HEAT KERNELS ON RIEMANNIAN POLYHEDRA AND HEAT FLOWS INTO NPC MANIFOLDS

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Abstract

We extend the results of [ES] to show that for a continuous initial map f with bounded pointwise energy from a flat, compact, admissible polyhedron to a smooth compact Riemannian manifold with non-positive sectional curvature, there exists a heat flow beginning at f that converges uniformly and in energy to a harmonic map. We show that this heat flow is in $C^{1+\alpha,1+\beta}$, $\alpha, \beta > 0$, on open sets bounded away from the $(n-2)$-skeleton, satisfies a natural balancing condition on the $(n-1)$-skeleton, and solves the harmonic map heat flow equation pointwise on the interior of top-dimensional simplexes. We develop Gaussian-type estimates for the gradient of heat kernel on a flat, compact, admissible polyhedron, and methods to address existence and regularity of partial differential equations on admissible polyhedra.

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Contents

1 Introduction

1.1 Overview

We extend the results of [ES] to show that for a continuous initial map f with bounded pointwise energy from a flat, compact, admissible polyhedron to a smooth compact Riemannian manifold with non-positive sectional curvature, there exists a heat flow beginning at f that converges uniformly and in $W^{1,2}$ to a harmonic map. We show that this heat flow is in $C^{1+\alpha,1+\beta}$, $\alpha, \beta > 0$, on open sets bounded away from the $(n-2)$-skeleton, satisfies a natural balancing condition on the $(n-1)$-skeleton, and solves the harmonic map heat flow equation pointwise on the interior of top-dimensional simplexes. We develop Gaussian-type estimates for the gradient of heat kernel on a flat, compact, admissible polyhedron, and methods to address existence and regularity of partial differential equations on admissible polyhedra.

1.2 Main Results

We consider an admissible Riemannian polyhedron X (see Definition 2.2 on page 22) and a compact smooth Riemannian manifold N with non-positive sectional curvature. In Proposition 5.45, Corollary 5.46 and Theorem 5.47, we obtain the following.

Theorem. *Let X be compact and simplex-wise flat and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in $C^1(X)$ and have bounded energy density. There exists a strong embedded solution W to the harmonic map heat flow problem with initial value F_0 on $X \times [0, \infty)$ with the following properties:*

i. W is continuous on $X \times [0, \infty)$; $W \in C^{1+\alpha,1+\beta}_{\text{loc}}(X \setminus X^{(n-2)} \times [0, \infty), N)$, $\alpha, \beta > 0$;

for each $t > 0$, W *is balanced (defined below); and* W *satisfies at manifold points*

$$\frac{\partial}{\partial t} W = \tau(W),$$

where $\tau = \mathrm{trace}_g \nabla dW$ *and* g *is the metric tensor.*

ii. *As* t *goes to infinity,* $W(\cdot, t)$ *converges uniformly and in* $W^{1,2}$ *to a harmonic map.*

iii. F_0 *is free-homotopic to a harmonic map.*

W *has the* balancing condition *if, at any point* p *on an* $(n-1)$*-face with adjoining* n*-simplexes* $s_1, \ldots, s_k$ *and for every coordinate* γ *in a neighborhood of* $W(p)$

$$\sum_{i=1}^{k} \frac{\partial W^\gamma}{\partial n_i}(p) = 0,$$

where $\frac{\partial}{\partial n_i}$ *is the normal to the edge of* s_i *that contains* p *(we presume the normals to point towards the interior of* s_i*).*

As there are many results involving energy flow methods where one begins with a specified map that converges under the flow to harmonic map, we refer to our subsequent section to describe the history and relevance of these results. As far as we are aware, this is the strongest regularity result that exists for heat flows in the context of an admissible polyhedron as the domain and a smooth nonpositively curved manifold as the target.

We also develop strong regularity results for the heat kernel on X and new Gaussian-type estimates for the gradient of the heat kernel. In particular we prove the following in Proposition 4.11 and Theorem 4.18 (on pages 62 and 67, respectively).

