Algebraic geometry over semi-structures and hyper-structures of characteristic one

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Abstract

In this thesis, we study algebraic geometry in characteristic one from the perspective of semirings and hyperrings. The thesis largely consists of three parts:

(1) We develop the basic notions and several methods of algebraic geometry over semirings. We first construct a semi-scheme by directly generalizing the classical construction of a scheme, and prove that any semiring can be canonically realized as a semiring of global functions on an affine semi-scheme. We then develop Čech cohomology theory for semi-schemes, and show that the classical isomorphism $\mathrm{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$ is still valid for a semi-scheme $(X, \mathcal{O}_X)$. In particular, we derive $\mathrm{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^*) \simeq \mathbb{Z}$ when $X = \mathbb{P}^1_{\mathbb{Q}_{max}}$. Finally, we introduce the notion of a valuation on a semiring, and prove that an analogue of an abstract curve by using the (suitably defined) function field $\mathbb{Q}_{max}(T)$ is homeomorphic to $\mathbb{P}^1_{\mathbb{F}_1}$.

(2) We develop algebraic geometry over hyperrings. The first motivation for this study arises from the following problem posed in [9]: if one follows the classical construction to define the hyper-scheme $(X = \mathrm{Spec}\, R, \mathcal{O}_X)$, where R is a hyperring, then a canonical isomorphism $R \simeq \mathcal{O}_X(X)$ does not hold in general. By investigating algebraic properties of hyperrings (which include a construction of a quotient hyperring and Hilbert Nullstellensatz), we give a partial answer for their problem as follows: when R does not have a (multiplicative) zero-divisor, the canonical isomorphism $R \simeq \mathcal{O}_X(X)$ holds for a hyper-scheme $(X = \mathrm{Spec}\, R, \mathcal{O}_X)$. In other words, R can be realized as a hyperring of global functions on an affine hyper-scheme.

We also give a (partial) affirmative answer to the following speculation posed by Connes and Consani in [7]: let $A = k[T]$ or $k[T, \frac{1}{T}]$, where $k = \mathbb{Q}$ or $\mathbb{F}_p$. When $k = \mathbb{F}_p$, the topological space $\mathrm{Spec}\, A$ is a hypergroup with a canonical hyper-operation $*$ induced from a coproduct of A. The similar statement holds with $k = \mathbb{Q}$ and $\mathrm{Spec}\, A \backslash \{\delta\}$, where δ is the generic point (cf. [7, Theorems 7.1 and 7.13]). Connes and Consani expected that the similar result would be true for Chevalley group schemes. We prove that when $X = \mathrm{Spec}\, A$ is an affine algebraic group scheme over arbitrary field, then, together with a canonical hyper-operation $*$ on X introduced in [7], $(X, *)$ becomes a slightly

general (in a precise sense) object than a hypergroup.

(3) We give a (partial) converse of S.Henry's symmetrization procedure which produces a hypergroup from a semigroup in a canonical way (cf. [21]). Furthermore, via the symmetrization process, we connect the notions of (1) and (2), and prove that such a link is closely related with the notion of real prime ideals.

Readers: Dr. Caterina Consani (advisor), Dr. Jack Morava

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The introduction of the cipher 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps ... *Alexander Grothendieck*

This thesis is dedicated to my family with respect and love.

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Introduction

The study of algebraic geometry in characteristic one was initiated from two completely separated motivations; (1) an interaction between algebraic geometry and combinatorics, and (2) an analogy between functions fields and number fields. In what follows, all semirings and hyperrings are assumed to be commutative.

The combinatorial approach to algebraic geometry often makes computations simpler. For example, a toric variety can be fully understood from the combinatorial structure of an associated fan, which is a more tractable object than a variety itself. More recently, it was noticed that one could build (combinatorial) geometry from algebraic geometry by means of a valuation of a ground field, which is known as tropical geometry. One of the main motivations of $\mathbb{F}_1$-geometry stems from such interaction.

The notion of 'the field $\mathbb{F}_1$ of characteristic one' first appeared in Jacques Tits' paper [45]. His goal was to give a geometric interpretation of a (split and semisimple) algebraic group $G(K)$ over an arbitrary field K, which was constructed by C.Chevalley in an algebraic way (cf. [5]). Tits' idea was to associate a projective geometry Γ_K (over K) to $G(K)$ so that $G(K)$ can be realized as a group of automorphisms of Γ_K. In his construction of a projective geometry Γ_K for a finite field $K = \mathbb{F}_q$, Tits observed that even though the algebraic structure of the field K vanishes as $q \to 1$, the projective geometry Γ_K associated to $G(K)$ does not degenerate completely. Thus, he thought that this limiting geometry should be built on the degenerate (mysterious) algebraic structure and referred it to 'the field of characteristic one', which is now known as $\mathbb{F}_1$. This indicates that an algebraic group $G(K)$ contains a (combinatorial) core, and the study of this limiting geometry is closely related to combinatorial geometry via the notion of $\mathbb{F}_1$.

Another (but entirely different) motivation for $\mathbb{F}_1$-geometry arises from the following observation (first appeared in [31]): by finding a proper notion of the geometry over $\mathbb{F}_1$ and by developing relevant tools, one looks for a way to interpret the affine scheme $\operatorname{Spec}\mathbb{Z}$ as 'the curve' over $\mathbb{F}_1$. Then, for example, the surface $C \times_{\mathbb{F}_q} C$, where C is a (smooth, projective) algebraic curve over a finite field $\mathbb{F}_q$, used in the geometric proof of Weil's conjecture for a curve C could be replaced with 'the surface' $\operatorname{Spec}\mathbb{Z} \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Z}$ over $\mathbb{F}_1$ and apply a similar argument to approach the Riemann Hypothesis.

In [41], C.Soulé gave the first mathematical definition of an algebraic variety over $\mathbb{F}_1$ by noticing that in order to realize $\operatorname{Spec}\mathbb{Z}$ as 'a curve' over $\mathbb{F}_1$, one has to develop algebraic geometry over various algebraic objects rather than commutative rings. His idea was to replace the category of commutative rings with the category of finite abelian groups by considering a scheme as a functor of points. He then introduced a zeta function of an algebraic variety over $\mathbb{F}_1$ when a counting function is given by a polynomial with integral coefficients. However, in [6], Connes and Consani pointed out that Soulé's definition is not compatible with the geometry of Chevalley groups as defined by Tits. They gave a more refined definition by imposing a graduation on Soulé's definition. This construction is compatible with Tits' geometry. More generally, Connes and Consani showed that Chevalley group schemes can be realized as algebraic varieties over (suitably defined) $\mathbb{F}_{1^2}$. Note that, in their subsequent paper [8], Connes and Consani merged their previous work [6], A. Deitmar's [16], and the functorial approach of B.Toën and M.Vaquié [46] (cf. [8]). The main idea is to replace the category of (graded) finite abelian groups with the category of pointed monoids. They also proved that there exists the real counting function $N(q)$ ($q \in [1, \infty)$) (as a distribution) for the completed 'curve' $\overline{\operatorname{Spec}\mathbb{Z}}$ over $\mathbb{F}_1$, whose corresponding (Hasse-Weil type) zeta function is the complete Riemann zeta function. What makes the story more interesting is the recent result [12] of the same authors; they construct the algebro-geometric space whose counting function (as a distribution) of points fixed by the (suitably defined) Frobenius action provides the complete Riemann zeta function. As we have seen, algebraic geometry over monoids has been initially the main interest (cf. [6], [8], [16], [17], [41], [46]). Another approach to the notion of $\mathbb{F}_1$-geometry, discovered later, is to consider algebraic structures which maintain an addition rather than loosing it completely. From this point of view, recently algebraic geometry over semirings has been studied in [11], [25], and in [18] in connection with tropical geometry. Also note that, in [29], Oliver Lorscheid unified monoids and semirings by means of his newly introduced structures, blueprints.

Our main goal in this thesis is to develop algebraic geometry over semirings and over somewhat exotic objects called 'hyperrings' (cf. §1.2 for the historical note on hyperrings). The main body of the thesis consists of five chapters. In the first chapter, we give a brief overview of the basic definitions and properties of semirings and hyperrings which will be used in the sequel.

Algebraic geometry over semirings

We investigate the basic notions of algebraic geometry over semirings. First, we define a (Hasse-Weil type) zeta function of a tropical variety. It has been known that all roots of a counting function (of lattice-points) of a special polytope have real part $-\frac{1}{2}$ and a counting fuction itself satisfies some functional equations (cf. [2, §2 and §4]). Since a tropical variety is a support of a polyhedron complex (moreover, sometimes it is a polytope), one is led to consider a possible link between a counting function of a polytope and a tropical variety. In [11], the authors initiated the study of semifields extension of the semifield $\mathbb{Z}_{max}$. In subsequent work [47], Jeffrey Tolliver proved that any semifield extension of $\mathbb{Z}_{max}$ is of the form $\mathbb{F}^{(n)} := \{q \in \mathbb{Q}_{max} \mid nq \in \mathbb{Z}\}$. These results suggest that a classical counting function (of lattice-points) can be understood as a (Hasse-Weil type) zeta function of a tropical variety. In this view point, we define a two variable zeta function $Z(X, t, v)$ of a tropical variety X and prove the following:

Theorem 1. *(cf. Proposition 2.1.27) Let X be a tropical variety. Suppose that X is a rational polytope. Then the zeta function $Z(X, t, v)$ of X is a rational function of t and v.*

Moreover, in Example 2.1.29, we provide evidence that an analogue of functional equation in characteristic one is valid for $\mathbb{P}^n$.

Next, we introduce the notion of a semi-scheme and a Picard group of a semi-scheme by directly generalizing the classical construction. We then generalize Čech cohomology to semi-schemes by using the result of [37]. We prove the following:

Theorem 2. *(cf. Proposition 2.2.4, Remark after Proposition 2.2.16, Proposition 2.3.22, Theorem 2.3.34, Example 2.3.35)*

1. *Let $(X = \operatorname{Spec} M, \mathcal{O}_X)$ be an affine semi-scheme, where M is a semiring. Then we have the following canonical isomorphism:*

$$M \simeq \mathcal{O}_X(X). \tag{0.0.1}$$

   In particular, the category of semirings and the category of affine semi-schemes are equivalent via the functors Spec and Γ.

2. *For a semi-scheme $(X, \mathcal{O}_X)$, the set $\operatorname{Pic}(X)$ of invertible sheaves of $\mathcal{O}_X$-semimodules on X is a group.*

3. *For a semi-scheme $(X, \mathcal{O}_X)$, we have $\Gamma(X, \mathcal{O}_X) \simeq \check{\mathrm{H}}^0(X, \mathcal{O}_X)$.*

4. *Let X be the projective line $\mathbb{P}^1_{\mathbb{Q}_{max}}$ over the semifield $\mathbb{Q}_{max}$. Then we have,*

$$\check{\mathrm{H}}^0(X, \mathcal{O}_X) \simeq \mathbb{Q}_{max}, \quad \check{\mathrm{H}}^n(X, \mathcal{O}_X) = 0 \text{ for } n \geq 2, \quad \operatorname{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^*) \simeq \mathbb{Z}.$$

In particular, an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-semimodules on X is isomorphic to $\mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$.

Finally, we define the notion of a valuation of a semiring and 'the function semifield' $\mathbb{Q}_{max}(T)$. Then we construct an abstract curve associated to a pair $(\mathbb{Q}_{max}(T), \mathbb{Q}_{max})$ and prove the following:

Theorem 3. *(cf. Remark 2.4.25) Let $k = \mathbb{Q}_{max}$ and $K = \mathbb{Q}_{max}(T)$. Then the set C_K of valuations on K which are trivial on k is homeomorphic (with suitably defined topology) to the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ over $\mathbb{F}_1$ introduced in [16].*

From semi-structures to hyper-structures

In [21], Simon Henry constructed a procedure which produces a hypergroup M_S from a semigroup M in a canonical way via a map $s : M \longrightarrow M_S$ which is called the symmetrization. We generalize Henry's construction to semirings (cf. Lemma 3.1.6, Proposition 3.1.10). Moreover, by implementing the notion of a good ordering (cf. Definition 3.1.2), we prove that a partial converse of Henry' construction holds as follows:

Theorem 4. *(cf. Proposition 3.1.8) Let R be a hyperring such that*

$$x + x = x \quad \forall x \in R; \quad x + y \in \{x, y\} \quad \forall x \neq -y \in R. \tag{0.0.2}$$

Let P be a good ordering on R. Then

1. *P is a totally ordered semiring (with a canonical order).*
2. *Under the symmetrization process, P_S is a hyperring with a multiplication given component-wise and P_S is isomorphic to R as hyperrings.*

We also investigate several properties of a symmetrization process. In particular, a symmetrization commutes with a localization (cf. Proposition 3.1.14).

Algebraic geometry over hyperrings

We first study algebraic properties of hyperrings. A construction of a quotient hyperring has been known only for a special class of (hyper) ideals of a hyperring (cf. [15]). We prove that, in fact, such construction works for any (hyper) ideal (cf. Proposition 4.1.6). Furthermore, we define the notion of a congruence relation on a hyperring and prove the following:

Theorem 5. *(cf. Propositions 4.1.15 and 4.1.17) There exists a one-to-one correspondence between the set of (hyper) ideals of a hyperring R and the set of congruence relations on R.*

We note that such a one-to-one correspondence is valid in the case of commutative rings; however, it is not in the case of semirings (cf. Example 4.1.10).

In [50], Oleg Viro tried to recast a tropical variety in the framework of hyper-structures. To realize his goal, we define an algebraic variety over a hyperring in the classical sense; a set of solutions of polynomial equations. As a byproduct, we obtain the following description of a tropical variety in terms of hyper-structures.

Theorem 6. *(cf. Proposition 4.2.31) Let $\mathbf{R} := (\mathbb{R}_{max})_S$ be the hyperring symmetrized by the tropical semifield $\mathbb{R}_{max}$. Let us define the map, $s^n : (\mathbb{R}_{max})^n \longrightarrow (\mathbf{R})^n, \quad (a_1, ..., a_n) \mapsto (s(a_1), ..., s(a_n))$. Let X be an n-dimensional tropical variety over $\mathbb{R}_{max}$. Then there exist a (suitably defined) algebraic variety X_S over the hyperring $\mathbf{R}$, and the following set bijection:*

$$\varphi : X \simeq (\mathrm{Img}(s^n) \cap X_S).$$

Next, we take the scheme-theoretic point view. The main obstacle is that, as Connes and Consani pointed out in [9], a canonical isomorphism as in (0.0.1) is no longer true for hyperrings (cf. Example 4.3.12). In fact, a priori if one follows the classical construction of a structure sheaf, such sheaf does not even have to be a sheaf of hyperrings (cf. Remark 4.3.8). However, we prove that when a hyperring does not have a (multiplicative) zero-divisor, the classical construction and results can be directly generalized to hyperrings. More precisely, we prove the following:

Theorem 7. *(cf. Theorem 4.3.11) Let R be a hyperring without a zero-divisor, $K = \mathrm{Frac}(R)$, and $X = \operatorname{Spec} R$. Let $\mathcal{O}_X$ be the sheaf of multiplicative monoids on X as in (4.3.6), equipped with the hyper-addition (4.3.9). Then, the following holds*

1. *$\mathcal{O}_X(D(f))$ is a hyperring isomorphic to R_f. In particular, if $f = 1$, we have $R \simeq \mathcal{O}_X(X)$.*

2. *For each open subset U of X, $\mathcal{O}_X(U)$ is a hyperring. More precisely, $\mathcal{O}_X(U)$ is isomorphic to the following hyperring:*

$$\mathcal{O}_X(U) \simeq Y(U) := \{u \in K \mid \forall \mathfrak{p} \in U, u = \frac{a}{b} \textit{ for some } b \notin \mathfrak{p}\}.$$

   Moreover, by considering the canonical map $R_f \hookrightarrow K$, we have

$$\mathcal{O}_X(U) \simeq \bigcap_{D(f) \subseteq U} \mathcal{O}_X(D(f)).$$

3. *For each $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ exists and is isomorphic to $R_{\mathfrak{p}}$.*

Note that (co)limits do not exist in the category of hyperrings in general, therefore one can not presume the existence of stalks in Theorem 7.

Next, we define a zeta function of an affine hyper-scheme (cf. Definition 4.3.39) and prove that a zeta function is invariant under 'the scalar extension' $-\otimes_{\mathbb{Z}} \mathbf{K}$, where $\mathbf{K}$ is the Krasner's hyperfield. More precisely, we show the following:

Theorem 8. *(cf. Theorem 4.3.44) Let k be a field, $G = k^\times$, and A be a reduced finitely generated (commutative) k-algebra. Let $R := A/G$ be the quotient hyperring. Then, R is a finitely generated hyper $\mathbf{K}$-algebra. Furthermore, if $X := \operatorname{Spec} A$ and $Y := \operatorname{Spec} R$, then we have the following:*

$$Z(Y,t) := \prod_{y \in |Y|} (1 - t^{\deg(y)})^{-1} = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}, \tag{0.0.3}$$

where $|X|$ and $|Y|$ are the sets of closed points of X and Y respectively. In particular, when k is a finite field of odd characteristic, we have $Z(Y,t) = Z(X,t)$, where $Z(X,t)$ is the classical Hasse-Weil zeta function attached to the algebraic variety $X = \operatorname{Spec} A$.

To link algebraic geometry over semirings and hyperrings, we first generalize the notion of real prime ideals in real algebraic geometry (cf. Definition 4.3.67). Then, an affine hyper-scheme is linked to an affine semi-scheme in the following sense:

Theorem 9. *(cf. Propositions 4.3.66, 4.3.68, and 4.3.69) Let M be a semiring and assume that M produces the hyperring M_S via the symmetrization process. Then $\operatorname{Spec} M_S$ is homeomorphic to the subspace of $\operatorname{Spec} M$ which consists of real prime ideals. Moreover, any (hyper) prime ideal of M_S is real.*

Finally, we give a (partial) affirmative answer to the speculation posed in [7]. For an affine group scheme $X = \operatorname{Spec} A$ over a field k, the set $\operatorname{Hom}(A, K)$ of homomorphisms has the canonical group structure induced from a coporudct of A for any field extension K of k. However, the underlying space $\operatorname{Spec} A$ itself does not carry any algebraic structure in general. In [7], the authors found the following identification (of sets):

$$\operatorname{Hom}(A, \mathbf{K}) = \operatorname{Spec} A, \tag{0.0.4}$$

where $\mathbf{K}$ is the Krasner's hyperfield. In other words, one can realize the underlying space $\operatorname{Spec} A$ as the set of '$\mathbf{K}$-rational points' of X. A natural question which arises from this perspective is whether $\operatorname{Spec} A$ is a hypergroup or not. Connes and Consani proved that the answer is affirmative when $A = k[T]$ or $k[T, \frac{1}{T}]$ and $k = \mathbb{Q}$ or $\mathbb{F}_p$, and expected that the similar development would hold when X is a Chevalley group scheme. We answer their expectation; to an affine algebraic group scheme, a similar argument can be applied. More precisely, we prove the following:

Theorem 10. *(cf. Theorem 5.1.12) Any affine algebraic group scheme $X = \operatorname{Spec} A$ over a field k has a canonical hyper-structure $*$ induced from the coproduct of A which satisfies the following conditions:*

1. $*$ *is weakly-associative, i.e.* $f*(g*h)\cap(f*g)*h\neq\emptyset\ \forall f,g,h\in X$.

2. $*$ *is equipped with the identity element* e*, i.e.* $f*e=e*f=f\ \forall f\in X$.

3. *For each* $f\in X$*, there exists a canonical element* $\tilde{f}\in X$ *such that* $e\in(f*\tilde{f})\cap(\tilde{f}*f)$.

4. *For* $f,g,h\in X$*, the following holds:* $f\in g*h\iff\tilde{f}\in\tilde{h}*\tilde{g}$.

1

Background and historical note

In the first subsection, we provide the basic definitions and properties of semirings and hyperrings, which are to be used in the subsequent chapters. Then we give a historical overview on theory of hyperrings.

1.1 Background on semi-structures and hyper-structures

1.1.1 Basic notions: Semi-structures

We introduce the basic notions and properties in semiring theory.

Definition 1.1.1. *A set M equipped with a binary operation $\cdot$ is called a semigroup if for $a, b, c \in M$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and there exists $1 \in M$ such that $1 \cdot a = a \cdot 1 = a$. When $a \cdot b = b \cdot a$ $\forall a, b \in M$, we say that M is a commutative semigroup.*

Definition 1.1.2. *A semiring $(M, +, \cdot)$ is a non-empty set M endowed with an addition $+$ and a multiplication $\cdot$ such that*

1. *$(M, +)$ is a commutative semigroup with the neutral element 0.*

2. *$(M, \cdot)$ is a semigroup with the identity 1.*

3. *$r(s + t) = rs + rt$ and $(s + t)r = sr + tr \quad \forall r, s, t \in M$.*

4. $r \cdot 0 = 0 \cdot r = 0 \quad \forall r \in M$.

5. $0 \neq 1$.

If $(M, \cdot)$ is a commutative semigroup, then we call M a commutative semiring. If $(M \backslash \{0\}, \cdot)$ is a group, then a semiring M is called a semifield.

Definition 1.1.3. *(cf. [19]) Let M_1, M_2 be semirings. A map $f : M_1 \longrightarrow M_2$ is a homomorphism of semirings if f satisfies the following conditions: $\forall a, b \in M_1$,*

$$f(a+b) = f(a) + f(b), \quad f(ab) = f(a)f(b), \quad f(0) = 0, \quad f(1) = 1.$$

Definition 1.1.4. *Let R be a semiring and T be a semigroup. We say that T is a R-semimodule if there exists a map $\varphi : R \times M \longrightarrow M$ which satisfies the following properties: $\forall r, r_1, r_2 \in R$, $\forall t, t_1, t_2 \in T$,*

1. $\varphi(1, r) = r$.

2. If $t = 0$ or $r = 0$, then $\varphi(t, r) = 0$.

3. $\varphi(t_1 + t_2, r) = \varphi(t_1, r) + \varphi(t_2, r), \quad \varphi(t, r_1 + r_2) = \varphi(t, r_1) + \varphi(t, r_2)$.

4. $\varphi(t_1 t_2, r) = \varphi(t_1, \varphi(t_2, r)), \quad \varphi(t, r_1 r_2) = \varphi(t, r_2) r_2$.

In what follows, we always assume that all semirings are commutative. We review the notion of (prime) ideals of a semiring M.

Definition 1.1.5. *(cf. [19]) Let M be a semiring.*

1. A non-empty subset I of M is an ideal if $(I, +)$ is a sub-semigroup of $(M, +)$ and for $a \in I, r \in M$, we have $r \cdot a \in I$.

2. An ideal $I \subsetneq M$ is prime if I satisfies the following property: if $xy \in I$, then $x \in I$ or $y \in I$ $\forall x, y \in I$.

3. An ideal $I \subsetneq M$ is maximal if I satisfies the following property: if $J \subsetneq M$ is an ideal and $I \subseteq J$, then $I = J$.

Proposition 1.1.6. *(cf. [19, §6]) Let M be a semiring.*

1. *Any maximal ideal $\mathfrak{m}$ of M is prime.*

2. *Any proper ideal I of M (i.e. $I \neq M$) is contained in a maximal ideal of M.*

Let M be a semiring and $X = \operatorname{Spec} M$ be the set of prime ideals of M. Then, as in the classical case, one can impose the Zariski topology on X as follows: a subset A of X is closed if and only if $A = V(I)$ for some ideal I of M, where $V(I) := \{\mathfrak{p} \in X \mid I \subseteq \mathfrak{p}\}$ (cf. [19, §6]). Moreover, the following Hilbert's Nullstellensatz holds: for an ideal I of M, we have

$$\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \{a \in M \mid a^n \in I \text{ for some } n \in \mathbb{N}\}. \tag{1.1.1}$$

The notion of localization can be directly generalized to a semiring. Let M be a semiring and S be a multiplicative subset of M, equivalently, S is a (multiplicative) submonoid. Then, as a set, $S^{-1}M$ is $(M \times S/\sim)$, where $\sim$ is a congruence relation on $M \times S$ such that

$$(m_1, r_1) \sim (m_2, r_2) \iff \exists s \in S \text{ such that } sm_1s_2 = sm_2s_1. \tag{1.1.2}$$

Note that by a congruence relation $\sim$ on a semiring M we mean an equivalence relation which satisfies the following condition: if $x \sim y$ and $x' \sim y'$, then $x + x' \sim y + y'$ and $xx' \sim yy'$ $\forall x, x', y, y' \in M$. We denote by $\frac{m}{s}$ the equivalence class of (m, s) under the congruence relation (1.1.2). Then, $S^{-1}M$ is a semiring and a localization map $S^{-1} : M \longrightarrow S^{-1}M$ sending m to $\frac{m}{1}$ is a homomorphism of semirings. Moreover, as in the classical case, for a (prime) ideal I of M such that $I \cap S = \emptyset$, the set $S^{-1}I := \{\frac{i}{s} \mid i \in I, s \in S\}$ is a (prime) ideal of $S^{-1}M$. Finally, when $S = M \backslash \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of M, the semiring $S^{-1}M$ has the unique maximal ideal, namely $S^{-1}\mathfrak{p}$ (cf. [19, §10]).

By an idempotent semiring, we mean a semiring M such that $x + x = x$ $\forall x \in M$.

Example 1.1.7. *Let $\mathbb{B} := \{0, 1\}$. We define an addition as: $1 + 1 = 1$, $1 + 0 =$*

$0+1=1$, and $0+0=0$. A multiplication is defined by $1\cdot 1=1$, $1\cdot 0=0$, and $0\cdot 0=0$. Then, $\mathbb{B}$ becomes the initial object in the category of idempotent semirings.

Example 1.1.8. *The tropical semifield $\mathbb{R}_{max}$ is $\mathbb{R}\cup\{-\infty\}$ as a set. An addition $\oplus$ is given by: $a\oplus b:=\max\{a,b\}\quad \forall a,b\in\mathbb{R}_{max}$, where $-\infty\leq a\ \forall a\in\mathbb{R}_{max}$. A multiplication $\odot$ is defined as the usual addition of $\mathbb{R}$ as follows: $a\odot b:=a+b$, where $+$ is the usual addition of real numbers and $(-\infty)\odot a=a\odot(-\infty)=(-\infty)$ $\forall a\in\mathbb{R}_{max}$. We denote by $\mathbb{Q}_{max}$, $\mathbb{Z}_{max}$ the sub-semifields of $\mathbb{R}_{max}$ with the underlying sets $\mathbb{Q}\cup\{-\infty\}$, $\mathbb{Z}\cup\{-\infty\}$ respectively.*

When M is an idempotent semiring, one can impose the following canonical partial order on M:

$$a\leq b \iff a+b=b \quad \forall a,b\in M. \tag{1.1.3}$$

Note that by a partial order on M we mean a binary relation $\leq$ on M which is reflexive, transitive, and antisymmetric.

1.1.2 Basic notions: Hyper-structures

In this subsection, we introduce the basic definitions and properties of hyperrings.

Definition 1.1.9. *(cf. [9]) A hyper-operation on a non-empty set H is a map*

$$+: H\times H\to\mathcal{P}(H)^*,$$

where $\mathcal{P}(H)^*$ is the set of non-empty subsets of H. In particular, $\forall A,B\subseteq H$, we also denote

$$A+B:=\bigcup_{a\in A,b\in B}(a+b).$$

Definition 1.1.10. *(cf. [9]) A canonical hypergroup $(H,+)$ is a non-empty pointed set with a hyper-operation $+$ which satisfies the following properties:*

1. *$x+y=y+x\quad\forall x,y\in H.\quad$ (commutativity)*

2. $(x+y)+z = x+(y+z) \quad \forall x,y,z \in H.$ *(associativity)*

3. $0+x = x = x+0 \quad \forall x \in H.$ *(neutral element)*

4. $\forall x \in H \quad \exists! y (:= -x) \in H \quad s.t. \quad 0 \in x+y.$ *(unique inverse)*

5. $x \in y+z \Longrightarrow z \in x-y.$ *(reversibility)*

Remark 1.1.11. *The uniqueness of* (4) *rules out the trivial choice of the inverse, e.g. the full set H as an inverse of any element. The reversibility property is meant to be the 'hyper'-subtraction.*

Note that a hypergroup is, in fact, more general object than a canonical hypergroup. However, throughout the thesis, by a hypergroup we will always mean a canonical hypergroup.

Definition 1.1.12. *(cf. [9]) A hyperring $(R,+,\cdot)$ is a non-empty set R with a hyperaddition $+$ and a usual multiplication $\cdot$ which satisfy the following conditions:*

1. $(R,+)$ *is a canonical hypergroup.*

2. $(R,\cdot)$ *is a monoid with* 1_R *(not necessarily commutative).*

3. A hyperaddition and a multiplication are compatible, i.e. $\forall x,y,z \in R$, $x(y+z) = xy+xz$, $(x+y)z = xz+yz$.

4. 0 *is an absorbing element, i.e.* $\forall x \in R$, $x \cdot 0 = 0 = 0 \cdot x$.

5. $0 \neq 1$.

When $(R \setminus \{0\}, \cdot)$ *is a group, we call* $(R,+,\cdot)$ *a hyperfield.*

Definition 1.1.13. *(cf. [9]) For hyperrings* $(R_1,+_1,\cdot_1)$, $(R_2,+_2,\cdot_2)$ *a map* $f : R_1 \longrightarrow R_2$ *is called a homomorphism of hyperrings if*

1. $f(a+_1 b) \subseteq f(a) +_2 f(b) \quad \forall a,b \in R_1.$

2. $f(a \cdot_1 b) = f(a) \cdot_2 f(b) \quad \forall a,b \in R_1.$

3. *We call f strict if* $f(a+_1 b) = f(a) +_2 f(b) \quad \forall a,b \in R_1$.

4. *We call f an epimorphism if*

$$x +_2 y = \bigcup \{f(a +_1 b) \mid f(a) = x, f(b) = y\} \quad \forall x, y \in R_2.$$

Example 1.1.14. *(cf. [9]) Let $\mathbf{K} := \{0,1\}$. A (commutative) multiplication of $\mathbf{K}$ is given by*

$$1 \cdot 1 = 1, \quad 0 \cdot 1 = 1 \cdot 0 = 0,$$

and a (commutative) hyperaddition is given by

$$0+1 = \{1\}, \quad 0+0 = \{0\}, \quad 1+1 = \{0,1\}.$$

Then $(\mathbf{K}, +, \cdot)$ is a hyperfield called the Krasner's hyperfield.

Let R be a hyperring. For $x, y \in R$, if $x+y$ consists of a single element z, we let $x+y = z$ rather than $x+y = \{z\}$. Another interesting example is the hyperfield of signs.

Example 1.1.15. *(cf. [9]) Let $\mathbf{S} = \{-1, 0, 1\}$. A multiplication is commutative and given by*

$$1 \cdot 1 = (-1) \cdot (-1) = 1, \quad (-1) \cdot 1 = (-1), \quad a \cdot 0 = 0 \quad \forall a \in \mathbf{S}.$$

A hyperaddition $+$ is commutative and given by

$$0+0 = 0, \quad 1+0 = 1+1 = 1, \quad (-1)+0 = (-1)+(-1) = (-1), \quad 1+(-1) = \{-1,0,1\}.$$

In other words, a hyperaddition is given by the rule of signs and hence we call $\mathbf{S}$ the hyperfield of signs.

We review the notion of (prime) ideals for hyperrings. In the sequel, all hyperrings are assumed to be commutative.

Definition 1.1.16. *(cf. [9]) Let R be a hyperring.*

1. *A non-empty subset I of R is a hyperideal if: $\forall a, b \in I \Longrightarrow a - b \subseteq I$ and $\forall a \in I, \forall r \in R \Longrightarrow r \cdot a \in I$.*

2. *A hyperideal $I \subsetneq R$ is prime if I satisfies the following property: if $xy \in I$, then $x \in I$ or $y \in I$ $\forall x, y \in I$.*

3. *A hyperideal $I \subsetneq R$ is maximal if I satisfies the following property: if $J \subsetneq R$ is a hyperideal of R which contains I, then $I = J$.*

Proposition 1.1.17. *(cf. [15]) Let R be hyperring.*

1. *Let I be a proper hyperideal of R (i.e. $I \neq R$). Then there exists a maximal hyperideal $\mathfrak{m}$ such that $I \subseteq \mathfrak{m}$.*

2. *Any maximal hyperideal $\mathfrak{m}$ is prime.*

Definition 1.1.18. *(cf. [39]) Let R be a hyperring. We denote by $\operatorname{Spec} R$ the set of prime hyperideals of R. One can impose the Zariski topology on $\operatorname{Spec} R$ as in the classical case. In other words,*

$$\text{a subset } A \subseteq \operatorname{Spec} R \text{ is closed } \iff A = V(I) \text{ for some hyperideal } I \text{ of } R, \tag{1.1.4}$$

where $V(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}$.

Proposition 1.1.19. *(cf. [39]) Let R be a hyperring and $X = \operatorname{Spec} R$.*

1. *Let $\{I_j\}_{j \in J}$ be a family of hyperideals of R. Then we have*

$$\bigcap_{j \in J} V(I_j) = V(< \bigcup_{j \in J} I_j >), \tag{1.1.5}$$

where $< \bigcup_{j \in J} I_j >$ is the smallest hyperideal containing $\bigcup_{j \in J} I_j$. Note that such hyperideal exists since an arbitrary intersection of hyperideals is a hyperideal.

2. *Let I and I' be hyperideals of R, then we have*

$$V(I)\bigcup V(I') = V(I \cap I'). \tag{1.1.6}$$

Next, we review the notion of a localization of a hyperring. This construction has been promoted by R.Procesi-Ciampi and R.Rota (cf. [39]).

For a (multiplicative) submonoid S of a hyperring R, one defines the localization $S^{-1}R$ as follows: as a set, $S^{-1}R$ is the set $(R \times S/\sim)$ of equivalence classes, where

$$(r_1, s_1) \sim (r_2, s_2) \Longleftrightarrow \exists x \in S \quad s.t. \quad xr_1s_2 = xr_2s_1. \tag{1.1.7}$$

Let $[(r, s)]$ be the equivalence class of $(r, s) \in R \times S$ under the equivalence relation (1.1.7). A hyperaddition of $S^{-1}R$ is given by

$$[(r_1, s_1)] + [(r_2, s_2)] = [(r_1s_2 + s_1r_2), s_1s_2] = \{[(y, s_1s_2)] \mid y \in r_1s_2 + s_1r_2\}.$$

A multiplication is naturally given as follows:

$$[(r_1, s_1)] \cdot [(r_2, s_2)] = [(r_1r_2, s_1s_2)].$$

We denote by $\frac{r}{s}$ a element $[(r, s)]$. Note that as in the classical case, the localization map, $S^{-1} : R \longrightarrow S^{-1}R$ sending r to $\frac{r}{1}$, is a homomorphism of hyperrings.

Proposition 1.1.20. *(cf. [15]) Let R be a hyperring and S be a multiplicative subset of R.*

1. *For a hyperideal I of R, the following set:*

$$S^{-1}I := \{\frac{i}{s} \mid i \in I, s \in S\}$$

is a hyperideal of $S^{-1}R$.

2. *If $\mathfrak{p}$ is a prime hyperideal of R such that $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}\mathfrak{p}$ is a prime*

hyperideal of $S^{-1}R$.

3. *If $S = R\backslash\mathfrak{p}$ for some prime hyperideal $\mathfrak{p}$ of R, then $S^{-1}R$ has the unique maximal hyperideal given by $S^{-1}\mathfrak{p}$.*

The following theorems provide a useful way to construct hyperrings from classical commutative algebras.

Theorem 1.1.21. *(cf. [9, Proposition 2.6]) Let A be a commutative ring and $G \subseteq A^\times$ be a subgroup of the multiplicative group $A^\times$. Then, the set A/G is a hyperring with the following operations:*

1. *$xG \cdot yG := xyG \quad \forall x, y \in A$. (multiplication)*

2. *$xG + yG := \{zG \mid z = xa + yb \text{ for some } a, b \in G\} \quad \forall x, y \in A$. (hyperaddition)*

A hyperring which arises in this way is called a quotient hyperring.

Note that, for a field k with $|k| \geq 3$, we can identify the Krasner's hyperfield $\mathbf{K}$ with the quotient hyperring $k/k^\times$. We recall the following interesting fact.

Theorem 1.1.22. *(cf. [9, Proposition 2.7]) Let A be a commutative ring, and let $G \subseteq A^\times$ be a subgroup of the multiplicative group $A^\times$. Assume further that $|G| \geq 2$. Then, the quotient hyperring A/G is an extension of the Krasner's hyperfield $\mathbf{K}$ if and only if $\{0\} \cup G$ is a subfield of A.*

1.2 Historical note on hyperrings

The notion of a hypergroup was first introduced by F.Marty in [34] and subsequently, in 1956, M.Krasner introduced the notion of hyperrings as a technical tool in his paper [24] on the approximation of valued fields. However, for decades, hyper-structure has been better known to computer scientists or applied mathematicians than those who work in pure mathematics; this is due to uses of hyper-structures in connection with fuzzy logic (a form of multi-valued logic), automata, cryptography, coding theory via associations schemes, and hypergraphs (cf. [13], [52]).

In [50], Oleg Viro wrote: "Probably, the main obstacle for hyperfields to become a mainstream notion is that a multivalued operation does not fit to the tradition of set-theoretic terminology, which forces to avoid multivalued maps at any cost. I believe the taboo on multivalued maps has no real ground, and eventually will be removed." In recent years, hyper-structure theory has been revitalized in connection with various fields. For example, in connection with number theory, A.Connes' adèle class space $\mathbb{H}_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^{\times}$ of a global field $\mathbb{K}$ is a hyperring extension of the Krasner's hyperfield $\mathbf{K}$ (cf. [9]). Moreover, the use of hyper-structures is essential in the archimedean (isotypical) Witt construction introduced in [10]. Also, in [50], the author found a link between hyper-structures and tropical geometry via dequantization. Finally, in [32], M.Marshall generalizes the Artin-Schreier theory for fields to hyperfields. Note that the weakness of semirings is that they do not posses additive inverses. This problem can be fixed by considering hyper-structures via Henry's symmetrization process (cf. [21]). Therefore, one might benefit by using both semi-structures and hyper-structures.

2

Algebraic geometry over semi-structures

We develop algebraic geometry over semi-structures in this chapter. In the first section, we take an elementary approach to investigate an algebraic variety over various sub-semifields of $\mathbb{R}_{max}$ considered as a set of solutions of polynomial equations. In the second section, we introduce the notion of a semi-scheme generalizing a scheme in such a way that a underlying algebra is that of semirings and develop Čech cohomology theory of semi-schemes. As a byproduct, we confirm that any invertible sheaf on $\mathbb{P}^1_{\mathbb{Q}_{max}}$ is isomorphic to $\mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$. Finally, in the last section, we introduce the notion of valuations over semirings and prove that the analogue of an abstract curve by using (suitably defined) $\mathbb{Q}_{max}(T)$ is provided by the projective line $\mathbb{P}^1_{\mathbb{F}_1}$.

2.1 Solutions of polynomial equations over semi-structures

2.1.1 Solutions of polynomial equations over $\mathbb{Z}_{max}$

In recent years, tropical geometry has become a young and popular subject of mathematics. Tropical geometry is, briefly speaking, the study of a tropical variety which is a set of 'solutions' of polynomial equations over the semifield $\mathbb{R}_{max}$ (cf. Example

1.1.8).

In the recent paper [11], A.Connes and C.Consani studied arithmetics of the subsemifield $\mathbb{Z}_{max}$ of $\mathbb{R}_{max}$. This suggests that one might study arithmetics of tropical varieties based on $\mathbb{Z}_{max}$. In this section we will briefly introduce the basic theorems and definitions of tropical geometry and explain how one can naturally replace $\mathbb{R}_{max}$ with $\mathbb{Z}_{max}$. We will follow notations and definitions in [30]. The only difference between this section and [30] is that we use the maximum convention instead of the minimum convention, but such choice makes no difference in developing the theory.

Recall that the semifield $\mathbb{Z}_{max}$ is a subsemifield of $\mathbb{R}_{max}$ with the underlying set $\mathbb{Z}_{max} = \mathbb{Z} \cup \{-\infty\}$. We also note that the set $\mathbb{R}_{max}[x_1, ..., x_n]$ of polynomials with coefficients in $\mathbb{R}_{max}$ is also a semiring with the operations induced from $\mathbb{R}_{max}$. To be specific, an element F of $\mathbb{R}_{max}[x_1, ..., x_n]$ is a finite formal sum of monomials using $\oplus$ and $\odot$. Furthermore, one defines $x_i \oplus -\infty = x_i$, $x_i \odot 0 = x_i$. Then, for $F \in \mathbb{R}_{max}[x_1, ..., x_n]$, one defines the following set:

$$V(F) := \{w \in \mathbb{R}^n \mid \text{ the maximum in } F \text{ is achieved at least twice}\}. \tag{2.1.1}$$

In what follows, we fix an algebraically closed field K with a nontrivial valuation ν. For $f = \sum_{u \in \mathbb{Z}^n} C_u x^u \in K[x_1^{\pm}, ..., x_n^{\pm}]$, one defines the tropicalization $trop(f)$ of f as follows:

$$trop(f) := \oplus_{u \in \mathbb{Z}^n} \nu(C_u) \odot x^{\odot u} = \max_u \{\nu(C_u) + u \cdot x\} \in \mathbb{R}_{max}[x_1, ... x_n]. \tag{2.1.2}$$

With the above notations, one has $trop(V(f)) = V(trop(f))$.

Example 2.1.1. *Let us compute an easy example. Let $F := 0 \oplus x \oplus y \in \mathbb{R}_{max}[x, y]$ be a tropical linear polynomial. It follows from the definition that $V(F)$ is the subset of $\mathbb{R}^2$ where the maximum in $F = 0 \oplus x \oplus y$ is achieved at least twice. Thus one can observe that $V(F)$ is the union of the sets X_1, X_2, X_3 by choosing each two of terms x, y, and 0 to be a maximum as follows:*

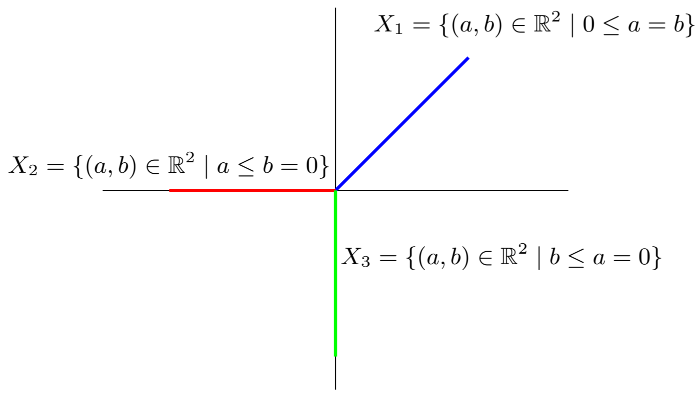

Figure 2.1: Tropical Line in $\mathbb{R}^2$

Example 2.1.2. *(cf. [30]) Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over $\mathbb{C}$. Then K can be written as*

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{n \ge 1} \mathbb{C}((t^{\frac{1}{n}})),$$

where $\mathbb{C}((t^{\frac{1}{n}}))$ is the field of Laurent series in the formal variable $t^{\frac{1}{n}}$. Note that K has a natural valuation ν such that for $c(t) \in K^*$, $\nu(c(t))$ is the lowest exponent that appears in the series expansion of $c(t)$. For example, the valuation $\nu(c_0(t))$ of $c_0(t) := \frac{t^2}{1-t} = t^2 + t^3 + t^4 ...$ is 2. Suppose that $f(x_1, x_2) = 5 + c_0(t)x_1 + x_1x_2$. Then, we have

$$trop(f) := \nu(5) \oplus \nu(c_0(t)) \odot x_1 \oplus \nu(1) \odot x_1 \odot x_2$$

$$= \max\{\nu(5), \nu(c_0(t)) + x_1, \nu(1) + x_1 + x_2\} = \max\{0, 2 + x_1, x_1 + x_2\}.$$

For $f \in K[x_1^{\pm}, ..., x_n^{\pm}]$, one defines the tropical hypersurface $trop(V(f))$ as the following set:

$$trop(V(f)) := \{w \in \mathbb{R}^n | \text{ the maximum in } trop(f) \text{ is achieved at least twice}\}.$$

Example 2.1.3. *Let f be as in Example 2.1.2. Then, $trop(V(f))$ is a union of the* *sets X_1, X_2, X_3, where*

$$X_1 := \{(x,y) \in \mathbb{R}^2 \mid 0 \leq 2+x_1 = x_1+x_2\}, \quad X_2 := \{(x,y) \in \mathbb{R}^2 \mid 2+x_1 \leq 0 = x_1+x_2\},$$

$$\textit{and} \quad X_3 := \{(x,y) \in \mathbb{R}^2 \mid x_1 + x_2 \leq 2 + x_1 = 0\}.$$

For a subset $X \subseteq \mathbb{R}^n$, let $\overline{X}$ be the (topological) closure of X in $\mathbb{R}^n$. One of the main theorems in tropical geometry is the following:

Theorem 2.1.4. *(Kapranov's theorem) Let K an algebraically closed field with a valuation ν. Suppose that $f = \sum_{u \in \mathbb{Z}^n} C_u x^u \in K[x_1^{\pm}, ..., x_n^{\pm}]$. Then,*

$$trop(V(f)) = \overline{\{(\nu(y_1), ..., \nu(y_n)) \in \mathbb{R}^n \mid y = (y_1, ..., y_n) \in V(f)\}}.$$

Example 2.1.5. *([30, Example 3.1.4]) Let K be an algebraically closed field with a valuation ν. Let $1 + x + y \in K[x^{\pm 1}, y^{\pm 1}]$. Then,*

$$V(f) = \{(z, -1-z) \in K^2 \mid z \neq 0, -1\}.$$

Moreover, we have

$$(\nu(z), \nu(-1-z)) = \begin{cases} (\nu(z), 0) & \textit{if } \nu(z) > 0 \\ (\nu(z), \nu(z)) & \textit{if } \nu(z) < 0 \\ (0, \nu(-1-z)) & \textit{if } \nu(z) = 0, \nu(-1-z) > 0 \\ (0,0) & \textit{otherwise.} \end{cases} \tag{2.1.3}$$

Since K is algebraically closed, the value group of ν is dense in $\mathbb{R}$. It follows from (2.1.3) that the closure of the set $\{(\nu(z), \nu(-1-z) \mid z \neq 0, -1\}$ is same as the set $V(F)$ in Example 2.1.1.

Remark 2.1.6. *The set $trop(V(f))$ is also same as a support of some Gröbner complex, however, we will not use that result in this chapter. For details we refer the readers to Chapter 3 of [30].*

Let X be the algebraic variety defined by an ideal $I \subseteq K[x_1^{\pm}, ..., x_n^{\pm}]$. One defines the tropicalization $trop(X)$ of X as follows:

$$trop(X) := \bigcap_{f \in I} trop(V(f)) \subseteq \mathbb{R}^n.$$

There are two main theorems in tropical geometry.

Theorem 2.1.7. *(Fundamental theorem of tropical algebraic geometry) Let I be an ideal of $K[x_1^{\pm}, ..., x_n^{\pm}]$ and $X := V(I)$. Then,*

$$trop(X) = \overline{\{(\nu(y_1), ..., \nu(y_n)) \in \mathbb{R}^n \mid y = (y_1, ..., y_n) \in X\}}. \tag{2.1.4}$$

Theorem 2.1.8. *(Structure theorem for tropical varieties) Let X be an irreducible d-dimensional subvariety of a torus T^n over K. Let Γ be the value group of a valuation ν on K. Then, $trop(X)$ is the support of a balanced, weighted Γ-rational polyhedral complex which is pure of dimension d. Moreover, the polyhedral complex is connected through codimension one.*

Example 2.1.9. *From Example 2.1.1, one observes that $trop(X)$ is the support of a polyhedral complex pure of dimension 1 connected through codimension 1, i.e. $trop(X)$ is a connected finite graph.*

When we replace $\mathbb{R}_{max}$ with a subsemifield M of $\mathbb{R}_{max}$, the most naive definition of a tropical variety over M is the following:

Definition 2.1.10. *Let M be a subsemifield of $\mathbb{R}_{max}$ and $M_1 := M \backslash \{-\infty\} (= M^*)$.*

1. *For $F \in M[x_1, ..., x_n]$, we define the set $V_M(F)$ of solutions of F over M as follows: $V_M(F) := \{w \in M_1^n \mid$ the maximum in F is achieved at least twice$\}$.*

2. *For an ideal $I \in M[x_1, ..., x_n]$, we define the set $V_M(I)$ as follows:*

$$V_M(I) := \bigcap_{F \in I} V_M(F).$$

Example 2.1.11. *Let $M = \mathbb{Z}_{max}$ and $F := 0 \oplus x \oplus y \in \mathbb{Z}_{max}[x, y]$. Then, the set $V_M(F)$ is the intersection of $V(F)$ in Example 2.1.1 with $\mathbb{Z}^2$.*

In fact, for a subsemifield M of $\mathbb{R}_{max}$ and an ideal $I \in M[x_1, ..., x_n]$, we obtain

$$V_M(I) = V(I) \cap M_1^n, \tag{2.1.5}$$

where $V(I)$ is a tropical variety defined by I. In the sequel, by the set of M-rational points of $V(I)$ or a tropical variety defined by I over M, we mean $V_M(I)$ in (2.1.5). In [18], Jeffrey Giansiracusa and Noah Giansiracusa proved that there is a (semi) scheme structure which one can associate to a tropical variety, and the set $V_M(F)$ can be understood as the set of M-rational points of that (semi) scheme. We explain their result succinctly here.

Fix a subsemifield M of $\mathbb{R}_{max}$ and let $S_M = M[x_1^{\pm}, ..., x_n^{\pm}]$. For $F = \max_u(a_u + x \cdot u) \in S_M$, one defines the set $supp(F) := \{u \in \mathbb{Z}^n \mid a_u \neq -\infty\}$. For $v \in supp(F)$, one defines

$$F_{\hat{v}} := \max_{u \neq v}(a_u + x \cdot u).$$

The bend relation of F is defined by: $\mathcal{B}(F) := \{F \sim F_{\hat{v}} : v \in supp(F)\}$. For example, if $F := 1 \oplus x \oplus y = \max\{1, x, y\}$, then we have

$$\mathcal{B}(F) = \{F \sim 1 \oplus x, F \sim 1 \oplus y, F \sim x \oplus y\}.$$

For an ideal I of S_M, the scheme-theoretic tropicalization of I is the congruence on S_M generated by $\{\mathcal{B}(trop(f)) : f \in I\}$ which they denote by $\mathcal{T}rop(I)$. Then, the quotient $S_M/\mathcal{T}rop(I)$ is a semiring and we have

$$V_M(I) = \mathrm{Hom}(S_M/\mathcal{T}rop(I), M), \tag{2.1.6}$$

where homomorphisms are semiring homomorphisms. In other words,

$$V_M(I) = V(I) \cap (M \backslash \{-\infty\})^n = \{w \in (M \backslash \{-\infty\})^n \mid F(w) = F_{\hat{v}}(w) \quad \forall F \in I, v \in supp(F)\}.$$

Thus, $V_M(I)$ can be considered as the set of M-rational points of $\mathrm{Spec}(S_M/\mathcal{T}rop(I))$. This justifies our notation.

Remark 2.1.12. *When the value group Γ is a subgroup of $\mathbb{Q}$, a polynomial F in $\Gamma[x_1, ..., x_n]$ always has a solution over $\mathbb{Q}_{max}$ since tropical polynomials are piecewise linear functions. Hence, the semifield $\mathbb{Q}_{max}$ can be considered as 'algebraically closed'.*

We close this subsection by claiming that the naive generalization of Galois theory does not behave well in this setting.

Proposition 2.1.13. *The only automorphism of $\mathbb{R}_{max}$ fixing $\mathbb{Z}_{max}$ is the identity map.*

Proof. Let φ be an automorphism of $\mathbb{R}_{max}$ fixing $\mathbb{Z}_{max}$. Then, φ also has to fix $\mathbb{Q}_{max}$. Indeed, for $\frac{a}{b} \in \mathbb{Q}_{max}$, we have $a = \varphi(a) = \varphi(b \cdot \frac{a}{b}) = \varphi(\frac{a}{b} + \frac{a}{b} + ... + \frac{a}{b}) = b \cdot \varphi(\frac{a}{b})$. It follows that $\varphi(\frac{a}{b}) = \frac{a}{b}$. Furthermore, since φ and φ^{-1} are order-preserving functions, they should be continuous with respect to Euclidean topology. Hence, φ also has to fix $\mathbb{R}_{max}$. $\square$

Remark 2.1.14. *Proposition 2.1.13 suggests that if one wants to understand the set of 'rational points' as the set of elements which are fixed by the action of a 'Galois group', then one needs to develop Galois theory which is not as naive as the above.*

2.1.2 Counting rational points

In the view of Theorem 2.1.8 (the structure theorem) and (2.1.6), algebraic geometry over $\mathbb{R}_{max}$ is the geometry of polyhedral complexes and algebraic geometry over $\mathbb{Z}_{max}$ is the geometry of lattice points (or integral points) of such polyhedral complexes.
In [11], the authors showed that for each $n > 1$, there is a Frobenius map Fr_n :

$\mathbb{Z}_{max} \longrightarrow \mathbb{Z}_{max}$ such that the image of Fr_n is isomorphic to the semifield extension $\mathbb{F}^{(n)} \simeq \{q \in \mathbb{Q}_{max} \mid nq \in \mathbb{Z}_{max}\}$ of $\mathbb{Z}_{max}$ of (suitable defined) degree n. Moreover, in [47], Jeffrey Tolliver showed that any finite semifield extension of $\mathbb{Z}_{max}$ of degree n is isomorphic to $\mathbb{F}^{(n)}$. In the sequel, we denote $\mathbb{F} := \mathbb{Z}_{max}$.

In the sense that $\mathbb{F}$ and $\mathbb{F}^{(n)}$ are characteristic one analogues of finite fields $\mathbb{F}_q$ and $\mathbb{F}_{q^n}$, one might be interested in counting the number of '$\mathbb{F}^{(n)}$-rational' points of a given tropical variety X over $\mathbb{Z}_{max}$. However, in general, a cardinality of a set of '$\mathbb{F}^{(n)}$-rational' points is not finite. In this subsection, we pose two different counting problems to overcome such obstruction.

Throughout this section, let K be an algebraically closed, complete non-archimedean field with a non-trivial valuation ν such that the value group Γ_K is a subgroup of $\mathbb{Q}$. Let X be an irreducible algebraic variety over K of dimension d defined by an ideal $I \subseteq K[X_1^{\pm}, ..., X_m^{\pm}]$. Let $trop(I) := \{trop(f) \mid f \in I\} \subseteq \Gamma_K[X_1^{\pm}, ..., X_m^{\pm}]$ and $Trop(X)$ be a tropical variety over Γ_K defined by $trop(I)$. Note that we consider $\Gamma_K \cup \{-\infty\}$ as the subsemifield of $\mathbb{Q}_{max}$ by imposing the idempotent operation induced from $\mathbb{R}_{max}$. From the structure theorem of tropical geometry (cf. Theorem 2.1.8 or [30, Theorem 3.3.6] for details), $Trop(X)$ is the support of a polyhedral complex of pure dimension d. Since X is a subvariety of a torus, counting $\mathbb{F}$-points or $\mathbb{F}^{(n)}$-points is indeed equivalent to counting $\mathbb{Z}$-points or $\frac{1}{n}\mathbb{Z}$-points of $Trop(X)$. By introducing such notions, our goal is to find a proper definition of a (Hasse-Weil type) zeta function of a tropical variety.

The first counting problem

Let X and K be as above. For $l \in \mathbb{R}_{>0}$, we define the following number:

$$N_n(X, l) := \#\{(x_1, ..., x_m) \in Trop(X) \cap (\mathbb{F}^{(n)})^m \mid \max(|x_1|, ..., |x_m|) \leq l\}.$$

In other words, $N_n(X, l)$ is the number of $\mathbb{F}^{(n)}$-rational points $x = (x_1, ..., x_m)$ of $Trop(X)$ such that $|x_i|$ is bounded by l. In particular, $N_1(X, l)$ is the number of

$\mathbb{F}$-points of $Trop(X)$ which are bounded by l. In general, $N_n(X,l)$ goes to infinity as l goes to infinity. Therefore, we will focus on the asymptotic behavior of the following (suitably normalized) number:

$$R(X,n) := \lim_{l\to\infty} \frac{N_n(X,l)}{N_1(X,l)}.$$

When $N_n(X,l) = N_1(X,l) = 0 \ \forall l \in \mathbb{R}_{>0}$, we define $R(X,n) := 0$. The main result in this subsection is Proposition 2.1.19: for an irreducible curve X in a torus over a suitable field, we have $R(X,n) = n$ for infinitely many $n \in \mathbb{Z}$.

As an example, consider $X = T^m = (K^*)^m$, an m-dimensional torus. We then have $Trop(X) = \mathbb{R}^m$. In fact, let $Y := \{(\nu(x_1),...,\nu(x_n) \mid x_i \in K^*\} = \Gamma_K^m$, where Γ_K is the value group of K. Since K is algebraically closed, Γ_K is dense in $\mathbb{R}$. It follows from Theorem 2.1.7 that $\overline{Y} = \mathbb{R}^m = Trop(X)$. Then, for $l \in \mathbb{Z}_{>0}$, $N_1(X,l) = (2l+1)^m$ and $N_n(X,l) = (2nl+1)^m$. Thus, if we follow the sequence of natural numbers, the limit $R(X,n)$ will be n^m. What is interesting is that if we consider an m-dimensional torus over a finite field $\mathbb{F}_q$, then the number of $\mathbb{F}_q$-rational points is $(q-1)^m$ and the number of $\mathbb{F}_{q^n}$-rational points is $(q^n-1)^m$. Then, we observe that the following limit

$$\lim_{q\to 1} \frac{(q^n-1)^m}{(q-1)^m} = \lim_{q\to 1} (\frac{q^n-1}{q-1})^m = n^m$$

gives the same number. In the above example, we computed $R(T^m,n)$ only with $l \in \mathbb{Z}_{>0}$. In fact, we have the following:

Proposition 2.1.15. *Let $X = T^m$ be an m-dimensional torus over K. Then the limit $R(X,n)$ exists and is equal to n^m.*

Proof. For $l \in \mathbb{R}_{>0}$, let $\lfloor l \rfloor$ be the greatest integer which is less than or equal to l and let $B_l := \{x = (x_1,...,x_m) \in \mathbb{R}^m \mid |x_i| \leq \lfloor l \rfloor\}$. Consider the following sets:

$$M_1(n) := \#\{x = (x_1,...,x_m) \in (\mathbb{F}^{(n)})^m \mid \max(|x_1|,...,|x_n|) \leq \lfloor l \rfloor\},$$

$$M_2(n) := \#\{x = (x_1, ..., x_m) \in (\mathbb{F}^{(n)})^m \mid \lfloor l \rfloor < |x_i| \leq l \text{ for some } i\}.$$

Then, $M_1(n) = (2n\lfloor l \rfloor + 1)^m$ and $N_n(X, l) = M_1(n) + M_2(n)$. Since $(l - \lfloor l \rfloor) \leq 1$, the number of $\mathbb{F}^{(n)}$-points in the closed interval $[\lfloor l \rfloor, l]$ is less than or equal to n. Because the number of facets of B_l is 2^m, we have the following bound:

$$0 \leq M_2(n) \leq 2^m n(2nl + 1)^{m-1}.$$

In particular, for $n = 1$, we have

$$N_1(X, l) = M_1(1) + M_2(1), \quad M_1(1) = (2\lfloor l \rfloor + 1)^m, \quad 0 \leq M_2(1) \leq 2^m(2l + 1)^{m-1}.$$

It follows from the definition that

$$R(X, n) := \lim_{l \to \infty} \frac{N_n(X, l)}{N_1(X, l)} = \lim_{l \to \infty} \frac{M_1(n) + M_2(n)}{M_1(1) + M_2(1)}.$$

Since m is a fixed number and M_2 is bounded by the polynomial in l of degree $(m-1)$, we have

$$\lim_{l \to \infty} \frac{M_2(n)}{M_1(n)} = \lim_{l \to \infty} \frac{M_2(1)}{M_1(n)} = 0, \quad \lim_{l \to \infty} \frac{M_1(1)}{M_1(n)} = \lim_{l \to \infty} \frac{(2\lfloor l \rfloor + 1)^m}{(2n\lfloor l \rfloor + 1)^m} = \frac{1}{n^m}.$$

Hence we have

$$R(X, n) := \lim_{l \to \infty} \frac{N_n(X, l)}{N_1(X, l)} = \lim_{l \to \infty} \frac{1}{\frac{M_1(1)}{M_1(n)}} = n^m.$$

□

Next, we consider the case of a plane tropical curve V. In fact, V is a finite (planar) graph in this case; the following is known.

Remark 2.1.16. *([30, Proposition 1.3.1]) A plane tropical curve V is a finite graph which is embedded in the plane $\mathbb{R}^2$. It has both bounded and unbounded edges, all edge slopes are rational, and this graph satisfies a balancing condition around each node.*

Unlike the torus case, when we deal with plane curves, a choice of n should be general enough as the following example illustrates.

Example 2.1.17. *Let V be the plane tropical curve defined by $X^{\odot t} \oplus Y^{\odot t} \oplus 1$, where $t > 1$. Then V is the graph with the three unbounded edges in $\mathbb{R}^2$; $X = \{(x, \frac{1}{t}) \mid x \leq \frac{1}{t}\}$, $Y = \{(\frac{1}{t}, y) \mid y \leq \frac{1}{t}\}$, and $Z = \{(z, z) \mid \frac{1}{t} \leq z\}$. Suppose that $n = t$. Then, on the edge X, we have infinitely many $\mathbb{F}^{(n)}$-points, but no $\mathbb{F}$-point. Thus, we have $R(V, n) = \infty$ in this case. On the other hand, if we choose n so that $t \nmid n$, then on edges X and Y, there is no $\mathbb{F}$ or $\mathbb{F}^{(n)}$-point. On the edge Z, the similar computation as in the torus case shows that $R(V, n) = n$. Thus, as long as $t \nmid n$, we have $R(V, n) = n$.*

In fact, this is true for any plane tropical curve.

Proposition 2.1.18. *Let V be a plane tropical curve. Then, for infinitely many integers n, $R(V, n)$ exists. Furthermore, we have $R(V, n) = n$ if at least one of the following conditions is satisfied:*

1. *V has an unbounded edge which is not parallel to a coordinate axis.*

2. *Each vertex of V is an element of $\mathbb{Z}^2$.*

Proof. This is actually an easy consequence of Remark 2.1.16. We examine each case of edges. Let $Y = r$ be a horizontal edge (i.e. parallel to the first coordinate axis) with a vertex (a, r) in $\mathbb{Q}^2$. If r is an integer, then we have $R(Y = r, n) = n$ $\forall n \in \mathbb{N}$ as in the case of torus. If $r \in \mathbb{F}^{(t)} \backslash \mathbb{F}$, for an integer n such that $\gcd(n, t) = 1$, we have no $\mathbb{F}$-point and $\mathbb{F}^{(n)}$-point. Therefore, in this case, $R(Y = r, n) = 0$. For the case of a vertical edge $X = r$, the exact same argument works. Finally, for an unbounded edge Z with a rational slope which is not parallel to a coordinate axis, we have infinitely many $\mathbb{F}$-points (hence, $\mathbb{F}^{(n)}$-points). Moreover, since Z has a slope which is not zero nor infinity, Z passes an integral point in finite length. However, the finite line segment of Z does not change the limit $R(Z, n)$ since Z has infinitely many $\mathbb{F}$ and $\mathbb{F}^{(n)}$-points. It follows that we may assume that the vertex (a, b) of the edge Z is in $\mathbb{Z}^2$ for computing the limit $R(Z, n)$. We may further assume that $(a, b) = (0, 0)$

since this will not change the number of $\mathbb{F}$ or $\mathbb{F}^{(n)}$-points. Therefore, we assume that $Z = \{(x, \frac{k}{m}x) \mid 0 \leq x\}$, where $m, k \in \mathbb{Z}\backslash\{0\}$. If $m \mid k$, then the counting argument is same as the torus case. Hence, we assume that $\gcd(m, k) = 1$. Suppose that $|k| < |m|$. Then, for $l \in \mathbb{R}_{>0}$, the $\mathbb{F}$-points on the ray Z are $(0,0), (m, k), (2m, 2k), (3m, 3k)$..... Since $|k| < |m|$, we have $(N_1(Z, l) - 1)|m| \leq l$. Hence, $N_1(Z, l) \leq (\frac{l}{|m|} + 1) := \tilde{l} + 1$ and $N_1(Z, l) = \left\lfloor \tilde{l} \right\rfloor + 1$. Similarly, we can find $\mathbb{F}^{(n)}$-points. In fact, since $\frac{k}{m}(\frac{\alpha}{n}) = (\frac{\beta}{n}) \iff m \mid \alpha$, one observes that $\mathbb{F}^{(n)}$-points are given by $(0,0), (\frac{m}{n}, \frac{k}{n}), (\frac{2m}{n}, \frac{2k}{n})$... Since $|k| < |m|$, we have $(N_n(Z, l) - 1)\frac{|m|}{|n|} \leq l$ and $N_n(Z, l) \leq (\frac{l}{|m|})n + 1 = \tilde{l}n + 1$. This implies that

$$N_n(Z, l) = \left\lfloor \tilde{l}n \right\rfloor + 1 = \left\lfloor \tilde{l} \right\rfloor n + C, \quad |C| < n + 1.$$

Thus, we have

$$R(Z, l) := \lim_{l \to \infty} \frac{N_n(Z, l)}{N_1(Z, l)} = n.$$

Now, let

$$V = P_1 \cup ... \cup P_s \cup f_1 \cup ... \cup f_t,$$

where P_i are unbounded edges and f_i are bounded edges. Assume that for each $i = 1, ..., s$, the limit $R(P_i, n)$ exists and $R(P_i, n) = n$ for at least one i. Then, since f_i are all bounded edges, there exists $0 < \delta$ such that $\forall x \in f_i$, $|x| < \delta$ $\forall i = 1, ..., t$. Let $G_1 := f_1 \cup ... \cup f_t$ and $G_2 := P_1 \cup ... \cup P_s$. Then, we have $N_n(V, l) = N_n(G_1, l) + N_n(G_2, l) - C$, where C is a finite number which is less than or equal to the number of vertices of V. Since G_1 is a union of bounded edges, for a large l, we have some finite numbers A and B such that

$$R(V, n) = \lim_{l \to \infty} \frac{N_n(G_1, l) + N_n(G_2, l) - C}{N_1(G_1, l) + N_1(G_2, l) - C} = \lim_{l \to \infty} \frac{A + N_n(G_2, l)}{B + N_1(G_2, l)}.$$

If $R(P_i, n)$ exists, then

$$|N_n(P_i, l) - nN_1(P_i, l)| < 2 \cdot n \quad \forall i = 1, ..., s. \tag{2.1.7}$$

In fact, suppose that $R(P_i, n)$ exists. Then, the numbers $N_n(P_i, l)$ and $N_1(P_i, l)$ are either both zero or both non-zero for $l >> 0$. Therefore, the only difference between $N_n(P_i, l)$ and $nN_1(P_i, l)$ happens at each side of the edge. Thus, we obtain (2.1.7). However, we proved that, in any case, $R(P_i, n)$ exists and is equal to either 0 or n. Thus, for $l >> 0$, we have

$$\begin{aligned} |\frac{A + \sum_{i=1}^{s} N_n(P_i, l)}{B + \sum_{i=1}^{s} N_1(P_i, l)} - n| &= |\frac{(A - nB) + \sum_{i=1}^{s}(N_n(P_i, l) - nN_1(P_i, l))}{B + \sum_{i=1}^{s} N_1(P_i, l)}| \\ &\leq |\frac{(A - nB) + 2ns}{B + \sum_{i=1}^{s} N_1(P_i, l)}|. \end{aligned} \tag{2.1.8}$$

Since we assumed that $R(P_i, n) = n$ for some i, RHS of (2.1.8) goes to zero when l goes to infinity. It follows that $R(V, n) = n$.

To sum up, when V has only unbounded edges which are parallel to coordinate axises, there are two possible sub-cases. The first is when at least one edge is emanated from an integral point. In this case, the above computations show that $R(V, n) = n$. The second case is when all edges are emanated from non-integral points. In this case, for infinitely many integer n, we have $R(V, n) = 0$. The last case is when V has an unbounded edge which is not parallel to a coordinate axis. In this case, the above computation shows that $R(V, n) = n$ for infinitely many integer n. This proves our proposition. □

In fact, Proposition 2.1.18 can be generalized as follows:

Proposition 2.1.19. *Let K be an algebraically closed field with a complete, nontrivial, non-archimedean valuation with a value group $\Gamma_K \subseteq \mathbb{Q}$. Let X be an irreducible curve over K in T^m and $V := Trop(X)$. Then, for infinitely many integer n, the limit $R(V, n)$ exists. In particular, $R(V, n) = n$ if V satisfies at least one of the following* *conditions:*

1. *V has an unbounded edge which is not parallel to a coordinate axis.*

2. *Each vertex of V is an element of $\mathbb{Z}^m$.*

Proof. The proof is similar to the proof of Proposition 2.1.18. From Theorem 2.1.8 (the structure theorem), V is a finite graph in $\mathbb{R}^m$. We investigate the possible cases of the edges of V. Let P be an unbounded edge which is not parallel to a coordinate axis. Then, P will both have infinitely many $\mathbb{F}$ and $\mathbb{F}^{(n)}$-points $\forall n \in \mathbb{N}$ since P is emanated from a point in $\mathbb{Q}^m$ and has a rational slope. Fix $l \in \mathbb{R}_{>0}$ and consider the following box B with the side length $2l$:

$$B := \{x = (x_1, ..., x_m) \in \mathbb{R}^m \mid |x_i| \leq l\}.$$

Let $\psi := B \cap P$ be a line segment in B. Suppose that l is large enough so that ψ has more than two of $\mathbb{F}$ and $\mathbb{F}^{(n)}$-points. This is possible since ψ contains infinitely many $\mathbb{F}$ and $\mathbb{F}^{(n)}$-points. Let Z, W be the integral points of ψ such that the distance between them is the largest among all pairs of integral points of ψ. We label the integral points on the line segment ψ as $Z = A_0, A_1, ..., A_{d-1} = W$ so that there is no integral point between A_i and A_{i+1}. In particular, $N_1(P, l) = d$. We claim that for each sub-segment $\overline{A_i A_{i+1}}$, we have $(n+1)$ of $\mathbb{F}^{(n)}$-points including both ends. For the notational convenience, let $A_i = R$ and $A_{i+1} = T$. Then, we have

$$S := \overline{RT} = \{(1-t)R + tT \mid t \in [0, 1]\}.$$

Since R and T are $\mathbb{F}$-points, it follows that S contains at least $(n+1)$ of $\mathbb{F}^{(n)}$-points given by $t = \frac{k}{n}$, where $k \in \{0, 1, ..., n\}$. Suppose that S contains more than $(n+1)$ of $\mathbb{F}^{(n)}$-points. Then, there exist $\mathbb{F}^{(n)}$-points $u = (1-t_1)R + t_1 T$ and $v = (1-t_2)R + t_2 T$ such that $|t_2 - t_1| < \frac{1}{n}$. Let $t_3 := n(t_2 - t_1)$. It follows that

$$(1-t_3)R + t_3 T = R + t_3(T - R) = R + n(t_2 - t_1)(T - R).$$

We observe that

$$u-v=(1-t_1)R+t_1T-(1-t_2)R-t_2T=R(t_2-t_1)+T(t_1-t_2)=(t_2-t_1)(R-T).$$

Since u and v are $\mathbb{F}^{(n)}$-points, the point $v-u=(t_2-t_1)(T-R)$ is also an $\mathbb{F}^{(n)}$-point and hence $n(v-u)=n(t_2-t_1)(T-R)$ is an $\mathbb{F}$-point. This implies that $(1-t_3)R+t_3T$ is an $\mathbb{F}$-point between R and T, and this gives a contradiction. Thus, there are exactly $(n+1)$ of $\mathbb{F}^{(n)}$-points on $\overline{RT}$. Therefore, if $N_1(P,l)=d$, then $N_n(P,l)=n(d-1)+1+C(l)$, where $C(l)$ is a constant such that $|C(l)|\leq 2(n+1)$ $\forall l\in\mathbb{R}_{>0}$. It follows that

$$R(P,n):=\lim_{l\to\infty}\frac{N_n(P,l)}{N_1(P,1)}=\lim_{d\to\infty}\frac{n(d-1)+C(l)}{d}=n.$$

The second case is when P is parallel to some coordinate axises. There are three sub-cases. The first case is when all coordinates x_i which are parallel to coordinate axises are of the form $x_i=m_i\in\mathbb{Z}$. In this case, the same argument as above gives us the number $R(P,n)=n$. The second case is when $x_i=m_i\in\mathbb{F}^{(e_i)}\backslash\mathbb{F}$ for some $e_i\in\mathbb{N}$. Then, by a choice of n such that $\gcd(n,e_i)=1$, we have $R(P,n)=n$ or $R(P,n)=0$. The case of $R(P,n)=0$ happens when all such x_i are in $m_i\in\mathbb{F}^{(e_i)}\backslash\mathbb{F}$. The final case is when none of x_i is in $\mathbb{F}^{(e_i)}$. Then, we have $R(P,n)=0$. For the general case of V, we can compute in the exact same way as in the plane curve case.
To sum up, if V has no unbounded edge, then $R(V,n)$ exists $\forall n\in\mathbb{N}$. If V has an unbounded edge which is not parallel to a coordinate axis, then for infinitely many (positive) integer n, we have $R(V,n)=n$. If V has unbounded edges and all of such edges are parallel to some coordinate axises with $x_i=m_i$, then as we analyzed above, for infinitely many $n\in\mathbb{N}$, the limit $R(P,n)$ exists and equal to 0 or n depending on values m_i. This completes our proof. $\square$

If a dimension of an algebraic variety X is greater than 1, in general, it seems hard to compute above number $R(X,n)$. Also, as we computed above, computing $R(V,n)$ is closely related to computing $\mathbb{F}$-points or, in general, $\mathbb{F}^{(n)}$-points of polytopes. Thus, in the next subsection, we pose the second counting problem which measures asymptotic behavior of the numbers of rational points by using a filtration of polytopes.

