N)$ obtained as $N$ runs over a fundamental system of neighborhoods of $x_1$. $B$ is a filter base since $Y(N)$ is never empty and $Y(N_1 \cap N_2) \subset Y(N_1) \cap Y(N_2)$. Since $(V \cup V') \cap Y$ is compact, there exists a filter containing $B$ and having a limit $x_2$ in $(V \cap V') \cap Y$.

Now a contradiction to $U \cap V = 0$, $x_1 \in U$, $x_2 \in V$, will be obtained by proving that $x_1$ and $x_2$ are in the same component of $V \cup V'$ and hence in the same component of $W$. If $x_1$ and $x_2$ are not connected in $V \cup V'$, there are disjoint open sets $U_1$ and $U_2$ containing $x_1$ and $x_2$ respectively such that $V \cup V' \subset U_1 \cup U_2$; cf. [28], p. 101, Corollary 1.5. It is clear from the definitions of $x_1$ and $x_2$ that there is a connected subset of $V$ which intersects $U_1$ and $U_2$. This is impossible because $U_1$ and $U_2$ are open and disjoint. This shows that $x_1$ and $x_2$ must be connected in $V \cup V'$ and in $W$, which proves the lemma.

3. <u>Proof of Theorem I.1</u>. For each $x$ in $S$, let $U_x$ be a fixed neighborhood such that all components of $U_x \cap S$ are convex. Suppose that $x$ and $y$, $x \neq y$, are arbitrary points of $S$. Let $\Omega$ denote the collection of subsets $S_1$ of $S$ with the properties: (i) $S_1$ contains $x$ and $y$, (ii) $S_1$ is compact, (iii) $S_1$ is connected, (iv) for every point $z$ in $S_1$, all of the components of $S_1 \cap U_z$ are convex. Then $\Omega$ is not empty since $S$ itself is in $\Omega$. The collection $\Omega$ contains a minimal elements, that is, an element $S_0$ such that $S_0$ does not contain any other element of $\Omega$ properly. This assertion follows from Zorn's lemma, since if $\Omega^*$ is a subcollection of $\Omega$ which is linearly ordered by set inclusion, the intersection of all of the elements of $\Omega^*$ is in $\Omega$.

Let $T$ denote the convex closure of a minimal set $S_0$. $T$ is compact; cf. [2], p. 81, corollarie. If $T$ is the segment $xy$, the proof is complete.

Otherwise, T contains an extremal point z distinct from x and y, by the Krein-Milman theorem; cf. [2], p. 84. $S_0$ contains z; cf. [2], p. 84, proposition 4. The neighborhood $U_z$ contains a convex neighborhood N of z, which contains neither x nor y. Let U denote the convex closure of T - N. It follows that U does not contain z; cf. [2], p. 84, proposition 4. Let $V = S_0 \cap U$. Then x and y are not in the same component of V, because a component of V containing x and y would be an element of $\Omega$ and this is impossible by the minimality of $S_0$ and $V \subsetneqq S_0$. Let X and Y be disjoint closed sets such that $x \in X$ and $y \in Y$ and $V = X \cup Y$. Let $W = S_0 \cap N$. Then $S_0 = W \cup X \cup Y$. Lemma I.1 shows that W contains a component Q which intersects both X and Y. Q is convex, since N is convex and $N \subset U_z$. Hence $Q \cap U \subset V = X \cup Y$ is convex, hence connected, and intersects X and Y. But this is impossible because X and Y are closed and disjoint. This completes the proof of Theorem I.1.

4. Proof of Theorem I.2. First it will be shown that there is a sequence $D_1, D_2, \ldots$ of convex, compact subsets of C such that

$$C = \bigcup_{n=1}^{\infty} D_n \text{ and } D_n \subset \text{Interior } D_{n+1}, \; n = 1,2,\ldots$$

It can be supposed that C contains the origin. Let D be a compact neighborhood of the origin and let $\lambda > 1$ be a real number such that $Q \subset D \subset \lambda Q$, where Q is an open convex neighborhood of the origin. Denote by $D_{n-1}$, $n = 2,3,\ldots$ the convex closure of the set $\lambda^n D \cap (1 - 1/n)C$. The sets $D_n$ are compact, since they are closed subsets of $\lambda^n D$. The remaining assertions about the sets $D_n$ follow from the relations $D_{n-2} \subset \lambda^n Q \cap (1 - 1/n)C \subset D_{n-1}$.

Let $S_n = D_n \cap S$. Each set $S_n$ is compact, hence by Theorem I.1 each component of $S_n$ is convex. To prove Theorem I.2, it therefore suffices to show that $S_n$ in connected (hence convex) for $n = 1,2,\ldots$ .

Suppose $S_m$ is not connected for some m. It can be supposed that $S_1$ is not connected. Suppose $S_1 = A_1 \cup B_1$ where $A_1$ and $B_1$ are disjoint non-void closed sets. Then it follows that $S_2$ has a decomposition $S_2 = A_2 \cup B_2$, where $A_1 \subset A_2$, $B_1 \subset B_2$, and $A_2$ and $B_2$ are closed and disjoint. Otherwise there is a component R of $S_2$ which intersects $A_1$ and $B_1$, and since R and $D_1$ are convex, $R \cap D_1 \subset S_1$ is convex, hence connected. This contradicts the definition of $A_1$ and $B_1$. Therefore there exist closed sets $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ such that $A_n \subset A_{n+1}$, $B_n \subset B_{n+1}$, $n = 1,2,\ldots$ and $A_n \cap B_n = 0$.

Let $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $S = A \cup B$ where A and B are disjoint and non-void. A and B are closed, because if, for example, $x \in B$, then $x \in B_k$ for some $k \geqq 1$. Hence, $x \in$ Interior $(D_{k+1})$ and therefore x is not in the closure of $A = \bigcup_{n \geqq k+1} A_k$. This shows that A and B are closed, hence S is not connected, contrary to hypothesis. This completes the proof.

5. A weaker form of local convexity. There is a notion of local convexity which is still weaker than almost convexity. Call a subset S of a topological vector space semi-convex if each point x of S has a neighborhood $U_x$ such that the component of $U_x \cap S$ containing x is convex. The proofs of the theorems above fail if semi-convexity replaces almost convexity. Nevertheless, it is possible that the theorems remain correct. For the case of subsets of the plane, $E^2$, a proof will be sketched, but the general case will remain undecided.

Theorem I.3. Let S be a connected, closed, semi-convex subset of $E^2$. Then S is convex.

This theorem is a corollary of the following:

Theorem (Straus and Valentine [24]). Let S be a closed, connected subset of $E^n (n \geq 2)$. Suppose that each point of S is contained in a unique maximal convex subset of S of dimension greater than n - 2. Then S is convex.

Incidentally, the proof given by Straus and Valentine of this theorem appears to require that S be strongly connected, that is that for every pair of points in S, there is a compact connected set containing both points. However, that the theorem is valid as stated may be seen by an argument similar to that used above to prove Theorem I.2.

Theorem I.3 follows from the theorem of Straus and Valentine because, under the conditions of Theorem I.3, each point x of S must be in a unique maximal convex subset of dimension one or two. To show this, let $S_x$ for every x in S denote the union of all subsets of x which are starlike with respect to x. By the hypothesis of semi-convexity, each set $S_x$ contains a convex subset containing x, which can be assumed to be of dimension one or two because S is connected. The proof is completed by showing that $S_x$ itself is convex. This can be done by an argument similar to Tietze's in [25], Section 4; cf. [2], p. 56, Example 22-b.

# Chapter II. On surfaces with a semi-definite second fundamental form

1. Preliminaries. A Riemannian manifold is said to be of class $C^k$ $(k \geq 1)$ if it is of class $C^k$ as a differentiable manifold and in any coordinate system the components of the metric tensor are functions of class $C^{k-1}$. Unless the contrary is stated a manifold will mean a manifold without a boundary. Let $M_1$ and $M_2$ be manifolds of class $C^k$ and of dimensions $m_1$ and $m_2$ $(m_1 < m_2)$ respectively. $M_1$ will be said to be $C^m$-immersed $(m \leq k)$ in $M_2$ if there is a single-valued map $F : M_1 \to M_2$ of class $C^m$ and of rank $m_1$ at every point of $M_1$. Such an immersion will be called isometric if $M_1$ and $M_2$ are Riemannian manifolds and the metric induced on the image $F(M_1)$ as a subset of $M_2$ is the same as the metric induced locally from the metric of $M_1$. If the map F is one-to-one, $M_1$ will be said to be $C^m$-imbedded in $M_2$. If the map F is one-to-one and open, that is, if the image of every open subset of $M_1$ is an open subset of $F(M_1)$, the imbedding will be said to be open. In this case, $M_1$ and $F(M_1)$ are homeomorphic. In the special case in which $M_1$ is of dimension m and $M_2 = E^{m+1}$, the image $F(M_1)$ under an immersion will be called an m-hypersurface, or if m = 2, a surface.

The first part of this chapter is concerned with the properties of hypersurfaces S satisfying the hypothesis below which will be denoted by (A):

(i) S is an open n + 1 $(n \geq 2)$ times differentiable n-hypersurface imbedded in $E^{n+1}$;

(ii) S is oriented and the second fundamental form of S is semi-definite at every point.