Theorem. *Let $h(z,v,t)$ be the heat kernel on X, an admissible Riemannian polyhedron. For $t > 0$,*

i. *$h(z_0, v_0, t)$ is $C^\infty((0,\infty))$ with respect to t for $z_0, v_0 \in X$.*

ii. *For any compactly contained open set $V \subset X$ bounded away from $X^{(n-2)}$, $h(z_0, v, t_0) \in C^\infty(\overline{V})$, for any $z_0 \in X$.*

iii. *For any compactly contained open set $V \subset X$ bounded away from $X^{(n-2)}$, $h(z, v_0, t_0) \in C^\infty(\overline{V})$, for any $v_0 \in X$.*

iv. *$h(z,v,t)$ is balanced in both z and v.*

Additionally, assume that X is compact and simplex-wise flat and let $\{Z_i\}_{i=1}^n$ be an orthonormal basis as in Definition 2.5. Then, for any $R > 0$, there exists positive constants $B, \{C_j\}$ only dependent on X and R such that for all $z, v \in X$,

$$\left| D_Z \frac{\partial^j}{\partial t^j} h(z,v,t) \right| \leq \frac{C_j}{\min\{t,R\}^{\frac{n}{2}}} t^{-j-\frac{1}{2}} e^{-\frac{d(z,v)^2}{Bt}}$$

where $\{C_j\}$ are dependent on j, X and R.

1.3 Approach

We show the existence of a flow between an admissible Riemannian polyhedron and a compact Riemannian manifold with nonpositive sectional curvature by following the results of [ES, PSC, St2]. In [St1, St2, St3], Sturm shows methods of defining weak solutions to the heat flow when the domain is a Dirichlet space with conditions and the target is $\mathbb{R}$; specifically, the domain must satisfy a volume doubling property and have a uniform lower bound for Poincaré constants on balls. In [PSC], Pivarski and Saloff-Coste show that an admissible Riemannian polyhedron with reasonable geometric restrictions satisfies these conditions. Also in [St1, St2, St3], Sturm shows the existence of a heat kernel that satisfies many of the properties expected in the case when the domain is a region in $\mathbb{R}^n$, and shows that this heat kernel satisfies a parabolic Harnack inequality and hence is Hölder continuous in time and space. Other Gaussian-type estimates for the heat kernel are given. Clearly, using heat kernels, which take values in $\mathbb{R}$, will not directly solve the heat flow problem when the target is a compact Riemannian manifold unless it is $\mathbb{R}^n$. By embedding the target isometrically in Euclidean space and following the arguments of [ES], we show that this heat kernel can be used to build a sequence of maps that converge to one that solves the heat flow problem in the style of [ES] and converges in time to a harmonic map. In the event that the energy density of the map under flow does not stay bounded, we appeal to [Ma]. We adapt the results and methods of [BSCSW] and [DM3] to obtain strong regularity results for the heat flow.

We presume that the target, N, a smooth compact Riemannian manifold with nonpositive sectional curvature is embedded in $\mathbb{R}^q$ for some $q \in \mathbb{N}$. We also presume that our initial map F_0 is continuous and has continuous, bounded first order derivatives on each n-simplex. The heart of the argument is the existence and properties of a solution $W\colon X \times [0, \infty) \to N \subset \mathbb{R}^q$ defined in each coordinate γ by

$$W^\gamma(z,t) = \int_0^t \int_X h(z,v,t-\tau) G^\gamma(v,\tau)\, d\mu(v) d\tau + \int_X h(z,v,t) F_0^\gamma(v)\, d\mu(v), \quad (1.1)$$

where, in local coordinates,

$$G^\gamma(v,\tau) := A^\gamma_{\alpha\beta}(W)\frac{\partial W^\alpha}{\partial v^i}\frac{\partial W^\beta}{\partial v^j}g^{ij},$$

and A is the trace of the second fundamental form of the nearest-point projection map (see Proposition 5.8) with $A^\gamma_{\alpha\beta}$ as the coefficients of $A^\gamma(dW, dW)$; also $h(z,v,t)$ is the heat kernel on defined on $X \times X \times (0,\infty)$, which we explore later.

We note that terms of W appear on both sides of equation (1.1), and it is not obvious at all that such a W should exist that satisfies it. However, once existence is established, we can show from the properties of the heat kernel that is a solution as noted in our main theorem.