The second counting problem

For a bounded subset X of $\mathbb{R}^m$, we define the following number:

$$N_n(X) = \#(X \cap (\mathbb{F}^{(n)})^m).$$

In particular, $N_1(X)$ is the number of integral points of X. In this subsection, we investigate a sequence $\{X_i\}$ of subsets of a tropical variety V which satisfies the following properties:

1. $$X_i \subseteq X_{i+1}, \quad \bigcup_{i \geq 1} X_i = V. \tag{2.1.9}$$

2. The limit
$$R(V, \{X_i\}, n) := \lim_{i \to \infty} \frac{N_n(X_i)}{N_1(X_i)} \tag{2.1.10}$$
makes sense.

The main result of this subsection is Corollary 2.1.22; if $V = Trop(X)$ is a support of a polyhedral fan which is pure of dimension d, then there exists a sequence $\{X_i\}$ of subsets of V which satisfies (2.1.9) and (2.1.10). In particular, $R(V, \{X_i\}, n) = n^d$. In the case when X is a rational polytope, a counting of lattice (i.e. integral) points has been studied and named Ehrhart theory (cf. [2], [44]). We briefly review the classical results of Ehrhart theory. Recall that by a quasi-polynomial f of degree d we mean a function $f : \mathbb{Z} \longrightarrow \mathbb{C}$ of the following form:

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + ... + c_0(n),$$

where $c_i(n)$ is a periodic function with an integer period and $c_d(n)$ is not identically zero. Equivalently, f is a quasi-polynomial if there exists $N > 0$ (namely, a common period of $c_0, ..., c_d$) and polynomials $f_0, ..., f_{N-1}$ such that $f(n) = f_i(n)$ if $n \equiv i$(mod N). An integer N (which is not unique) is called a quasi-period of f. Let P be a convex rational polytope in $\mathbb{R}^m$. For $M \in \mathbb{N}$, we define the following nonnegative integer:

$$i(P, M) = \#(MP \cap \mathbb{Z}^m),$$

where $MP := \{Mx \mid x \in P\}$. Then, for each convex rational polytope P, there exists a quasi-polynomial f such that $f(M) = i(P, M)$. Furthermore, the leading coefficient c_d is known to be the (suitably normalized) volume of P. In particular, c_d is indeed a constant. Let us further recall some definitions. By a polyhedral cone P in $\mathbb{R}^m$ we mean a set of the following form:

$$P = \{\sum_{i=1}^{k} \lambda_i v_i \mid 0 \leq \lambda_i\} \text{ for some fixed } v_1, ..., v_k \in \mathbb{R}^m.$$

A polyhedral cone P is called a rational polyhedral cone if $v_1, ...v_k \in \mathbb{Q}^m$. The following result can be easily derived.

Lemma 2.1.20. *For a d-dimensional rational polyhedral cone P in $\mathbb{R}^m$, there exists a sequence $\{P_i\}$ of convex rational polytopes in P such that $P_j \subseteq P_{j+1}$, $\bigcup_{j\geq 1} P_j = P$, and*

$$R(P, \{P_j\}, n) := \lim_{j\to\infty} \frac{N_n(P_j)}{N_1(P_j)} = n^d.$$

Proof. By the definition, there exist $v_1, ..., v_k \in \mathbb{Q}^m$ such that $P = \{\sum_{i=1}^{k} \lambda_i v_i \mid 0 \leq \lambda_i\}$. Consider the following subset of P:

$$P_1 := \{\sum_{i=1}^{k} \lambda_i v_i \mid 0 \leq \lambda_i \leq 1\}.$$

We then have $P_1 \subseteq P$. One can further clearly observe that P_1 is a convex rational polytope. Fix an integer $N > 1$ and for each $j \in \mathbb{N}$, we define the following set:

$$P_j := N^{j-1}P_1 = \{N^{j-1}\alpha \mid \alpha \in P_1\}.$$

Since P_j is a rescaling of P_1 by a natural number, we know that P_j is a convex rational polytope $\forall j \in \mathbb{N}$. We claim that $P_j \subseteq P_{j+1}$. In fact, it is enough to show that $P_1 \subseteq P_2$. We have $\alpha \in P_1 \iff \alpha = \sum_{i=1}^k \lambda_i v_i$ for some $0 \le \lambda_i \le 1$. Let $\beta := \frac{1}{N}\alpha = \sum_{i=1}^k \frac{\lambda_i}{N} v_i$. Since $\lambda_i \le 1 < N$, we have $\frac{\lambda_i}{N} < 1$ and $\beta \in P_1$. Therefore, $N\beta = \alpha \in P_2$ and hence $P_1 \subseteq P_2$. For the second assertion, for $\alpha = \sum_{i=1}^k \lambda_i v_i \in P$, there exists j such that $\lambda_i \le N^{j-1}$ $\forall i = 1, ..., k$. It follows that $\alpha \in P_j$ and hence $\bigcup_{j \ge 1} P_j = P$. For the last assertion, we first observe that for a bounded set Q of $\mathbb{R}^m$, there is a set bijection φ as follows:

$$\varphi : X := (Q \cap (\mathbb{Z}[\frac{1}{n}])^m) \longrightarrow Y := (nQ \cap \mathbb{Z}^m), \quad \alpha \mapsto n\alpha.$$

In fact, φ is well-defined since for $\alpha \in X$, we have $n\alpha \in Y$. Clearly, φ is an injection, and the inverse map φ^{-1} is given by sending β to $\frac{1}{n}\beta$. From this bijection, we obtain

$$i(P_j, n) = N_n(P_j).$$

It follows from Ehrhart's theory that there exists a quasi-polynomial $f(x) = a_d x^d + a_{d-1}x^{d-1} + ... + a_0$ such that $f(M) = i(P_1, M) = N_M(P_1)$. Since $P_j = N^{j-1}P_1$, we have

$$i(P_j, n) = i(N^{j-1}P_1, n) = i(P_1, nN^{j-1}).$$

Thus,

$$\frac{N_n(P_j)}{N_1(P_j)} = \frac{i(P_j, n)}{i(P_j, 1)} = \frac{i(P_1, N^{j-1}n)}{i(P_1, N^{j-1})} = \frac{a_d(N^{j-1}n)^d + a_{d-1}(N^{j-1}n)^{d-1} + ... + a_0}{a_d(N^{j-1})^d + a_{d-1}(N^{j-1})^{d-1} + ... + a_0}.$$

Since a_d and n are fixed, a_i are bounded, and $N > 1$, we have

$$\lim_{j\to\infty} \frac{N_n(P_j)}{N_1(P_j)} = \frac{a_d(N^{j-1}n)^d + a_{d-1}(N^{j-1}n)^{d-1} + ... + a_0}{a_d(N^{j-1})^d + a_{d-1}(N^{j-1})^{d-1} + ... + a_0} = n^d.$$

This proves our lemma. $\square$

Recall that by a finite polyhedral fan Σ we mean a finite collection of polyhedral cones such that the intersection of any two is a face of each. The support $|\Sigma|$ of Σ is the set, $\{\alpha \in \mathbb{R}^m \mid \alpha \in P \text{ for some } P \in \Sigma\}$. A polyhedral fan Σ is said to be pure of dimension d if every polyhedral cone in Σ that is not the face of other cones in Σ has dimension d.

Theorem 2.1.21. *Let Σ be a finite rational polyhedral fan which is pure of dimension d in $\mathbb{R}^m$. Then, there exists a sequence of subsets $X_i \subseteq |\Sigma|$ such that $X_j \subseteq X_{j+1}$, $\bigcup_{j\geq 1} X_j = |\Sigma|$, and*

$$\lim_{j\to\infty} \frac{N_n(X_j)}{N_1(X_j)} = n^d.$$

Proof. Let $P_1, ..., P_r$ be all of d-dimensional rational cones in Σ. Fix an integer $N > 1$. For each $P_i = \{\sum_{i=1}^k \lambda_i v_i \mid 0 \leq \lambda_i\}$, we define a sequence of polytopes $Q_{i,j} \subset P_i$ as follows:

$$Q_{i,1} := \{\sum_{i=1}^{k} \lambda_i v_i \mid 0 \leq \lambda_i \leq 1\}, \quad Q_{i,j} := N^{j-1} Q_{i,1} \text{ for } j \geq 2.$$

We then define the following set:

$$X_j := \bigcup_{i=1}^{r} Q_{i,j}.$$

Clearly, we have $X_j = N^{j-1}X_1$. By the exact same argument as in Lemma 2.1.20, we have $X_j \subseteq X_{j+1}$ and $\bigcup_{j\geq 1} X_j = |\Sigma|$. Thus, all we have to prove is the last assertion. Let $f_i(x)$ be the quasi-polynomial of degree d associated to $Q_{i,1}$ as in Lemma 2.1.20. Then, we have

$$i(X_1, M) = (\sum_{i=1}^{r} f_i(M)) + g(M),$$

where $g(x)$ is a quasi-polynomial of degree less than or equal to $(d-1)$ which we obtain from an inclusion-exclusion computation by using Lemma 2.1.20 since a face of a cone is a cone. It follows that

$$i(X_j, n) = i(N^{j-1}X_1, n) = i(X_1, N^{j-1}n) = (\sum_{i=1}^{r} f_i(N^{j-1}n)) + g(N^{j-1}n).$$

Since the degree of $g(x)$ is less than or equal to $(d-1)$ and $N > 1$, we have

$$\lim_{j\to\infty} \frac{N_n(X_j)}{N_1(X_j)} = \lim_{j\to\infty} \frac{i(X_1, N^{j-1}n)}{i(X_1, N^{j-1})} = \lim_{j\to\infty} \frac{(\sum_{i=1}^{r} f_i(N^{j-1}n)) + g(N^{j-1}n)}{(\sum_{i=1}^{r} f_i(N^{j-1})) + g(N^{j-1})} = n^d.$$

□

Corollary 2.1.22. *Let X be an irreducible algebraic variety contained in a torus T^m over K. Suppose that $Trop(X)$ is a support of polyhedral fan Σ. Then, there exists a sequence of subsets $X_i \subseteq Trop(X)$ such that $X_j \subseteq X_{j+1}$, $\bigcup_{j\geq 1} X_j = Trop(X)$, and*

$$\lim_{j\to\infty} \frac{N_n(X_j)}{N_1(X_j)} = n^d.$$

Proof. This is straightforward. □

Example 2.1.23. *Let SL_2 be the algebraic variety defined by a polynomial $xy - zw - 1 \in K[x, y, z, w]$. Consider $X := SL_2 \cap T^4$, where T^4 is a torus. Then, $Trop(X)$ consists of the following three cones:*

$$X_1 := \{(x, y, z, w) \in \mathbb{R}^4 \mid 0 \leq x + y = z + w\},$$

$$X_2 := \{(x, y, z, w) \in \mathbb{R}^4 \mid z + w \leq x + y = 0\},$$

$$X_3 := \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y \leq z + w = 0\}.$$

Each X_i is indeed a cone since we can write them in the matrix form. For example,

X_1 can be written as follows:

$$X_1 = \{\alpha = (x, y, z, w) \in \mathbb{R}^4 \mid A\alpha \leq 0\}, \text{ where } A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

It follows that $Trop(X)$ is a support of polyhedral fan and hence we can apply our corollary to X.

Example 2.1.24. *The tropicalization $Trop(X)$ of an irreducible curve X in T^m over K is a finite connected graph, and this is a special case of a polyhedral fan. Therefore, we can apply our corollary to $Trop(X)$.*

Example 2.1.25. *Consider the Grassmannian $X := G(d, m) \cap T^{\binom{m}{d}}$ (in a torus) as an algebraic variety defined by the Plücker ideal $I_{d,m}$. Let $Trop(X)$ be the tropicalization of X. Then, for $d = 2$, $Trop(X)$ is a polyhedral fan in $\mathbb{R}^{\binom{m}{2}}$ (cf. [42, Corollary 3.1]).*

Remark 2.1.26.

1. *Let X be a hypersurface defined by $f = \sum_{\alpha \in \mathbb{Z}^m} c_\alpha X^\alpha \in K[X_1^{\pm 1}, ..., X_m^{\pm 1}]$. If the values $\nu(c_\alpha)$ of c_α occurring in f are all same, then $Trop(V(f))$ is a polyhedral fan. Furthermore, if a valuation of a field K is trivial, then for any irreducible (algebraic) subvariety X of T^m, $Trop(X)$ is a finite polyhedral fan (cf. [30]).*

2. *In some cases, a collection of convex rational polytopes $P_1, ..., P_r$ totally determines $Trop(X)$. Since the number of $\mathbb{F}^{(n)}$-points in a convex rational polytope P_i is finite, one is induced to consider a generating function of the following type:*

$$F(\lambda) = 1 + \sum_{j=1}^{r} \sum_{n \geq 1} N_n(P_j)\lambda^n.$$

Since P_j is a convex rational polytope, we have $i(P_j, n) = N_n(P_j)$, hence

$$F(\lambda) = 1 + \sum_{j=1}^{r} \sum_{n \geq 1} i(P_j, n)\lambda^n.$$

In fact, the function of the type $g(\lambda) = 1 + \sum_{n\geq 1} i(P,n)\lambda^n$ is known to be a rational function for any polytope P (cf. Theorem 4.6.25, [44]). For example, if $Trop(X)$ is defined by $x \oplus y \oplus 1$, this is a union of three rays; $Q_1 = \{(x,0) \mid x \leq 0\}$, $Q_2 := \{(0,y) \mid y \leq 0\}$, $Q_3 := \{(x,x) \mid 0 \leq x\}$. Thus, three integral vectors $v_1 = (-1,0)$, $v_2 = (0,-1)$, $v_3 = (1,1)$ contain all information about $Trop(X)$. Let P_i be a line segment connecting the origin and v_i and $P = P_1 \cup P_2 \cup P_3$. Then, $i(P,n) = (3n+1)$ and

$$F(\lambda) = 1 + \sum_{n\geq 1}(3n+1)\lambda^n = 1 + \frac{3\lambda}{1-\lambda} + \frac{\lambda}{(1-\lambda)^2}.$$

We will explain more about this idea in the next subsection.

2.1.3 A zeta function of a tropical variety

Recall that all finite semifield extensions of $\mathbb{F} = \mathbb{Z}_{max}$ are of the forms $\mathbb{F}^{(n)} := \frac{1}{n}\mathbb{Z} \cup \{-\infty\}$ for some positive integer n (cf. [47]). Intuitively, the relation between $\mathbb{F}$ and $\mathbb{F}^{(n)}$ is the characteristic one analogue of the relation between a finite field $\mathbb{F}_q$ and its finite extension $\mathbb{F}_{q^n}$. Therefore, one might consider a zeta function, in characteristic one, of a tropical variety V as a generating function of numbers of $\mathbb{F}^{(n)}$-points of V. However, a tropical variety is a support of a polyhedral complex; hence it has infinitely many $\mathbb{F}^{(n)}$-points in general.

In this section, we define a two variable (Hasse-Weil type) zeta function which encodes all information about $\mathbb{F}^{(n)}$-points of a tropical variety. Then we compute toy examples. Fix an integer $d \in \mathbb{N}$. For $m \in \mathbb{N}$, we define $B_m := [-m,m]^d \subseteq \mathbb{R}^d$. For a subset S of $\mathbb{R}^d$, we let $S_m := (S \bigcap B_m)$ and define the following number:

$$i(S_m,n) := \#\{S_m \bigcap (\mathbb{F}^{(n)})^d\} = \#\{nS_m \bigcap (\mathbb{F})^d)\}.$$

Furthermore, we define the following function Φ_S:

$$\Phi_S : \mathbb{Z}_{>0} \longrightarrow \mathbb{Q}[[t]], \quad m \mapsto 1 + \sum_{n\geq 1} i(S_m, n)t^n.$$

Finally, for a subset $S \subseteq \mathbb{R}^d$, we define a two variable zeta function $Z(S, v, t)$ as a formal series as follows:

$$Z(S, v, t) := \sum_{m\geq 1} \Phi_S(m)v^m = \sum_{m\geq 1}(\sum_{n\geq 0} i(S_m, n)t^n)v^m, \quad i(S_m, 0) := 1.$$

Proposition 2.1.27. *Let P be a convex rational polytope in $\mathbb{R}^d$. Then, $Z(P, t, v)$ is a rational function of t and v.*

Proof. Since P is a polytope, there exists $m_0 \in \mathbb{N}$ such that $P_m = P\ \forall m \geq m_0$. Then, for $m \geq m_0$, we have

$$\Phi(m) = 1 + \sum_{n\geq 1} i(P, n)t^n.$$

However, $\Phi(m)$ is named the Ehrhart series of P and known to be a rational function (cf. [44]). Let us denote this function by $Ehr_P(t)$. Then, we have

$$Z(P, v, t) = \sum_{m=1}^{m_0-1} \Phi(m)v^m + \sum_{m\geq m_0} Ehr_P(t)v^m.$$

Since $Ehr_P(t)$ is a rational function and

$$\sum_{m\geq m_0} Ehr_P(t)v^m = Ehr_P(t) \sum_{m\geq m_0} v^m = Ehr_P(t)(\frac{v^{m_0}}{1-v}),$$

we observe that $Z(P, v, t)$ is a rational function if and only if $\sum_{m=1}^{m_0-1} \Phi(m)v^m$ is a rational function. However, we have

$$\sum_{m=1}^{m_0-1} \Phi(m)v^m = \sum_{m=1}^{m_0-1} Ehr_{P_m}(t)v^m. \tag{2.1.11}$$

It follows from Ehrhart theory that each $Ehr_{P_m}(t)$ is a rational function. Since only finitely many m are involved, (2.1.11) is a rational function. $\square$

The next example shows that not only for polytopes but also for some polyhedra P, a zeta function $Z(P,t,v)$ is a rational function.

Example 2.1.28. *A d-dimensional tropical torus is $\mathbb{R}^d$. Let $P=\mathbb{R}^d$. Then, the zeta function $Z(P,t,v)$ is a rational function. Indeed, for each $m\in\mathbb{N}$, we have $P_m=B_m$ and $i(P_m,n)=(2nm+1)^d$. Since $(\sum_{n\geq 1}n^kt^n)'=\sum_{n\geq 1}n^{k+1}t^{n-1}=\frac{1}{t}\sum_{n\geq 1}n^{k+1}t^n$, from the induction argument, we can see that, for each $k\in\mathbb{N}$, the series $\sum_{n\geq 1}n^kt^n$ is a rational function. We denote this function by $f_k(t)$. We then have*

$$\Phi(m)=1+\sum_{n\geq 1}(2nm+1)^dt^n=1+\sum_{n\geq 1}(\sum_{k=0}^{d}2^kn^km^k)t^n=1+\sum_{k=0}^{d}2^km^k\sum_{n\geq 1}n^kt^n. \tag{2.1.12}$$

The last term of (2.1.12) is equal to $\sum_{k=0}^{d}2^km^kf_k(t)$. Hence, we have

$$Z(P,t,v)=\sum_{m\geq 1}(1+\sum_{k=0}^{d}2^km^kf_k(t))v^m=\sum_{m\geq 1}v^m+\sum_{m\geq 1}\sum_{k=0}^{d}2^km^kf_k(t)v^m$$

$$=\frac{v}{1-v}+\sum_{k=0}^{d}2^kf_k(t)\sum_{m\geq 1}m^kv^m=\frac{v}{1-v}+\sum_{k=0}^{d}2^kf_k(t)f_k(v).$$

Thus, in this case, $Z(P,t,v)$ is a rational function.

Example 2.1.29. *The tropicalization of the projective space $\mathbb{P}^n$ can be thought as the standard simplex Δ in dimension n (cf. [40]). In this case, it is known that $Ehr_\Delta(t)=\frac{1}{(1-t)^{d+1}}$. Therefore, one obtains that*

$$Z(\Delta,v,t)=\sum_{m\geq 1}\Phi_\Delta(m)v^m=\sum_{m\geq 1}\frac{1}{(1-t)^{n+1}}v^m=\frac{1}{(1-t)^{n+1}}\frac{1}{(1-v)}. \tag{2.1.13}$$

Let X be smooth, geometrically connected, projective variety of dimension n over a finite field $\mathbb{F}_q$. Let $Z(X,t)=Z(t)$ be the classical Hasse-Weil zeta function of X.

Then one has the following functional equation:

$$Z(\frac{1}{q^n t}) = \pm q^{\frac{qE}{2}} t^E Z(t), \tag{2.1.14}$$

where E is the Euler characteristic of X.

From (2.1.13), one also obtains the following functional equations:

$$Z(\Delta, \frac{1}{v}, t) = -vZ(\Delta, v, t), \quad Z(\Delta, v, \frac{1}{t}) = (-1)^{n+1} t^{d+1} Z(\Delta, v, t). \tag{2.1.15}$$

In characteristic one, we would have '$q = 1$'. Since $n+1$ is the Euler characteristic of $\mathbb{P}^n$, (2.1.15) can be thought as a characteristic one analogue of (2.1.14) for $X = \mathbb{P}^n$.

2.2 Construction of semi-schemes

In this section, we show that the classical construction of schemes can be directly generalized to the category of commutative semirings. Throughout this section, all semirings are assumed to be commutative. Also, by a semiring of characteristic one we mean a semiring M such that $x + y \in \{x, y\}$ $\forall x, y \in M$.

Recall that for a semiring M, by a prime ideal $\mathfrak{p}$ of M we mean an ideal $\mathfrak{p}$ of a semiring M such that if $xy \in \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. The set $X = \operatorname{Spec} M$ is a topological space equipped with Zariski topology. Then, as in the classical case, we can implement the structure sheaf $\mathcal{O}_X$ of X. For more details, see §1.1.1.

The first main result in this section is Proposition 2.2.4 stating that $\mathcal{O}_X(X) \simeq M$ for an affine semi-scheme $(X = \operatorname{Spec} M, \mathcal{O}_X)$.

Recall that a (multiplicatively) cancellative semiring M is a semiring such that: $\forall x, y, z \in M$, $xy = xz$ implies $y = z$ if $x \neq 0_M$. Note that this is different from that M has no (multiplicative) zero-divisor due to the lack of additive inverses.

The second main result is that, when M is a multiplicatively cancellative semiring of characteristic one, the structure sheaf $\mathcal{O}_X$ of the semi-scheme $(X = \operatorname{Spec} M, \mathcal{O}_X)$ is a sheaf of semirings of characteristic one.

Finally, we show that several notions of $\mathcal{O}_X$-modules can be generalized to $\mathcal{O}_X$-semimodules. In particular, we show that the classical construction of a Picard group can be generalized to semi-structures.

Lemma 2.2.1. *Let M be a semiring and S be a multiplicative subset of M. Let $N := S^{-1}M$ and $S^{-1} : M \longrightarrow N$ be a localization map (cf. §1 for the definitions). Then, we have the following universal property: let L be a semiring and $\varphi : M \longrightarrow L$ be a homomorphism of semirings such that each element of $\varphi(S)$ is multiplicatively invertible in L. Then, there exists a unique homomorphism $h : N \longrightarrow L$ of semirings such that $\varphi = h \circ S^{-1}$. Furthermore, if M is of characteristic one, then so is N.*

Proof. The proof of the universal property is well known in the theory of semirings. For example, see page 116 of [19]. For the last statement, if M is of characteristic one, then we have $x + y \in \{x, y\}$ $\forall x, y \in M$. Therefore, for $\frac{x}{a}, \frac{y}{b} \in N$, we have

$$\frac{x}{a} + \frac{y}{b} = \frac{bx + ay}{ab} \in \{\frac{bx}{ab}, \frac{ay}{ab}\} = \{\frac{x}{a}, \frac{y}{b}\}.$$

□

Lemma 2.2.2. *Let M be a semiring and $\mathfrak{p}$ be a prime ideal of M. Let $S := M\backslash\mathfrak{p}$. Then, the semiring $S^{-1}M(:= M_{\mathfrak{p}})$ has a unique maximal ideal, namely $S^{-1}\mathfrak{p}$.*

Proof. This is well known in the theory of semirings. For example, see [19, §10]. □

Lemma 2.2.3. *If $\varphi : N \longrightarrow M$ is a homomorphism of semirings, then for a prime ideal $\mathfrak{p}$ of M, φ induces a homomorphism of semirings $\varphi_{\mathfrak{p}}$ as follows:*

$$\varphi_{\mathfrak{p}} : N_{\mathfrak{q}} \longrightarrow M_{\mathfrak{p}}, \quad \frac{a}{b} \mapsto \frac{\varphi(a)}{\varphi(b)}, \text{ where } \mathfrak{q} = \varphi^{-1}(\mathfrak{p}).$$

Furthermore, if $\mathfrak{m}_2$ is the maximal ideal of $M_{\mathfrak{p}}$ and $\mathfrak{m}_1$ is the maximal ideal of $N_{\mathfrak{q}}$, then $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_2) = \mathfrak{m}_1$.

Proof. First, $\varphi_{\mathfrak{p}}$ is well defined. In fact, if $\frac{a}{b} = \frac{c}{d}$, then we have $sad = sbc$ in N for some $s \in N\backslash\mathfrak{q}$. It follows that $\varphi(s)\varphi(a)\varphi(d) = \varphi(s)\varphi(b)\varphi(c)$, and $\varphi(s) \notin \mathfrak{p}$. Thus, we have $\frac{\varphi(a)}{\varphi(b)} = \frac{\varphi(c)}{\varphi(d)}$. Furthermore, $\varphi_{\mathfrak{p}}$ is clearly a homomorphism of semirings. For the last assertion, we know that $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_2) \subseteq \mathfrak{m}_1$. However, from Lemma 2.2.2, $\mathfrak{m}_1 = S^{-1}\mathfrak{q}$, where $S = N \backslash \mathfrak{q}$. Suppose that $\frac{a}{b} \in \mathfrak{m}_1$, where $a \in \mathfrak{q}$ and $b \in S = N\backslash \mathfrak{q}$. Then, we can write $a = \varphi^{-1}(c)$ for some $c \in \mathfrak{p}$ and $b = \varphi^{-1}(d)$ for some $d \in M\backslash \mathfrak{p}$ since $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$. It follows that $\frac{a}{b} \in \varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_2)$ and hence $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_2) = \mathfrak{m}_1$. □

Let M be a semiring and $X = \operatorname{Spec} M$. We follow the classical construction of a structure sheaf. For an open subset U of X, we define

$$\mathcal{O}_X(U) := \{s : U \to \bigsqcup_{\mathfrak{p}\in U} M_{\mathfrak{p}}\}, \tag{2.2.1}$$

where $s \in \mathcal{O}_X(U)$ are sections such that $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ which also satisfies the following condition: for each $\mathfrak{p} \in U$, there exists an open neighborhood $V_{\mathfrak{p}} \subseteq U$ of $\mathfrak{p}$ and $a, f \in M$ such that

$$\forall \mathfrak{q} \in V_{\mathfrak{p}}, \quad f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = \frac{a}{f} \text{ in } M_{\mathfrak{q}}. \tag{2.2.2}$$

Clearly, $\mathcal{O}_X$ is a sheaf of sets. In fact, $\mathcal{O}_X(U)$ is a semiring under the following operations: for $s, t \in \mathcal{O}_X(U)$,

$$s \cdot t : U \to \bigsqcup M_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto s(\mathfrak{p})t(\mathfrak{p}), \quad s + t : U \to \bigsqcup M_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto s(\mathfrak{p}) + t(\mathfrak{p}). \tag{2.2.3}$$

By an affine semi-scheme we mean a pair $(X = \operatorname{Spec} M, \mathcal{O}_X)$ for a semiring M. Note that for a non-zero element $f \in M$, we let $M_f := S^{-1}M$, where $S = \{1, f, f^2, ..., \}$. Recall that, for an ideal I of M, we denote $V(I) := \{\mathfrak{p} \in \operatorname{Spec} M \mid I \subseteq \mathfrak{p}\}$ and $D(f) := \{\mathfrak{p} \in \operatorname{Spec} M \mid f \notin \mathfrak{p}\}$. In the sequel, by an affine semi-scheme $X = \operatorname{Spec} M$ we always mean a pair $(X = \operatorname{Spec} M, \mathcal{O}_X)$ of a topological space $\operatorname{Spec} M$ and a structure sheaf $\mathcal{O}_X$ unless otherwise stated.

Proposition 2.2.4. *Let M be a semiring and $X = \operatorname{Spec} M$ be an affine semi-scheme. Then, for a non-zero element $f \in M$, we have $M_f \simeq \mathcal{O}_X(D(f))$. In particular,*

$M \simeq \mathcal{O}_X(X)$. Furthermore, if M is (additively) idempotent, then so is $\mathcal{O}_X(U)$ for an open subset U of X.

Proof. The proof is similar to the classical case. For example, as in the classical case, if we take $f = 1$ then $M \simeq M_f$. Indeed, consider the following map:

$$\varphi : M \longrightarrow M_f, \quad a \mapsto \frac{a}{f}.$$

Then, φ is clearly a homomorphism of semirings and injective since $\frac{a}{f} = \frac{b}{f}$ if and only if there exits some $n \in \mathbb{N}$ such that $f^{n+1}a = f^{n+1}b$. This implies that $a = b$ since $f = 1$. Furthermore, φ is surjective; $\frac{a}{f^n} = \frac{a}{f} = \varphi(a)$ since $f = 1$. It follows that once we prove that $M_f \simeq \mathcal{O}_X(D(f))$, then the isomorphism $M \simeq \mathcal{O}_X(X)$ follows.
We first define the following map ψ from M_f to $\mathcal{O}_X(D(f))$:

$$\psi : M_f \longrightarrow \mathcal{O}_X(D(f)), \quad \frac{a}{f^n} \mapsto s, \tag{2.2.4}$$

where $s(\mathfrak{p}) = \frac{a}{f^n}$ in $M_\mathfrak{p}$. Then, ψ is well defined. Indeed, from the definition, we have $s(\mathfrak{p}) \in M_\mathfrak{p}$ for each $\mathfrak{p} \in D(f)$ and s as in (2.2.4) clearly satisfies the local representability condition (2.2.2). Furthermore, ψ is a homomorphism of semirings.
Next, we claim that ψ is injective. Suppose that $\psi(\frac{a}{f^n}) = \psi(\frac{b}{f^m})$. Then, $\frac{a}{f^n} = \frac{b}{f^m}$ in $M_\mathfrak{p}$ $\forall \mathfrak{p} \in D(f)$. This implies that $\exists\ h \notin \mathfrak{p}$ such that $hf^m a = hf^n b$ in M for each $\mathfrak{p} \in D(f)$. Let $J = \{\alpha \in M \mid \alpha f^m a = \alpha f^n b\}$. Then J is an ideal of M, and for $\mathfrak{p} \in D(f)$, we have $J \not\subseteq \mathfrak{p}$. It follows that $V(J) \cap D(f) = \emptyset$. However, for an ideal I of M, we have $\bigcap_{I \subseteq \mathfrak{p}} \mathfrak{p} = \sqrt{I}$ (cf. [19, Proposition 6.19]). Thus, $V(J) \bigcap D(f) = \emptyset$ implies that $V(J) \subseteq (D(f))^c = V(f)$. In particular, $f \in \sqrt{J}$ and hence $f^l \in J$ for some $l \in \mathbb{N}$ by Hilbert's Nullstellensatz for semirings (cf. Equation (1.1.1)). It follows that $f^{l+m}a = f^{l+n}b$ and $\frac{a}{f^n} = \frac{b}{f^m}$ in M_f. This shows that ψ is injective. The proof of surjectivity is also similar to the classical case since basic algebras of ideals of semirings are same as those of commutative rings (cf. [19]). The last assertion follows from the fact that if M is (additively) idempotent, then so is any localization

of M. $\square$

Recall that as in the category of commutative rings, direct limits and inverse limits exist in the category of semirings. For details, we refer the readers to [19]. It follows that the notion of stalks can be directly generalized to semi-structures.

Proposition 2.2.5. *Let M be a semiring. Then, for $\mathfrak{p} \in X = \operatorname{Spec} M$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ of the sheaf $\mathcal{O}_X$ is isomorphic to the local semiring $M_\mathfrak{p}$. Furthermore, if M is of characteristic one, then so is $\mathcal{O}_{X,\mathfrak{p}}$.*

Proof. The proof is exactly same as the classical case, but we include the proof here for the completeness. For an open neighborhood U of $\mathfrak{p}$, we define the map $\psi_U : \mathcal{O}_X(U) \longrightarrow M_\mathfrak{p}$ sending s to $s(\mathfrak{p})$. Clearly, ψ_U is a homomorphism of semirings which is compatible with restriction maps. Since $\mathcal{O}_{X,\mathfrak{p}}$ is the direct limit of the directed system $\{\mathcal{O}_X(U)\}_{U \ni \mathfrak{p}}$, there exists a unique homomorphism $\varphi : \mathcal{O}_{X,\mathfrak{p}} \longrightarrow M_\mathfrak{p}$ of semirings. We observe that φ is surjective. Indeed, from Proposition 2.2.4, each element $\frac{a}{f}$ of $M_\mathfrak{p}$ for which $f \notin \mathfrak{p}$ can be understood as an element of $\mathcal{O}_X(D(f))$. Thus, all we have to prove is that φ is an injection. For an open neighborhood U of $\mathfrak{p}$ and $s, t \in \mathcal{O}_X(U)$, suppose that $s(\mathfrak{p}) = t(\mathfrak{p})$ at $\mathfrak{p}$. Then, by shrinking U if necessary, we may assume that $s = \frac{a}{f}$ and $t = \frac{b}{g}$ on U, where $a, b, f, g \in M$ and $f, g \notin \mathfrak{p}$. Since $\frac{a}{f} = \frac{b}{g}$ in $M_\mathfrak{p}$, there exists $h \in M \backslash \mathfrak{p}$ such that $hag = hbf$ in M. Hence, s and t are equal on $U \cap D(f) \cap D(g) \cap D(h)$ which contains $\mathfrak{p}$. This implies that $s = t$ on some neighborhood of $\mathfrak{p}$ and hence they have the same stalk at $\mathfrak{p}$. The last assertion follows from the isomorphism φ and Lemma 2.2.1. $\square$

Let M be a semiring. An affine semi-scheme $(X = \operatorname{Spec} M, \mathcal{O}_X)$ is a locally semiringed space in the sense that it is a pair of a topological space X and the structure sheaf $\mathcal{O}_X$ of semirings such that $\mathcal{O}_{X,\mathfrak{p}}$ is a local semiring $\forall\ \mathfrak{p} \in X$. A semi-scheme is a locally semiringed space covered by affine semi-schemes. A morphism from a semi-scheme $(Y, \mathcal{O}_Y)$ to a semi-scheme $(X, \mathcal{O}_X)$ is a pair $(f, f^\#)$; a continuous map $f : Y \longrightarrow X$ and a map $f^\# : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ of sheaves of semirings such that the induced map of local semirings is local as in the classical case. Then one obtains the following:

Proposition 2.2.6. *Let M, N be semirings and let $X = (\operatorname{Spec} M, \mathcal{O}_X)$, $Y = (\operatorname{Spec} N, \mathcal{O}_Y)$ be affine semi-schemes. Then, we have the following set identification:*

$$\operatorname{Hom}(M, N) = \operatorname{Hom}(Y, X), \tag{2.2.5}$$

where $\operatorname{Hom}(M, N)$ *is the set of homomorphisms of semirings and* $\operatorname{Hom}(Y, X)$ *is the set of morphisms of semi-schemes.*

Proof. Let $\varphi : M \longrightarrow N$ be a homomorphism of semirings. From Lemma 2.2.3, φ induces a local homomorphism $\varphi_{\mathfrak{p}} : M_{\varphi^{-1}(\mathfrak{p})} \longrightarrow N_{\mathfrak{p}}$ for each $\mathfrak{p} \in \operatorname{Spec} N$. Moreover, φ induces a continuous map $f : \operatorname{Spec} N \longrightarrow \operatorname{Spec} M$ such that $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. Then f induces a morphism $f^{\#} : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ of sheaves. Indeed, for an open subset V of $\operatorname{Spec} M$, we have $\mathcal{O}_X(V) = \{s \mid s : V \longrightarrow \bigsqcup_{\mathfrak{p} \in V} M_{\mathfrak{p}}\}$ and $f_*\mathcal{O}_Y(V) := \mathcal{O}_Y(f^{-1}(V)) = \{t \mid t : f^{-1}(V) \longrightarrow \bigsqcup_{\mathfrak{q} \in f^{-1}(V)} N_{\mathfrak{q}}\}$ such that s and t satisfy the local condition (2.2.2). Consider the following maps:

$$\psi := \bigsqcup_{\mathfrak{p} \in V} \varphi_{\mathfrak{p}} : \bigsqcup_{\mathfrak{p} \in f^{-1}(V)} M_{\varphi^{-1}(\mathfrak{p})} \longrightarrow \bigsqcup_{\mathfrak{p} \in f^{-1}(V)} N_{\mathfrak{p}},$$

$$f^{\#}(V) : \mathcal{O}_X(V) \longrightarrow \mathcal{O}_Y(f^{-1}(V)), \quad s \mapsto t := \psi \circ s \circ f.$$

We first claim that $f^{\#}(V)$ is well defined. We have $t(\mathfrak{p}) = \psi \circ s(f(\mathfrak{p})) = \psi \circ s(\varphi^{-1}(\mathfrak{p}))$. However, $s(\varphi^{-1}(\mathfrak{p})) \in M_{\varphi^{-1}(\mathfrak{p})}$ and $\psi(M_{\varphi^{-1}}(\mathfrak{p})) \subseteq N_{\mathfrak{p}}$, thus $t(\mathfrak{p}) \in N_{\mathfrak{p}}$. Moreover, t satisfies the condition (2.2.2). In fact, let $\mathfrak{p} \in f^{-1}(V)$ such that $f(\mathfrak{p}) = \mathfrak{q} \in V$. Since $s \in \mathcal{O}_X(V)$, there exists a neighborhood $V_1 \subseteq V$ of $\mathfrak{q}$ and elements $a, f \in M$ which satisfy the following: $\forall\ \mathfrak{r} \in V_1$ with $f \notin \mathfrak{r}$, we have $s(\mathfrak{r}) = \frac{a}{f}$ in $M_{\mathfrak{r}}$. We define $V_2 := f^{-1}(V_1) \subseteq f^{-1}(V)$ which is a neighborhood of $\mathfrak{p}$. Then $\forall\ \mathfrak{u} \in V_2$ such that $\varphi(f) \notin \mathfrak{u}$, we have $t(\mathfrak{u}) = \psi \circ s(f(\mathfrak{u})) = \psi \circ s(\varphi^{-1}(\mathfrak{u}))$. However, since $f \notin \varphi^{-1}(\mathfrak{u}) \in V_1$, we have $t(\mathfrak{u}) = \psi \circ s(\varphi^{-1}(\mathfrak{u})) = \psi(\frac{a}{f}) = \frac{\varphi(a)}{\varphi(f)}$ in $M_{\mathfrak{u}}$ by Lemma 2.2.3. It follows that t

is in $\mathcal{O}_Y(f^{-1}(V))$ and hence $f^{\#}(V)$ is well defined.

Secondly, we show that $f^{\#}(V)$ is compatible with an inclusion $V \hookrightarrow U$ of open sets of $\operatorname{Spec} M$; this is clear from the construction.

Thirdly, we show that $f^{\#}(V)$ is indeed a homomorphism of semirings. Let $s_i \mapsto t_i$ for $i = 1, 2$. Suppose that $s_1 s_2 \mapsto t$. Then, since ψ is a homomorphism, we have $t(\mathfrak{p}) = \psi \circ s \circ f(\mathfrak{p}) = \psi(s_1 s_2(\varphi^{-1}(\mathfrak{p}))) = \psi(s_1(\varphi^{-1}(\mathfrak{p}))s_2(\varphi^{-1}(\mathfrak{p})) = \psi(s_1(\varphi^{-1}(\mathfrak{p}))\psi(s_2(\varphi^{-1}(\mathfrak{p})) = t_1(\mathfrak{p})t_2(\mathfrak{p})$. The addition can be similarly checked.

Finally, we show that $f^{\#}(V)$ is local; this directly follows from Lemma 2.2.3 since $f^{\#}_{\mathfrak{p}} = \varphi_{\mathfrak{p}}$. This shows that an element of $\operatorname{Hom}(M, N)$ induces an element of $\operatorname{Hom}(Y, X)$.

Conversely, let $(f, f^{\#}) : Y \longrightarrow X$ be a morphism of affine semi-schemes. By Proposition 2.2.4, we have the homomorphism $f^{\#}(X) : \mathcal{O}_X(X) = M \longrightarrow \mathcal{O}_Y(f^{-1}(X)) = \mathcal{O}_Y(Y) = N$ of semirings. Let $\varphi := f^{\#}(X)$. We only have to show that the map $(g, g^{\#})$ induced from φ is equal to $(f, f^{\#})$. Since $\varphi = f^{\#}(X)$, φ is compatible with local homomorphisms of stalks. In other words, we have

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & N \arrow[d] \\ M_{f(\mathfrak{p})} \arrow[r, "f^{\#}_{\mathfrak{p}}"] & N_{\mathfrak{p}} \end{tikzcd} \tag{2.2.6}$$

In particular, we have $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$. But, our previous construction of $(g, g^{\#})$ from φ also gives $g(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. It follows that g and f agree and $g^{\#}_{\mathfrak{p}} = f^{\#}_{\mathfrak{p}}$ $\forall$ $\mathfrak{p} \in \operatorname{Spec} N$. This means that $g^{\#}$ and $f^{\#}$ locally agree and hence $g^{\#} = f^{\#}$. $\square$

The condition $x + x = x$ on a semiring M is transfered to a structure sheaf $\mathcal{O}_X$ as we have observed in Proposition 2.2.4. On the other hand, the condition $x + y \in \{x + y\}$ on M does not have to be transfered to $\mathcal{O}_X$. In the next proposition, we prove that if M is a multiplicatively cancellative semiring of characteristic one, then for $X = \operatorname{Spec} M$, the structure sheaf $\mathcal{O}_X$ is a sheaf of semirings of characteristic one. In other words, the condition $x + y \in \{x, y\}$ on M can be transfered if M is multiplicatively cancellative.

Lemma 2.2.7. *If M is a multiplicatively cancellative semiring, then* $\operatorname{Spec} M$ *is irreducible.*

Proof. Let $X = \operatorname{Spec} M$ and $H = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. Then H is a prime ideal. Indeed, clearly H is an ideal. As in the classical case, we have $H = \{a \in M \mid a^n = 0$ for some $n \in \mathbb{N}\}$ (cf. [19, Proposition 6.21]). Suppose that $a^n b^n = (ab)^n = 0$. Since M is multiplicatively cancellative, we have $a^n = 0$ or $b^n = 0$. This shows that H is a prime ideal. Next, suppose that $X = V(I) \bigcup V(J)$ for some ideals I, J of M. Since H is a prime ideal, we have $H \in V(J)$ or $H \in V(I)$. This implies that $J \subseteq H$ or $I \subseteq H$. Therefore, $X = V(I)$ or $X = V(J)$. □

Proposition 2.2.8. *Let M be a multiplicatively cancellative semiring of characteristic one. Let $X = (\operatorname{Spec} M, \mathcal{O}_X)$, an affine semi-scheme. Then, for an open subset U of X, $\mathcal{O}_X(U)$ is a semiring of characteristic one.*

Proof. Since $\mathcal{O}_X(U)$ is a semiring, all we have to show is that $\mathcal{O}_X(U)$ is of characteristic one. Since M is multiplicatively cancellative, $K := \operatorname{Frac}(M)$ is a semifield and for an non-zero element $f \in M$, M_f can be considered as a subsemiring of $\operatorname{Frac}(M)$. Under this identification, we claim that

$$\mathcal{O}_X(U) \simeq \bigcap_{D(f) \subseteq U} M_f \subseteq \operatorname{Frac}(M). \tag{2.2.7}$$

Once we prove (2.2.7), since K is of characteristic one, the conclusion follows. In fact, for $s \in \mathcal{O}_X(U)$, we can find a cover $U = \bigcup D(h_i)$ such that $s = \frac{a_i}{h_i}$ on $D(h_i)$. Since M is multiplicatively cancellative, $X = \operatorname{Spec} M$ is irreducible from Lemma 2.2.7. Hence, U is also irreducible and $D(h_i) \bigcap D(h_j) \neq \emptyset \ \forall i, j$. This implies that $\frac{a_i}{h_i} = \frac{a_j}{h_j}$ on $D(h_i) \bigcap D(h_j)$, therefore, $s_{ij} a_i h_j = s_{ij} a_j h_i$ for some non-zero elements $s_{ij} \in M$. It follows that $\frac{a_i}{h_i} = \frac{a_j}{h_j}$ as elements of K and each $s \in \mathcal{O}_X(U)$ uniquely determines an element of K. Consider the following map:

$$\varphi : \mathcal{O}_X(U) \longrightarrow X(U) := \{u \in K \mid \forall \mathfrak{p} \in U, \text{ we can write } u = \frac{a}{b} \text{ for some } b \notin \mathfrak{p}\} \subseteq K,$$

where $\varphi(s)$ is a unique element $\frac{a}{b}$ of K determined by s as we discussed above. Then φ is well defined; for each $\mathfrak{p} \in U$, $\mathfrak{p}$ is in some $D(h_i)$ and $\frac{a}{b} = \frac{a_i}{h_i}$, thus $u = \frac{a}{b} \in X(U)$. Moreover, φ is a bijection. Indeed, each element $\frac{x}{y}$ of $X(U)$ can be considered as an element s of $\mathcal{O}_X(U)$ by letting $s(\mathfrak{p}) = \frac{x}{y}$ in $M_{\mathfrak{p}}$. Therefore, φ is surjective. Also, φ is clearly injective since $\mathcal{O}_X$ is a sheaf. From the definition of φ, it follows that $\varphi(st) = \varphi(s)\varphi(t)$, $\varphi(s+t) = \varphi(s) + \varphi(t)$. This shows that $\mathcal{O}_X(U) \simeq X(U)$. Furthermore, for $D(f) \subseteq U$, we have $X(U) \subseteq X(D(f)) \subseteq K$. Thus $X(U) \subseteq \bigcap_{D(f)\subseteq U} X(D(f))$. Conversely, suppose that $u = \frac{a}{b} \in \bigcap_{D(f)\subseteq U} X(D(f))$ and $\mathfrak{p} \in U$. Then $\mathfrak{p}$ is in some $D(f)$. Thus, $u \in X(D(f))$ implies that $u \in X(U)$. This completes our proof. $\square$

Remark 2.2.9. *In the papers, [25], [26], [27], Paul Lescot considered a topological space of prime congruences instead of prime ideals. Let M be a semiring. A congruence on M is an equivalence relation preserving operations of M. More precisely, if $x \sim y$ and $a \sim b$, then $xa \sim yb$ and $x + a \sim y + b$ $\forall x, y, a, b \in M$. A prime congruence is a congruence $\sim$ which satisfies the following condition: if $xy \sim 0$, then $x \sim 0$ or $y \sim 0$. In the theory of commutative rings, there is a one to one correspondence between congruences on a commutative ring A and ideals of A. However, such correspondence no longer holds for semirings (cf. Example 4.1.10). In general, one only obtains an ideal from a congruence as follows:*

$$I_\sim := \{a \in M \mid a \sim 0\}. \tag{2.2.8}$$

The main advantage of a congruence over an ideal is that in the theory of semirings a quotient by an ideal does not behave well, however, a quotient by a congruence behaves well.

Similar to the construction of a prime spectrum $\operatorname{Spec} M$, *one can define the set X of prime congruences and impose Zariski topology on X. Each ideal $I_\sim$ arises from a congruence $\sim$ as in (2.2.8) is called a saturated ideal. In his papers, Paul Lescot had not considered a structure sheaf on the topological space X. However, one can mimic* *the construction of a structure sheaf on semi-schemes by using saturated prime ideals. This gives the notion of a congruence semi-scheme $(X, \mathcal{O}_X)$. It seems, however, that a semiring $\mathcal{O}_X(X)$ of global sections of an 'affine congruence semi-scheme $(X, \mathcal{O}_X)$' might not be isomorphic to a semiring M since Hilbert's Nullstellensatz which is the main ingredient in the proof of the classical case does not hold in the case of congruences. If every ideal of a semiring M is saturated, then an affine semi-scheme induced from M and an affine congruence semi-scheme induced from M are isomorphic as locally semiringed spaces. For example, this is the case when M is a commutative ring.*

For a given semi-scheme X, one defines a sheaf of $\mathcal{O}_X$-semimodules to be a sheaf $\mathcal{F}$ of sets on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-semimodule, and restriction maps $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ and $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ are compatible for open sets $V \subseteq U$ of X. A morphism of sheaves of $\mathcal{O}_X$-semimodules is also defined in the same way as in the classical case.

Example 2.2.10. *Clearly, a structure sheaf $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-semimodules. Furthermore, let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-semimodules. Then, as in the classical case, the sheaf* $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ *becomes a sheaf of $\mathcal{O}_X$-semimodules.*

For a semimodule M over a semiring A, one can associate a sheaf of $\mathcal{O}_X$-semimodules $\widetilde{M}$ as in the classical theory as follows:

$$\widetilde{M}(U) := \{s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}\},$$

where $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ and s is locally representable by fractions as in (2.2.2). Then, clearly $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-semimodules. Furthermore, by the exact same arguments in the classical case, one obtains $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $\widetilde{M}(D(f)) = M_f$. In particular, $\Gamma(X, \widetilde{M}) = M$ when X is an affine semi-scheme.

Definition 2.2.11. *Let $(X, \mathcal{O}_X)$ be a semi-scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-semimodules is called quasi-coherent if each $x \in X$ has an affine neighborhood $U \simeq$* Spec A *such that $\mathcal{F}|_U \simeq \widetilde{M}$ for some $\mathcal{O}_X(U)$-semimodule M.*

Next, we construct the tensor product $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ of sheaves of $\mathcal{O}_X$-semimodules. Note that when we define a tenor product of semimodules, we need to be careful. There are several ways one can generalize the classical construction of a tensor product to semimodules, and some generalizations might not work well. For example, the generalization as in the Golan's book [19] is not a proper generalization. In fact, if we follow the generalization in [19], for a semiring A and an A-semimodule M, we have

$$A\otimes_A M \simeq (M/\sim), \tag{2.2.9}$$

where $\sim$ is a congruence relation on M such that $a\sim b$ if and only if $\exists\, c\in M$ such that $a+c=b+c$. When A is an idempotent semiring in which our main interest lies, the tensor product of [19] does not behave well. For example, we have $\mathbb{Z}_{max}\otimes_{\mathbb{Z}_{max}}\mathbb{R}_{max}\simeq\{0\}$. Furthermore, we have

$$\{0\}=\mathrm{Hom}(\mathbb{Z}_{max}\otimes_{\mathbb{Z}_{max}}\mathbb{Z}_{max},\mathbb{Z}_{max})\neq\mathrm{Hom}(\mathbb{Z}_{max},\mathrm{Hom}(\mathbb{Z}_{max},\mathbb{Z}_{max}))=\mathbb{Z}_{max}.$$

This implies that we can not have the Hom-Tensor duality at the level of sheaves of $\mathcal{O}_X$-semimodules with the Golan's notion. Therefore, one can not generalize directly the construction of Picard groups. To this end, we use the definition of a tensor product which is proposed in [36]. Then we recover usual isomorphisms which one can expect from a tensor product. More precisely, we have $R\otimes_R M\simeq M\otimes_R R\simeq M$ and $\mathrm{Hom}(M\otimes_R N,P)\simeq\mathrm{Hom}(M,\mathrm{Hom}(N,P))$ for a semiring R and R-semimodules, M,N,P. By appealing to such results, we can define the Picard group $\mathrm{Pic}(X)$ of a semi-scheme X. The construction is exactly same as the classical case, but we include the proof here for the completeness.

Lemma 2.2.12. *Let X be a semi-scheme. Let $\mathcal{F},\mathcal{G}$ be sheaves of $\mathcal{O}_X$-semimodules. Then, for each $\mathfrak{p}\in X$, we have*

$$(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})_{\mathfrak{p}}\simeq\mathcal{F}_{\mathfrak{p}}\otimes_{\mathcal{O}_{X,\mathfrak{p}}}\mathcal{G}_{\mathfrak{p}}. \tag{2.2.10}$$

Proof. This follows from the corresponding fact of semimodules. Let U be an open neighborhood of $\mathfrak{p}$. Since $\mathcal{F}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_\mathfrak{p}$ is an $\mathcal{O}_{X,\mathfrak{p}}$-semimodule, via the homomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\mathfrak{p}}$, we know that $\mathcal{F}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_\mathfrak{p}$ carries the $\mathcal{O}_X(U)$-semimodule structure. One can observe that the following map

$$\varphi_U : \mathcal{F}(U) \times \mathcal{G}(U) \longrightarrow \mathcal{F}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_\mathfrak{p}, \quad (s,t) \mapsto s_\mathfrak{p} \otimes t_\mathfrak{p}$$

is $\mathcal{O}_X(U)$-bilinear. Thus, from the universal property of a tensor product (cf. [36, §6]), we have the following induced homomorphism (also, denoted by φ_U):

$$\varphi_U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \longrightarrow \mathcal{F}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_\mathfrak{p}, \quad s \otimes t \mapsto s_\mathfrak{p} \otimes t_\mathfrak{p}.$$

Let $\mathcal{H}$ be the presheaf such that $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Then, by the definition of stalks, φ_U induces the following homomorphism:

$$h : \mathcal{H}_\mathfrak{p} \longrightarrow \mathcal{F}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_\mathfrak{p}.$$

Consider the following map:

$$\psi : \mathcal{F}_\mathfrak{p} \times \mathcal{G}_\mathfrak{p} \longrightarrow \mathcal{H}_\mathfrak{p}, \quad (s_\mathfrak{p}, t_\mathfrak{p}) \mapsto (s|_{U\cap V} \otimes t|_{U\cap V})_\mathfrak{p},$$

where $s \in \mathcal{F}(U)$, $t \in \mathcal{G}(V)$, and $\mathfrak{p} \in U \cap V$. Then, clearly ψ is $\mathcal{O}_{X,\mathfrak{p}}$-bilinear and hence ψ induces the following homomorphism (also, denoted by ψ):

$$\psi : \mathcal{F}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{G}_\mathfrak{p} \longrightarrow \mathcal{H}_\mathfrak{p}, \quad s_\mathfrak{p} \otimes t_\mathfrak{p} \mapsto (s|_{U\cap V} \otimes t|_{U\cap V})_\mathfrak{p}.$$

It is clear that h and ψ are inverses to each other. Moreover, $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_\mathfrak{p} \simeq \mathcal{H}_\mathfrak{p}$ as in the classical case. This completes the proof. $\square$

By an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-semimodules we mean a sheaf of $\mathcal{O}_X$-semimodules which is locally isomorphic to $\mathcal{O}_X$.

Lemma 2.2.13. *Let X be a semi-scheme. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-semimodules on X. Then, we have*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}). \tag{2.2.11}$$

Proof. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}_X$-semimodules defined by

$$\mathcal{G}(U) := \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U, \mathcal{O}_X|_U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}(U) \quad \text{for an open subset } U \subseteq X.$$

For an open subset U of X, we define

$$\varphi_U : \mathcal{G}(U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U, \mathcal{L}|_U), \quad \beta \otimes a \mapsto \hat{\beta},$$

where $\hat{\beta}(V) : \mathcal{L}(V) \longrightarrow \mathcal{L}(V)$, $t \mapsto a|_V \cdot \beta(V)(t)$ for an open subset V of U. One can easily check that $\hat{\beta} \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U, \mathcal{L}|_U)$ and hence φ_U is well defined. Since the construction is functorial, φ_U and φ_V agree on $U \cap V$. Thus, we can glue $\{\varphi_U\}_{U \subseteq X}$ to construct a morphism, $\varphi : \mathcal{G} \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$. Let $\mathcal{G}^+$ be the sheafification of $\mathcal{G}$ together with a morphism $\alpha : \mathcal{G} \longrightarrow \mathcal{G}^+$. In fact, by the definition, we have $\mathcal{G}^+ = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L}$. Then, there exists a unique morphism $\varphi^+ : \mathcal{G}^+ \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ which satisfies the following diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & \mathcal{G}^+ \arrow[dl, dotted, "\varphi^+"] \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) & \end{tikzcd}$$

However, φ^+ induces a homomorphism on stalks. It follows from Lemma 2.2.12 that, for each $\mathfrak{p} \in X$, we obtain

$$\varphi^+_{\mathfrak{p}} : (\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L})_{\mathfrak{p}} \simeq (\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{L}_{\mathfrak{p}}) \longrightarrow (\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}))_{\mathfrak{p}}.$$

Since $\mathcal{L}$ is an invertible sheaf, equivalently, we have

$$\varphi_{\mathfrak{p}}^{+} : (\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathcal{O}_{X,\mathfrak{p}}) \longrightarrow (\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X))_{\mathfrak{p}}, \quad f_{\mathfrak{p}} \otimes a_{\mathfrak{p}} \mapsto f_{\mathfrak{p}} \cdot a_{\mathfrak{p}}.$$

From [36, Theorem 7.6], $\varphi_{\mathfrak{p}}^{+}$ is an isomorphism. In other words, we have the morphism $\varphi^{+} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ such that the induced map $\varphi_{\mathfrak{p}}^{+}$ on stalks is an isomorphism $\forall \mathfrak{p} \in X$. Hence, φ^{+} is an isomorphism. $\square$

Proposition 2.2.14. *Let X be a semi-scheme. Then we have an isomorphism*

$$\varphi : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X.$$

Proof. Let U be an open subset of X. For a morphism $f : \mathcal{O}_X|_U \longrightarrow \mathcal{O}_X|_U$, we have $f(U) : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)$. In particular, each f determines an element $f(U)(1) \in \mathcal{O}_X(U)$. Consider the following map:

$$\varphi_U : \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{O}_X|_U) \longrightarrow \mathcal{O}_X(U), \quad \varphi_U(f) = f(U)(1).$$

We claim that φ_U is injective. In fact, suppose that $f(U)(1) = g(U)(1)$. Then, for an open subset $V \subseteq U$, we have $f(V)(1) = g(V)(1)$. Since $f(V)$ and $g(V)$ are homomorphisms of $\mathcal{O}_X(V)$-semimodules, we have $f(V) = g(V)$. This implies that $f = g$. Furthermore, for $t \in \mathcal{O}_X(U)$, we define a homomorphism $t : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)$ of $\mathcal{O}_X(U)$-semimodules by sending 1 to t. Let $f : \mathcal{O}_X|_U \longrightarrow \mathcal{O}_X|_U$ be a morphism of $\mathcal{O}_X$-semimodules such that for an open subset $V \subseteq U$, $f(V) : \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(V)$ defined by $1 \mapsto t|_V$. Then, clearly $\varphi_U(f) = t$. This proves that φ_U is a surjection and hence an isomorphism. Moreover, one can observe that for open sets U, V of X, isomorphisms φ_U and φ_V agree on $W := U \cap V$. Hence, we can define a morphism $\varphi : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$ such that $\varphi(U) := \varphi_U$ and φ becomes our desired isomorphism. $\square$

Proposition 2.2.15. *Let X be a semi-scheme. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-*

semimodules on X. Then we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{L}) \simeq \mathcal{O}_X.$$

Proof. Let $\varphi : \mathcal{O}_X \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})$ be a morphism of sheaves such that for an open subset U of X, we have $\varphi(U) : \mathcal{O}_X(U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U,\mathcal{L}|_U), \quad \alpha \mapsto \hat{\alpha}$, where for an open subset $V \subseteq U$,

$$\hat{\alpha}(V) : \mathcal{L}(V) \longrightarrow \mathcal{L}(V), \quad t \mapsto \alpha|_V \cdot t.$$

Then, clearly $\hat{\alpha} \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{L}|_U,\mathcal{L}|_U)$ and φ is compatible with the restriction maps. Hence, φ is well defined. For $\mathfrak{p} \in X$, there exists an open neighborhood $U_\mathfrak{p}$ of $\mathfrak{p}$ such that $\mathcal{L}|_{U_\mathfrak{p}} \simeq \mathcal{O}_X|_{U_\mathfrak{p}}$. We can further assume that $U_\mathfrak{p} = \mathrm{Spec}\, M$ for some semiring M. Then, we have

$$\varphi|_{U_\mathfrak{p}} : \mathcal{O}_X|_{U_\mathfrak{p}} \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X)|_{U_\mathfrak{p}}.$$

It follows from Proposition 2.2.14 that

$$\varphi(U_\mathfrak{p}) : M \longrightarrow \mathrm{Hom}_M(M,M) \simeq M, \quad m \mapsto m.$$

Therefore $\varphi_\mathfrak{p} : M_\mathfrak{p} \simeq M_\mathfrak{p}$ and hence φ is an isomorphism. □

Proposition 2.2.16. *Let X be a semi-scheme. Let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-semimodules on X. Then the sheaf $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)$ is also an invertible sheaf of $\mathcal{O}_X$-semimodules. Furthermore, we have the following isomorphism:*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X.$$

Proof. We first claim that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)$ is an invertible sheaf of $\mathcal{O}_X$-semimodules. In fact, we can find an open cover $\mathcal{U} = \{U_i\}$ of X such that for each i, $\mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$

and $U_i = \operatorname{Spec} R_i$ for some semiring R_i. It follows from Proposition 2.2.14 that

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)|_{U_i} \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)|_{U_i} \simeq \mathcal{O}_X|_{U_i}.$$

For the second assertion, from Lemma 2.2.13, we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}).$$

Then, the conclusion follows from Proposition 2.2.15. □

The set $\mathrm{Pic}(X)$ of isomorphism classes of invertible sheaves (of $\mathcal{O}_X$-semimodules) on a semi-scheme X is indeed a group with a group operation $\otimes_{\mathcal{O}_X}$. In fact, the isomorphism class of $\mathcal{O}_X$ is the identity. The inverse of the isomorphism class of $\mathcal{L}$ is the isomorphism class of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. The associativity of the group operation follows from the associativity of the tensor product (cf. [36, Theorem 7.6]). In the next subsection, we will construct Čech cohomology theory for a semi-scheme X, and derive the following classical result:

$$\mathrm{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^*).$$

2.3 Cohomology theories of semi-schemes

In this section, we investigate the notion of cohomology theories of semimodules. In the first subsection, we construct an injective resolution of an idempotent semimodule. When we work over semi-structures, one of the main flaws is that a kernel being zero does not give the full insight of a map being injective. For example, consider the following sequence:

$$0 \xrightarrow{i} \mathbb{Z}_{max} \xrightarrow{f} \mathbb{B}, \quad \text{where } f(x) = 0 \Longleftrightarrow x = 0. \tag{2.3.1}$$

Then, we have $\mathrm{Img}(i) = \{0\} = \mathrm{Ker}(f)$, but clearly f is not one-to one. Furthermore, for a semimodule homomorphism $f : A \longrightarrow B$, the semimodule $\mathrm{Img}(f)$ does not have to be the kernel of the projection $B \longrightarrow B/\mathrm{Img}(f)$ as one can see in Lemma 2.3.3. Therefore, to define the notion of exactness over semi-structures, one might not want to simply impose the condition Img = Ker. To this end, we introduce three possible definitions.

In the second subsection, we generalize Čech cohomology to semi-structures. We will make use of the idea in [37] which interprets an alternating sum as the sum of two sums such that one represents the positive sums and the other represents the negative sums. Then, we compute the Čech cohomology of the projective line $\mathbb{P}^1_{\mathbb{Q}_{max}}$ over $\mathbb{Q}_{max}$. Furthermore, we show that the classical cohomological interpretation of a Picard group holds, i.e. for a semi-scheme X, we have $\mathrm{Pic}(X) = \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$.

2.3.1 An injective resolution of idempotent semimodules

In the first subsection, we test several possible definitions of exactness over semimodules. Then, in the second subsection, we construct an injective resolution of an idempotent semimodule and sheafify the construction. Finally, we explain the difficulty of the derived functors approach toward a cohomology theory over semi-structures.

Exactness of semimodules

To correct the problems we explained (for example, (2.3.1)), we introduce the following definition from the paper [1].

Definition 2.3.1. *(cf. [1]) Let R be a semiring. Let A, B be R-semimodules, and $f : A \longrightarrow B$ be a homomorphism of semimodules.*

1. *f is k-uniform if for $x, y \in A$ such that $f(x) = f(y)$, there exists $t_1, t_2 \in \mathrm{Ker}(f)$ such that $x + t_1 = y + t_2$.*

2. $\overline{\mathrm{Img}(f)} := \{y \in B \mid \exists t_1, t_2 \in A$ *such that* $y + f(t_1) = f(t_2)\}$.

Remark 2.3.2. *The first part of Definition 2.3.1 is designed to fix the injectivity issue and the second part is to fix the surjectivity issue.*

Lemma 2.3.3. *Let R be a semiring. Let A, B be R-semimodules, and $f : A \longrightarrow B$ be a homomorphism of semimodules. Then $B/\operatorname{Img}(f) \simeq B/\overline{\operatorname{Img}(f)}$ as R-semimodules.*

Proof. It is enough to show that the congruence relations induced by $\operatorname{Img}(f)$ and $\overline{\operatorname{Img}(f)}$ are same. Let $\sim_f$ and $\sim_{\overline{f}}$ be the congruence relations induced by $\operatorname{Img}(f)$ and $\overline{\operatorname{Img}(f)}$ respectively. Suppose that $x \sim_f y$. Then $x + f(t_1) = y + f(t_2)$ for some $t_1, t_2 \in A$. Since $\operatorname{Img}(f) \subseteq \overline{\operatorname{Img}(f)}$, this implies that $x \sim_{\overline{f}} y$. Conversely, suppose that $x \sim_{\overline{f}} y$. Then, $x + r_1 = y + r_2$ for some $r_1, r_2 \in \overline{\operatorname{Img}(f)}$. However, by the definition of $\overline{\operatorname{Img}(f)}$, we have $r_1 + f(d_1) = f(d_2)$ and $r_2 + f(g_1) = f(g_2)$ for some $d_1, d_2, g_1, g_2 \in A$. Hence, $x + r_1 + f(d_1 + g_1) = x + f(d_2 + g_1) = y + r_2 + f(d_1 + g_1) = y + f(d_1 + g_2)$. Therefore, $x \sim_f y$. $\square$

Lemma 2.3.4. *Let R be a semiring. Let A, B be R-semimodules and $f : A \longrightarrow B$ be a semimodule homomorphism. Then the canonical projection $\pi : B \longrightarrow B/\operatorname{Img}(f)$ is k-uniform.*

Proof. Suppose that $\pi(x) = \pi(y)$. Then we have $x \sim_f y$. This means that there exists $t_1, t_2 \in A$ such that $x + f(t_1) = y + f(t_2)$. However, clearly $f(t_1), f(t_2) \in \operatorname{Ker}(\pi)$ and hence π is k-uniform. $\square$

Definition 2.3.5. *Let R be a semiring. Let A, B, C be R-semimodules. Consider the following sequence of R-semimodules.*

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.3.2}$$

1. *We say that (2.3.2) is weak exact at B if $\overline{\operatorname{Img}(f)} = \operatorname{Ker}(g)$.*

2. *We say that (2.3.2) is half exact at B if $\operatorname{Img}(f) = \operatorname{Ker}(g)$.*

3. *We say that (2.3.2) is strong exact at B if it is weak exact at B and g is k-uniform.*

If $\operatorname{Img}(f) = \operatorname{Ker}(g)$, then we have $\overline{\operatorname{Img}(f)} = \operatorname{Ker}(g)$. Indeed, for $y \in \overline{\operatorname{Img}(f)}$, we have $y + f(t_1) = f(t_2)$, thus $g(y) = 0$ and $y \in \operatorname{Ker}(g)$. Hence, half exactness implies weak exactness. This implies that if g is k-uniform, then half exactness implies strong exactness. However, strong exactness does not imply half exactness in general. For example, consider the following sequence:

$$\mathbb{N}_{max} \xrightarrow{i} \mathbb{Z}_{max} \xrightarrow{g} 0, \tag{2.3.3}$$

where i is an injection and g is the zero map. Clearly, g is k-uniform. We can see that $\operatorname{Img}(i)$ is a proper subset of $\mathbb{Z}_{max}$ and $\operatorname{Ker}(g) = \mathbb{Z}_{max}$, thus (2.3.3) is not half exact at $\mathbb{Z}_{max}$. On the other hand, we have

$$\overline{\operatorname{Img}(i)} = \{y \in \mathbb{Z}_{max} \mid \exists t_1, t_2 \in \mathbb{N}_{max} \text{ such that } y + i(t_1) = i(t_2)\} = \mathbb{Z}_{max}.$$

Therefore, (2.3.3) is strong exact at $\mathbb{Z}_{max}$.

Proposition 2.3.6. *Let R be a semiring. Let A, B, C be R-semimodules. Consider the following sequence:*

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.3.4}$$

Then (2.3.4) is strong exact at B if and only if the homomorphism g induces the (well-defined) injection $\hat{g} : B/\operatorname{Img}(f) \longrightarrow C$ defined by $\hat{g}(\bar{x}) = g(x)$, where $\bar{x}$ is the equivalence class of $x \in B$ in $B/\operatorname{Img}(f)$.

Proof. Suppose that (2.3.4) is strong exact at B. We first show that the map $\hat{g}$ is well defined. Indeed, if $\overline{\alpha} = \overline{\beta}$, then $\alpha + t_1 = \beta + t_2$ for some $t_1, t_2 \in \operatorname{Img}(f)$. Since (2.3.4) is strong exact at B, we have $\operatorname{Img}(f) \subseteq \operatorname{Ker}(g)$. It follows that $g(\alpha) = g(\beta)$ and hence $\hat{g}$ is well defined. Clearly, $\hat{g}$ is an R-semimodule homomorphism. Moreover, suppose that $\hat{g}(\overline{\alpha}) = \hat{g}(\overline{\beta})$. Then we have $g(\alpha) = g(\beta)$. Since g is k-uniform, this implies that $\alpha + t_1 = \beta + t_2$ for some $t_1, t_2 \in \operatorname{Ker}(g)$. However, since $\operatorname{Ker}(g) = \overline{\operatorname{Img}(f)}$, there exists $r_1, r_2, s_1, s_2 \in A$ such that $t_1 + f(r_1) = f(r_2)$ and $t_2 + f(s_1) = f(s_2)$. It follows that

$\alpha + t_1 + f(r_1 + s_1) = \beta + t_2 + f(r_1 + s_1)$ and $\alpha + f(s_1 + r_2) = \beta + f(r_1 + s_2)$. This implies that $\overline{\alpha} = \overline{\beta}$ and hence $\hat{g}$ is one-to-one.