More precisely, (i) means that S is a connected subset of $E^{n+1}$, and if x is any point on S there is an n + 1 times differentiable homeomorphism $x = X(n)$ from the n-cell $\left\{ u = (u^1,\ldots,u^n): \ |u|^2 = \sum_{i=1}^{n} (n^i)^2 < 1 \right\}$ to a subset of S containing x. Furthermore, the n(n + 1) matrix whose i-th row is $X_i = \partial X/\partial u^i$ is required to be of rank n at every point. The unit normal to S at X(u) will be denoted by N(u) and the second fundamental form of S by $-dX.dN = h_{ij}du^i du^j$. Of course, N(u) and $h_{ij}$ are only determined up to multiplication by -1; however, this ambiguity is unimportant because in (ii) it is permissible for the second fundamental form to be either non-negative or non-positive semi-definite. In fact, much of the investigation centers about the fact that, whereas the second fundamental form of a hypersurface satisfying (A) can change type locally (cf. the example after Theorem II.5 in Section below), this cannot happen "in the large" under certain conditions. The main result below, Theorem II.5, is a variant of Stoker's [23] theorem on the nature of a complete surface of positive curvature. Stoker's result is in turn an extension of Hadamard's theorem [7] that a closed surface with positive curvature is an ovaloid.

If S satisfies (A), S is the union of three disjoint sets $H_+, H_0$, and $H_-$, where $H_0$ is the subset of S where $h_{ij} = 0$ for all i,j, $1 \leqq i,\ j \leqq n$ and $H_+[H_-]$ is the subset of S where $-dX.dN$ is non-negative [non-positive] semi-definite.

2. Surfaces of the form z = z(x). In this section let $x = (x^1,\ldots,x^n)$ and let points of $E^{n+1}$ have the coordinates $(x,z) = (x^1,\ldots,x^n,z)$. Suppose S is in $E^{n+1}$, satisfies (A), and is given in the form $z = z(x)$ for x in a domain D. Let $D_+, D_0, D_-$, denote respectively the orthogonal

projections of the sets $H_+, H_0, H_-$, of S on the hyperplane $z = 0$. The first lemma is a corollary of a theorem of Sard [20]; cf. Sion [22].

_Lemma II.1. Let S satisfy_ (A) _and be defined by a function_ $z = z(x)$ _for_ $x = (x^1, \ldots, x^n)$ _in a convex_ x _domain_ D. _Then if_ U _is a component of_ $D_0$, $z(x)$ _is linear on_ U.

_Proof._ The vector grad $z \equiv (z_1(x), \ldots, z_n(x))$ is of class $C^n$. Since the second partial derivatives of z satisfy $z_{ij}(x) = 0$ for $1 \leqq i, j \leqq n$ and x in U, grad z is constant on U by [20], Theorem 6.1, p. 888. Suppose grad $z \equiv (a^1, \ldots, a^n)$ on U and let $F(x) = z(x) - \sum_{i=1}^{n} a^i x^i$. The first partial derivatives of F are identically zero on U, hence by a theorem of A. P. Morse [14], p. 68, F is constant on U. This proves that z is linear on U.

_Remark._ It is only for the purpose of obtaining this lemma that S is assumed to be of class $C^{n+1}$ rather than of class $C^2$ in (A). Note also that the semi-definiteness of $-dX.dN$ is not used in the proof.

_Theorem II.1. Assume_ (A) _and suppose that, in the_ $E^{n+1}$ _space_ $(x^1, \ldots, x^n, z)$, S _can be represented in the form_ $z = z(x^1, \ldots, x^n)$, _where_ z _is defined on a convex_ x-_domain_ D. _Let_ T _be a component of the set_ $D_+$ _(or_ $D_-$_) and let_ U _be a component of_ $D - T$. _Then_ U _is convex._

_Proof._ Let $U_a$ denote any component of $D - T$ and $U_a'$ the boundary of $U_a$. Then $U_a' \cap D$ is a subset of $D_0$. Since D is homeomorphic to $E^n$, $U_a' \cap D$ is connected. This assertion follows from Theorem 14.5, p. 124, and the remark on p. 137 in Newman [16], which show that each component of $D - T$ contains only one component of $T' \cap D = (D - T)' \cap D$; hence $U_a' \cap D = (D - T)' \cap U_a$ is connected.

By Lemma II.1, there is a linear function of x, say $z = z(x,a)$, such that

(1) $z(x,a) \equiv z(x),\ z_j(x,a) \equiv z_i(x)$ on $U_a' \cap D \subset D_0$, $i = 1,\ldots,n$.

Furthermore,

(2) $z_{ij}(x,a) \equiv z_{ij}(x) \equiv 0$ on $U_a' \cap D \subset D_0$, $1 \leqq i,\ j \leqq n$.

Define the function $w = w(x)$ in D by

(3) $w(x) = z(x)$ if $x \in T$ and $w(x) = z(x,a)$ if $x \in U_a$.

It is clear that W is defined and of class $C^2$ on D because of (1) and (2). In view of (3) and $T \subset D_+$, w also satisfies

(4) the matrix $(w_{ij})$ is non-negative definite.

Let $V_a$ denote the convex hull of $U_a$ and let $J(x,a) \equiv w(x) - z(x,a)$ on D. It will first be shown that

(5) $J(x,a) \equiv 0$ on $V_a$.

$J(x,a)$ is a convex function of x because of (4), hence (1)-(4) imply that $J(x,a) \geqq 0$ for all x in D. On the other hand, by a theorem of Caratheodory (cf. [5], p. 35) if $x \in V_a$, $x = \sum_{i=1}^{n} \lambda^i x_i$ where $\sum \lambda^i = 1$, $\lambda^i \geqq 0$, and $x_i \in U_a$ for $i = 1,2,\ldots,n$. The convexity of $J(x,a)$ shows that $J(x,a) \leqq \sum_{i=1}^{n} \lambda^i J(x_i,a) = 0$ because $J(x,a) \equiv 0$ on $U_a$. This proves (5).

Now it will be shown that

(6) $w_{ij} = 0$ on $V_a$ for $1 \leqq i,\ j \leqq n$.

Clearly, because of (4) and $z_{ij}(x,a) \equiv 0$, (6) follows if it is verified that

(7) $J_{jj}(x,a) = 0$ on $V_a$ for $1 \leqq j \leqq n$.

If the dimension of $V_a$ is n, i.e., if $V_a$ has interior points, (7) follows from (5).

Suppose that the dimension of $V_a$ is less than n, then $U_a$ has no interior points and $U_a = U_a' \cap D$. In this case (2) and (3) show that

(6) and (7) hold for x in $U_a$. Let x be a point of $V_a$. Then, as in the proof of (5), $x = \sum_{i=1}^{n} \lambda^i x_i$ for $x_i$ in $U_a$ and suitable $\lambda_i$. Let $\Delta$ be a real number, $\Delta^j$ the n-vector whose j-th component is $\Delta$ and other components are 0. Put

$$F(\Delta) = \sum_{i=1}^{n} \lambda^i (J(x_i + \Delta^j, a) - J(x + \Delta^j, a)).$$

Since $x + \Delta^j = \sum_{i=1}^{n} \lambda^i (x_i + \Delta^j)$ and J(x,a) is convex, $F(\Delta)$ is defined for small $|\Delta|$ and satisfies $F(\Delta) \geqq 0$. By (5), F(0) = 0 so that F has a minimum at $\Delta = 0$. Therefore,

$$F''(0) = \sum_{i=1}^{n} \lambda^i (J_{jj}(x_i, a) - J_{jj}(x, a)) \geqq 0.$$

Hence, $J_{jj}(x,a) \leqq \sum_{i=1}^{n} \lambda^i J_{jj}(x_i, a) = 0$, because $x_i$ is in $U_a$ and (7) holds for $x_i$ in $U_a$. On the other hand, $J_{jj}(x,a) \geqq 0$ because of (4) and $z_{jj}(x,a) \equiv 0$. This proves (7), hence (6).

The conclusion of the theorem now follows. For if x is in T, then $w_{ij}(x) \neq 0$ for some pair i,j. Hence (6) shows that $V_a$ does not intersect T. Therefore $U_a = V_a$ and $U_a$ is convex.

Corollary II.1. *Let S be as in Theorem II.1. Then the components of $D_0$, $D_0 \cup D_+$, and $D_0 \cup D_-$ are convex.*

*Proof.* Let $\{S_a\}$ be the components of $D_+$ and $\{T_{ab}\}$ the components of $D - S_a$. Let U be a component of $D_0 \cup D_-$ and $V = \bigcap \{T_{ab} : U \subset T_{ab}\}$. Clearly, $U \subset V$. It will be shown that U = V.

By Theorem II.1, each set $T_{ab}$ is convex. Hence V is convex and therefore connected. Since $V \subset D_0 \cup D_-$, it follows that $V \subset U$. This proves V = U and U is convex. This completes the proof of the assertion about $D_0 \cup D_-$. The assertions about $D_0$ and $D_0 \cup D_+$ are proved similarly.

3. Surfaces satisfying (A). In what follows $E^{n+1}$ is the space of points $x = (x_1, \ldots, x^{n+1})$. The requirement that S be defined by a function $x^{n+1} = z(x^1, \ldots, x^n)$ will be dropped. A subset C of $E^{n+1}$ will be called a bounding set for a surface S if $S \subset C$ and $S' - S \subset C$. In particular, if the set $S' - S$ is empty, $E^{n+1}$ is a bounding set for S.