We break the approach into four steps.

i. Linear case: we consider the case where the target is $\mathbb{R}$ and show the existence and regularity of weak solutions and of a heat kernel that will be a fundamental tool in the non-linear case. See Sections 3 and 4.

ii. Short time existence: we show for a continuous initial map $F_0\colon X \to N \subset \mathbb{R}^q$ in C^1, there exists an $\epsilon > 0$ dependent on the energy density of F_0 such that a solution W exists on the time interval $[0,\epsilon)$. We also establish a Gaussian-type gradient estimate for the heat kernel. See Section 5.4.

iii. Long time existence: we show that if there exists a solution on an open time interval $[0,T)$ it can be extended to a longer interval $[0, T+\delta)$, which give existence of a solution in infinite time. See Section 5.6

iv. Convergence to a harmonic map: by long time existence, we show that as

$t \to \infty$, W must converge to a harmonic map. We show that this constitutes a free homotopy from the initial map to the harmonic map. See Section 5.6.

Each of these steps deserves a more through explanation, which we give below, and we reserve discussion regularity for the end of this section.

The linear case We use the results of [St2, PSC] to show that for $f \in L^2(X)$ there exists a solution $u\colon X \times [0, \infty) \to \mathbb{R}$ that weakly satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0.$$

and $\lim_{t\to 0} u = f$ in L^2 (see Section 3.1 for definitions, and Section 4.1 for proofs). Such solutions are given by an integral kernel $h\colon X \times X \times (0, \infty) \to \mathbb{R}$ such that

$$u(z,t) := \int_X h(z,v,t) f(v)\, dv.$$

h is called *the heat kernel* (see Section 4.2 for definition and existence). The existence of such solutions are dependent on the existence of a Dirichlet form corresponding to energy (see Section 2.2), and on X having a volume doubling property and a lower bound on Poincaré inequality of balls of a fixed radius (see Section 2.3). In [PSC], Pivarski and Saloff-Coste show that an admissible polyhedron X has both properties and that there exists a Dirichlet form that corresponds to the Korevaar-Schoen-type energy functional (see [DM2] for an example). We develop regularity results for this linear, homogeneous setting (see Section 3) to show that the heat kernel and weak solutions of the heat equation are balanced and highly regular away from the $(n-2)$-skeleton of X. Our regularity approach follows from the work of [BSCSW], where a so-called *strip complex* is considered.

Short time existence We now consider the non-linear case where the target is a smooth compact manifold, N, with non-positive sectional curvature. To show the existence of a solution to the heat flow on a small interval, we do not attempt to solve equation (1.1) directly. Instead, we follow the approach of [ES] and show there is always a small interval $[0, \epsilon)$, where $\epsilon > 0$ is dependent on the energy density of the initial map, such that a sequence of approximating maps converge in energy to a continuous limit. We define our sequence of approximating maps, $\left\{W^l\right\}_{l=0}^{\infty}$, as follows. In each coordinate and for each $l \in \mathbb{N}$, let

$$\left.\begin{aligned} W^{0,\gamma}(z,t) &= \int_X h(z,v,t) F_0^{\gamma}(v)\, dv, \\ W^{l,\gamma}(z,t) &= \int_0^t \int_X h(z,v,t-\tau) G^{l-1,\gamma}(v,\tau)\, dv\, d\tau + W^{0,\gamma}(z,t), \end{aligned}\right\}$$

where

$$G^{l,\gamma}(v,\tau) = A^{\gamma}_{\alpha\beta}(W^l) \left(\frac{\partial W^{l,\alpha}}{\partial v^i}\right) \left(\frac{\partial W^{l,\beta}}{\partial v^j}\right) g^{ij}.$$

We describe the details of the convergence and the existence of a positive ϵ in Section 5.4. Crucial to the convergence of these maps is a Gaussian-type estimate for the gradient of the heat kernel. Specifically, for a fixed $R > 0$, there exists $B, C > 0$ such that the following holds:

$$|\nabla_z h(z,v,t)| \leq \frac{C}{\min\left\{t, R\right\}^{\frac{n}{2}}} t^{-\frac{1}{2}} e^{-\frac{d(z,v)^2}{Bt}},$$

where ∇_z denotes the gradient with respect to the z-slot. This is a new result and we give proofs in Section 4.4.

Long time existence Our approach to show that a solution as in equation (1.1) can be extended to exist on all of $[0, \infty)$ works by contradiction. We show that if a solution exists on an open-ended interval $[0, T)$, then it must exist on $[0, T]$ which means that by our result for short term existence, it must exist on $[0, T + \epsilon)$. To show this convergence of a solution on a closed interval, we develop regularity results in Section 4.5 that we can use with the Arzelà-Ascoli theorem. Specifically, if W is a solution on some interval $[0, T)$, then we show that for any open set A bounded away from the $(n-2)$-skeleton,

$$W|_{\overline{A}} \in C^{1+\alpha, 1+\beta}(\overline{A} \times [0, T), N),$$

for some $\alpha, \beta > 0$. We also show that, in finite time, the pointwise energy of the flow must stay bounded.