Conversely, assume that (2.3.4) satisfies the given condition. We first show that (2.3.4) is weak exact at B. If $y \in \overline{\mathrm{Img}(f)}$, then $y + f(t_1) = f(t_2)$ for some $t_1, t_2 \in A$. It follows that $\overline{y} = \overline{0}$ in $B/\mathrm{Img}(f)$. We also have that $\hat{g}(\overline{y}) = g(y) = \hat{g}(\overline{0}) = g(0) = 0$. Hence, $\overline{\mathrm{Img}(f)} \subseteq \mathrm{Ker}(g)$. On the other hand, if $y \in \mathrm{Ker}(g)$, then $\hat{g}(\overline{y}) = g(y) = 0$. Since $\hat{g}$ is one-to-one, we have $\overline{y} = \overline{0}$ in $B/\mathrm{Img}(f)$. This implies that $y + f(t_1) = f(t_2)$ for some $t_1, t_2 \in A$, hence $y \in \overline{\mathrm{Img}(f)}$. This proves that (2.3.4) is weak exact at B. We next claim that g is k-uniform. Indeed, if $g(\alpha) = g(\beta)$, then $\hat{g}(\overline{\alpha}) = \hat{g}(\overline{\beta})$. Since $\hat{g}$ is one-to-one, we have $\overline{\alpha} = \overline{\beta}$ and hence $\alpha + t_1 = \beta + t_2$ for some $t_1, t_2 \in \mathrm{Img}(f) \subseteq \mathrm{Ker}(g)$. This proves that (2.3.4) is strong exact at B. $\square$

Definition 2.3.7. *Let R be a semiring.*

1. *A cochain complex $A^{\cdot}$ of R-semimodules is a family $\{A^i\}_{i\in\mathbb{Z}}$ of R-semimodules, together with R-semimodule maps $\partial^i : A^i \longrightarrow A^{i+1}$ such that each composition $\partial^{i+1} \circ \partial^i$ is the zero map.*

2. *The semimodule of i-cocycles of $A^{\cdot}$, denoted by $Z^i = Z^i(C^{\cdot})$, is the kernel of ∂^i. The semimodule of i-coboundaries of $A^{\cdot}$ is $\overline{\mathrm{Img}(\partial^{i-1})}$ and denoted by $B^i = B^i(A^{\cdot})$. Furthermore, we define the n-th cohomology semimodule as $H^n(A^{\cdot}) := \mathrm{Ker}(\partial^n)/\overline{\mathrm{Img}(\partial^{n-1})}$.*

3. *A morphism between two cochain complexes $A = (A^i, \partial^i), B = (B^i, d^i)$ is a family of R-semimodule homomorphisms $f^i : A^i \longrightarrow B^i$ such that $d^i \circ f^i = f^{i+1} \circ \partial^i$. We similarly defines chain complexes of semimodules and a map between them.*

Remark 2.3.8. *In Definition 2.3.7, i-coboundaries $B^i(A^{\cdot})$ is not $\mathrm{Img}(\partial^{i-1})$, but $\overline{\mathrm{Img}(\partial^{i-1})}$. Hence, one might wonder whether the condition $\partial^{i+1} \circ \partial^i = 0$ is enough to force $B^i(A^{\cdot})$ to be a sub-semimodule of $Z^i(A^{\cdot})$. However, the condition $\partial^{i+1} \circ \partial^i = 0$ implies that $\mathrm{Img}(\partial^{i-1}) \subseteq Z^i(A^{\cdot})$. Then, for $y \in \overline{\mathrm{Img}(\partial^{i-1})}$, we have $t_1, t_2 \in A^{i-1}$ such that $y + \partial^{i-1}(t_1) = \partial^{i-1}(t_2)$. Hence, $\partial^i(y) = 0$ and $\overline{\mathrm{Img}(\partial^{i-1})} \subseteq Z^i(A^{\cdot})$.*

As in the classical case, we say that a sequence of (co)chain complexes,

$$0 \longrightarrow A^{\cdot} \xrightarrow{f} B^{\cdot} \xrightarrow{g} C^{\cdot} \longrightarrow 0$$

is weak, half, strong exact if and only if the corresponding sequence, for each n,

$$0 \longrightarrow A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \longrightarrow 0$$

is weak, half, strong exact respectively.

Definition 2.3.9. *Let R be a semiring. Let $(A^{\cdot}, \partial^{\cdot})$, $(B^{\cdot}, d^{\cdot})$ be cochain complexes of R-semimodules. Let $f = (f^i)$, $g = (g^i)$ be morphisms from $(A^{\cdot}, \partial^{\cdot})$ to $(B^{\cdot}, d^{\cdot})$. We say that f and g are homotopic, denoted by $f \simeq g$, if there exist two collections of homomorphisms $h = (h^i : A^i \longrightarrow B^{i-1})$, $s = (s^i : A^i \longrightarrow B^{i-1})$ such that*

$$h^{i+1} \circ \partial^i + d^{i-1} \circ h^i + f^i = s^{i+1} \circ \partial^i + d^{i-1} \circ s^i + g^i. \tag{2.3.5}$$

Remark 2.3.10. *It is clear that Definition 2.3.9 generalizes the classical notion by considering $h - s$ or $s - h$ as a homotopy.*

Proposition 2.3.11. *Let R be a semiring. Let $(A^{\cdot}, \partial^{\cdot})$, $(B^{\cdot}, d^{\cdot})$ be cochain complexes of R-semimodules. Let $f = (f^i) : (A^{\cdot}, \partial^{\cdot}) \longrightarrow (B^{\cdot}, d^{\cdot})$ be a morphism. Then, f induces the following homomorphism for each n:*

$$H^n(f) : H^n(A^{\cdot}) \longrightarrow H^n(B^{\cdot}), \quad \bar{x} \mapsto \overline{f^n(x)},$$

where $\bar{x}$ is the equivalence class of $x \in Z^n(A^{\cdot})$ in $H^n(A^{\cdot})$. Moreover, if $f \simeq g$, then $H^n(f) = H^n(g)$.

Proof. First, we show that $H^n(f)$ is well defined. In fact, we have $\bar{a} = \bar{b} \Longleftrightarrow a + \partial^{n-1}(t_1) = b + \partial^{n-1}(t_2)$. It follows that $f^n(a) + f^n \circ \partial^{n-1}(t_1) = f^n(b) + f^n \circ \partial^{n-1}(t_2)$ for some $t_1, t_2 \in A^{n-1}$. Since f is a chain map, we have $f^n \circ \partial^{n-1} = d^{n-1} \circ f^{n-1}$ and therefore $\overline{f^n(a)} = \overline{f^n(b)}$. It is clear that $H^n(f)$ is a homomorphism. If $f \simeq g$, then for $\bar{x} \in H^n(A^{\cdot})$, we have

$$h^{n+1} \circ \partial^n(x) + d^{n-1} \circ h^n(x) + f^n(x) = s^{n+1} \circ \partial^n(x) + d^{n-1} \circ s^n(x) + g^n(x). \quad (2.3.6)$$

Since $x \in \operatorname{Ker} \partial^n$, (2.3.6) is equivalent to the following:

$$d^{n-1} \circ h^n(x) + f^n(x) = d^{n-1} \circ s^n(x) + g^n(x).$$

Hence, it follows that $H^n(f)(\bar{x}) = \overline{f^n(x)} = \overline{g^n(x)} = H^n(g)(\bar{x})$. $\square$

In the classical theory, the global sections functor Γ is left exact. The following proposition is an analogue of that fact over semi-structures. We fist define the notion of exactness of a sequence of sheaves in terms of stalks.

Definition 2.3.12. *Let R be a semiring, and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of R-semimodules on a topological space X. We say that the sequence,*

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is weak, half, strong exact at $\mathcal{G}$ if the following induced map

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is weak, half, strong exact at $\mathcal{G}_x$ $\forall x \in X$ in the sense of Definition 2.3.5.

Proposition 2.3.13. *Let R be a semiring and let*

$$0 \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \quad (2.3.7)$$

be a sequence of sheaves of R-semimodules on a topological space X. Then the following holds.

1. *If (2.3.7) is strong exact at $\mathcal{F}$ and $\mathcal{G}$, then for an open subset U of X, the homomorphism $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ of R-semimodules is one-to-one and $\operatorname{Img}(\varphi_U) \subseteq$*

$\operatorname{Ker}(\psi_U)$.

2. *If* (2.3.7) *is strong exact at* $\mathcal{F}$ *and* $\mathcal{G}$*, and also half exact at* $\mathcal{F}$ *and* $\mathcal{G}$*, then the following sequence of* R*-semimodules, for an open subset* U *of* X*,*

$$0 \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U) \tag{2.3.8}$$

is half exact at at $\mathcal{F}(U)$ *and* $\mathcal{G}(U)$*.*

Proof. Suppose that (2.3.7) is strong exact at $\mathcal{F}$ and $\mathcal{G}$, then for $x \in U$, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r, "\alpha_U"] \arrow[d] & \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathcal{G}(U) \arrow[r, "\psi_U"] \arrow[d] & \mathcal{H}(U) \arrow[d] \\
0 \arrow[r, "\alpha_x"] & \mathcal{F}_x \arrow[r, "\varphi_x"] & \mathcal{G}_x \arrow[r, "\psi_x"] & \mathcal{H}_x
\end{tikzcd}$$

such that the second row is strong exact at $\mathcal{F}_x$ and $\mathcal{G}_x$. Assume that $\varphi_U(s) = \varphi_U(t)$. Then $\varphi_x(s_x) = \varphi_x(t_x)$. However, φ_x is one-to-one because the second row is strong exact. It follows that $s_x = t_x$. This implies that, for $x \in U$, there exists an open neighborhood V_x of x in U such that $s|_{V_x} = t|_{V_x}$. Thus $\{V_x\}_{x \in U}$ form an open cover of U. Hence, $s = t$ since $\mathcal{F}$ is a sheaf. This proves that φ_U is injective. If $t = \varphi_U(s)$, then $\varphi(s_x) = t_x$. Since $\overline{\operatorname{Img}(\varphi_x)} = \operatorname{Ker}\psi_x$, we have $\psi_x(t_x) = 0$. This implies that if $q = \psi_U(t)$, then $q_x = 0$ at each $x \in U$. Therefore, $q = 0$. This shows that $\operatorname{Img}(\varphi_U) \subseteq \operatorname{Ker}\psi_U$. In particular, $\overline{\operatorname{Img}(\varphi_U)} \subseteq \operatorname{Ker}\psi_U$.
For the second part, suppose that (2.3.7) is both strong and half exact at $\mathcal{F}$ and $\mathcal{G}$. From the first part of the proposition, φ_U is injective and thus (2.3.8) is half exact at $\mathcal{F}(U)$. Also, the same argument shows that $\operatorname{Img}(\varphi_U) \subseteq \operatorname{Ker}\psi_U$. Conversely, if $t \in \operatorname{Ker}\psi_U$, then $t_x \in \operatorname{Ker}\psi_x = \operatorname{Img}\varphi_x$. It follows that there exists an open

neighborhood V_x of x in U satisfying the following commutative diagram:

$$\begin{tikzcd} s|_{V_x} \arrow[r, mapsto, "\varphi_{V_x}"] \arrow[d, mapsto] & t|_{V_x} \arrow[r, mapsto, "\psi_{V_x}"] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\ s_x \arrow[r, mapsto, "\varphi_x"] & t_x \arrow[r, mapsto, "\psi_x"] & 0 \end{tikzcd}$$

We know that for an open subset V, φ_V is one-to-one. It follows that $\varphi_{V_x \cap V'_x}$ is one-to-one $\forall x, x' \in U$. Therefore, we can glue $s|_{V_x}$ to obtain a section $s \in \mathcal{F}(U)$ such that $\varphi_U(s) = t$. Thus, $\operatorname{Ker} \psi_U \subseteq \operatorname{Img}(\varphi_U)$. ∎

Remark 2.3.14. *In Proposition 2.3.13, the failure of* $\operatorname{Ker} \psi_U \subseteq \overline{\operatorname{Img}(\varphi_U)}$ *in the first part comes from the definition;* $y \in \operatorname{Img}(\varphi_x) \iff y + \varphi_x(t_1) = \varphi_x(t_2)$*. In other words, such local data* t_i *can not be glued in general since a choice is involved.*

An injective resolution of an idempotent semimodule

Let us recall the definition of an injective semimodule. Let R be a semiring. A R-semimodule I is injective if and only if, for any pair (M, N) of a semimodule M and its sub-semimodule N, any R-homomorphism from N to I can be extended to a R-homomorphism from M to I. It is known that a semimodule over an (additively) idempotent semirings can be embedded in an injective semimodule (cf. [51]). In other words, for an idempotent semiring R, the category of R-semimodules has enough injectives. In fact, we have the following:

Proposition 2.3.15. *Let R be an (additively) idempotent semiring. Then, for an R-semimodule M, we have a strong exact sequence,*

$$0 \longrightarrow M \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \dots \tag{2.3.9}$$

such that each I^j is an injective R-semimodule.

Proof. The proof is exactly same as the classical construction. We only emphasize that (2.3.9) is strong exact. First, since each R-semimodule can be embedded in an injective R-semimodule, we have an injective R-semimodule I^0 and a sequence of R-semimodules as follows:

$$0 \longrightarrow M \xrightarrow{\epsilon_0} I^0 \xrightarrow{P_0} I^0/\operatorname{Img}(\epsilon_0) \longrightarrow 0. \tag{2.3.10}$$

Since ϵ_0 is one-to-one, ϵ_0 is k-uniform and (2.3.10) is strong exact at M. Let $M_1 := I^0/\operatorname{Img}(\epsilon_0)$. Then, since M_1 is an R-semimodule, there exists an injective semimodule I^1 and an one-to-one R-homomorphism ϵ_1 which satisfy the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\epsilon_0"] & I^0 \arrow[r, "P_0"] \arrow[dr, "d^0"'] & I^0/\operatorname{Img}(\epsilon_0) := M_1 \arrow[r] \arrow[d, "\epsilon_1"] & 0 \\
 & & & I^1 \arrow[d, "P_1"] & \\
 & & & I^1/\operatorname{Img}(\epsilon_1) := M_2 \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Hence, we derive the following sequence of R-semimodules:

$$0 \longrightarrow M \xrightarrow{\epsilon_0} I^0 \xrightarrow{d_0} I^1 \xrightarrow{P_1} I^1/\operatorname{Img}(\epsilon_1) := M_2 \longrightarrow 0. \tag{2.3.11}$$

At I^0, we can first observe that $\operatorname{Img}(\epsilon_0) \subseteq \operatorname{Ker}(P_0)$, hence $\overline{\operatorname{Img}(\epsilon_0)} \subseteq \operatorname{Ker}(P_0)$. On the other hand, for $x \in \operatorname{Ker}(P_0)$, we have $P_0(x) = 0$. It follows that $x + t_1 = t_2$ for some $t_i \in \operatorname{Img}(\epsilon_0)$, hence $x \in \overline{\operatorname{Img}(\epsilon_0)}$. Therefore, we have $\overline{\operatorname{Img}(\epsilon_0)} = \operatorname{Ker}(P_0)$. However, since $d^0 = \epsilon_1 \circ P_0$ and ϵ_1 is injective, we have $\operatorname{Ker}(P_0) = \operatorname{Ker}(d^0)$. This shows that (2.3.11) is weak exact at I^0. Furthermore, for $x, y \in I^0$, suppose that $d^0(x) = d^0(y)$. Then, we have $\epsilon_1(P_0(x)) = \epsilon_1(P_0(y))$. Since ϵ_1 is one-to-one, it follows that $P_0(x) = P_0(y)$. This implies that $x + \epsilon_0(t_1) = y + \epsilon_0(t_2)$ for some $t_1, t_2 \in I^0$. However, since $\overline{\operatorname{Img}(\epsilon_0)} \subseteq \operatorname{Ker}(P_0)$, we have $\epsilon_0(t_1), \epsilon_0(t_2) \in \operatorname{Ker}(P_0) \subseteq \operatorname{Ker}(d^0)$. This shows that d^0 is

k-uniform and hence (2.3.11) is strong exact at I_0. One can inductively define I^j and this gives the desired (strong) injective resolution. $\square$

Next, we construct an injective resolution of a sheaf of idempotent semimodules.

Proposition 2.3.16. *Let R be a semiring. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of R-semimodules on a topological space X. Then a morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is an isomorphism if and only if the induced map $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism for each $x \in X$. In particular, if φ is injective, then φ_x is injective for each $x \in X$.*

Proof. The proof is identical to the classical case. $\square$

Let R be a semiring. For sheaves $\mathcal{F}$, $\mathcal{G}$ of R-semimodules, the sheaf hom $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is again a sheaf of R-semimodules. A subsheaf and a quotient sheaf are defined as in the classical case. We define a sheaf $\mathcal{I}$ of R-semimodules is injective if $\mathcal{I}$ satisfies the following condition: let $(\mathcal{G}, \mathcal{F})$ be a pair of a sheaf $\mathcal{G}$ and a subsheaf $\mathcal{F}$. Then, for any morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{I}$ of sheaves, there exists a morphism $\psi : \mathcal{G} \longrightarrow \mathcal{I}$ such that $\psi \circ i = \varphi$, where i is an inclusion from $\mathcal{F}$ to $\mathcal{G}$. We have the following:

Proposition 2.3.17. *Let $(X, \mathcal{O}_X)$ be a locally semiringed space such that $\mathcal{O}_{X,x}$ is an idempotent semiring for each $x \in X$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-semimodules. Then, we have the following strong exact sequence of $\mathcal{O}_X$-semimodules:*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \mathcal{I}^3 \longrightarrow \dots$$

such that each $\mathcal{I}^j$ is an injective sheaf of $\mathcal{O}_X$-semimodules.

Proof. Since the category of idempotent semimodules has enough injectives (Proposition 2.3.15) and has limits, products, the proof is same as the classical case. More precisely, each stalk $\mathcal{F}_x$ can be embedded in an injective $\mathcal{O}_{X,x}$-semimodule I_x from Proposition 2.3.15. As in the classical construction of an injective resolution, we define $\mathcal{I}^0 := \prod_{x \in X} j_*(I_x)$, where $j_*(I_x)$ is the sheaf such that $j_*(I_x)(U) = I_x$ if $x \in U$ and 0 otherwise. The exact same argument as in the classical case shows that $\mathcal{I}^0$ is an injective sheaf of $\mathcal{O}_X$-semimodules and the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\epsilon} \mathcal{I}^0$$

is a strong exact sequence from the definition and Proposition 2.3.15. By using the quotient sheaf $\mathcal{I}^0/\mathcal{F}$, we can define inductively $\mathcal{I}^j$ and hence obtain the desired injective resolution which is strong exact. $\square$

Corollary 2.3.18. *Let X is a topological space. Then, the category of sheaves of idempotent semigroups on X has enough injectives.*

Proof. One can impose the constant sheaf $\mathcal{B}$ of the idempotent semifield $\mathbb{B}$ on X. Then, $(X, \mathcal{B})$ satisfies the condition of Proposition 2.3.17 and the category of sheaves of idempotent semigroups is indeed the category of sheaves of $\mathcal{B}$-semimodules. $\square$

Remark 2.3.19. *Assume that a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-semimodules has an injective resolution which is both strong and half. Then, it follows from Proposition 2.3.13 that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. However, by far, Proposition 2.3.17 is the best result we have. Moreover, even if we can find an injective resolution which is both strong and half, we have to show that two such resolutions are homotopic in order to properly define the sheaf cohomology. There is some evidence that the derived functors approach to sheaf cohomology might not be a good way to pursue. More precisely, in [28], Oliver Lorscheid computed the sheaf cohomology of the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ over $\mathbb{F}_1$ via an injective resolution, however, the computation is not in accordance with the classical result. For example, $\mathrm{H}^1(\mathbb{P}^1_{\mathbb{F}_1}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}})$ is an infinite-dimensional $\mathbb{F}_1$-vector space whereas classically, we have $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Although this is the case of a monoid scheme, this suggests that one might have to look for other possible approaches.*
In the next subsection, we directly generalize Čech cohomology theory and show that several classical properties are still valid in such framework. In particular, the generalized Čech cohomology of the projective line $\mathbb{P}^1_{\mathbb{Q}_{max}}$ over $\mathbb{Q}_{max}$ is similar to the classical case. Moreover, for a semi-scheme $(X, \mathcal{O}_X)$, we verify the classical cohomological

interpretation of a Picard group; $\mathrm{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$.

2.3.2 Čech cohomology

In [37], Alex Patchkoria generalized the notion of a chain complex of modules to semimodules. The main idea is that one may consider an alternating sum as the sum of two sums for which stand for a positive sum and a negative sum respectively. In this subsection, we use this idea to define Čech cohomology with values in sheaves of semimodules. Then we compute the simple example of the projective line $\mathbb{P}^1_{\mathbb{Q}_{max}}$ over $\mathbb{Q}_{max}$.

Definition 2.3.20. *(cf. [37])*

1. *Let R be a semiring. One says that a sequence of R-semimodules and R-homomorphisms,*

$$X: \cdots \underset{\partial^-_{n-2}}{\overset{\partial^+_{n-2}}{\rightrightarrows}} X^{n-1} \underset{\partial^-_{n-1}}{\overset{\partial^+_{n-1}}{\rightrightarrows}} X^n \underset{\partial^-_{n}}{\overset{\partial^+_{n}}{\rightrightarrows}} X^{n+1} \underset{\partial^-_{n+1}}{\overset{\partial^+_{n+1}}{\rightrightarrows}} \cdots, \quad n \in \mathbb{Z},$$

written $X = \{X^n, \partial^+_n, \partial^-_n\}$ for short, is a cochain complex if

$$\partial^+_{n+1} \circ \partial^+_n + \partial^-_{n+1} \circ \partial^-_n = \partial^-_{n+1} \circ \partial^+_n + \partial^+_{n+1} \circ \partial^-_n, \quad n \in \mathbb{Z}. \tag{2.3.12}$$

2. *For a cochain complex X, one defines the following R-semimodule:*

$$Z^n(X) := \{x \in X^n \mid \partial^+_n(x) = \partial^-_n(x)\}$$

as n-cocycles, and the n-th cohomology as an R-semimodule

$$H^n(X) := Z^n(X)/\rho^n,$$

where ρ^n is a congruence relation on $Z^n(X)$ such that $x\rho^n y$ if and only if

$$x + \partial^+_{n-1}(u) + \partial^-_{n-1}(v) = y + \partial^+_{n-1}(v) + \partial^-_{n-1}(u) \text{ for some } u, v \in X^{n-1}. \tag{2.3.13}$$

Suppose that $X = \{X^n, d_n^+, d_n^-\}$ and $Y = \{Y^n, \partial_n^+, \partial_n^-\}$ are cochain complexes of semimodules. Then, by a $\pm$-morphism from X to Y one means a collection $f = \{f^n\}$ of homomorphisms of semimodules which satisfies the following condition:

$$f^{n+1} \circ d_n^+ = \partial_n^+ \circ f^n, \quad f^{n+1} \circ d_n^- = \partial_n^- \circ f^n. \tag{2.3.14}$$

In [37], it is proven that a $\pm$-morphism $f = \{f^n\}$ from $X = \{X^n, d_n^+, d_n^-\}$ to $Y = \{Y^n, \partial_n^+, \partial_n^-\}$ induces a canonical homomorphism $H^n(f)$ of cohomology semimodules as follows:

$$H^n(f) : H^n(X) \longrightarrow H^n(Y), \quad [x] \mapsto [f^n(x)], \quad n \in \mathbb{Z}, \tag{2.3.15}$$

where $[x]$ is the equivalence class of $x \in Z^n(X)$ in $H^n(X)$.

Remark 2.3.21. *As pointed out in [37], a sequence $G = \{G^n, d_n^+, d_n^-\}$ of modules is a cochain complex in the sense of Definition 2.3.20 if and only if $G' = \{G^n, \partial^n := d_n^+ - d_n^-\}$ is a cochain complex of modules in the classical sense. Clearly, in this case, the cohomology semimodules of G as in Definition 2.3.20 is the cohomology modules of G' in the classical sense.*

By means of Definition 2.3.20, we introduce Čech cohomology with values in sheaves of semimodules which generalizes the classical construction. Let R be a semiring, X be a topological space, and $\mathcal{F}$ be a sheaf of R-semimodules on X. Suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of X, where I is a totally ordered set. Let $U_{i_0, i_1 \ldots, i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$. Then, as in the classical case, we define the following set:

$$C^n = C^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_n} \mathcal{F}(U_{i_0, i_1, \ldots, i_n}), \quad n \in \mathbb{N}. \tag{2.3.16}$$

Let $x_{i_0, \ldots, i_n}$ be the coordinate of $x \in C^n$ in $\mathcal{F}(U_{i_0, i_1, \ldots, i_n})$. The differentials are given as follows:

$$(d_n^+(x))_{i_0, i_1, \ldots, i_{n+1}} = \sum_{k=0, k=\text{even}}^{n+1} x_{i_0, \ldots \hat{i_k}, \ldots, i_{n+1}}|_{U_{i_0, i_1, \ldots, i_{n+1}}}, \tag{2.3.17}$$

$$(d_n^-(x))_{i_0,i_1,\dots,i_{n+1}} = \sum_{k=0,k=\text{odd}}^{n+1} x_{i_0,\dots\hat{i_k},\dots,i_{n+1}}|_{U_{i_0,i_1,\dots,i_{n+1}}}, \tag{2.3.18}$$

where the notation $\hat{i_k}$ means that we omit that index. One can directly use the classical computation to show that $C = \{C^n, d_n^+, d_n^-\}$ is a cochain complex in the sense of Definition 2.3.20. We denote the n-th cohomology semimodule (with respect to an open covering $\mathcal{U}$) of C by $\check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F})$.

Proposition 2.3.22. *Let R be semiring, X be a topological space, and $\mathcal{F}$ be a sheaf of R-semimodules on X. Let $\mathcal{U}$ be an open covering of X. Then we have*

$$\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X).$$

Proof. By the definition, we have $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) := Z^0(\mathcal{U}, \mathcal{F})/\rho^0$. Moreover, $x\rho^0 y \iff x + d_{-1}^+(u) + d_{-1}^-(v) = y + d_{-1}^+(v) + d_{-1}^-(u)$ for some $u, v \in C^{-1}$. Since $C^{-1} := 0$, we have $x\rho^0 y \iff x = y$. It follows that $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = Z^0(\mathcal{U}, \mathcal{F})$. Consider the following:

$$C^0 = \prod_{i\in I} \mathcal{F}(U_i) \overset{d_0^+}{\underset{d_0^-}{\rightrightarrows}} C^1 = \prod_{i<j\in I} \mathcal{F}(U_{ij}),$$

where d_0^+ is the product of maps $\mathcal{F}(U_j) \longrightarrow \mathcal{F}(U_{ij})$ induced by the inclusion $U_{ij} \longrightarrow U_j$ and d_0^- is the product of maps $\mathcal{F}(U_i) \longrightarrow \mathcal{F}(U_{ij})$ induced by the inclusion $U_{ij} \longrightarrow U_i$. Clearly, we have $Z^0(\mathcal{U}, \mathcal{F}) \subseteq C^0$. It follows from the inclusion $U_i \hookrightarrow X$ that we have a homomorphism $r_i : \mathcal{F}(X) \longrightarrow \mathcal{F}(U_i)$, hence the following homomorphism:

$$r = (r_i) : \mathcal{F}(X) \longrightarrow C^0.$$

Since $\mathcal{F}$ is a sheaf, we have $\mathrm{Img}(r) \subseteq Z^0(\mathcal{U}, \mathcal{F})$. Conversely, suppose that

$$y = (y_i) \in Z^0(\mathcal{U}, \mathcal{F}) = \{y \in C^0 = \prod_{i\in I} \mathcal{F}(U_i) \mid d_0^+(y) = d_0^-(y)\}.$$

Then we have $y_i|_{U_{ij}} = y_j|_{U_{ij}}$. It follows that there exists a unique global section

$y_X \in \mathcal{F}(X)$ such that $(y_X)|_{U_i} = y_i$. Consider the following map:

$$s : Z^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{F}(X), \quad y \mapsto y_X.$$

Then s is clearly an R-homomorphism. Furthermore, $r \circ s$ and $s \circ r$ are identity maps. This shows that $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for an open covering $\mathcal{U}$ of X. $\square$

Proposition 2.3.23. *Let R be semiring, X be a topological space, and $\mathcal{F}$ be a sheaf of R-semimodules on X. Let $\mathcal{U}$ be an open covering of X which consists of n proper open subsets of X. Then $\check{\mathrm{H}}^m(\mathcal{U}, \mathcal{F}) = 0 \ \forall m \geq n$.*

Proof. The proof is identical to that of the classical case since $C^m = 0$ for $m \geq n$. $\square$

We say that a covering $\mathcal{V} = \{V_j\}_{j \in J}$ of a topological space X is a refinement of another covering $\mathcal{U} = \{U_i\}_{i \in I}$ if there exists a map $\sigma : J \longrightarrow I$ such that $V_j \subseteq U_{\sigma(j)}$ for each $j \in J$. Suppose that $X^n := C^n(\mathcal{U}, \mathcal{F})$ and $Y^n := C^n(\mathcal{V}, \mathcal{F})$. Then the map σ induces the following $\pm$-morphism:

$$\sigma^n : X^n \longrightarrow Y^n, \quad \sigma^n(x)_{j_0,\dots,j_n} = x_{\sigma(j_0),\dots,\sigma(j_n)}|_{V_{j_0,\dots,j_n}}. \tag{2.3.19}$$

In fact, let $X = \{X^n, d_n^+, d_n^-\}$ and $Y = \{Y^n, \partial_n^+, \partial_n^-\}$. We have

$$(\sigma^{n+1} \circ d_n^+(x))_{j_0,\dots,j_{n+1}} = (d_n^+(x))_{\sigma(j_0),\dots,\sigma(j_{n+1})}|_{V_{j_0,\dots,j_{n+1}}}$$

$$= (\sum_{k=0,k=even}^{n+1} x_{\sigma(j_0),\dots,\sigma(\hat{j}_k),\sigma(j_{n+1})}|_{U_{\sigma(j_0),\dots,\sigma(j_{n+1})}})|_{V_{j_0,\dots,j_{n+1}}}$$

$$= (\sum_{k=0,k=even}^{n+1} x_{\sigma(j_0),\dots,\sigma(\hat{j}_k),\sigma(j_{n+1})})|_{V_{j_0,\dots,j_{n+1}}} = \sum_{k=0,k=even}^{n+1} \sigma^n(x)_{j_0,\dots,\hat{j}_k,\dots,j_{n+1}}|_{V_{j_0,\dots,j_{n+1}}}$$

$$= (\partial_n^+ \circ \sigma^n(x))_{j_0,\dots,j_{n+1}}.$$

Hence, we obtain $\sigma^{n+1} \circ d_n^+ = \partial_n^+ \circ \sigma^n$. Similarly one can prove that $\partial^{n+1} \circ d_n^- = \partial_n^- \circ \sigma^n$. The $\pm$-morphism $\sigma = \{\sigma^n\}$ induces a homomorphism, $\check{\mathrm{H}}^n(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathrm{H}}^n(\mathcal{V}, \mathcal{F})$.

The collection of open coverings of a topological space X becomes a directed system (with a refinement as a partial order). Since (co)limits exist in the category of semimodules, the following definition is well defined.

Definition 2.3.24. *Let R be a semiring. Let X be a topological space and $\mathcal{F}$ be a sheaf of R-semimodules on X. We define the n-th Čech cohomology of X with values in $\mathcal{F}$ as follows:*

$$\check{\mathrm{H}}^n(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^n(\mathcal{U},\mathcal{F}).$$

Note that from Proposition 2.3.22, we have $\check{\mathrm{H}}^0(X,\mathcal{F}) = \mathcal{F}(X)$.

Example 2.3.25. *Consider the projective line $X = \mathbb{P}^1_{\mathbb{Q}_{max}}$ over $\mathbb{Q}_{max}$. More precisely, we consider X as the semi-scheme with two open affine charts $U_0 := \operatorname{Spec} \mathbb{Q}_{max}[T]$ and $U_1 := \operatorname{Spec} \mathbb{Q}_{max}[\frac{1}{T}]$ glued along $T \mapsto \frac{1}{T}$. As in the classical case, one observes that $\mathcal{O}_X(X) = \mathbb{Q}_{max}$. From Proposition 2.3.22, we have $\check{\mathrm{H}}^0(X,\mathcal{O}_X) = \mathbb{Q}_{max}$. Furthermore, since X has the open covering $\mathcal{U} = \{U_0, U_1\}$ which consists of two proper open subsets of X, we have $\check{\mathrm{H}}^n(X,\mathcal{O}_X) = 0$ for $n \geq 2$ from Proposition 2.3.23. Finally, with respect to the covering $\mathcal{U} = \{U_0, U_1\}$, we have*

$$C:\ \mathcal{O}_X(U_0) \oplus \mathcal{O}_X(U_1) \underset{d_0^-}{\overset{d_0^+}{\rightrightarrows}} \mathcal{O}_X(U_{01}) \underset{d_1^-}{\overset{d_1^+}{\rightrightarrows}} 0\,.$$

In other words, we have

$$C:\ \mathbb{Q}_{max}[T] \oplus \mathbb{Q}_{max}[\tfrac{1}{T}] \underset{d_0^-}{\overset{d_0^+}{\rightrightarrows}} \mathbb{Q}_{max}[T, \tfrac{1}{T}] \underset{d_1^-}{\overset{d_1^+}{\rightrightarrows}} 0\,,$$

where $d_0^+(a,b) = b$ and $d_0^-(a,b) = a$. It follows that $Z^1(\mathcal{U},\mathcal{O}_X) = \mathbb{Q}_{max}[T,\frac{1}{T}]$. Let $x, y \in Z^1(\mathcal{U},\mathcal{O}_X)$. Then, we can write $x = x_0 + x_1$, $y = y_0 + y_1$, where $x_0, y_0 \in \mathbb{Q}_{max}[T]$ and $x_1, y_1 \in \mathbb{Q}_{max}[\frac{1}{T}]$. Let $u = (x_0, y_1)$, $v = (y_0, x_1)$. Then, we have

$$x + d_0^+(u) + d_0^-(v) = y + d_0^+(v) + d_0^-(u).$$

It follows that $x\rho^1 y$ and hence $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X) = 0$. However, this computation depends on the specific covering $\mathcal{U}$ since different from the classical case, we do not know yet whether $\check{\mathrm{H}}^1(X, \mathcal{O}_X) = \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X)$ or not. We remark that the above computation is also valid when we replace $\mathbb{Q}_{max}$ with other totally ordered semifields.

Next, we prove that the Picard group $\mathrm{Pic}(X)$ of a semi-scheme X is isomorphic to the first Čech cohomology group of the sheaf $\mathcal{O}_X^*$. The proof is not much different from the classical case, however, we include the proof for the completeness. Note that $\mathcal{O}_X^*$ is the sheaf such that $\mathcal{O}_X^*(U) = \{a \in \mathcal{O}_X(U) \mid ab = 1 \text{ for some } b \in \mathcal{O}_X(U)\}$ for an open subset U of X. Even though $\mathcal{O}_X$ is a sheaf of semirings, $\mathcal{O}_X^*$ is a sheaf of (multiplicative) abelian groups. Hence, $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$ is an abelian group. We use the multiplicative notation for $\mathcal{O}_X^*$.

In what follows, let X be a semi-scheme, $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-semimodules on X, and $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X such that $\varphi_i : \mathcal{O}_X|_{U_i} \simeq \mathcal{L}|_{U_i}$ $\forall i \in I$. Let $e_i \in \mathcal{L}(U_i)$ be the image of $1 \in \mathcal{O}_X(U_i)$ under $\varphi_i(U_i)$. Through the following lemmas, we define a corresponding cocyle in $\check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$ for an invertible sheaf $\mathcal{L}$ on X.

Lemma 2.3.26. *For $i < j \in I$ and $U_{ij} = U_i \cap U_j$, there exists $f_{ij} \in \mathcal{O}_X^*(U_{ij})$ such that*

$$e_i|_{U_{ij}} = (e_j|_{U_{ij}}) f_{ij}.$$

Proof. This is clear since $e_i|_{U_{ij}}$ and $e_j|_{U_{ij}}$ are invertible elements in $\mathcal{O}_X^*(U_{ij})$. □

We fix f_{ij} in Lemma 2.3.26. We have the following:

Lemma 2.3.27. *Let $f := (f_{ij}) \in C^1(\mathcal{U}, \mathcal{O}_X^*)$. Then we have $d_1^+(f) = d_1^-(f)$ and hence $f \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$. In particular, f has the canonical image in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$.*

Proof. For $i < j < k$, we have $e_i|_{U_{ij}} = (e_j|_{U_{ij}}) f_{ij}$, $e_j|_{U_{jk}} = (e_k|_{U_{jk}}) f_{jk}$. Thus we have

$$e_i|_{U_{ijk}} = (e_j|_{U_{ijk}})(f_{ij})|_{U_{ijk}} = (e_k|_{U_{ijk}})(f_{jk})|_{U_{ijk}}(f_{ij})|_{U_{ijk}} = e_k|_{U_{ijk}}(f_{ik})|_{U_{ijk}}.$$

This implies that $(f_{jk})|_{U_{ijk}}(f_{ij})|_{U_{ijk}} = (f_{ik})|_{U_{ijk}}$. It follows that $(d_1^+(f))_{ijk} = (d_1^-(f))|_{ijk}$

and hence $f = (f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$. Therefore, f has the canonical image in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$.

□

Lemma 2.3.28. *The canonical image of $f \in C^1(\mathcal{U}, \mathcal{O}_X^*)$ in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$ as in Lemma 2.3.27 does not depend on the choice of e_i.*

Proof. Let $\{e_i'\}_{i\in I}$ be another choice with $\{f_{ij}'\}$. We can take $\{g_i\}_{i\in I}$, where $g_i \in \mathcal{O}_X^*(U_i)$ such that $e_i' = g_i e_i$. Then, we have $e_i|_{U_{ij}} = f_{ij} e_j|_{U_{ij}}$, $e_i'|_{U_{ij}} = f_{ij}' e_j'|_{U_{ij}}$. It follows that $g_i|_{U_{ij}} e_i|_{U_{ij}} = f_{ij}' g_j|_{U_{ij}} e_j'|_{U_{ij}} = f_{ij}' g_j|_{U_{ij}} e_j|_{U_{ij}}$ and $g_i|_{U_{ij}} e_i|_{U_{ij}} = g_i|_{U_{ij}} f_{ij} e_j|_{U_{ij}}$. Therefore, $f_{ij} g_i|_{U_{ij}} = f_{ij}' g_j|_{U_{ij}}$. This implies that $f \cdot d_0^-(g) = f' \cdot d_0^+(g)$. In other words, f and f' give the same canonical image in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$. □

We denote the canonical image of $f \in C^1(\mathcal{U}, \mathcal{O}_X^*)$ in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$ by $\phi_{\mathcal{U}}(\mathcal{L})$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{V_j\}_{j\in J}$ be two open coverings of X such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$ and $\mathcal{L}|_{V_j} \simeq \mathcal{O}_X|_{V_j}$ $\forall i \in I$, $j \in J$. We define a new covering $\mathcal{U} \cap \mathcal{U}' := \{U_i \cap U_j\}_{(i,j)\in I\times J}$ of X. Then, clearly $\mathcal{U} \cap \mathcal{U}'$ is a refinement of $\mathcal{U}$. It follows that $\phi_{\mathcal{U}}(\mathcal{L})$ has a canonical image in $\check{\mathrm{H}}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*)$.

Lemma 2.3.29. *Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{V_j\}_{j\in J}$ be two open coverings of X such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$ and $\mathcal{L}|_{V_j} \simeq \mathcal{O}_X|_{V_j}$ $\forall i \in I$, $j \in J$. Let $f \in C^1(\mathcal{U}, \mathcal{O}_X^*)$ and $f' \in C^1(\mathcal{U}', \mathcal{O}_X^*)$ (as in Lemma 2.3.27). Then the canonical images of f and f' are same in $\check{\mathrm{H}}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*)$. In particular, each invertible sheaf $\mathcal{L}$ determines a unique element $\phi(\mathcal{L})$ in $\check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$.*

Proof. Let $\{e_i\}_{i\in I}$, $\{f_{ij}\}_{i,j\in I}$ for $\mathcal{U}$ and $\{e_j'\}_{j\in J}$, $\{f_{kl}'\}_{k,l\in J}$ for $\mathcal{U}'$ as in Lemma 2.3.26. We claim that the images of $\phi_{\mathcal{U}}(\mathcal{L})$ and $\phi_{\mathcal{U}'}(\mathcal{L})$ in $\check{\mathrm{H}}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*)$ are equal. Indeed, we can find $g_{ik} \in \mathcal{O}_X^*(U_i \cap V_k)$ such that $e_k'|_{U_i\cap V_k} = (g_{ik}) e_i|_{U_i \cap V_k}$. Hence, from the relation $e_k'|_{U_{ij}\cap V_{kl}} = (f_{kl}')|_{U_{ij}\cap V_{kl}} \cdot e_l'|_{U_{ij}\cap V_{kl}}$, we have that $(g_{ik})|_{U_{ij}\cap V_{kl}} \cdot e_i|_{U_{ij}\cap V_{kl}} = (f_{kl}')|_{U_{ij}\cap V_{kl}} \cdot e_l'|_{U_{ij}\cap V_{kl}} = (f_{kl}')|_{U_{ij}\cap V_{kl}} \cdot (g_{jl})|_{U_{ij}\cap V_{kl}} \cdot e_j|_{U_{ij}\cap V_{kl}}$. It follows that

$$(g_{ik})|_{U_{ij}\cap V_{kl}} \cdot (f_{ij})|_{U_{ij}\cap V_{kl}} = (f_{kl}')|_{U_{ij}\cap V_{kl}} \cdot (g_{jl})|_{U_{ij}\cap V_{kl}}. \tag{2.3.20}$$

Let $g = (g_{ik})$ for $i \in I, k \in J$. Then, we have $g \in C^0(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*)$. Give the set $I \times J$ a dictionary order. Then we have

$$(d_0^+(g))|_{(i,k)\times(j,l)} = g_{jl}|_{U_{ij}\cap V_{kl}} \text{ and } (d_0^-(g))|_{(i,k)\times(j,l)} = g_{ik}|_{U_{ij}\cap V_{kl}}. \tag{2.3.21}$$

Let $\alpha : Z^1(\mathcal{U}, \mathcal{O}_X^*) \longrightarrow Z^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*)$ be the $\pm$-morphism as in (2.3.19). Then α induces the map $\hat{\alpha}$:$\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*) \longrightarrow$$\check{\mathrm{H}}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*)$. Similarly, for $\mathcal{U}'$, we obtain

$$\beta : Z^1(\mathcal{U}', \mathcal{O}_X^*) \longrightarrow Z^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*), \quad \hat{\beta} : \check{\mathrm{H}}^1(\mathcal{U}', \mathcal{O}_X^*) \longrightarrow \check{\mathrm{H}}^1(\mathcal{U} \cap \mathcal{U}', \mathcal{O}_X^*).$$

In particular, if $\phi_{\mathcal{U}}(\mathcal{L}) = [f]$, then $\hat{\alpha}([f]) = [\alpha(f)]$, where $[f]$ is the equivalence class of $f \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$ in $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$. To complete the proof, we have to show that $[\alpha(f)] = [\beta(f')]$. We know that $\alpha(f)_{(i,k)\times(j,l)} = f_{ij}|_{U_{ij}\cap V_{kl}}$ and $\beta(f')_{(i,k)\times(j,l)} = f'_{kl}|_{U_{ij}\cap V_{kl}}$. It follows from (2.3.20) and (2.3.21) that

$$(\alpha(f) \cdot d_0^-(g))|_{U_{ij}\cap V_{kl}} = (\beta(f') \cdot d_0^+(g))|_{U_{ij}\cap V_{kl}}.$$

This proves that $[\alpha(f)] = [\beta(f')]$. Thus, f and f' have the same image in $\check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$. We denote this image by $\phi(\mathcal{L})$. □

Consider the following map:

$$\phi : \mathrm{Pic}(X) \longrightarrow \check{\mathrm{H}}^1(X, \mathcal{O}_X^*), \quad [\mathcal{L}] \mapsto \phi(\mathcal{L}),$$

where $[\mathcal{L}]$ is the isomorphism class of $\mathcal{L}$ in $\mathrm{Pic}(X)$.

Lemma 2.3.30. *ϕ is well defined.*

Proof. Suppose that $\mathcal{L} \simeq \mathcal{L}'$. We have to show that $\phi(\mathcal{L}) = \phi(\mathcal{L}')$. Let us fix an isomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{L}'$. We can find an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that on U_i both $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic to $\mathcal{O}_X$. Let $\{e_i\}$ and $\{f_{ij}\}$ be as in Lemma 2.3.26 for $\mathcal{L}$. Then we have $\varphi_{U_{ij}}(e_i)|_{U_{ij}} = f_{ij} \cdot \varphi_{U_{ij}}(e_j)|_{U_{ij}}$. Since $\phi(\mathcal{L}')$ does not depend on the choice of $\{e'_i\}$, we let $e'_i = \varphi_{U_i}(e_i)$ as in Lemma 2.3.26 for $\mathcal{L}'$. Then the desired property follows. $\square$

Lemma 2.3.31. *ϕ is a group homomorphism.*

Proof. Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are invertible sheaves of $\mathcal{O}_X$-semimodules. Then, so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ (directly follows from Lemma 2.2.12). Therefore, we can find an affine open covering $\mathcal{U} = \{U_i = \operatorname{Spec} R_i\}_{i \in I}$ of X such that $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}')(U_i) \simeq \mathcal{O}_X(U_i) \simeq \mathcal{L}(U_i) \simeq \mathcal{L}'(U_i) \simeq R_i$. In particular, we have $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}')(U_i) \simeq (\mathcal{L}(U_i) \otimes_{\mathcal{O}_X} \mathcal{L}'(U_i))$. Let $\{e_i\}_{i \in I}$, $\{f_{ij}\}_{i,j \in I}$ for $\mathcal{L}$ and $\{e'_j\}_{j \in J}$, $\{f'_{kl}\}_{k,l \in J}$ for $\mathcal{L}'$ as in Lemma 2.3.26 on the open covering $\mathcal{U}$. Then, we can take $\{e_i \otimes e'_i\}$ as a basis for $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}')(U_i)$ and the corresponding transition map is $F = (f_{ij} \cdot f'_{ij})$. It follows that $\phi(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}') = \phi(\mathcal{L})\phi(\mathcal{L}')$. $\square$

Lemma 2.3.32. *$\phi([\mathcal{L}]) = 1$ if and only if $[\mathcal{L}]$ is the isomorphism class of $\mathcal{O}_X$. In particular, ϕ is injective.*

Proof. Suppose that $\phi(\mathcal{L}) = 1$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$ $\forall i \in I$ and let f and e_i be as in Lemma 2.3.26. Since the canonical image of f does not depend on the choice of an open covering $\mathcal{U}$, we may assume that $[f] = [1] \in \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$. This implies that there exists $g \in C^0(\mathcal{U}, \mathcal{O}_X^*)$ such that $d_0^+(g) = f \cdot d_0^-(g)$. Hence, $(d_0^+(g))_{ij} = (f \cdot d_0^-(g))_{ij}$ and $f_{ij} \cdot g_i|_{U_{ij}} = g_j|_{U_{ij}}$. It follows that $(g_i e_i)|_{U_{ij}} = g_i|_{U_{ij}} e_i|_{U_{ij}} = g_i|_{U_{ij}} f_{ij} e_j|_{U_{ij}} = g_j|_{U_{ij}} e_j|_{U_{ij}} = (g_j e_j)|_{U_{ij}}$. Thus, $e_i g_i$ and $e_j g_j$ agree on U_{ij} and hence we can glue them to obtain the global isomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{O}_X$. Conversely, if $\mathcal{L} \simeq \mathcal{O}_X$, then clearly $\phi(\mathcal{L}) = 1$. In fact, one can take $e_i = e|_{U_i}$, where e is the identity in $\mathcal{O}_X(X)$. $\square$

Lemma 2.3.33. *ϕ is surjective.*

Proof. Notice that $\alpha \in \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$ comes from $[f] \in \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$ for an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X. Let $\mathcal{L}_i := \mathcal{O}_X|_{U_i}$ for each $i \in I$. Let $f = (f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$. Then, for $i < j$, each f_{ij} defines the following isomorphism:

$$\phi_{ij} : \mathcal{L}_i|_{U_{ij}} \longrightarrow \mathcal{L}_j|_{U_{ij}}, \quad s \mapsto f_{ij} \cdot s.$$

We define $\phi_{ii} := id$. Since $f \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$, we have $d_1^+(f) = d_1^-(f)$. It follows that $(d_1^+(f))_{ijk} = f_{jk} \cdot f_{ij} = (d_1^-(f))_{ijk} = f_{ik}$, and $f_{ij} \cdot f_{jk} = f_{ik}$. This implies that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ and therefore one can glue $\mathcal{L}_i$ to obtain the invertible sheaf $\mathcal{L}$. Let e_i be the image of 1 under the isomorphism $\mathcal{O}_X(U_i) \simeq \mathcal{L}(U_i)$. Then, we obtain the corresponding $f = (f_{ij})$. This implies that $\phi([\mathcal{L}]) = \alpha$, hence ϕ is surjective. $\square$

Finally, we conclude the following theorem via the isomorphism ϕ.

Theorem 2.3.34. $\mathrm{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^*)$ *for a semi-scheme* $(X, \mathcal{O}_X)$.

Example 2.3.35. *Consider the semifield $\mathbb{Q}_{max}(T)$. We first compute the invertible elements of the semiring $B := \mathbb{Q}_{max}[T]$. If $f(T) = \sum_{i=0}^n a_i T^i$ is an invertible element of B, then there exists $g(T) = \sum_{i=0}^m b_i T^i$ such that*

$$f(T) \odot g(T) = \sum_{i=0}^{n+m} (\max_{r+l=i}\{a_r + b_l\}) T^i = 1_B = 0.$$

This implies that $\max_{r+l=i}\{a_r + b_l\} := c_i = -\infty$ for $i \geq 1$. Hence, $a_j = b_j = -\infty$ for $j \geq 1$ and $f(T) \in \mathbb{Q}$.

Next, let $A := \mathbb{Q}_{max}[T, \frac{1}{T}]$ and A^* be the set of elements in A which is multiplicatively invertible (In particular, A^* is an abelian group). If $f(T) \in A^*$, then there exists $k \in \mathbb{N}$ such that $T^k f(T) \in B$. This implies that $T^k f(T) \in \mathbb{Q}$ from the first case. Since T^k for $k \in \mathbb{Z}$ is invertible in A, we conclude that $A^* = \{qT^n \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}$.

Let $X := \mathbb{P}^1_{\mathbb{Q}_{max}}$ and $\mathcal{U} = \{U_1, U_2\}$ be an open covering of X such that $U_1 \simeq \mathrm{Spec}\,\mathbb{Q}_{max}[T]$ and $U_2 \simeq \mathrm{Spec}\,\mathbb{Q}_{max}[\frac{1}{T}]$. From the above computation, we have $\mathcal{O}_X^*(U_i) = \mathbb{Q}$ and $\mathcal{O}_X^*(U_1 \cap U_2) = A^* = \{qT^n \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}$. Then, we have the following Čech complex:

$$C: \; C^0 = \mathbb{Q} \times \mathbb{Q} \underset{d_0^-}{\overset{d_0^+}{\rightrightarrows}} C^1 = A^* \underset{d_1^-}{\overset{d_1^+}{\rightrightarrows}} 0\,,$$

where $d_0^+(a, b) = b$, $d_0^-(a, b) = a$. Clearly, we have $C^1 = Z^1(\mathcal{U}, \mathcal{O}_X^*)$. Two elements qT^n and $q'T^{n'}$ in A^* are equivalent if and only if there exist $c = (a, b)$, $c' = (a', b') \in C^0$

such that

$$qT^n \odot d_0^+(c) \odot d_0^-(c') = q'T^{n'} \odot d_0^+(c') \odot d_0^-(c). \tag{2.3.22}$$

However, (2.3.22) holds if and only if $n = n'$. Therefore, we have

$$\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*) = C^1/\rho^1 = \{T^n \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}.$$

This is coherent with the classical result.

Example 2.3.36. *Note that, different from the classical case, $\mathbb{Q}_{max}[T]$ is not multiplicatively cancellative. Therefore the canonical map, $S^{-1} : \mathbb{Q}_{max}[T] \longrightarrow S^{-1}\mathbb{Q}_{max}[T]$ does not have to be injective. In tropical geometry, rather than working directly with $\mathbb{Q}_{max}[T]$, one works with the semiring $\overline{\mathbb{Q}_{max}[T]} := \mathbb{Q}_{max}[T]/\sim$, where $\sim$ is a congruence relation such that $f(T) \sim g(T) \iff f(x) = g(x)\ \forall x \in \mathbb{Q}_{max}$ (see, §2.4.2 for details about $\overline{\mathbb{Q}_{max}[T]}$ together with the classification of valuations on it). Let $B := \overline{\mathbb{Q}_{max}[T]}$. If $\overline{f(T)} \in B$ is multiplicatively invertible, then there exists $\overline{g(T)}$ such that $\overline{f(T) \odot g(T)} = \overline{1_B} = \overline{0}$. However, for $l \in \mathbb{Q}_{max}$, the set $\overline{l}$ consists of a single element l. It follows that $f(T) \odot g(T) = 0$. From Example 2.3.35, this implies that $f(T) \in \mathbb{Q}$ and hence $B^* = \mathbb{Q}$. Let $S = \{\overline{1}, \overline{T}, \overline{T}^2, ...\}$ be a multiplicative subset of B, and $A := S^{-1}B$. Since B is multiplicatively cancellative (cf. Corollary 2.4.17), B is canonically embedded into A. Moreover, similar to Example 2.3.35, one can observe that $A^* = \{q\overline{T}^n \mid q \in \mathbb{Q}, n \in \mathbb{Z}\}$.*
Suppose that the projective line $X := \mathbb{P}^1$ over $\mathbb{Q}_{max}$ is the semi-scheme such that two affine semi-schemes $\operatorname{Spec} \overline{\mathbb{Q}_{max}[T]}$ and $\operatorname{Spec} \overline{\mathbb{Q}_{max}[\frac{1}{T}]}$ are glued along $\operatorname{Spec} A$. The exact same argument as in Example 2.3.35 shows the following:

$$\check{\mathrm{H}}^1(X, \mathcal{O}_X^*) = \mathbb{Z}.$$

Remark 2.3.37. *One can observe that Example 2.3.35 also shows that any invertible sheaf $\mathcal{L}$ on $\mathbb{P}^1_{\mathbb{Q}_{max}}$ should be isomorphic to $\mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$. This classifies all invertible sheaves on $\mathbb{P}^1_{\mathbb{Q}_{max}}$ as in the classical case.*

Remark 2.3.38. *Since differential maps of many (co)homology theories are defined by alternating sums, it seems that many of those theories can be directly generalized by using the above framework. For example, if k is a semifield, then Hochschild homology can be computed via the above framework and the result is same as classical case, i.e. $HH_0(k) = k$ and $HH_n(k) = 0$ for all $n > 0$.*

2.4 Valuation theory over semi-structures

As in the classical case, one might expect that a theory of valuations over semi-structures provides some geometric information. To shape a theory of valuations over semi-structures, one first needs to find proper definitions. We will provide three possible approaches and compute toy examples for each. The first definition directly extends the definition of classical valuation. The second definition comes from the observation that for a valuation ν, we have $\nu(a+b) \in \{\nu(a), \nu(b)\}$ if $\nu(a) \neq \nu(b)$. In the last approach, we shall make use of hyperfields instead of the field $\mathbb{R}$ of real numbers. This is related with the probabilistic intuition: when $\nu(a) = \nu(b)$, the value $\nu(a+b)$ is not solely determined by $\nu(a)$ and $\nu(b)$. In the sequel, by an idempotent semiring we mean a semiring S such that $x + x = x \ \forall x \in S$. An idempotent semiring S has a canonical partial order $\leq$ such that $x \leq y \iff x + y = y \ \forall x, y \in S$.

Remark 2.4.1. *In fact, a theory of valuations over semirings has been introduced in [22], but has not been studied in the perspective of $\mathbb{F}_1$-geometry. Our goal in this section is to find an analogue of abstract curves in characteristic one. Furthermore, the authors of [22] had more concentrated on supertropical semirings which are more generalized objects than semirings.*

Definition 2.4.2. *Let S be an idempotent semiring. A valuation on S is a function $\nu : S \longrightarrow \mathbb{R} \cup \{\infty\}$ which satisfies the following conditions:*

1. *$\nu(x) = \infty \iff x = 0_S$.*

2. *$\nu(xy) = \nu(x) + \nu(y)$, where $+$ is the usual addition of $\mathbb{R}$.*

3. $\min\{\nu(x), \nu(y)\} \leq \nu(x+y)$ $\forall x, y \in S$.

Remark 2.4.3. *The third condition is redundant in some cases. For example, if S is a semiring of characteristic one, i.e. $x+y \in \{x, y\}$ $\forall x, y \in S$, then the third condition is automatic.*

Definition 2.4.4. *([22, Definition 2.2]) Let S be an idempotent semiring. A strict valuation on S is a function $\nu : S \longrightarrow \mathbb{R}_{max}$ which satisfies the following conditions:*

1. $\nu(x) = -\infty \iff x = 0_S$.

2. $\nu(xy) = \nu(x) + \nu(y)$, *where $+$ is the usual addition of $\mathbb{R}$.*

3. $\nu(x+y) = \max\{\nu(x), \nu(y)\}$ $\forall x, y \in S$.

In other words, a strict valuation ν is a homomorphism of a semiring S to the semi-fields $\mathbb{R}_{max}$ which has a trivial kernel.

As we mentioned earlier, Definition 2.4.4 can be justified in the sense that $\nu(a+b) \in \{\nu(a), \nu(b)\}$ for $\nu(a) \neq \nu(b)$ for a valuation ν on a commutative ring. Classically, the third condition is a subadditivity condition. However, we force the third condition to be an additivity condition and hence it is named a strict valuation. One can think of the similar generalization over a hyperfield. To this end, we introduce the following hyperfield.

Definition 2.4.5. *The hyperfield $\mathbb{R}_{+,val}$ has an underlying set as $\mathbb{R} \cup \{-\infty\}$. The addition $\oplus$ is defined as follows: for $x, y \in \mathbb{R}_{+,val}$,*

$$x \oplus y = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ [-\infty, x] & \text{if } x = y \end{cases}$$

The multiplication $\odot$ is given by the usual addition of real numbers with $a \odot (-\infty) = -\infty$ for $a \in \mathbb{R} \cup \{-\infty\}$.

The addition of $\mathbb{R}_{+,val}$ is designed to capture the information we loose when $\nu(x) =$

$\nu(y)$ since, in this case, $\nu(x+y)$ can be any number less than or equal to $\nu(x)$. We first have to show that the above definition makes sense, i.e. $\mathbb{R}_{+,val}$ is indeed a hyperfield.

Proposition 2.4.6. *$\mathbb{R}_{+,val}$ is a hyperfield.*

Proof. To avoid any notational confusion, let us use $\oplus$, $\odot$ for the addition and the multiplication of $\mathbb{R}_{+,val}$. First, we show that $\mathbb{R}_{+,val}$ is a canonical hypergroup. The addition is clearly commutative. We show that

$$(x \oplus y) \oplus z = x \oplus (y \oplus z).$$

The first case is when $x = y = z$; this is clear. The second case is when x, y, z are all different. Then we have $LHS = RHS = \max\{x, y, z\}$. The third case is when $x = y$ is different from z. In this case, we have $LHS = [-\infty, x] \oplus z$. As the first sub-case of this, if $x < z$, then we have $LHS = z$. On the other hand, in this case, we have $RHS = x \oplus (y \oplus z) = x \oplus z = z$. As the second sub-case of this, if $z < x$, then $LHS = [-\infty, x]$ and $RHS = x \oplus (y \oplus z) = x \oplus y = [-\infty, x]$. The fourth case is when $y = z$ is different from x; this is similar to the third case. The last case is when $x = z$ is different from y. As the first sub-case, if $x < y$, then $LHS = y = RHS$. As the second sub-case, if $y < x = z$, then we have $LHS = [-\infty, x] = RHS$. This shows that $\oplus$ is associative. One can observe that $-\infty$ is the additive identity, and the additive inverse of x is x itself. For the reversibility property, let us assume that $x \in y \oplus z$. If $y \neq z$, then $y \oplus z = \max\{y, z\}$. Hence, we may assume that $y < z$. Then, $x \in y \oplus z$ means that $x = z$. Therefore, we have $z \in x \oplus y = x$. If $y = z$, then we have $x \in [-\infty, y]$. This implies that $x \leq y$. In this case, we have $z \in x \oplus y$. This shows that $\mathbb{R}_{+,val}$ is a canonical hypergroup. From the definition, $\odot$ is invertible. All we have to show is the following:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

This is trivial if $x = -\infty$, hence we may assume that $x \neq -\infty$. The first case is when

$y \oplus z$ is a single element. We may assume that $y < z$. Then $LHS = x \odot z$. On the other hand, if $y < z$, then we have $x \odot y < x \odot z$. It follows that $RHS = x \odot y \oplus x \odot z = x \odot z$. The second case is when $y = z$, then we have $y \oplus z = [-\infty, y]$. Hence, we have $LHS = x \odot (y \oplus z) = [-\infty, x \odot y]$. On the other hand, since $x \odot y = x \odot z$ in this case, we have $RHS = [-\infty, x \odot y]$. Therefore, $\mathbb{R}_{+,val}$ is a hyperfield. $\square$

Next, we define a valuation of an idempotent semiring with values in $\mathbb{R}_{+,val}$.

Definition 2.4.7. *Let S be an idempotent semiring and $H = \mathbb{R}_{+,val}$. A valuation of S with values in H is a function $\nu : S \longrightarrow H$ which satisfies the following conditions:*

$$\nu(x+y) \in \nu(x) \oplus \nu(y), \quad \nu(xy) = \nu(x) \odot \nu(y), \quad \nu(x) = -\infty \Longleftrightarrow x = 0_S. \quad (2.4.1)$$

We next define absolute values on an idempotent semiring which has values in hyperfields. First, we recall the following three hyperfields (cf. [50]).

Definition 2.4.8. *1. The hyperfield $\mathcal{T}\mathbb{R}$ has an underlying set as $\mathbb{R}$. The addition is defined as follows: for $x, y \in \mathcal{T}\mathbb{R}$,*

$$x + y = \begin{cases} x & \text{if } |x| > |y| \\ y & \text{if } |x| < |y| \\ y & \text{if } x = y \\ [-|x|, |x|] & \text{if } x = -y \end{cases}$$

and the multiplication is the usual multiplication of $\mathbb{R}$.

2. The hyperfield $\mathbb{R}_{+,\triangle}$ has the underlying set $\mathbb{R}_{\geq 0}$. The addition is defined as follows: for $x, y \in \mathbb{R}_{+,\triangle}$,

$$x + y = \{c \in \mathbb{R}_{+,\triangle} \mid |x - y| \leq c \leq x + y\},$$

and the multiplication is the usual multiplication of real numbers.

3. The hyperfield $\mathbb{R}_{+,Y}$ has the underlying set $\mathbb{R}_{\geq 0}$. The addition is defined as *follows: for* $x, y \in \mathbb{R}_{+,Y}$*,*

$$x + y = \begin{cases} \max\{x, y\} & \text{if } x \neq y \\ [0, x] & \text{if } x = y \end{cases}$$

and the multiplication is the usual multiplication of real numbers.

Definition 2.4.9. *Let* H *be any of hyperfields in Definition 2.4.8 and* S *be an idempotent semiring. An absolute value on* S *with values in* H *is a function* $|-| : S \longrightarrow H$ *which satisfies the following conditions:*

$$|x| = 0_H \Longleftrightarrow x = 0_S, \quad |xy| = |x||y|, \quad |x + y| \in |x| + |y| \quad \forall x, y \in S. \tag{2.4.2}$$

Note that in Definition 2.4.2, 2.4.4, and 2.4.7, we say that two valuations ν_1, ν_2 are equivalent if there exists $\rho > 0$ such that $\nu_1(x) = \rho\nu_2(x)$ $\forall x \in S$, where $\rho\nu_2(x)$ is the usual multiplication of real numbers. For Definition 2.4.9, since the second condition is multiplicative, we say that two absolute values $|-|_1$, $|-|_2$ are equivalent if there exists $\rho > 0$ such that $|x|_1 = |x|_2^\rho$ $\forall x \in S$, where $|x|_2^\rho$ is the usual exponent of real numbers.

Next, we let $M = \mathbb{Q}_{max}$ or $\mathbb{Q}_{max}(T)$ and classify valuations and absolute values on M up to equivalence.

2.4.1 The first example, $\mathbb{Q}_{max}$

Proposition 2.4.10. *Let* $M = \mathbb{Q}_{max}$*. Then,*

1. *With Definition 2.4.2, the set of valuations on* M *is equal to* $\mathbb{R}$*. There are exactly three valuations on* M *up to equivalence.*

2. *With Definition 2.4.4, the set of strict valuations on* M *is equal to* $\mathbb{R}_{\geq 0}$*. There are exactly two strict valuations on* M *up to equivalence.*

3. *With Definition 2.4.7, the set of valuations on* M *with values in* $\mathbb{R}_{+,val}$ *is equal to* $\mathbb{R}_{\geq 0}$*. There are exactly two valuations on* M *up to equivalence.*

4. *With Definition 2.4.9 together with any hyperfield in Definition 2.4.8, the set of absolute values on M is equal to $\mathbb{R}_{\geq 1}$. There are exactly two absolute values of M up to equivalence.*

Proof. To avoid any possible confusion, let us denote by $\oplus$, $\odot$ the addition and the multiplication of M respectively.

1. In this case, as we previously remarked, the third condition is redundant since M is of characteristic one. We claim that any valuation ν on M only depends on the value $\nu(1)$. In fact, since $\mathbb{Z}$ is (multiplicatively) generated by 1 in $\mathbb{Q}_{max}$, it follows from the second condition that the value $\nu(1)$ determines $\nu(m)$ $\forall m \in \mathbb{Z}$. Moreover, for $\frac{1}{n}$, we have $\nu(1) = \nu(\frac{1}{n} \odot ... \odot \frac{1}{n}) = n\nu(\frac{1}{n})$ and hence $\nu(\frac{1}{n}) = \frac{1}{n}\nu(1)$. This implies that for $\frac{m}{n} \in \mathbb{Q}$, we have $\nu(\frac{m}{n}) = \frac{m}{n}\nu(1)$. Conversely, let $\nu : M \longrightarrow \mathbb{R} \cup \{\infty\}$ be a function such that $\nu(\frac{a}{b}) := \frac{a}{b}\nu(1)$ for some $\nu(1) \neq \infty$. Then, clearly ν is a valuation on M. It follows that the set of valuations on M is equal to $\mathbb{R}$. Next, suppose that ν_1, ν_2 are valuations on M such that $\nu(1) > 0, \nu(2) > 0$, then they are equivalent. In fact, if we take $\rho := \frac{\nu_1(1)}{\nu_2(1)}$, then for $x \in \mathbb{Q}_{max} \backslash \{-\infty\}$, we have $\nu_1(x) = x\nu_1(1) = x\rho\nu_2(1) = \rho\nu_2(x)$. Similarly, valuations ν_1 and ν_2 on M with $\nu_i(1) < 0$ are equivalent. Finally, $\nu(1) = 0$ gives a trivial valuation. Therefore, we have exactly three valuations up to equivalence.

2. In this case, we claim that a strict valuation ν is an order-preserving map. Indeed, we have $x \leq y \iff x \oplus y = y$. Suppose that $x \leq y$. Then we have $\nu(y) = \nu(x \oplus y) = \nu(x) + \nu(y) \iff \nu(x) \leq \nu(y)$. On the other hand, as in the above case, a strict valuation ν only depends on $\nu(1)$. Since ν is an order-preserving map and $\nu(0) = 0$, it follows that $\nu(1) \geq 0$. Therefore, the set of valuations on M is equal to $\mathbb{R}_{\geq 0}$. Moreover, if $\nu(1) = 0$, then we have a trivial valuation and strict valuations ν on M such that $\nu(1) > 0$ are equivalent as in the above case. Thus, in this case, there are exactly two strict valuations on M up to equivalence.

3. In this case, a valuation ν on M is determined by $\nu(1)$ and $\nu(1) \geq 0$. In fact, suppose that $x \leq y$. Then we have

$$\nu(x \oplus y) = \nu(y) \in \nu(x) + \nu(y). \tag{2.4.3}$$

Assume that $\nu(y) < \nu(x)$. Then we have $\nu(x) + \nu(y) = \nu(x)$ and it follows from (2.4.3) that $\nu(x) = \nu(y)$ which is a contradiction. This shows that ν is an order-preserving map. Furthermore, we have $\nu(0) = 0$ since $\nu(0 \odot 0) = \nu(0) = \nu(0) + \nu(0)$ ($\cdot$ is the usual addition of real numbers). It follows that $\nu(1) \geq 0 (= 1_{\mathbb{R}_{+,val}})$. Finally, similar to the first case, we have $\nu(\frac{a}{b}) = \frac{a}{b}\nu(1)$. Conversely, it is clear that all maps which satisfy such properties are valuations on M. Hence, the set of valuations on M is equal to $\mathbb{R}_{\geq 0}$. Furthermore, two valuations ν_1, ν_2 on M with $\nu_1(1)$, $\nu_2(1) > 0$ are equivalent as in the first case. Hence, there are exactly two valuations on M up to equivalence.

4. First, consider when $H = \mathcal{T}\mathbb{R}$. Let $|-|$ be an absolute value on M with values in H. One can observe that $|1| \geq 0$. Indeed, if $|1| = t < 0$, then we have $|\frac{1}{2} \odot \frac{1}{2}| = |\frac{1}{2}|^2 = |1| = t < 0$. However, this is impossible since $|\frac{1}{2}|$ is a real number. Thus, $|1| \geq 0$. This implies that for $x \in \mathbb{Q}_{max} \backslash \{-\infty\}$, we have $|x| \geq 0$. Next, we claim that the condition $|x \oplus y| \in |x| + |y|$ forces $|-|$ to be an order-preserving map. Indeed, if $x \leq y$, then $|x \oplus y| = |y| \in |x| + |y|$. From the reversibility property of a hyperfield, we have $|x| \in |y| - |y|$, where $|y| - |y| = [-|y|, |y|]$. Since $|x|, |y| \geq 0$, it follows that $|x| \leq |y|$. Finally, we claim that $1 \leq |1|$. In fact, let $|1| = \alpha$. Then we have $\alpha = |1| \leq |n| = \alpha^n$. From the first condition of the definition, we have $\alpha \neq 0$ and hence $1 \leq \alpha$. Therefore, as in the first case, an absolute value $|-|$ on M is totally determined by the value $|1|$. Conversely, any map $|-| : M \longrightarrow H$ which satisfies the following conditions: $|1| \geq 1$, $|\frac{m}{n}| := |1|^{\frac{m}{n}}$ for $\frac{m}{n} \in \mathbb{Q}$, and $|0_M| = 0$ is an absolute value on M. Two absolute values on M with $|1|_1 = \alpha > 1, |1|_2 = \beta > 1$ are equivalent

with $\rho = \frac{\log\alpha}{\log\beta}$. When $|1| = \alpha = 1$, we have the trivial absolute value. Thus, there are exactly two absolute values on M up to equivalence.

Next, consider when $H = \mathbb{R}_{+,\triangle}$. In this case, the third condition implies that if $x \leq y$, then $|y| = |x \oplus y| \in |x| + |y|$. From the reversibility property, we have $|x| \in |y| - |y| = [0, 2|y|]$. In particular,

$$|x| \leq 2|y| \quad \text{if } x \leq y. \tag{2.4.4}$$

Let $|1| = \alpha$ and $0 \leq m < n$ for $m, n \in \mathbb{Z}$. Then, we have $|m| = \alpha^m$, $|n| = \alpha^n$. Moreover, it follows from (2.4.4) that $|m| = \alpha^m \leq 2|n| = 2\alpha^n$ and hence $\frac{1}{2} \leq \alpha^r$ $\forall r > 0$. This implies that $|1| = \alpha \geq 1$. Conversely, any such map is an absolute value on M. Thus, the set of absolute values on M is equal to $\mathbb{R}_{\geq 1}$. Furthermore, two absolute values with $|1|_1 = \alpha > 1, |1|_2 = \beta > 1$ are equivalent with $\rho = \frac{\log\alpha}{\log\beta}$. Similarly, when $|1| = 1$, we obtain the trivial absolute value. Therefore, there are exactly two absolute values on M up to equivalence.

When $H = \mathbb{R}_{+,Y}$, it is similar to the above cases. For example, an absolute value $|-|$ on M is an order-preserving map and $|1| \geq 1$. Furthermore, the exact same argument shows that any two absolute values with $|1|_1 > 1, |1|_2 > 1$ are equivalent. Similarly, when $|1| = 1$, we have the trivial valuation. Therefore, there are exactly two absolute values up to equivalence.