Theorem II.2. *Assume (A). Let U be a component of $H_0$. Suppose that S is openly imbedded and has a convex bounding set. Then U is convex.*

Proof. Let $y = (y^1, \ldots, y^{n+1})$ be any point of S. The openness of the imbedding implies that an $x = (x^1, \ldots, x^{n+1})$ coordinate system can be chosen such that all points on S near y, say all points on S in the n-cell $S(y) : \{x : |x^i - y^i| < r(y) \text{ for } i = 1, \ldots, n+1\}$, can be represented in the form $x^{n+1} = z(x^1, \ldots, x^n)$. Corollary II.1 shows that each component of the orthogonal projection of $U \cap S(y)$ onto the hyperplane $x^{n+1} = 0$ is convex. Lemma II.1 shows that every component of $U \cap S(y)$ lies on a hyperplane, hence every component of $U \cap S(y)$ is convex. This shows that the set U is almost convex in the sense of Chapter I. Consequently, Theorem I.2 shows that U itself is convex.

Corollary II.2. *Under the hypotheses of Theorem II.2, the normal N is constant $(N \equiv N(U))$ on U, and U lies in a hyperplane orthogonal to N(U).*

This corollary is a consequence of the proof of Theorem II.2.

Lemma II.2. *Let D be a compact, convex body in $E^n$ $(n \geqq 2)$. Suppose C is a closed, convex subset of D and $C' \cap D'$ is convex. Then D - C is arcwise connected.*

The proof of this lemma is simple and will be omitted.

Let P be a hyperplane in $E^{n+1}$. A surface S will be said to approach P from one side if there is an open half space $P_+$ bounded by P and such that $S \subset P_+$, $S' - S \subset P$, and for every $\epsilon > 0$ the sets of points on S at a distance greater than $\epsilon$ from P is bounded. In particular, $P_+$ will be a bounding set for S.

Theorem II.3. Assume (A). Suppose S is openly imbedded, is homeomorphic to $E^n$, and approaches a hyperplane P from one side. Then one of the sets $H_+, H_-$ is empty.

It can be supposed that P is the hyperplane $x^{n+1} = 0$ and that S lies in the half space $x^{n+1} > 0$. If the theorem is false, there are points $x_+$ in $H_+$ and $x_-$ in $H_-$. Then there is some component U of $H_0$ such that $x_+$ and $x_-$ are in different components of S - U, cf. [16], p. 123, Theorem 14.3, and p. 137. By Theorem II.2, U is convex.

Let $\gamma$ be an arc on S connecting $x_+$ to $x_-$, and let $\epsilon$ satisfy $0 < \epsilon < \min \{x^{n+1} : x = (x^1, \ldots, x^{n+1}) \in \gamma\}$. Suppose Y is the hyperplane containing U and orthogonal to the surface normal N(U) in U. Introduce new coordinates $(y^1, \ldots, y^n, z) = (y, z)$ in $E^{n+1}$ such that Y is the hyperplane $z = 0$. Let $U_\epsilon$ be the intersection of U with the half space $x^{n+1} \geqq \epsilon$. Since $U_\epsilon$ is compact and convex, there is a compact n-dimensional convex subset V of Y such that $U_\epsilon$ is in the interior (as a subset of Y) of V and S can be represented in the (y,z) coordinate system by $z = z(y)$ for $(y,0) \in V$. Finally, let $W = V \cap \{x : x^{n+1} \geqq \epsilon\}$ and let $S_\epsilon$ be the subset of S represented in the (y,z) coordinates by $z = z(y)$ for $(y,0) \in W$. W is clearly convex, compact, and n-dimensional.

$W - U_\epsilon$ is not connected. For let $x_1$ denote the first point of $\gamma$ not in the component of $S - U$ containing $x_+$ and let $x_2$ denote the last point on $\gamma$ not in the component of $S - U$ containing $x_-$. By the choice of $\epsilon$, $U_\epsilon$ contains $x_1$ and $x_2$ and the $(n + 1)$st coordinates of these points are greater than $\epsilon$. Hence there are points $x_3$ and $x_4$ of $S_\epsilon - U_\epsilon$ which preceed $x_1$ and follow $x_2$ respectively on $\gamma$. $x_3$ and $x_4$ cannot be connected in $S_\epsilon - U_\epsilon$, since if this were possible $x_+$ and $x_-$ would be connected in $S - U$. It follows that $W - U_\epsilon$ is not connected.

On the other hand, $U_\epsilon' \cap W' = U_\epsilon \cap \{x : x^{n+1} = \epsilon\}$ is convex. Hence with the identification $D = W$ and $C = U_\epsilon$, Lemma II.2 shows that $W - U_\epsilon$ is connected. This contradiction completes the proof.

4. Lemmas. This section is devoted to some results which will be needed in the proof of Theorem II.5.

Lemma II.3. *Let X be a locally compact metric space and F a local homeomorphism from X to a topological space Y. Suppose M is a compact subset of X such that F is restricted to M is a homeomorphism. Then there is a relatively compact open set O containing M such that F restricted to O is a homeomorphism.*

The proof of this lemma is easy and will be omitted.

Lemma II.4. *Let S be an n-hypersurface of class $C^1$.*

(i) *Then if S is openly imbedded, $E^{n+1} - S$ has at most two components.*

(ii) *If S is homeomorphic to $E^n$ and is a closed subset of $E^{n+1}$, then $E^{n+1} - S$ has at least two components.*

Proof. Suppose that S is openly imbedded. It is easy to verify that this implies that for every point x of S, there is neighborhood B of x such that B - S is homeomorphic to a solid open sphere with its equitorial plane removed.

To verify (i) let $D_1, D_2, \ldots$ be the components of $E^{n+1} - S$. Let $T_i$ be the subset of S consisting of limit points of $D_i$. Clearly $T_i$ is closed in S. By the remark at the beginning of the proof, $T_i$ must also be open in S. Since S is connected each $T_i$ must be either empty or all of S. It follows from the remark at the beginning of the proof that each point of S is in at most two of the sets $T_i$, hence there are at most two non-empty sets $T_i$. This shows that there are at most two non-empty sets $D_i$. This proves (i).

To verify (ii), suppose that S is homeomorphic to $E^n$ and is a closed subset of $E^{n+1}$. If S is a hyperplane it is obvious that S separates $E^{n+1}$, i.e., $E^{n+1} - S$ has at least two components. The proof can therefore be completed by verifying the following proposition.

_A closed subset $S_1$ of $E^{n+1}$ separates $E^{n+1}$ if some subset $S_2$ of $E^{n+1}$ homeomorphic to $S_1$ separates $E^{n+1}$._

To verify this proposition, let $S_1$ and $S_2$ be homeomorphic closed subsets of $E^{n+1}$ and suppose $S_2$ separates $E^{n+1}$. Let $E^{n+1}$ be compactified in the usual way by adding the point $\infty$ and let $S_1{}^*$ and $S_2{}^*$ denote the images of $S_1$ and $S_2$ in $E^{n+1} \cup \{\infty\}$ respectively, where $\infty$ will be considered a point of $S_i{}^*$ if $S_i$ is unbounded. Clearly $S_1{}^*$ and $S_2{}^*$ are homeomorphic and compact. $S_2{}^*$ separates $E^{n+1} \cup \{\infty\}$ because $S_2$ separates $E^{n+1}$. This implies that $S_1{}^*$ separates $E^{n+1} \cup \{\infty\}$ (cf. [11], p. 101, Corollary 3), hence $S_1$ separates $E^{n+1}$. This completes the proof of (ii).

_Corollary II.4. Let S be a hypersurface of class $C^1$ openly_ imbedded in $E^{n+1}$. Suppose S is homeomorphic to $E^n$ and approaches a hyperplane P from one side. Let $P_+$ denote the open half space containing S. Then $P_+ - S$ has exactly two components and every point of S is a boundary point of both components.

Proof. Suppose $P_+ = \{x : x^{n+1} > 0\}$. The proof is completed by applying Lemma II.4 to the image of S under the homeomorphism $x = (x^1,\ldots,x^{n+1}) \to (x^1,\ldots,x^n,\log x^{n+1})$ of $P_+$ to $E^{n+1}$, noting because S is closed in $P_+$, the image of S will be closed in $E^{n+1}$.

Theorem II.4. Assume (A). Suppose S openly imbedded, is homeomorphic to $E^n$, and approaches a hyperplane P from one side. Then S is the part of the boundary of a bounded convex body lying on one side of P.

Proof. Let $P_+$ denote the open half-space containing S and let $D_1$ and $D_2$ denote the two components of $P_+ - S$ which exist by Corollary II.4. These components can be distinguished by the fact that the unit normal N is directed toward one of them, say $D_1$, at every point of S. By Theorem II.3, it can be supposed that the set $H_+$ is empty. Then the set $D_1$ is locally convex, i.e., every point of $D_1'$ is at the center of a solid sphere B such that $D_1 \cap B$ is a subset of a hemisphere in B. This assertion is obvious for points of $D_1'$ which are on P. For other points of $D_1'$, it follows from the nonpositiveness of $-dX.dN$. A theorem of E. Schmidt (cf. [26], p. 241) shows that $D_1$ is convex. Finally by an argument given by Van Heijenoort ([26], p. 231), it can be shown that $D_1$ is bounded. Since $S \subset D_1'$ and approaches P from one side, this completes the proof.

5. The main theorem. A Riemannian manifold becomes a metric space

if the distance between two points is defined to be the greatest lower bound of the lengths of the arcs connecting them. A manifold is said to be complete if the metric space obtained in this way is complete. There are a number of equivalent definitions of completeness, cf. [15]. The image of a complete n-manifold isometrically immersed in $E^{n+1}$ is called a complete n-hypersurface.