Convergence to a harmonic map To show convergence to a harmonic map as t goes to ∞, we split our consideration into two cases. In the first case, we presume that the energy density remains bounded as $t \to \infty$ and, in the second case, we allow the possibility that the supremeum of energy density goes to infinity, but the total energy remains bounded.

In the first case, we use an approach similar to the one used to show long time existence. However, to be able to use the Arzelà-Ascoli theorem, we need to develop Schauder-type estimates for regions bounded away from the $(n-2)$-skeleton of X. Specifically, for an open set Ω bounded away from the $(n-2)$-skeleton of X and for

$T > 0$, we have for a solution W to the flow,

$$|u(\cdot,t)|_{C^{1+\alpha}(Q'_T,\mathbb{R}^q)} \leq C,$$

where $\Omega' \subset \Omega$ is compactly contained, $Q_T := \Omega \times (0,T)$, $Q'_T := \Omega' \times (0,T)$, and C is dependent on $X, N, \alpha, \operatorname{dist}(\partial\Omega, \Omega')$, $|u|_{C^0(Q_T)}$, $|\nabla u|_{C^0(Q_T)}$, $|\nabla f_0|_{C^\beta(\Omega)}$. Most importantly, we show that C is not dependent on T and that the other terms on which C is dependent are bounded. See Section 5.5. Our approach is based on a technique used in [DM2]. This result along with the Arzelà-Ascoli theorem allows us to show convergence in energy to a harmonic map, which is proven in Section 5.6.

In the case that the supremum of the energy density goes to ∞ (but the density is in $L^1(X)$), we show that the flow is identical to a Gradient-of-Energy flow defined in [Ma] and thus use results about this flow to get convergence to a harmonic map in energy. The regularity results away from the $(n-2)$-skeleton remain the same as in the case with bounded energy density.

Regularity For regularity for linear, homogeneous elliptic- and parabolic-type equations of the form

$$\left(\frac{\partial}{\partial t} - \Delta\right) f = 0 \quad \text{and} \quad \Delta u = 0,$$

we adopt the results of [BSCSW] nearly directly. See Section 3. Their main tool is a hypoellipticity-type result for manifolds that is then adapted for neighborhoods on the singular set, which are manifolds of codimension 1. Specifically, they are able to show the following: let M be a manifold, $f \in L^2(M)$ and F a distribution defined by

$$F = \left(\mathrm{Id} + \sqrt{-\Delta_M}\right) f,$$

where Δ_M is the Laplacian on M and $\sqrt{-\Delta_M}$ is a hypoelliptic operator. Also, let $\Omega \subset \Omega'$ both be compactly contained open sets in M. If $F(\phi) = 0$ for all $\phi \in C_c^\infty(\Omega)$ and there exists $u \in L^2(M)$ such that

$$F(u) = \int_M \phi u \, dM$$

for all $\phi \in C_c^\infty(M \setminus \overline{\Omega'})$, then $f \in C_{\text{loc}}^\infty(\Omega)$. We apply this result to the setting of the domain being an admissible complex to neighborhoods of $(n-1)$-skeleton, bounded away from the $(n-2)$-skeleton. We extend this result to show that it applies in our setting to nonhomogeneous and non-linear equations such as the harmonic map equation and the heat flow equation. See Sections 4.5 and 3.5.

We also find it necessary to develop Schauder estimates for neighborhoods on the $(n-1)$-skeleton but away from the $(n-2)$-skeleton. It is tempting to use global Schauder estimates where a region of the $(n-1)$-skeleton may be considered the boundary. This has the disadvantage of requiring boundedness of high-order derivatives of the solution on the $(n-1)$-skeleton. We instead use a "folding" technique of [DM3] to redefine the heat flow on such a neighborhood, and transform it to a heat-type equation in a region of $\mathbb{R}^n$, where n is the dimension of our domain. This is achieved by taking a solution to the flow, showing it is balanced, and then constructing a different solution to a differential equation on a ball in Euclidean space by taking linear combinations of the nonlinear solution near a point on the $(n-1)$-skeleton. We can then apply standard results from [LSU] to achieve Schauder estimates that do not require control of high order derivatives of solutions on the $(n-1)$-skeleton. We present this approach in Section 5.5.