$\square$

Remark 2.4.11. *Note that the hyperfields $\mathcal{T}\mathbb{R}$ and $\mathbb{R}_{+,Y}$ are defined to recast the archimedean information on M and the hyperfield $\mathbb{R}_{+,\Delta}$ is defined to recast non-archimedean information on M.*

2.4.2 The second example, $\mathbb{Q}_{max}(T)$

We begin with investigating $\mathbb{Q}_{max}[T]$, the idempotent semiring of polynomials with coefficient in $\mathbb{Q}_{max}$. In the sequel, we use the notations $+$ and $\cdot$ for the usual operations of $\mathbb{Q}$. We use the notations $\oplus, \odot$ for the operations of $\mathbb{Q}_{max}[T]$ and $+_t, \cdot_t$ for $\mathbb{Q}_{max}$. For $f(T) = \sum_{i=0}^{n} a_i T^i$, $g(T) = \sum_{i=0}^{m} b_i T^i \in \mathbb{Q}_{max}[T]$, suppose that $n \leq m$. The addition and the multiplication of $\mathbb{Q}_{max}[T]$ are given as follows:

$$(f+g)(T) = \sum_{i=0}^{n} \max\{a_i, b_i\} T^i + \sum_{i=n+1}^{m} b_i T^i, \tag{2.4.5}$$

$$(fg)(T) = \sum_{i=0}^{n+m} (\sum_{r+l=i} a_r b_l) T^i = \sum_{i=0}^{n+m} (\max_{r+l=i}\{a_r + b_l\}) T^i. \tag{2.4.6}$$

Note that we can consider the semifield $\mathbb{Q}_{max}$ as an algebraic closure of $\mathbb{Z}_{max}$ since any polynomial equation with coefficients in $\mathbb{Z}_{max}$ has a (tropical) solution in $\mathbb{Q}_{max}$. However, different from the classical case, any polynomial in $\mathbb{Q}_{max}[T]$ does not have to be factored into linear polynomials. Consider the following example.

Example 2.4.12. *Let $P(T) = T^{\odot 2} \oplus T \oplus 3 \in \mathbb{Q}_{max}[T]$. Then, $T = \frac{3}{2}$ is a tropical solution of $P(T)$. Suppose that $T^{\odot 2} \oplus T \oplus 3 = (T \oplus a) \odot (T \oplus b)$. Then, we have $(T \oplus a) \odot (T \oplus b) = T^{\odot 2} \oplus \max\{a, b\} \odot T \oplus (a + b)$. Thus, for $P(T)$ to be factored into linear polynomials, we should have $\max\{a, b\} = 1$ and $a + b = 3$, however, this is impossible. Hence, $P(T)$ can not be factored into linear polynomials.*

To remedy this issue, in tropical geometry, one imposes a functional equivalence relation on $\mathbb{Q}_{max}[T]$ (cf. [18]). Recall that polynomials $f(T), g(T) \in \mathbb{Q}_{max}[T]$ are functionally equivalent, denoted by $f(T) \sim g(T)$, if $f(t) = g(t)$ $\forall t \in \mathbb{Q}_{max}$.

Proposition 2.4.13. *For $M = \mathbb{Q}_{max}[T]$, a functional equivalence relation $\sim$ on M is a congruence relation.*

Proof. Clearly, $\sim$ is an equivalence relation. Suppose that $f(T) \sim g(T)$ and $h(T) \sim q(T)$. Then, we have to show that $f(T) \oplus h(T) \sim g(T) \oplus q(T)$ and $f(T) \odot h(T) \sim g(T) \odot q(T)$. Let $f(T) = \sum_{i=0}^{n} a_i T^i$, $g(T) = \sum_{i=0}^{m} b_i T^i$. It is enough to show that

$$(f \oplus g)(x) = f(x) +_t g(x), \quad (f \odot g)(x) = f(x) \cdot_t g(x) \quad \forall x \in \mathbb{Q}_{max}.$$

We may assume that $n \le m$. Then we have

$$(f \oplus g)(T) = \sum_{i=0}^{n} \max\{a_i, b_i\}T^i \oplus \sum_{i=n+1}^{m} b_i T^i.$$

For $x \in \mathbb{Q}_{max}$, we have, by letting $a_i = -\infty$ for $i = n+1, ..., m$,

$$(f \oplus g)(x) = \max_{i=0,\dots,m}\{\max\{a_i, b_i\} + ix\}.$$

However, $f(x) = \max_{i=0,\dots n}\{a_i + ix\}$ and $g(x) = \max_{i=0,\dots m}\{b_i + ix\}$, thus

$$f(x) +_t g(x) = \max\{f(x), g(x)\} = \max\{\max_{i=0,\dots,n}\{a_i + ix\}, \max_{i=0,\dots,m}\{b_i + ix\}\}$$

$$= \max_{i=0,\dots,m}\{\max\{a_i, b_i\} + ix\} = (f \oplus g)(x).$$

This proves the first part. Next, we have

$$(f \odot g)(T) = \sum_{i=0}^{n+m}(\sum_{r+l=i} a_r b_l)T^i = \sum_{i=0}^{n+m}(\max_{r+l=i}\{a_r + b_l\})T^i.$$

It follows that for $x \in \mathbb{Q}_{max}$, we have

$$(f \odot g)(x) = \max_{0 \le i \le n+m}\{\max_{r+l=i}\{a_r + b_l\} + ix\} = \max_{0 \le i \le n+m}\{\max_{r+l=i}\{a_r + rx + b_l + lx\}\}.$$

On the other hand, we have

$$f(x) \cdot_t g(x) = \max_{0 \le i \le n}\{a_i + ix\} + \max_{0 \le j \le m}\{b_j + jx\}.$$

Thus, if $f(x) = a_{i_0} + i_0 x$ and $g(x) = b_{j_0} + j_0 x$ for some i_0 and j_0, then we have

$$f(x) \cdot_t g(x) = (a_{i_0} + i_0 x) + (b_{j_0} + j_0 x) = (a_{i_0} + b_{j_0}) + (i_0 + j_0)x.$$

It follows that $f(x) \cdot_t g(x) \le (f \odot g)(x)$. But, if $(f \odot g)(x) = (a_{r_0} + r_0 x) + (b_{l_0} + l_0 x)$, then $(f \odot g)(x) \le f(x) \cdot_t g(x)$. Hence, $(f \odot g)(x) = f(x) \cdot_t g(x)$. $\square$

From Proposition 2.4.13, the set $\overline{\mathbb{Q}_{max}[T]} := \mathbb{Q}_{max}[T]/\sim$ is an idempotent semiring. In fact, $\overline{\mathbb{Q}_{max}[T]}$ is a semiring since $\sim$ is a congruence relation. Furthermore, for $f(T) \in \mathbb{Q}_{max}[T]$, we have $f(x) +_t f(x) = f(x)\ \forall x \in \mathbb{Q}_{max}$. This implies that $f(T) \oplus f(T) \sim f(T)$ and hence $\overline{\mathbb{Q}_{max}[T]}$ is an idempotent semiring. It is known that, for $\overline{\mathbb{Q}_{max}[T]}$, the fundamental theorem of tropical algebra holds. i.e. a polynomial $\overline{P(T)} \in \overline{\mathbb{Q}_{max}[T]}$ can be uniquely factored into linear polynomials in $\overline{\mathbb{Q}_{max}[T]}$ (cf. [43] or [48]). In particular, this implies that the notion of a degree of $\overline{f(T)} \in \overline{\mathbb{Q}_{max}[T]}$ is well-defined. Furthermore, $\overline{\mathbb{Q}_{max}[T]}$ does not have any multiplicative zero-divisor. Indeed, suppose that $\overline{f(T)} \cdot \overline{g(T)} = \overline{(fg)(T)} \sim (-\infty)$. Then, for $x \in \mathbb{Q}_{max}$, we have $f(x) \cdot_t g(x) = f(x) + g(x) = -\infty$. In other words, for $x \in \mathbb{Q}_{max}$, we have $f(x) = -\infty$ or $g(x) = -\infty$. However, this only happens when $f(T) = -\infty$ or $g(T) = -\infty$. Thus, $\overline{\mathbb{Q}_{max}[T]}$ does not have a multiplicative zero-divisor. In fact, in Corollary 2.4.17, we shall prove that $\overline{\mathbb{Q}_{max}[T]}$ satisfies the stronger condition: $\overline{\mathbb{Q}_{max}[T]}$ is multiplicatively cancellative.

Next, we prove several lemmas to classify valuations on $\mathbb{Q}_{max}(T)$.

Lemma 2.4.14. *Let $M := \overline{\mathbb{Q}_{max}[T]}$. For $\overline{f(T)} \in M$, let r_f be the maximum natural number such that $\overline{T}^{r_f}$ can divide $\overline{f(T)}$. Then, for $\overline{f(T)}, \overline{g(T)} \in M$, we have*

$$r_{f\oplus g} = \min\{r_f, r_g\}, \quad r_{f\odot g} = r_f + r_g.$$

Proof. Let $f(T), g(T) \in \mathbb{Q}_{max}[T]$. We first claim that if $f(T)$ has a constant term and $g(T)$ does not have a constant term, then $f(T)$ and $g(T)$ are not functionally equivalent. Indeed, if $f(T) = \sum a_i T^i$ and $g(T) = \sum b_i T^i$, then $f(-\infty) = a_0 \neq -\infty = g(-\infty)$. One can further observe that if $f(T) \sim T$, then $f(T) = T$. In fact, from the fundamental theorem of tropical algebra, we know that the degree of $f(T)$ should be one. Hence, $f(T) = a \odot T \oplus b$ for some $a, b \in \mathbb{Q}_{max}$. Then $b = -\infty$ since, otherwise, $f(-\infty) = b \neq -\infty$ and therefore $f(T) \not\sim T$. Furthermore, $a = 0$ since, otherwise, we have $f(-a) = 0$. However, this is different from the evaluation of T at $-a$.

Next, we claim that $\overline{f(T)} \in M$ has the factor $\overline{T}$ if and only if any representative of $\overline{f(T)}$ does not have a constant term. To see this, suppose that $\overline{f(T)}$ has the factor $\overline{T}$. Then, $f(T) \sim T \odot g(T)$ for some $g(T) \in \mathbb{Q}_{max}[T]$. Since $T \odot g(T)$ does not have a constant term, from the first claim, $f(T)$ also does not have a constant term. Conversely, suppose that any representative of $\overline{f(T)}$ does not have a constant term. We can write $f(T) = T \odot g(T)$ for some $g(T) \in \mathbb{Q}_{max}$. Hence, $\overline{f(T)}$ has a factor $\overline{T}$. From the fundamental theorem of tropical algebra, r_f is well defined. Moreover, for $\overline{f(T)}, \overline{g(T)} \in M$, we can write $\overline{f(T)} = \overline{T}^l \odot \overline{h(T)}$, $\overline{g(T)} = \overline{T}^m \odot \overline{p(T)}$ for some $\overline{h(T)}$, $\overline{p(T)}$ such that $\overline{h(T)}$ and $\overline{p(T)}$ do not have $\overline{T}$ as a factor. From our previous claim, this is equivalent to that $\overline{h(T)}$ and $\overline{p(T)}$ do have a constant term. Assume that $l \leq m$, then we have

$$\overline{f(T)} \oplus \overline{g(T)} = \overline{T}^l \odot (\overline{h(T)} \oplus \overline{T}^{(m-l)} \overline{p(T)}).$$

Since $\overline{h(T)}$ has a constant term, it follows that $\overline{h(T)} \oplus \overline{T}^{(m-l)} \overline{p(T)}$ has a constant term and therefore $\overline{h(T)} \oplus \overline{T}^{(m-l)} \overline{p(T)}$ does not have a factor $\overline{T}$. This shows that $r_{f \oplus g} = \min\{r_f, r_g\}$. The second assertion $r_{f \odot g} = r_f + r_g$ is clear from the fundamental theorem of tropical algebra. $\square$

Remark 2.4.15. *Lemma 2.4.14 is different from the classical case. Essentially, this is due to the absence of additive inverses. In the classical case, if* $f(T) = T^l h(T)$, $g(T) = T^m p(T) \in \mathbb{Q}[T]$ *with* $l < m$, *then* $f(T) + g(T) = T^l(h(T) + T^{(m-l)} p(T))$. *Hence, we have* $r_{f+g} = \min\{r_f, r_g\}$. *The problem is when* $l = m$. *For example, if* $f(T) = T(T+1)$, $g(T) = T(T-1) \in \mathbb{Q}[T]$, *then* $r_f = r_g = 1$. *However,* $f(T) + g(T) = 2T^2$ *and hence* $r_{f+g} = 2 > \min\{r_f, r_g\} = 1$ *from the additive cancellation which is impossible in the case of idempotent semirings.*

Lemma 2.4.16. *Let* $M := \overline{\mathbb{Q}_{max}[T]}$. *Then, for* $\overline{f(T)} \in M$, $\deg \overline{f(T)}$ *is well defined. Furthermore, for* $\overline{f(T)}, \overline{g(T)} \in M$, *we have*

$$\deg(\overline{f(T)} \oplus \overline{g(T)}) = \max\{\deg \overline{f(T)}, \deg \overline{g(T)}\}, \quad \deg(\overline{f(T)} \odot \overline{g(T)}) = \deg \overline{f(T)} + \deg \overline{g(T)}.$$

Proof. This is straightforward from the fundamental theorem of tropical algebra and the fact that no additive cancellation happens when we add two tropical polynomials. $\square$

Corollary 2.4.17. *Let $M := \overline{\mathbb{Q}_{max}[T]}$. Then M is multiplicatively cancellative.*

Proof. For $\overline{f(T) \odot h(T)} = \overline{g(T) \odot h(T)}$ with $h(T) \neq -\infty$, we have to show that $\overline{f(T)} = \overline{g(T)}$. We keep using the notation as in Lemma 2.4.14. We know that $\overline{f(T) \odot h(T)} = \overline{g(T) \odot h(T)}$ is equivalent to the following condition:

$$f(x) + h(x) = g(x) + h(x) \quad \forall x \in \mathbb{Q}_{max}, \tag{2.4.7}$$

where $+$ is the usual addition. Thus, if $h(x) \neq -\infty$, we have $f(x) = g(x)$. Since $h(x) = -\infty$ happens only when $x = -\infty$, it follows that $f(x) = g(x)$ as long as $x \neq -\infty$. Hence, all we have to show is that $f(-\infty) = g(-\infty)$. From Lemma 2.4.14, we have $r_f + r_h = r_g + r_h$ and therefore $r_f = r_g$. Fix a representative $f(T) = \sum a_i T^i$ of $\overline{f(T)}$. We then have $f(-\infty) = a_0$ if $r_f = 0$ and $f(-\infty) = -\infty$ if $r_f \neq 0$. Thus, we may assume that $r_f = r_g = 0$. Fix a representative $g(T) = \sum b_i T^i$ of $\overline{g(T)}$. From [48, Lemma 3.2], there exists a real number M such that if $x > M$, then $f(x) = a_0$ and $g(x) = b_0$. Since we know that $f(T)$ and $g(T)$ agree on all elements of $\mathbb{Q}_{max}$ but $-\infty$, we conclude that $f(x) = a_0 = b_0 = g(x)$ for $x > M$. Therefore, we have $f(-\infty) = a_0 = b_0 = g(-\infty)$ and hence $\overline{f(T)} = \overline{g(T)}$. $\square$

Let $M := \overline{\mathbb{Q}_{max}[T]}$, $S := M \backslash \{\overline{-\infty}\}$, and $\mathbb{Q}_{max}(T) := S^{-1}M$. It follows from Corollary 2.4.17 that the localization map $S^{-1} : M \longrightarrow S^{-1}M$ is injective and $\mathbb{Q}_{max}(T)$ is an idempotent semifield.

Proposition 2.4.18. *Let M be a multiplicatively cancellative idempotent semiring. Let $S := M \backslash \{0_M\}$ and $N := S^{-1}M$. Let ν be a valuation (or an absolute value) on N in the sense of any of Definitions 2.4.2, 2.4.4, 2.4.7, and 2.4.9. Then, a valuations (or an absolute value) ν on N only depends on the image $i(M)$ of the canonical injection $i : M \longrightarrow S^{-1}M = N$, $m \mapsto \frac{m}{1}$.*

Proof. Since i is an injection, one can identify an element $m \in M$ with $\frac{m}{1} \in S^{-1}M = N$ under the canonical injection i. We have $1_N = \frac{a}{a}$ $\forall a \in S = M^{\times}$. Then, with Definitions 2.4.2, 2.4.4, and 2.4.7, we have $\nu(1_N) = \nu(a) + \nu(\frac{1}{a}) = 0$, where $+$ is the usual addition of real numbers. It follows that $\nu(\frac{1}{a}) = -\nu(a)$ and hence $\nu(\frac{a}{b}) = \nu(a) - \nu(b)$. In the case of Definition 2.4.9, we have $\nu(1_N) = \nu(a)\nu(\frac{1}{a}) = 1$. It follows that $\nu(\frac{1}{a}) = \frac{1}{\nu(a)}$ and hence $\nu(\frac{a}{b}) = \frac{\nu(a)}{\nu(b)}$. $\square$

Remark 2.4.19. *In the theory of commutative rings, to be multiplicatively cancellative and to have no (multiplicative) zero divisors are equivalent conditions whereas, in the theory of semirings, the first condition implies the second condition and not conversely in general. However, even when M is only a semiring without (multiplicative) zero divisors, one can derive the statement as in Proposition 2.4.18 in the following sense. Let M be a semiring without (multiplicative) zero divisors and $Val(M)$ be the set of valuations on M (with respect to Definition 2.4.4 or 2.4.7). Then, there exists a set bijection between $Val(M)$ and $Val(S^{-1}M)$. Indeed, for $\nu \in Val(M)$, one can define a valuation $\tilde{\nu} \in Val(S^{-1}M)$ such that $\tilde{\nu}(\frac{a}{b}) = \nu(a)\nu(b)^{-1}$. Conversely, for $\nu \in Val(S^{-1}M)$, we define $\hat{\nu} = \nu \circ i \in Val(M)$, where $i : M \longrightarrow S^{-1}M$. One can easily check that these two are well defined and inverses to each other. For absolute values (Definition 2.4.9), one also derives the similar result.*

Proposition 2.4.20. *Let $M := \overline{\mathbb{Q}_{max}[T]}$, $S := M \backslash \{\overline{-\infty}\}$, and $\mathbb{Q}_{max}(T) := S^{-1}M$. Then, with Definition 2.4.4, the set of strict valuations on $\mathbb{Q}_{max}(T)$ which are trivial on $\mathbb{Q}_{max}$ is equal to $\mathbb{R}$. There are exactly three strict valuations on $\mathbb{Q}_{max}(T)$ which are trivial on $\mathbb{Q}_{max}$ up to equivalence.*

Proof. From Proposition 2.4.18 and Corollary 2.4.17, a strict valuation ν on $\mathbb{Q}_{max}(T)$ only depends on values of ν on M. Let $\overline{f(T)} \in M$. Then, from the fundamental theorem of tropical algebra, we have

$$\overline{f(T)} = \overline{l_1(T)} \odot \overline{l_2(T)} \odot ... \odot \overline{l_n(T)},$$

where $l_i(T) = a_iT \oplus b_i$ for some $a_i \in \mathbb{Q}, b_i \in \mathbb{Q}_{max}$. It follows that

$$\nu(\overline{f(T)}) = \nu(\overline{l_1(T)}) + \nu(\overline{l_2(T)}) + ... + \nu(\overline{l_n(T)}).$$

Let us first assume that $\nu(\overline{T}) < 0$. If $b_i \neq -\infty$, since ν is trivial on $\mathbb{Q}_{max}$, we have

$$\nu(a_i\overline{T} \oplus b_i) = \max\{(\nu(a_i) + \nu(\overline{T})), \nu(b_i)\} = \max\{\nu(\overline{T}), 0\} = 0.$$

Thus, we have

$$\nu(\overline{f(T)}) = r_f(\nu(\overline{T})), \tag{2.4.8}$$

where r_f is as in Lemma 2.4.14. Conversely, any map $\nu : \mathbb{Q}_{max}(T) \longrightarrow \mathbb{R}_{max}$ satisfying the following conditions:

$$\nu(q) = 0 \quad \forall q \in \mathbb{Q}, \quad \nu(-\infty) = -\infty, \quad \nu(\overline{T}) < 0, \quad \nu(\overline{f(T)}) = r_f(\nu(\overline{T}))$$

is indeed a strict valuation. In fact, from Lemma 2.4.14, we know that $r_{f\oplus g} = \min\{r_f, r_g\}$. Since $\nu(\overline{T}) < 0$ and $r_f, r_g \in \mathbb{N}$, this implies that

$$\nu(\overline{f(T)} \oplus \overline{g(T)}) = \nu(\overline{f(T) \oplus g(T)}) = r_{f\oplus g}\nu(\overline{T}) = \min\{r_f, r_g\}\nu(\overline{T})$$

$$= \max\{r_f\nu(\overline{T}), r_g\nu(\overline{T})\} = \max\{\nu(\overline{f(T)}), \nu(\overline{g(T)})\}.$$

Moreover, $\nu(\overline{f(T)} \odot \overline{g(T)}) = \nu(\overline{f(T) \odot g(T)}) = r_{f\odot g}\nu(\overline{T}) = (r_f + r_g)\nu(\overline{T}) = r_f\nu(\overline{T}) + r_g\nu(\overline{T}) = \nu(\overline{f(T)}) + \nu(\overline{g(T)})$. Furthermore, all such valuations on $\mathbb{Q}_{max}(T)$ are equivalent. Indeed, let ν_1, ν_2 be strict valuations on $\mathbb{Q}_{max}(T)$ such that $\nu_1(\overline{T}) = \alpha$ and $\nu_2(\overline{T}) = \beta$. Since α, β are negative numbers, $\rho := \frac{\beta}{\alpha}$ is a positive number and $\nu_2(\overline{f(T)}) = r_f\beta = (r_f\rho)\alpha = \rho\nu_1(\overline{f(T)})$.

Secondly, suppose that $\nu(\overline{T}) = 0$. Then, we have

$$\nu(a_i\overline{T} \oplus b_i) = 0.$$

In other words, ν is a trivial valuation since $0 = 1_{\mathbb{R}_{max}}$. Clearly, this is not equivalent to the first case.

The final case is when $\nu(\overline{T}) > 0$. Then we have

$$\nu(a_i\overline{T} \oplus b_i) = \max\{(\nu(a_i) + \nu(\overline{T})), \nu(b_i)\} = \max\{\nu(\overline{T}), 0\} = \nu(\overline{T}).$$

It follows that

$$\nu(\overline{f(T)}) = \deg(\overline{f(T)})\nu(\overline{T}).$$

Conversely, any map $\nu : \mathbb{Q}_{max}(T) \longrightarrow \mathbb{R}_{max}$ satisfying the following conditions:

$$\nu(q) = 0 \quad \forall q \in \mathbb{Q}, \quad \nu(-\infty) = -\infty, \quad \nu(\overline{T}) > 0, \quad \nu(\overline{f(T)}) = \deg(\overline{f(T)})(\nu(\overline{T}))$$

is indeed a strict valuation from Lemma 2.4.16. Furthermore, suppose that ν_1, ν_2 are strict valuations on $\mathbb{Q}_{max}(T)$ such that $\nu_1(\overline{f(T)}) = \alpha > 0, \nu_2(\overline{f(T)}) = \beta > 0$. Then, with $\rho = \frac{\beta}{\alpha}$, ν_1, ν_2 are equivalent. Furthermore, this case is not equivalent to any of the above. To sum up, the set of strict valuations on $\mathbb{Q}_{max}(T)$ which are trivial on $\mathbb{Q}_{max}$ is equal to $\mathbb{R}$ (by sending ν to $\nu(\overline{T})$). There are exactly three strict valuations depending on a sign of a value of $\overline{T}$. $\square$

Proposition 2.4.21. *Let $M := \overline{\mathbb{Q}_{max}[T]}$, $S := M\backslash\{\overline{-\infty}\}$, and $\mathbb{Q}_{max}(T) := S^{-1}M$. Then, with Definition 2.4.7, the set of valuations on $\mathbb{Q}_{max}(T)$ with values in $\mathbb{R}_{+,val}$ which are trivial on $\mathbb{Q}_{max}$ is equal to $\mathbb{R}$. There are exactly three valuations on $\mathbb{Q}_{max}(T)$ which are trivial on $\mathbb{Q}_{max}$ up to equivalence.*

Proof. To avoid the notational confusion, we denote by $\oplus, \odot$ the addition and the multiplication of idempotent semirings and by $\vee, \cdot$ the addition and the multiplication of $\mathbb{R}_{+,val}$. From Proposition 2.4.18, a valuation ν on $\mathbb{Q}_{max}(T)$ only depends on values of ν on M. Let ν be a valuation on $\mathbb{Q}_{max}(T)$ which is trivial on $\mathbb{Q}_{max}$. For $\overline{f(T)} \in M$, from the fundamental theorem of tropical algebra, we have $\overline{f(T)} = \overline{l_1(T)} \odot \overline{l_2(T)} \odot \dots \odot \overline{l_n(T)}$, where $l_i(T) = a_iT \oplus b_i$ for some $a_i \in \mathbb{Q}, b_i \in \mathbb{Q}_{max}$. Hence, ν is entirely determined by values on linear polynomials. Similar to Proposition 2.4.20, we divide the cases up to a sign of $\nu(\overline{T})$. The first case is when $\nu(\overline{T}) < 0$. Since ν is trivial on $\mathbb{Q}_{max}$, if $b \neq -\infty$, we have

$$\nu(a\overline{T} \oplus b) \in \nu(a\overline{T}) \vee \nu(b) = (\nu(a) \cdot \nu(\overline{T})) \vee \nu(b) = \max\{\nu(\overline{T}), 0\} = 0.$$

With the same notation as in Lemma 2.4.14, we have

$$\nu(\overline{f(T)}) = r_f \nu(\overline{T}). \tag{2.4.9}$$

Conversely, any map $\nu : \mathbb{Q}_{max}(T) \longrightarrow \mathbb{R}_{+,val}$ given by (2.4.9) is indeed a valuation. Indeed, from the fundamental theorem of tropical algebra, we have $\nu(\overline{f(T)} \odot \overline{g(T)}) = (r_f + r_g)\nu(\overline{T}) = r_f\nu(\overline{T}) + r_g\nu(\overline{T}) = \nu(\overline{f(T)}) \cdot \nu(\overline{g(T)})$. Moreover, from Lemma 2.4.14, we have $\nu(\overline{f(T)} \oplus \overline{g(T)}) = r_{(f \oplus g)}\nu(\overline{T}) = \min\{r_f, r_g\}\nu(\overline{T}) = \max\{r_f\nu(T), r_g\nu(T)\} = \max\{\nu(\overline{f(T)}), \nu(\overline{g(T)})\} \in \nu(\overline{f(T)}) \vee \nu(\overline{g(T)})$. Similar to Proposition 2.4.20, all these cases are equivalent.

The second case is when $\nu(\overline{T}) = 0$. Then we have $\nu(a_i\overline{T} + b_i) = 0$ and this case gives us a trivial valuation since $0 = 1_{\mathbb{R}_{+,val}}$. Clearly this is not equivalent to the first case. The final case is when $\nu(\overline{T}) > 0$. Then, as in Proposition 2.4.20, we have $\nu(\overline{f(T)}) = \deg(\overline{f(T)})(\nu(\bar{T}))$. Conversely, any map $\nu : \mathbb{Q}_{max}(T) \longrightarrow \mathbb{R}_{+,val}$ given in this way is indeed a valuation by Lemma 2.4.16. These are all equivalent from the exact same argument in Proposition 2.4.20. $\square$

Proposition 2.4.22. *Let $M := \overline{\mathbb{Q}_{max}[T]}$, $S := M \backslash \{\overline{-\infty}\}$, and $\mathbb{Q}_{max}(T) := S^{-1}M$. Then, with Definition 2.4.9 and the hyperfield $\mathbb{R}_{+,Y}$, the set of absolute values on $\mathbb{Q}_{max}(T)$ which are trivial on $\mathbb{Q}_{max}$ is equal to $\mathbb{R}_{>0}$. There are exactly three absolute values on $\mathbb{Q}_{max}(T)$ which are trivial on $\mathbb{Q}_{max}$ up to equivalence.*

Proof. To avoid the notational confusion, we denote by $\oplus, \odot$ the addition and the multiplication of idempotent semirings and by $\vee, \cdot$ the addition and the multiplication of $\mathbb{R}_{+,Y}$. From Proposition 2.4.18, an absolute value $|-|$ on $\mathbb{Q}_{max}(T)$ only depends on values on M. There are three possibilities. The first case is when $|\overline{T}| = \alpha > 1$. Let $a_i\overline{T} \oplus b_i$ be a linear polynomial. i.e. $a_i \neq -\infty$. Since $|-|$ is trivial on $\mathbb{Q}_{max}$, if $b_i \neq -\infty$, we have

$$|a_i\overline{T} \oplus b_i| \in |a_i| \cdot |\overline{T}| \vee |b_i| = |\overline{T}| \vee 1 = \alpha \vee 1.$$

Since $\alpha > 1$, we have $\alpha \vee 1 = \alpha$ and hence $|a_i\overline{T} \oplus b_i| = \alpha$. In other words, for $\overline{f(T)} \in M$, we have $|\overline{f(T)}| = \alpha^{\deg(\overline{f(T)})}$. Conversely, any map $|-| : \mathbb{Q}_{max}(T) \longrightarrow \mathbb{R}_{+,Y}$ given in this way is an absolute value which is trivial on $\mathbb{Q}_{max}$; since $\alpha > 1$, it directly follows from Lemma 2.4.16. Furthermore, any two absolute values $|-|_1, |-|_2$ such that $|\overline{T}|_1 = \alpha > 1, |\overline{T}|_2 = \beta > 1$ are equivalent with $\rho = \frac{\log \alpha}{\log \beta}$.
The second case is when $|\overline{T}| = \alpha < 1$. In this case, for $a \in \mathbb{Q}_{max} \backslash \{-\infty\}$, we have

$$|\overline{T} \oplus a| \in |\overline{T}| \vee 1 = \alpha \vee 1 = 1.$$

This implies that for $\overline{f(T)} \in M$, we have $|\overline{f(T)}| = \alpha^{r_f}$, where r_f is as in Lemma 2.4.14. Conversely, one can observe that this condition defines an absolute value which is trivial on $\mathbb{Q}_{max}$. Indeed, from Lemma 2.4.14, we have $|\overline{f(T)} \oplus \overline{g(T)}| = \alpha^{r_{f \oplus g}} = \alpha^{\min\{r_f, r_g\}}$. Since $\alpha < 1$, we have $\alpha^{\min\{r_f, r_g\}} = \max\{\alpha^{r_f}, \alpha^{r_g}\} \in \alpha^{r_f} \vee \alpha^{r_g} = |\overline{f(T)}| \vee |\overline{g(T)}|$. Furthermore, clearly $|\overline{f(T)} \odot \overline{g(T)}| = |\overline{f(T)}| \cdot |\overline{g(T)}|$. In this case, for absolute values $|-|_1, |-|_2$ such that $|\overline{T}|_1 = \alpha, |\overline{T}|_2 = \beta$ and $\alpha, \beta < 1$, since $\log \alpha$, $\log \beta < 0$, we have $\rho := \frac{\log \alpha}{\log \beta} > 0$ and $|-|_2^\rho = |-|_1$. This shows that all such $|-|_1$ and $|-|_2$ are equivalent.
The final case is when $|\overline{T}| = 1$. We have $|\overline{T} \oplus a| \in |\overline{T}| \vee |a| = |\overline{T}| \vee 1 = 1 \vee 1 = [0, 1]$ for $a \neq -\infty$. Since $\overline{T} \oplus (\overline{T} \oplus a) = \overline{T} \oplus a$, we have

$$|\overline{T} \oplus a| = |\overline{T} \oplus (\overline{T} \oplus a)| \in |\overline{T}| \vee |\overline{T} \oplus a|. \tag{2.4.10}$$

Suppose that $|\overline{T} \oplus a| = \beta < 1$. Then, since $|\overline{T}| = 1$, the right hand side of (2.4.10) is equal to $1 \vee \beta = 1$. This implies that $|\overline{T} \oplus a| = 1$ which is a contradiction. Therefore,

$|\overline{T} \oplus a| = 1$. It follows that for $\overline{f(T)} \in M$, we have $|\overline{(f(T)}| = 1$. In other words, this is the case of a trivial absolute value. $\square$

Our motivation in developing a valuation theory of semi-structures is to make an analogue of abstract curves in characteristic one. To explain this connection, let us recall several classical definitions and results of abstract curves (cf. [20, §1.5]).

Let k be an algebraically closed field and K be a finitely generated field extension of k of transcendence degree 1, i.e. a function field of dimension 1. By a valuation ν of K/k is a valuation on K which is trivial on k. In other words, ν is a valuation on K such that $\nu(x) = 1 \ \forall x \in k \backslash \{0\}$. A valuation ν is discrete if the value group of ν is isomorphic to the abelian group $\mathbb{Z}$ of integers, and the corresponding valuation ring is called a discrete valuation ring. Let C_K be the set of all discrete valuation rings of K/k. For $\mathfrak{p} \in C_K$, we denote by $R_\mathfrak{p}$ the discrete valuation ring corresponding to $\mathfrak{p}$. One makes the set C_K into a topological space by defining the closed sets to be the finite subsets of C_K and C_K itself. Furthermore, if U is an open subset of C_K, one defines the ring of regular functions on U to be $\mathcal{O}(U) := \bigcap_{\mathfrak{p} \in U} R_\mathfrak{p}$. Note that this is motivated by the same property when X is an integral scheme. An element $f \in \mathcal{O}(U)$ defines a function $f : U \longrightarrow k$ such that $f(\mathfrak{p})$ is the residue of f modulo the maximal ideal of $R_\mathfrak{p}$.

An abstract nonsingular curve over k is an open subset $U \subseteq C_K$ with the induced topology and the induced notion of regular functions. A morphism between two abstract nonsingular curves X and Y over k is a continuous function $\varphi : X \longrightarrow Y$ such that for each open subset $V \subseteq Y$ and every regular function $f : V \longrightarrow k$, $f \circ \varphi$ is a regular function on $\varphi^{-1}(V)$. The following theorem is one of the main theorems in the theory of abstract curves.

Theorem 2.4.23. *([20, Theorem 6.9, §1.5]) Let K be a function field of dimension 1 over an algebraically closed field k. An abstract nonsingular curve defined above is isomorphic to a nonsingular projective curve over k.*

Since two valuations are equivalent if and only if they have the same valuation ring, the set C_K can be considered as the set of discrete valuations of K/k up to equivalence. From Propositions 2.4.20 and 2.4.21, the direct analogue of the set C_K with $K = \mathbb{Q}_{max}(T)$ and $k = \mathbb{Q}_{max}$ is the set $Val(\mathbb{Q}_{max}(T)) := \{\nu_+, \nu_-\}$, where ν_+ is the class of valuations ν such that $\nu(\overline{T}) > 0$ and ν_- is the class of valuations ν such that $\nu(\overline{T}) < 0$. Furthermore, since their image is the integers as a set, they can be considered as discrete valuations. In the spirit of the construction of abstract curves, one can expect that the set of valuations $Val(\mathbb{Q}_{max}(T))$ gives some geometric information about the projective line $\mathbb{P}^1$ over $\mathbb{Q}_{max}$. However, one can observe that $X := \mathrm{Spec}(\overline{\mathbb{Q}_{max}[T]})$ contains many points. For example, in [18], the authors proved that there is one-to-one correspondence between principle prime ideals of $\overline{\mathbb{Q}_{max}[T]}$ and points of $\mathbb{Q}_{max}$. Hence, the points of $\mathbb{P}^1$ over $\mathbb{Q}_{max}$ are a lot more than the points of $Val(\mathbb{Q}_{max}(T))$. It seems more interesting connection of $Val(\mathbb{Q}_{max}(T))$ is with the projective line $\mathbb{P}^1$ over $\mathbb{F}_1$ rather than over $\mathbb{Q}_{max}$. Let us first recall the construction of the projective line $\mathbb{P}^1$ over $\mathbb{F}_1$.

Example 2.4.24. *(An example from [16]) One constructs the projective line $\mathbb{P}^1$ over $\mathbb{F}_1$ as follows. Let $C_\infty := \{..., t^{-1}, 1, t, ...\}$ be an infinite cyclic group generated by t and let $C_{\infty,+} := \{1, t, t^2, ...\}$, $C_{\infty,-} := \{1, t^{-1}, t^{-2}, ...\}$ be sub-monoids of C_∞. Let $U_+ := \mathrm{Spec}(C_{\infty,+})$, $U_- := \mathrm{Spec}(C_{\infty,-})$, and $U := \mathrm{Spec}(C_\infty)$ (see [16] for the notion of monoid spectra). One defines the projective line $\mathbb{P}^1$ over $\mathbb{F}_1$ by gluing U_+ and U_- along U. The space U_+ has two points, a generic point c_0 and a closed point c_+ containing t. Similarly, the space U_- has two points, a generic point c_0 and a closed point c_- containing t^{-1}. Hence, the projective line $\mathbb{P}^1_{\mathbb{F}_1}$ over $\mathbb{F}_1$ consists of three points $\{c_+, c_0, c_-\}$.*

Remark 2.4.25. *We can observe that the number of closed points of $\mathbb{P}^1_{\mathbb{F}_1}$ is exactly same as the number of points of $Val(\mathbb{Q}_{max}(T)) = \{\nu_+, \nu_-\}$. Furthermore, ν_+ corresponds to c_+ which is the prime ideal containing t and ν_- corresponds to c_- which is the prime ideal containing t^{-1}. In fact, one may consider that ν_0, which is an* *equivalence class of a trivial valuation, corresponds to* c_0 *which is the prime ideal consists of* $\{1 = t^0\}$*. This correspondence can be justified since Theorem 2.4.23 only concerns closed points of a projective nonsingular curve. Hence,* $Val(\mathbb{Q}_{max}(T))$ *can be considered as the projective line* $\mathbb{P}^1_{\mathbb{F}_1}$ *understood as an abstract curve.*

On the other hand, one can think of an absolute value with the hyperfield $\mathbb{R}_{+,Y}$ *as an analogue of a non-archimedean absolute value. In fact, from the definition of the hyperfield* $\mathbb{R}_{+,Y}$ *and Definition 2.4.9, we have an analogue of the ultrametric inequality:* $|x+y| \in |x| + |y| = \max\{|x|, |y|\}$ *if* $|x| \neq |y|$*. Furthermore, classically there is a natural one-to-one correspondence between the set of equivalence classes of valuations and the set of equivalence classes of non-archimedean absolute values. Hence, the set of absolute values of* $\mathbb{Q}_{max}(T)$ *as in Definition 2.4.9 with the hyperfield* $\mathbb{R}_{+,Y}$ *might be considered as the set of equivalence classes of valuations of* $\mathbb{Q}_{max}(T)$ *with values in the hyperfield. Let* $X(\mathbb{Q}_{max}(T))$ *be a set of equivalence class of such absolute values such that the image is isomorphic to the integers. Then, from Proposition 2.4.22, we have* $X(\mathbb{Q}_{max}(T)) = \{\mu_+, \mu_-\}$*, where* $\mu_+(\overline{(T)}) > 1, \mu_-(\overline{(T)}) < 1$*. Therefore, in this case, we are also able to give a similar correspondence to* $\mathbb{P}^1_{\mathbb{F}_1} = \{c_+, c_0, c_-\}$*.*

3

From semi-structures to hyper-structures

In this chapter, we first review the symmetrization process introduced in [21], then we investigate some algebraic properties of this process which will be applied in the next chapter to link geometries over semi-structures and hyper-structures. Throughout this chapter, by a semiring of characteristic one we mean a semiring M such that

$$x + y \in \{x, y\} \quad \forall x, y \in M. \tag{3.0.1}$$

We recall that $\mathbb{B} = \{0, 1\}$ is the smallest semifield of characteristic one such that

$$1 + 1 = 1, \quad 0 + 1 = 0 = 1 + 0, \quad 1 \cdot 1 = 1, \quad 1 \cdot 0 = 0.$$

We denote by $\mathbf{S}$ the hyperfield of signs (cf. §1.1.2).

3.1 The symmetrization functor $- \otimes_{\mathbb{B}} \mathbf{S}$

In his paper [21], S.Henry introduced the symmetrization process which generalizes in a suitable way the construction of the Grothendieck group completion of a multiplicative monoid. This process allows one to encode the structure of a $\mathbb{B}$-semimodule as the ‘positive’ part of a hypergroup interpreted as a **S**-hypermodule.

Next, we briefly recall this symmetrization process. Let B be a commutative monoid denoted additively and endowed with a neutral element 0. One can define the following canonical partial order on B:

$$x \leq y \Longleftrightarrow x + y = y. \tag{3.1.1}$$

By a partial order on B we mean a binary relation on B which is reflexive, transitive, and antisymmetric. A partial order is said to be total if for any $x, y \in B$, we have $x \leq y$ or $y \leq x$. We claim that when B satisfies the condition (3.0.1), such order is total. In fact, we know that $x + y = x$ or $x + y = y$ $\forall x, y \in B$, hence $x \leq y$ or $y \leq x$. We introduce the following notation

$$s(B) := \{(s, 1), (s, -1), 0 = (0, 1) = (0, -1) \mid s \in B\backslash\{0\}\}.$$

To minimize the notation we denote $(s, 1) := s$, $(s, -1) := -s$, and $|(s, 1))| = |(s, -1)| = s$. For any $X = (x, p) \in s(B)$, we define $\mathrm{sign}(X) = p$. $s(B)$ is a hypergroup (cf. §1.1.2 for the definition) with the addition given by

$$x + y = \begin{cases} x & \text{if } |x| > |y| \text{ or } x = y \\ y & \text{if } |x| < |y| \text{ or } x = y \\ [-x, x] = \{(t, \pm 1) \mid t \leq |x|)\} & \text{if } y = -x \end{cases} \tag{3.1.2}$$

We denote with $s : B \longrightarrow s(B)$, $s \mapsto (s, 1)$ the associated map.

Let H be a hypergroup and B be a commutative monoid. We say that a map $f : B \longrightarrow H$ is additive if

$$f(0) = 0 \text{ and } f(a + b) \in f(a) + f(b) \subseteq H \quad \forall a, b, \in B.$$

We claim that the construction of $s(B)$ determines the minimal hypergroup associated to a commutative monoid B as the following universal property states.

(Universal Property): Let B be a commutative monoid such that the canonical order as in (3.1.1) is total. Then, for any hypergroup K and an additive map $g : B \longrightarrow K$, there exists a unique homomorphism $h : s(B) \longrightarrow K$ of hypergroups such that $g = h \circ s$.

In fact, such $h : s(B) \longrightarrow K$ is given by $h(X) = \mathrm{sign}(X)g(x)$, $\forall X = (x, p) \in s(B)$ (cf. [21, Theorem 5.1]).

Remark 3.1.1. *1. Let B be a commutative monoid such that the canonical order as in (3.1.1) is total. Assume also that B is equipped with a smallest element. Then B can be upgraded to a semiring by defining the addition law as the maximum(with respect to the canonical order) and the multiplication as the usual addition. For example, $\mathbb{R}_{max}$ is the semifield obtained from the (multiplicative)commutative monoid $(\mathbb{R} \cup \{-\infty\}, +)$.*

2. The symmetrization process can be applied to a general class of monoids (cf. [21, Theorem 5.1]). In fact, for a commutative monoid B, $s(B)$ is a hypergroup if and only if B satisfies the following condition: for all $x, y, z, w \in B$,

$$x + y = z + w \Longrightarrow \exists b \in B \ s.t. \begin{cases} x + b = z; \quad b + w = y, \\ or\ x = z + b; \quad w = b + y \end{cases} \tag{3.1.3}$$

In [21], it is also proved that when B is an idempotent monoid, B fulfills the condition (3.1.3) if and only if the canonical order of B as in (3.1.1) is total (cf. [21, Proposition 6.2]).

Let $(B, +, \cdot)$ be an idempotent semiring. It follows that the additive monoid $(B, +)$ allows for the symmetrization process if and only if $(B, +, \cdot)$ is, in fact, of characteristic one. Since our main interest lies in idempotent semirings, to this end, we will mostly focus on a semiring of characteristic one.

As Connes and Consani pointed out in [11], the symmetrization process can be understood in terms of the functor "extension of scalars". In this section we investigate how this process relates some algebraic properties of semirings to those of hyperrings. To begin with, we shall provide (cf. Proposition 3.1.8) a partial converse of Henry's construction.

Definition 3.1.2. *(cf. [32, §5]) Let R be a hyperring. A good ordering on R is a subset $P \subseteq R$ satisfying the following properties:*

$$P + P \subseteq P, \quad PP \subseteq P, \quad P \cup -P = R, \text{ and } P \cap -P = \{0\}.$$

Example 3.1.3. *Let $R = \mathcal{T}\mathbb{R}$ be Viro's hyperfield of real numbers as in Definition 2.4.8. A good ordering on R is provided by the subset $P = \{x \in \mathcal{T}\mathbb{R} \mid x \geq 0\} \subseteq R$. The easiest example of a good ordering on a hyperfield is given by choosing $R = \mathbf{S}$, the hyperfield of signs, then $P := \{0, 1\}$.*

Remark 3.1.4.

1. *In general, the definition of an ordering $P \subseteq R$ on a commutative ring R only requires $P \cap -P$ to be a prime ideal of R. In the above definition this condition is replaced by imposing that $P \cap -P = \{0\}$. This is done to encode P as the 'positive' part of R. Note that if P is a good ordering on R, then $-P$ is also a good ordering on R.*

2. *The conditions $P \cup -P = R$, $P \cap -P = \{0\}$ mean: $x = -x \iff x = 0$.*

3. *One can easily see that a hyperring R has a good ordering if and only if there exists a hyperring homomorphism $g : R \longrightarrow \mathbf{S}$ such that $g^{-1}(0) = \{0\}$. Indeed, suppose that R has a good ordering P. We define $g : R \longrightarrow \mathbf{S}$ such that*

$$g(x) = \begin{cases} 1 & \text{if } x \in P\backslash\{0\} \\ -1 & \text{if } x \in -P\backslash\{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly this is a homomorphism of hyperrings such that $g^{-1}(0) = \{0\}$. Conversely, suppose that $g : R \longrightarrow \mathbf{S}$ is a homomorphism of hyperrings such that $g^{-1}(0) = \{0\}$. Then the set $P := g^{-1}(\{0, 1\})$ becomes a good ordering on R.

If $B = (B, +, \cdot)$ is a semiring such that the symmetrization process can be applied to the additive monoid $(B, +)$, the multiplicative structure of B induces the corresponding multiplicative structure on $s(B)$ in the component-wise way. In other words, one can define the multiplication law on $s(B)$ such that:

$$(x,p)\cdot(y,q) := (xy, pq), \quad p,q \in \{-1,1\}; \quad 1\cdot 1 = (-1)\cdot(-1) = 1, \quad 1\cdot(-1) = -1.$$

Remark 3.1.5. *Let M be a semiring allowing for the symmetrization process. We will prove in Lemma 3.1.6 that under the component-wise multiplication, $s(M)$ is not a hyperring but only a multiring(cf. [32]). A multiring is a weaker version of a hyperring in the sense that a hyperring fulfills the distributive law $x(y+z) = xy + xz$ whereas the notion of a multiring only assumes the weak distributive property*

$$x(y+z) \subseteq xy + xz.$$

For example, let M be the semiring whose underlying set is $\mathbb{Z}_{\geq 0}$ with the addition given by $x + y := \max\{x, y\}$, and the multiplication given by the usual multiplication. Then $s(M)$ does not satisfy the distributive law. For example, $2(3-3) \neq 6-6 = [-6, 6]$. Indeed, we have $5 \in 6-6 = [-6, 6]$, but 5 can not be an element of $2(3-3) = 2\cdot[-3, 3]$ because 2 can not divide 5 in usual sense. For $s(M)$ to satisfy the distributive law it seems necessary to add a suitable divisibility condition on the multiplication of $s(M)$. A particular case will be studied in Proposition 3.1.10.

Let M be a semiring. Since $s(M)$ can be understood as a scalar extension $M \otimes_{\mathbb{B}} \mathbf{S}$, we denote $M_S := s(M)$ from now on. If M_S is not just a multiring but a hyperring, then M_S together with $i : M \longrightarrow M_S$ is indeed a universal pair among all hyperrings in the sense of [21].

Lemma 3.1.6. *Let B be a semiring of characteristic one. Then B_S is a multiring with the component-wise multiplication. In particular, if B is a semifield of characteristic one, then B_S is a hyperfield.*

Proof. We first note that $x \leq y$ implies $xz \leq yz$ for all $z \in B$. In fact, it follows from $x \leq y \Longleftrightarrow x + y = y$ that $xz + yz = yz \Longleftrightarrow xz \leq yz$. For the first assertion, all we have to show is that $X(Y+Z) \subseteq XY + XZ$ for all $X, Y, Z \in B_S$. If $X = 0$, then there is nothing to prove. Therefore we may assume that $X \neq 0$. Let $Y = (y, p)$, $Z = (z, q)$, $X = (x, r)$. When $\#(Y+Z) = 1$, it follows from (3.1.2) that there are three possible cases. The first case is when $Y = Z$. In this case, we have $X(Y+Z) = XY = XY + XY = XY + XZ$. The second case is when $p = q$, but $y \neq z$. Since B is of characteristic one, we may further assume that $y > z$. Therefore, we have $xy \geq xz$ and $X(Y+Z) = XY \in XY + XZ$. The final case is when $p \neq q$ and $y \neq z$. But, in this case, the similar argument as the second case shows that $X(Y+Z) \subseteq XY + XZ$. When $\#(Y+Z) \neq 1$, from (3.1.2), we may assume that $Y = (y, 1)$, $Z = (y, -1)$, and $X = (x, r)$. Take any $T = (t, p) \in (Y+Z)$. It follows from (3.1.2) that $t \leq y$, hence $xt \leq xy$. Therefore we have $XT = (xt, pr) \in XY + XZ$. When B is a semifield, each non-zero element of B_S has a multiplicative inverse. Therefore B_S is a multifield and it is well-known that any multifield is a hyperfield(and vice versa). □

Lemma 3.1.7. *Let R be a multiring and H be a hyperring. Suppose that there exists an isomorphism $\varphi : (R, +) \longrightarrow (H, +)$ of hypergroups such that $\varphi(xy) = \varphi(x)\varphi(y)$ $\forall x, y \in R$. Then R is a hyperring and φ is an isomorphism of hyperrings.*

Proof. First we claim that $xy + xz \subseteq x(y+z)$ for all $x, y, z \in R$. We have $\varphi(xy+xz) = \varphi(xy) + \varphi(xz) = \varphi(x)\varphi(y) + \varphi(x)\varphi(z) = \varphi(x)(\varphi(y) + \varphi(z)) = \varphi(x)\varphi(y+z) = \varphi(x(y+z))$. By taking φ^{-1}, we obtain our claim and so R is a hyperring. To show that φ is an isomorphism of hyperrings, we have to prove that $\varphi^{-1}(ab) = \varphi^{-1}(a)\varphi^{-1}(b)$. This is clear by taking $a = \varphi(x), b = \varphi(y)$. □

Proposition 3.1.8. *Let R be a hyperring such that*

$$x + x = x \quad \forall x \in R; \quad x + y \in \{x, y\} \quad \forall x \neq -y \in R. \tag{3.1.4}$$

Let P be a good ordering on R. Then

1. *P is a semiring such that the canonical order deduced from the addition as in (3.1.1) is a total order and $x + x = x$ for all $x \in P$. i.e. P is a semiring of characteristic one.*

2. *Under the symmetrization process, P_S is a hyperring with the multiplication given component-wise and P_S is isomorphic to R as hyperrings.*

Proof. We first prove that P is a semiring satisfying the properties stated in 1. Trivially we have $0 \in P$. If $1 \notin P$ then $-1 \in P$ and since $PP \subseteq P$, this implies $(-1)(-1) = 1 \in P$ which is a contradiction. Hence $1 \in P$. Furthermore, the addition on P induced from R is single-valued since we assumed that for any $x, y \in R$ with $x \neq -y$, $x + y$ is a single element. As we mentioned in Remark 3.1.4, if x, y are non-zero elements of P then they can not be the additive inverse of each other. The first two conditions of a good ordering imply that the induced addition and multiplication are closed. Thus P is a semiring. Furthermore, we have $x + x = x$ for all $x \in P$. Since $x + y \in \{x, y\}$, it follows that the canonical order is total. Moreover, this order is compatible with the multiplication. In fact, $x \leq y \Longleftrightarrow x + y = y$. Then for any $z \in P$ we have $zx + zy = zy \Longrightarrow zx \leq zy$. This proves the first part of the proposition.

Because P satisfies the sufficient condition of having the symmetrization, P_S is a hypergroup. In fact, it follows from Lemma 3.1.6 that P_S is a multiring. We claim that together with the inclusion map $i : P \hookrightarrow R$, (R, i) is the universal pair. Indeed, let K be a hypergroup and $g : P \longrightarrow K$ be an additive map. Define $h : R \longrightarrow K$ such that

$$h(x) = \begin{cases} g(x) & \text{if } x \in P \\ -g(-x) & \text{if } x \in -P \end{cases}$$

This is well-defined since $P \cup -P = R$, $P \cap -P = \{0\}$, and $g(0) = 0$. We observe that $h(0) = 0$. If $x, y \in P$, then $h(x + y) = g(x + y) \in g(x) + g(y) = h(x) + h(y)$. If

$x, y \in -P$, then so is for $x+y$, hence $h(x+y) = -g(-x-y) \in -(g(-x)+g(-y)) = -g(-x) - g(-y) = h(x) + h(y)$. Finally, if $x \in P$, $y \in -P$, then let $t = -y \in P$. If $z \in x+y$, we want to show that $h(z) \in h(x) + h(y)$. The first case is when $z \in P$. Then we have $z \in x + y = x - t = -t + x$. From the reversibility property it follows that $x \in z - (-t) = z + t$. Since $x, z, t \in P$ we can use the property of g to deduce that $g(x) \in g(z+t) \in g(z) + g(t)$. Again from the reversibility we derive that $g(z) \in g(x) - g(t)$, equivalently we have that $h(z) \in h(x) + h(-t) = h(x) + h(y)$. The second case is when $z \in -P$. We let $z = -w, w \in P$. Then we have $z \in x + y = x - t \iff -w \in x - t \iff w \in t - x = -x + t$. Again from the reversibility we have $t \in w + x$, then since $t, w, x \in P$, it follows that $g(t) \in g(w) + g(x) = g(x) + g(w)$. From the reversibility, $g(w) \in g(t) - g(x) \iff -g(w) \in g(x) - g(t)$. Therefore, we conclude that $h(-w) \in h(x) + h(-t)$, or $h(z) \in h(x) + h(y)$. This shows that h is a homomorphism of hypergroups.

It follows from the construction that $g = h \circ i$, and such h is unique. Indeed, suppose that $g = f \circ i$. Then for any $x \in P$, we have $g(x) = f(i(x)) = f(x) = h(x)$. For any $x \in -P$ we know that $-x \in P$ and $0 \in x - x$. Hence $f(0) = 0 \in f(x-x) \in f(x) + f(-x)$. From the uniqueness of the inverse, $f(x) = -f(-x) = -g(-x) = h(x)$. Since (R, i) is also the universal pair, as hypergroups, R is isomorphic to P_S. Furthermore, this isomorphism is also a homomorphism of multirings which is an isomorphism of hypergroups. Thus, from Lemma 3.1.7, it follows that P_S is a hyperring which is isomorphic to R. $\square$

Remark 3.1.9. *The above proposition has an easier interpretation when we restrict to the case of a hyperfield R satisfying the condition (3.1.4). In fact, in this case, the notion of a good ordering on R given in Definition 3.1.2 coincides with the notion of an ordering on R given in [32]. In that paper, M.Marshall defined real hyperfields as hyperfields F such that $-1 \notin \sum F^2$ and proved that F is a real hyperfield if and only if F has an ordering. Let M be a semifield of characteristic one, then we have $-1 \notin \sum M_S^2$. Thus it follows that M_S is a real hyperfield. Conversely, suppose* *that R is a real hyperfield satisfying the condition* (3.1.4). *Since any real hyperfield has a good ordering $P \subset R$, it follows that $R \simeq P_S$ from Proposition 3.1.8. If we let $- \otimes_{\mathbb{B}} S$ be a functor from the category of semifields of characteristic one to the category of hyperrings, it follows that the category of real hyperfields satisfying the condition* (3.1.4) *is the essential image of the functor $- \otimes_{\mathbb{B}} \mathbf{S}$.*

Proposition 3.1.10. *Let M be a semiring of characteristic one such that*

$$x < y \Longrightarrow xz < yz \quad \forall x, y, z \in M \backslash \{0_M\}, \tag{3.1.5}$$

where $<$ is the canonical order as in (3.1.1). *Suppose that M satisfies the following condition*

$$\forall x, y \in M, \quad \exists \alpha, \beta \in M \text{ s.t. } x\alpha = y, \quad x = y\beta. \tag{3.1.6}$$

Then M_S is a hyperring. Conversely, let us further assume that $1_M \leq x$ for all $x \in M \backslash \{0_M\}$. If M_S is a hyperring then M satisfies the condition (3.1.6).

Proof. From Lemma 3.1.6, we know that M_S is a multiring. Therefore, to prove that M_S is a hyperring under the condition (3.1.6), it is enough to show that $XY + XZ \subseteq X(Y + Z)$ $\forall X, Y, Z \in M_S$. If $|Y| \neq |Z|$ or $|Y| = |Z|$ and $\text{sign}(Y) = \text{sign}(Z)$, then it is straightforward. In fact, in this case, we would only have single-valued operations. Hence, an inclusion is indeed an equality. The only nontrivial case is when $|Y| = |Z|$, $\text{sign}(Y) \neq \text{sign}(Z)$, and $X \neq 0$. Therefore, we may assume that $Y = (y, 1)$, $Z = (y, -1)$, and $X \neq 0$. Let $X = (x, r)$ and $T = (t, p) \in XY + XZ$, then $t \leq xy$. It follows from the divisibility condition (3.1.6) on M that $t = x\beta$ for some $\beta \in M$. Then we have $\beta \leq y$. Otherwise we would have $y < \beta$, but from the condition (3.1.5), this implies that $xy < x\beta = t$ which is a contradiction(we assumed that $x \neq 0$). Thus $T = (t, p) = (x\beta, p) \in X(Y + Z)$.
For the second assertion, for any $x, y \in M$, let $X = (x, 1)$, $Y = (y, 1)$, $-Y = (y, -1)$. Since we assumed that M_S is a hyperring we know that $X(Y - Y) = XY - XY$. Furthermore, since $1_M \leq x$ for all $x \neq 0$, we have $y \leq xy$. This implies $(y, 1) \in$

$(XY - XY) = X(Y - Y)$, and $y = x\alpha$ for some $\alpha \leq y$. Similarly we can find β such that $x = y\beta$ using $Y(X - X) = YX - YX$. Therefore, M satisfies the condition (3.1.6). $\square$

Remark 3.1.11. *Any semifield M of characteristic one always satisfies the condition (3.1.5) and the divisibility condition (3.1.6). Hence it follows from the above proposition that M_S is a hyperfield. One can observe that this agrees with the statement of Lemma 3.1.6.*

Surprisingly, if M_S is a hyperring then M_S automatically satisfies the following stronger condition.

Proposition 3.1.12. *Let M be a semiring of characteristic one. Suppose that M_S is a hyperring. Then M_S is doubly distributive. i.e. for any $X, Y, Z, W \in M_S$ we have*

$$(X + Y)(Z + W) = XZ + XW + YZ + YW. \tag{3.1.7}$$

Proof. In general, one only has

$$(X + Y)(Z + W) \subseteq XZ + XW + YZ + YW.$$

Thus we have to show the other inclusion. There are two possible cases depending upon the cardinalities of $(X + Y)$ and $(Z + W)$. The first case is when at least one of $(X + Y)$ and $(W + Z)$ consists of a single element. If $\#(X + Y) = 1$, then we may assume that $X + Y = X$(cf. (3.1.2) for the definition of the addition in M_S). Then $XW + YW = (X + Y)W = XW$ and $XZ + YZ = (X + Y)Z = XZ$, hence $XZ + XW + YZ + YW = XW + XZ = X(W + Z) = (X + Y)(W + Z)$. If $\#(W+Z) = 1$, then the argument is similar. The second case is when neither $(X+Y)$ nor $(W + Z)$ consists of a single element. Hence we may let that $X = -Y$, $Z = -W$, and $X = (x, 1), Z = (z, 1)$. Thus we have $(X+Y) = [-X, X]$ and $(Z+W) = [-Z, Z]$. It follows that $(X + Y)(Z + W) = [-X, X] \cdot [-Z, Z]$. If $T \in XZ + XW + YZ +$

$YW = X(Z+W) + Y(Z+W)$ then there exist $\alpha, \beta \in Z+W = [-Z, Z]$ such that $T \in X\alpha + Y\beta$. Since $X = -Y$ we can rewrite $T \in X\alpha - X\beta = X(\alpha - \beta)$. We know that $X \in (X+Y) = [-X, X]$. Furthermore, for $\alpha, \beta \in (Z+W) = [-Z, Z]$, we have $-\beta \in [-Z, Z]$ since $|\beta| \leq |Z|$. In particular, $(\alpha - \beta) \subseteq [-Z, Z]$. We conclude $T \in (X+Y)(Z+W)$, therefore $XZ + XW + YZ + YW \subseteq (X+Y)(Z+W)$. $\square$

The following corollary shows that M_S has many Frobenius endomorphisms.

Corollary 3.1.13. *Let M, M_S be the same as in Proposition 3.1.12. Then for any $m \in \mathbb{N}$ we have*

$$(X+Y)^m = X^m + Y^m \quad \forall X, Y \in M_S. \tag{3.1.8}$$

Proof. Let $X = (x, p)$, $Y = (y, q)$. We prove this by induction. If $m = 1$, then there is nothing to prove. Let us assume that (3.1.8) holds for $m = n$. For $n+1$, it follows from the above proposition and the inductive assumption that

$$(X+Y)^{n+1} = X^{n+1} + X^n Y + Y^n X + Y^{n+1}. \tag{3.1.9}$$

If we have $\#(X+Y) = 1$ then it is clear. In fact, one of the following $X = Y$, $x < y$, $y < x$ should hold. When $x < y$ we have $X + Y = Y$. Therefore the left hand side of (3.1.8) is Y^{n+1}. On the other hand we have $(X^n + Y^n) = Y^n$, thus the right hand side of (3.1.8) is Y^{n+1}. The case when $y < x$ is similar. When $X = Y$ the outcome is trivial. It follows that the only non-trivial case is when $x = y$, $p = -q$. We may assume that $p = 1$, hence $Y = -X$. Then the left hand side of (3.1.9) is $[-X, X]^{n+1}$. Moreover, we have that $X^{n+1} + X^n Y + Y^n X + Y^{n+1} = X^{n+1} - X^{n+1} + Y^{n+1} - Y^{n+1} = [-X^{n+1}, X^{n+1}] + [-X^{n+1}, X^{n+1}]$. Therefore the right hand side of (3.1.9) is that $[-X^{n+1}, X^{n+1}] + [-X^{n+1}, X^{n+1}]$. We claim that $[-X, X]^{n+1} = [-X^{n+1}, X^{n+1}]$. Let $t \in [-X, X]^{n+1}$. This means $t = t_1 t_2 ... t_{n+1}$ for some $t_i \in [-X, X]$. Since each $|t_i| \leq X$, we have $|t| \leq X^{n+1}$ and $t \in [-X^{n+1}, X^{n+1}]$. Conversely, let $t \in [-X^{n+1}, X^{n+1}]$. Then $|t| \leq X^{n+1}$. Since M_S is a hyperring, we have $[-X^{n+1}, X^{n+1}] = X^{n+1} - X^{n+1} = X(X^n - X^n)$. Therefore, we can find

$t_1 \in X^n - X^n = [-X^n, X^n]$ such that $t = Xt_1$. Inductively we can write $t = X^n t_n$ with $|t_n| \leq X$. This means $t \in [-X, X]^{n+1}$. This proves our claim. All we have to show to complete our proof is the following:

$$[-Z, Z] + [-Z, Z] = [-Z, Z] \quad \forall Z = (z, 1) \in M_S.$$

By choosing $0 \in [-Z, Z]$ we clearly have $[-Z, Z] \subseteq [-Z, Z] + [-Z, Z]$. Conversely, if $\alpha \in [-Z, Z] + [-Z, Z]$ then $\alpha \in t + q$ for some $t, q \in [-Z, Z]$. But for $\alpha \in t + q$ we have $|\alpha| \leq \max\{|t|, |q|\} \leq Z$. It follows that $\alpha \in [-Z, Z]$. This completes the proof. $\square$

The next proposition shows that the localization commutes with the symmetrization.

Proposition 3.1.14. *Let M be a semiring of characteristic one and $s : M \longrightarrow M_S$ be the symmetrization map. Assume that $M_S = s(M)$ is a hyperring. Suppose S is a multiplicative subset of M. Then $\tilde{S} := s(S)$ is a multiplicative subset of M_S. Furthermore, the following conclusions hold.*

1. *$S^{-1}M$ is a semiring of characteristic one.*
2. *$s(S^{-1}M) \simeq \tilde{S}^{-1}(M_S)$.*

Proof. The fact that $s(S) = \tilde{S}$ is a multiplicative subset of M_S is straightforward. For the first assertion, since clearly $S^{-1}M$ is a semiring, all we have to prove is that $S^{-1}M$ is of characteristic one. In other words, we have to show that $\alpha + \beta \in \{\alpha, \beta\}$ $\forall \alpha, \beta \in S^{-1}M$. In fact, for any $\frac{x}{s}, \frac{y}{t} \in S^{-1}M$, we have $\frac{x}{s} + \frac{y}{t} = \frac{xt+sy}{st}$. Since $xt + sy \in \{xt, sy\}$ it follows that $\frac{xt+sy}{st} \in \{\frac{xt}{st}, \frac{sy}{st}\} = \{\frac{x}{s}, \frac{y}{t}\}$. Therefore $S^{-1}M$ is a semiring of characteristic one.
For the second assertion, we prove that the map

$$i : S^{-1}M \longrightarrow \tilde{S}^{-1}(M_S) \quad \frac{\alpha}{s} \mapsto \frac{(\alpha, 1)}{(s, 1)}$$

is a well-defined additive map and that $(i, \tilde{S}^{-1}(M_S))$ is the universal pair. Then it follows from the universality that $s(S^{-1}M) \simeq \tilde{S}^{-1}(M_S)$ as hypergroups. We will prove that such isomorphism is also an isomorphism of hyperrings. We first show that i is well-defined. In fact, if $\frac{\alpha}{s} = \frac{\beta}{t} \in S^{-1}M$ then we have $g\alpha t = g\beta s$ for some $g \in S$. It follows that $(g, 1)(\alpha, 1)(t, 1) = (g\alpha t, 1) = (g\beta s, 1) = (g, 1)(\beta, 1)(s, 1)$. Since $(g, 1) \in \tilde{S}$, we have $i(\frac{\alpha}{s}) = \frac{(\alpha,1)}{(s,1)} = \frac{(\beta,1)}{(t,1)} = i(\frac{\beta}{t})$. Therefore, i is well-defined. One can clearly see that $i(0) = 0$. For any $\frac{x}{s}, \frac{y}{t} \in S^{-1}M$,

$$i(\frac{x}{s} + \frac{y}{t}) = i(\frac{xt + ys}{st}) = \frac{(xt + ys, 1)}{(st, 1)} \in \frac{(x, 1)}{(s, 1)} + \frac{(y, 1)}{(t, 1)} = i(\frac{x}{s}) + i(\frac{y}{t}).$$

This shows that i is an additive map. Next, we prove that $(i, \tilde{S}^{-1}(M_S))$ is a universal pair. Let K be a hypergroup and $g : S^{-1}M \longrightarrow K$ be an additive map. We have to show that there exists a unique homomorphism $h : \tilde{S}^{-1}(M_S) \longrightarrow K$ of hypergroups such that $g = h \circ i$. Let us define a map $h : \tilde{S}^{-1}(M_S) \longrightarrow K$ as

$$h(\frac{(x, p)}{(s, 1)}) = \begin{cases} g(\frac{x}{s}) & \text{if } p = 1 \\ -g(\frac{x}{s}) & \text{if } p = -1 \end{cases} \tag{3.1.10}$$

In other words, $h(\frac{(x,p)}{(s,1)})$ =sign$(p)g(\frac{x}{s})$. Then h is well-defined. In fact, for any $\frac{(x,p)}{(s,q)}$, we may assume that $q = 1$ by multiplying $1 = \frac{(1,q)}{(1,q)}$. Thus the definition makes sense. We claim that if $\frac{(x,1)}{(s,1)} = \frac{(y,p)}{(t,1)}$, then $p = 1$. This is because $\frac{(x,1)}{(s,1)} = \frac{(y,p)}{(t,1)}$ is equivalent to the statement that $(g, 1)(t, 1)(x, 1) = (gtx, 1) = (gys, p) = (g, 1)(y, p)(s, 1)$ for some $(g, 1) \in \tilde{S}$. Furthermore, suppose that $\frac{(x,1)}{(s,1)} = \frac{(y,1)}{(t,1)}$. Then for some $(g, 1) \in \tilde{S}$, we have $(gtx, 1) = (gys, 1)$. But since the symmetrization map is injective we have that $gtx = gys$, hence $\frac{x}{s} = \frac{y}{t}$ and $h(\frac{(x,1)}{(s,1)}) = g(\frac{x}{s}) = g(\frac{y}{t}) = h(\frac{(y,1)}{(t,1)})$. The exact same argument shows that for any $\frac{(x,-1)}{(s,1)} = \frac{(y,-1)}{(t,1)}$ we have $h(\frac{(x,-1)}{(s,1)}) = h(\frac{(y,-1)}{(s,1)})$. Therefore h is well-defined. Next, we prove that h is a homomorphism of hypergroups. We have to show that $h(X + Y) \subseteq h(X) + h(Y)$ for all $X, Y \in \tilde{S}^{-1}(M_S)$. We divide cases depending upon the signs of elements X, Y. The first case is when $X = \frac{(x,1)}{(s,1)}$,

$Y = \frac{(y,1)}{(t,1)}$. In this case $X+Y$ is a single element, $Z = \frac{(xt+ys,1)}{(st,1)}$. Since g is additive, it follows that $h(X+Y) = h(Z) = g(\frac{xt+ys}{st}) = g(\frac{x}{s} + \frac{y}{t}) \in g(\frac{x}{s}) + g(\frac{y}{t}) = h(X) + h(Y)$. The second case is when $X = \frac{(x,-1)}{(s,1)}$, $Y = \frac{(y,-1)}{(t,1)}$. But the exact same argument shows that we also have $h(X+Y) \subseteq h(X) + h(Y)$ in this case. The third case is when $X = \frac{(x,1)}{(s,1)}$, $Y = \frac{(y,-1)}{(t,1)}$ with $tx \neq sy$. Since M is totally ordered, it follows that either $tx < sy$ or $tx > sy$. We may assume that $tx > sy$ since the argument would be symmetric. Since $tx > sy$ we have $(tx,1) + (sy,-1) = (tx,1)$. It follows that $X + Y = X - Y = X$, hence $h(X+Y) = h(X-Y) = h(X)$. What we want to show is that $h(X) = h(X+Y) \in h(X) + h(Y)$, equivalently $g(\frac{x}{s}) \in g(\frac{x}{s}) - g(\frac{y}{t})$. It follows from the reversibility property of K that it is again equivalent to $g(\frac{x}{s}) \in g(\frac{x}{s}) + g(\frac{y}{t})$. But since g is additive and $tx > sy$, we have $g(\frac{x}{s}) = g(\frac{x}{s} + \frac{y}{t}) \in g(\frac{x}{s}) + g(\frac{y}{t})$. The fourth case is when $X = \frac{(x,1)}{(s,1)}$, $Y = \frac{(y,-1)}{(t,1)}$ with $tx = sy (:= d)$. We want to show $h(X+Y) \subseteq h(X) + h(Y)$, where $h(X) = g(\frac{x}{s}), h(Y) = -g(\frac{y}{t})$. We have

$$X + Y = \{\frac{c}{(st,1)} \mid c \in (tx,1) + (sy,-1)\} = \{\frac{c}{(st,1)} \mid c \in [(d,-1),(d,1)]\}.$$

The first sub-case of this case is when $c = (f,1)$, $f \leq d$, $Z = \frac{c}{(st,1)} \in X + Y$. Let $W = -Y = \frac{(y,1)}{(t,1)}$. It follows from the reversibility property of $\tilde{S}^{-1}(M_S)$,

$$Z \in X + Y = X - W = -W + X \Longleftrightarrow X \in Z - (-W) = Z + W.$$

Since X, Z, W all belong to the first case, we know

$$h(X) \in h(Z) + h(W) \Longleftrightarrow g(\frac{x}{s}) \in g(\frac{f}{st}) + g(\frac{y}{t}).$$

It follows again from the reversibility of K, the above is equivalent to $g(\frac{f}{st}) \in g(\frac{x}{s}) - g(\frac{y}{t})$. Hence we have $h(Z) \in h(X) + h(Y)$. The second sub-case is when $c = (f,-1)$, $f \leq d$, $Z = \frac{c}{st} \in X + Y$. Similarly let $W = -Y = \frac{(y,1)}{(t,1)}$, $D = -Z = \frac{(f,1)}{(st,1)}$. It follows from the reversibility property,

$$Z \in X + Y \iff -D \in X - W \iff D \in W - X = -X + W \iff W \in D + X.$$

Since X, D, W belong to the first case we know

$$h(W) \in h(D) + h(X) \iff g(\frac{y}{t}) \in g(\frac{f}{st}) + g(\frac{x}{s}) \iff g(\frac{f}{st}) \in g(\frac{y}{t}) - g(\frac{x}{s}).$$

The above is equivalent to the following.

$$-g(\frac{f}{st}) \in g(\frac{x}{s}) - g(\frac{y}{t}) \iff h(\frac{(f,-1)}{(st,1)}) \in h(\frac{(x,1)}{(s,1)}) + h(\frac{(y,-1)}{(t,1)}) \iff h(Z) \in h(X) + h(Y).$$

This proves h is a homomorphism of hypergroups. One can observe that from the construction and the condition $g = h \circ i$, h is unique. It follows from the uniqueness of a universal pair, when $K = s(S^{-1}M)$, h is an isomorphism of hypergroups. From Lemma 3.1.7, all we have to prove is that h also preserves multiplicative structure. Indeed, for any $X = \frac{(x,p)}{(s,1)}$ and $Y = \frac{(y,q)}{(t,1)}$, it follows from (3.1.10) that $h(XY) = h(\frac{(xy,pq)}{(st,1)}) =\mathrm{sign}(pq)s(\frac{xy}{st}) =\mathrm{sign}(pq)(\frac{xy}{st}, 1)$. But we know that $\mathrm{sign}(pq)(\frac{xy}{st}, 1) =\mathrm{sign}(p)\mathrm{sign}(q)(\frac{x}{s}, 1)(\frac{y}{t}, 1) = (\mathrm{sign}(p)(\frac{x}{s}, 1)(\mathrm{sign}(q)(\frac{y}{t}, 1)) = h(\frac{(x,p)}{(s,1)})h(\frac{(y,q)}{(t,1)}) = h(X)h(Y)$. Thus we have that $s(S^{-1}M) \simeq \tilde{S}^{-1}(M_S)$ as hyperrings. □

Corollary 3.1.15. *Let M be a semiring of characteristic one. Suppose that $s(M) = M_S$ is a hyperring. For any non-zero element $f \in M$, let $\hat{f} = (f, 1) \in M_S$. Then we have the following isomorphism of hyperrings.*

$$s(M_f) \simeq (M_S)_{\hat{f}}.$$

Proof. This is straightforward from Proposition 3.1.14 with $S = \{1, f, f^2, ...\}$. □

4

Algebraic geometry over hyper-structures

This chapter consists of three parts. In the first section we study congruence relations on a hyperring and introduce the notion of a quotient hyperring. Unlike the case of commutative rings, there is no one-to-one correspondence between ideals and congruence relations on a semiring while such correspondence is valid in the case of hyperrings (cf. Example 4.1.10, Proposition 4.1.15 and 4.1.17).