The main theorem is of the same type as a theorem of Stoker [23], which has been generalized by Van Heijenoort [26] (and which, in turn, is a generalization of a classical theorem of Hadamard [7]).

*Theorem II.5. Let the hypersurface S be a $C^{n+1}$ ($n \geq 2$) immersion of a complete Riemannian manifold M in $E^{n+1}$. Suppose that the second fundamental form of S is semi-definite at every point of S and is definite at some point of S. Then the immersion is an imbedding, S bounds a convex body, and is homeomorphic to either $S^n$ or $E^n$.*

As usual, $S^n$ denotes the surface of the unit sphere in $E^{n+1}$.

The requirement that the second fundamental form of S be semi-definite (or definite) depends only on M and not on the imbedding. In fact, if local coordinates are chosen such that the second fundamental form is diagonalized at a point, it is clear that the products $h_{ii}h_{jj}$ are the components $R_{ijij}$ of the Riemann-Christoffel tensor.

Theorem II.5 neither contains nor is contained in the theorems of Stoker and Van Heijenoort. It fails to contain their theorems because of its stricter smoothness requirements. Theorem II.5 does not follow from Van Heyenoort's theorem because the local condition that -dX.dN be semi-definite does not imply that S is locally convex, for instance, a cylinder need not be convex. A less trivial example is given by the 2-dimensional surface in the (x,y,z) space $E^3$ defined by $z = x^3(1 + y^2)$ for $-\infty < x < +\infty$, $y^2 < \frac{1}{2}$. This surface has a semi-definite second fundamental form, and in fact the form is positive definite for $x > 0$ and negative definite for $x < 0$. The surface is not locally convex along $x = 0$. It follows from Theorem II.5 that no neighborhood of the origin on this surface can be a part of complete surface of class $C^3$ with non-negative Gaussian curvature.

Aside from smoothness assumptions, the relation of Theorem II.5 to known results is as follows: Hadamard's result [7], p. 352, corresponds to the case where M is compact, $n = 2$, and the second fundamental form is definite (instead of semi-definite). Chern and Lashof [3], p. 6, Theorem 4, generalize Hadamard's result to the case where the second fundamental form is semi-definite. Stoker's theorem [23] generalizes Hadamard's result in another direction by removing the restriction that M is compact but retaining the assumption that the second fundamental form is definite. Finally, Van Heijenoort [26] considers all $n \geqq 2$ and non-compact hypersurfaces, but replaces the assumption that the second fundamental form is semi-definite (and definite at some point) by a local convexity condition. Theorem II.5 contains all of these results when sufficient smoothness is assumed.

Van Heijenoort gives examples which illustrate the role of the hypotheses that S is complete and that the second fundamental form of S is definite at one point.

In the proof of Theorem II.5 and of Lemma II.4 below the immersion map will be denoted by F, $F : M \to S$. If A is a subset of M, F/A and F(A) will denote respectively the map F restricted to A and image of A in S under F. Our proof uses some of the devices of Stoker and Van Heijenoort. However, with the two exceptions noted below a complete proof which is independent of their papers is given.

6. A lemma. The proof of Theorem II.5 depends on the following lemma.

*Lemma II.5. Assume the conditions of Theorem II.5. Suppose that an open non-void subset $N_0$ of M is given with properties that (i) $F/N_0$ is a homeomorphism, (ii) $F(N_0)$ is the part of the boundary of a bounded convex body in an open half-space determined by a hyperplane P. Then there is a closed subset N of M such that, (a) $N \supset N_0$, (b) $F/N$ is a homeomorphism, (c) $F(N)$ is the boundary of a convex body or is the closure of the part of the boundary of a bounded convex body in an open half-space determined by a hyperplane parallel to P; in the latter case, the normal to $F(N)$ is parallel to the normal to P at some point of $F(N')$.*

*Proof.* Suppose $P = \{x : x^{n+1} = 0\}$ $F(N_0) \subset \{x : x^{n+1} < 0\}$, and $F(N_0)$ has $x^{n+1} = z_0$ as a supporting hyperplane. Let $z \geq 0$ and let $N(z)$ denote an open subset of M (if one exists) satisfying (i) $N(z) \supset N_0$, (ii) the map $F/N(z)$ is topological, (iii) there is a compact convex body $B(z)$ which has $x^{n+1} = z$ as a supporting hyperplane and is such that $F(N(z))$ is the part $C(z)$ of the boundary $B'(z)$ in the half-space $x^{n+1} < 0$. Such a set exists for some z because $N_0$ itself satisfies these conditions.

The notation $N(z)$, $B(z)$, $C(z)$ is justified by the fact that these sets are uniquely determined by z if they exist. This assertion is an immediate consequence of the following simple topological proposition, which will be used again in Section 7 below.

(*) Let $O_1$ and $O_2$ be open subsets of a topological space. Suppose $O_1 \cap O_2 \neq 0$, $O_1 \cap O_2' = 0$, and $O_1$ is connected. Then $O_1 \subset O_2$.

To verify (*), note that $O_1 \cap O_2' = 0$ means that $O_1 - O_2$ is open. Then, the three conditions $O_1 \cap O_2 \neq 0$, $O_1$ is connected, and $O_1 = (O_1 \cap O_2) \cup (O_1 - O_2)$ imply that $O_1 - O_2$ is empty, i.e., $O_1 \subset O_2$.

The uniqueness of $N(z)$ follows from (*), for if $N(z_1)$ and $N(z_2)$ both satisfy (i), (ii), (iii) for $z_1 \leqq z_2$, $N(z_1) \subset N(z_2)$ follows from (*) by identifying $N(z_i)$ with $O_i$ for $i = 1,2$. In particular, if $z_1 = z_2$, $N(z_1) = N(z_2)$.

Let $z^*$, $0 \leqq z^* \leqq \infty$ denote the least upper bound of the z values such that the set $N(z)$ satisfying (i), (ii), (iii) exists. Let $N_1 = \bigcup \{N(z) : 0 \leqq z < z^*\}$ B the closure of $\bigcup \{B(z) : 0 \leqq z < z^*\}$ and $C_1 = \bigcup \{C(z) : 0 \leqq z < z^*\}$. Clearly if $N(z)$ exists, $C(z) = C_1 \cap \{x : x^{n+1} < z\}$. This shows that $F/N_1$ is a homeomorphism and $C_1 = F(N_1)$. Also, B is a convex body and $C_1$ is the part of the boundary of B in the half-space $x^{n+1} < z^*$. In particular, if $z^* = \infty$, $C_1$ is the boundary of B.

In case $z^* = \infty$, the proof is complete because $F(N_1) = C_1 = C$ is the boundary of a convex body. Hence $N_1$ is complete and $N \equiv N_1 = M$.

In case $z^* < \infty$, an argument of Van Heijenoort ([26], p. 229 and the remark on p. 241) which we shall not repeat shows that $F/N_1$ can be extended to a homeomorphism from $N \equiv N_1 \cup N_1'$ onto $C \equiv C_1 \cup C_1'$. Let D be the set $B \cap \{x : x^{n+1} = z^*\}$. D is a convex set of dimension smaller than $n + 1$. If the dimension of D is less than n, then $C = B'$ and the proof is complete in this case also.

The proof is now complete except in the case where $z^* < \infty$ and the dimension of D is n. It can be shown by a simple argument that D is bounded; cf. [26], p. 231.

It remains to verify the part of the conclusion of Lemma II.5 beginning "in the latter case ...". If m is a point of M and z(m) is the (n + 1)st coordinate of F(m), the gradient, $\mathrm{grad}_u z \equiv (z_1, z_2, \ldots, z_n)$ can be defined in terms of local coordinates $(u_1, \ldots, u_n)$ on M. If $M_0$ denotes the subset of $N_1$ whose F-image is on the supporting hyperplane $x^{n+1} = z_0$, then grad z $\equiv 0$ on $M_0$. But grad z $\neq 0$ at any point of $N_1 - M_0$, for if grad z = 0 at a point $m_1 \in N_1 - M_0$, $x^{n+1} = z(m_1) < z^*$ would be a supporting hyperplane for B at $F(m_1)$, which is impossible.

The assertion to be verified can now be stated as follows

(1) grad z = 0 at some point of N'.

This will be proved by showing that the falsity of (1) implies that there exists a set W, $N \subset W \subset M$, such that the hypersurface F(W) satisfies the hypotheses of Theorem II.4 with respect to the hyperplane $x^{n+1} = z^* + \delta$ for some $\delta > 0$. The conclusion of Theorem II.4 will then contradict the definition of the set N, hence it will follow that (1) must be true.

Suppose that (1) is false. Consider the system of ordinary differential equations

(2) $du^i/dt = g^{ij}\,\partial z/\partial u^j\,(g^{kl}\,\partial z/\partial u^k\,\partial z/\partial u^l)^{-1}$, $i = 1, \ldots, n$

where $u = (u^1, \ldots, u^n)$ are local parameters on M and the matrix $(g^{ij}(u))$ is the inverse of the metric tensor $(g_{ij}(u))$ of M relative to the u coordinate system. The right side of (2) is defined in a vicinity of any point of N' if (1) is false. (2) are the differential equations for the orthogonal trajectories to z = const..