The second section is devoted to the development of algebraic geometry over hyper-structures. We take the view point of an algebraic variety as the set of solutions of polynomial equations and study several basic notions. Then we use the symmetrization process described in Chapter 3 to interpret in a suitable way a tropical variety as the 'positive part' of an algebraic variety over hyper-structures (cf. Proposition 4.2.31). Finally, we study an analogue in characteristic one of the analytification of an affine algebraic variety.

In the third section, we continue our development of algebraic geometry over hyper-structures. This time, we take the scheme theoretic point of view. We prove that some classical results, which are essential in development of the scheme theory, are still valid. Then we define the notion of an integral hyper-scheme. We observe that in the case of hyperrings, the construction of a structure sheaf is subtle (cf. Remark

4.3.8).

In Theorem 4.3.11, we prove that for any hyperring R without (multiplicative) zero divisors, one recovers the important result: $\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R$.

In the second subsection, we provide a notion of Hasse-Weil zeta function attached to an algebraic variety over hyper-structures and prove that it agrees with the classical Hasse-Weil zeta function in some special case (cf. Theorem 4.3.44). Finally, in §4.3.4, we use the symmetrization process to link algebraic geometry over semi-structures and hyper-structures in the scheme theoretic sense.

Throughout this chapter we follow basic definitions in the hyperring theory given in §1.1.2. Also we use the term ideals for hyperideals if there is no possible confusion.

4.1 Quotients of hyperrings

In algebra, the construction of a quotient object is usually essential to develop an algebraic theory. A particular case of quotient construction for hyperrings has been studied by means of the notion of normal hyperideals (cf. [14], [15]). Next, we review the definition of a normal hyperideal.

Definition 4.1.1. *(cf. [15]) Let R be a hyperring. A non-empty subset $I \subseteq R$ is a hyperideal if*

$$a - b \subseteq I, \quad ra \in I \quad \forall a, b \in I, \forall r \in R.$$

A hyperideal $I(\neq R)$ is prime if

$$xy \in I \Longrightarrow x \in I \text{ or } y \in I \quad \forall x, y \in R.$$

A hyperideal is normal if

$$x + I - x \subseteq I \quad \forall x \in R.$$

Remark 4.1.2. *In [15], B.Davvaz and A.Salasi introduced the notion of a normal* *hyperideal I of a hyperring R so that the following relation*

$$x \equiv y \iff (x-y) \cap I \neq \emptyset \tag{4.1.1}$$

becomes an equivalence relation. One may observe that when R is a commutative ring, any ideal of R is normal. In other words, in the classical case, the normal condition is redundant.

The definition of a normal hyperideal looks too restrictive for applications. For example, suppose that R is a hyperring extension of the Krasner hyperfield $\mathbf{K}$. Then for any $x \in R$ we have $x + x = \{0, x\}$, therefore $x = -x$. It follows that the only normal hyperideal of R is R itself.
In the following subsection, we prove that the relation (4.1.1) is, in fact, an equivalence relation without appealing to the normal condition on a hyperideal I. Furthermore, we show that one can canonically construct a quotient hyperring R/I for any hyperideal I of a hyperring R.

4.1.1 Construction of quotients

Let R be a hyperring and I an ideal of R. We introduce the following relation on R (cf. [15])

$$x \sim y \iff x + I = y + I \tag{4.1.2}$$

where $x+I := \bigcup_{a\in I}(x+a)$ and the equality on the right side of (4.1.2) is meant as an equality of sets. Clearly, the relation (4.1.2) is reflexive and symmetric. Also $x \sim y$ and $y \sim z$ imply $x + I = y + I$ and $y + I = z + I$, therefore $x + I = z + I$. Hence $x \sim z$. This shows that $\sim$ is an equivalence relation.

Remark 4.1.3. *When R is a commutative ring, (4.1.2) is the classical equivalence relation obtained from an ideal I: $x \sim y \iff x - y \in I$.*

The following lemma provides an equivalent description of (4.1.2).

Lemma 4.1.4. *Let R be a hyperring and I be an ideal of R. Let $\sim$ be the relation on R as in (4.1.2). Then*

$$x \sim y \iff (x-y) \cap I \neq \emptyset, \quad \forall x, y \in R. \tag{4.1.3}$$

Proof. Notice that $(x-y) \cap I \neq \emptyset \iff (y-x) \cap I \neq \emptyset$. Suppose that $x \sim y$. Then by definition we have $x+I = y+I$. By choosing $0 \in I$, it follows that $x+0 = x \in y+I$. Thus, $x \in y+a$ for some $a \in I$. By the reversibility property of R, we know that $x \in y+a$ is equivalent to $a \in x-y$. Thus we derive that $a \in (x-y) \cap I$, hence $(x-y) \cap I \neq \emptyset$.
Conversely, suppose that $(x-y) \cap I \neq \emptyset$. We need to show that $x+I = y+I$. Since the argument is symmetric, it is enough to show that $x+I \subseteq y+I$. For any $t \in x+I$, there exists $\alpha \in I$ such that $t \in x+\alpha$. Since $(x-y) \cap I \neq \emptyset$, it follows that there exists $\beta \in (x-y) \cap I$. From the reversibility, this implies that $x \in y+\beta$. Therefore, we have $t \in x+\alpha \subseteq (y+\beta)+\alpha = y+(\alpha+\beta)$. This implies that there exists $\gamma \in (\alpha+\beta)$ such that $t \in y+\gamma$. But since $\alpha, \beta \in I$ we have $\gamma \in I$, thus $t \in y+I$. $\square$

Next, we use the equivalence relation (4.1.2) to define quotient hyperrings. We will use the notations $[x]$ and $x+I$ interchangeably for the equivalence class of x under (4.1.2). We will also use frequently the reversibility property of a hyperring without explicitly mentioning it.

Definition 4.1.5. *Let R be a hyperring and I be an ideal of R. We define*

$$R/I := \{[x] | x \in R\}$$

to be the set of equivalence classes of (4.1.2) on R. We impose on R/I two binary operations: an addition:

$$[a] \oplus [b] = (a+I) \oplus (b+I) := \{c+I | c \in a+b\} \tag{4.1.4}$$

and a multiplication:

$$[a] \odot [b] := a \cdot b + I. \tag{4.1.5}$$

Proposition 4.1.6. *With the notation as in Definition 4.1.5, R/I is a hyperring with an addition $\oplus$ and a multiplication $\odot$.*

Proof. We first prove that operations $\oplus$ and $\odot$ are well-defined. For the addition, it is enough to show that $(a + I) \oplus (b + I) = (a' + I) \oplus (b + I)$ for any $[a] = [a']$. In fact, we only have to show one inclusion since the argument is symmetric. Thus, we show that $[a] \oplus [b] \subseteq [a'] \oplus [b]$. If $z + I \in (a + I) \oplus (b + I)$, then we may assume $z \in a + b$. We need to show that there exists $w \in a' + b$ such that $[z] = [w]$. But if $z \in a + b = b + a$ then $a \in z - b$. In particular,

$$(a - a') \subseteq (z - b) - a' = z - (a' + b). \tag{4.1.6}$$

Since $[a] = [a']$, it follows from Lemma 4.1.4 that there exists $\delta \in (a - a') \cap I$. It also follows from (4.1.6) that we have $\delta \in z - w$ for some $w \in a' + b$ and this implies $(z - w) \cap I \neq \emptyset$. Therefore, we have $[z] = [w]$. For the multiplication, we need to show that $a \cdot b + I = a' \cdot b + I$. Since $(a - a') \cap I \neq \emptyset$, we have $\delta \in (a - a') \cap I \subseteq (a - a')$ which implies that $(a - a')b \cap I \neq \emptyset$. Therefore, $[a \cdot b] = [a' \cdot b]$ from Lemma 4.1.4. Hence, $\oplus$ and $\odot$ are well-defined.

Next, we prove that $(R/I, \oplus)$ is a (canonical) hypergroup. Clearly $\oplus$ is commutative. We claim that

$$X := ([a] \oplus [b]) \oplus [c] = \{[d] = d + I | d \in a + b + c\} := Y.$$

If $[w] \in X$, then $[w] \in [r] \oplus [c]$ for some $[r] \in [a] \oplus [b]$. We may assume $w \in r + c$ and $r \in a + b$. Then we have $w \in r + c \subseteq (a + b) + c = a + b + c$. Thus $[w] \in Y$. Conversely, if $[z] \in Y$ then we may assume $z \in a + b + c = (a + b) + c$. This means $z \in t + c$ for some $t \in a + b$. In turn, this implies $[z] \in [t] \oplus [c]$, $[t] \in [a] \oplus [b]$. Hence $[z] \in X$. It follows from the same argument with $[a] \oplus ([b] \oplus [c])$ that the operation $\oplus$ is associative. The class

$[0]$ is the unique neutral element. In fact, we have $[0]\oplus[x] = \{[d] | d \in 0+x = x\} = [x]$. Suppose that we have $[w]$ such that $[w] \oplus [x] = [x]$ for all $[x] \in R/I$. For $x \in I$, we have $x \in w + x = x + w$. Hence $w \in x - x \subseteq I$. But one can see that $[w] = [0]$ for all $w \in I$ from Lemma 4.1.4. Therefore, the neutral element is unique. Next, we claim that $[0] \in [x] \oplus [y] \iff [y] = [-x]$. Since $0 \in (x - x)$ we have $[0] \in [x] \oplus [-x]$. Conversely, suppose that $0 + I \in (x + I) \oplus (y + I)$ for some $y \in R$. We need to show that $y + I = -x + I$. Since $0 + I \in (x + I) \oplus (y + I)$, there exists $c \in x + y$ such that $c + I = I$. It follows that $c \in I$. Moreover, from $c \in x + y = y - (-x)$, we have that $c \in (y - (-x)) \cap I$. Thus $(y - (-x)) \cap I \neq \emptyset$ and $[y] = [-x]$. For the reversibility property, if $[x] \in [y] \oplus [z]$, then we need to show that $[z] \in [x] \oplus [-y]$. But $[x] \in [y] \oplus [z] \iff (x + I) \in (y + I) \oplus (z + I) \iff x + I = c + I$ for some $c \in y + z$. From the reversibility property of R, $z \in c - y$. Thus $[z] \in [c] \oplus [-y]$. But we have $[x] = [c]$, hence $[z] \in [x] \oplus [-y]$. Finally, we only have to prove that $\oplus, \odot$ are distributive. i.e.

$$([a] \oplus [b]) \odot [c] = ([a] \odot [c]) \oplus ([b] \odot [c]).$$

But this directly follows from that of R. This completes the proof. $\square$

In the sequel, we consider R/I as a hyperring with the addition $\oplus$ and the multiplication $\odot$.

Next, we recall (from §1.1.2) the definition of a strict homomorphism of hyperrings. By a strict homomorphism $f : R \longrightarrow H$ of hyperrings we mean a homomorphism of hyperrings such that

$$f(x + y) = f(x) + f(y) \qquad \forall x, y \in R. \tag{4.1.7}$$

Proposition 4.1.7. *Let R be a hyperring and I be an ideal of R. The projection map*

$$\pi : R \longrightarrow R/I, \quad x \mapsto [x]$$

is a strict, surjective homomorphism of hyperrings with $\operatorname{Ker}\pi = I$.

Proof. Clearly, π is surjective and $\pi(xy) = \pi(x)\pi(y)$. By the definition of a hyperaddition (4.1.4), we have $\pi(x + y) = [x + y] \subseteq [x] \oplus [y]$. This shows that π is a homomorphism of hyperrings. For the strictness, take $[c] \in [x] \oplus [y]$. Then there exists $z \in x+y$ such that $[z] = [c]$. It follows that $\pi(z) = [z] = [c]$, thus π is strict. For the last assertion, suppose that $\pi(x) = [0]$. This implies that $[x] = x+I = 0+I = [0]$, hence $x \in I$. Therefore $\operatorname{Ker}\pi = I$. $\square$

The next proposition shows that a quotient hyperring satisfies the universal property as in the classical case.

Proposition 4.1.8. *Let R and H be hyperrings and $\varphi : R \longrightarrow H$ be a homomorphism of hyperrings. Suppose that I is an ideal of R such that $I \subseteq Ker\varphi$. Then there exists a unique hyperring homomorphism $\tilde{\varphi} : R/I \longrightarrow H$ such that $\varphi = \pi \circ \tilde{\varphi}$, where $\pi : R \longrightarrow R/I$ is the projection map as in Proposition 4.1.7.*

Proof. Let us define

$$\tilde{\varphi} : R/I \longrightarrow H, \quad \tilde{\varphi}([x]) = \varphi(x) \quad \forall [x] \in R/I.$$

We first have to show that $\tilde{\varphi}$ is well-defined. Let $[x] = [y]$ for $x, y \in R$. Then we have $x + I = y + I$, hence $x \in y + c$ for some $c \in I$. Since $c \in I \subseteq Ker\varphi$, it follows that

$$\varphi(x) \in \varphi(y + c) \subseteq \varphi(y) + \varphi(c) = \varphi(y) + 0 = \varphi(y).$$

Therefore, $\varphi(x) = \varphi(y)$ and $\tilde{\varphi}$ is well-defined. Furthermore, since φ is a hyperring homomorphism, $\tilde{\varphi}$ is also a hyperring homomorphism. By the construction, we have $\varphi = \pi \circ \tilde{\varphi}$. The uniqueness is clear. $\square$

Remark 4.1.9. *One can easily see that if f and g are strict homomorphisms, then so is $f \circ g$. In particular, in Proposition 4.1.8, if φ is a strict hyperring homomorphism, then so is $\tilde{\varphi}$, since π is strict.*

4.1.2 Congruence relations

In this subsection, we define a congruence relation on a hyperring R and prove that there is a one-to-one correspondence between ideals and congruence relations on R. Note that in the theory of semirings, this correspondence fails in general as the following example shows.

Example 4.1.10. *Let $M := \mathbb{Q}_{\geq 0}$ be the semifield of nonnegative rational numbers with the usual addition and the usual multiplication. Since M is a semifield, $\{0\}$ and M are the only ideals of M. One can easily see that $\{0\}$ corresponds to the congruence relation:*

$$x \equiv 0 \qquad \forall x \in M$$

and M corresponds to the congruence relation:

$$x \equiv y \Longleftrightarrow x = y \qquad \forall x, y \in M.$$

However there are more congruence relations. For example, one may consider the following relation:

$$x \equiv_2 y \Longleftrightarrow \exists k \in 2\mathbb{Z} + 1 \ s.t. \ k(x - y) \in 2\mathbb{Z} \qquad \forall x, y \in M.$$

Clearly, $\equiv_2$ is reflexive and symmetric. Furthermore, suppose that $x \equiv_2 y$ and $y \equiv_2 z$. Then there exist odd integers k_1 and k_2 such that $k_1(x - y)$, $k_2(y - z) \in 2\mathbb{Z}$. One can easily check that $k_1k_2(x - z) \in 2\mathbb{Z}$. Therefore $\equiv_2$ is an equivalence relation. Next, when $x \equiv_2 y$ and $\alpha \equiv_2 \beta$, $\exists$ odd integers k and t such that $k(x - y), t(\alpha - \beta) \in 2\mathbb{Z}$. It follows that

$$kt((x + \alpha) - (y + \beta)) = tk(x - y) + kt(\alpha - \beta) \in 2\mathbb{Z}.$$

Also, one can easily see that $kt(x\alpha - y\beta) \in 2\mathbb{Z}$. *Hence, we conclude that*

$$x \equiv_2 y \text{ and } \alpha \equiv_2 \beta \Longrightarrow x+\alpha \equiv_2 y+\beta \text{ and } x\alpha \equiv_2 y\beta.$$

Therefore $\equiv_2$ *is a congruence relation on* M *which does not have a corresponding ideal of* M*. This example shows that a one-to-one correspondence between ideals and congruence relations fails in this case. In fact, it is well-known that if* M *is a semiring having no nontrivial proper congruence relations then either* $M = \mathbb{B}$ *or a field (cf. [19, §7]).*

We notice that in hyperring theory, a sum of two element is no longer an element in general but a set. Therefore, to define a congruence relation on a hyperring R, we need a suitable notion stating when two subsets of R are equivalent. The following definition provides such notion.

Definition 4.1.11. *Let* R *be a hyperring and* $\equiv$ *be an equivalence relation on* R*. Let* A, B *be two subsets of* R*. We write* $A \equiv B$ *when the following condition holds:*

$$\forall a \in A, \forall b \in B \qquad \exists a' \in A \text{ and } \exists b' \in B \text{ s.t. } a \equiv b' \text{ and } a' \equiv b. \tag{4.1.8}$$

Definition 4.1.12. *Let* R *be a hyperring. A congruence relation* $\equiv$ *on* R *is an equivalence relation on* R *satisfying the following property:*

$$\forall x_1, x_2, y_1, y_2 \in R, \quad x_1 \equiv y_1, x_2 \equiv y_2 \quad \Longrightarrow \quad x_1x_2 \equiv y_1y_2, \quad x_1 + x_2 \equiv y_1 + y_2. \tag{4.1.9}$$

The following proposition shows that when a congruence relation $\equiv$ is defined on R, then there is a canonical hyperring structure on the set $R/\equiv$ of equivalence classes. We let $[r]$ denote an equivalence class of $r \in R$ under $\equiv$.

Proposition 4.1.13. *The set* $(R/\equiv) := \{[r] | r \in R\}$ *is a hyperring, where the* *addition is defined by*

$$[x]+[y] := \{[t] \mid t \in x'+y' \quad \forall [x']=[x],[y']=[y]\} \quad \forall x,y,x',y' \in R, \tag{4.1.10}$$

and the multiplication law is given by

$$[x]\cdot[y] := [xy] \quad \forall x,y \in R. \tag{4.1.11}$$

Proof. Firstly, we prove that the addition and the multiplication are well-defined. One easily sees that (4.1.10) does not depend on representatives since it is already defined by all possible representatives. Also it follows from (4.1.9) that the multiplication is well-defined.

Secondly, we claim that $(R/\equiv,+)$ is a (canonical) hypergroup. We first show that $+$ is associative by proving the following equality

$$X := \{[t] | t \in x'+y'+z', [x']=[x],[y']=[y],[z']=[z]\} = ([x]+[y])+[z] := Y.$$

Indeed, if $t \in x'+y'+z'$ then $t \in \alpha+z'$ for some $\alpha \in x'+y'$. This implies that $[t] \in [\alpha]+[z]$ and $[\alpha] \in [x]+[y]$, hence $[t] \in Y$. Conversely, if $[t] \in ([x]+[y])+[z]$ then $[t] \in [\alpha]+[z]$ for some $[\alpha] \in [x]+[y]$. From (4.1.10), we have $t \in \alpha'+z'$ for some $\alpha', z' \in R$ such that $[\alpha']=[\alpha],[z']=[z]$. Also $[\alpha'] \in [x]+[y]$ since $[\alpha]=[\alpha']$. This implies that $\alpha' \in x'+y'$ and $t \in x'+y'+z'$ for some $x',y' \in R$ such that $[x']=[x],[y']=[y]$. The operations are trivially commutative. The class $[0]$ works as the zero element. Indeed, if $[t] \in [x]+[0]$ then $t \in x'+y'$ with $x' \equiv x$ and $y' \equiv 0$. It follows from (4.1.9) that $x'+y' \equiv x$, hence $t \equiv x$. Thus $[x]+[0]=[x]$. An additive inverse of $[x]$ is $[-x]$. Indeed, since $0 \in x-x$ it is clear that $[0] \in [x]+[-x]$. Next, we show that an inverse is unique. If $[0] \in [x]+[y]$ then we have $0 \in x'+y'$ with $x' \equiv x$ and $y' \equiv y$. It follows that $y'=-x'$ and $-x \equiv -x'$, therefore $y \equiv y' = -x' \equiv -x$. Thus an additive inverse uniquely exists. The reversibility property directly follows from that of R and the fact that $[x+y] \subseteq [x]+[y]$. This proves that $(R/\equiv,+)$ is a (canonical) hypergroups.

Finally, one can observe that [1] works as the identity element. Therefore, all we have to show is the distributive property:

$$[z]([x]+[y]) = [z][x]+[z][y] = [zx]+[zy], \quad \forall [x],[y],[z] \in R/\equiv.$$

If $[\alpha] \in [x]+[y]$, then $\alpha \in x'+y'$ with $[x']=[x], [y']=[y]$. This implies $z\alpha \in zx'+zy'$. But since $[zx']=[zx], [zy']=[zy]$, it follows that $[z\alpha] \in [zx]+[zy]$. Conversely if $[t] \in [zx]+[zy]$ then $t \in \alpha+\beta$ with $[\alpha]=[zx]$, $[\beta]=[zy]$. Thus $\alpha+\beta \equiv zx+zy = z(x+y)$, and $t \equiv z\gamma$ for some $\gamma \in x+y$. This completes the proof. □

In what follows, for a hyperring R and a congruence relation $\equiv$ on R, we always consider $R/\equiv$ as a hyperring with the structure defined in Proposition 4.1.13.

Proposition 4.1.14. *Let R be a hyperring and $\equiv$ be a congruence relation on R. Then the map*

$$\pi : R \longrightarrow R/\equiv, \quad r \mapsto [r] \quad \forall r \in R$$

is a strict surjective hyperring homomorphism.

Proof. The map π is clearly a surjective hyperring homomorphism. We prove that π is also strict by showing that $[x]+[y] \subseteq [x+y]$. If $[t] \in [x]+[y]$ then $t \in x'+y'$ for some $x', y' \in R$ such that $x' \equiv x$ and $y' \equiv y$. It follows from (4.1.9) that $x+y \equiv x'+y'$. From (4.1.8), there exists $\alpha \in x+y$ such that $[\alpha]=[t]$. Therefore, $[t]=[\alpha] \in [x+y]$ and π is strict. □

Proposition 4.1.15. *Let $\pi : R \longrightarrow R/\equiv$ be the canonical projection as in Proposition 4.1.14. Let $I = \operatorname{Ker}\pi$. Then*

$$\varphi : R/I \longrightarrow R/\equiv, \quad <r> \mapsto [r] \quad \forall r \in R$$

is an isomorphism of hyperrings, where $<r>$ is an equivalence class of r in R/I under the equivalence relation (4.1.2) and $[r]$ is an equivalence class of r in $R/\equiv$

under $\equiv$.

Proof. This follows from Proposition 4.1.14 and Proposition 2.11 of [15] which states that the first isomorphism theorem for hyperrings holds when a given homomorphism is strict. □

It follows from Proposition 4.1.15 that for a congruence relation $\equiv$ on R, one can find an ideal I of R such that $R/I \simeq (R/\equiv)$. Conversely, in the next proposition, we prove that for any hyperideal I, one can find a congruence relation $\equiv$ such that $R/I \simeq (R/\equiv)$.

Remark 4.1.16. *Note that some of the algebraic properties of a hyperring differ greatly from those of a commutative ring. For example, a hyperring does not satisfy doubly distributive property (cf. Remark 4.3.2). Thus one should be careful when generalizing classical results of commutative rings to hyperrings.*

Proposition 4.1.17. *Let R be a hyperring and I be an ideal of R. Then the relation $\equiv$ such that*

$$x \equiv y \iff x + I = y + I$$

is a congruence relation and $R/I \simeq (R/\equiv)$.

Proof. Clearly $\equiv$ is an equivalence relation. If $x_1 \equiv y_1$ and $x_2 \equiv y_2$, we have

$$x_i + I = y_i + I, \quad i = 1, 2. \tag{4.1.12}$$

Thus we can find $\alpha, \beta \in I$ such that $x_1 \in y_1 + \alpha$, $x_2 \in y_2 + \beta$. By multiplying these two, one obtains

$$x_1x_2 \in (y_1 + \alpha)(y_2 + \beta) \subseteq y_1y_2 + y_1\beta + y_2\alpha + \alpha\beta.$$

Therefore, for any $t \in I$, we have $x_1x_2 + t \subseteq y_1y_2 + (y_1\beta + y_2\alpha + \alpha\beta + t)$. But since $\alpha, \beta, t \in I$, it follows that $(y_1\beta + y_2\alpha + \alpha\beta + t) \subseteq I$. Hence, $x_1x_2 + t \subseteq y_1y_2 + I$ and

$x_1x_2 + I \subseteq y_1y_2 + I$. Since the argument is symmetric, we have

$$x_1x_2 + I = y_1y_2 + I \Longleftrightarrow x_1x_2 \equiv y_1y_2.$$

For the other condition of a congruence relation, we need to show $(x_1+x_2) \equiv (y_1+y_2)$. It is enough to show that $\forall\ t \in x_1 + x_2$, there exists $y \in y_1 + y_2$ such that $t \equiv y$. We can take $\alpha, \beta \in I$ such that $x_1 \in y_1 + \alpha$, $x_2 \in y_2 + \beta$ from (4.1.12). It follows that

$$t \in (x_1 + x_2) \subseteq (y_1 + y_2) + (\alpha + \beta).$$

Hence, $t \in y + \gamma$ for some $y \in y_1 + y_2$, $\gamma \in \alpha + \beta \subseteq I$. This implies that $t \equiv y$ from (4.1.3) and the reversibility property of R. It is clear that in this case the kernel of a canonical projection map $\pi : R \longrightarrow R/\equiv$ is I. It follows from the first isomorphism theorem of hyperrings (cf. [15, Proposition 2.11]) that $R/I \simeq R/\equiv$ since π is strict. $\square$

Remark 4.1.18. *Let R be a hyperring and I be an ideal of R. In a quotient hyperring R/I, we defined the addition as*

$$a \oplus b = \{[c] | c \in a + b\}$$

and we proved that $x \sim y \Longleftrightarrow x + I = y + I$ is a congruence relation. In this case, we defined the addition as

$$a + b = \{[c] | c \in a' + b' \quad \forall [a'] = [a], [b'] = [b]\}.$$

At first glance, $a \oplus b$ and $a + b$ seem different, but in fact they are the same sets. Clearly $a \oplus b \subset a + b$. Conversely, assume that $t' \in a' + b'$ for some $[a'] = [a], [b'] = [b]$. Since $a' + I = a + I, b' + I = b + I$, we can find $\alpha, \beta \in$ such that $a' \in a + \alpha, b' \in b + \beta$. This implies that $t' \in a' + b' \subseteq (a + b) + (\alpha + \beta)$. But since $(\alpha + \beta) \subseteq I$, it follows that $t' \in t + \gamma$ for some $t \in (a + b), \gamma \in I$. By the reversibility property of R, $\gamma \in t' - t$.

In other words, $(t - t') \cap I \neq \emptyset$, hence $[t] = [t']$. This shows that $[a] + [b] \subseteq [a] \oplus [b]$.

4.2 Solutions of polynomial equations over hyper-structures

In this section, we study the set of solutions of polynomial equations over hyper-structures. We also investigate on the notion, in characteristic one, of the analytification of a classical algebraic variety. Two are the goals which motivate this study. Firstly, we would like to link the classical geometric construction to hyper-structures, while the second goal is to interpret a tropical algebraic variety, in a suitable way, as the 'positive part' of an an algebraic variety over hyper-structures in view of the symmetrization process described in §3.

In §4.2.1, we shall pursue the first goal. Let A be an integral domain and G be a multiplicative subgroup of $A^\times$. To construct such link, we will use the projection map $\pi : A \longrightarrow A/G$ from A to the quotient hyperring A/G. Our construction is motivated by the result, [9, Proposition 6.1], which states that for any commutative ring A containing the field $\mathbb{Q}$ of rational numbers, we have

$$A \otimes_{\mathbb{Z}} \mathbf{K} = A/\mathbb{Q}^\times, \quad A \otimes_{\mathbb{Z}} \mathbf{S} = A/\mathbb{Q}^\times_+.$$

Therefore, when $\mathbf{K}$ and $\mathbf{S}$ are respectively the Krasner's hyperfield and the hyperfield of signs, and for $G = \mathbb{Q}^\times$, solutions of polynomials equations over A/G can be roughly considered as the definition of a suitable scalar extension or equivalently stated passing from an algebraic variety over A to an algebraic variety over A/G.

In §4.2.2, we will investigate the second goal. In particular we shall study the basic notion of a polynomial equation '$f = 0$' in n variables and with coefficients in a hyperring R. We will consider f as a function

$$f : L^n \longrightarrow P^*(L)$$

(where $P^*(L)$ is the set of non-empty subsets of L) under a suitable equivalence relation which depends on a hyperring extension L of R rather than considering f as a polynomial (cf. Equation (4.2.21)). Then, we define the set of solutions of '$f = 0$' on L as the set

$$\{x = (x_1, ..., x_n) \in L^n \mid 0 \in f(x)\}$$

by defining an appropriate notion of values $f(x)$ (cf. Definition 4.2.18). Using this framework, we can reinterpret a tropical variety as the 'positive part' of an algebraic variety over hyper-structures via the symmetrization procedure of Chapter 3.

Finally, in the last subsection, we define the notion of a multiplicative seminorm on a commutative ring with values in either a semifield or a hyperfield (cf. Definitions 4.2.33, 4.2.35). By means of these definitions, we introduce the notion of the analytification, in characteristic one, of an affine variety $X = \operatorname{Spec} A$ over a field K. We prove (cf. Proposition 4.2.38) that the underlying space of X can be understood as the analytification of X in characteristic one over the semifield $\mathbb{B}$ or the hyperfield $\mathbf{S}$. We also prove that the analytification of X is equipped with a topology which is stronger than the Zariski topology provided that $\mathbb{B}$ and $\mathbf{S}$ have the discrete topology (cf. Proposition 4.2.39).

4.2.1 Solutions of polynomial equations over quotient hyperrings

In this subsection, we consider the quotient hyperring $R = B/G$ for some fixed commutative ring B and a multiplicative subgroup G of the group of units $B^\times$ of B. Through the implementation of the quotient hyperring $R = B/G$ one can link classical algebra and hyper-structure theory via the canonical projection map $\pi : B \longrightarrow B/G$. We denote $[b] = \pi(b)$.

Let A be an integral domain and $G \leq A^\times$ be a multiplicative subgroup. Let B be an integral domain containing A. Then one can interpret the quotient B/G as a hyperring extension of A/G: by that we mean that there exists an injective homomorphism $\varphi : A/G \longrightarrow B/G$ of hyperrings.

In the classical case, with $f = \sum a_I X^I \in A[X_1, ..., X_n]$ a polynomial, we define the set of solutions of the equation $f = 0$ over B as:

$$\{b = (b_1, ..., b_n) \in B^n \mid 0 = f(b_1, ..., b_n)\}. \tag{4.2.1}$$

To extend this classical definition for hyper-structures, we introduce the set:

$$\tilde{f}(t) := \{\alpha \in B/G \mid \alpha \in \sum [a_I][t^I], \text{ for all presentations of } f = \sum a_I X^I\}. \tag{4.2.2}$$

In general, for $f \in A[X_1, ..., X_n]$, there are several ways to write $f = \sum a_I X^I$ so that they represent the same element of $A[X_1, ..., X_n]$. For example, one can write $x^2 - 1 \in A[x]$ as $(x + 1)(x - 1)$ or $x^2 + x - x - 1$. Then the condition $\alpha \in \sum [a_I][t^I]$ in (4.2.2) should hold for all these presentations. In the trivial case of $G = \{e\}$, we have $\tilde{f}(t) = \{f(t)\}$. In other words, $\tilde{f}(t)$ is the evaluation of f at t in the classical sense.

Example 4.2.1. *Let $A = B = \mathbb{Q}$, $G = \mathbb{Q}^\times$, and $f(x, y) = 3x - y \in \mathbb{Q}[x, y]$. Take $t = ([1], [1])$, $d = ([0], [1])$, and $r = ([1], [0])$ in $(B/G)^2 = \mathbf{K}^2$. Then we have*

$$\tilde{f}(t) \subseteq \{[0], [1]\}, \quad \tilde{f}(d) \subseteq \{[1]\}, \quad \tilde{f}(r) \subseteq \{[1]\}.$$

Next, we provide two possible definitions for the notion of a solution of a polynomial equation over a hyperring of type B/G and show that such definitions do not depend on the choice of the generators of an ideal $I \subseteq A[x_1, ..., x_n]$. We shall also prove that the two definitions agree under certain conditions. We keep the same notation as above. In particular, A is an integral domain.

Definition 4.2.2.

1. *Let $f \in A[X_1, .., X_n]$ be a polynomial. By a solution of f over B/G we mean an element $t = ([t_1], ..., [t_n]) \in (B/G)^n$ such that $0 \in \tilde{f}(t)$. We denote by $V(f)$ the set of solutions of f over B/G.*

2. *For a subset $X \subseteq A[X_1, ..., X_n]$, let $< X >$ be the ideal generated by X. We say that $t = ([t_1], ..., [t_n]) \in (B/G)^n$ is a common solution of X over B/G if for any*

finite subsets $\{f_1, ..., f_r\} \subseteq < X >$ *and* $\{g_1, ..., g_r\} \subseteq A[X_1, ...X_n]$*, the following condition is satisfied:*

$$0 \in \tilde{f}_1(t)\tilde{g}_1(t) + ... + \tilde{f}_r(t)\tilde{g}_r(t). \tag{4.2.3}$$

We denote by $V(X)$ *the set of common solutions of* X *over* B/G *.*

Alternately, one can introduce the following definition:

Definition 4.2.3. *1. Let* $f \in A[X_1, ...X_n]$ *be a polynomial, we say that* $t = ([t_1], ..., [t_n]) \in (B/G)^n$ *is a solution of* f *over* B/G *if*

$$\forall i = 1, ..., n \quad \exists y_i \in B \text{ s.t. } \begin{cases} [y_i] = [t_i] \text{ and} \\ f(y_1, ..., y_n) = 0. \end{cases} \tag{4.2.4}$$

We denote by $V(f)$ *the set of solutions of* f *over* B/G*.*

2. For a subset $X \subseteq A[X_1, ..., X_n]$*, we say that* $t = ([t_1], ..., [t_n]) \in (B/G)^n$ *is a common solution of* X *if for any finite subset* $\{f_1, ..., f_r\} \subseteq X$*, the following condition holds.*

$$\forall i = 1, ..., n, \quad \exists y_i \in B \text{ s.t. } \begin{cases} [y_i] = [t_i] \quad \forall i = 1, ..., n \text{ and} \\ f_j(y_1, ..., y_n) = 0 \quad \forall j = 1, ..., r. \end{cases} \tag{4.2.5}$$

We denote by $V(X)$ *the set of common solutions of* X *over* B/G*.*

One may observe that a solution in the sense of Definition 4.2.3 is a classical solution up to twists by the multiplication of elements of G. For example, consider the polynomials $f_g = x - g \in A[x]$ for $g \in G$. Then the set of classical solutions of f_g consists of a single element g. However, the set of solutions of f_g over A/G in the sense of Definition 4.2.3 is $\{[1]\}$ for all $g \in G$. In other words, over A/G, all f_g has the same set of solutions in the sense of Definition 4.2.3.

Example 4.2.4. *Let* $A = B = \mathbb{Q}$*,* $G = \mathbb{Q}^\times$*, and* $f = 3x - y \in \mathbb{Q}[x, y]$*. Then* $t = ([a], [b])$ *is a solution of* f *over* B/G *in the sense of Definition 4.2.3 if and only* *if there exist* $q_1, q_2 \in G = \mathbb{Q}^\times$ *such that* $3q_1a - q_2b = 0$*. This holds if and only if* a *and* b *are both non-zero. Hence* $t = ([1],[1])$ *is the only solution of* f *over* $B/G = \mathbf{K}$*. When* $G = \mathbb{Q}_+^\times$*, the signs of* a *and* b *should coincide. It follows that* $t = ([1],[1])$ *and* $t' = ([-1],[-1])$ *are the only solutions of* f *over* $B/\mathbb{Q}_+^\times = \mathbf{S}$*.*

Remark 4.2.5. *1. When a set* $X \subseteq A[X_1, ..., X_n]$ *consists of a single polynomial* f*, we have* $V(X) = V(f)$*, using either of Definition 4.2.2 and 4.2.3.*

2. Let I *be an ideal of* $A[X_1, ..., X_n]$ *and* $X \subseteq I$ *be a set of generators of* I*. Then* $V(I) = V(X)$ *in the sense of Definition 4.2.2 since it is already defined in terms of an ideal of* $A[X_1, ..., X_n]$*.*

3. Following Definition 4.2.3, the set of solutions of an ideal I *of* $A[X_1, ...X_n]$ *does not depend on the choice of generators of* I*. Indeed, let* $I =< X >$ *be the ideal generated by* X*. Then, by the definition,* $V(I) \subseteq V(X)$*. Conversely, let us choose any finite subset* $\{h_1, ...h_s\}$ *of* I *and* $t = ([t_1], ..., [t_n]) \in V(X)$*. We need to show that* t *is a common solution of* $\{h_1, ..., h_s\}$*. Because* $I =< X >$*, there exist* $g_{ij} \in A[X_1, ..., X_n]$ *and* $f_j \in X$ *such that* $h_i = \sum_i g_{ij} f_j$*. However, since there exist* $[y_i] = [t_i]$ *such that* $f_j(y_1, ...y_n) = 0 \ \forall j$*, it follows that* t *is also a common solution of* $\{h_1, ...h_s\}$*. Therefore,* $t \in V(I)$*.*

When $G = \{e\}$, both definitions recover the classical meaning of a solution of a polynomial equation $f = 0$. In particular, they agree when $G = \{e\}$. While Definition 4.2.2 is more intuitive, Definition 4.2.3 can be easily linked to classical results and is easier to work with. Our next goal is to investigate more in details these two definitions. In Proposition 4.2.8, we will prove that they agree in a particular case.

In the sequel, we let A be an integral domain, B is an integral domain containing A, and G is a non-trivial multiplicative subgroup of $A^\times$. We let $R := B/G$ be the quotient hyperring.

To start with, we associate a matrix M to each polynomial $f \in A[X_1, ...X_n]$. Let us write $f = a_0 + a_1 X^{I_1} + ... + a_k X^{I_k}$ such that $I_j \neq I_t$ if $j \neq t$. Then we define

$$M := (m_{ij}) \in M_{k\times n}(\mathbb{Z}), \text{ where } m_{ij} \text{ is the power of } X_j \text{ in } I_i. \tag{4.2.6}$$

Note that the rank of M is independent of the ordering of I_j since a choice of a different ordering will simply permute the rows of M.

Example 4.2.6. *Let $f = X_1X_2 - X_3X_4 - 1 \in A[X_1, X_2, X_3, X_4]$. Note that $f = 0$ can be considered as the polynomial equation defining SL_2. Then the matrix associated to f is given by*

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Example 4.2.7. *Let $f = X_1^2X_2^3 + X_3^2X_4 - X_1X_3 + 1 \in\in A[X_1, X_2, X_3, X_4]$. Then*

$$M = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Proposition 4.2.8. *Let $f = a_0 + a_1X^{I_1} + ... + a_kX^{I_k} \in A[X_1, ..., X_n]$ such that $I_j \neq I_t$ if $j \neq t$. Suppose that a matrix M associated to f as in (4.2.6) has full rank and that $k \leq n$. If one of the following conditions holds then Definition 4.2.2 and 4.2.3 agree on $I =< f >$ over $R = B/G$.*

1. *For any $q \in G$ and $u \in \mathbb{N}$, there exists $\gamma \in G$ such that $\gamma^u = q$.*
2. *M is a square matrix ($k = n$) and $M^{-1} \in M_{n\times n}(\mathbb{Z})$.*
3. *M is not a square matrix ($k < n$) and one can add more rows to M to make a square matrix N so that N^{-1} exists and becomes an element of $M_{n\times n}(\mathbb{Z})$.*

Remark 4.2.9. *Before we prove Proposition 4.2.8, we mention that the matrix M of Example 4.2.6 satisfies the third condition, thus the two definitions agree for SL_2 considered here as the set of solutions of the polynomial equation $X_1X_2 - X_3X_4 - 1 = 0$.*

It follows from Proposition 4.2.8 that the same conclusion fails for SL_n with $n > 3$ since a matrix M will never be of full rank in this case.

Proof. We will use capital letters X, Y, and T to refer to multi-index notation. Take $f = \sum_s a_s X^{I_s} \in A[X_1, ..., X_n]$, and let $T = ([t_1], ..., [t_n]) \in (B/G)^n$ be a solution of f in the sense of Definition 4.2.3. Then:

$$\text{For each } i = 1, ..., n, \quad \exists y_i \text{ such that } [y_i] = [t_i] \text{ and } f(y_1, ..., y_n) = 0. \tag{4.2.7}$$

Thus, for any presentation $\sum_s a_s X^{I_s}$ of f, we have $f(y_1, ..., y_n) = \sum_s a_s Y^{I_s} = 0$. Therefore,

$$0 = [f(y_1, ..., y_n)] = [\sum_s a_s Y^{I_s}] \in \sum_s [a_s Y^{I_s}] = \sum_s [a_s][Y^{I_s}] = \sum_s [a_s][T^{I_s}].$$

It follows that $0 \in \tilde{f}(T)$.

Conversely, let $T = ([t_1], ..., [t_n]) \in (B/G)^n$ be a solution of $f = \sum_s a_s X^{I_s}$ in the sense of Definition 4.2.2. This means that for any presentation $f = \sum a_s X^{I_s}$, we have

$$0 \in [a_0] + [a_1][T^{I_1}] + ... + [a_k][T^{I_k}]. \tag{4.2.8}$$

It follows that

$$\exists q_0, q_1, ... q_k \in G \text{ such that } 0 = q_0 a_0 + q_1 a_1 T^{I_1} + ... + q_k a_k T^{I_k}. \tag{4.2.9}$$

We may assume that $q_0 = 1$ by dividing both side of (4.2.9) by q_0. We need to show the following:

$$\text{for each } i = 1, ..., n \quad \exists y_i \in B \text{ s.t. } [y_i] = [t_i] \text{ and } a_0 + a_1 Y^{I_1} + ... + a_k Y^{I_k} = 0. \tag{4.2.10}$$

Firstly, let us assume that the first condition is satisfied. Suppose that $n = k$. Since M is of full rank with integer entries, it follows that M^{-1} exists and has only rational entries. Denote by $q^{\frac{1}{s}}$ any $\gamma \in G$ such that $\gamma^s = q$ and also denote by $q^{\frac{m}{s}}$ an element

γ^m, where $q \in G$. Such element exists for all $q \in G$ and $s, m \in \mathbb{N}$ by the assumption of the first condition. Note that the choice of γ is not canonical. Then, for $M = (m_{ij})$ and $M^{-1} = (b_{ij})$, we define

$$y_i := (\prod_{j=1}^{k} q_j^{b_{ij}}) t_i.$$

It follows that

$$y_i^{m_{si}} = (\prod_{j=1}^{k} q_j^{b_{ij}})^{m_{si}} t_i^{m_{si}} = (\prod_{j=1}^{k} q_j^{b_{ij} m_{si}}) t_i^{m_{si}} = (\prod_{j=1}^{k} q_j^{m_{si} b_{ij}}) t_i^{m_{si}}.$$

Thus we have

$$Y^{I_s} = y_1^{m_{s1}} ... y_n^{m_{sn}} = (t_1^{m_{s1}} ... t_n^{m_{sn}}) \prod_{i=1}^{n} (\prod_{j=1}^{k} q_j^{m_{si} b_{ij}}) = T^{I_s} \prod_{i=1}^{n} (\prod_{j=1}^{k} q_j^{m_{si} b_{ij}}).$$

Furthermore,

$$T^{I_s} \prod_{j=1}^{k} (\prod_{i=1}^{n} q_j^{m_{si} b_{ij}}) = T^{I_s} (\prod_{j=1}^{k} q_j^{\sum_{i=1}^{n} m_{si} b_{ij}}) = T^{I_s} (\prod_{j=1}^{k} q_j^{\delta_{sj}}) = T^{I_s} q_s = q_s T^{I_s}.$$

In other words, for each $s = 1, ..., k$, we have $Y^{I_s} = q_s T^{I_s}$ and $[y_i] = [t_i]$ $\forall i$. Therefore, these y_i satisfy the condition (4.2.10). It follows that T is a solution of f in the sense of Definition 4.2.3.

When $k < n$, one can add more rows to make M into an invertible matrix N since we assumed that M has full rank. Then we apply the same change of variable to N as above. This proves the proposition under the first condition.

When the second condition or the third condition holds, since we only have integer entries, all such $q_j^{b_{ij}}$ are well-defined without the further assumption on G as in the first case. The conclusion follows from the same argument. This completes the proof. $\square$

We remark that in the proof of Proposition 4.2.8, one can see that Definition 4.2.3 implies Definition 4.2.2 in general, but not conversely.

The Hasse-Weil zeta function over hyper-structures

The Hasse-Weil zeta function is the generating function of solutions of polynomial equations over finite fields extensions. More precisely, let X be an algebraic variety over the finite field $\mathbb{F}_q$ and $|X(\mathbb{F}_{q^m})|$ be the number of solutions of X over the finite field extension $\mathbb{F}_{q^m}$. The Hasse-Weil zeta function $Z(X,t)$ of X is defined by

$$Z(X,t) := \exp(\sum_{m\geq 1} \frac{N_m}{m} t^m), \quad N_m = |X(\mathbb{F}_{q^m})|. \tag{4.2.11}$$

To mimic (4.2.11) in hyper-structures, we need the appropriate notions of a 'hyper'-solution and a 'finite hyperfield extension'. We use Definition 4.2.2 or 4.2.3 of the previous subsection as the definition of a 'hyper-solution', however, there is no natural analogue of $\mathbb{F}_{q^m}$ in hyper-structures. In fact, in the theory of hyperfields, finite hyperfield extensions of the same (suitably defined) degree do not have to be isomorphic (cf. [9, Remark 3.7]). In this subsection, we will mostly focus on finite extensions of the Krasner's hyperfield $\mathbf{K}$ of the type $R_m := \mathbb{F}_{p^m}/\mathbb{F}_p^\times$ by considering it as the analogue of the finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ of degree m. Then, either by applying Definition 4.2.2 or Definition 4.2.3, the direct analogue of (4.2.11) would be to define

$$Z_H(X,t) = \exp(\sum_{m\geq 1} \frac{N_m}{m} t^m), \quad N_m = |X(R_m)|, \tag{4.2.12}$$

where $X(R_m)$ is the set of solutions of X over R_m.

Recall that a real-valued function $N : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a counting function of solutions of an algebraic variety X over $\mathbb{F}_q$ when $|X(\mathbb{F}_{q^m})| = N(q^m)$ for all $m \in \mathbb{Z}_{>0}$. Let p be an odd prime number, X be an affine algebraic variety over $\mathbb{F}_p$, and $X(R_m)$ be the set of solutions of X over $R_m = \mathbb{F}_{p^m}/\mathbb{F}_p^\times$ in the sense of Definition 4.2.2 or 4.2.3 with $A = \mathbb{F}_p$ and $B = \mathbb{F}_{p^m}$. We shall restrict to the affine case since we do not have yet defined the gluing notion in relation to our definitions.

Definition 4.2.10. *Let X be an affine algebraic variety over $\mathbb{F}_p$. A real-valued func-*

tion $N : \mathbb{R} \longrightarrow \mathbb{R}$ is called a counting function of X over the Krasner's hyperfield $\mathbf{K}$ *(with respect to Definition 4.2.2 or 4.2.3) if*

$$|X(R_m)| = N(|R_m|) \quad \forall m \in \mathbb{Z}_{>0}. \tag{4.2.13}$$

Example 4.2.11. *Suppose that $X = \mathbb{A}^n$ or $\mathbb{G}_m^n$ over $\mathbb{F}_p$. Then, with any of Definition 4.2.2 and 4.2.3, we obtain the counting functions $N(y) = y^n$ and $(y-1)^n$ respectively. These agree with the counting functions of $\mathbb{A}^n$ and $\mathbb{G}_m^n$ in the classical case.*

The next proposition shows that not only simple cases like $\mathbb{A}^n$ and $\mathbb{G}_m^n$, but also for some case a counting function over $\mathbf{K}$ agrees with a classical one. Note that the similar observation to the next proposition has been explained in §5.4 of [49].

Proposition 4.2.12. *Let X be an affine algebraic variety defined by a polynomial $f = y_1^{a_1}...y_n^{a_n} - y_1^{b_1}...y_n^{b_n} \in \mathbb{F}_p[y_1, ..., y_n]$.*

1. *The counting function of X over* $\mathbf{K}$ *(with respect to Definition 4.2.3) exists and agrees with the classical counting function of X over $\mathbb{F}_p$.*
2. *Let $c_i = a_i - b_i$. If the row vector $[c_1...c_n]$ satisfies one of the conditions given in Proposition 4.2.8, then the counting function of X over* $\mathbf{K}$ *(with respect to any of Definition 4.2.2 and 4.2.3) agrees with the classical counting function of X over $\mathbb{F}_p$.*

Proof. We prove the first assertion. The second assertion directly follows from Proposition 4.2.8 and the first assertion. We will compare ways to count solutions in each case in terms of $|\mathbb{G}_m|$. We divide the proof in two cases: when at least one of y_i is 0 and when none of y_i is 0.

If a (hyper)solution $y = (y_1, ...y_n)$ contains k zeros, the number s_k of such (hyper)solutions is given by

$$s_k = \binom{n}{k} t^{n-k}, \text{ where } t = |\mathbb{G}_m|.$$

Therefore, in this case, both the numbers of hyper-solutions and classical solutions agree in terms of $t = |\mathbb{G}_m|$.

Next, we compute s_0, the number of (hyper)solutions without a zero. With respect to multiplications, $\mathbb{F}_{p^m}^\times$ and $R_m^\times$ are cyclic groups of order $t = |\mathbb{G}_m|$. Let us first consider the classical case. For notational convenience, let $k := \mathbb{F}_{p^m}$, $k^\times :=< \alpha >$ with $|\alpha| = t$. Suppose that $y = (y_1, ..., y_n)$ is a solution such that $y_i \neq 0\ \forall i$. Then solving $f = y_1^{a_1}...y_n^{a_n} - y_1^{b_1}...y_n^{b_n}$ is equivalent to solving $y_1^{c_1}...y_n^{c_n} - 1$. However, since we are solving $y_1^{c_1}...y_n^{c_n} - 1$ over k, this is equivalent to finding $\lambda = (\lambda_1, ..., \lambda_n)$ with $1 \leq \lambda_i \leq t$ such that

$$\sum \lambda_i c_i \equiv 0(\bmod t). \tag{4.2.14}$$

This is because we may write $y_i = \alpha^{\lambda_i}$ for each $i = 1, ..., n$. Then $y_1^{c_1}...y_n^{c_n} = \alpha^{\lambda_1 c_1}...\alpha^{\lambda_n c_n} = \alpha^{\sum \lambda_i c_i}$, and $|\alpha| = t$. Hence s_0 is the number of distinct solutions of (4.2.14). In the case of hyper-solutions, we can count in the similar manner. Since we are counting solutions do not contain zeros, in this case, solving $f = y_1^{a_1}...y_n^{a_n} - y_1^{b_1}...y_n^{b_n}$ is equivalent to solving $y_1^{c_1}...y_n^{c_n} - 1$. One can easily observe that $y = (y_1, ...y_n) \in (R_m^\times)^n$ is a solution of $y_1^{c_1}...y_n^{c_n} - 1$ in the sense of Definition 4.2.3 if and only if $0 \in y_1^{c_1}...y_n^{c_n} - 1$. That is equivalent to solving $y_1^{c_1}...y_n^{c_n} = 1$. But we can write $R_m^\times =< \beta >$ with $|\beta| = t$. Therefore, same as the classical case, it reduces to finding $\lambda = (\lambda_1, ..., \lambda_n)$ with $1 \leq \lambda_i \leq t$ such that $\sum \lambda_i c_i \equiv 0(\bmod t)$. This proves our proposition. □

Suppose that there exists a counting function $N(y)$ of X over $\mathbf{K}$. Then the Hasse-Weil zeta function attached to X over $\mathbf{K}$ as in (4.2.12) becomes the following:

$$Z_H(X, t) = \exp(\sum_{m \geq 1} \frac{N(|R_m|)}{m} t^m). \tag{4.2.15}$$

Example 4.2.13. *Let $X = \mathbb{G}_m$ over $\mathbb{F}_p$. Then we have the counting function $N(y) =$*

$y-1$ *over* $\mathbf{K}$. *One observes that* $|R_m| = \frac{p^m-1}{p-1}+1$. *It follows that*

$$Z_H(\mathbb{G}_m,t) = \exp(\sum_{m\geq 1}\frac{N(|R_m|)}{m}t^m) = \exp(\sum_{m\geq 1}\frac{\frac{p^m-1}{p-1}}{m}t^m) = \exp(\frac{1}{p-1}(\sum_{m\geq 1}\frac{(pt)^m}{m} - \sum_{m\geq 1}\frac{t^m}{m}))$$

$$= \exp(\frac{1}{p(p-1)}\ln(\frac{1}{1-pt}) - \frac{1}{p-1}\ln(\frac{1}{1-t})) = (\frac{1-t}{(1-pt)^p})^{\frac{1}{p-1}}.$$

Remark 4.2.14. *Example 4.2.13 shows that even if a classical counting function and a counting function over* $\mathbf{K}$ *agree, their Hasse-Weil zeta functions do not have to agree. Furthermore, it seems hard to derive interesting properties of* $Z_H(X,t)$ *due to the difficulty in dealing with Definitions 4.2.2 and 4.2.3. However, in §4.3, we define an integral hyper-scheme and the Hasse-Weil zeta function attached to it. Then, we generalize several properties of a classical Hasse-Weil zeta function.*

4.2.2 A tropical variety over hyper-structures

In this subsection, we recast a tropical variety as the 'positive part' of an algebraic variety over hyper-structures. The basic notion of tropical geometry that we need in this subsection is reviewed in §2.1.1. For more details about tropical geometry we refer the reader to [30]. Note that we use the generalized notion of a tropical variety in this subsection (cf. Equation (4.2.23), Remarks 4.2.30 and 4.2.32), and that such choice makes no difference in further study.

The main motivation for the study proposed in this subsection comes from the following observation. The definition of a tropical variety does not seem natural in the sense that it is not defined as the set of solutions of polynomial equations, but as the set of points where a maximum is attained at least twice. Recently, there have been several attempts to build an algebraic foundation of tropical geometry: e.g. [18], [23], [35], [50].

Next, Proposition 4.2.31 shows that there exists a more natural description of a tropical variety by applying a symmetrization procedure and the definition of an algebraic variety over hyper-structures.

We use the multi-index notation: for $I=(i_1,...,i_n)\in\mathbb{N}^n$, $X^I := x_1^{i_1}\cdots x_n^{i_n}$. Let us first define the notion of a polynomial equation with coefficients in a hyperring R.

Definition 4.2.15. *Let R be a hyperring.*

1. *By a monomial f with n variables over R we mean a formal sum consisting of a single term:*

$$f := a_I X^I, \qquad a_I \in R. \tag{4.2.16}$$

2. *By a polynomial f with n variables over R we mean a finite formal sum:*

$$f := \sum_{I\in\mathbb{N}^n} a_I X^I, \qquad a_I\in R, \quad a_I = 0 \text{ for all but finitely many } I \tag{4.2.17}$$

such that there is no repetition of monomials with the same multi-index I. We denote by $R[x_1,...,x_n]$ the set of polynomials with n variables over R.

One can be easily mislead in hyperring theory. For example, $(x-x)$ is not a polynomial over the hyperfield of signs $\mathbf{S}$ since the term x is repeated. The reason why we do not want $(x-x)$ to be a polynomial is that whenever a repetition of a monomial occurs, an ambiguity follows. For instance, we may have $(x-x)=(1-1)x=\{-x,0,x\}$. In other words, $(x-x)$ does not represent a single element.

Furthermore, one can not perform the basic arithmetic in general. For example, (x^2-1) differs from $(x+1)(x-1)$ as an element of $\mathbf{S}[x]$ (cf. Example 4.2.16). Note that $(x+1)(x-1)$ is not even a polynomial over $\mathbf{S}$ since it is not of the form (4.2.17). However, in (4.2.18), we shall explain the meaning of such type. Therefore, for $f,g\in R[x_1,...,x_n]$, we say $f=g$ only if they are identical.

We directly generalize the classical addition and multiplication of polynomial equations to $R[x_1,...,x_n]$. For example, for $f=\sum_{i=0}^n a_i x^i$, $g=\sum_{j=0}^m b_j x^j \in R[x]$, the addition and the multiplication of f and g are given by

$$f+g := \sum_{i=0}^{n}(a_i+b_i)x^i + \sum_{i=n+1}^{n} b_ix^i, \qquad fg := \sum_{i=0}^{n+m}(\sum_{r+l=i} a_rb_l)x^i. \quad (n \leq m) \tag{4.2.18}$$

However, $(a_i + b_i)$ and $\sum_{r+l=i} a_rb_l$ are not elements, but subsets of R in general. Therefore, the addition and the multiplication defined in this way are in general multi-valued as the following example shows.

Example 4.2.16. *Let $R = \mathbf{S}$, the hyperfield of signs. For $x-1, x+1 \in \mathbf{S}[x]$, we have*

$$(x+1)(x-1) = x^2 + (1-1)x - 1 = \{x^2-1, x^2+x-1, x^2-x-1\}.$$

$$(x+1)+(x-1) = (x+x)+(1-1) = \{x, x+1, x-1\}.$$

Remark 4.2.17. *We emphasize that $R[x_1, ..., x_n]$ is only a set with two multi-valued binary operations. However it appears, in some circumstance, that $R[x_1, ..., x_n]$ behaves like a hyperring as Example 4.2.26 shows.*

Throughout this subsection, we will simply write a polynomial over R instead of a polynomial with n variables when there is no possible confusion.

Definition 4.2.18. *Let R be a hyperring and $R[x_1, ..., x_n]$ be the set of polynomials over R. Let L be a hyperring extension of R. By an evaluation $f(\alpha)$ of $f = \sum_I a_IX^I \in R[x_1, ..., x_n]$ at $\alpha = (\alpha_1, ..., \alpha_n) \in L^n$ we mean the following set:*

$$f(\alpha) := \sum_I a_I\alpha^I \subseteq L. \tag{4.2.19}$$

Example 4.2.19. *Let $R = L = \mathbf{S}$, the hyperfield of signs. Suppose that $f = x^2 - x \in R[x]$. Then: $f(1) = \mathbf{S}$, $\quad f(0) = \{0\}$, $\quad f(-1) = \{1\}$.*
Let $L = \mathcal{T}\mathbb{R}$, Viro's hyperfield. Then: $f(1) = [-1, 1]$, $\quad f(0) = \{0\}$, $\quad f(-1) = \{1\}$.

An intuitive definition of a set of solutions of $f \in R[x_1, ..., x_n]$ over a hyperring extension L of R would be the following set:

$$\{\alpha = (\alpha_1, ..., \alpha_n) \in L^n \mid 0 \in f(\alpha)\}. \tag{4.2.20}$$

However, (4.2.20) may depend on the way one writes f (cf. [49, §5.2]). Moreover, for two different elements $f, g \in R[x_1, ..., x_n]$, we may have $f(\alpha) = g(\alpha)$ $\forall \alpha \in L^n$. For example, suppose that $f = x^2 - 1$, $g = x^4 - 1 \in \mathbf{S}[x]$. Then $f(a) = g(a)$ $\forall a \in \mathcal{T}\mathbb{R}$, but $f \neq g$ as elements of $\mathbf{S}[x]$. To resolve these issues, we introduce the following relation on $R[x_1, ..., x_n]$.

Definition 4.2.20. *Let R be a hyperring and $R[x_1, ..., x_n]$ be the set of polynomials over R. Let L be a hyperring extension of R. For $f, g \in R[x_1, ..., x_n]$, we define*

$$f \equiv_L g \Longleftrightarrow f(\alpha) = g(\alpha) \text{ (as sets)} \quad \forall \alpha \in L^n. \tag{4.2.21}$$

Remark 4.2.21. *The relation (4.2.21) depends on a hyperring extension L of R. However, we note that if H is a hyperring extension of L then $f \equiv_H g \Longrightarrow f \equiv_L g$.*

The following statement is clear in view of the above definition.

Proposition 4.2.22. *Let R be a hyperring and L be a hyperring extension of R. Then the relation (4.2.21) on $R[x_1, ..., x_n]$ is an equivalence relation.*

Proof. This is straightforward since (4.2.21) is defined in terms of an equality of sets. □

Under the equivalence relation (4.2.21), we can consider each polynomial $f \in R[x_1, ..., x_n]$ as the following function:

$$f : L^n \longrightarrow P^*(L), \qquad \alpha = (\alpha_1, ..., \alpha_n) \mapsto f(\alpha), \tag{4.2.22}$$

where $P^*(L)$ is the set of non-empty subsets of L.

Definition 4.2.23. *Let R be a hyperring and $R[x_1, ..., x_n]$ be the set of polynomials over R. Let L be a hyperring extension of R. By a solution of f over L we mean an* *element $a = (a_1, ..., a_n) \in L^n$ such that $0 \in f(a)$ where we consider f as in (4.2.22) under $\equiv_L$. We denote by $V_L(f)$ the set of solutions of f over L.*

Remark 4.2.24. *Suppose that H is a hyperring extension of L. It clearly follows from the definition that $V_L(f) \subseteq V_H(f)$.*

Let $R[x_1, ..., x_n]/\equiv_L$ be the set of equivalence classes of $R[x_1, ..., x_n]$ under $\equiv_L$. When L is doubly distributive (hence, so is R), the multiplication on $R[x_1, ..., x_n]/\equiv_L$ induced from the multi-valued multiplication on $R[x_1, ..., x_n]$ is well-defined and single-valued. In fact, suppose that $f, g \in R[x_1, ..., x_n]$. If $h \in f \cdot g$ then it follows from the doubly distributive property of L that $h(\alpha) = f(\alpha) \cdot g(\alpha)$ for all $\alpha \in L^n$ (cf. [50] the remark after Theorem 4.B.). Therefore, under $\equiv_L$, the set $f \cdot g$ becomes a single equivalence class. This is one of the advantages of working with $R[x_1, ..., x_n]/\equiv_L$ rather than working directly with $R[x_1, ..., x_n]$.

Example 4.2.25. *Let $R = \mathbf{S}$, the hyperfield of signs and $L = \mathcal{T}\mathbb{R}$, Viro's hyperfield. Then L satisfies the doubly distributive property (cf. [50, Theorem 7.B.]). In Example 4.2.16, we computed $(x+1)(x-1) = \{x^2 - 1, x^2 + x - 1, x^2 - x - 1\}$ in $\mathbf{S}[x]$. One can easily see that*

$$\forall a \in \mathcal{T}\mathbb{R}, \quad (a^2 - 1) = (a^2 + a - 1) = (a^2 - a - 1) = \begin{cases} a^2 & \textit{if } |a| > 1 \\ [-1, 1] & \textit{if } |a| = 1 \\ -1 & \textit{if } |a| < 1 \end{cases}$$

Therefore $(x^2 - 1) \equiv_{\mathcal{T}\mathbb{R}} (x^2 + x - 1) \equiv_{\mathcal{T}\mathbb{R}} (x^2 - x - 1)$, and $x = 1, -1$ are the only solutions of the equivalence class of $(x^2 - 1)$ under $\equiv_{\mathcal{T}\mathbb{R}}$.

However, the following example shows that in general one can not expect $R[x_1, ..., x_n]/\equiv_L$ to be a hyperring even when L satisfies the doubly distributive property.

Example 4.2.26. *Let $R = L = \mathbf{K}$, the Krasner's hyperfield. Let $[f]$ be the equivalence class of $f \in \mathbf{K}[x]$ under $\equiv_{\mathbf{K}}$. Then any two non-constant polynomials over $\mathbf{K}$ with* *the same constant term are equivalent under* $\equiv_{\mathbf{K}}$. *It follows that*

$$(\mathbf{K}[x]/\equiv_{\mathbf{K}}) = \{[0],[1],[1+x],[x]\}.$$

For the notational convenience, let $0 := [0], 1 := [1], a := [1+x]$, *and* $b := [x]$. *Then we have the following tables:*

$+$	0	1	a	b
0	0	1	a	b
1	1	$\{0,1\}$	$\{a,b\}$	a
a	a	$\{a,b\}$	$\{0,1,a,b\}$	$\{1,a\}$
b	b	a	$\{1,a\}$	$\{0,b\}$

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	b	b

One can check by using the above tables that $(\mathbf{K}[x]/\equiv_{\mathbf{K}},+)$ *is a canonical hypergroup, but it fails to satisfy the distributive law. For example, we have*

$$a(1+b) = \{a\} \subsetneq a+ab = \{1,a\}.$$

However, one sees that $(\mathbf{K}[x]/\equiv_{\mathbf{K}},+,\cdot)$ *still satisfies the weak version of the distributive law we previously mentioned (cf. Remark 3.1.5). It follows that* $(\mathbf{K}[x]/\equiv_{\mathbf{K}},+,\cdot)$ *is not a hyperring but a multiring.*

We recall that if A *and* B *are multirings, one defines a homomorphism of multirings as a map* $\varphi : A \longrightarrow B$ *such that*

$$\varphi(a+b) \subseteq \varphi(a)+\varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(1_A) = 1_B, \quad \varphi(0_A) = 0_B \qquad \forall a,b \in A.$$

Consider the set $\mathbb{A}^1_{\mathbf{K}}(\mathbf{K}) := \mathrm{Hom}_{multi}(\mathbf{K}[x]/\equiv_{\mathbf{K}},\mathbf{K})$ *of multiring homomorphisms from* $\mathbf{K}[x]/\equiv_{\mathbf{K}}$ *to* $\mathbf{K}$. *If* $\varphi \in \mathbb{A}^1_{\mathbf{K}}(\mathbf{K})$ *then one has* $\varphi(a) = 1$ *since* $a+a = (\mathbf{K}[x]/\equiv_{\mathbf{K}})$. *One can also easily check that* $\varphi(b)$ *can be any point of* $\mathbf{K}$. *It follows that* $\mathbb{A}^1_{\mathbf{K}}(\mathbf{K}) = \{\varphi_0,\varphi_1\}$, *where* $\varphi_0(b) = 0$ *and* $\varphi_1(b) = 1$. *This suggests that one might consider* $\mathbf{K}[x]/\equiv_{\mathbf{K}}$ *as the 'coordinate ring' of an affine line over* $\mathbf{K}$.

In the sequel, we will always consider an element f of $R[x_1, ..., x_n]$ under the equivalence relation (4.2.21) with a predesignated hyperring extension L of R.

We begin with the case of a hypersurface. Recall that a tropical variety (or tropical hypersurface) $trop(V(f))$ defined by a polynomial equation $f \in \mathbb{Z}_{max}[x_1, ..., x_n]$ is the following set:

$$\{a \in (\mathbb{Z}_{max})^n \mid \text{ the maximum of } f \text{ is achieved at least twice}\}. \tag{4.2.23}$$

For the notational convenience, let $M = \mathbb{Z}_{max}$ and $M_S = R = s(\mathbb{Z}_{max})$, the symmetrization of M. Note that M_S is a hyperfield since M is a semifield (cf. Lemma 3.1.6).
Let $f(x_1, ..., x_n) \in M[x_1, ..., x_n]$. Write $f(x_1, ..., x_n) = \sum_i m_i(x_1, ..., x_n)$ as a sum of distinct monomials and fix this presentation. Then we define

$$f_{\hat{i}}(x_1, ..., x_n) = \sum_{j \neq i} m_j(x_1, ..., x_n) \in M[x_1, ..., x_n] \quad \text{ for each } i.$$

By identifying an element $a \in M$ with the element $(a, 1) \in M_S = R$, we define

$$\tilde{f}_{\hat{i}}(x_1, ..., x_n) := (\sum_{j \neq i} (m_j(x_1, ..., x_n), 1)) + (m_i(x_1, ..., x_n), -1) \in R[x_1, ..., x_n].$$

With these notations we have the following description of a tropical hypersurface.

Proposition 4.2.27. *With the same notation as above, we let*

$$V(\tilde{f}_{\hat{i}}) := \{z \in R^n \mid 0_R \in \tilde{f}_{\hat{i}}(z)\}, \quad HV(f) := \bigcup_i V(\tilde{f}_{\hat{i}}).$$

Then, with $\varphi = s^n : M^n \longrightarrow R^n$, we have a set bijection:

$$trop(V(f)) \simeq (HV(f) \cap \mathrm{Img}(\varphi)),$$

where $trop(V(f))$ is the tropical variety defined by $f \in M[x_1, ..., x_n]$.

Remark 4.2.28. *Even though we started by fixing one presentation of a polynomial equation $f \in M[x_1, ..., x_n]$, the set $trop(V(f))$ does not depend on the chosen presentation of f. Therefore, even though $HV(f)$ may vary depending on a presentation of f, the set $HV(f) \bigcap \mathrm{Img}(\varphi)$ is invariant of the presentation as long as there is no repetition of monomials.*

Before we prove Proposition 4.2.27, we present an example to show how this procedure works.