Locally unique solutions exist through every point $u_0$ of the parameter space. In particular, suppose $u_0$ corresponds to a point $p_0$ on N'. Let $u(t,u_0)$ be the solution of (2) through $u_0$ and let $p(t,u_0)$ be the image of $u(t,u_0)$ in M. The arc $p(t,u_0)$ does not depend on the local coordinates. The compactness of N' and the falsity of (1) imply that there is a $\delta > 0$ such that $p(t,p_0)$ exists for all t $0 \leqq t \leqq \delta$ and all $p_0 \in N'$. The map $p : [0,\delta] \times N' \to M$ is continuous by a standard theorem on the behavior of solutions of differential equations. Let Q denote the image of $[0,\delta] \times N'$ in M. It can be assumed that $\delta$ is so small that if $V = Q \cup N$, then F/V is a homeomorphism. This is a consequence of Lemma II.3.

The map $p : [0,\delta] \times N' \to Q$ is one to one. To see this, suppose that $p(t_1,p_1) = p(t_2,p_2)$ for some $(t_i,p_i) \in [0,\delta] \times N'$, i = 1,2 and $(t_1,p_1) \neq (t_2,p_2)$. Then $p_1 = p_2$ is impossible because dz/dt = 1 along a solution of (2). Therefore $p_1 \neq p_2$. Let $t_3$ denote the smallest t value such that $p(t_3,p_1)$ lies on the arc $p(t,p_2)$. It follows from dz/dt = 1 along a solution of (2) that $p(t_3,p_1) = p(t_3,p_2)$, but $p(t,p_1) \neq p(t,p_2)$ for $0 \leq t < t_3$. This contradicts the local uniqueness of solutions of (1), which proves that p is one to one.

The map p is a homeomorphism, being a one to one continuous map of the compact Hausdorff space $[0,\delta] \times N'$ to Q. It follows that Q is homeomorphic to the closure of the region between two concentric n-spheres. Since $Q \cap N = N'$, $V \equiv Q \cup N$ is homeomorphic to a closed n cell. Let W be the interior of V. W is homeomorphic to $E^n$ as is its image F(W).

The relation $dz/dt = 1$ along a solution of (2) and the definition of W shows that F(W) approaches the hyperplane $x^{n+1} = z^* + \delta$ from the half-space $x^{n+1} < z^* + \delta$. This shows that F(W) satisfies the hypotheses of Theorem II.4 as asserted. This completes the proof of Lemma II.5.

7. <u>Proof of Theorem II.5</u>. Note that a set fulfilling the conditions required of $N_0$ in Lemma II.5 exists. This follows easily from the assumption that $-dX.dN$ is definite at some point of S. For some particular choice of such a set $N_0$, let R denote the set N of the conclusion of Lemma II.5. If F(R) is the boundary of a convex body, R, the isometric image of F(R) is a complete manifold. Since a complete manifold cannot be extended, this proves R = M, and the proof of the theorem is complete in this case.

In the remaining case, F(R) is part of the boundary of a compact convex body which will be denoted by K. In the notation established in the proof of Lemma II.5, F(R) is the part of K in the half-space $x^{n+1} < z^*$. Let D denote as above the convex set $K \cap \{x : x^{n+1} = z^*\}$ and note that $D' = F(R')$. If grad z = 0 on all of R', let $m_1$ be any point of R'. Otherwise let $m_1$ be a point of R' such that grad z = 0 at $m_1$, but grad $z \neq 0$ at some point of R' in every neighborhood of $m_1$. It can be supposed that $x^n = y_0$ is a supporting hyperplane for D at $m_1$, and $D \subset \{x : x^n \geqq y_0\}$. K has supporting hyperplanes $x^n = y_1$ and $x^n = y_2$, where $y_1 < y_0 < y_2$.

Apply Lemma II.5 again, taking as $N_0$ the part of R which maps into $x^n < y_0$. If the set F(N) of the conclusion of Lemma II.5 is the boundary of a convex body, the proof can be completed as before. Therefore suppose F(N) is the part of a compact convex body in $x^n < y^*$.

Clearly $y^* \geqq y_0$. In fact, $y^* > y_0$ because the normal to $F(N_0)$ is not parallel to the $x^n$-axis at any point of $F(N')$ because of $y_2 > y_0$.

Let $Q(y^*)$ denote the set of points p in the interior of R such that $F(p) \in \{x : x^n < y^*\}$. Then $Q(y^*) \subset N_1 \equiv$ interior N; in particular, $m_1$ is an interior point of N because of $y^* > y_0$. This assertion follows from the proposition (*) of Section 6 by taking $O_1 = Q(y^*)$ and $O_2 = N_1$.

Now it will be shown that grad $z = 0$ everywhere on R'. Otherwise, the definition of $m_1$ and $m_1 \in N_1$ imply that there is a point $m_2$ in $R' \cap N_1$ where grad $z \neq 0$. Hence there is a point $m_3 \in N_1$ such that $z(m_3) > z^*$. But this is impossible because $x^{n+1} = z^*$ is a supporting hyperplane for B at $F(m_1)$. This proves grad $z = 0$ everywhere on R'.

It follows that grad $y = 0$ on N' if $y = g(p)$ denotes the n-th coordinate of $F(p)$.

The results above show that $y^* = y_2$. For suppose $y^* > y_2$. Then there is a point m in the interior of R such that $x^n = y_2$ is a supporting hyperplane to K at $F(m)$. But $Q(y^*) \subset N_1$ shows that m is in $N_1$ and $x^n = y_2$ is a supporting plane to B at $F(m)$, which contradicts $y^* > y_2$. Also $y^* < y_2$ is impossible because $Q(y^*)'$ contains no points where grad $y = 0$ if $y^* < y_2$. This is a contradiction because $N' \cap Q(y^*)'$ is obviously non-void. This proves $y^* = y_2$.

Now it is clear that B = K. Let $M_1 = N \cup R$. Then $F(M_1) = B'$. For, on the one hand, $F(M_1) \subset B'$ is clear. On the other hand, suppose $x = (x^1,\ldots,x^{n+1}) \in B'$ and $x \notin F(R)$. Then x is interior to D and $x^n < y_2$. Hence $x \in B' \cap \{x : x^n < y_2\} \subset F(N)$ which proves $F(M_1) = B'$.

$M_1$ is the union of the three disjoint sets $N \cap R$, $N - R$, $R - N$, and $F$ is one-to-one on each of these. Since $F(N \cap R) \subset \{x : x^{n+1} < z^*, x^n < y^*\}$, $F(N - R) \subset \{x : x^n = y^*, x^{n+1} < z^*\}$ and $F(R - N) \subset \{x : x^n < y^*, x^{n+1} = z^*\}$, $F$ is one-to-one on $M_1$. This proves $M_1 = M$ and the proof of the theorem is complete.

8. Applications. First, note that a slightly stronger version of Theorem II.5 has actually been proved. For, the assumption that the isometry $F$ is of class $C^{n+1}$ was only used to prove that $F(M)$ is of class $C^{n+1}$. It would have been sufficient to assume that the isometry is of class $C^2$ and $F(M)$ is of class $C^{n+1}$ as a differentiable manifold.

In view of this remark, the proof of Theorem II.5 has the following corollary.

*Corollary II.5. Let $S_1$ be a $C^2$ n-hypersurface which bounds a convex body in $E^{n+1}$. Let $S_2$ be an n-hypersurface of class $C^{n+1}$ which is a $C^2$ isometric immersion of $S_1$ in $E^{n+1}$. Then $S_2$ bounds a convex body in $E^{n+1}$.*

The statement of Corollary II.5 has meaning even if $S_2$ and the isometry are only continuous. This raises the questions: For which $k$, $1 < k < n + 1$ is Corollary II.5 correct if $S_2$ is of class $C^k$ rather than of class $C^{n+1}$? For which $k$, $0 < k < n + 1$ is Corollary II.5 correct if $S_2$ is of class $C^k$ and the isometry is of class $C^1$? The analogous question is false if $S_2$ and the isometry are only continuous, since, for example, a cap can be cut from a sphere, inverted and replaced. It seems likely that Theorem II.5 and Corollary II.5 are correct if $C^k$ replaces $C^{n+1}$ for $k \geq 2$. On the other hand, the possibility that the statements become false for $k = 1$ is suggested by the results of Kuiper [13] which show that if $n = 2$ imbeddings of class $C^1$ can have surprising properties.

Corollary II.5 can be used to show that in the statements of some theorems, the requirement that a smooth surface be convex is superfluous. This point will be illustrated by a rigidity theorem of Pogorelov (cf. [18]).

Rigidity Theorem. Let $S_1$ be a 2-dimensional surface which bounds a convex body in $E^3$. Suppose $S_1$ has a spherical image $2\pi$. Then, if $S_2$ is a convex surface isometric to $S_1$, $S_2$ is congruent to $S_1$.

If $S_1$ and $S_2$ are required to be of class $C^2$ and $C^3$ respectively and the isometry is of class $C^2$, it is not necessary to assume that $S_2$ is convex or even without self-intersections because these properties follow from Corollary II.5.

## Chapter III. Maximum principles for partial differential operators and their applications in the theory of surfaces

1. Introduction. Let A, B, C, D, and E be defined on an (x,y) domain T with boundary T'. The partial differential operator

(1) $$Lz \equiv Az_{xx} + 2Bz_{xy} + Cz_{yy} + Dz_x + Ez_y$$

will be said to have a weak maximum principle if

(2) $$\max_{T'} z = \max_{T \cup T'} z \equiv m$$

holds for every function $z = z(x,y)$ which is continuous on $T \cup T'$ and is of class $C^2$ on T and satisfies

(3) $$Lz \geqq 0$$

on T.

The first three sections of this chapter are concerned with the proofs of some weak maximum principles which generalize some of the theorems proved by Hartman and the author in [8]. The method used here will differ from that employed in [8] in the respect that maximum principles will be proved somewhat more directly and consequently no analogue of the main local Theorem I, p. 219 in [8] will be obtained or needed. The maximum principles proved here differ from those of [8] mainly in that here the coefficients of the operator L will be permitted to be discontinuous. In Section 4, some examples are given to demonstrate the need for some of the hypotheses made on the operator L.

In Section 5, it is shown that operators with discontinuous coefficients can arise in a rather natural way. The possibility of sharpening some of A. D. Aleksandrov's results in the theory of surfaces is also discussed. The remaining two sections are devoted to theorems of Hilbert and Weyl on the extrema of the curvatures of a surface which have been used to obtain uniqueness theorems in the theory of surfaces. Examples are given to show that Hilbert's and Weyl's theorems fail to hold for surfaces which are not sufficiently smooth. This indicates that maximum principles can give sharper results in the theory of surfaces than can be obtained by consideration of the extrema of the curvatures of a surface.

2. Preliminaries. Let L be an operator of the form (1). A vector $(x',y') \neq (0,0)$ will be said to be in a characteristic direction at a point $(x,y)$ of $T \cup T'$ if there is a sequence of points $\{(x_n,y_n)\}$ in $T \cup T'$ such that

(4) $$\lim (x_n,y_n) = (x,y) \qquad \text{as } n \to \infty,$$

(5) $$\lim [C(x_n,y_n)x'^2 - 2B(x_n,y_n)x'y' + A(x_n,y_n)y'^2] = 0 \text{ as } n \to \infty.$$

The following conditions (i)-(iii) on the $(x,y)$ set T and real valued functions A, B, C, D, and E will be used below and will be referred to as hypothesis (H).

(i) T is a bounded domain with boundary T',

(ii) A, B, C, D, E satisfy

(6) $$\text{l.u.b.} (|A|,|B|,|C|,|D|,|E|) \leqq Q < \infty \text{ on } T$$

for some constant Q,

(7) $$AC - B^2 \geqq 0,$$

(8) $$A \geqq 0 \text{ and } C \geqq 0,$$

(iii) at every point of T there is a vector which is not in a characteristic direction.

Remark. A definition of a characteristic direction which is equivalent to the one given, provided (6) holds, is obtained if (5) above is replaced by

$$\lim [C(x_n,y_n)x_n'^2 - 2B(x_n,y_n)x_n'y_n' + C(x_n,y_n)y_n'^2] = 0 \text{ as } n \to \infty$$

where $(x_1',y_1')$, $(x_2',y_2')$,... is any sequence of vectors satisfying

$$\lim (x_n',y_n') = (x',y') \qquad \text{as } n \to \infty.$$

The equivalence of the two definition follows easily from (6), and will be used several times below.

A point (x,y) of T is called elliptic or parabolic according as there does not or does exist a vector in a characteristic direction at (x,y). Note that, since A, B, C are not assumed to be continuous, the condition $AC - B^2 > 0$ is necessary but not sufficient for a point to be elliptic.

The proofs of the maximum principles depend on a number of simple lemmas.

Lemma III.1. Assume (H). Suppose $(x_0,y_0)$ is a point of $T \cup T'$ and that $(x',y')$ is not in a characteristic direction at $(x_0,y_0)$. Then there is a positive constant $\delta$ such that if (x,y) is a point of $T \cup T'$ satisfying $(x - x_0)^2 + (y - y_0)^2 \leqq \delta$ and the vector $(\xi,\eta) \neq (0,0)$ satisfies $(\xi - x')^2 + (\eta - y')^2 \leqq \delta$, then

$$C(x,y)\xi^2 - 2B(x,y)\xi\eta + A(x,y)\eta^2 \geqq \delta.$$

The proof of Lemma III.1 follows immediately from the Remark following the statement of the hypothesis (H).

The following remark, which is a corollary of Lemma III.1, justifies the definition given of an "elliptic point."

Remark. Assume (H) and let $(x_0,y_0)$ be an elliptic point. Then there is a neighborhood of $(x_0,y_0)$ in which L is strongly elliptic. The last statement means that (6) holds and there is a constant $\delta > 0$

such that

$$A(x,y)\xi^2 + 2B(x,y)\xi\eta + C(x,y)\eta^2 \geqq \delta(\xi^2 + \eta^2)$$

holds for arbitrary $(\xi, \eta)$ if $(x,y)$ is a point of T satisfying $(x - x_0)^2 + (y - y_0)^2 \leqq \delta$.

*Lemma* III.2. *Assume* (H). *Let* S *be a compact subset of* $T \cup T'$ *and suppose there is a continuous vector function* $(x'(x,y), y'(x,y))$ *defined on* S *such that for all* $(x,y)$ *in* S *the vector* $(x'(x,y), y'(x,y)) \neq 0$ *is not in a characteristic direction at* $(x,y)$. *Then there is an analytic vector function* $(\xi(x,y), \eta(x,y))$ *defined on* S *with the same properties*.

*Proof*. For every $n = 1,2,\ldots$ there exist polynomials $\xi_n(x,y)$, $\eta_n(x,y)$ satisfying $(\xi_n(x,y) - x'(x,y))^2 + (\eta_n(x,y) - y'(x,y))^2 \leqq 1/n$ on S. If n is large enough, for all $(x,y)$ in S, the vector $(\xi_n(x,y), \eta_n(x,y))$ is not in a characteristic direction at $(x,y)$. This assertion is proved by a simple argument which employs only the compactness of S and the definition of a characteristic direction. Thus, for some large n, the vector $(\xi_n(x,y), \eta_n(x,y))$ has the properties asserted for $(\xi(x,y), \eta(x,y))$.

*Lemma* III.3. *Assume* (H). *Let* $z(x_0,y_0) = m$ *and let* R *be a closed disk in* T *with* $(x_0,y_0)$ *on its boundary*. *Suppose* $z < m$ *in the interior of* R. *Then* $(x_0,y_0)$ *is a parabolic point and a tangent vector to* R *at* $(x_0,y_0)$ *is in a characteristic direction*.

*Proof*. The assertion that $(x_0,y_0)$ is a parabolic point is a consequence of the remark following Lemma III.1 and the strong maximum principle for strongly elliptic operators; cf. [10]. The assertion concerning the tangent to R at $(x_0,y_0)$ can be proved as Lemma 1, [8],

p. 219, by a modification of the arguments of E. Hopf [10] and Nirenberg [17].

The last lemma is a special case of a theorem of Kamke [12], Satz 1, p. 287.

Lemma III.4. *Suppose that the function* $f_i(x,y)$, $g_i(x,y)$ $(i = 1,2)$ *are of class* $C^k$ $(k \geqq 1)$ *on a simply connected* $(x,y)$-*domain* U *and satisfy*

(10) $\qquad f_1g_2 - f_2g_1 > 0.$

*Then if* S *is a bounded subdomain of* U *whose closure lies in* U, *there are functions* $U(x,y)$, $V(x,y)$ *which are of class* $C^k$ *on* S, *satisfy*

(11) $\qquad f_1(x,y)U_x + g_1(x,y)U_y \equiv 0$

(12) $\qquad f_2(x,y)V_x + g_2(x,y)V_y \equiv 0$

(13) $\qquad \partial(U,V)/\partial(x,y) = U_xV_y - U_yV_x > 0$

*on* S, *and the transformation* $(x,y) \rightarrow (U,V)$ *maps* S *onto its* $(U,V)$-*image in a one-to-one manner.*

3. Maximum principles. The first theorem to be proved is

Theorem III.1. *Assume* (H) *and suppose that some fixed vector* $(x',y')$ *is not in a characteristic direction at any point of* T. *Then* L *has a weak maximum principle on* T.

*Proof.* It can be supposed that $(x',y') = (1,0)$. Suppose the theorem is false and that all points of $T \cup T'$ where $z = m$ are in T. Let $y_0$ be the smallest number such that the line $y = y_0$ contains a point $(x_0,y_0)$ where $z = m$. Then there is a small closed disk R in T tangent to $y = y_0$ at $(x_0,y_0)$ and in the half plane $y \leqq y_0$. R will contain no points where $z = m$ other than $(x_0,y_0)$. Lemma III.3 implies that the tangent to R at $(x_0,y_0)$ is in a characteristic direction, but this contradicts $(x',y') = (1,0)$.

Corollary III.1. Assume (H). Let S be a subdomain of T such that at every point (x,y) of the closure of S there is a vector $(\xi,\eta) \neq 0$ which depends on (x,y) and is not in a characteristic direction. Then there is an $\epsilon > 0$ such that L has a weak maximum principle on any subdomain of S with diameter less than $\epsilon$.

Proof. By Lemma II.1, each point $(x_0,y_0)$ of $S \cup S'$ has a neighborhood $(x - x_0)^2 + (y - y_0)^2 \leqq \delta(x_0,y_0)$ in which some fixed non-trivial vector is not in a characteristic direction. The union of all such neighborhoods covers $S \cup S'$. Let $\epsilon$ be a Lebesgue number for the covering. Then for every subdomain U of S of diameter less than $\epsilon$, there is a vector not in a characteristic direction at any point of U. Theorem III.1 shows that L has a weak maximum principle on U, which completes the proof.

Corollary III.1 can be used to obtain strong maximum principles of the type found in [8], Section 9, for operators with discontinuous coefficients; cf. [8], Section 10.