Example 4.2.29. *Let $f(x, y) = x + y + 1 \in \mathbb{Z}_{max}[x, y]$. Then $trop(V(f))$ consists of three rays (cf. Example 2.1.1):*

$$trop(V(f)) = \{(x, y) \in \mathbb{Z}_{max} \times \mathbb{Z}_{max} \mid 1 \leq y = x, \ \ or \ y \leq x = 1, \ \ or \ x \leq y = 1\}.$$

With the above notations, we have

$$f_x(x, y) := y + 1, \quad \tilde{f}_x(x, y) := (1, 1)y + (1, 1) + (1, -1)x. \tag{4.2.24}$$

Since we only consider the 'positive' solutions, x and y should be of the form $(t, 1)$. Therefore, in this case, we have

$$\tilde{f}_x(x, y) = (y, 1) + (1, 1) + (x, -1) = (y + 1, 1) + (x, -1). \tag{4.2.25}$$

By the definition of symmetrization (cf. Equation (3.1.2)), we have

$$0_R \in \tilde{f}_x(x, y) \Longleftrightarrow y + 1 = x \ \ in \ \mathbb{Z}_{max}.$$

Thus, we obtain

$$\{(x, y) \in \mathbb{Z}_{max} \times \mathbb{Z}_{max} \mid 1 \leq y = x, \ \ or \ y \leq x = 1\} = V(\tilde{f}_x(x, y)) \bigcap (s(\mathbb{Z}_{max}) \times s(\mathbb{Z}_{max})).$$

Similarly with $f_y(x,y) = x+1$ *we have*

$$0_R \in \tilde{f}_y(x,y) \Longleftrightarrow x+1 = y \text{ in } \mathbb{Z}_{max}.$$

This time, we obtain

$$\{(x,y) \in \mathbb{Z}_{max} \times \mathbb{Z}_{max} \mid 1 \leq x = y, \text{ or } x \leq y = 1\} = V(\tilde{f}_y(x,y)) \bigcap (s(\mathbb{Z}_{max}) \times s(\mathbb{Z}_{max})).$$

Finally, with $f_1(x,y) = x+y$, *we have*

$$0_R \in \tilde{f}_1(x,y) \Longleftrightarrow x+y = 1 \text{ in } \mathbb{Z}_{max}.$$

This gives

$$\{(x,y) \in \mathbb{Z}_{max} \times \mathbb{Z}_{max} \mid y \leq x = 1, \text{ or } x \leq y = 1\} = V(\tilde{f}_1(x,y)) \bigcap (s(\mathbb{Z}_{max}) \times s(\mathbb{Z}_{max})).$$

By taking the union of all three we recover

$$trop(V(f)) = (\bigcup_{z \in \{x,y,1\}} V(\tilde{f}_z(x,y))) \bigcap (s(\mathbb{Z}_{max}) \times s(\mathbb{Z}_{max})).$$

Now we give the proof of Proposition 4.2.27.

Proof. When f is a single monomial, the result is clear since 0_M and 0_R will be the only solution for each. Thus we may assume that f is not a monomial. If $z = (z_1, ..., z_n) \in trop(V(f))$ then there exist $m_i(x_1, ..., x_n)$, $m_j(x_1, ..., x_n)$ with $i \neq j$ such that the value $m_i(z) = m_i(z)$ attains the maximum among all $m_r(z)$. Then we have $f(z) = m_i(z) = m_j(z) \in M$. It follows that

$$0_R \in (f(z), 1) + (m_i(z), -1) = (f_{\hat{i}}(z), 1) + (m_i(z), -1) = \tilde{f}_{\hat{i}}(\varphi(z)).$$

Thus we have $\varphi(z) \in HV(f)$.

Conversely, suppose that $\varphi(z) \in HV(f) \cap \mathrm{Img}(\varphi)$. Let $\varphi(z) = (\varphi(z_i))$, where $(z_i, 1) \in$

$M \times \{1\} \subseteq R$. Then, by the definition of $HV(f) = \bigcup V(\tilde{f}_{\hat{i}})$, $\varphi(z)$ is an element of $V(\tilde{f}_{\hat{i}})$ for some i. In other words,

$$0_R \in (\sum_{j\neq i}(m_j(z),1)) + (m_i(z),-1) = \tilde{f}_{\hat{i}}(\varphi(z)).$$

Therefore, there exists some $r \neq i$ such that

$$\sum_{j\neq i}(m_j(z),1) = (m_r(z),1) = (m_i(z),1) \quad \text{and} \quad m_j(z) \leq m_r(z) = m_i(z) \qquad \forall j \neq i,r.$$

It follows that $z \in trop(V(f))$. So far we have showed that

$$\varphi(trop(V(f))) = HV(f) \cap \mathrm{Img}(\varphi).$$

Since φ is one-to-one, we conclude that $trop(V(f)) \simeq HV(f) \cap \mathrm{Img}(\varphi)$ as sets. In other words, $trop(V(f))$ is the 'positive' part of $HV(f)$. □

Remark 4.2.30. *Our definition (4.2.23) of $trop(V(f))$ may contain a point $a = (a_1,...,a_n)$ such that $a_i = -\infty(= 0_M)$ for some i. This is little different from the conventional definition of a tropical hypersurface in which one excludes such points. However, from the proof of Proposition 4.2.27, one can observe that the subset of $trop(V(f))$ which does not have 0_M at any coordinate maps bijectively onto the subset of $H(V(f))\bigcap \mathrm{Img}(\varphi)$ which does not have 0_R at any coordinate.*

When I is an ideal of $\mathbb{Z}_{max}[x_1,..,x_n]$ one defines a tropical variety defined by I as follows:

$$trop(V(I)) := \bigcap_{f\in I} trop(V(f)). \tag{4.2.26}$$

One has to be careful with (4.2.26) since the intersection is over all polynomials in I not just over a set of generators of I (cf. [30]).

To understand (4.2.26) as the 'positive' part of an algebraic variety over hyperstructures, we extend the previous proposition as follows.

Proposition 4.2.31. *Let I be an ideal of $\mathbb{Z}_{max}[x_1, .., x_n]$. Then, with the same notation as Proposition 4.2.27, we have a set bijection via $\varphi = s^n$:*

$$trop(V(I)) := \bigcap_{f \in I} trop(V(f)) \simeq (\bigcap_{f \in I} HV(f)) \bigcap \operatorname{Img} \varphi.$$

Proof. Take any $z \in trop(V(I)) \subseteq (\mathbb{Z}_{max})^n$, then by definition, $z \in \bigcap_{f \in I} trop(V(f))$. That is $z \in trop(V(f))\ \forall f \in I$. It follows from the previous proposition that $\varphi(z) \in HV(f)\ \forall f \in I$, thus $\varphi(z) \in (\bigcap_{f \in I} HV(f)) \bigcap \operatorname{Img} \varphi$.

Conversely, if $\varphi(z) \in (\bigcap_{f \in I} HV(f)) \bigcap \operatorname{Img} \varphi$ then $\varphi(z) \in HV(f)\ \forall f \in I$. From the previous proposition it follows that $z \in trop(V(f))\ \forall f \in I$, hence $z \in trop(V(I))$. Thus we have

$$\varphi(trop(V(I))) = (\bigcap_{f \in I} HV(f)) \bigcap \operatorname{Img} \varphi.$$

The conclusion follows from the injectivity of φ. □

Remark 4.2.32.

1. *One can replace $\mathbb{Z}_{max}$ with any semifield M of characteristic one. Then the same statement holds with $R = M_S$. In particular, when $M = \mathbb{R}_{max}$, the subset of $trop(V(I))$ which consists of points without $-\infty$ at any coordinate is exactly same as the tropical variety defined by I as in [30].*

2. *Let K be a field with a non-archimedean valuation v and a value group Γ_K. Suppose that K is complete with respect to v. Since Γ_K is an additive subgroup of $\mathbb{R}$, we can consider $\Gamma_K \cup \{-\infty\}$ as the subsemifield of $\mathbb{R}_{max}$ by defining an addition law as the maximum and a multiplication law as the usual addition (cf. Remark 3.1.1).*

   Let $f \in K[x_1, ..., x_n]$, $F = trop(f) \in \Gamma_K[x_1, ..., x_n]$ as in [30](or see §2.1.1). Then the following is well-known (cf. Theorem 2.1.4).

$$trop(V(F)) = \{(v(x_1), ..., v(x_n)) | x = (x_1, ..., x_n) \in V(f) \subseteq K^n\}.$$

Thus, together with the above proposition, we have a set bijection:

$$\{(v(x_1),...,v(x_n))|x=(x_1,...,x_n)\in V(f)\subseteq K^n\}\simeq (HV(F)\bigcap \operatorname{Img}\varphi).$$

4.2.3 Analytification of affine algebraic varieties in characteristic one

Let us first review the definition of the Berkovich analytification X^{an} of an affine algebraic variety X over K, where K is an algebraically closed field which is complete with respect to a non-archimedean absolute value v. Note that by an algebraic variety over K we mean a reduced scheme of finite type over K (possibly reducible).

A multiplicative seminorm $|-|$ on a commutative ring A is a multiplicative monoid map $|-|: A \longrightarrow \mathbb{R}_{\geq 0}$ such that $|0_A| = 0$ and $|a+b| \leq |a| + |b|$ $\forall a,b \in A$. We call that a multiplicative seminorm $|-|$ is non-archimedean if $|a+b| \leq \max\{|a|,|b|\}$ $\forall a,b \in A$. When A is a commutative K-algebra, we say that $|-|$ is compatible with v if $|k| = v(k)$ $\forall k \in K$.

The Berkovich analytification X^{an} is a topological space whose underlying set consisting of multiplicative seminorms on the coordinate ring $\mathcal{O}_X(X)$ which are compatible with a non-archimedean absolute value v on the ground field K. The topology is given by the coarsest topology such that $\forall a \in \mathcal{O}_X(X)$, the following map is continuous.

$$ev_a : X^{an} \longrightarrow \mathbb{R}_{\geq 0}, \quad |-| \mapsto |a|, \tag{4.2.27}$$

where $\mathbb{R}_{\geq 0}$ is endowed with the Euclidean topology. For more details about multiplicative seminorms or the Berkovich analytification we refer the reader to the first chapter of [3].

In this subsection, inspired by [18] where authors extended the notion of valuations on a commutative ring so that a value group is no longer a group but an idempotent semiring, we generalize the notion of a multiplicative seminorm on a commutative ring A so that it has values in a semifield of characteristic one or a hyperfield with a good ordering (cf. Definition 3.1.2). Furthermore, by appealing to such generalization, we construct the Berkovich analytificaiton, in characteristic one, of a (classical) affine algebraic variety.

We remark that the situation is different from §2.4 where we investigate a valuation of a semiring which has values in semifields or hyperfields. However, in this section, we study a multiplicative seminorm of classical objects which has values in semifields of characteristic one or hyperfields with good orderings.

Note that in what follows we use the canonical order $\leq$ on a semifield S of characteristic one reviewed in Chapter 3 unless otherwise stated.

Definition 4.2.33. *A multiplicative seminorm on a commutative ring A with values in a semifield S of characteristic one is a map $|-|: A \longrightarrow S$ such that*

$$0_S \leq |a|, \quad |ab| = |a||b|, \quad |a+b| \leq |a| + |b|, \quad |0_A| = 0_S \qquad \forall a, b \in A. \tag{4.2.28}$$

Let K be a field. When A is a commutative K-algebra and $v: K \longrightarrow S$ is a multiplicative seminorm on K with values in S, we say $|-|$ is compatible with v if $|a| = v(a)$ $\forall a \in K$.

Example 4.2.34. *Let A be a commutative ring and $\varphi : A \longrightarrow \mathbb{R}_{\geq 0}$ be a non-archimedean multiplicative seminorm in the classical sense. Let $S = \mathbb{R}_{max}$. Then*

$$\ln(\varphi) : A \longrightarrow \mathbb{R}_{max}, \quad a \mapsto \ln(\varphi(a)), \quad \ln(0) := -\infty$$

is a multiplicative seminorm with values in $\mathbb{R}_{max}$.

Conversely, suppose that $\psi : A \longrightarrow \mathbb{R}_{max}$ is a multiplicative seminorm in the sense of Definition 4.2.33. Then

$$\exp(\psi) : A \longrightarrow \mathbb{R}_{\geq 0}, \quad a \mapsto \exp(\psi(a)), \quad \exp(-\infty) := 0$$

is a non-archimedean multiplicative seminorm on A in the classical sense.

Definition 4.2.35. *Let R be a hyperring which has a good ordering P. A multi-*

plicative seminorm on a commutative ring A with values in a pair (R, P) is a map $|-| : A \longrightarrow R$ such that $\forall a, b \in A$ the following holds.

$$|a| \in P, \quad |ab| = |a||b|, \quad (|a+b| + |a| + |b|) \bigcap (|a| + |b|) \neq \emptyset, \quad |0_A| = 0_R. \quad (4.2.29)$$

Let K be a field. When A is a commutative K-algebra and $v : K \longrightarrow R$ is a multiplicative seminorm on K with values in a pair (R, P), we say that $|-|$ is compatible with v if $|a| = v(a)$ $\forall a \in K$.

We will say interchangeably a multiplicative seminorm with values in a pair (R, P) and a multiplicative seminorm with values in a good ordering P when there is no possible confusion.

Remark 4.2.36. *1. Since a classical multiplicative seminorm maps a commutative ring to a nonnegative real numbers, we impose the condition $0_S \leq |a|$ for a semifield S of characteristic one and the condition $|a| \in P$ for a hyperring R with a good ordering P. However, the condition $0_S \leq |a|$ is redundant since S is totally ordered with the canonical order.*

2. The semi-algebraic condition $|a+b| \leq |a| + |b|$ is equivalent to the algebraic condition $|a+b| + |a| + |b| = |a| + |b|$ provided that $|a+b| \leq max\{|a|, |b|\}$ (cf. [18]). This motivates the condition $(|a+b| + |a| + |b|) \bigcap (|a| + |b|) \neq \emptyset$ in Definition 4.2.35.

Definition 4.2.37. *Let $X = \operatorname{Spec} A$ be an affine algebraic variety over a field K. Let P be either a semifield of characteristic one or a good ordering of a hyperfield R. Let v be a multiplicative seminorm on K with values in P. We denote by $X^{an}_{P,v}$ the set of multiplicative seminorms on A with values in P which are compatible with v. We call $X^{an}_{P,v}$ the analytification of X with values in P with respect to v.*

In the sequel, we will denote by X^{an}_P when there is no ambiguity about a multiplicative seminorm v on a ground field K with values in P. For example, when P is either $\mathbb{B}$ or the good ordering $\{0,1\}$ of the hyperfield $\mathbf{S}$ of signs, there exists a unique non-trivial multiplicative seminorm on K with values in P. Therefore, in this case, we simply denote by X_P^{an}.

Proposition 4.2.38. *Let P be either $\mathbb{B}$ or the good ordering $\{0,1\}$ of $\mathbf{S}$, and A be a commutative K-algebra. Then there exists a canonical one-to-one correspondence between the set X_P^{an} and the set of prime ideals of A where $X = \operatorname{Spec} A$.*

Proof. We first consider when $P=\mathbb{B}$. Let φ be a multiplicative seminorm on A with values in $\mathbb{B}$. We claim that $\mathfrak{p} := \operatorname{Ker}(\varphi)$ is the prime ideal of A. In fact, $0 \in \mathfrak{p}$. If $a,b \in \mathfrak{p}$ then we have

$$0 \leq \varphi(a+b) \leq \varphi(a)+\varphi(b) = 0+0 = 0.$$

It follows that $\varphi(a+b)=0$, hence $a+b \in \mathfrak{p}$. Next, for $r \in A$, $a \in \mathfrak{p}$, clearly $ra \in \mathfrak{p}$ from the multiplicative condition of φ. Furthermore, for multiplicative seminorms φ,ψ with values in $\mathbb{B}$, one sees that $\varphi=\psi$ if and only if $\operatorname{Ker}(\varphi)=\operatorname{Ker}(\psi)$ since $\mathbb{B}$ consists of two points. Therefore, each φ uniquely determines the prime ideal of A. Conversely, for $\mathfrak{p} \in \operatorname{Spec} A$, let us define $\varphi_{\mathfrak{p}}$ to be the map from A to $\mathbb{B}$ such that $\varphi_{\mathfrak{p}}(a) = 0$ if and only if $a \in \mathfrak{p}$. We claim that $\varphi_{\mathfrak{p}}$ is a multiplicative seminorm with values in $\mathbb{B}$. In fact, we have $\varphi_{\mathfrak{p}}(0) = 0$. There are four possible pairs of $(\varphi_{\mathfrak{p}}(a),\varphi_{\mathfrak{p}}(b))$; $(1,1),(1,0),(0,1),(0,0)$. If it is $(1,1)$ then $a,b \notin \mathfrak{p}$. Thus $ab \notin \mathfrak{p}$, and we have $\varphi_{\mathfrak{p}}(ab) = 1 = \varphi_{\mathfrak{p}}(a)\varphi_{\mathfrak{p}}(b)$. Furthermore, since $\varphi_{\mathfrak{p}}(a)+\varphi_{\mathfrak{p}}(b)=1$, the second condition is satisfied. If it is $(1,0)$ then we have $\varphi_{\mathfrak{p}}(ab) = 0 = 1\cdot 0 = \varphi_{\mathfrak{p}}(a) = \varphi_{\mathfrak{p}}(b)$. Furthermore, since $\varphi_{\mathfrak{p}}(a+b)$ is either 0 or 1, the second condition easily follows. The case of $(0,1)$ is same as that of $(1,0)$. Finally, if the pair is $(0,0)$, it is straightforward. The proof when $P=\{0,1\}$ of $\mathbf{S}$ is similar. □

Classically, the Berkovich analytification X^{an} of an affine algebraic variety X is equipped with the topology induced from Euclidean topology of $\mathbb{R}$ (cf. (4.2.27)). However, in Definition 4.2.37, we only define an analytification as a set. This is because we do not assume that a semifield of characteristic one or a hyperfield is equipped with a topology in general. But, one can mimic the classical construction of the Berkovich analytification to impose the topology on $X^{an}_{P,v}$ as long as P is a topological space. In other words, as in (4.2.27), we give the weakest topology on $X^{an}_{P,v}$ such that $\forall a \in \mathcal{O}_X(X)$ the following map is continuous:

$$ev_a : X^{an}_{P,v} \longrightarrow P, \quad |-| \mapsto |a|. \tag{4.2.30}$$

Let P be either $\mathbb{B}$ or the good ordering $\{0,1\}$ of $\mathbf{S}$ with the discrete topology and $X = \operatorname{Spec} A$ be an affine algebraic variety over a field K. The topology on X^{an}_P as above is the subspace topology induced from the product topology of $\prod_{a\in A} P$. Let us denote this topology on X^{an}_P as $\mathcal{T}$. Since we have $X^{an}_P = X = \operatorname{Spec} A$ (as sets), one may wonder the relationship between the Zariski topology on X^{an}_P and $\mathcal{T}$. In fact, $\mathcal{T}$ is finer than the Zariski topology. In the next proposition, we use the correspondence between X and X^{an}_P of Proposition 4.2.38.

Proposition 4.2.39. *Let P be either $\mathbb{B}$ or the good ordering $\{0,1\}$ of the hyperfield $\mathbf{S}$ and $X = \operatorname{Spec} A$ be an affine algebraic variety over a field K. If $U \subseteq X^{an}_P$ is a Zariski open subset then U is open with the topology $\mathcal{T}$.*

Proof. For each $a \in A$, we define

$$B_a := \prod_{i\in A} P_i, \qquad P_i = \begin{cases} P & \text{if } i \neq a \\ \{1\} & \text{if } i = a. \end{cases}$$

Then B_a is an open subset of $\prod_{i\in A} P$ with respect to the product topology. Since U is Zariski open, $U = D(I)$ for some ideal I of A. Let $V := \bigcup_{a\in I} B_a$. Then V is open with respect to the product topology. Therefore, it is enough to show that $U = U \cap V$. In fact, clearly we have $U \cap V \subseteq U$. Conversely, if $\varphi \in U$ then $I \not\subseteq \operatorname{Ker}(\varphi)$. It follows that $\exists a \in I \backslash \operatorname{Ker}(\varphi)$. This implies that $\varphi(a) = 1$, hence $\varphi \in B_a \subseteq V$. This proves the other inclusion. $\square$

Remark 4.2.40. *1. The topology $\mathcal{T}$ and the Zariski topology are not same in general. Let $\alpha, \beta \subseteq A$ be finite subsets. Let us define*

$$B_{\alpha,\beta} := \prod_{i \in A} P_i, \qquad P_i = \begin{cases} 1 & \text{if } i \in \alpha \\ 0 & \text{if } i \in \beta \\ P & \text{otherwise.} \end{cases}$$

Then $B_{\alpha,\beta}$ form a basis of $\prod_{a \in A} P$ with respect to the product topology. However, we have $B_{\alpha,\beta} \bigcap X_P^{an} = V(<\beta>) \bigcap D(\alpha)$, and this is not an open set with respect to the Zariski topology in general.

2. Let M be a semifield of characteristic one and $R = M_S$ be the symmetrization of M. The analytification of an affine algebraic variety $X = \operatorname{Spec} A$ with values in R depends on the choice of a good ordering of R. However, by using the symmetrization map $s : M \longrightarrow R$, we see that the analytification with values in M and the analytification with values in $(R, s(M))$ can be identified (in fact, one may choose $-s(M)$ as a good ordering of R to identify).

For the rest of this subsection we fix the following notations: $P = \mathbb{R}_{max}$ and $R = P_S$, the symmetrization of $\mathbb{R}_{max}$.

Lemma 4.2.41. *Let K be a field with a non-archimedean multiplicative seminorm $\nu : K^* \longrightarrow \mathbb{R}_{\geq 0}$. Then ν can be uniquely extended to a multiplicative seminorm on K with values in $\mathbb{R}_{max}$.*

Proof. This is straightforward by the exact same argument in Example 4.2.34. □

Lemma 4.2.42. *Let K be a field and A be a commutative K-algebra. Suppose that $f, g \in \operatorname{Hom}_{alg}(A, K)$. Then*

$$f = g \Longleftrightarrow \operatorname{Ker}(f) = \operatorname{Ker}(g).$$

Proof. $f = g$ obviously implies the same kernel. Conversely, suppose that $f \neq g$.

Then there exists $a \in A$ such that $f(a) \neq g(a)$. Since f and g have the same kernel, it follows that $f(a) = b \neq 0$. Then we have $f(b^{-1}a) = 1$ since b is a non-zero element of K and $f(1) = g(1) = 1$. This implies that $1 - b^{-1}a \in \operatorname{Ker}(f) = \operatorname{Ker}(g)$, hence $g(a) = b$. This contradicts our assumption, thus $f = g$. □

As in [38], we give the topology on $\mathbb{R}_{max}$ which extends the topology of $\mathbb{R}$ by defining the completed rays $[-\infty, a)$ for $a \in \mathbb{R}$ as a basis of neighborhoods of $-\infty$. Let $X = \operatorname{Spec} A$ be an affine algebraic variety over a field K and v be a multiplicative seminorm on K with values in P. We give the topology on $X^{an}_{P,v}$ the weakest topology such that all maps

$$\psi_f : X^{an}_{P,v} \longrightarrow \mathbb{R}_{max}, \quad |-| \mapsto |f|$$

are continuous for each $f \in A$ (cf. (4.2.30)). We note that this topology is equivalent to the topology induced by the product topology on $\prod_{a \in A} \mathbb{R}_{max}$.

Proposition 4.2.43. *Let ν be a non-archimedean multiplicative seminorm on a field K and v be the multiplicative seminorm extending ν as in Lemma 4.2.41. Let $X = \operatorname{Spec} A$ be an affine algebraic variety over K. Then the following holds.*

1. *$x^{-1}(\{-\infty\})$ is a proper prime ideal of A $\forall x \in X^{an}_{P,v}$.*
2. *There exists a canonical injection from $X(K)$ into $X^{an}_{P,v}$ where $X(K)$ is the set of K-rational points of X.*

Proof. 1. Let $I := x^{-1}(\{-\infty\})$, then trivially $0_A \in I$. Suppose that $\alpha, \beta \in I$, $\gamma \in A$. It follows from $x(\alpha+\beta) \leq \max\{x(\alpha), x(\beta)\} = -\infty$ that $\alpha + \beta \in I$. Since $x(\gamma\beta) = x(\gamma) + x(\beta) = -\infty$, we have $\gamma\beta \in I$. This shows that I is an ideal. Suppose that $pq \in I$. Then $x(pq) = x(p) + x(q) = -\infty$, therefore $x(p) = -\infty$ or $x(q) = -\infty$. It follows that I is a proper prime ideal since $1_A \notin I$.

2. Let us define the following map:

$$\psi : X(K) = \operatorname{Hom}(A, K) \longrightarrow X^{an}_{P,v}, \quad p \mapsto v \circ p.$$

We have to prove that $v \circ p : A \longrightarrow \mathbb{R}_{max}$ is an element of $X^{an}_{P,v}$. Clearly $v(p(0_A)) = v(0_K) = -\infty$ and $0_{\mathbb{R}_{max}} \leq v(p(a))$. The multiplicative property is also straightforward from the definition of p and v. Furthermore, we have

$$v(p(a+b)) = v(p(a)+p(b)) \leq \max\{v(p(a)), v(p(b))\}.$$

Therefore, all that remains is to show that $v \circ p$ is compatible with v. However, for $a \in K$, we have $v(p(a)) = v(a)$. Thus ψ is well-defined.

Next, for the injectivity, suppose that $x := v \circ P = v \circ Q := y \in X^{an}_{P,v}$ with $P, Q \in X(K) = \mathrm{Hom}(A, K)$. It follows from the first assertion that $J := x^{-1}(\{-\infty\}) = y^{-1}(\{-\infty\})$ is a proper prime ideal of A. We observe that $\mathrm{Ker}(P) \subseteq J$ because $x = v \circ P$. Since P is a K-rational point of X, $\mathrm{Ker}(P)$ is a maximal ideal. Thus $\mathrm{Ker}(P) = J$ and similarly $\mathrm{Ker}(Q) = J$. Then $\mathrm{Ker}(P) = \mathrm{Ker}(Q)$, hence $P = Q$ from Lemma 4.2.42.

□

4.3 Construction of hyper-schemes

In this section, we study several notions of algebraic geometry over hyper-structures from the scheme theoretic point of view. In the first subsection, we prove that several classical results in commutative algebra can be generalized to hyper-structures. By means of these results, in the second subsection, we construct an integral hyper-scheme and prove that $\Gamma(X, \mathcal{O}_X) \simeq R$ for an affine integral hyper-scheme $(X, \mathcal{O}_X)$. Then, we propose a definition for the Hasse-Weil zeta function of an integral hyper-scheme and explain how this definition generalizes the classical notion (cf. §4.3.3). Finally, we link the theories of semi-schemes and hyper-schemes using the symmetrization process of §3.

4.3.1 Analogues of classical lemmas

In this subsection, we reformulate several basic results in commutative algebra in terms of hyperrings. Throughout this subsection, we denote by R a hyperring and by $V(I)$ the set of of prime hyperideals of R containing a hyperideal I. We also denote by $Nil(R)$ the intersection of all prime hyperideals of R.

Lemma 4.3.1. *Let $I \subseteq R$ be a hyperideal. Then the following set:*

$$\sqrt{I} := \{r \in R \mid \exists n \in \mathbb{N} \text{ such that } r^n \in I\}$$

is a hyperideal.

Proof. Trivially we have $0 \in \sqrt{I}$. Suppose that $a \in \sqrt{I}$, then $a^n \in I$ for some $n \in \mathbb{N}$. Since I is a hyperideal, for $r \in R$, we have $r^n a^n = (ra)^n \in I$. It follows that $ra \in \sqrt{I}$. Clearly, $(-a)^n$ is either a^n or $-a^n$. Since both a^n and $-a^n$ are in I, it follows that $-a \in \sqrt{I}$. Finally, suppose that $a, b \in \sqrt{I}$ and $a^n, b^m \in I$. Then, for $l \geq (n+m)$, we have $(a+b)^l \subseteq \sum \binom{l}{k} a^k b^{l-k} \subseteq I$. This implies that $(a+b) \subseteq \sqrt{I}$; therefore, $\sqrt{I}$ is a hyperideal. ☐

Remark 4.3.2. *In general, a hyperring does not satisfy the doubly distributive property (cf. [50, pp 13-14]), in other words, the following identity:*

$$(a+b)(c+d) = ac + ad + bc + bd$$

is in general not fulfilled. Instead, the following identity:

$$(a+b)(c+d) \subseteq ac + ad + bc + bd$$

holds.

Lemma 4.3.3. *Let R be a hyperring and I a hyperideal of R. Then*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}.$$

Proof. Suppose that $a \in \sqrt{I}$, then $a^n \in I \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in V(I)$. Since $\mathfrak{p}$ is a prime hyperideal, it follows that $a \in \mathfrak{p}$; hence, $\sqrt{I} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in V(I)$.
Conversely, suppose that $f \in \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$ and $f \notin \sqrt{I}$. This implies that

$$S := \{1, f, f^2, ...\} \cap I = \emptyset.$$

Let Σ be the set of hyperideals J of R such that $S \cap J = \emptyset$ and $I \subseteq J$. Then $\Sigma \neq \emptyset$ since we have $\sqrt{I} \in \Sigma$. By Zorn's lemma (ordered by inclusion), Σ has a maximal element $\mathfrak{q}$. Then $\mathfrak{q}$ is a prime hyperideal. Indeed, by definition, $\mathfrak{q}$ is a hyperideal. Therefore, all we have to prove is that $\mathfrak{q}$ is prime. One can easily check, for $x \in R$, the following set:

$$\mathfrak{q} + xR := \bigcup \{a + b \mid a \in \mathfrak{q}, b \in xR\}$$

is a hyperideal. If $x, y \notin \mathfrak{q}$ then $\mathfrak{q}$ is properly contained in $\mathfrak{q} + xR$ and $\mathfrak{q} + yR$. Thus, $\mathfrak{q} + xR$, $\mathfrak{q} + yR \notin \Sigma$ from the maximality of $\mathfrak{q}$ in Σ. It follows that $f^n \in \mathfrak{q} + xR$ and $f^m \in \mathfrak{q} + yR$ for some $n, m \in \mathbb{N}$. In other words, $f^n \in a_1 + xr_1$, $f^m \in a_2 + yr_2$ for some $a_1, a_2 \in \mathfrak{q}$ and $r_1, r_2 \in R$. Therefore, we have

$$f^{n+m} \in (a_1 + xr_1)(a_2 + yr_2) \subseteq a_1 a_2 + a_1 y r_2 + a_2 x r_1 + x y r_1 r_2 \subseteq \mathfrak{q} + xyR.$$

This implies that $xy \notin \mathfrak{q}$ because if $xy \in \mathfrak{q}$ then $f^{n+m} \in \mathfrak{q}$, and we assumed that $f^l \notin \mathfrak{q}$ for all $l \in \mathbb{N}$. It follows that $\mathfrak{q}$ is a prime hyperideal containing I such that $S \cap \mathfrak{q} = \emptyset$. However, this is impossible since we took $f \in \cap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. This completes the proof. $\square$

For a family $\{X_\alpha\}_{\alpha \in J}$ of subsets $X_\alpha \subseteq R$, we denote by $< X_\alpha >_{\alpha \in J}$ the smallest hyperideal of R containing $(\bigcup_{\alpha \in J} X_\alpha)$. Note that $< X_\alpha >_{\alpha \in J}$ always exists since an intersection of hyperideals is a hyperideal as in the classical case. We call $< X_\alpha >_{\alpha \in J}$ the hyperideal generated by $\{X_\alpha\}_{\alpha \in J}$.

Lemma 4.3.4. *Let J be an index set.*

1. *Let $h \in R$. Then the hyperideal generated by h is*

$$hR := \{hr \mid r \in R\}. \tag{4.3.1}$$

2. *Suppose that I_i is the principal hyperideal generated by an element $h_i \in R$ for each $i \in J$. Then*

$$< I_i >_{i \in J} = \{r \in R \mid r \in \sum_{i=1}^{n} b_i h_i, b_i \in R, i \in J, n \in \mathbb{N}\}. \tag{4.3.2}$$

3. *Let $\{I_i\}_{i \in J}$ be a family of hyperideals $I_j \subseteq R$. Then*

$$< I_i >_{i \in J} = \{r \in R \mid r \in \sum_{i=1}^{n} b_i h_i, b_i \in R, h_i \in I_i, i \in J, n \in \mathbb{N}\}. \tag{4.3.3}$$

Proof. 1. Trivially hR is a hyperideal, and any hyperideal I containing h should contain hR by definition. It follows that $< h >= hR$.

2. Let $I := \{r \in R \mid r \in \sum b_i h_i, b_i \in R\}$ be the right hand side of (4.3.2). Then any hyperideal containing all I_i should contain I since a hyperideal is closed under an addition. Thus, it is enough to show that I is a hyperideal. In fact, we have $0 = 0 \cdot h_i \in I$. Suppose that $a \in I$, then $a \in \sum b_i h_i$ for some $b_i \in R$. Therefore, for $r \in R$, we have $ra \in \sum r b_i h_i$ and $-a \in \sum (-b_i) h_i$. Finally, suppose that $a, b \in I$ with $a \in \sum b_i h_i$ and $b \in \sum c_i h_i$. It follows that $a + b \subseteq \sum (b_i + c_i) h_i \subseteq I$. Hence I is a hyperideal.

3. The proof is similar to the above case.

□

Let $R^\times := \{r \in R \mid rr' = 1 \text{ for some } r' \in R\}$ and $J(R)$ be the intersection of all maximal hyperideals of R. By a maximal hyperideal of R we mean a hyperideal $\mathfrak{m} \subsetneq R$ which is properly contained in no other hyperideal but R. The following lemma has been proven in [15].

Lemma 4.3.5. *([15, Proposition 2.12, 2.13, 2.14])*

1. $x \in J(R) \Longleftrightarrow (1 - xy) \subseteq R^\times \quad \forall y \in R$.

2. *For any hyperideal $I \subsetneq R$, we have $V(I) \neq \emptyset$.*

One imposes the Zariski topology on the set $\operatorname{Spec} R$ of prime hyperideals of R as in the classical case (cf. §1.1.2). In what follows, we consider $X = \operatorname{Spec} R$ as a topological space equipped with the Zariski topology. Then, as in classical algebraic geometry, we have the following.

Proposition 4.3.6. *$X = \operatorname{Spec} R$ is a disconnected topological space if and only if R has a (multiplicative) idempotent element different from $0, 1$.*

Proof. Suppose that $e \neq 0, 1$ is an idempotent element of R. Then we have $e^2 = e$, and it follows that $0 \in e(e - 1)$. Therefore there is an element $f \in e - 1$ such that $ef = 0$. Moreover, $f \neq 0$ since $e \neq 1$. Similarly, f can not be 1 since $ef = 0$ and $e \neq 0$. Together with Lemma 4.3.5, it follows that $V(e)$ and $V(f)$ are non-empty subsets of X. Since $ef = 0$, we have $X = \operatorname{Spec} R = V(e) \bigcup V(f)$. Moreover, $V(e) \bigcap V(f) = \emptyset$. Indeed, if $p \in V(e) \bigcap V(f)$, then $e, f \in p$. This implies that $-e, -f, (f - e) \subseteq p$. However, we have $f \in e - 1 = -1 + e \Longleftrightarrow -1 \in f - e$. Therefore, we should have $1 \in p$ and it is impossible. It follows that $\{V(e)^c, V(f)^c\}$ becomes the disjoint open cover of X, hence X is disconnected.
Conversely, suppose that $X = \operatorname{Spec} R = U_1 \bigcup U_2$, where U_1 and U_2 are disjoint open subsets of X. This means that U_1 and U_2 are also closed. Therefore, we may assume

$$X = \operatorname{Spec} R = V(I) \bigcup V(J) = V(IJ), \quad V(I) \bigcap V(J) = V(< I, J >) = \emptyset$$

for some hyperideals I and J (cf. Proposition 1.1.19). Let $Nil(R) := \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. It follows from Lemma 4.3.1 and 4.3.3 that $Nil(R)$ is the set of all nilpotent elements of R. Since $V(IJ) = X = V(Nil(R))$, we have $\sqrt{IJ} = Nil(R)$ from Lemma 4.3.3. Moreover, the fact $V(< I, J >) = \emptyset$ implies that $\sqrt{< I, J >}$ contains 1. Otherwise, $\sqrt{< I, J >}$ does not contain any unit element, and $V(< I, J >) \neq \emptyset$ from Lemma 4.3.5. It follows that $1 \in \sqrt{< I, J >}$, hence $1 \in < I, J >$. From Lemma 4.3.4, there exist $a \in I$ and $b \in J$ such that $1 \in a + b$. However, we also have $a, b \notin Nil(R)$. Indeed, suppose that $a \in Nil(R)$. Then $a \in \mathfrak{p}$ for all $\mathfrak{p} \in X$, but this implies that for $\mathfrak{p} \in V(J)$, $\mathfrak{p}$ contains both a and b. It follows that $1 \in (a + b) \subseteq \mathfrak{p}$. Therefore, we have $a, b \notin Nil(R)$.

Next, since $V(a) \supseteq V(I) \iff D(a) \subseteq (V(I))^c = V(J)$, we have $D(a) \subseteq V(J)$, $D(b) \subseteq V(I)$. This implies $(V(a) \cup V(b))^c = D(a) \cap D(b) \subseteq V(I) \cap V(J) = \emptyset$, thus $V(a) \cup V(b) = X$. Suppose that $A =< a >$ and $B =< b >$. Then $ab \in A \cap B$, and it follows that $V(a) \cup V(b) = V(A) \cup V(B) = V(AB)$. Thus we have $AB \subseteq \sqrt{AB} = Nil(R)$. Therefore, $ab \in Nil(R)$, in turn, $(ab)^n = a^n b^n = 0$ for some $n \in \mathbb{N}$. However, $a^n, b^n \neq 0$ since $a, b \notin Nil(R)$. We observe the following:

$$1 \in a + b \Longrightarrow 1 \in (a + b)^n \subseteq \sum d_k a^k b^{n-k} \Longrightarrow 1 \in (a^n + b^n) + abf, \tag{4.3.4}$$

for some $f \in R$. Since $ab \in Nil(R)$, clearly $abf \in Nil(R) \subseteq J(R)$. It follows from (4.3.4) that

$$1 \in \alpha + abf, \qquad \text{for some } \alpha \in a^n + b^n.$$

This implies that $\alpha \in 1 - abf$. But since $abf \in J(R)$, from Lemma 4.3.5, α is a unit. Let $\beta = \alpha^{-1}$. Then

$$\alpha \in a^n + b^n \Longrightarrow \alpha\beta = 1 \in (a^n + b^n)\beta = a^n\beta + b^n\beta \Longrightarrow b^n\beta \in 1 - a^n\beta.$$

One observes that $a^n\beta, b^n\beta \neq 0$ since $a^n, b^n \neq 0$ and β is a unit. Furthermore, $a^n\beta, b^n\beta \neq 1$. Since $a^n\beta = 1 \iff a^n = \beta^{-1} = \alpha$, it would imply that $a^n = \alpha \in I$. Therefore $V(I) = \emptyset$. But we assumed that $V(I) \neq \emptyset$. Finally let us define an element

$e = a^n\beta$. Then we know, from the above, $e \neq 0, 1$. Furthermore, we have

$$e^2 - e = e(e-1) = a^n\beta(a^n\beta - 1).$$

Since we have $b^n\beta \in 1 - a^n\beta$ and $b^n\beta a^n\beta = a^n b^n \beta^2 = 0$, it follows that

$$0 \in e(e-1) = e^2 - e.$$

Hence, from the uniqueness of an inverse, we have $e^2 = e$ and $e \neq 0, 1$. □

Proposition 4.3.7. *$X = \operatorname{Spec} R$ is irreducible if and only if $Nil(R)$ is a prime hyperideal.*

Proof. Suppose that X is irreducible. If $ab \in Nil(R)$ then from Lemma 4.3.3 there exists $n \in \mathbb{N}$ such that $(ab)^n = a^n b^n = 0$. It follows that $X = V(a^n) \cup V(b^n)$. We know that $V(a^n) = V(a)$ and $V(b^n) = V(b)$. Since X is irreducible, we have either $X = V(a)$ or $X = V(b)$, it follows that $a \in Nil(R)$ or $b \in Nil(R)$. Conversely, suppose that $X = V(I) \cup V(J)$. Since $Nil(R)$ is prime, we should have $Nil(R) \in V(I)$ or $Nil(R) \in V(J)$. This implies that $X = V(I)$ or $X = V(J)$. Therefore, X is irreducible. □

4.3.2 Construction of an integral hyper-scheme

In classical algebraic geometry, a scheme is a pair $(X, \mathcal{O}_X)$ of a topological space X and the structure sheaf $\mathcal{O}_X$ on X. The implementation of the notion of structure sheaf is essential to link local and global algebraic data.

Let A be a commutative ring and $(X = \operatorname{Spec} A, \mathcal{O}_X)$ be an affine scheme. One of important results in classical algebraic geometry is the following:

$$\mathcal{O}_X(X) \simeq A. \tag{4.3.5}$$

In other worlds, a commutative ring A can be understood as the ring of functions on the topological space $X = \operatorname{Spec} A$. When we directly generalize the construction of the structure sheaf of a commutative ring to a hyperring, (4.3.5) no longer holds (cf. Example 4.3.12). Furthermore, in this case, $\mathcal{O}_X$ does not even have to be a sheaf of hyperrings (cf. Remark 4.3.8). To this end, we construct the structure sheaf on the topological space $X = \operatorname{Spec} R$ only when R is a hyperring without (multiplicative) zero-divisors. We follow the classical construction.

Let R be a hyperring and $X = \operatorname{Spec} R$. For an open subset $U \subseteq X$, we define

$$\mathcal{O}_X(U) := \{s : U \to \bigsqcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}\} \tag{4.3.6}$$

where $s \in \mathcal{O}_X(U)$ are sections such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ which also satisfying the following property: for each $\mathfrak{p} \in U$, there exist a neighborhood $V_{\mathfrak{p}} \subseteq U$ of $\mathfrak{p}$ and $a, f \in R$ such that

$$\forall \mathfrak{q} \in V_{\mathfrak{p}}, \quad f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = \frac{a}{f} \text{ in } R_{\mathfrak{q}}. \tag{4.3.7}$$

A restriction map $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$ is given by sending s to $s \circ i$, where $i : V \hookrightarrow U$ is an inclusion map. Then, clearly $\mathcal{O}_X$ is a sheaf of sets on X. Moreover, one can define the multiplication $s \cdot t$ of sections $s, t \in \mathcal{O}_X$ as follows:

$$s \cdot t : U \to \bigsqcup R_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto s(\mathfrak{p})t(\mathfrak{p}). \tag{4.3.8}$$

Equipped with the above multiplication, one can easily see that $\mathcal{O}_X$ becomes a sheaf of (multiplicative) monoids on X. Furthermore, $\mathcal{O}_X(U)$ is equipped with the following hyper-structure:

$$s + t = \{r \in \mathcal{O}_X(U) \mid r(\mathfrak{p}) \in s(\mathfrak{p}) + t(\mathfrak{p}), \quad \forall \mathfrak{p} \in U\}. \tag{4.3.9}$$

Remark 4.3.8. *This construction is essentially same as in [39]. However, in [39], the proof is incomplete in the sense that the authors did not prove that (4.3.9) is* *associative and distributive with respect to* (4.3.8). *Moreover, the main purpose of this subsection is to recover a hyperring R as the hyperring of global sections on a topological space* $\operatorname{Spec} R$ *while the authors of [39] have not considered such property. In Theorem 4.3.11, we prove that when R does not have (multiplicative) zero-divisors, $\mathcal{O}_X$ is indeed the sheaf of hyperrings, and $\mathcal{O}_X(X) \simeq R$.*

Definition 4.3.9. *A hyperring R is called a hyperdomain if R does not have (multiplicative) zero-divisors. In other words, for $x, y \in R$, if $xy = 0$ then either $x = 0$ or $y = 0$.*

Let R be a hyperdomain and $S := R^{\times}$ the largest multiplicative subset of R. Then, clearly $K := \operatorname{Frac}(R) = S^{-1}R$ is a hyperfield and the canonical homomorphism $S^{-1} : R \longrightarrow K$ of hyperrings sending r to $\frac{r}{1}$ is strict and injective.
Let $S_f = \{1, f \neq 0, ..., f^n, ...\}$ be the multiplicative subset of R and $R_f := S_f^{-1}R$, then we have the canonical homomorphisms of hyperrings $R \hookrightarrow R_f \hookrightarrow K$ which are injective and strict. Therefore, in the sequel, we consider R_f as the hyperring extension of R and K as the hyperring extension of both R and R_f via the above canonical maps. For $\mathfrak{p} \in \operatorname{Spec} R$, we denote by $R_{\mathfrak{p}}$ the hyperring $S^{-1}R$, where $S = R \backslash \mathfrak{p}$.

Lemma 4.3.10. *Let A be a set equipped with the two binary operations:*

$$+_A : A \times A \longrightarrow P^*(A), \quad \cdot_A : A \times A \longrightarrow A,$$

where $P^*(A)$ is the set of nonempty subsets of A. Suppose that R is a hyperring and there exists a set bijection $\varphi : A \longrightarrow R$ such that $\varphi(a +_A b) = \varphi(a) + \varphi(b)$ and $\varphi(a \cdot_A b) = \varphi(a)\varphi(b)$. Then, A is a hyperring isomorphic to R.

Proof. The proof is straightforward. For example, $\varphi^{-1}(0_R) := 0_A$ is the neutral element. In fact, for $a \in A$, we have $\varphi(0_A +_A a) = \varphi(0_A) + \varphi(a) = 0_R + \varphi(a) = \varphi(a)$. Since φ is bijective, it follows that $0_A +_A a = a \ \forall a \in A$. Similarly, $1_A := \varphi^{-1}(1_R)$ is the identity element. For $a \in A$, we can write $-\varphi(a) = \varphi(b)$ for some $b \in A$. Then, we have $\varphi(a +_A b) = \varphi(a) + \varphi(b) = \varphi(a) - \varphi(a)$. It follows that $0_R \in \varphi(a +_A b)$, hence $0_A \in a +_A b$. The other properties can be easily checked. Clearly, if A is a hyperring, then A and R are isomorphic via φ. □

Theorem 4.3.11. *Let R be a hyperdomain, $K = \mathrm{Frac}(R)$, and $X = \operatorname{Spec} R$. Let $\mathcal{O}_X$ be the sheaf of multiplicative monoids on X as in (4.3.6), equipped with the hyper-addition (4.3.9). Then, the following holds*

1. *$\mathcal{O}_X(D(f))$ is a hyperring isomorphic to R_f. In particular, if $f = 1$, we have $R \simeq \mathcal{O}_X(X)(= \Gamma(X, \mathcal{O}_X))$.*
2. *For each open subset U of X, $\mathcal{O}_X(U)$ is a hyperring. More precisely, $\mathcal{O}_X(U)$ is isomorphic to the following hyperring:*

$$\mathcal{O}_X(U) \simeq Y(U) := \{u \in K \mid \forall \mathfrak{p} \in U, u = \frac{a}{b} \textit{ for some } b \notin \mathfrak{p}\}.$$

   Moreover, by considering the canonical map $R_f \hookrightarrow K$, we have

$$\mathcal{O}_X(U) \simeq \bigcap_{D(f) \subseteq U} \mathcal{O}_X(D(f)).$$

3. *For each $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ exists and is isomorphic to $R_{\mathfrak{p}}$.*

Proof. 1. The proof is similar to the classical case (cf. [20]). Consider the following map:

$$\psi : R_f \to \mathcal{O}_X(D(f)), \quad \frac{a}{f^n} \mapsto s, \text{ where } s(\mathfrak{p}) = \frac{a}{f^n} \text{ in } R_{\mathfrak{p}}. \tag{4.3.10}$$

Clearly, ψ is well-defined since the map s defined as in (4.3.10) satisfies the condition (4.3.7). It also follows from the definition that

$$\psi(\frac{a}{f^n} \cdot \frac{b}{f^m}) = \psi(\frac{a}{f^n}) \cdot \psi(\frac{b}{f^m}), \quad \psi(\frac{a}{f^n} + \frac{b}{f^m}) \subseteq \psi(\frac{a}{f^n}) + \psi(\frac{b}{f^m}).$$

First, we claim that ψ is one-to-one. Indeed, suppose that $\psi(\frac{a}{f^n}) = \psi(\frac{b}{f^m})$. Then,

$\frac{a}{f^n} = \frac{b}{f^m}$ as elements of $R_{\mathfrak{p}}$ $\forall \mathfrak{p} \in D(f)$. Hence there is an element $h \notin \mathfrak{p}$ such that $hf^m a = hf^n b$ in R. This implies that $0 \in hf^m a - hf^n b = h(f^m a - f^n b)$. However, since $h \notin \mathfrak{p}$ (hence, $h \neq 0$) and R is a hyperdomain, it follows that $f^m a = f^n b$. This implies that $\frac{a}{f^n} = \frac{b}{f^m}$ in R_f, thus ψ is one-to-one.

Next, we claim that ψ is onto. Take $s \in \mathcal{O}_X(D(f))$. Then, we can cover $D(f)$ with open sets V_i so that s is represented by a quotient $\frac{a_i}{g_i}$ on V_i with $g_i \notin \mathfrak{p}$ $\forall \mathfrak{p} \in V_i$ from (4.3.7). Since open subsets of the form $D(h)$ form a basis, we may assume that $V_i = D(h_i)$ for some $h_i \in R$. Let (h_i) and (g_i) be the hyperideals generated by h_i and g_i. Since s is represented by $\frac{a_i}{g_i}$ on $D(h_i)$, we have $D(h_i) \subseteq D(g_i)$; hence, $V((h_i)) \supseteq V((g_i))$. It follows from Lemma 4.3.3 that $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$. In particular, $h_i^n \in (g_i)$ for some $n \in \mathbb{N}$. Then, from Lemma 4.3.4, we have $h_i^n = cg_i$ for some $c \in R$. Hence $\frac{a_i}{g_i} = \frac{ca_i}{h_i^n}$. If we replace h_i by h_i^n (since $D(h_i) = D(h_i^n)$) and a_i by ca_i, we may assume that $D(f)$ is covered by the open subsets $D(h_i)$ on which s is represented by $\frac{a_i}{h_i}$. Moreover, as in the classical case, we observe that $D(f)$ can be covered by finitely many $D(h_i)$. In fact,

$$D(f) \subseteq \bigcup D(h_i) \Longleftrightarrow V((f)) \supseteq \bigcap V((h_i)). \tag{4.3.11}$$

Let $I_i = (h_i)$, $I =< I_i >$, and $J = (f)$. Then, (4.3.11) can be written as follows:

$$\bigcap V(I_i) = V(I), \quad D(f) \subseteq \bigcup D(h_i) \Longleftrightarrow V(J) \supseteq V(I). \tag{4.3.12}$$

It follows from Lemma 4.3.3 that $\sqrt{J} \subseteq \sqrt{I}$, thus $f^n \in I$ for some $n \in \mathbb{N}$. Then, from Lemma 4.3.4, we have

$$f^n \in \sum_{i=1}^{r} b_i h_i \text{ for some } b_i \in R. \tag{4.3.13}$$

We claim that $D(f)$ can be covered by $D(h_1) \cup ... \cup D(h_r)$. Indeed, this is equivalent to

$$V((f)) \supseteq \bigcap_{i=1}^{r} V((h_i)) = V(< (h_i) >_{i=1,...,r}).$$

Let $I :=< h_i >_{i=1,...,r}$. Suppose that $\mathfrak{p} \in V(I)$. Since $I \subseteq \mathfrak{p}$, it follows from (4.3.13) that $f^n \in \mathfrak{p}$, hence $f \in \mathfrak{p}$. This implies that $(f) \subseteq \mathfrak{p}$, thus $\mathfrak{p} \in V((f))$. From now on, we fix the elements $h_1, ..., h_r$ such that $D(f) \subseteq D(h_1) \cup ... \cup D(h_r)$. Then, on $D(h_ih_j)$, we have two elements $\frac{a_i}{h_i}, \frac{a_j}{h_j}$ of $R_{h_ih_j}$ which represent the same element s. It follows from the injectivity of ψ we proved, applied to $D(h_ih_j)$, one should have $\frac{a_i}{h_i} = \frac{a_j}{h_j}$ in $R_{h_ih_j}$. Therefore, $(h_ih_j)^n h_j a_i = (h_ih_j)^n h_i a_j$ for some $n \in \mathbb{N}$. However, since R is a hyperdomain, we have $(h_ih_j)^n \neq 0$. It follows that $h_j a_i = h_i a_j$ $\forall i, j = 1, ..., r$ from the uniqueness of an additive inverse.
Write $f^n \in \sum_{i=1}^{r} b_i h_i$ as in (4.3.13). Then, for each $j \in \{1, ..., r\}$, we have

$$f^n a_j \in (\sum_i b_i h_i) a_j = \sum_i b_i a_j h_i = \sum_i b_i a_i h_j = (\sum_i b_i a_i) h_j.$$

It follows that for each $j = 1, ..., r$, there exists $\beta_j \in \sum_i b_i a_i$ such that $f^n a_j = \beta_j h_j$. Hence, we have

$$\frac{\beta_j}{f^n} = \frac{a_j}{h_j} \quad \text{on } D(h_j). \tag{4.3.14}$$

However, $\beta_i = \beta_j$ $\forall i, j = 1, ..., r$. Indeed, on $D(h_ih_j)$, we proved that $\frac{a_j}{h_j} = \frac{a_i}{h_i}$. Together with (4.3.14) and the injectivity of ψ, we have

$$\frac{\beta_j}{f^n} = \frac{a_j}{h_j} = \frac{a_i}{h_i} = \frac{\beta_i}{f^n} \text{ on } D(h_ih_j).$$

Therefore, $\exists m \in \mathbb{N}$ such that $(h_ih_j)^m f^n \beta_j = (h_ih_j)^m f^n \beta_i$. Equivalently, we have $0 \in (h_ih_j)^m f^n (\beta_j - \beta_i)$. However, we know that $(h_ih_j)^m, f^n \neq 0$ since $h_ih_j, f \neq 0$ and R is a hyperdomain. It follows that $0 \in \beta_i - \beta_j$, thus $\beta_i = \beta_j$ from the uniqueness of an additive inverse. Let β be this common value β_i. Then, we have $f^n a_j = \beta h_j$ $\forall j = 1, ..., r$. Therefore, $\frac{\beta}{f^n} = \frac{a_j}{h_j}$ on $D(h_j)$. In other words, $\psi(\frac{\beta}{f^n}) = s$. This shows that ψ is onto. This, however, does not complete the proof. We need to show that $\psi(ab) = \psi(a)\psi(b)$ and $\psi(a+b) = \psi(a)+\psi(b)$, then the result follows from Lemma 4.3.10. Clearly, we have $\psi(ab) = \psi(a)\psi(b)$ and $\psi(\frac{a}{f^n} + \frac{b}{f^m}) \subseteq \psi(\frac{a}{f^n}) + \psi(\frac{b}{f^m})$. We show the following:

$$\psi(\frac{a}{f^n}) + \psi(\frac{b}{f^m}) \subseteq \psi(\frac{a}{f^n} + \frac{b}{f^m}).$$

Let $s = \psi(\frac{a}{f^n})$, $t = \psi(\frac{b}{f^m})$. Then, we have

$$s + t = \{r \in \mathcal{O}_X(D(f)) \mid r(\mathfrak{p}) \in s(\mathfrak{p}) + t(\mathfrak{p}) \quad \forall \mathfrak{p} \in D(f)\}.$$

For $r \in s+t$, since ψ is onto, $r = \psi(\frac{c}{f^l})$ for some $\frac{c}{f^l} \in R_f$. It follows from $r(\mathfrak{p}) \in s(\mathfrak{p}) + t(\mathfrak{p})$ that $\frac{c}{f^l} \in \frac{a}{f^n} + \frac{b}{f^m}$ in $R_{\mathfrak{p}}$, and this is equivalent to the following:

$$\frac{c}{f^l} = \frac{d}{f^{n+m}} \text{ for some } d \in (af^m + bf^n) \text{ in } R_{\mathfrak{p}}.$$

Therefore, $ucf^{n+m} = udf^l$ for some $u \in R\backslash\mathfrak{p}$. Since $u \neq 0$, we have $cf^{n+m} = df^l$. Equivalently, $\frac{c}{f^l} = \frac{d}{f^{n+m}}$ in R_f. However, $\frac{d}{f^{n+m}} \in \frac{a}{f^n} + \frac{b}{f^m}$, therefore $s + t \subseteq \psi(\frac{a}{f^n} + \frac{b}{f^m})$. This shows the other inclusion. The conclusion follows from Lemma 4.3.10.

2. One can easily see that $Y(U)$ is a hyperring (in fact, a sub-hyperring of K). We show that there exists a bijection φ of sets from $\mathcal{O}_X(U)$ to $Y(U)$ such that $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$. Then, the first assertion will follow from Lemma 4.3.10. Indeed, if $s \in \mathcal{O}_X(U)$, then from the same argument (in the proof of 1), we can find a cover $U = \bigcup D(h_i)$ such that $s = \frac{a_i}{h_i}$ on $D(h_i)$. However, since R has no (multiplicative) zero-divisor, $X = \operatorname{Spec} R$ is irreducible from Proposition 4.3.7. Thus, $D(h_i) \cap D(h_j) \neq \emptyset \ \forall i, j$. It follows that $\frac{a_i}{h_i} = \frac{a_j}{h_j}$ on $D(h_i) \cap D(h_j)$, equivalently $0 \in s_{ij}(a_ih_j - h_ia_j)$ for some $s_{ij} \neq 0 \in R$. Since

$s_{ij} \neq 0$ and R is a hyperdomain, it follows that

$$0 \in (a_i h_j - h_i a_j) \Longleftrightarrow a_i h_j = a_j h_i \Longleftrightarrow \frac{a_i}{h_i} = \frac{a_j}{h_j} \text{ as elements of } K = \mathrm{Frac}(R).$$

Let $u = \frac{a}{b}$ be this common value in K. Then, for each $\mathfrak{p} \in U$, we have $\mathfrak{p} \in D(h_i)$ for some $h_i \notin \mathfrak{p}$ and $\frac{a}{b} = \frac{a_i}{h_i}$ on $D(h_i)$. It follows that $u \in Y(U)$. Since $\mathcal{O}_X$ is a sheaf, $u \in Y(U)$ is uniquely determined by s. We let $\varphi(s) := u$. Then, we have

$$\varphi : \mathcal{O}_X(U) \longrightarrow Y(U) := \{u \in K \mid \forall \mathfrak{p} \in U, u = \frac{a}{b}, b \notin \mathfrak{p}\} \subseteq K.$$

φ is well-defined and one-to-one since $\mathcal{O}_X$ is a sheaf (of sets). We claim that φ is onto. In fact, for $u = \frac{a}{b} \in Y(U)$, we define $s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ such that $s(\mathfrak{p}) = \frac{a}{b} = \frac{a'}{b'}$ for $b' \notin \mathfrak{p}$ from the definition of $Y(U)$. Then $s \in \mathcal{O}_X(U)$. Next, it follows from the definition that $\varphi(s \cdot t) = \varphi(s) \cdot \varphi(t)$. Furthermore, we have $\varphi(s+t) \subseteq \varphi(s) + \varphi(t)$. Indeed, we have

$$\alpha \in s + t \Longleftrightarrow \alpha(\mathfrak{p}) \in s(\mathfrak{p}) + t(\mathfrak{p}) \quad \forall \mathfrak{p} \in U. \tag{4.3.15}$$

However, since φ is bijective, each section is globally represented by an element of K. Suppose that α, s, t are globally represented by $\frac{g}{f}, \frac{a}{h}, \frac{b}{m}$ respectively. Then, (4.3.15) is equivalent to the following:

$$\alpha(\mathfrak{p}) \in s(\mathfrak{p}) + t(\mathfrak{p}) \Longleftrightarrow \frac{g}{f} \in \frac{a}{h} + \frac{b}{m} \Longleftrightarrow \varphi(\alpha) \in \varphi(s) + \varphi(t).$$

Conversely, for $\frac{g}{f} \in \varphi(s) + \varphi(t) = \frac{a}{h} + \frac{b}{m}$, we have $\alpha \in \mathcal{O}_X(U)$ such that $\alpha(\mathfrak{p}) = \frac{g}{f}$ at $R_{\mathfrak{p}}$. It follows that $\alpha \in s + t$, and $\varphi(s) + \varphi(t) \subseteq \varphi(s+t)$. This shows that $\mathcal{O}_X(U)$ is isomorphic to $Y(U)$ via φ from Lemma 4.3.10.

Next, we prove the second assertion. For $D(f) \subseteq U$, we have $Y(U) \subseteq Y(D(f)) \subseteq K$. This implies that $Y(U) \subseteq \bigcap_{D(f) \subseteq U} Y(D(f))$. Conversely, suppose that $u = \frac{a}{b} \in \bigcap_{D(f) \subseteq U} Y(D(f))$. Then, for each $\mathfrak{p} \in U$, we have $\mathfrak{p} \in D(f)$ for some

$D(f) \subseteq U$. Since $u \in Y(D(f))$, u can be written as $\frac{x}{y}$ so that $y \notin \mathfrak{p}$. It follows that $u \in Y(U)$, and $Y(U) = \bigcap_{D(f)\subseteq U} Y(D(f))$. One observes that $Y(D(f)) = R_f \subseteq K$. Thus, under $\mathcal{O}_X(D(f)) \simeq R_f$, we have $\mathcal{O}_X(U) \simeq \bigcap_{D(f)\subseteq U} \mathcal{O}_X(D(f))$.

3. In general, direct limits do not exist in the category of hyperrings. Thus, one should be careful. Since open sets of the form $D(f)$ form a basis of X, it is enough to show that

$$\varinjlim_{D(f)\ni\mathfrak{p}} \mathcal{O}_X(D(f)) = R_{\mathfrak{p}}. \tag{4.3.16}$$

For each $f \in R$, let $\psi_f : \mathcal{O}_X(D(f)) \longrightarrow R_{\mathfrak{p}}$, $s \mapsto s(\mathfrak{p})$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_X(D(f)) \arrow[rr, "\rho"] \arrow[dr, "\psi_f"'] & & \mathcal{O}_X(D(g)) \arrow[dl, "\psi_g"] \\
& R_{\mathfrak{p}} &
\end{tikzcd},$$

where ρ is a restriction map of the structure sheaf $\mathcal{O}_X$. Let H be a hyperring and suppose that we have another commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_X(D(f)) \arrow[rr, "\rho"] \arrow[dr, "\psi_f"] \arrow[ddr, "\varphi_f"'] & & \mathcal{O}_X(D(g)) \arrow[dl, "\psi_g"'] \arrow[ddl, "\varphi_g"] \\
& R_{\mathfrak{p}} & \\
& H &
\end{tikzcd}.$$

Let us define the map ψ as follows:

$$\psi : R_{\mathfrak{p}} \longrightarrow H, \quad \frac{b}{t} \mapsto \varphi_t(\frac{b}{t}),$$

where $\frac{b}{t}$ is considered as an element of $\mathcal{O}_X(D(t))$ such that $\frac{b}{t}(\mathfrak{q}) = \frac{b}{t}$ in $R_{\mathfrak{q}}$ for each $\mathfrak{q} \in D(t)$. We show that ψ is a well-defined homomorphism of hyperrings. Then, the uniqueness of such map easily follows. Indeed, suppose that $\varphi : R_{\mathfrak{p}} \longrightarrow H$

is the homomorphism of hyperrings such that $\varphi_f = \varphi \circ \psi_f \ \forall f \in R$ for $\mathfrak{p} \in D(f)$. A section s of $\mathcal{O}_X(D(f))$ is represented by $\frac{b}{f^n}$ (from the first part of the proposition). Therefore, $\psi_f(s) = \psi_f(\frac{b}{f^n})$, and $\varphi_f(s) = \varphi \circ \psi_f(s) = \varphi \circ \psi_f(\frac{b}{f^n}) = \varphi(\frac{b}{f^n})$. However, we have $\varphi_f(s) = \psi \circ \psi_f(s) = \psi(\frac{b}{f})$. Thus, such ψ is unique if it exists.

Next, we show that ψ is well-defined. Indeed, if $\frac{b}{t} = \frac{b'}{t'}$, then we have $bt' = b't$ since R is a hyperdomain. It follows that $D(bt') = D(b't) \subseteq D(t)$, and we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_X(D(t)) \arrow[rr, "\rho"] \arrow[dr, "\varphi_t"'] & & \mathcal{O}_X(D(b't)) = \mathcal{O}_X(D(bt')) \arrow[dl, "\varphi_{b't}"] \\
& H &
\end{tikzcd}$$

From the similar argument with t', we have $\varphi_t(\frac{b}{t}) = \varphi_{t'}(\frac{b'}{t'})$. This shows that ψ does not depend on the choice of t, hence ψ is well-defined. For $\frac{a}{f}, \frac{b}{g} \in R_{\mathfrak{p}}$, by considering φ_{fg}, we have $\psi(\frac{a}{f}) = \varphi_{fg}(\frac{a}{f})$, $\psi(\frac{b}{g}) = \varphi_{fg}(\frac{b}{g})$, and $\psi(\frac{ab}{fg}) = \varphi_{fg}(\frac{ab}{fg})$. Thus, $\psi(\frac{a}{f}\frac{b}{g}) = \psi(\frac{a}{f})\psi(\frac{b}{g})$. Similarly, we have $\psi(\frac{a}{f} + \frac{b}{g}) \subseteq \psi(\frac{a}{f}) + \psi(\frac{b}{g})$. Hence, ψ is a homomorphism of hyperrings.

Finally, since $\{D(f)\}$ is a basis of X, we have

$$\varinjlim_{U \ni \mathfrak{p}} \mathcal{O}_X(U) = \varinjlim_{D(f) \ni \mathfrak{p}} \mathcal{O}_X(D(f)).$$

Therefore, we conclude that $\mathcal{O}_{X,\mathfrak{p}} \simeq R_{\mathfrak{p}}$.

□

When R is a hyperdomain, we call the pair $(X = \operatorname{Spec} R, \mathcal{O}_X)$ as in Theorem 4.3.11 an integral affine hyper-scheme. The following example shows that if R has zero divisors, then in general $R \neq \Gamma(X, \mathcal{O}_X)$.

Example 4.3.12. *Consider the following quotient hyperring R:*

$$R = \mathbb{Q} \oplus \mathbb{Q}/G, \ \textit{where } G = \{(1,1), (-1,-1)\}.$$

Then, $\operatorname{Spec} R = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ *with* $\mathfrak{p}_1 =< [(1,0)] >$ *and* $\mathfrak{p}_2 =< [(0,1)] >$*. Each* $\mathfrak{p}_j$ *becomes one point open and closed subset of* X *and the intersection* $\mathfrak{p}_1 \cap \mathfrak{p}_2$ *is empty. Furthermore, one can easily check that* $R_{\mathfrak{p}_i} \simeq \mathbb{Q}/H$*, where* $H = \{1, -1\}$*. Therefore,*

$$\Gamma(X, \mathcal{O}_X) \simeq (\mathbb{Q}/H) \oplus (\mathbb{Q}/H) \neq R.$$

Remark 4.3.13. *One can construct other examples of affine hyper-schemes* $X = \operatorname{Spec} R$ *for which* $R \neq \Gamma(X, \mathcal{O}_X)$*, but all such examples are disconnected. We do not have yet any example of a connected topological space* $X = \operatorname{Spec} R$ *with* $R \neq \Gamma(X, \mathcal{O}_X)$*. On the other hand, being connected is not a necessary condition. In fact, let* $A = \mathbb{Z}_{12}$ *and* $G = \{1, 5\} \subseteq (\mathbb{Z}_{12})^{\times}$*. Then, with the quotient hyperring* $R = A/G$*, the space* $X = \operatorname{Spec} R$ *is disconnected (consist of two points), however, one can easily check that* $R \simeq \Gamma(X, \mathcal{O}_X)$*.*

Let R be a hyperring, $X = \operatorname{Spec} R$, and $\mathcal{O}_X$ be the structure sheaf (of sets) of X. Then, as we previously mentioned in Remark 4.3.8, $\Gamma(X, \mathcal{O}_X)$ does not have to be a hyperring. Moreover, even if $\Gamma(X, \mathcal{O}_X)$ is a hyperring, Example 4.3.12 shows that the natural map $R \longrightarrow \Gamma(X, \mathcal{O}_X)$ is not even an injective map in general. By appealing to the classical construction of Cartier divisors, we define the presheaf $\mathcal{F}_X$ of hyperrings on $X = \operatorname{Spec} R$ which slightly generalizes $\mathcal{O}_X$ (cf. Remark 4.3.15 and Proposition 4.3.17).

Let $S := \{\alpha \in R \mid \alpha \text{ is not a zero-divisor}\}$. In other words, S is the set of regular elements of R. Then, $S \neq \emptyset$ since $1 \in S$. Furthermore, S is a multiplicative subset of R, therefore one can define $K := S^{-1}R$. In what follows, we denote by R a hyperring, and S, K as above. Note that by a sub-hyperring R of a hyperring L we mean a subset R of L which is a hyperring with the induced operations.

Lemma 4.3.14. *Let* S *be the set as above, and* $f \in S$*. Let* $\varphi : R \longrightarrow R_f$*,* $\varphi(a) = \frac{a}{1}$ *be the natural map of localization and* $\psi : R_f \longrightarrow K := S^{-1}R$ *be a homomorphism of hyperrings such that* $\psi(\frac{a}{f^n}) = \frac{a}{f^n}$*. Then* φ *and* ψ *are strict, injective homomorphisms* *of hyperrings.*

Proof. We only prove the case of φ since the proof for ψ is similar. If $\frac{a}{1} = \frac{b}{1}$ then $f^n a = f^n b$ for some $n \in \mathbb{N}$. This implies that $0 \in f^n(a-b)$. Hence, we have $c \in (a-b)$ such that $0 = f^n c$. Since $f^n \in S$ can not be a zero-divisor, we have $c = 0$, therefore $a = b$. This shows that φ is injective. Furthermore, if $\gamma \in \varphi(a) + \varphi(b) = \frac{a}{1} + \frac{b}{1}$, then $\gamma = \frac{t}{1}$ for some $t \in a + b$. Therefore $\gamma = \varphi(t)$, and φ is strict. $\square$

For each open subset U of $X = \operatorname{Spec} R$, we define the following set:

$$\mathcal{F}_X(U) := \{u \in K \mid \forall \mathfrak{p} \in U, u = \frac{a}{b}, b \in S \cap \mathfrak{p}^c\}. \tag{4.3.17}$$

In other words, $u \in K$ is an element of $\mathcal{F}_X(U)$ if u has a representative $\frac{a}{b}$ such that $b \notin \mathfrak{p}$ for each $\mathfrak{p} \in U$. The restriction map is given by the natural injection. i.e. if $V \subseteq U$, then we have $\mathcal{F}_X(U) \hookrightarrow \mathcal{F}_X(V)$. Then, one can easily observe that $\mathcal{F}_X(U)$ is a hyperring. Thus, $\mathcal{F}_X$ becomes a presheaf of hyperrings on $X = \operatorname{Spec} R$.

Remark 4.3.15. *It follows from Theorem 4.3.11 that when R is a hyperdomain, we have $\mathcal{O}_X(U) \simeq \mathcal{F}_X(U)$ for each open subset U of X. Therefore, in this case, $\mathcal{F}_X$ is indeed a sheaf of hyperring and $\mathcal{F}_X(X) = R$.*

Proposition 4.3.16. *Let R be a hyperring. If $X = \operatorname{Spec} R$ is irreducible, then $\mathcal{F}_X$ is a sheaf of hyperrings.*

Proof. Since $\mathcal{F}_X(U)$ is clearly a hyperring, we only have to prove that $\mathcal{F}_X$ is a sheaf. Suppose that $U = \bigcup V_i$ is an open covering of U. Firstly, if $s \in \mathcal{F}_X(U)$ is an element such that $s|_{V_i} = 0$ for all i, then we have to show that $s = 0$. However, this is clear since the restriction map is injective. Secondly, let $s_i \in \mathcal{F}_X(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i, j. Since X is irreducible, it follows that $V_i \cap V_j \neq \emptyset \ \forall i, j$. Moreover, the condition $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ means that $s_i = s_j$ as elements of K. Let s be this common element of K. Then, $\{s_i\}$ can be glued to s. Clearly, s is an element of $\mathcal{F}_X(U)$. $\square$

The following proposition shows that $\mathcal{F}_X$ behaves more nicely than $\mathcal{O}_X$ in some cases.

Proposition 4.3.17. *Let R be a hyperring and assume that $X = \operatorname{Spec} R$ is irreducible. Then, for $f \in S$, there exists a canonical injective and strict homomorphism $\varphi : R_f \longrightarrow \mathcal{F}_X(D(f))$. In particular, R is a sub-hyperring of $\mathcal{F}_X(X)$. Furthermore, if R has a unique maximal hyperideal, then $R \simeq \mathcal{F}_X(X)$.*

Proof. From Lemma 4.3.14, there exists a canonical injective and strict homomorphism $\psi : R_f \longrightarrow K$. From the definition of $\mathcal{F}_X(D(f))$, one sees that the image of ψ lies in $\mathcal{F}_X(D(f)) \subseteq K$. Therefore, ψ becomes our desired φ. When R has a unique maximal hyperideal, we have to show that any element u of $\mathcal{F}_X(X)$ is of the form $\frac{a}{1}$ for some $a \in R$. Suppose that $\mathfrak{m}$ is the maximal ideal of R. Then, $u \in \mathcal{F}_X(X)$ means that $u = \frac{a}{b}$ for some $b \in S - \mathfrak{m}$. Since $\mathfrak{m}$ is the only maximal hyperideal of R, it follows from Lemma 4.3.4 that $1 \in a + b$ for some $a \in \mathfrak{m}$. Therefore, $b \in 1 - a$ and b is a unit by Lemma 4.3.5. Thus, $u = \frac{a}{b} = \frac{ab^{-1}}{1}$ and $R \simeq \mathcal{F}_X(X)$. $\square$

Next, we prove that the category of hyperdomains and the category of integral affine hyper-schemes are equivalent via the contravariant functor, Spec. Note that one can directly generalize the notion of a ringed space to define a hyperringed space. However, the notion of a locally hyperringed space should be treated with greater care since the category of hyperrings does not have (co)limits in general. Nevertheless, an integral affine hyper-scheme $(X, \mathcal{O}_X)$ can be considered as a locally hyperringed space thank to Theorem 4.3.11. Thus, in what follows we consider $(X, \mathcal{O}_X)$ as a locally hyperringed space in the sense of the direct generalization of the classical notion. We will simply write X instead of $(X, \mathcal{O}_X)$ if there is no possible confusion. The following lemma has been proven in [15] and [39], and will be mainly used.