Theorem III.2. Assume (H). Suppose that T is simply connected and that there exist continuous functions $\xi(x,y)$, $\eta(x,y)$ defined on T and such that, for every (x,y) in T, the vector $(\xi(x,y), \eta(x,y))$ is not a characteristic direction at (x,y). Then L has a weak maximum principle.

Proof. It is clearly sufficient to prove that L has a weak maximum principle on each bounded subdomain S of T such that the closure of S is in T. Let S be such a domain. Lemma III.2 shows that it can be supposed that $\xi(x,y)$ and $\eta(x,y)$ are smooth, say, of class $C^2$ on the closure of a simply connected domain U containing the closure of S.

Apply Lemma III.4, taking $f_1 = g_2 = \xi$ and $-f_2 = +g_1 = \eta$. This gives a one-to-one $C^2$-transformation with non-vanishing Jacobian of S onto a (u,v)-domain S*. Equation (11) becomes

$$\xi(x,y)U_x + \eta(x,y)U_y = 0$$

which implies that

(13) $$(U_x, U_y) = \lambda(x,y)(\eta, -\xi)$$

with $\lambda$ continuous and non-zero on S since $\partial(U,V)/\partial(x,y) \neq 0$. After the transformation $(x,y) \to (u,v)$, Lz becomes

(14) $$L^*z = A^*z_{vu} + 2B^*z_{uv} + C^*z_{vv} + D^*z_u + E^*z_v,$$

where, for instance,

(15) $$A^*(u,v) \equiv A(x,y)U_x^2 + 2B(x,y)U_xU_y + C(x,y)U_y^2.$$

Since $(\xi, \eta)$ is not in a characteristic direction, (13) and (15) imply

(16) $$A^*(u,v) \equiv \lambda^2(x,y)(A\eta^2 - 2B\xi\eta + C\xi^2) > 0.$$

Now it will be verified that the vector (0,1) is not in a characteristic direction at any point (u,v) of S* with respect to L*. Otherwise, there is a sequence of points $(u_1,v_1)$, $(u_2,v_2)$, ... of S* satisfying

(17) $$\lim\, (u_n,v_n) = (u_0,v_0) \in S^* \quad \text{as } n \to \infty$$

and

(18) $$\lim A^*(u_n,v_n) = 0 \quad \text{as } n \to \infty.$$

If $(x_n,y_n)$ is the point of S corresponding to $(u_n,v_n)$, (16) shows that (18) and (17) are equivalent respectively to

(19) $$\lim(x_n,y_n) = (x_0,y_0) \in S^* \quad \text{as } n \to \infty$$

and

(20) $$\lim \lambda_n^2(A_n\eta_n^2 - 2B_n\eta_n\xi_n + C_n\xi_n^2) = 0 \quad \text{as } n \to \infty,$$

where the subscript n on a function means that the argument of the function is $(x_n, y_n)$. By the remark at the end of Section 2 and $\lambda^2(x_0, y_0) > 0$, (19) and (20) show that the vector $(\xi(x_0,y_0), \eta(x_0,y_0))$ is in a characteristic direction at $(x_0,y_0)$. This contradiction proves the assertion that (0,1) is not a characteristic direction at any point of S*.

Let V be a domain with its closure in S, and let V* be the image of V in S*. The operator L* then satisfies the hypotheses of Theorem III.1 on V*, hence L* has a weak maximum principle on V*. This shows that L has a weak maximum principle on V, hence on S and on T. This completes the proof of Theorem III.3.

4. Counterexamples. Let $L_1$ denote the operator

$$L_1 z \equiv y^2 z_{xx} - 2xyz_{xy} + x^2 z_{yy} - xz_x - yz_y.$$

If T is the domain $1 < x^2 + y^2 < 2$, all of the hypotheses of Theorem III.2 are satisfied except the hypothesis that T is simply connected. However, $L_1$ does not have a weak maximum principle on T. For if $f = f(t)$ is a function of class $C^2$ for $1 \leqq t \leqq 2$, then $z \equiv f(x^2 + y^2)$ satisfies $L_1 z \equiv 0$ on T and it is clear that f can be chosen such that $\max_T z > \max_{T'} z$. This shows that the hypothesis that T is simply connected is essential in Theorem III.2.

In the operator

$$L_2 z \equiv g(x^2 + y^2)(z_{xx} + z_{yy}),$$

let $g = g(t)$ be a continuous function on $0 \leqq t \leqq 1$ satisfying $g(1) = 0$, $g(t) > 0$ if $0 \leqq t < 1$. Put

$$Lz \equiv \begin{cases} L_1 z + L_2 z & \text{for } x^2 + y^2 < 1 \\ L_1 z & \text{for } 1 \leqq x^2 + y^2 \end{cases}$$

and let T be the domain $x^2 + y^2 < 2$. Then Lz satisfies all of the hypotheses of Theorem III.2 except the hypothesis that there is a continuous vector function $(\xi(x,y), \eta(x,y))$ such that for every point $(x,y)$ in $T$, $(\xi(x,y), \eta(x,y))$ is not in a characteristic direction at $(x,y)$. $Lz$ fails to satisfy this hypothesis in spite of the fact that it is possible to find a neighborhood of every point of $T \cup T'$ in which continuous vector functions of the required type do exist. $Lz$ fails to have a weak maximum principle, for let

$$z(x,y) = \begin{cases} 0 & \text{if } x^2 + y^2 < 1 \\ f(x^2 + y^2) & \text{if } 1 \leqq x^2 + y^2 \leqq 2 \end{cases}$$

where $f$ is of class $C^2$ and satisfies $f(1) = f'(1) = f''(1) = 0$. Again it is clear that $f$ can be chosen such that $\max_{T'} z < \max_{T} z$.

5. Remarks on the application of the maximum principles. The purpose of this section is to motivate the introduction of some of the hypotheses of the theorems in Section 3.

Let $L_1, \ldots, L_N$ be a finite set of operators of the form

$$L_i z \equiv A_i z_{xx} + 2B_i z_{xy} + C_i z_{yy} + D_i z_x + E_i z_y$$

with continuous coefficients defined on the $(x,y)$ domain $T$. Suppose the conditions

$$A_i C_i - B_i^2 \geqq 0 \quad \text{and} \quad A_i + C_i > 0$$

hold everywhere in $T$ for $i = 1,2,\ldots,N$. If $z$ is a function which is defined and continuous on $T \cup T'$ and of class $C^2$ on $T$, let $k = k(x,y,z)$ denote the smallest integer such that

$$L_k z \geqq L_i z; \quad i = 1,2,\ldots,N$$

holds at $(x,y)$. Then the operator $L \equiv L_k$ is easily seen to satisfy the hypothesis (H) on $T$, hence the maximum principles of Section 3 can be applied to $L$. Thus operators with discontinuous coefficients arise in a rather natural way from operators with continuous coefficients.

Suppose that $\varphi(t_1,\ldots,t_N)$ is a real-valued function which is increasing in each variable and satisfies $\varphi(0,0,\ldots,0) \leqq 0$. Then maximum principles for the non-linear operator $\varphi z \equiv \varphi(L_1 z,\ldots,L_N z)$ can be deduced from the theorems of Section 3. For, $\varphi z \geqq 0$ implies $Lz \geqq 0$, where $L \equiv L_k$ is as in the preceding paragraph. Note that this is true even if $\varphi$ is not differentiable or even continuous.

A. D. Aleksandrov [1] has employed maximum principles for operators of the form $\varphi(L_1 z,\ldots,L_N z)$ to give a very general and unified treatment of uniqueness theorems in the theory of surfaces. However, he requires that $\varphi$ not only be increasing in each variable but also be differentiable. The considerations above suggest that the assumption that $\varphi$ be differentiable may not be essential in his theorems. (He has formulated some theorems in which differentiability of $\varphi$ is omitted, but at the expense of other types of hypothesis.) However, the maximum principles obtained here are not adequate to generalize, or even to obtain, Aleksandrov's theorems in this way. Such generalizations would require strong rather than weak maximum principles.

To illustrate the possibility of extending Aleksandrov's results by employing the remarks above we consider the following known proposition.

(*) Let S be a closed, two-dimensional surface of class $C^2$ with positive Gaussian curvature. Suppose $\varphi(t_1,t_2)$ is monotone increasing in both variables, and the mean and Gaussian curvatures H and K respectively satisfy $\varphi(H,K) \equiv 0$ on S. Then S is a sphere.

Grotemeyer [6] remarked that (*) is a corollary of a theorem of Chern [4] if S is of class $C^4$. Similarly, if S is of class $C^2$,

(*) follows from Pogorelov's sharper version [19] of Chern's result. Alexandrov's methods are not quite adequate to prove (*) directly because $\varphi$ is not assumed to be differentiable. However, it is easy to verify that his methods together with the remarks above on maximum principles for operators $\varphi z = \varphi(L_1 z, L_2 z)$ are sufficient to prove (*).

6. <u>The theorems of Hilbert and Weyl</u>. Let S be a piece of two-dimensional surface of class $C^2$ in $E^3$. If p is a point on S, $k_1(p)$ and $k_2(p)$ will denote the principal curvatures of S at p, which are determined up to a factor $\pm 1$. $H(p) = \frac{1}{2}(k_1 + k_2)$ and $K(p) = k_1 k_2$ will denote respectively the mean and Gaussian curvatures of S at p. S will be called *locally convex* if $K > 0$ everywhere on S and S has no self-intersections. In this case it will be supposed that the normal to S is directed in such a way that $H \geqq K^{\frac{1}{2}} > 0$ and $k_1 \geqq k_2 > 0$. A function $f = f(p)$ defined on S will be said to have a *local maximum [minimum] at $p_0$* if there is a neighborhood U of $p_0$ such that $f(p) \leqq f(p_0)$ $[f(p) \geqq f(p_0)]$ for all p in U.