Lemma 4.3.18. *([15, Theorem 3.6], [39, Proposition 8]) Let $\varphi : R \longrightarrow H$ be a homomorphism of hyperrings. Then, for $\mathfrak{p} \in \operatorname{Spec} H$, φ induces the following* *homomorphism $\varphi_{\mathfrak{p}}$ of hyperrings:*

$$\varphi_{\mathfrak{p}} : R_{\mathfrak{q}} \longrightarrow H_{\mathfrak{p}}, \quad \frac{a}{b} \mapsto \frac{\varphi(a)}{\varphi(b)}, \qquad \textit{where } \mathfrak{q} = \varphi^{-1}(\mathfrak{p})$$

such that if $\mathfrak{m}_{\mathfrak{p}}, \mathfrak{m}_{\mathfrak{q}}$ are unique maximal hyperideals of $H_{\mathfrak{p}}$ and $R_{\mathfrak{q}}$ respectively, then $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{\mathfrak{p}}) = \mathfrak{m}_{\mathfrak{q}}$.

Proposition 4.3.19. *Let R and H be hyperdomains, and $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} H$. Then, we have*

$$\operatorname{Hom}(R, H) = \operatorname{Hom}(Y, X), \tag{4.3.18}$$

where $\operatorname{Hom}(R, H)$ is the set of homomorphisms of hyperrings and $\operatorname{Hom}(Y, X)$ is the set of morphisms of locally hyperringed spaces.

Proof. Clearly, a homomorphism $\varphi : R \longrightarrow H$ of hyperdomains induces the continuous map

$$f : Y = \operatorname{Spec} H \longrightarrow X = \operatorname{Spec} R, \quad \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}).$$

Then, f induces the morphism of sheaves: $f^{\#} : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$. Indeed, for an open subset $V \subseteq X$, we have $f_*\mathcal{O}_Y(V) := \mathcal{O}_Y(f^{-1}(V)) = \{t \mid t : f^{-1}(V) \longrightarrow \bigsqcup_{\mathfrak{q}\in f^{-1}(V)} H_{\mathfrak{q}}\}$, where t satisfies the local condition (4.3.7). First, we define

$$\psi_V := \bigsqcup_{\mathfrak{p}\in V} \varphi_{\mathfrak{p}} : \bigsqcup_{\mathfrak{p}\in f^{-1}(V)} R_{\varphi^{-1}(\mathfrak{p})} \longrightarrow \bigsqcup_{\mathfrak{p}\in f^{-1}(V)} H_{\mathfrak{p}}, \tag{4.3.19}$$

where $\varphi_{\mathfrak{p}}$ is the map induced from φ at $\mathfrak{p}$ as in Lemma 4.3.18. We also define

$$f^{\#}(V) : \mathcal{O}_X(V) \longrightarrow \mathcal{O}_Y(f^{-1}(V)), \quad s \mapsto t := \psi_V \circ s \circ f. \tag{4.3.20}$$

We need to check four things. Firstly, we show that t as in (4.3.20) is an element of $\mathcal{O}_Y(f^{-1}(V))$. Since $t(\mathfrak{p}) = \psi_V \circ s(f(\mathfrak{p})) = \psi_V \circ s(\varphi^{-1}(\mathfrak{p}))$ and $s(\varphi^{-1}(\mathfrak{p})) \in R_{\varphi^{-1}(\mathfrak{p})}$, it follows from $\psi_V(R_{\varphi^{-1}}(\mathfrak{p})) \subseteq H_{\mathfrak{p}}$ that $t(\mathfrak{p}) \in H_{\mathfrak{p}}$. Moreover, for $\mathfrak{p} \in f^{-1}(V)$, suppose that $f(\mathfrak{p}) = \mathfrak{q} \in V$. Then, since $s \in \mathcal{O}_X(V)$, there exists a neighborhood $V_1 \subseteq V$ of $\mathfrak{q}$

and elements $a, f \in R$ such that $f \notin \mathfrak{r}\ \forall \mathfrak{r} \in V_1$ and $s(\mathfrak{r}) = \frac{a}{f}$ in $R_\mathfrak{r}$. In other words, s is locally representable by $\frac{a}{f}$ near $\mathfrak{q}$. We claim that t is locally representable by $\frac{\varphi(a)}{\varphi(f)}$ near $f^{-1}(\mathfrak{p})$. Let us define $V_2 := f^{-1}(V_1) \subseteq f^{-1}(V)$, the neighborhood of $\mathfrak{p}$. Then, $\varphi(f) \notin \mathfrak{u}\ \forall \mathfrak{u} \in V_2$ and we have $t(\mathfrak{u}) = \psi_V \circ s(f(\mathfrak{u})) = \psi_V \circ s(\varphi^{-1}(\mathfrak{u}))$. However, since $f \notin \varphi^{-1}(\mathfrak{u}) \in V_1$, we have $t(\mathfrak{u}) = \psi_V \circ s(\varphi^{-1}(\mathfrak{u})) = \psi_V(\frac{a}{f}) = \varphi_\mathfrak{u}(\frac{a}{f}) = \frac{\varphi(a)}{\varphi(f)}$ in $R_\mathfrak{u}$ by Lemma 4.3.18. Therefore, $t \in \mathcal{O}_Y(f^{-1}(V))$.

Secondly, we show that $f^\#$ is compatible with an inclusion $V \hookrightarrow U$ of open sets of X; this is clear from the construction.

Thirdly, we show that $f^\#(V)$ is a homomorphism of hyperrings. Suppose that $s = s_1 s_2 \mapsto t$, where $s_i \mapsto t_i$, $i = 1, 2$. Then, $t(\mathfrak{p}) = \psi_V \circ s \circ f(\mathfrak{p}) = \psi_V(s_1 s_2(\varphi^{-1}(\mathfrak{p}))) = \psi_V(s_1(\varphi^{-1}(\mathfrak{p})) s_2(\varphi^{-1}(\mathfrak{p})) = \psi_V(s_1(\varphi^{-1}(\mathfrak{p}))\psi_V(s_2(\varphi^{-1}(\mathfrak{p})) = t_1(\mathfrak{p}) t_2(\mathfrak{p})$. For the addition, if $s \in s_1 + s_2$, then we have $s(\mathfrak{q}) \in s_1(\mathfrak{q}) + s_2(\mathfrak{q})\ \forall \mathfrak{q} \in V$. Suppose that $s \mapsto t$ and $s_i \mapsto t_i$, $i = 1, 2$. Then, for $\mathfrak{p} \in f^{-1}(V)$, we have $t(\mathfrak{p}) = \psi_V \circ s \circ (f(\mathfrak{p})) = \psi_V \circ s(\varphi^{-1}(\mathfrak{p})) \in \psi_V(s_1(\varphi^{-1}(\mathfrak{p})) + s_2(\varphi^{-1}(\mathfrak{p}))) \subseteq \psi_V(s_1(\varphi^{-1}(\mathfrak{p}))) + \psi_V(s_2(\varphi^{-1}(\mathfrak{p}))) = t_1(\mathfrak{p}) + t_2(\mathfrak{p})$.

Finally, we show that $f^\#(V)$ is local. It easily follows from the third statement of Theorem 4.3.11 and Lemma 4.3.18.

Conversely, suppose that a morphism

$$(g, g^\#) : Y = (\mathrm{Spec}\, H, \mathcal{O}_Y) \longrightarrow X = (\mathrm{Spec}\, R, \mathcal{O}_X)$$

of integral affine hyper-schemes is given. Since R and H are hyperdomains, we can recover a homomorphism of hyperrings $\varphi : R \longrightarrow H$ by taking global sections thanks to Theorem 4.3.11. Therefore, all we have to prove is that the map $(f, f^\#)$ induced from φ as in (4.3.20) is same as $(g, g^\#)$. But, taking global sections should be compatible with local homomorphisms of stalks. Thus, we have

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & H \arrow[d] \\ R_{g(\mathfrak{p})} \arrow[r, "g_{\mathfrak{p}}^{\#}"] & H_{\mathfrak{p}} \end{tikzcd}$$

This implies that $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p}) = g(\mathfrak{p})$ and $f_{\mathfrak{p}}^{\#} = g_{\mathfrak{p}}^{\#}$. Thus, we have $(g, g^{\#}) = (f, f^{\#})$. $\square$

Let R be a hyperdomain. Then, the hyperring $R_{\mathfrak{p}}$ has a unique maximal hyperideal for each $\mathfrak{p} \in \operatorname{Spec} R$. We define $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ for $x \in \operatorname{Spec} R$, where $\mathfrak{m}_x$ is a unique maximal hyperideal of $\mathcal{O}_{X,x}$.

Proposition 4.3.20. *Let R be a hyperdomain containing the Krasner's hyperfield $\mathbf{K}$. We fix an odd prime number p and let $R_m := \mathbb{F}_{p^m}/\mathbb{F}_p^{\times}$ be the hyperfield extension of $\mathbf{K}$. Then, to give a morphism (of locally hyperringed spaces) from $\operatorname{Spec} R_m$ to $X = (\operatorname{Spec} R, \mathcal{O}_X)$ is equivalent to give a point $x \in \operatorname{Spec} R$ and $\varphi_x : k(x) \longrightarrow R_m$, a homomorphism of hyperrings.*

Proof. Suppose that $(f, f^{\#}) : \operatorname{Spec} R_m \longrightarrow X$, $0 \mapsto x$ is a morphism of integral affine hyper-schemes. Then, $(f, f^{\#})$ induces the map on stalks; $f_x^{\#} : \mathcal{O}_{X,x} \longrightarrow R_m$. Since $f_x^{\#}$ is a local homomorphism of hyperrings, we have $f_x^{\#-1}\{0\} = \mathfrak{m}_x$, where $\mathfrak{m}_x$ is the unique maximal hyperideal of $\mathcal{O}_{X,x}$. Since $\mathfrak{m}_x \subseteq \operatorname{Ker} f_x^{\#}$, $f_x^{\#}$ induces a homomorphism of hyperrings $\varphi_x : \mathcal{O}_{X,x}/\mathfrak{m}_x \longrightarrow R_m$.
Conversely, suppose that $x \in X$ and a homomorphism of hyperrings $\varphi_x : k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \longrightarrow R_m$ are given. Let $f : \operatorname{Spec} R_m \longrightarrow X$ sending 0 to x. Then, trivially f is continuous. Next, we define the map of sheaves of hyperrings $f^{\#} : \mathcal{O}_X \longrightarrow f_*\mathcal{O}_{\operatorname{Spec} R_m}$. We observe the following:

$$\mathcal{O}_{\operatorname{Spec} R_m}(f^{-1}(U)) = \begin{cases} R_m & \text{if } x \in U \subseteq X \\ 0 & \text{if } x \notin U \subseteq X. \end{cases}$$

Thus, for each $x \in U$, we define

$$f^{\#}(U) := \varphi_x \circ \pi \circ f^{\#}_{U,x} : \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_{\operatorname{Spec} R_m}(U) = R_m,$$

where φ_x is given, $\pi : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the canonical projection map, and $f^{\#}_{U,x} : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$ is the canonical map to the stalk. If $x \notin U$, we simply define $f^{\#}(U)$ as the zero map. We have to show that $f^{\#}$ is indeed a map of sheaves. Since we already know each $f^{\#}(U)$ is a homomorphism of hyperrings, we only have to check the compatibility condition. Suppose that $V \subseteq U \subseteq X$. If $x \notin U$ then $x \notin V$, hence nothing to prove. If $x \in U \cap V^c$ then $\mathcal{O}_X(f^{-1}(U)) = R_m$ and $\mathcal{O}_X(f^{-1}(V)) = 0$, hence it is also clearly compatible. If $x \in V$ then $\mathcal{O}_{\operatorname{Spec} R_m}(f^{-1}(U)) = \mathcal{O}_{\operatorname{Spec} R_m}(f^{-1}(V)) = \mathcal{O}_{\operatorname{Spec} R_m}(\{0\}) = R_m$, and the restriction map $res_{f^{-1}(U),f^{-1}(V)} : \mathcal{O}_{\operatorname{Spec} R_m}(f^{-1}(U)) \longrightarrow \mathcal{O}_{\operatorname{Spec} R_m}(f^{-1}(V))$ is the identity map from R_m to R_m. Therefore, we first have to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r, "res_{U,V}"] \arrow[d, "f^{\#}(U)"'] & \mathcal{O}_X(V) \arrow[d, "f^{\#}(V)"] \\
R_m \arrow[r, "id"] & R_m
\end{tikzcd}$$

However, it follows from $f^{\#}_{U,x} = f^{\#}_{V,x} \circ res_{U,V}$ that $f^{\#}(U) = \varphi \circ \pi \circ f^{\#}_{U,x} = \varphi \circ \pi \circ f^{\#}_{V,x} \circ res_{U,V} = f^{\#}(V) \circ res_{U,V}$. Secondly, we have to show that $f^{\#}_0$ is a local homomorphism of hyperrings. By taking global sections, we have the following commutative diagram.

$$\begin{tikzcd}
R \arrow[r, "f^{\#}(X)"] \arrow[d] & R_m \arrow[d, "id"] \\
R_x \arrow[r, "f^{\#}_0"] & R_m
\end{tikzcd}$$

Then, one can observe that $f^{\#}_0$ is a local homomorphism of local hyperrings since $f^{\#}(X)^{-1}(\{0\}) = x \in \operatorname{Spec} R$ and $R \longrightarrow R_x$ sends x to the unique maximal hyperideal of R_x. $\square$

Remark 4.3.21. *Proposition 4.3.19 works for any hyperfield extension L of $\mathbf{K}$. We use R_m only because we will use the exact same statement in §4.3.3 to construct the Hasse-Weil zeta function attached to an integral hyper-scheme over $\mathbf{K}$.*

Next, we provide an example showing that an integral hyper-scheme can be linked to the classical theory, this is the scheme theoretic version of §4.2.1. Let A be an integral domain containing the field $\mathbb{Q}$ of rational numbers, $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec}(A/\mathbb{Q}^\times) = \operatorname{Spec}(A \otimes_{\mathbb{Z}} \mathbf{K})$. We prove that there exists a canonical homeomorphism $\varphi : Y \longrightarrow X$ such that $\mathcal{O}_Y(\varphi^{-1}(U)) \simeq \mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathbf{K}$ for an open subset $U \subseteq X$. Indeed, such homeomorphism is very predictable from the following observation. Let B an integral domain containing the field $\mathbb{Q}$ of rational numbers. Then, a polynomial $f \in B[X_1, ..., X_n]$ vanishes if and only if qf vanishes $\forall q \in \mathbb{Q}^\times$.

Lemma 4.3.22. *Let A be an integral domain containing the field $\mathbb{Q}$ of rational numbers. Let $A \ni f \neq 0$ and $\tilde{f}$ be the image of f under the canonical projection map $\pi : A \longrightarrow R := A/\mathbb{Q}^\times$. Then, we have*

$$A_f/\mathbb{Q}^\times \simeq R_{\tilde{f}},$$

where $R_{\tilde{f}}$ is the localization of R at $\tilde{f}$.

Proof. Since A is an integral domain, A_f contains A (hence, contains $\mathbb{Q}$). Thus $A_f/\mathbb{Q}^\times$ is well-defined. Let us define the following map:

$$\psi : A_f/\mathbb{Q}^\times \longrightarrow R_{\tilde{f}}, \quad [\frac{a}{f^n}] \mapsto \frac{[a]}{\tilde{f}^n},$$

where $[\frac{a}{f^n}]$ is the equivalence class of $\frac{a}{f^n} \in A_f$ in $A_f/\mathbb{Q}^\times$ and $[a]$ is the equivalence class of $a \in A$ in $R = A/\mathbb{Q}^\times$. We prove that ψ is an isomorphism of hyperrings. First, we claim that ψ is well-defined. Indeed, we have

$$[\frac{a}{f^n}] = [\frac{b}{f^m}] \Longleftrightarrow \frac{aq}{f^n} = \frac{b}{f^m} \text{ for some } q \in \mathbb{Q}^\times. \tag{4.3.21}$$

But, (4.3.21) is equivalent to $aqf^m = bf^n$. In other words, we have $\pi(a)\pi(f)^m = \pi(b)\pi(f)^n \iff [a]\tilde{f}^m = [b]\tilde{f}^n$. It follows that $\frac{[a]}{\tilde{f}^n} = \frac{[b]}{\tilde{f}^m}$, and ψ is well-defined. From the construction, ψ is clearly a map of monoids. Hence, to show that ψ is a homomorphism of hyperrings, we only have to show the following:

$$[\frac{c}{f^l}] \in [\frac{a}{f^n}] + [\frac{b}{f^m}] \Longrightarrow \frac{[c]}{\tilde{f}^l} \in \frac{[a]}{\tilde{f}^n} + \frac{[b]}{\tilde{f}^m}.$$

However, $[\frac{c}{f^l}] \in [\frac{a}{f^n}] + [\frac{b}{f^m}]$ implies that $\frac{c}{f^l} = \frac{q_1 a}{f^n} + \frac{q_2 b}{f^m} \in A_f$ for some $q_i \in \mathbb{Q}^\times$. Therefore, it follows from $\frac{q_1 a}{f^n} + \frac{q_2 b}{f^m} = \frac{q_1 a f^m + q_2 b f^n}{f^{n+m}}$ that

$$\psi([\frac{c}{f^l}]) = \psi([\frac{q_1 a f^m + q_2 b f^n}{f^{n+m}}]) = \frac{[q_1 a f^m + q_2 b f^n]}{\tilde{f}^{n+m}}.$$

Since we have $[q_1 a f^m + q_2 b f^n] \in [af^m] + [bf^n] = [a]\tilde{f}^m + [b]\tilde{f}^n$, this implies that

$$\psi([\frac{c}{f^l}]) \in \frac{[a]}{\tilde{f}^n} + \frac{[b]}{\tilde{f}^m} = \psi([\frac{a}{f^n}]) + \psi([\frac{b}{f^m}]).$$

Next, we prove that ψ is strict. We have to show the following:

$$\frac{[c]}{\tilde{f}^l} \in \frac{[a]}{\tilde{f}^n} + \frac{[b]}{\tilde{f}^m} \Longrightarrow [\frac{c}{f^l}] \in [\frac{a}{f^n}] + [\frac{b}{f^m}].$$

By the definition, we have

$$\frac{[a]}{\tilde{f}^n} + \frac{[b]}{\tilde{f}^m} = \{\frac{[c]}{\tilde{f}^{n+m}} \quad | \quad [c] \in [a]\tilde{f}^m + [b]\tilde{f}^n\}.$$

Hence, without loss of generality, we may assume that $l = n + m$. Furthermore, it follows from $[c] \in [a]\tilde{f}^m + [b]\tilde{f}^n = [a][f^m] + [b][f^n] = [af^m] + [bf^n]$ that $c = q_1 a f^m + q_2 b f^n \in A$ for some $q_i \in \mathbb{Q}^\times$. Therefore, we have

$$[\frac{c}{f^{n+m}}] \in [\frac{a}{f^n}] + [\frac{b}{f^m}],$$

and this shows that ψ is strict. Clearly, ψ is surjective. If $[\frac{a}{f^n}] \in \operatorname{Ker}\psi$ then $\frac{[a]}{\tilde{f}^n} = \frac{[0]}{\tilde{f}}$, thus $[a]\tilde{f} = [af] = [0]$ and $af = 0$, $\frac{a}{f^n} = 0$. Finally, it follows from the first isomorphism theorem of hyperrings (cf. [15, Proposition 2.11]) that ψ is an isomorphism of hyperrings. $\square$

Lemma 4.3.23. *Let A be an integral domain containing the field $\mathbb{Q}$ of rational numbers and $R = A/\mathbb{Q}^\times$. Then, we have*

$$\mathrm{Frac}(A)/\mathbb{Q}^\times \simeq \mathrm{Frac}(R) = \mathrm{Frac}(A/\mathbb{Q}^\times).$$

Proof. The proof is similar to Lemma 4.3.22. For the notational convenience, let us define the following map:

$$\psi : \mathrm{Frac}(A)/\mathbb{Q}^\times \longrightarrow \mathrm{Frac}(R), \quad [\frac{b}{a}] \mapsto \frac{[b]}{[a]}.$$

Again, we have to show that this is well-defined, bijective, and a strict homomorphism of hyperrings. The proof is almost identical to that of Lemma 4.3.22. $\square$

Proposition 4.3.24. *Let A be an integral domain containing the field $\mathbb{Q}$ of rational numbers. Let $R := A/\mathbb{Q}^\times$, $X = (\mathrm{Spec}\, A, \mathcal{O}_X)$, $Y = (\mathrm{Spec}\, R, \mathcal{O}_Y)$, and $\pi : A \longrightarrow A/\mathbb{Q}^\times$ be the canonical projection map. Then, the following holds.*

1. $$\varphi : \mathrm{Spec}\, R \longrightarrow \mathrm{Spec}\, A, \quad \mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$$

   is a homeomorphism.

2. *For an open subset $U \subseteq X$, we have*

   $$\mathcal{O}_Y(\varphi^{-1}(U)) \simeq \mathcal{O}_X(U)/\mathbb{Q}^\times.$$

Proof. The first assertion will be proved in the next section in more general form (cf. Lemma 4.3.45). Note that the map induced from the canonical projection $\pi : A \longrightarrow A/\mathbb{Q}^\times$ is the desired homeomorphism φ.

For the second claim, we use the following classical identification:

$$\mathcal{O}_X(U) = \bigcap_{D(f)\subseteq U} A_f \subseteq K := \mathrm{Frac}(A). \tag{4.3.22}$$

Each $\mathcal{O}_X(U)$ is an integral domain containing $\mathbb{Q}$, hence $\mathcal{O}_X(U)/\mathbb{Q}^\times$ is well-defined. From (4.3.22), we may assume that

$$\mathcal{O}_X(U)/\mathbb{Q}^\times \text{ and } A_f/\mathbb{Q}^\times \text{ are subsets of } K/\mathbb{Q}^\times. \tag{4.3.23}$$

Also, from Lemma 4.3.22 and 4.3.23, we have

$$A_f/\mathbb{Q}^\times \simeq R_{\tilde{f}}, \quad K/\mathbb{Q}^\times \simeq L := \mathrm{Frac}(R). \tag{4.3.24}$$

It follows from (4.3.23) and (4.3.24) that

$$\mathcal{O}_X(U)/\mathbb{Q}^\times = (\bigcap_{D(f)\subseteq U} A_f)/\mathbb{Q}^\times = \bigcap_{D(f)\subseteq U} (A_f/\mathbb{Q}^\times). \tag{4.3.25}$$

In fact, the first equality simply follows from (4.3.22). It remains to show the second equality. Indeed, we know that $[a] \in \mathcal{O}_X(U)/\mathbb{Q}^\times \iff qa \in \mathcal{O}_X(U) = \bigcap_{D(f)\subseteq U} A_f \iff qa \in A_f \ \forall f \in A$ such that $D(f) \subseteq U$ for some $q \in \mathbb{Q}^\times$. This implies that $[a] \in A_f/\mathbb{Q}^\times \ \forall f \in A$ such that $D(f) \subseteq U$. It follows that

$$\mathcal{O}_X(U)/\mathbb{Q}^\times = (\bigcap_{D(f)\subseteq U} A_f)/\mathbb{Q}^\times \subseteq \bigcap_{D(f)\subseteq U} (A_f/\mathbb{Q}^\times).$$

Conversely, if $[a] \in \bigcap_{D(f)\subseteq U}(A_f/\mathbb{Q}^\times)$ then $[a] \in A_f/\mathbb{Q}^\times \ \forall f \in A$ such that $D(f) \subseteq U$. In other words, for each f, there exists $q_f \in \mathbb{Q}^\times$ such that $aq_f \in A_f$. However, $\mathbb{Q} \subseteq A_f \ \forall f \in A$, hence $a \in A_f$ and $a \in \bigcap_{D(f)\subseteq U} A_f$. It follows that

$$[a] \in (\bigcap_{D(f)\subseteq U} A_f)/\mathbb{Q}^\times = \mathcal{O}_X(U)/\mathbb{Q}^\times,$$

and this shows (4.3.25).

Finally, let $\tilde{f} = \pi(f)$ as in Lemma 4.3.22. Then, there is a one-to-one correspondence between the following sets (cf. the proof of Lemma 4.3.45):

$$\mathcal{A} := \{f \in A | D(f) \subseteq U\}, \quad \mathcal{B} := \{\tilde{f} \in R | D(\tilde{f}) \subseteq \varphi^{-1}(U)\}, \tag{4.3.26}$$

where φ is the canonical homeomorphism in the first assertion. Therefore, together with Theorem 4.3.11, we have

$$\mathcal{O}_X(U)/\mathbb{Q}^\times = \bigcap_{D(f)\subseteq U} (A_f/\mathbb{Q}^\times) \simeq \bigcap_{D(f)\subseteq U} R_{\tilde{f}} = \bigcap_{D(\tilde{f})\subseteq \varphi^{-1}(U)} R_{\tilde{f}} \simeq \mathcal{O}_Y(\varphi^{-1}(U)).$$

This proves the second assertion. □

Remark 4.3.25. *Proposition 4.3.24 states that if A is an integral domain containing $\mathbb{Q}$, then $X := \operatorname{Spec} A \simeq X_{\mathbf{K}} = \operatorname{Spec} A_{\mathbf{K}} = \operatorname{Spec}(A/\mathbb{Q}^\times)$. In other words, the spaces are homeomorphic, but their functions(sections) are different. In fact, what the second assertion states is that sections of $X_{\mathbf{K}}$ can be derived from sections of X by tensoring them with $\mathbf{K}$ in the sense of [9].*

4.3.3 The Hasse-Weil zeta function revisited

In §4.2.1, we naively constructed the Hasse-Weil zeta function attached to an algebraic variety over hyper-structures viewed as a set of solutions of polynomial equations. In this subsection, we construct the Hasse-Weil zeta function attached to an integral affine hyper-scheme which behaves better, in a way, with respect to the one we constructed before. By an algebraic variety over a field k we mean a reduced scheme of finite type over k. We make use of the following well-known product formula (4.3.27).

Theorem 4.3.26. *(Product formula) Let X be an algebraic variety over the finite field $k = \mathbb{F}_q$. Then, we have*

$$Z(X,t) := \exp\left(\sum_{m\geq 1} \frac{N_m}{m} t^m\right) = \prod_{x\in|X|} (1 - t^{\deg(x)})^{-1}, \tag{4.3.27}$$

where $|X|$ is the set of closed points of X, $k(x)$ is the residue field at x, and $\deg(x) := [k(x) : k]$ *is the degree of the residue field at x.*

First, we introduce the notions of residue field and degree in hyper-structures.

Definition 4.3.27. *Let T be a hyperfield. By a (left) hyper T-algebra we mean a pair (R, φ) of a hyperring R and a map $\varphi : T \times R \longrightarrow R$ satisfying the following properties: $\forall r, r_1, r_2 \in R$, $\forall t, t_1, t_2 \in T$,*

1. $\varphi(1, r) = r$.
2. $\varphi(t, r) = 0 \Longleftrightarrow t = 0$ *or* $r = 0$.
3. $\varphi(t_1 + t_2, r) = \varphi(t_1, r) + \varphi(t_2, r), \quad \varphi(t, r_1 + r_2) = \varphi(t, r_1) + \varphi(t, r_2)$.
4. $\varphi(t_1 t_2, r) = \varphi(t_1, \varphi(t_2, r)), \quad \varphi(t, r_1 r_2) = \varphi(t, r_2) r_2$.

We denote $\varphi(t, r) := tr$ if there is no possible confusion. Then, the above definition can be written as:

$$1r = r, \quad tr = 0 \Longleftrightarrow t = 0 \text{ or } r = 0, \quad (t_1 + t_2)r = t_1 r + t_2 r,$$

$$t(r_1 + r_2) = tr_1 + tr_2, \quad (t_1 t_2)r = t_1(t_2 r), \text{ and } t(r_1 r_2) = (tr_1)r_2.$$

Furthermore, if R_1, R_2 are hyper T-algebras with associated maps φ_1, φ_2 respectively, we say that a hyperring homomorphism $\psi : R_1 \longrightarrow R_2$ is a hyper T-algebra homomorphism if $\psi(\varphi_1(t, r)) = \varphi_2(t, \psi(r))$, or $\psi(tr) = t\psi(r)$.

Note that, in the third condition, $t_1 + t_2$ and $r_1 + r_2$ are sets in general. Hence, the equality means that they are equal as sets. In the sequel, for the notational convenience, we will simply say that R is a hyper T-algebra assuming that φ is given.

Example 4.3.28. *Let A be a commutative ring containing the field $\mathbb{Q}$ of rational numbers. Then, $A/\mathbb{Q}^\times$ is a hyper $\mathbf{K}$-algebra and $A/\mathbb{Q}^\times_{>0}$ is a hyper $\mathbf{S}$-algebra (cf. [9]).*

Remark 4.3.29. *Let T, R, and φ be as in Definition 4.3.27. Consider the following map:*

$$\psi : T \longrightarrow R, \quad \psi(t) \mapsto \varphi(t,1). \tag{4.3.28}$$

We claim that ψ is a strict and injective homomorphism of hyperrings. In fact, we first observe that $\varphi(t-t,r) = \varphi(t,r)+\varphi(-t,r)$. Thus, we have $0 \in \varphi(t,r)+\varphi(-t,r)$, and it follows that $-\varphi(t,r) = \varphi(-t,r)$. Similarly, we have $-\varphi(t,r) = \varphi(t,-r)$. Next, we have $\psi(t_1+t_2) = \varphi(t_1+t_2,1) = \varphi(t_1,1)+\varphi(t_2,1) = \psi(t_1)+\psi(t_2)$. Furthermore, $\psi(t_1t_2) = \varphi(t_1t_2,1) = \varphi(t_1,\varphi(t_2,1)) = \varphi(t_1,1\cdot\varphi(t_2,1)) = \varphi(t_1,1)\varphi(t_2,1) = \psi(t_1)\psi(t_2)$. This shows that ψ is indeed a strict homomorphism of hyperrings. Also, $\psi(t_1) = \psi(t_2) \iff \varphi(t_1,1) = \varphi(t_2,1) \iff 0 \in \varphi(t_1-t_2,1)$. Therefore, $0 \in t_1-t_2$, and $t_1 = t_2$. This shows that ψ is injective.
Note that one might define the third condition as follows:

$$\varphi(t_1+t_2,r) \subseteq \varphi(t_1,r)+\varphi(t_2,r), \quad \varphi(t,r_1+r_2) \subseteq \varphi(t,r_1)+\varphi(t,r_2). \tag{4.3.29}$$

However, by forcing (4.3.29) to be equalities as in Definition 4.3.27, one can identify T to a sub-hyperfield of R via ψ. Then, an element $\varphi(t,r)$ is simply a multiplication of $\psi(t)$ and r as the elements of R. For this reason, we take the 'equality' approach instead of the 'inclusion' approach.

Definition 4.3.30. *Let T be a hyperfield and R be a hyper T-algebra. By a set X of generators of R over T we mean a subset $X \subseteq R$ satisfying the following condition:*

$$\forall r \in R \quad \exists n \in \mathbb{N}, \{a_1,...,a_n\} \subseteq T, \{x_1,...,x_n\} \subseteq X \ s.t. \ r \in \sum_{i=1}^{n} a_i x_i. \tag{4.3.30}$$

If there exists a finite set X of generators of R over T, then we say that R is finitely generated over T. When R is finitely generated over T, we define the degree $[R:T]$ of R over T as the smallest number among the cardinalities of finite sets of generators of R over T.

Lemma 4.3.31. *Let T be a hyperfield and R be a hyper T-algebra. Suppose that I is a hyperideal of R. Then, R/I has the canonical hyper T-algebra structure induced from R. Furthermore, if R is finitely generated over T, then so is R/I.*

Proof. Let $\varphi : T \times R \longrightarrow R$ be the hyper T-algebra structure of R and $[r]$ be the equivalence class of $r \in R$ in R/I. Let us define the following map:

$$\varphi_I : T \times R/I \longrightarrow R/I, \quad (t, [r]) \mapsto [\varphi(t, r)].$$

First, we claim that φ_I is well-defined. Suppose that $[r'] = [r]$. This implies that there exists $\alpha \in (r - r') \bigcap I$ (cf. Lemma 4.1.4). It follows that $\varphi(t, \alpha) = \varphi(t, \alpha \cdot 1) = \varphi(t, 1)\alpha \in I$. However, we have $\varphi(t, r) - \varphi(t, r') = \varphi(t, r) + \varphi(t, -r') = \varphi(t, r - r') \ni \varphi(t, \alpha)$, thus $[\varphi(t, r)] = [\varphi(t, r')]$. Next, we show that φ_I satisfies the conditions in Definition 4.3.27. We have $\varphi_I(t, [r]) = 0 \iff \varphi(t, r) \in I$. If $t = 0$, then this clearly satisfies the first and the second conditions. If $t \neq 0$, then we have $r = \varphi(1, r) = \varphi(tt^{-1}, r) = \varphi(t, \varphi(t^{-1}, 1 \cdot r)) = \varphi(t, \varphi(t^{-1}, 1)r) = \varphi(t, r)\varphi(t^{-1}, 1)$. But, $\varphi(t^{-1}, 1)$ is a unit since we have $\varphi(t, 1)\varphi(t^{-1}, 1) = \varphi(t, \varphi(t^{-1}, 1)) = \varphi(tt^{-1}, 1) = \varphi(1, 1) = 1$. It follows that $r \in I \iff \varphi(t, r) \in I$. Furthermore, clearly $\varphi_I(1, [r]) = [\varphi(1, r)] = [r]$. This proves the first and the second conditions. Next, we have $\varphi_I(t_1+t_2, [r]) = [\varphi(t_1+t_2, r)] = [\varphi(t_1, r)+\varphi(t_2, r)]$. Since the canonical projection map is strict (cf. Proposition 4.1.7), we have $[\varphi(t_1, r) + \varphi(t_2, r)] = [\varphi(t_1, r)] + [\varphi(t_2, r)] = \varphi_I(t_1, [r])+\varphi_I(t_2, [r])$. We also have $\varphi_I(t, [r_1]+[r_2]) = \varphi_I(t, [r_1+r_2]) = [\varphi(t, r_1+r_2)] = [\varphi(t, r_1) + \varphi(t, r_2)] = \varphi_I(t, [r_1]) + \varphi_I(t, [r_2])$. This proves the third condition, and the fourth condition can be proven similarly. Finally, suppose that R is finitely generated over T and $\{x_1, ..., x_n\}$ is a set of generators of R over T. Then $\{[x_1], ..., [x_n]\}$ is the set of generators of R/I over T. In fact, if $[r] \in R/I$ then $r \in \sum \varphi(a_i, x_i)$ for some $a_i \in T$. Therefore, $[r] \in \sum[\varphi(a_i, x_i)] = \sum \varphi_I(a_i, [x_i])$. □

Lemma 4.3.32. *Let T be a hyperfield and R be a hyper T-algebra. Then, for a multiplicative subset $S \subseteq R$, the set $S^{-1}R$ has the canonical hyper T-algebra structure* *induced from R.*

Proof. Let $\varphi : T \times R \longrightarrow R$ be the hyper T-algebra structure of R. Let us define

$$\varphi_S : T \times S^{-1}R \longrightarrow S^{-1}R, \quad (t, \frac{r}{l}) \mapsto \frac{\varphi(t,r)}{l}.$$

We claim that φ_S is well-defined. Indeed, for $\frac{r}{l} = \frac{r'}{l'}$, we have $qrl' = qr'l$ for some $q \in S$. Therefore $\varphi(t,r)ql' = \varphi(t,qrl') = \varphi(t,qr'l) = \varphi(t,r')ql$, and $\varphi_S(t,\frac{r}{l}) = \frac{\varphi(t,r)}{l} = \frac{\varphi(t,r')}{l'} = \varphi_S(t,\frac{r'}{l'})$. For the first and the second conditions, $\varphi_S(t,\frac{r}{l}) = \frac{\varphi(t,r)}{l} = 0$ if and only if $q\varphi(t,r) = \varphi(t,qr) = 0$. It follows that $t = 0$ or $qr = 0 \iff \frac{r}{l} = 0$. Furthermore, $\varphi_S(1,\frac{r}{l}) = \frac{\varphi(t,r)}{l} = \frac{r}{l}$. The third condition easily follows since $\frac{r}{l} + \frac{r'}{l'} = \frac{r+r'}{l}$ in $S^{-1}R$. The fourth condition is also immediate from that of φ. $\square$

Remark 4.3.33. *When R is finitely generated over T, the induced hyper T-algebra structure on $S^{-1}R$ does not have to be finitely generated because this is not even true in general in the classical case.*

Lemma 4.3.34. *([39, Proposition 8]) Let H be a hyperring and S be a multiplicative subset of H. Let $S^{-1} : H \longrightarrow S^{-1}H$ be the homomorphism of hyperrings sending h to $\frac{h}{1}$. Let f be a homomorphism of hyperrings $f : H \longrightarrow K$ such that any element y of $f(S)$ is invertible in K. Then, $\exists! \tilde{f} : S^{-1}H \longrightarrow K$ such that $\tilde{f} \circ S^{-1} = f$. In particular, $\tilde{f}(\frac{a}{b}) = f(a)f(b)^{-1}$.*

Lemma 4.3.35. *Let T be a hyperfield and R_1, R_2 be hyper T-algebras. Suppose that $f : R_1 \longrightarrow R_2$ is a surjective homomorphism of hyper T-algebras and R_1 is finitely generated over T. Then, R_2 is also finitely generated over T.*

Proof. Suppose that $\{x_1, ..., x_n\}$ generates R_1 over T. We claim that $\{f(x_1), ..., f(x_n)\}$ generates R_2 over T. Since f is surjective, for $\beta \in R_2$, there exists $\alpha \in R_1$ such that $f(\alpha) = \beta$. Then, we can find $a_i \in T$ such that $\alpha \in \sum_{i=1}^n a_i x_i$. It follows that

$$f(\alpha) = \beta \in f(\sum_{i=1}^n a_i x_i) \subseteq \sum_{i=1}^n f(a_i x_i) = \sum_{i=1}^n a_i f(x_i).$$

□

Remark 4.3.36. *Note that Lemma 4.3.35 implies that if I is a hyperideal of a finitely generated hyper T-algebra R, then R/I is also finitely generated hyper T-algebra with the induced hyper T-algebra structure as in Lemma 4.3.31.*

Lemma 4.3.37. *Let R be a hyperring, $\mathfrak{m}$ be a maximal hyperideal of R, and $\pi : R \longrightarrow R/\mathfrak{m}$ be the canonical projection map. Then, $K := R/\mathfrak{m}$ is a hyperfield and the set of nonzero elements of K is $\tilde{S} := \pi(S)$, where $S = R\backslash\mathfrak{m}$.*

Proof. We know that $R/\mathfrak{m}$ is a hyperring. Suppose that $[r] \in K\backslash\{0\}$. Since $r \notin \mathfrak{m}$ and $\mathfrak{m}$ is maximal, it follows that $R =< \mathfrak{m}, r >$. Therefore, $1 \in rt + q$ for some $t \in R$ and $q \in \mathfrak{m}$. This implies that $q \in 1 - rt$, therefore $[1] = [rt] = [r][t]$ since $[q] = 0$. This shows that K is a hyperfield. The second assertion is clear. □

Let T be a hyperfield, R be a finitely generated hyper T-algebra, $X = \operatorname{Spec} R$, and $|X|$ be the set of closed points of X. For $x \in |X|$, we define the residue field at x as $k(x) := R_x/\mathfrak{m}_x$, where R_x is the localization of R at x and $\mathfrak{m}_x$ is the maximal hyperideal of R_x (cf. Proposition 1.1.20). From Lemma 4.3.31 and 4.3.32, we can impose to $k(x)$ the canonical hyper T-algebra structure induced from R. In the sequel, we always consider $k(x)$ as a hyper T-algebra with this induced structure. The next proposition shows that $k(x)$ is finitely generated over T if R is finitely generated over T.

Proposition 4.3.38. *Let T be a hyperfield and R be a finitely generated hyper T-algebra. Then, for a maximal hyperideal $\mathfrak{m} \subseteq R$, $k(\mathfrak{m}) := R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ is a finitely generated hyper T-algebra and $[k(\mathfrak{m}) : T] = [R/\mathfrak{m} : T]$.*

Proof. We note that $[R/\mathfrak{m} : T]$ makes sense since $R/\mathfrak{m}$ is finitely generated over T (cf. Remark 4.3.36). Let $\pi : R \longrightarrow R/\mathfrak{m}$, $S = R\backslash\mathfrak{m}$, and $\tilde{S} = \pi(S)$. Clearly, $\tilde{S}$ is a multiplicative subset of $R/\mathfrak{m}$. We define $\tilde{S}^{-1} : R/\mathfrak{m} \longrightarrow \tilde{S}^{-1}(R/\mathfrak{m})$, the localization map. Let $f := \tilde{S}^{-1} \circ \pi : R \longrightarrow \tilde{S}^{-1}(R/\mathfrak{m})$. We observe that for $t \in S$, $f(t)$ is invertible in $\tilde{S}^{-1}(R/\mathfrak{m})$. It follows from Lemma 4.3.34 that there exists a unique map $h : S^{-1}R \longrightarrow \tilde{S}^{-1}(R/\mathfrak{m})$ satisfying the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "S^{-1}"] \arrow[d, "\pi"'] & S^{-1}R \arrow[d, "\exists! h"] \\ R/\mathfrak{m} \arrow[r, "\tilde{S}^{-1}"] & \tilde{S}^{-1}(R/\mathfrak{m}) \end{tikzcd} \tag{4.3.31}$$

Let $K = R/\mathfrak{m}$. It follows from Lemma 4.3.37 that K is a hyperfield and $K^\times = \tilde{S}$. This implies that $\tilde{S}^{-1}$ is indeed the identity map on $K = R/\mathfrak{m}$. In particular, $\tilde{S}^{-1}$ is surjective. Since π is also surjective, from $\tilde{S}^{-1} \circ \pi = h \circ S^{-1}$, we conclude that h is surjective.

Next, we show that h is strict. From the commutative diagram (4.3.31), we have $h(\frac{a}{b}) = \frac{\pi(a)}{\pi(b)}$ for $\frac{a}{b} \in S^{-1}R$. Let us denote $\pi(a) = [a]$ for the notational convenience. We have to show the following: for all $\frac{a}{b}, \frac{c}{d} \in S^{-1}R$,

$$h(\frac{a}{b}) + h(\frac{c}{d}) = \frac{[a]}{[b]} + \frac{[c]}{[d]} \subseteq h(\frac{a}{b} + \frac{c}{d}). \tag{4.3.32}$$

Take $y \in \frac{[a]}{[b]} + \frac{[c]}{[d]}$. Then y can be written as $y = \frac{[z]}{[bd]}$ for some $[z] \in [ad] + [bc]$. Since π is a strict homomorphism, we have $[z] \in [ad] + [bc] = [ad + bc]$. Thus, there exists $\alpha \in ad + bc$ such that $[\alpha] = [z]$. It follows that $\frac{\alpha}{bd} \in \frac{a}{b} + \frac{c}{d}$ and $h(\frac{\alpha}{bd}) = \frac{[\alpha]}{[bd]} = \frac{[z]}{[bd]} = y$, hence h is strict.

Finally, we show that $\mathrm{Ker}(h) = S^{-1}\mathfrak{m}$. If $\frac{\alpha}{\beta} \in S^{-1}\mathfrak{m}$, then $h(\frac{\alpha}{\beta}) = \frac{[\alpha]}{[\beta]}$. Since $\frac{\alpha}{\beta} \in S^{-1}\mathfrak{m}$, we may assume $\alpha \in \mathfrak{m}$. This implies that $[\alpha] = 0$ and $h(\frac{\alpha}{\beta}) = 0$, hence $S^{-1}\mathfrak{m} \subseteq \mathrm{Ker}(h)$. Conversely, if $\frac{r}{a} \in \mathrm{Ker}(h)$, then $h(\frac{r}{a}) = \frac{[r]}{[a]} = 0$. Since $\tilde{S}^{-1}(R/\mathfrak{m})$ is a hyperfield, this implies that $[r] = 0$. Therefore, $r \in \mathfrak{m}$ and $\frac{r}{a} \in S^{-1}\mathfrak{m}$.

To sum up, h is a strict surjective homomorphism with $\mathrm{Ker}(h) = S^{-1}\mathfrak{m}$. Then, we have the following isomorphism of hyper T-algebras:

$$\tilde{h} : S^{-1}R/S^{-1}\mathfrak{m} \simeq \tilde{S}^{-1}(R/\mathfrak{m}) = R/\mathfrak{m}, \quad \tilde{h}([t]) = h(t), \tag{4.3.33}$$

where $[t]$ is the equivalence class of $t \in S^{-1}R$ in $S^{-1}R/S^{-1}\mathfrak{m}$. In fact, from the first isomorphism theorem of hyperrings (cf. [15, Proposition 2.11]), $\tilde{h}$ is the isomorphism of hyperrings. Moreover, we have

$$\tilde{h}(t \cdot [\frac{b}{a}]) = \tilde{h}([t \cdot \frac{b}{a}]) = \tilde{h}([\frac{t \cdot b}{a}]) = \frac{[t \cdot b]}{[a]} = \frac{t \cdot [b]}{[a]} = t \cdot \frac{[b]}{[a]}.$$

This shows that $\tilde{h}$ is, in fact, an isomorphism of hyper T-algebras.
It remains to show $[k(\mathfrak{m}) : T] = [R/\mathfrak{m} : T]$. However, from (4.3.33), $k(\mathfrak{m}) \simeq R/\mathfrak{m}$ as hyper T-algebras. This completes our proof. $\square$

Next, we define a zeta function attached to $X = \operatorname{Spec} R$, where R is a hyperring. Note that we consider X solely as a topological space since we know that $X = (\operatorname{Spec} R, \mathcal{O}_X)$ is a locally hyperringed space only when R is a hyperdomain.

Definition 4.3.39. *Let T be a hyperfield and R be a finitely generated hyper T-algebra. Let $X = \operatorname{Spec} R$ and $|X|$ be the set of closed points of X. We define the zeta function $Z(X,t)$ attached to X as follows:*

$$Z(X,t) := \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}, \tag{4.3.34}$$

where $\deg(x) := [k(x) : T]$.

Remark 4.3.40.

1. *It follows from Proposition 4.3.38 that if R is finitely generated then $\deg(x)$ is finite $\forall x \in |X|$, hence (4.3.34) is well defined.*

2. *In the classical case, the Hasse-Weil zeta function is defined only for an algebraic variety over a finite field. We will similarly consider when T is a finite hyperfield, however, (4.3.34) makes sense for any hyperfield T.*

3. *Let $Y \subseteq X$ be a closed subset and $U = X \backslash Y$. Let us define the following zeta*

function attached to U:

$$Z(U,t) := \prod_{x\in\{|X|\cap U\}} (1 - t^{\deg(x)})^{-1}.$$

Then, as in the classical case, we obtain

$$Z(X,t) = Z(Y,t)\cdot Z(U,t).$$

We refer to [9] for details about hyperrings in the following examples.

Example 4.3.41. *Let $R = \mathbf{K}[H]$ be the hyperring extension of $\mathbf{K}$ of dimension 2, where H is an abelian group of the order greater than 3. Then, $X = \operatorname{Spec} R = \{pt\}$ since R is a hyperfield. Thus,*

$$Z(X,t) = (1-t^2)^{-1}.$$

Example 4.3.42. *Let $R = \mathbf{K}[H]\cup\{a\}$ be the hyperring extension of $\mathbf{K}$ of dimension 2, where H is an abelian group of the order greater than 3 and $a^2 = 0$, $au = ua = a$ $\forall u \in H$. Then, $X = \operatorname{Spec} R$ has a unique maximal hyperideal $\mathfrak{p} = \{0,a\}$, and $k(\mathfrak{p}) = \mathbf{K}$. Hence, we obtain*

$$Z(X,t) = (1-t)^{-1}.$$

Example 4.3.43. *Let $R = \mathbf{K}[H]\cup\{e,f\}$, where H is an abelian group of the order greater than 3 and $e^2 = e$, $f^2 = f$, $ef = fe = 0$, $au = ua = a$ $\forall u \in H$ and $a \in \{e,f\}$. Then, R has two maximal hyperideals, $\mathfrak{m}_1 = \{0,e\}$ and $\mathfrak{m}_2 = \{0,f\}$. One can easily check that $k(\mathfrak{m}_1) = k(\mathfrak{m}_2) = \mathbf{K}$, hence we obtain,*

$$Z(X,t) = (1-t)^{-1}(1-t)^{-1} = (1-t)^{-2}.$$

Let k be a field, $G = k^\times$, and A be a commutative k-algebra. Then, the quotient hyperring $R := A/G$ carries a canonical hyper $\mathbf{K}$-algebra structure since R contains $\mathbf{K}$

(cf. [9, Proposition 2.7]). The next theorem illustrates an interesting link between the classical Hasse-Weil zeta function attached to $\operatorname{Spec} A$ and the zeta function attached to $\operatorname{Spec} A/G$ as in (4.3.34).

Theorem 4.3.44. *Let k be a field, $G = k^\times$, and A be a reduced finitely generated (commutative) k-algebra. Let $R := A/G$ be the quotient hyperring. Then, R is a finitely generated hyper $\mathbf{K}$-algebra. Furthermore, if $X := \operatorname{Spec} A$ and $Y := \operatorname{Spec} R$, then we have the following:*

$$Z(Y,t) := \prod_{y \in |Y|} (1 - t^{\deg(y)})^{-1} = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}. \tag{4.3.35}$$

In particular, when k is a finite field of odd characteristic, we have $Z(Y,t) = Z(X,t)$, where $Z(X,t)$ is the classical Hasse-Weil zeta function attached to the algebraic variety $X = \operatorname{Spec} A$.

We prove the following lemma before we prove Theorem 4.3.44.

Lemma 4.3.45. *Let A be a commutative ring, $G \subseteq A^\times$ be a multiplicative subgroup, and A/G be the quotient hyperring. Then, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec}(A/G)$ are homeomorphic (under the Zariski topology).*

Proof. If $G = \{1\}$ then there is nothing to prove. Thus, we may assume that $|G| \geq 2$. We define the following map:

$$\sim : X \longrightarrow Y, \quad \mathfrak{q} \mapsto \tilde{\mathfrak{q}} := \{\alpha G \mid \alpha \in \mathfrak{q}\}.$$

We claim that the map $\sim$ is well-defined. Indeed, we have

$$\alpha G = \beta G \iff \alpha = \beta u, u \in G \subseteq A^\times \iff \alpha, \beta \in \mathfrak{q} \text{ or } \alpha, \beta \notin \mathfrak{q},$$

therefore $\tilde{\mathfrak{q}}$ is uniquely determined by $\mathfrak{q}$. Furthermore, $\tilde{\mathfrak{q}}$ is a hyperideal. In fact, we have $0G \in \tilde{\mathfrak{q}}$. If $aG \in \tilde{\mathfrak{q}}$ then $(-a)G = -(aG) \in \tilde{\mathfrak{q}}$. For $rG \in A/G$ and $aG \in \tilde{\mathfrak{q}}$, since

$(rG)(aG) = raG$ and $ra \in \mathfrak{q}$, it follows that $(rG)(aG) \in \tilde{\mathfrak{q}}$. Suppose that $aG, bG \in \tilde{\mathfrak{q}}$. One can observe that $aG, bG \in \tilde{\mathfrak{q}} \Longleftrightarrow a, b \in \mathfrak{q}$ since $G \subseteq A^\times$ and $\mathfrak{q}$ is a prime ideal. Therefore, for $zG \in \tilde{\mathfrak{q}}$, we may assume that $z = at + bh$ for some $t, h \in G$. It follows that $z \in \mathfrak{q}$, hence $zG \in \tilde{\mathfrak{q}}$. This shows that $\tilde{\mathfrak{q}}$ is a hyperideal. Next, we show that $\tilde{\mathfrak{q}}$ is prime. Suppose that $(aG)(bG) = (abG) \in \tilde{\mathfrak{q}}$ and $aG \notin \tilde{\mathfrak{q}}$. This implies that $ab \in \mathfrak{q}$ and $au \notin \mathfrak{q}$ $\forall u \in G$, hence $a \notin \mathfrak{q}$. Since $\mathfrak{q}$ is prime, this implies that $b \in \mathfrak{q}$, and $bG \in \tilde{\mathfrak{q}}$.
Next, we claim that the map $\sim$ is continuous. Let $\varphi := \sim$ for the notational convenience. It is enough to show that $\varphi^{-1}(D(fG))$ is open. We have the following:

$$\varphi^{-1}(D(fG)) = D(f). \tag{4.3.36}$$

Indeed, if $\mathfrak{q} \in D(f)$, then $\varphi(\mathfrak{q}) = \tilde{\mathfrak{q}}$ can not contain fG by definition. Hence, $D(f) \subseteq \varphi^{-1}(D(fG))$. Conversely, suppose that $\mathfrak{p} \in \varphi^{-1}(D(fG))$, then $\varphi(\mathfrak{p}) \in D(fG)$. Since $\varphi(\mathfrak{p}) = \tilde{\mathfrak{p}} = \{\alpha G \mid \alpha \in \mathfrak{p}\}$ and $f \notin \mathfrak{p}$, it follows that $\mathfrak{p} \in D(f)$, hence $\varphi^{-1}(D(fG)) \subseteq D(f)$. This proves (4.3.36), hence $\sim$ is continuous.
Finally, we construct the inverse of the map $\varphi = \sim$. The canonical projection map $\pi : A \longrightarrow A/G$ induces the following canonical map:

$$\psi : Y \longrightarrow X, \quad \mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p}).$$

Clearly, ψ is continuous since $\psi^{-1}(D(f)) = D(fG)$. We claim that φ and ψ are inverses to each other. Since both φ and ψ are continuous, it is enough to show that φ is bijective and $\varphi \circ \psi = id_Y$. First, we show that φ is injective. Assume that $\varphi(\mathfrak{q}) = \varphi(\mathfrak{p})$ for $\mathfrak{p}, \mathfrak{q} \in X$. Then, for $x \in \mathfrak{q}$, we have $y \in \mathfrak{p}$ such that $xG = yG$. It follows that $x = yg$ for some $g \in G$, hence $x \in \mathfrak{p}$. Since the argument is symmetric, we have $\mathfrak{p} = \mathfrak{q}$.
For the surjectivity of φ, take an element $\wp \in \operatorname{Spec} A/G$. We consider αG as the

subset $\alpha G := \{\alpha g \mid g \in G\} \subseteq A$ and define

$$\mathfrak{p} := \bigcup_{\alpha G \in \wp} \alpha G.$$

We have to show that $\mathfrak{p}$ is a prime ideal of A. We have $0 \in \mathfrak{p}$. Moreover, $a \in \mathfrak{p} \iff a \in \alpha G$ for some $\alpha G \in \wp$. It follows that $-\alpha G \in \wp$ and hence $-a \in \mathfrak{p}$. Furthermore, for $a \in \mathfrak{p}$ and $r \in A$, we have $aG \in \wp$ and $rG \in A/G$. It follows from $(rG)(aG) = (raG) \in \wp$ that $ra \in \mathfrak{p}$. If $a, b \in \mathfrak{p}$ then $aG, bG \in \wp$. This implies that $aG + bG \subseteq \wp$ and hence $a + b \in \mathfrak{p}$. This proves that $\mathfrak{p}$ is an ideal. We observe that $\mathfrak{p}$ can not be A since that implies $1 \in \mathfrak{p}$ and $1G \in \wp$, but $\wp \neq A/G$. One further observes that $\mathfrak{p}$ is prime since for $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$, we have $(aG)(bG) \in \wp$ and $aG \notin \wp$. This implies that $bG \in \wp$, hence $b \in \mathfrak{p}$. Obviously, we have $\varphi(\mathfrak{p}) = \wp$. This shows that φ is surjective. In fact, one can see that $\mathfrak{p} = \psi(\wp)$. Thus, we have $\varphi(\mathfrak{p}) = \varphi \circ \psi(\wp) = \wp$ and therefore $\varphi \circ \psi = id_Y$. This completes our proof. $\square$

Now, we are ready to prove Theorem 4.3.44.

Proof of Theorem 4.3.44. The second assertion follows from the first assertion (4.3.35) and Theorem 4.3.26 (Product formula). From Lemma 4.3.45, we know that the map $\varphi : X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} R$ is a homeomorphism, where $R = A/G$. Therefore, it is enough to show that

$$[k(\varphi(x)) : \mathbf{K}] = [k(x) : k] \quad \forall x \in |X|. \tag{4.3.37}$$

Let us fix the homeomorphism φ of Lemma 4.3.45. Let $x \in |X|$ and $\mathfrak{m}$ be the maximal ideal of A corresponding to x regarded as the point of X. Since φ is a homeomorphism, it follows that $y := \varphi(x) \in |Y|$. Let $\mathfrak{n}$ be the maximal hyperideal of R corresponding to y. In the classical case, we have $A_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}} = (A/\mathfrak{m})_{\mathfrak{m}} = A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m} \subseteq A$. Similarly, in the proof of Proposition 4.3.38 (cf. Equation (4.3.33)), we showed that for a maximal hyperideal $\mathfrak{n} \subseteq R$, we have $R_{\mathfrak{n}}/\mathfrak{n}_{\mathfrak{n}} = (R/\mathfrak{n})_{\mathfrak{n}} = R/\mathfrak{n}$.

Therefore, to prove (4.3.37), we only have to show the following:

$$[A/\mathfrak{m} : k] = [R/\mathfrak{n} : \mathbf{K}]. \tag{4.3.38}$$

One can observe that $\mathfrak{n} = \{aG \mid a \in \mathfrak{m}\} = \mathfrak{m}/G \subseteq A/G$. We let $L := A/\mathfrak{m}$ and $H := (A/G)/(\mathfrak{m}/G) = R/\mathfrak{n}$. For the national convenience, we write $\bar{a} := \pi(a)$ and $[aG] := \pi'(aG)$, where $\pi : A \longrightarrow A/\mathfrak{m}$ and $\pi' : A/G \longrightarrow H = (A/G)/(\mathfrak{m}/G)$ are the canonical projection maps.

Let $\{\bar{a}_i\}$ be any smallest finite set of generators of L over $k = G \cup \{0\}$. This choice is possible since $\deg(x)$ is finite. We claim that $\{[a_iG]\}$ becomes the set of generators of H over $\mathbf{K}$. Indeed, if $[aG] \in H$, then $\bar{a} = \sum_{i=1}^{n} \beta_i\bar{a}_i$ for some $\beta_i \in k$. It follows that $a - \sum_{i=1}^{n} \beta_i a_i \in \mathfrak{m}$, thus $a = \sum_{i=1}^{n} \beta_i a_i + l$ for some $l \in \mathfrak{m}$. This implies that

$$aG = (\sum_{i=1}^{n} \beta_i a_i + l)G \in (\sum_{i=1}^{n} \beta_i a_i)G + lG \quad (\text{ in } R = A/G), \tag{4.3.39}$$

therefore, we have

$$[aG] \in \sum_{i=1}^{n} [\beta_i a_i G] = \sum_{i=1}^{n} [\beta_i G][a_i G] \quad (\text{ in } H = R/\mathfrak{n}). \tag{4.3.40}$$

Since $\beta_i \in k^\times = G$, we have

$$[\beta_i G] = \begin{cases} 0 & \text{if } \beta_i = 0 \\ 1 & \text{if } \beta_i \neq 0. \end{cases}$$

It follows that

$$[aG] \in \sum_{i=1}^{n} [\beta_i G][a_i G] = \sum_{i=1}^{n} b_i [a_i G], \text{ where } b_i \in \mathbf{K} = \{0, 1\}, \tag{4.3.41}$$

and hence $\{[a_iG]\}$ generates H over $\mathbf{K}$. This implies that $\deg(y) \leq \deg(x)$.

Conversely, suppose that $\{[a_iG]\}$ is a smallest finite set of generators of H over $\mathbf{K}$. Note that the choice is possible since $\deg(y)$ is finite by Proposition 4.3.38. We claim that $\{\bar{a}_i\}$ generates L over k. Indeed, for $\bar{\beta} \in L = A/\mathfrak{m}$, there exist $b_i \in \mathbf{K}$ such that

$$[\beta G] \in \sum_{i=1}^{n} b_i[a_i G], \text{ where } b_i \in \mathbf{K}.$$

We may assume that $b_i = 1 \ \forall i = 1, ..., n$. Then we have

$$[\beta G] \in \sum_{i=1}^{n} [a_i G] = \{[bG] \mid bG \in \sum_{i=1}^{n} a_i G\},$$

therefore $[\beta G] = [bG]$ for some $bG \in \sum_{i=1}^{n} a_i G$. However, by definition, we have the following:

$$bG \in \sum_{i=1}^{n} a_i G \Longleftrightarrow b = \sum_{i=1}^{n} a_i g_i \text{ for some } g_i \in G.$$

On the other hand, since $H = R/\mathfrak{n}$ is the quotient of the quotient hyperring $R = A/G$, we have the following:

$$[\beta G] = [bG] \Longleftrightarrow (\beta G - bG) \bigcap \mathfrak{n} \neq \emptyset, \quad \beta G, bG \in R.$$

It follows that there exists $lG \in \mathfrak{n}$ such that $lG \in \beta G - bG$, and $l = \beta g - bh$ for some $g, h \in G = k^{\times}$. Thus, we derive

$$\beta = bhg^{-1} + lg^{-1} = (\sum_{i=1}^{n} a_i g_i)hg^{-1} + lg^{-1} = \sum_{i=1}^{n} (hg^{-1})g_i a_i + lg^{-1}. \tag{4.3.42}$$

Since $lG \in \mathfrak{n}$, it follows that $l \in \mathfrak{m}$. Then, (4.3.42) implies the following:

$$\bar{\beta} = \overline{\sum_{i=1}^{n} (hg^{-1})g_i a_i} = \sum_{i=1}^{n} \overline{(hg^{-1}g_i)}\bar{a}_i = \sum_{i=1}^{n} (hg^{-1}g_i)\bar{a}_i.$$

This shows that $\{\bar{a}_i\}$ generates L over k and hence $\deg(x) \leq \deg(y)$. $\square$

Corollary 4.3.46. *Let A be a (reduced) finitely generated commutative $\mathbb{Q}$-algebra containing the field $\mathbb{Q}$ of rational numbers. Let $R := A \otimes_{\mathbb{Z}} \mathbf{K}$, $X = \operatorname{Spec} A$ be the algebraic variety over $\mathbb{Q}$, and $Y = \operatorname{Spec} R$ be the affine hyper-scheme over $\mathbf{K}$. Then,* *we have*

$$Z(Y,t) = \prod_{x\in|X|}(1-t^{\deg(x)})^{-1}.$$

Proof. It follows from $A\otimes_{\mathbb{Z}}\mathbf{K} = A/\mathbb{Q}^\times$ and Theorem 4.3.44. $\square$

The Hasse-Weil zeta function $Z(X,t)$ attached to an algebraic variety X over a finite field $\mathbb{F}_q$ is the generating function of the numbers of rational points of X over $\mathbb{F}_{q^m}$ for various $m\in\mathbb{N}$. However, in Definition 4.3.39, we define the zeta function attached to a topological space $X = \operatorname{Spec} R$ by directly applying the product formula at the price of losing the information about the size of 'rational points' of X.

Let R be a hyperdomain containing $\mathbf{K}$. We consider R as a hyper $\mathbf{K}$-algebra. Let $X = (\operatorname{Spec} R, \mathcal{O}_X)$ be an integral affine hyper-scheme over $\mathbf{K}$. We first need a suitable notion of a 'finite extension' of $\mathbf{K}$. To this end, as in §4.2.1, we only focus on the case $R_m := \mathbb{F}_{p^m}/\mathbb{F}_p^\times$ with an odd prime p regarded as an analogue of $\mathbb{F}_{p^m}$ over $\mathbf{K}$. If we naively extend the notion of rational points, we may define the set $X(R_m)$ of rational points of X over R_m as follows:

$$X(R_m) := \operatorname{Hom}_{\operatorname{Spec}\mathbf{K}}(\operatorname{Spec} R_m, X). \tag{4.3.43}$$

From Proposition 4.3.19, we have

$$X(R_m) := \operatorname{Hom}_{\operatorname{Spec}\mathbf{K}}(\operatorname{Spec} R_m, X) = \operatorname{Hom}(R, R_m). \tag{4.3.44}$$

However, unlike the classical case, the set $\operatorname{Hom}(R, R_m)$ is generally an infinite set. For example, if $m = 1$, then $R_m = \mathbf{K}$. However, it is known that $\operatorname{Hom}(R,\mathbf{K}) = \operatorname{Spec} R$ (cf. [9, §2]). This suggests that the set of rational points as in (4.3.43) is too large and hence we need to somehow reduce the size.

In the following subsections, we provide a more suitable notion of rational points (cf. Definition 4.3.61) and define another zeta function $Z(\tilde{X},t)$ as the generating function of the numbers of 'rational' points. Then, we prove that $Z(\tilde{X},t)$ is, in a suitable way, a part of the zeta function $Z(X,t)$ as in Definition 4.3.39. The following subsection is dedicated to prove several lemmas which will be used to prove the aforementioned result on $Z(\tilde{X},t)$.

Some lemmas

Throughout this subsection, by p we always mean a prime number of odd characteristic. We let $R_m := \mathbb{F}_{p^m}/\mathbb{F}_p^\times$ and $[a] := \pi(a)$, where $\pi : \mathbb{F}_{p^m} \longrightarrow R_m$ is the canonical projection map.

Lemma 4.3.47. *Let A, B be hyperrings and $\varphi : A \longrightarrow B$, $\psi : B \longrightarrow A$ be homomorphisms of hyperrings such that $\varphi \circ \psi = id_B$, $\psi \circ \varphi = id_A$. Then φ and ψ are strict.*

Proof. Since the argument is symmetric, we only show that φ is strict. For $a, b \in A$, let $x = \varphi(a)$, $y = \varphi(b)$. We need to prove that $x + y \subseteq \varphi(a+b)$. If $t \in x+y$ then $\psi(t) \in \psi(x+y) \subseteq \psi(x) + \psi(y) = \psi(\varphi(a)) + \psi(\varphi(b)) = a + b$, and it follows that $\varphi(\psi(t)) = t \in \varphi(a+b)$. □

Lemma 4.3.48. *Let A be a cyclic group of order mn, $B \subseteq A$ be a subgroup of order n, and $G := A/B$. Suppose that $G =< [\beta] >$, where $\beta \in A$ and $[\beta]$ is the equivalence class of β in G. Then there exists $\gamma \in B$ such that A is generated by $\beta\gamma$.*

Proof. Without loss of generality, we may assume that $A = \mathbb{Z}/mn\mathbb{Z} = \{0, 1, ..., mn\}$, $B = m\mathbb{Z}/mn\mathbb{Z} = \{m, 2m, ..., nm = 0\}$, and $G = \{0, 1, ..., m\} = \mathbb{Z}/m\mathbb{Z}$. Since $[\beta]$ generates G, it follows that $\gcd(\beta, m) = 1$. By Dirichlet theorem, there are infinitely many prime numbers of the form $\alpha_j = \beta + jm$. Let us pick any one of them so that $\gcd(\alpha_j, mn) = 1$. If $\alpha = \bar{\alpha_j} \equiv \alpha_j (\text{mod } mn)$ then A is generated by α. □

Lemma 4.3.49. *Let $\varphi : R_m \longrightarrow R_m$ be an isomorphism of hyperrings. Then, there exists $\tilde{\varphi} \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$ such that $\varphi([a]) = [\tilde{\varphi}(a)]$.*

Proof. Let α be a generator of the cyclic group $\mathbb{F}_{p^m}^\times$ and suppose that $\varphi([\alpha]) = [\beta]$. Since φ is an isomorphism and $[\alpha]$ generates $R_m^\times$, $[\beta]$ should generates $R_m^\times$. As a group, we have $R_m^\times = \mathbb{F}_{p^m}^\times/\mathbb{F}_p^\times$. It follows from Lemma 4.3.48 that there exists $x \in \mathbb{F}_p^\times$ such that βx generates $\mathbb{F}_{p^m}^\times$. Then α and βx generate $\mathbb{F}_{p^m}^\times$ and hence there exists $\tilde{\varphi} \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$ such that $\tilde{\varphi}(\alpha) = \beta x$. For $a \in \mathbb{F}_{p^m}^\times$, we have $[a] = [b] \iff a = by, y \in \mathbb{F}_p^\times$. Since $\tilde{\varphi} \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$, it follows that $\tilde{\varphi}(a) = \tilde{\varphi}(by) = y\tilde{\varphi}(b)$, and $[\tilde{\varphi(a)}] = [y\tilde{\varphi}(b)] = [\tilde{\varphi}(b)]$. This implies that the homomorphism

$$[\tilde{\varphi}] : R_m \longrightarrow R_m, \quad [\tilde{\varphi}]([a]) := [\tilde{\varphi}(a)]$$

is a well-defined homomorphism and $[\tilde{\varphi}]([\alpha]) = [\tilde{\varphi}(\alpha)] = [\beta x] = [\beta]$. Hence, $\varphi = [\tilde{\varphi}]$ since $[\alpha]$ generates $R_m^\times$. $\square$

Proposition 4.3.50. *1. Let $\varphi_p : R_m \longrightarrow R_m, \quad [a] \mapsto [a]^p$. Then, φ_p is an automorphism of R_m.*

2. Let $G := \mathrm{Aut}(R_m)$ be the group of automorphisms of R_m. Then, G is a cyclic group of order m generated by φ_p.

Proof. 1. Trivially, φ_p is a monoid map. For $[a], [b] \in R_m$, if $[c] \in [a] + [b]$, then $c = q_1 a + q_2 b$ for some $q_1, q_2 \in \mathbb{F}_p^\times$. It follows that $\varphi_p([c]) = [c]^p = [c^p] = [(q_1a + q_2b)^p] = [q_1^p a^p + q_2^p b^p] \in [a^p] + [b^p] = [a]^p + [b]^p = \varphi_p([a]) + \varphi_p([b])$. Hence φ_p is a homomorphism of hyperrings. Next, we claim that φ_p is strict. Suppose that $[c] \in [a^p]+[b^p] = [a]^p+[b]^p = \varphi_p([a])+\varphi_p([b])$. Then, we have $c = q_1a^p+q_2b^p$ for some $q_1, q_2 \in \mathbb{F}_p^\times$. However, since the Frobenius map is an automorphism of $\mathbb{F}_p$ and $q_i \neq 0$, there exist $d_1, d_2 \in \mathbb{F}_p^\times$ such that $d_1^p = q_1, d_2^p = q_2$. It follows that $c = d_1^p a^p + d_2^p b^p = (d_1 a + d_2 b)^p$. Therefore, we have $\varphi_p([d_1a + d_2b]) = [c]$, $[d_1a+d_2b] \in [a]+[b]$, and it follows that $\varphi_p([a])+\varphi_p([b]) \subseteq \varphi_p([a]+[b])$. Since R_m is a hyperfield and φ_p is a non-trivial map, we have $\mathrm{Ker}(\varphi_p) = \{0\}$ and hence φ_p is an injection since φ_p is strict. Then φ_p is an injection from the finite set R_m to R_m, therefore φ_p is surjective as well. It follows that φ_p is an automorphism.

2. Let H be the subgroup of $G = \mathrm{Aut}(R_m)$ generated by φ_p. We first show that $H = G$. Suppose that $\varphi : R_m \longrightarrow R_m$ is an isomorphism. Then, from Lemma

4.3.49, we know that φ is induced by some $\tilde{\varphi} \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$. However, $\mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$ is the cyclic group generated by the Frobenius map, ψ. Thus, $\tilde{\varphi} = (\psi)^l$ for some $1 \leq l \leq m$. Clearly, $[\psi] = \varphi_p$, where $[\psi] \in \mathrm{Aut}(R_m)$ is induced by ψ as in Lemma 4.3.49. It follows that $\varphi = \varphi_p^l$, thus $H = G$.

Secondly, we show that $|H| = m$. Suppose that the order $|\varphi_p|$ of φ_p is r. Then, for $[a] \in R_m$, we have $\varphi_p^m([a]) = [a]^{p^m} = [a^{p^m}] = [a]$ since $a \in \mathbb{F}_{p^m}$. This implies that φ_p^m is the identity map. It follows that $r \mid m$, in particular, $r \leq m$. Next, fix a generator α (as a group) of $\mathbb{F}_{p^m}^\times$. Since $|\varphi_p| = r$, we have $\varphi_p^r([\alpha]) = [\alpha]^{p^r} = [\alpha^{p^r}] = [\alpha]$. Therefore, there exists $x \in \mathbb{F}_p^\times$ such that $\alpha^{p^r} = \alpha x$, and $\alpha^{p^r - 1} = x$ for some $x \in \mathbb{F}_p^\times$. It follows that $(\alpha^{p^r-1})^{(p-1)} = \alpha^{(p^r-1)(p-1)} = 1$. This implies the following:

$$(p^m - 1) \mid (p^r - 1)(p - 1). \tag{4.3.45}$$

However, if $r < m$ and $3 \leq p$, then the following function:

$$f(p) := (p^m - 1) - (p^r - 1)(p - 1) = p^{(r+1)}(p^{(m-r-1)} - 1) + p^r + (p - 2)$$

is always nonnegative. Thus, it follows from (4.3.45) that $m \leq r$ and hence $r = m$.

□

Example 4.3.51. *(cf. [9, Example 2.8]) Let $F := \mathbb{F}_9/\mathbb{F}_3^\times = \{0, 1, \alpha, \alpha^2, \alpha^3\}$. Then, $\mathrm{Aut}(F) \simeq \mathbb{Z}/2\mathbb{Z}$. In fact, if $g \in \mathrm{Aut}(F)$, then the only possible images of α under g is α and α^3. One can check that both of them are indeed automorphisms of F, and the later map is the Frobenius of F.*

Proposition 4.3.52. *Let $e \mid m$ for $e, m \in \mathbb{N}$ and suppose that a homomorphism of hyperfields $\varphi : R_e \longrightarrow R_m$ satisfies the following condition:*

$$\exists \tilde{\varphi} : \mathbb{F}_{p^e} \longrightarrow \mathbb{F}_{p^m} \ \textit{s.t.}\ \tilde{\varphi}(a) = a \quad \forall a \in \mathbb{F}_p \ \textit{and}\ \varphi([b]) = [\tilde{\varphi}(b)] \quad \forall b \in \mathbb{F}_{p^e}. \tag{4.3.46}$$

Then, φ is strict.

Proof. We have to show that $\varphi([a]) + \varphi([b]) \subseteq \varphi([a] + [b])$. Suppose that $[c] \in \varphi([a]) + \varphi([b]) = [\tilde{\varphi}(a)] + [\tilde{\varphi}(a)]$, then $c = q_1\tilde{\varphi}(a) + q_2\tilde{\varphi}(b)$ for some $q_i \in \mathbb{F}_p^\times$. However, since $\tilde{\varphi}$ fixes $\mathbb{F}_p$, it follows that $c = \tilde{\varphi}(q_1 a + q_2 b)$. Therefore $[q_1 a + q_2 b] \in [a] + [b]$, and $[c] \in \varphi([a] + [b])$. $\square$

When $e \mid m$, we have the canonical injection $\tilde{\varphi} : \mathbb{F}_{p^e} \longrightarrow \mathbb{F}_{p^m}$ and $\tilde{\varphi}$ satisfies the condition (4.3.52). It follows from Proposition 4.3.52 that $\tilde{\varphi}$ induces the strict homomorphism $\varphi : R_e \longrightarrow R_m$ of hyperfields. Therefore, we may assume that R_m contains R_e and further may consider R_m as a finitely generated hyper R_e-algebra. With this justification, the notation $[R_e : R_m]$ makes sense. Then, we have the following.

Proposition 4.3.53. *$[R_m : R_e] = [\mathbb{F}_{p^m} : \mathbb{F}_{p^e}]$ for $e, m \in \mathbb{N}$ such that $e \mid m$.*

Proof. Let $\alpha := [R_m : R_e]$ and $\beta := [\mathbb{F}_{p^m} : \mathbb{F}_{p^e}]$. Let $\{[x_i]\}$ be a smallest finite set of generators of R_m over R_e. In other words, for $[a] \in R_m$, there exist $\{[d_i]\} \in R_e$ such that $[a] \in \sum_{i=1}^{\alpha}[d_i][x_i]$. We claim that $\{x_i\}$ is the set of generators of $\mathbb{F}_{p^m}$ over $\mathbb{F}_{p^e}$. Indeed, for $a \in \mathbb{F}_{p^m}$, we have $[a] \in \sum_{i=1}^{\alpha}[d_i][x_i] = \sum_{i=1}^{\alpha}[d_i x_i]$, thus $a = \sum_{i=1}^{\alpha} q_i d_i x_i$ for some $q_i \in \mathbb{F}_p^\times$. However, $q_i d_i \in \mathbb{F}_{p^e}$. Thus, we have $\beta \leq \alpha$.
Conversely, let $\{y_i\}$ be a smallest finite set of generators of $\mathbb{F}_{p^m}$ over $\mathbb{F}_{p^e}$. We show that $\{[y_i]\}$ is the set of generators of R_m over R_e. In fact, for a $[a] \in R_m$, a can be written as $a = \sum_{i=1}^{\beta} d_i y_i$, where $d_i \in \mathbb{F}_{p^e}$. It follows that $[a] \in \sum_{i=1}^{\beta}[d_i][y_i]$. Thus, $\alpha \leq \beta$. $\square$

When $e \mid m$, under the identification $R_e \subseteq R_m$, we define the subgroup

$$\mathrm{Aut}_{R_e}(R_m) := \{g \in \mathrm{Aut}(R_m) \mid g(r) = r \quad \forall r \in R_e\} \subseteq \mathrm{Aut}(R_m).$$

Then, one derives the following.

Proposition 4.3.54. *Suppose that $m = el$. Then, $\mathrm{Aut}_{R_e}(R_m) \simeq \mathbb{Z}/l\mathbb{Z}$.*

Proof. Let $H := \mathrm{Aut}_{R_e}(R_m)$ and $\varphi_p \in \mathrm{Aut}(R_m)$ be as in Proposition 4.3.50. We observe that $\varphi_p^e \in H$. Indeed, for $\beta \in \mathbb{F}_{p^e}$, we have $\varphi_p^e([\beta]) = [\beta]^{p^e} = [\beta^{p^e}] = [\beta]$. We claim that $K :=< \varphi_p^e >= H$. Clearly, we have $K \subseteq H$. Let $f \in H$. Since $\mathrm{Aut}(R_m)$ is generated by φ_p, there exists $r \in \mathbb{N}$ such that $f = \varphi_p^r$. We let β be a generator of $\mathbb{F}_{p^e}^\times$. Then, by definition, $f([\beta]) = \varphi_p^r([\beta]) = [\beta^{p^r}] = [\beta]$ since f fixes R_e. This implies that there exists $x \in \mathbb{F}_p^\times$ such that $\beta^{p^r} = x\beta$. Thus, $\beta^{p^r-1} = x \in \mathbb{F}_p^\times$, and $(\beta^{(p^r-1)})^{(p-1)} = \beta^{(p^r-1)(p-1)} = 1$. Since β is a generator of $\mathbb{F}_{p^e}^\times$, it follows that

$$(p^e-1)|(p^r-1)(p-1).$$

From the following Lemma 4.3.55, we conclude that $e \mid r$, hence $f = \varphi_p^r = (\varphi_p^e)^t$, where $r = et$. Thus, $f \in K$. Since the order of φ_p is el, the order of φ_p^e is l. This implies that $K = \mathrm{Aut}_{R_e}(R_m) \simeq \mathbb{Z}/l\mathbb{Z}$. $\square$

Lemma 4.3.55. *Let p be an odd prime number satisfying the following:*

$$(p^e-1) \mid (p^r-1)(p-1).$$

Then, e divides r.

Proof. Let $M := \frac{p^e-1}{p-1}$. Then $0, (p-1), (p^2-1), ..., (p^{(e-1)}-1)$ are all distinct modulo M since they are different numbers strictly less than M, and $(p^e-1) \equiv 0(\mod M)$. It follows that $(p^{ne}-1) \equiv 0(\mod M)\ \forall n \in \mathbb{N}$. Suppose that $r = ne+t$ and $0 \leq t < e$. Since $p^{ne+t}-p^t = p^t(p^{ne}-1) \equiv 0(\mod M)$, we have $(p^r-1) \equiv 0 \equiv (p^t-1)(\mod M)$. However, for $0 \leq t < e$, each p^t-1 is distinct modulo M. It follows that $t = 0$, thus $e \mid r$. $\square$

Let $S := \{f \in \mathrm{Hom}(R_e, R_m) \mid f \text{ is strict}\}$ be the subset of $\mathrm{Hom}(R_e, R_m)$. Then, the group $G = \mathrm{Aut}(R_m)$ acts on S in such a way that $g.f := g \circ f$ for $g \in G$ and $f \in S$. By using such action of G on S, we prove that $|S| = e$ (cf. Corollary 4.3.60).

Remark 4.3.56. *When $e \geq 3$, Corollary 4.3.60 can be derived more easily. Indeed,* *let us define the following set:*

$$Y := \{f \in \mathrm{Hom}(R_e, R_m) \mid \textit{the range of } f \textit{ has } \mathbf{K}\textit{-dimension} > 2\}.$$

From Theorem 3.13 *of [9], there exists a unique* $\tilde{f} : \mathbb{F}_{p^e} \longrightarrow \mathbb{F}_{p^m}$ *which fixes* $\mathbb{F}_p$*. Then, it follows from Proposition 4.3.52 that* f *is strict and hence* $Y \subseteq S$*.*
Conversely, suppose that $f \in S$*. Since* f *is strict and* $e \geq 3$*, it follows that the range of* f *has* $\mathbf{K}$*-dimension* > 2 *from Proposition 4.3.53. Therefore, we have* $S = Y$*. However, by Theorem* 3.13 *of [9], we have* $|Y| = \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^e}, \mathbb{F}_{p^m}) = e$ *and we conclude that* $|S| = e$*.*

For the rest of the subsection, we let $e, m \in \mathbb{N}$ such that $e \mid m$.

Lemma 4.3.57. *Let* $\varphi : R_e \longrightarrow R_m$ *be a strict homomorphism of hyperrings. Then, there exists a homomorphism* $\tilde{\varphi} : \mathbb{F}_{p^e} \longrightarrow \mathbb{F}_{p^m}$ *of fields fixing* $\mathbb{F}_p$ *such that* $\varphi([a]) = [\tilde{\varphi}(a)]$*.*

Proof. Let us fix a generator α of $\mathbb{F}_{p^e}^\times$. If $\varphi([\alpha]) = [\beta]$ then the order of $[\alpha]$ as an element of $R_e^\times$ is same as the order of $[\beta]$ as an element of $R_m^\times$. Indeed, if not, there exist i, j such that $\varphi([\alpha]^{i-j}) = [1]$. In other words, there exists l such that $0 < l < |R_e^\times|$ and $\varphi([\alpha]^l) = [\beta]^l = 1$. Since φ is strict, we should have $\varphi([1] + [\alpha]^l) = \varphi([1]) + \varphi([\alpha]^l) = [1] + [1] = \{0, 1\}$. We also have $\mathrm{Ker}(\varphi) = \{0\}$ because R_e is a hyperfield and hence we have $0 \in [1] + [\alpha^l]$. However, from the uniqueness of an additive inverse, we have $[\alpha^l] = [1]$. It follows that $\alpha^l = x \in \mathbb{F}_p^\times$, thus $(\alpha^l)^{p-1} = \alpha^{l(p-1)} = 1$. Since α generates $\mathbb{F}_{p^e}^\times$, this implies the following:

$$p^e - 1 \mid l(p-1), \text{ or } (\frac{p^e - 1}{p - 1}) \mid l.$$

But, this is impossible since $|R_e^\times| = \frac{p^e-1}{p-1}$ and $0 < l < |R_e^\times|$, therefore $|[\alpha]| = |[\beta]|$ as we claimed. Next, from Lemma 4.3.48, there exists $x \in \mathbb{F}_p^\times$ such that $|\alpha| = |\beta x|$. It follows that we have a homomorphism $\tilde{\varphi} : \mathbb{F}_{p^e} \longrightarrow \mathbb{F}_{p^m}$ of fields which maps α to βx and fixes $\mathbb{F}_p$. Then, one can observe that $\varphi([a]) = [\tilde{\varphi}(a)]$ $\forall [a] \in R_e$ as we desired. □

Proposition 4.3.58. *The group $G = \mathrm{Aut}(R_m)$ acts transitively on the set $S := \{f \in \mathrm{Hom}(R_e, R_m) \mid f$ is strict$\}$.*

Proof. If $\varphi, \psi \in S$ then there exist $\tilde{\varphi}, \tilde{\psi} \in \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^e}, \mathbb{F}_{p^m})$ which induce φ, ψ respectively as in Lemma 4.3.57. Since the group $\mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$ acts transitively on the set $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^e}, \mathbb{F}_{p^m})$, it follows that there exists $\tilde{g} \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^m})$ such that $\tilde{\varphi} = \tilde{g} \circ \tilde{\psi}$. However, $\tilde{g}$ induces the element $g \in \mathrm{Aut}(R_m)$ in such a way that $g([a]) := [\tilde{g}(a)]$. We obtain $\varphi = g \circ \psi$ and hence G acts transitively on S. $\square$

Proposition 4.3.59. *For each $f \in S := \{f \in \mathrm{Hom}(R_e, R_m) \mid f$ is strict$\}$, the stabilizer of f in $G = \mathrm{Aut}(R_m)$ is isomorphic to the subgroup $\mathrm{Aut}_{R_e}(R_m)$ of G.*

Proof. As we previously mentioned, we may assume that $R_e \subseteq R_m$ via the canonical strict and injective homomorphism $\varphi : R_e \longrightarrow R_m$ of hyperfields. Let α be a generator of $\mathbb{F}_{p^e}$. Suppose that $f : R_e \longrightarrow R_m$ is an element of S sending $[\alpha]$ to $[\beta]$. Then, it follows from the proof of Lemma 4.3.57 that $[\beta]$ is a generator of $R_e^\times \subseteq R_m^\times$. Let H_f be the stabilizer of f in $G = \mathrm{Aut}(R_m)$. Then, we have $g.f = g \circ f = f$. It follows that $g \circ f([\alpha]) = f([\alpha]) \Longleftrightarrow g([\beta]) = [\beta]$ and hence g fixes R_e. Conversely, if $g \in \mathrm{Aut}_{R_e}(R_m)$, then $g \circ f([\alpha]) = g([\beta]) = [\beta] = f([\alpha])$. Hence, g stabilizes f since $[\alpha]$ generates R_e. $\square$

Corollary 4.3.60. *Let $S := \{f \in \mathrm{Hom}(R_e, R_m) | f$ is strict$\}$ be the subset of $\mathrm{Hom}(R_e, R_m)$. Then, $|S| = e$*

Proof. Let $m = el$. The group $G = \mathrm{Aut}(R_m)$ acts transitively on $\mathrm{Hom}(R_e, R_m)$ by Proposition 4.3.58 and for each $f \in S$ the stabilizer of f in $\mathrm{Aut}(R_m)$ has l elements from Proposition 4.3.54 and 4.3.59. Thus, we obtain $|S| = e$. $\square$

The zeta function

Let R be a hyperdomain containing $\mathbf{K}$ and $X = (\operatorname{Spec} R, \mathcal{O}_X)$ be the integral affine hyper-scheme over $\mathbf{K}$. Recall that the zeta function attached to X is the following:

$$Z(X,t) := \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}, \tag{4.3.47}$$

where $\deg(x) := [k(x) : \mathbf{K}]$. We also have, from Propositions 4.3.19 and 4.3.20, the following:

$$X(R_m) := \operatorname{Hom}_{\operatorname{Spec}\mathbf{K}}(\operatorname{Spec} R_m, X) = \bigsqcup_{x \in X} \operatorname{Hom}(k(x), R_m). \tag{4.3.48}$$

In this subsection, we shows that the zeta function as in (4.3.47) contains, in a suitable sense, the information of the number $|X(R_m)|$ of 'R_m-rational points' of X $\forall m \in \mathbb{N}$. To this end, we first introduce some definitions. By $|X|$ we mean the set of closed points of X.

Definition 4.3.61. *1. $\tilde{X} := \{x \in |X| \mid k(x) \simeq R_e$ for some $e \in \mathbb{N}\}$.*

2. Let R_1 and R_2 be hyperrings. Then, we define

$$S\operatorname{Hom}(R_1, R_2) := \{f \in \operatorname{Hom}(R_1, R_2) \mid \ f \textit{ is strict}\}.$$

3. Let $\tilde{X}(R_m)$ be the following subset of $X(R_m)$:

$$\tilde{X}(R_m) := \bigsqcup_{x \in \tilde{X}} S\operatorname{Hom}(k(x), R_m) \subseteq \bigsqcup_{x \in X} \operatorname{Hom}(k(x), R_m) = X(R_m).$$

Remark 4.3.62. *Suppose that $f : R_e \longrightarrow R_m$ is a strict homomorphism, then we have $e \mid m$. In fact, since $|R_e^\times|$ should divide $|R_m^\times|$, we have*

$$\frac{p^e - 1}{p-1} \Big| \frac{p^m - 1}{p-1} \iff (p^e - 1)|(p^m - 1). \tag{4.3.49}$$

However, by the exact same argument as in Lemma 4.3.55, one can see that (4.3.49) *happens only when* $e \mid m$.

In the sequel, we assume that the number $a_r := |\{x \in \tilde{X} \mid [k(x) : \mathbf{K}] = r\}|$ is finite for each $r \in \mathbb{N}$. Let $N_m := |\tilde{X}(R_m)|$ be the cardinality of the set $\tilde{X}(R_m)$. Then, from Proposition 4.3.53, Remark 4.3.62, and Corollary 4.3.60, N_m is a finite number and one can further observe that $N_m = \sum_{r|m} r a_r$ as in the classical case. Let us define the new zeta function:

$$Z(\tilde{X}, t) := \exp(\sum_{m \geq 1} \frac{N_m}{m} t^m). \tag{4.3.50}$$

Example 4.3.63. *Let* $R = \mathbf{K}[H] \cup \{a\}$ *be the hyperring in Example 4.3.42 and* $X := \operatorname{Spec} R$. *Then,* $\tilde{X} = \{\mathfrak{p}\}$ *and* $k(\mathfrak{p}) = \mathbf{K}$. *Therefore,* $N_m = 1$ *for all* $m \in \mathbb{N}$. *We derive*

$$Z(\tilde{X}, t) := \exp(\sum_{m \geq 1} \frac{N_m}{m} t^m) = \exp(\sum_{m \geq 1} \frac{t^m}{m}) = (1 - t)^{-1} = Z(X, t).$$

Example 4.3.64. *Let* $R = \mathbf{K}[H] \cup \{e, f\}$ *be the hyperring in Example 4.3.43 and* $X := \operatorname{Spec} R$. *Then,* $\tilde{X} = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ *and* $k(\mathfrak{m}_1) = k(\mathfrak{m}_2) = \mathbf{K}$. *Therefore,* $N_m = 2$ *for all* $m \in \mathbb{N}$. *We obtain*

$$Z(\tilde{X}, t) := \exp(\sum_{m \geq 1} \frac{N_m}{m} t^m) = \exp(\sum_{m \geq 1} \frac{2t^m}{m}) = (1 - t)^{-2} = Z(X, t).$$

In general, we have

$$\log(Z(\tilde{X}, t)) = \sum_{m \geq 1} \frac{N_m}{m} t^m = \sum_{m \geq 1} (\sum_{r|m} r a_r) \frac{t^m}{m} = \sum_{m \geq 1} \sum_{r|m} \frac{r a_r}{m} t^m$$

$$= \sum_{r \geq 1} a_r \sum_{l \geq 1} \frac{t^{lr}}{l} = \sum_{r \geq 1} (-a_r) \log(1 - t^r) = \sum_{r \geq 1} \log(1 - t^r)^{-a_r} = \log(\prod_{r \geq 1} (1 - t^r)^{-a_r}).$$

Thus, we obtain the following theorem.

Theorem 4.3.65.

$$Z(\tilde{X},t) := \exp(\sum_{m\geq 1}\frac{N_m}{m}t^m) = \prod_{r\geq 1}(1-t^r)^{-a_r} = \prod_{x\in\tilde{X}}(1-t^{\deg(x)})^{-1}.$$

Since $\tilde{X} \subset |X|$, we see that, on one hand, $Z(\tilde{X},t)$ is the part of the zeta function $Z(X,t)$ as in (4.3.47). On the other hand, $Z(\tilde{X},t)$ contains the information, in a suitable sense, about the size of the sets of rational points of X. What looks more interesting is the following observation. When we construct $Z(\tilde{X},t)$, we fix an odd prime number p and hence $Z(\tilde{X},t)$ depends on the choice of such odd prime number. We can construct possibly different $Z(\tilde{X},t)$ by using various odd prime numbers, however, each of them should be a part of $Z(X,t)$ from Theorem 4.3.65. This suggests that $Z(X,t)$ encodes the information, in a suitable way, about all odd primes.

4.3.4 Connections with semi-structures

In this subsection, we use the symmetrization process of §3 to link a semi-scheme and a hyper-scheme. Throughout this subsection, we always assume that M is a semiring of characteristic one, M_S is the hyperring symmetrizing M, and $s : M \longrightarrow M_S$ is the symmetrization map unless otherwise stated.

We show that the topological space $\operatorname{Spec} M_S$ is homeomorphic to the subset X of real prime ideals (cf. Definition 4.3.67) of the topological space $\operatorname{Spec} M$ with the induced topology.

Note that the condition of semirings being of characteristic one is somewhat restrictive even thought it is natural for some applications. For example, $\mathbb{R}_{max}[T]$ is not of characteristic one since $T \oplus T^2 \notin \{T, T^2\}$. Ours though is the first attempt to link semi-scheme theory and hyper-scheme theory.

Proposition 4.3.66. *Let*

$$X := \{\mathfrak{q} \in \operatorname{Spec} M \mid \forall x \in \mathfrak{q}, \forall t \in M \ \textit{if}\ t \leq x \ \textit{then}\ t \in \mathfrak{q}\}.$$

Let X be equipped with the topology induced from $\operatorname{Spec} M$. *Then, X is homeomorphic to* $\operatorname{Spec} M_S$

Proof. Recall the definition of $M_S = s(M)$, and the symmetrization map:

$$s : M \longrightarrow M_S, \quad x \mapsto (x, 1).$$

We claim that if $\mathfrak{p} \in \operatorname{Spec} M_S$, then $\mathfrak{q} := s^{-1}(\mathfrak{p}) \in \operatorname{Spec} M$. Indeed, we have $0 \in \mathfrak{q}$. Since s is an injection, it follows that $x, y \in \mathfrak{q} \iff (x,1),(y,1) \in p$. Therefore, if $x, y \in \mathfrak{q}$, then $(x,1),(y,1) \in \mathfrak{p}$, and $(x,1)+(y,1) = (x+y,1) \in \mathfrak{p}$. Hence, $x+y \in \mathfrak{q}$. For $m \in M, x \in \mathfrak{q}$, we have $(m,1) \in M_S, (x,1) \in \mathfrak{p}$. It follows that $(mx,1) \in \mathfrak{p}$, thus $mx \in \mathfrak{q}$. This shows that $\mathfrak{q}$ is an ideal of M. Finally, $xy \in \mathfrak{q} \iff (xy,1) = (x,1)(y,1) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we know that $(x,1) \in \mathfrak{p}$ or $(y,1) \in \mathfrak{p}$. Equivalently, $x \in \mathfrak{q}$ or $y \in \mathfrak{q}$. Therefore, s induces the following well-defined map $s^{\#}$:

$$s^{\#} : \operatorname{Spec} M_S \longrightarrow \operatorname{Spec} M, \quad \mathfrak{p} \mapsto s^{-1}(\mathfrak{p}).$$

Clearly, $s^{\#}$ is continuous for the Zariski topology on $\operatorname{Spec} M_S$ and $\operatorname{Spec} M$. We first claim that $s^{\#}$ is one-to-one. This easily follows from the fact that s is an injection. Indeed, we have $s^{\#}(I) = s^{\#}(J) \iff s^{-1}(I) = s^{-1}(J)$. If $(a,1) \in I$, then $a \in s^{-1}(I) = s^{-1}(J)$. Thus, $(a,1) \in J$. Since I is a hyperideal, for $(a,-1) \in I$, we have $(a,1) \in I$. Therefore, $(a,1) \in J$. Because J is a hyperideal, we have $(a,-1) \in J$. This shows that $I \subseteq J$. Since the argument is symmetric, we also have $J \subseteq I$. Thus, we have $I = J$. Secondly, we observe that

$$s^{\#}(\operatorname{Spec} M_S) \subseteq X = \{\mathfrak{q} \in \operatorname{Spec} M \mid \forall x \in \mathfrak{q}, \forall t \in M \text{ if } t \leq x, \text{ then } t \in \mathfrak{q}\}.$$

To see this, take $\mathfrak{p} \in \operatorname{Spec} M_S$. Let $s^{\#}(\mathfrak{p}) = s^{-1}(\mathfrak{p}) := \mathfrak{q}$. Assume that $x \in \mathfrak{q}, t \in M$ with $t \leq x$. Then, $(x,1) \in \mathfrak{p}$. This implies that $(x,-1) \in \mathfrak{p}$, therefore $[(x,-1),(x,1)] \subseteq \mathfrak{p}$. Furthermore, $t \leq x$ implies that $(t,1) \in [(x,-1),(x,1)] \subseteq \mathfrak{p}$. Hence, $t \in s^{-1}(\mathfrak{p}) = \mathfrak{q}$, and we conclude that $s^{\#}(\operatorname{Spec} M_S) \subseteq X$.

Next, consider the following map ψ:

$$\psi : X \longrightarrow \operatorname{Spec} M_S, \quad \mathfrak{q} \mapsto s(\mathfrak{q}) \cup -s(\mathfrak{q}).$$

We claim that ψ is well-defined; $\mathfrak{p} := s(\mathfrak{q}) \cup -s(\mathfrak{q})$ is a prime hyperideal of M_S if $\mathfrak{q} \in X$. Indeed, we have $0 \in \mathfrak{p}$ since $0 \in \mathfrak{q}$ and $s(0) = 0$. Moreover, if $x \in \mathfrak{p}$, then either $x = s(a) = (a, 1)$ or $x = -s(a) = (a, -1)$ for some $a \in \mathfrak{q}$. If $x = s(a) = (a, 1)$, then $-s(a) = (a, -1) = -x \in \mathfrak{p}$. Similarly, if $x = -s(a) = (a, -1)$, then we have $s(a) = -x \in \mathfrak{p}$. Hence, for $x \in \mathfrak{p}$, we have $-x \in \mathfrak{p}$. Furthermore, for $T = (t, w) \in M_S$ and $X = (x, r) \in \mathfrak{p}$, we have $TX = (tx, 1)$ or $TX = (tx, -1)$. Since $x \in \mathfrak{q}, t \in M$, we have $tx \in \mathfrak{q}$. Thus, $TX \in s(\mathfrak{q}) \cup -s(\mathfrak{q}) = \mathfrak{p}$. For $x, y \in \mathfrak{p} = s(\mathfrak{q}) \cup -s(\mathfrak{q})$, we have to show that $x + y \subseteq \mathfrak{p}$. If $x = (a, 1)$ and $y = (b, 1)$, then this is trivial since $x + y \in \{x, y\}$ in this case. Similarly, when $x = (a, -1), y = (b, -1)$, this is clear. When $x = (a, 1), (b, -1)$ with $a < b$ or $b < a$, we also have $x + y \in \{x, y\}$. The only non-trivial case occurs when $x = (a, 1), y = (a, -1)$. In this case, $x + y = [(a, -1), (a, 1)]$. If $(t, 1) \in x + y$, then $t \leq a$. Since $a \in \mathfrak{q}$ and $\mathfrak{q} \in X$, it follows that $t \in \mathfrak{q}$. Hence, $(t, 1) \in \mathfrak{p}$. Similarly, for $(t, -1) \in x + y$, we have $t \leq a$. Since $a \in \mathfrak{q}$ and $\mathfrak{q} \in X$, we have $t \in \mathfrak{q}$ and $(t, 1) \in s(\mathfrak{q})$. Thus, $(t, -1) \in \mathfrak{p}$. Hence, we have $x + y \subseteq \mathfrak{p}$. This shows that $\mathfrak{p}$ is a hyperideal of M_S. Finally, suppose that $xy \in \mathfrak{p}$ with $x = (a, w), y = (b, r)$, where $w, r \in \{-1, 1\}$. Then, $xy \in \mathfrak{p}$ implies that $ab \in \mathfrak{q}$. Hence, $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. This means that $(a, 1), (a, -1) \in \mathfrak{p}$ or $(b, 1), (b, -1) \in \mathfrak{p}$ since $\mathfrak{p} = s(\mathfrak{q}) \cup -s(\mathfrak{q})$. In any case, we have $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. This proves our claim.

Next, one can observe that ψ is continuous. In fact, let I be a hyperideal of M_S. Then, for a closed subset $V(I)$ of $\operatorname{Spec} M_S$, $s^{-1}(I)$ is an ideal of M. Furthermore, $\psi^{-1}(V(I)) = V(s^{-1}(I)) \cap X$. Indeed, clearly $s^{-1}(I)$ is an ideal of M. For $\mathfrak{q} \in \psi^{-1}(V(I))$, we let $\psi(\mathfrak{q}) = s(\mathfrak{q}) \cup -s(\mathfrak{q}) := \mathfrak{p} \in V(I)$. Then, $I \subseteq s(\mathfrak{q}) \cup -s(\mathfrak{q})$. For $t \in s^{-1}(I)$, we have $(t, 1) \in I$. It follows that $(t, 1) \in s(\mathfrak{q})$ and $t \in \mathfrak{q}$. Thus, $s^{-1}(I) \subseteq \mathfrak{q}$ and $\mathfrak{q} \in V(s^{-1}(I))$. Since $\mathfrak{q} \in \psi^{-1}(V(I))$, trivially $\mathfrak{q} \in X$. Therefore, $\psi^{-1}(V(I)) \subseteq V(s^{-1}(I)) \cap X$. Conversely, if $\mathfrak{q} \in V(s^{-1}(I)) \cap X$, then $s^{-1}(I) \subseteq \mathfrak{q}$ and

$I \subseteq s(\mathfrak{q}) \cup -s(\mathfrak{q}) = \psi(\mathfrak{q})$. Thus, $\mathfrak{q} \in \psi^{-1}(V(I))$.

Now all we have to prove is that $s^{\#}$ and ψ are inverses to each other. We have

$$s^{\#}(\psi(q\mathfrak{q})) = s^{\#}(s(\mathfrak{q}) \cup -s(\mathfrak{q})) = s^{-1}(s(\mathfrak{q}) \cup -s(\mathfrak{q})) = \mathfrak{q}.$$

On the other hand, we also have

$$\psi(s^{\#}(\mathfrak{p})) = \psi(s^{-1}(\mathfrak{p})).$$

We claim that $\psi(s^{-1}(\mathfrak{p})) = \mathfrak{p}$. Indeed, let $T := (t, w) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a hyperideal, we may assume that $w = 1$. Then, $s^{-1}(T) = t \in s^{-1}(\mathfrak{p})$. However, since $\psi(s^{-1}(\mathfrak{p}))$ contains both $(t, 1)$ and $(t, -1)$, we have $T \in \psi(s^{-1}(\mathfrak{p}))$. Conversely, if $T = (t, w) \in \psi(s^{\#}(\mathfrak{p}))$, then $t \in s^{\#}(\mathfrak{p}) = s^{-1}(\mathfrak{p})$. Hence, we have $(t, 1) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a hyperideal, we also have $(t, -1) \in \mathfrak{p}$. Thus, $T \in \mathfrak{p}$. This completes our proof. □

At first glance, the definition of the set X of Proposition 4.3.66 seems to be rather obscure. However, the prime ideals in X are, in fact, real primes. Let us recall the definition (cf. [4]). For a commutative ring A, an ideal $p \subseteq A$ is a real ideal if $\sum_{i=1}^{n} r_i^2 \in p$, then $r_i \in p$. A real prime ideal of A is a prime ideal which is real. Real prime ideals are of main interest in real algebraic geometry since their notion is intimately related with that of an ordering. For more details about real prime ideals in relation with hyper-structures, see [32], [33].

We generalize the notion of a real ideal to semi-structures and hyper-structures, and show that the set X of Proposition 4.3.66 is indeed the set of real prime ideals of M. In other words, the topological space $\operatorname{Spec} M_S$ captures the 'real part' of the topological space $\operatorname{Spec} M$.

Definition 4.3.67. *1. Let M be a semiring, then an ideal $I \subseteq M$ is said to be a* *real ideal if*

$$\sum_{i=1}^{s} r_i^2 \in I \Longrightarrow r_i \in I \quad \forall i = 1, 2, ..., s, \forall r_i \in M, \forall s \in \mathbb{N}.$$

2. *Let R be a hyperring, then a hyperideal $I \subseteq R$ is said to be a real hyperideal if*

$$(\sum_{i=1}^{s} r_i^2) \cap I \neq \emptyset \Longrightarrow r_i \in I \quad \forall i = 1, 2, ..., s, \forall r_i \in R, \forall s \in \mathbb{N}.$$

3. *In either case of the above, a prime (hyper)ideal which is also a real (hyper)ideal is said to be a real prime (hyper)ideal.*

Proposition 4.3.68. *The set*

$$X := \{\mathfrak{q} \in \operatorname{Spec} M \mid \forall x \in \mathfrak{q}, \forall t \in M \ \textit{if}\ t \leq x, \ \textit{then}\ t \in \mathfrak{q}\}$$

coincides with the set of real prime ideals of M.

Proof. Let $\mathfrak{p}$ be an element of X. Suppose that $\sum_{i=1}^{s} r_i^2 \in \mathfrak{p}$. We have to show that $r_i \in \mathfrak{p}$ $\forall i$. Since M is of characteristic one, we have $x + y \in \{x, y\}$ $\forall x, y \in M$. It follows that $\sum_{i=1}^{s} r_i^2 = r_j^2$ for some $j \in \{1, 2, ..., s\}$. This implies that $r_i^2 \leq r_j^2$ for all i, where $\leq$ is the canonical order of M. Since $\mathfrak{p} \in X$, this implies that $r_i^2 \in \mathfrak{p}$ for all $i \in \{1, 2, ..., s\}$. However, $r_i^2 \in \mathfrak{p}$ implies that $r_i \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. This shows that $\mathfrak{p}$ is a real prime ideal.
Conversely, suppose that $\mathfrak{q}$ is a real prime ideal. Then, for $x \in \mathfrak{q}$ and $t \in M$ with $t \leq x$, we have

$$t \leq x \Longrightarrow t^2 \leq xt \leq x^2.$$

Therefore, $t^2 + x^2 = x^2 \in \mathfrak{q}$. Since $\mathfrak{q}$ is a real prime ideal, this implies that $t \in \mathfrak{q}$. Hence, $\mathfrak{q} \in X$. □

Proposition 4.3.69. *Any prime hyperideal of M_S is real.*

Proof. Let $\mathfrak{p}$ be a prime hyperideal of M_S. Suppose that $(\sum_{i=1}^{s} r_i^2) \cap \mathfrak{p} \neq \emptyset$. We know that any $r_i \in M_S$ is of the form $r_i = (c_i, w)$, where $c_i \in M$ and $w \in \{-1, 1\}$. Hence, $r_i^2 = (c_i^2, 1)$. It follows that $(\sum_{i=1}^{s} r_i^2)$ is a single element. In fact, we have

$$(\sum_{i=1}^{s} r_i^2) = r_j^2 \text{ for some } j \in \{1, 2, ..., s\}.$$

This implies that $r_j^2 \in \mathfrak{p}$. Since $\mathfrak{p}$ is a hyperideal, we have also $-(r_j^2) \in \mathfrak{p}$. It follows that $[-(r_j^2), r_j^2] \subseteq \mathfrak{p}$. Furthermore, for $i \in \{1, 2, ..., s\}$, we have $c_i^2 \leq c_j^2$ since $(\sum_{i=1}^{s} r_i^2) = r_j^2$. Hence, $r_i^2 \in [-(r_j^2), r_j^2] \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is a prime hyperideal, this implies that $r_i \in \mathfrak{p}$ for all i. Thus, $\mathfrak{p}$ is a real prime hyperideal. $\square$

In Proposition 4.3.66, we proved that the symmetrization map $s : M \longrightarrow M_S$ induces the continuous map $s^{\#} : \operatorname{Spec} M_S \longrightarrow \operatorname{Spec} M$. In what follows, we denote $s^{\#}$ by s for the notational convenience and also assume that M is multiplicatively cancellative. Note that such assumption on M implies that M_S is a hyperdomain. Let $X = (\operatorname{Spec} M_S, \mathcal{O}_X), Y = (\operatorname{Spec} M, \mathcal{O}_Y)$. From §2.2, we know that for each open subset $U \subseteq Y$, $\mathcal{O}_Y(U)$ is a semiring of characteristic one, hence $\mathcal{O}_Y(U)$ allows for the symmetrization process. Let $S_U : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_Y(U)_S$ be the symmetrization map for an open subset $U \subseteq Y$.

Lemma 4.3.70. *For an open subset $U \subseteq Y$, we have an isomorphism of hyperrings:*

$$S_U(\mathcal{O}_Y(U)) \simeq \mathcal{O}_X(s^{-1}(U)).$$

Proof. This follows from the fact that the symmetrization commutes with the localization. Let $R := M_S$, $V := s^{-1}(U)$, and $\hat{f} = (f, 1) \in R$ for $f \in M$. Then, by Proposition 2.2.8 and Theorem 4.3.11, we have

$$\mathcal{O}_Y(U) \simeq \bigcap_{D(f) \subseteq U} M_f, \quad \mathcal{O}_X(s^{-1}(U)) = \mathcal{O}_X(V) \simeq \bigcap_{D(\hat{f}) \subseteq V} R_{\hat{f}}. \tag{4.3.51}$$

Under the isomorphisms of (4.3.51), we may assume that $\mathcal{O}_Y(U) \subseteq \operatorname{Frac}(M)$ and $\mathcal{O}_X(s^{-1}(U)) \subseteq \operatorname{Frac}(R)$. On the other hand, from Proposition 3.1.14, we have an isomorphism:

$$h : s(\mathrm{Frac}(M)) \simeq \mathrm{Frac}(M_S) = \mathrm{Frac}(R).$$

Furthermore, by the isomorphism h and Corollary 3.1.15, we have

$$h(\bigcap_{D(f)\subseteq U} M_f) \simeq \bigcap_{D(f)\subseteq U} h(M_f) \simeq \bigcap_{D(\hat{f})\subseteq V} R_{\hat{f}}.$$

Since $h|_U = S_U$, we derive the desired result. $\square$

By combining Proposition 4.3.66 and Lemma 4.3.70, we derive the following

Theorem 4.3.71. *Let M be a (multiplicatively) cancellative semiring of characteristic one and M_S be the hyperring symmetrizing M. Then, the symmetrization map $s : M \longrightarrow M_S := R$ induces a pair of maps $(s, s^\#)$ between the hyper-scheme $X = (\mathrm{Spec}\, M_S, \mathcal{O}_X)$ and the semi-scheme $Y = (\mathrm{Spec}\, M, \mathcal{O}_Y)$ such that*

1. *$s : \mathrm{Spec}\, M_S \longrightarrow \mathrm{Spec}\, M$ is a continuous map*

2. *$s^\# : \mathcal{O}_Y \longrightarrow s_*\mathcal{O}_X$ is a morphism of sheaves (of sets) such that*

$$s^\#(U) = S_U : \mathcal{O}_Y(U) \longrightarrow s_*\mathcal{O}_X(U) = \mathcal{O}_X(s^{-1}(U))$$

and $\mathcal{O}_X(s^{-1}(U))$ is the hyperring symmetrizing the semiring $\mathcal{O}_Y(U)$.

5

Connections and Applications

5.1 Algebraic structure of affine algebraic group schemes

Let (A, Δ, m) be a commutative Hopf algebra over a field k, where $\Delta : A \longrightarrow A \otimes_k A$ is a coproduct and $m : A \otimes_k A \longrightarrow A$ is a multiplication. Let K be any field extension of k. Then, the set $X(K) = \mathrm{Hom}(\mathrm{Spec}\, K, \mathrm{Spec}\, A) = \mathrm{Hom}(A, K)$ of K-rational points of the affine group scheme $X = \mathrm{Spec}\, A$ over k has a group structure. More precisely, the group multiplication $*$ on the set $X(K)$ comes from the coproduct Δ of A. To be specific, for $f, g \in \mathrm{Hom}(A, K)$, one defines

$$f * g := m \circ (f \otimes g) \circ \Delta. \tag{5.1.1}$$

In this way, $(X(K), *)$ becomes a group.

In [7], the authors generalize the group operation (5.1.1) to hyper-structures as follows.

Definition 5.1.1. *([7, Definition 6.1]) Let $(\mathcal{H}, \Delta)$ be a commutative ring with a coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{H}$ and let R be a hyperring. Let $X = \mathrm{Hom}(\mathcal{H}, R)$ be the set of homomorphisms of hyperrings (by considering $\mathcal{H}$ as a hyperring). For $\varphi_j \in X$, $j = 1, 2$, one defines*

$$\varphi_1 *_\Delta \varphi_2 := \{\varphi \in X \mid \varphi(x) \in \sum \varphi_1(x_{(1)})\varphi_2(x_{(2)}), \quad \forall \Delta(x) = \sum x_{(1)} \otimes x_{(2)}\}. \tag{5.1.2}$$

In general, $\Delta(x)$ can have many presentations as an element of $\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{H}$, and the condition in (5.1.2) is required to hold for all presentations of $\Delta(x)$.

Lemma 5.1.2. *([7, Lemma 6.4]) Let $(\mathcal{H}, \Delta)$ be a commutative ring with a coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{H}$ and J_j be ideals of $\mathcal{H}$ for $j = 1, 2$. Then, the set*

$$J := J_1 \otimes_{\mathbb{Z}} \mathcal{H} + \mathcal{H} \otimes_{\mathbb{Z}} J_2 \tag{5.1.3}$$

is an ideal of $\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{H}$ as well as the set

$$J_1 *_{\Delta} J_2 := \{x \in \mathcal{H} \mid \Delta(x) \in J\} \tag{5.1.4}$$

is an ideal of $\mathcal{H}$. Furthermore, for $\varphi \in \varphi_1 *_{\Delta} \varphi_2$, we have

$$\mathrm{Ker}(\varphi_1) *_{\Delta} \mathrm{Ker}(\varphi_2) \subseteq \mathrm{Ker}(\varphi). \tag{5.1.5}$$

In [7], the authors prove that for a commutative ring A and for the Krasner's hyperfield $\mathbf{K}$, one has the following identification (of sets):

$$\mathrm{Hom}(A, \mathbf{K}) = \mathrm{Spec}\, A, \quad \varphi \mapsto \mathrm{Ker}(\varphi). \tag{5.1.6}$$

Thus, the underlying topological space $\mathrm{Spec}\, A$ can be considered as the set of '$\mathbf{K}$-rational points' of the affine scheme $X = \mathrm{Spec}\, A$. We also report the following

Theorem 5.1.3. *([7, Theorems 7.1 and 7.13]) Let $\mathbf{K}$ be the Krasner's hyperfield.*

1. *Let δ be the generic point of $\mathrm{Spec}\, \mathbb{Q}[T] = \mathrm{Hom}(\mathbb{Q}[T], \mathbf{K})$. Then, $\mathrm{Spec}\, \mathbb{Q}[T] \backslash \{\delta\}$ and $\mathrm{Spec}\, \mathbb{Q}[T, \frac{1}{T}] \backslash \{\delta\}$ are hypergroups via (5.1.2) and (5.1.6). Moreover, we have*

$$\mathrm{Spec}\, \mathbb{Q}[T] \backslash \{\delta\} \simeq \bar{\mathbb{Q}} / \mathrm{Aut}(\bar{\mathbb{Q}}), \quad \mathrm{Spec}\, \mathbb{Q}[T, \frac{1}{T}] \backslash \{\delta\} \simeq \bar{\mathbb{Q}}^{\times} / \mathrm{Aut}(\bar{\mathbb{Q}}).$$

2. *Let Ω be an algebraic closure of $\mathbb{F}_p[T]$. Then, $\mathrm{Spec}\, \mathbb{F}_p[T]$ and $\mathrm{Spec}\, \mathbb{F}_p[T, \frac{1}{T}]$ are*

hypergroups via (5.1.2) *and* (5.1.6)*. We also have*

$$\operatorname{Spec}\mathbb{F}_p[T] \simeq \Omega/\operatorname{Aut}(\Omega), \quad \operatorname{Spec}\mathbb{F}_p[T,\frac{1}{T}] \simeq \Omega^\times/\operatorname{Aut}(\Omega).$$

Let $(X = \operatorname{Spec} A, \mathcal{O}_X)$ be an affine group scheme. In general, the underlying topological space $\operatorname{Spec} A$ does not carry any algebraic structure. However, from (5.1.2) and (5.1.6), the authors define the hyper-operation $*$ on $X = \operatorname{Spec} A$, and show that in some cases, $(X, *)$ is a hypergroup (cf. Theorem 5.1.3).

In this section, we generalize Theorem 5.1.3 in a suitable way. Let A be a finitely generated (commutative) Hopf algebra over a field k. We show that $(X = \operatorname{Spec} A, *)$ is an algebraic object which satisfies the following conditions.

1. $(f*(g*h)) \cap ((f*g)*h) \neq \emptyset \ \forall f,g,h \in X$. (weak-associativity)
2. $\exists! e \in X$ s.t. $f*e = e*f = f \ \forall f \in X$. (the identity element)
3. For each $f \in X$, there exists (not necessarily unique) a canonical element $\tilde{f} \in X$ such that $e \in (\tilde{f}*f) \cap (f*\tilde{f})$. (an inverse element)
4. $f \in g*h \Longleftrightarrow \tilde{f} \in \tilde{h}*\tilde{g} \ \forall f,g,h \in X$. (an inversion property)

In other words, $(X = \operatorname{Spec} A, *)$ is an algebraic object which is more general than a hypergroup.

Note that in general, we can not expect the hyper-operation $*$ on $X = \operatorname{Spec} A$ to be commutative. Thus, the reversibility property of a hypergroup should be restated as an inversion property as in 4 above. Furthermore, for a Hopf ring A and $f,g \in \operatorname{Hom}(A,\mathbf{K})$, we have $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$ ([7, Lemma 6.2]), otherwise $f*g$ would be an empty set. In other words, the hyper-operation $*$ is non-trivial only within the fibers of the following restriction map

$$\Phi : \operatorname{Hom}(A,\mathbf{K}) \to \operatorname{Hom}(\mathbb{Z},\mathbf{K}) = \operatorname{Spec}\mathbb{Z}, \quad f \mapsto f|_{\mathbb{Z}}.$$

As explained in [7], one can easily check that for the generic point $\delta \in \operatorname{Spec}\mathbb{Z}$, we have the identification $\Phi^{-1}(\delta) = \mathrm{Hom}(A \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbf{K})$ which is compatible with the hyper-operations. Also, for $\wp = (p) \in \operatorname{Spec}\mathbb{Z}$, we have the identification $\Phi^{-1}(\wp) = \mathrm{Hom}(A \otimes_{\mathbb{Z}} \mathbb{F}_p, \mathbf{K})$ which is also compatible with the hyper-operations. In the following, we will focus on the case of a commutative Hopf algebra over a field k rather than a Hopf ring. In the sequel, all Hopf algebras will be assumed to be commutative. We begin with a lemma showing that if we work over a field, our hyper-operation is always non-trivial.

Lemma 5.1.4. *Let A be a Hopf algebra over a field k with a coproduct $\Delta : A \to A \otimes_k A$. If $f, g \in \mathrm{Hom}(A, \mathbf{K})$, then the set*

$$P := \Delta^{-1}(\mathrm{Ker}(f) \otimes_k A + A \otimes_k \mathrm{Ker}(g))$$

is a prime ideal of A.

Proof. Trivially, P is an ideal by being an inverse image of an ideal. Hence, all we have to show is that P is prime. Suppose that $\alpha\beta \in P$. Then, by definition, $\Delta(\alpha\beta) \in \mathrm{Ker}(f) \otimes_k A + A \otimes_k \mathrm{Ker}(g)$. This implies that for any decomposition $\Delta(\alpha\beta) = \sum \gamma_{(1)} \otimes_k \gamma_{(2)}$, we have $\sum f(\gamma_{(1)})g(\gamma_{(2)}) = 0$. Assume that $\alpha \notin P$. Then, there is a decomposition $\Delta\alpha = \sum a_i \otimes_k b_i$ such that $\sum f(a_i)g(b_i) = 1$ or $\{0, 1\}$. If $\beta \notin P$, then we also have a decomposition $\Delta\beta = \sum c_j \otimes_k d_j$ such that $\sum f(c_j)g(d_j) = 1$ or $\{0, 1\}$. For these two specific decompositions, we have

$$\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta) = (\sum a_i \otimes_k b_i)(\sum c_j \otimes_k d_j) = \sum_{i,j} a_i c_j \otimes_k b_i d_j. \qquad (5.1.7)$$

Since $\alpha\beta \in P$, we should have

$$\begin{aligned} \sum_{i,j} f(a_i c_j) g(b_i d_j) &= \sum_{i,j} f(a_i) f(c_j) g(b_i) g(d_j) \\ &= \sum_{i,j} f(a_i) g(b_i) f(c_j) g(d_j) = \sum_i [(f(a_i) g(b_i)) \sum_j f(c_j) g(d_j)] = 0. \qquad (5.1.8) \end{aligned}$$

However, since we know that $\sum_i f(a_i)g(b_i) = 1$ or $\{0,1\}$ and $\sum_j f(c_j)g(d_j) = 1$ or $\{0,1\}$, we only can have

$$\sum_i [(f(a_i)g(b_i)) \sum_j f(c_j)g(d_j)] = 1 \text{ or } \{0,1\}.$$

This contradicts to (5.1.8). Hence, either α or β should be in P. $\square$

Lemma 5.1.5. *Let A be a Hopf algebra over a field k. If $f, g \in \mathrm{Hom}(A, \mathbf{K})$, then the set $f * g$ is not empty.*

Proof. We use the same notation as in Lemma 5.1.4. For a non-zero element $a \in k$, we have $f(a) = g(a) = 1$. It follows that $k \not\subseteq P$ and hence $P \neq A$. Thus, in this case, P is a proper prime ideal. From the identification $\mathrm{Hom}(A, \mathbf{K}) = \mathrm{Spec}\, A$ of (5.1.6), we have the homomorphism $\varphi : A \to \mathbf{K}$ of hyperrings such that $\mathrm{Ker}(\varphi) = P$. We claim that $\varphi \in f * g$. Indeed, for $\alpha \in A$, suppose that $\alpha \in P$. Then, $\varphi(\alpha) = 0$ by Lemma 5.1.2. On the other hand, for any decomposition $\Delta(\alpha) = \sum a_i \otimes b_i$, we have $\sum f(a_i)g(b_i) = 0$. If $\alpha \notin P$, then $\varphi(\alpha) = 1$. However, we should also have $\sum f(a_i)g(b_i) = 1$ or $\{0,1\}$ in this case. This proves that $\varphi \in f * g$. $\square$

Remark 5.1.6. *Under the same notation as above, we consider the case of a Hopf ring A. Let p and q be distinct prime numbers and suppose that $p \in \mathrm{Ker}(f)$ and $q \in \mathrm{Ker}(g)$, where $f, g \in \mathrm{Hom}(A, \mathbf{K})$. Then, one can easily see that $p, q \in P$. This implies that $1 \in P$ and hence $P = A$. Furthermore, for $\varphi \in f * g$, we have $P \subseteq \mathrm{Ker}(\varphi)$ from Lemma 5.1.2. It follows that the only possible element φ in $f * g$ is the zero map. However, this is impossible since $\varphi(1) = 1$. Thus, in this case, we have $f * g = \emptyset$ as previously mentioned.*

Proposition 5.1.7. *Let A be a finitely generated Hopf algebra over a field k. Let H be a closed subgroup scheme of the affine algebraic group scheme $G = \mathrm{Spec}\, A$ and let $B := \Gamma(H, \mathcal{O}_H)$ be the Hopf algebra of global sections of H. Then, there exists an* *injection (of sets):*

$$\sim: \mathrm{Hom}(B, \mathbf{K}) \hookrightarrow \mathrm{Hom}(A, \mathbf{K})$$

which preserves the hyper-operations. i.e. for $f, g \in \mathrm{Hom}(B, \mathbf{K})$*, we have*

$$\widetilde{f \star g} = \tilde{f} * \tilde{g}, \tag{5.1.9}$$

where $\star$ *is the hyper-operation on* $\mathrm{Hom}(B, \mathbf{K})$ *and* $*$ *is the hyper-operation on* $\mathrm{Hom}(A, \mathbf{K})$ *as in Definition 5.1.1.*

Proof. Since H is a closed subgroup scheme of G, we know that $B \simeq A/I$ for some Hopf ideal I of A. Consider the following set:

$$X_I = \{\varphi \in \mathrm{Hom}(A, \mathbf{K}) \mid \varphi(i) = 0 \quad \forall i \in I\}.$$

Let $\pi : A \to A/I$ be a canonical projection map. We define the following map:

$$\sim: \mathrm{Hom}(B, \mathbf{K}) = \mathrm{Hom}(A/I, \mathbf{K}) \longrightarrow X_I, \quad \varphi \mapsto \tilde{\varphi},$$

where $\tilde{\varphi}$ is an element of $\mathrm{Hom}(A, \mathbf{K})$ such that $\mathrm{Ker}(\tilde{\varphi}) := \pi^{-1}(\mathrm{Ker}\,\varphi)$. Note that from the identification (5.1.6), the map $\sim$ is well-defined. Furthermore, since there is an one-to-one correspondence between the set of prime ideals of A containing I and the set of prime ideals of $B \simeq A/I$ given by $\wp \mapsto \wp/I$, the map $\sim$ is a bijection (of sets). We remark the following two facts:

1. If $\varphi \in \mathrm{Hom}(A/I, \mathbf{K})$ then $\tilde{\varphi}(r) = \varphi([r])$ for $r \in A$, where $[r] = \pi(r)$. In other words, $\tilde{\varphi} = \varphi \circ \pi$. In fact, since $\mathrm{Ker}\,\varphi = \mathrm{Ker}(\tilde{\varphi})/I$, we have

$$\tilde{\varphi}(r) = 0 \Longleftrightarrow r \in \mathrm{Ker}(\tilde{\varphi}) \Longleftrightarrow \varphi([r]) = \varphi(r/I) = 0.$$

2. For $\tilde{f}$, $\tilde{g} \in X_I$, we have $\tilde{f} * \tilde{g} \subseteq X_I$. Indeed, suppose that $\phi \in \tilde{f} * \tilde{g}$. Then, we

have to show that for $i \in I$, $\phi(i) = 0$. However, since I is a Hopf ideal, we have

$$\Delta(I) \subseteq I \otimes_k A + A \otimes_k I.$$

This implies that $\phi(i) \in \sum \tilde{f}(i_{(1)})\tilde{g}(i_{(2)}) = \{0\}$ for any decomposition $\Delta(i) = \sum i_{(1)} \otimes_k i_{(2)}$ since $\tilde{f}(a) = \tilde{g}(a) = 0\ \forall a \in I$.

Next, we prove that the map $\sim$ is compatible with hyper-operations. i.e. $\widetilde{f \star g} = \tilde{f} * \tilde{g}$. Let Δ_A be a coproduct of A and Δ_I be a coproduct of $B \simeq A/I$. Suppose that $\varphi \in f \star g$ and let $\Delta_A(r) = \sum r_{(1)} \otimes r_{(2)}$ be a decomposition of $r \in A$. We have to show that

$$\tilde{\varphi}(r) \in \sum \tilde{f}(r_{(1)})\tilde{g}(r_{(2)}).$$

Since I is a Hopf ideal, we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "\pi"'] & A \otimes_k A \arrow[d, "\pi \otimes \pi"] \\ A/I \arrow[r, "\Delta_I"] & A/I \otimes_k A/I \end{tikzcd} \tag{5.1.10}$$

It follows that $\Delta_I([r]) = \sum [r_{(1)}] \otimes_k [r_{(2)}]$. However, since $\varphi \in f \star g$, we have

$$\varphi([r]) \in \sum f([r_{(1)}])g([r_{(2)}]).$$

From the above remark 1, this implies that $\tilde{\varphi}(r) \in \sum \tilde{f}(r_{(1)})\tilde{g}(r_{(2)})$. Hence, $\tilde{\varphi} \in \tilde{f} * \tilde{g}$. Conversely, let $\tilde{f}, \tilde{g} \in X_I$ and suppose that $\psi \in \tilde{f} * \tilde{g}$. Since $\sim$ is a bijection, from the above remark 2, $\psi = \tilde{\varphi}$ for some $\varphi \in \mathrm{Hom}(B, \mathbf{K})$. We claim that $\varphi \in f \star g$. In other words, for $[r] \in A/I$ and a decomposition $\Delta_I([r]) = \sum [r_{(1)}] \otimes_k [r_{(2)}]$,

$$\varphi([r]) \in \sum f([r_{(1)}])g([r_{(2)}]).$$

Since π is surjective, we have $\mathrm{Ker}(\pi \otimes_k \pi) \subseteq \mathrm{Ker}\,\pi \otimes_k A + A \otimes_k \mathrm{Ker}\,\pi$. Therefore, from (5.1.10), we can find the following decomposition of r:

$$\Delta_A(r) = \sum r_{(1)} \otimes_k r_{(2)} + \sum i_{(1)} \otimes_k a_{(2)} + \sum a_{(1)} \otimes_k i_{(2)},$$

where $i_{(1)}, i_{(2)} \in I$ and $a_{(1)}, a_{(2)} \in A$. Since $\tilde{\varphi} \in \tilde{f} * \tilde{g}$, we have

$$\tilde{\varphi}(r) \in \sum \tilde{f}(r_{(1)})\tilde{g}(r_{(2)}) + \sum \tilde{f}(i_{(1)})\tilde{g}(a_{(2)}) + \sum \tilde{f}(a_{(1)})\tilde{g}(i_{(2)}).$$

However, it follows from the definition of $\tilde{f}$, $\tilde{g} \in X_I$ that

$$\sum \tilde{f}(i_{(1)})\tilde{g}(a_{(2)}) = \sum \tilde{f}(a_{(1)})\tilde{g}(i_{(2)}) = 0.$$

Therefore, we have $\tilde{\varphi}(r) \in \sum \tilde{f}(r_{(1)})\tilde{g}(r_{(2)})$. From the above remark 1, this implies that $\varphi([r]) \in \sum f([r_{(1)}])g([r_{(2)}])$. Hence, $\varphi \in f \star g$. $\square$

Let GL_n be the general linear group scheme. We will prove the following statements:

1. The hyper-structure $*$ on $GL_n(\mathbf{K})$ as in Definition 5.1.1 is weakly-associative.
2. The identity of $(GL_n(\mathbf{K}), *)$ is given by $e = \varphi \circ \varepsilon$, where ε is the counit of the Hopf algebra $\mathcal{O}_{GL_n}$ and $\varphi : k \to k/k^\times = \mathbf{K}$ is a canonical projection map.
3. For $f \in GL_n(\mathbf{K})$, a canonical inverse $\tilde{f}$ of f is given by $\tilde{f} = f \circ S$, where $S : \mathcal{O}_{GL_n} \longrightarrow \mathcal{O}_{GL_n}$ is the antipode map. Furthermore, we have

$$f \in h * g \Longleftrightarrow \tilde{f} \in \tilde{g} * \tilde{h}.$$

Any affine algebraic group scheme G is a closed subgroup scheme of the group scheme GL_n for some $n \in \mathbb{N}$. Assume that the above statements are true. Then, from Proposition 5.1.7, we can derive that the set $G(\mathbf{K})$ of '$\mathbf{K}$-rational points' of an affine algebraic group scheme G has the hyper-structure induced from GL_n which is weakly-associative equipped with a canonical inverse (not unique) and the identity, and also satisfies the inversion property.

In what follows, we let $A = \mathcal{O}_{GL_n} = k[X_{11}, X_{12}, ..., X_{nn}, 1/d]$ be the Hopf algebra of the global sections of the general linear group scheme GL_n over a field k, where d is the determinant of an $n \times n$ matrix. We first prove the statement 2.

Lemma 5.1.8. *The identity of the hyper-operation $*$ on $\mathrm{Hom}(A, \mathbf{K})$ is given by $e = \varphi \circ \varepsilon$, where ε is the counit of $A = \mathcal{O}_{GL_n}$ and $\varphi : k \to k/k^\times = \mathbf{K}$ is a canonical projection map.*

Proof. Let $f \in \mathrm{Hom}(A, \mathbf{K})$. We first claim that $f \in e * f$. Indeed, let $P \in A$. Then, for a decomposition $\Delta P = \sum a_i \otimes_k b_i$, we have $P = \sum \varepsilon(a_i) b_i$ since ε is the counit. It follows that

$$f(P) = f(\sum \varepsilon(a_i)b_i) \in \sum f(\varepsilon(a_i)b_i) = \sum f(\varepsilon(a_i))f(b_i).$$

Moreover, we have $f(\varepsilon(a_i)) = e(a_i)$ since

$$f(\varepsilon(a_i)) = 0 \iff \varepsilon(a_i) = 0 \iff a_i \in \mathrm{Ker}(\varepsilon) \iff e(a_i) = 0.$$

Therefore, $f(P) \in \sum f(\varepsilon(a_i))f(b_i) = \sum e(a_i)f(b_i)$. This shows that $f \in e * f$.

Next, we claim that if $g \in e * f$, then $g(P) = f(P)\ \forall P \in k[X_{ij}]$ (P does not contain a term involving $1/d$). Take such P and let $\Delta P = \sum a_t \otimes_k b_t$ be a decomposition. Let δ_{ij} be the Kronecker delta. Then, we can write a_t as $a_t = \alpha_t + \beta_t$, where $\alpha_t = \sum_l [b_l \prod_{i,j}(X_{ij} - \delta_{ij})^{m_{l,i,j}}]$ for some $b_l \in k$, $m_{l,i,j} \in \mathbb{Z}_{>0}$, and $\beta_t \in k$. Then, since $\beta_t \in k$, it follows that

$$\Delta P = \sum (\alpha_t + \beta_t) \otimes_k b_t = \sum \alpha_t \otimes_k b_t + \sum \beta_t \otimes_k b_t = \sum \alpha_t \otimes_k b_t + 1 \otimes_k (\sum \beta_t b_t).$$

However, since the ideal $< X_{ij} - \delta_{ij} >$ is contained in $\mathrm{Ker}(e)$, we have $e(\alpha_t) = 0\ \forall t$. This implies that for this specific decomposition $\Delta P = \sum \alpha_t \otimes_k b_t + 1 \otimes_k (\sum \beta_t b_t)$, we have

$$\sum e(\alpha_t)f(b_t) + e(1)f(\sum \beta_t b_t) = f(\sum \beta_t b_t).$$

Therefore, we have $g(P) = f(P) = f(\sum \beta_t b_t)$ since $g, f \in e * f$. In general, for $q \in A = k[X_{ij}, 1/d]$, there exists $N \in \mathbb{N}$ such that $d^N q \in k[X_{ij}]$. Then, from the previous claim, we have

$$f(d^N)f(q) = f(d^N q) = g(d^N q) = g(d^N)g(q).$$

However, since d is invertible, we have $f(d^N) = f(d)^N = g(d^N) = g(d)^N = 1$. It follows that $f(q) = g(q)\ \forall q \in k[X_{ij}, 1/d] = A$. Thus $f = g$, and $\{f\} = e * f$. Similarly, one can show that $\{f\} = f * e$. This completes our proof. □

Next, we prove the existence of a canonical inverse.

Lemma 5.1.9. *Let $S : A \longrightarrow A$ be the antipode map. Then, for $f \in GL_n(\mathbf{K})$, we have $e = \varphi \circ \varepsilon \in (f * \tilde{f}) \cap (\tilde{f} * f)$, where $\tilde{f} = (f \circ S)$.*

Proof. Let $f \in \mathrm{Hom}(A, \mathbf{K})$ and $\tilde{f} = f \circ S$. Suppose that $a \in A$. Then, for a decomposition $\Delta a = \sum a_i \otimes_k b_i$, we have $\varepsilon(a) = \sum a_i S(b_i)$ since ε is the counit and S is an antipode map. This implies that

$$f(\varepsilon(a)) = f(\sum a_i S(b_i)) \in \sum f(a_i S(b_i)) = \sum f(a_i)f(S(b_i)) = \sum f(a_i)\tilde{f}(b_i).$$

However, we know that $f(\varepsilon(a)) = 1$ if $\varepsilon(a)$ is non-zero and $f(\varepsilon(a)) = 0$ if $\varepsilon(a)$ is zero. It follows that $e(a) = \varphi(\varepsilon(a)) = f(\varepsilon(a))$. Hence, $e(a) \in \sum f(a_i)\tilde{f}(b_i)$. This shows that $e \in f * \tilde{f}$. Similarly, one can show that $e \in \tilde{f} * f$. □

Next, we prove the inversion property.

Lemma 5.1.10. *Let $S : A \longrightarrow A$ be the antipode map and $f, g, h \in \mathrm{Hom}(A, \mathbf{K})$. Let $\tilde{f} = f \circ S$, $\tilde{g} = g \circ S$, $\tilde{h} = h \circ S$. Then, $h \in f * g$ if and only if $\tilde{h} \in \tilde{g} * \tilde{f}$.*

Proof. Suppose that $\tilde{h} \in \tilde{g} * \tilde{f}$. Let $a \in A$ and $\Delta a = \sum a_i \otimes_k b_i$ be a decomposition of a.

Let $t : A\otimes_k A \longrightarrow A\otimes_k A$ be the twist homomorphism of Hopf algebras. i.e. $t(a\otimes_k b) = b \otimes_k a$. Then, since $\Delta \circ S = t \circ (S \otimes_k S) \circ \Delta$, we have $\Delta(S(a)) = \sum S(b_i) \otimes_k S(a_i)$. This implies that $\tilde{h}(S(a)) \in \sum \tilde{g}(S(b_i))\tilde{f}(S(a_i)) = \sum \tilde{f}(S(a_i))\tilde{g}(S(b_i))$ since $S^2 = id$. However, we have $\tilde{h}(S(a)) = h \circ S(S(a)) = h(a)$. Similarly, $\tilde{g}(S(b_i)) = g(b_i)$ and $\tilde{f}(S(a_i)) = f(a_i)$. Thus, $h(a) \in \sum f(a_i)g(b_i)$. This shows that $h \in f * g$.
Conversely, suppose that $h \in f * g$. Then, for $a \in A$ and a decomposition $\Delta a = \sum a_i \otimes_k b_i$, we have $\tilde{h}(a) \in \tilde{g}(b_i)\tilde{f}(a_i)$. However, by the exact same argument as above and the fact that $S = S^{-1}$, one can conclude that $\tilde{h} \in \tilde{g} * \tilde{f}$. $\square$

Finally, we prove that the hyper-operation $*$ on $\mathrm{Hom}(A, \mathbf{K})$ is weakly-associative.

Lemma 5.1.11. *Let A be a Hopf algebra over a field k, Δ be a coproduct of A, and $H := (\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta : A \longrightarrow A \otimes_k A \otimes_k A$. For $f, g, h \in \mathrm{Hom}(A, \mathbf{K})$, we let $J := \mathrm{Ker}(f) \otimes_k A \otimes_k A + A \otimes_k \mathrm{Ker}(g) \otimes_k A + A \otimes_k A \otimes_k \mathrm{Ker}(h)$. Then, the set $P := H^{-1}(J)$ is a proper prime ideal of A. Moreover, if φ is an element of $\mathrm{Hom}(A, \mathbf{K})$ determined by P, then $\varphi \in f * (g * h) \cap (f * g) * h$.*

Proof. The proof is similar to Lemma 5.1.4. For the first assertion, since J is clearly an ideal by being an inverse image of an ideal, we only have to prove that P is prime. Let $\alpha\beta \in P$. Then, since $H(\alpha\beta) \in J$, for any decomposition $H(\alpha\beta) = \sum \gamma_{(1)}\otimes_k\gamma_{(2)}\otimes_k\gamma_{(3)}$, we have

$$\sum f(\gamma_{(1)})g(\gamma_{(2)})h(\gamma_{(3)}) = 0. \tag{5.1.11}$$

Suppose that $\alpha, \beta \notin P$. Then, there exist decompositions $H(\alpha) = \sum a_i \otimes_k b_i \otimes_k c_i$ and $H(\beta) = \sum x_j \otimes_k y_j \otimes_k z_j$ such that

$$\sum f(a_i)g(b_i)h(c_i) = 1 \text{ or } \{0, 1\}, \quad \sum f(x_j)g(y_j)h(z_j) = 1 \text{ or } \{0, 1\}. \tag{5.1.12}$$

With these two specific decompositions, we have

$$H(\alpha\beta) = H(\alpha)H(\beta) = (\sum_i a_i \otimes_k b_i \otimes_k c_i)(\sum_j x_j \otimes_k y_j \otimes_k z_j) = \sum_{i,j} a_ix_j \otimes_k b_iy_j \otimes_k c_iz_j.$$

Since $\alpha\beta \in P$, we should have

$$\sum_{i,j} f(a_i x_j) g(b_i y_j) h(c_i z_j) = \sum_{i,j} f(a_i) g(b_i) h(c_i) f(x_j) g(y_j) h(z_j)$$
$$= \sum_i [f(a_i) g(b_i) h(c_i) \sum_j f(x_j) g(y_j) h(z_j)] = 0. \quad (5.1.13)$$

However, (5.1.13) contradicts to (5.1.12). It follows that $\alpha \in P$ or $\beta \in P$. Furthermore, since $H(1) = 1 \otimes 1 \otimes 1 \notin J$, P is proper. This proves the first assertion.
For the second assertion, it is enough to show that $\varphi \in f * (g * h)$ since the argument for $\varphi \in (f * g) * h$ will be symmetric. Let $\psi \in g * h$ such that $\mathrm{Ker}(\psi) = \Delta^{-1}(\mathrm{Ker}(g) \otimes_k A + A \otimes_k \mathrm{Ker}(h))$. This choice is possible by Lemma 5.1.4. We claim that $\varphi \in f * \psi$. Indeed, we have to check two cases. The first case is when $a \in A$ has a decomposition $\sum a_i \otimes_k b_i$ such that $\sum f(a_i)\psi(b_i) = 0$. Then, we have to show that $\varphi(a) = 0$. But, since $\sum f(a_i)\psi(b_i) = 0$, we know that $\sum a_i \otimes_k b_i \in \mathrm{Ker}(f) \otimes_k A + A \otimes_k \mathrm{Ker}(\psi)$. Since $\mathrm{Ker}(\psi) = \Delta^{-1}(\mathrm{Ker}(g) \otimes_k A + A \otimes_k \mathrm{Ker}(h))$, we have $H(a) = (id \otimes_k \Delta)(\sum a_i \otimes_k b_i) \in \mathrm{Ker}(f) \otimes_k A \otimes_k A + A \otimes_k \mathrm{Ker}(g) \otimes_k A + A \otimes_k A \otimes_k \mathrm{Ker}(h)$. Thus, $\varphi(a) = 0$. The second case is when $a \in A$ has a decomposition $\sum x_j \otimes_k y_j$ such that $\sum f(x_j)\psi(y_j) = 1$. Then, there exist x_i, y_i such that $f(x_i) = \psi(y_i) = 1$ and $f(x_j)\psi(y_j) = 0\ \forall j \neq i$. We may assume that $i = 1$. Then, $\sum_{i \geq 2} x_i \otimes_k y_i \in \mathrm{Ker}(f) \otimes_k A + A \otimes_k \mathrm{Ker}(\psi)$. This implies that $(id \otimes_k \Delta)(\sum_{i\geq 2} x_i \otimes_k y_i) \in J$. On the other hand, $(id \otimes_k \Delta)(x_1 \otimes_k y_1) \notin J$ since $x_1 \notin \mathrm{Ker}(f)$ and $y_1 \notin \mathrm{Ker}(\psi)$. It follows that $H(a) \notin J$, hence $\varphi(a) = 1$ as we desired. $\square$

By combining the above lemmas, we obtain the following result.

Theorem 5.1.12. *Any affine algebraic group scheme $X = \operatorname{Spec} A$ over a field k has a canonical hyper-structure $*$ induced from the coproduct of A which is weakly-associative and it is equipped with the identity element e. For each $f \in X$, there exists a canonical element $\tilde{f} \in X$ such that $e \in (f * \tilde{f}) \cap (\tilde{f} * f)$. Furthermore, for $f, g, h \in X$, the following holds: $f \in g * h \Longleftrightarrow \tilde{f} \in \tilde{h} * \tilde{g}$.*

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Curriculum Vitae

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