This section is concerned with the assertions

$(H_n)$ *Let S be a locally convex piece of surface of class $C^n$ $(n \geqq 2)$. Suppose $k_1$ has a local maximum and $k_2$ a local minimum at a point $p_0$ on S. Then in a neighborhood of $p_0$ S is a part of the surface of a sphere.*

$(W_n)$ *Let S be a locally convex piece of surface of class $C^n$ $(n \geqq 2)$. Suppose H has a local maximum and K a local minimum at a point $p_0$ on S. Then, in a neighborhood of $p_0$, S is a part of the surface of a sphere.*

The assertion $(H_4)$ is due to Hilbert [9], Anhang V, p. 238, although he did not explicitly formulate $(H_4)$. The proof fails if $n < 4$ because the existence and continuity of the second derivatives of $k_1$ and $k_2$ are used. Weyl proved $(W_4)$ in [27], p. 72; cf. Chern [3], p. 287, for another proof. Again both proofs fail if $n < 4$. Actually, $(W_n)$ follows from $(H_n)$. In fact, if H has maximum and K has a minimum at a point $p_0$ on S, then $k_1 = H + (H^2 - K)^{\frac{1}{2}}$ has a maximum and $k_2 = K/k_1$ a minimum at $p_0$. Therefore, a counterexample to $(W_n)$ for any n is also a counterexample to $(H_n)$.

Hilbert employed his theorem $(H_4)$ to prove the rigidity of the sphere, and Chern used $(H_4)$ to prove that all "special" Weingarten surfaces are spheres. Grotemeyer's corollary to Chern's theorem cited in Section 5 follows from $(W_4)$ just as Chern's theorem follows from $(H_4)$. Both Chern's theorem and Grotemeyer's corollary are now known to be correct for surfaces of class $C^2$; cf. Pogorelov [19] and Aleksandrov [1]. In view of this, it is somewhat surprising that, as will be shown by the examples in Section 7 below,

(*) <u>the assertions</u> $(H_2)$, $(H_3)$, <u>and</u> $(W_2)$ <u>are false</u>.

It will remain undecided whether or not the assertion $(W_3)$ is correct.

7. <u>Counterexamples</u>. The counterexamples to $(H_3)$ and $(W_2)$ are both surfaces defined by a function of the form

$$z(x,y) = + ax^2 + y^2 - w(x,y,\lambda) + bw(y,x,\lambda) \tag{1}$$

for $x^2 + y^2 < R_0^2$, where

$$w(x,y,\lambda) = \tfrac{1}{2}x^2(x^2 + y^2)^\lambda$$

is of class $C^2$ if $0 < \lambda \leqq \frac{1}{2}$ and of class $C^3$ if $\lambda > \frac{1}{2}$. $R_0, a, b, \lambda$ are positive constants which will be specified more precisely later.

The curvatures of S are given by the formulae

$$\begin{aligned} H &= \tfrac{1}{2}\left\{(1 + q^2)r - 2pqs + (1 + p^2)t\right\}/(1 + p^2 + q^2)^{3/2} \\ K &= (rt - s^2)/(1 + p^2 + q^2)^2 \end{aligned} \tag{2}$$

and

(3) $$k_1, k_2 = H \pm (H^2 - K)^{\frac{1}{2}}$$

where, as usual, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$. It follows easily from (1) - (3) that the curvatures H, K, $k_1$, and $k_2$ are positive for $x^2 + y^2 < R_0^2$ if $R_0$ is sufficiently small. In order to determine whether the functions defined in (2), (3) have maxima or minima at the origin, their partial derivatives with respect to $\rho$ for small $\rho > 0$ will be calculated. Here, $(\rho, \theta)$ are polar coordinates in the $(x,y)$ plane.

A simple calculation shows that

(4) $$p = 2a\rho \cos\theta + O(\rho^{2\lambda+1}), \quad q = 2\rho \sin\theta + O(\rho^{2\lambda+1})$$

and

(5) $$r = 2a - \rho^{2\lambda} f_1(\theta,\lambda,b), \quad s = O(\rho^{2\lambda}), \quad t = 2 - \rho^{2\lambda} f_2(\theta,\lambda,b),$$

where the estimates hold as $\rho \to 0$ uniformly in $\theta, a, b$, for $0 < a, b \leqq$ const., with $\lambda$ fixed. The functions $f_1$ and $f_2$ are the trigonometric polynomials

(6) $$f_1(\theta,\lambda,b) = 1 + 5\lambda\cos^2\theta + 2\lambda(\lambda - 1)\cos^4\theta - b\lambda\sin^2\theta(1 + 2(\lambda - 1)\cos^2\theta)$$
$$f_2(\theta,\lambda,b) = \lambda\cos^2\theta(1 + 2(\lambda - 1)\sin^2\theta) - b(1 + 5\lambda\sin^2\theta + 2\lambda(\lambda - 1)\sin^4\theta).$$

Also, for $\rho \neq 0$,

(7) $$p_\rho = 2a\cos\theta + O(\rho^{2\lambda}), \quad q_\rho = 2\sin\theta + O(\rho^{2\lambda})$$

and

(8) $$r_\rho = -2\lambda\rho^{2\lambda - 1} f_1(\theta,\lambda,b), \quad s_\rho = O(\rho^{2\lambda - 1}), \quad t_\rho = -2\lambda\rho^{2\lambda - 1} f_2(\theta,\lambda,b).$$

It is not difficult to see that, for $\rho \neq 0$, (2) - (8) imply

$$H_\rho = \tfrac{1}{2}(r_\rho + t_\rho) + O(\rho), \quad K_\rho = 2(r_\rho + a t_\rho) + O(\rho) + O(\rho^{4\lambda - 1})$$

and for $a > 1$

$$k_{1\rho} = r_\rho + O(\rho) + O(\rho^{4\lambda - 1}), \quad k_{2\rho} = t_\rho + O(\rho) + O(\rho^{4\lambda - 1}).$$

Hence

(9) $$H_\rho = -\lambda\rho^{2\lambda - 1}(f_1 + f_2) + 0(\rho)$$

(10) $$K_\rho = -4\lambda\rho^{2\lambda - 1}(f_1 + af_2) + 0(\rho) + 0(\rho^{4\lambda - 1})$$

(11) $$k_{1\rho} = -2\lambda\rho^{2\lambda - 1}f_1 + 0(\rho) + 0(\rho^{4\lambda - 1})$$

(12) $$k_{2\rho} = -2\lambda\rho^{2\lambda - 1}f_2 + 0(\rho) + 0(\rho^{4\lambda - 1}).$$

To obtain a counterexample to $(W_2)$, let a,b be fixed, $0 < b < 1$, $ab > 1$. If $\lambda = 0$,

$$f_1 + f_2 \equiv 1 - b > 0 \quad \text{and} \quad f_1 + af_2 = 1 - ab < 0.$$

The forms of $f_1$, $f_2$ show that if $\lambda = \lambda(a,b)$ is sufficiently small, then

(13) $$f_1 + f_2 > 0 \quad \text{and} \quad f_1 + af_2 < 0 \text{ for all } \theta.$$

It can be supposed that $\lambda < 1$. Then (9), (10), and (13) show that H has a relative maximum and K a relative minimum at the origin. A more detailed computation shows that (13) cannot hold unless $\lambda < \frac{1}{2}$, hence a counterexample to $(W_3)$ cannot be found in this manner.

A counterexample to $(H_3)$ is obtained by choosing $a > 1$ and b and $\lambda$ such that $\lambda > \frac{1}{2}$ and

(14) $$f_1 > 0 \quad \text{and} \quad f_2 < 0 \quad \text{for all } \theta.$$

Such a choice is possible because if $\lambda = \frac{1}{2}$ and $\frac{1}{2} < b < 2$ then $f_1 > 1 - b/2 > 0$ and $f_2 < \frac{1}{2} - b < 0$ for all $\theta$. Then (11), (12), and (14) show that $k_1$ has a relative maximum and $k_2$ a relative minimum at the origin. This shows that $(H_3)$ is false.

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## Richard Carl Sacksteder

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Ph.D. The Johns Hopkins University 1960

Dissertation: *On Local and Global Properties of Convex Sets and Hypersurfaces*

Mathematics Subject Classification: 52—Convex and discrete geometry

Advisor: Philip Hartman

Students:
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|---|---|---|---|
| Balsim, Igor | City University of New York | 2001 | |
| Boccio, Dona | City University of New York | 1988 | |
| Evenchick-Berger, Elinor | City University of New York | 1969 | |
| Harris, Jr., Whitney | City University of New York | 1981 | |
| Katzen, Martin | City University of New York | 1969 | |
| Lerohl, Randi | City University of New York | 1993 | |
| Reller, Austin | City University of New York | 1993 | |
| Rothschild, Joseph | City University of New York | 1971 | |
| Rywkin, Richard | City University of New York | 1984 | |
| Saadia-Otero, Marina | City University of New York | 1998 | |
| Sava-Goldstein, Eleanor | City University of New York | 1969 | |
| Tischler, David | City University of New York | 1969 | |
| Tischler, Rosamond | City University of New York | 1971 | |

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