1.4 History and Relevance

As of the writing of this paper, there are a number of results on heat flows between metric spaces of various smoothness. They may roughly be divided into a set of classical results, where the domains are smooth manifolds, and more modern results on spaces with a variety of singularities. We give a history of results and end this section with an exposition on the relevance of our results.

Classical results The classical result is the heat flow described by Eells and Sampson in [ES]. The assumptions are that the domain and target are smooth, compact Riemannian manifolds. Additionally, there are assumptions of nonpositive sectional curvature on the target and that the domain has no boundary. In this case, given a smooth initial map f_0, they show the existence of a smooth heat flow u that satisfies pointwise

$$\frac{\partial}{\partial t}u(z,t) = \tau(u(z,t)),$$

(τ is the torsion field of the map $u(\cdot,t)$) and that converges to a harmonic map with strong regularity properties. Given additional or weaker assumptions, a variety of other results can be shown. This method, however, does not include the case of a domain with boundary. In [H], Richard Hamilton showed that flow methods can be used in this case, too, to achieve similar results, including boundary regularity.

Modern Results Recent results aim to extend the flow-like properties of the approach of [ES] from the setting of manifolds to a setting where the domain and target may have singularities. As in the case with the setting of [ES], the goal is to show that flows exist in long time and that as time goes to infinity a limit exists and is harmonic. Specifically, there are the results of [C], where the domain is an orbifold and the target is a compact Riemannian manifold with nonpositive sectional curvature. Also, in [CR], Chiang and Ratto consider the case where the domain is a compact manifold with a finite number of conical singularities and the target is a compact Riemannian manifold with nonpositive sectional curvature. In [Ma], Mayer shows a very general result for a gradient-of-energy flow where the domain is merely an Alexandrov space with non-positive curvature (in the sense of Alexandrov) supplied with a lower-semicontinuous functional. The works of [AGS1, AGS2, St1, St2, St3, St4] can be divided into two categories. In one case, flows are considered when the domain is singular and the target is $\mathbb{R}$ and the other when the target is a locally compact length space with nonpositive curvature (a subset of the cases considered by Mayer in [Ma]). We give more details of each approach below.

Remark 1.1. We note that although we cite [AGS1, AGS2, St1, St2, St3, St4] as examples of results for flows when the domain is singular and the target is $\mathbb{R}$, it is hardly exhaustive. Rather we choose these as seminal representations of the expansive body of literature on the matter.

The Results of Mayer Mayer's results in [Ma] only assume that there is a complete length space $(\mathcal{M}, d)$ nonpositively curved in the Alexandrov sense and that there is a lower semicontinous, convex functional $F\colon (\mathcal{M}, d) \to \mathbb{R} \cup \infty$. From this it is possible to define a flow on $(\mathcal{M}, d)$ such that, given an an element $u \in L$ with $F(u) < \infty$, there is a flow $\{u_t\}_{t>0} \subset \mathcal{M}$ that satisfies some global properties similar to those satisfied by a map between two compact, smooth manifolds where the target as nonnegative sectional curvature. This is the Gradient-of-Energy Flow, which we review in more detail in Section 5.3. As part of this program, Mayer defines a so-called *norm of the gradient vector*. We summarize the results as follows.

Definition. Let $(\mathcal{M}, d)$ be a complete, nonpositively curved length space and let $F\colon \mathcal{M} \to \mathbb{R} \cup \infty$ be a lower semi-continuous, convex functional. We define the *norm of the gradient vector* at f_0 as

$$|\nabla_{-}F|(f_0) := \max\left\{ \limsup_{f \to f_0, f \in \mathcal{M}} \frac{F_\epsilon(f_0) - F_\epsilon(f)}{d(f_0, f)}, 0 \right\}.$$

Theorem. *For a complete NPC space $(\mathcal{M}, d)$ and a lower semi-continuous, convex functional $F\colon \mathcal{M} \to \mathbb{R} \cup \infty$, there exists a map*

$$(\cdot)_t \colon \mathcal{M} \times \mathbb{R}_{\geq 0} \to \mathcal{M}$$

that has the following properties:

i. *For $f \in \mathcal{M}$, $f_0 = f$*

ii. $\lim_{s \to 0} \frac{d_{L^2}(f_{t+s}, f_s)}{s} = |\nabla_{-}F|(f_t)$*, for all t*

iii. $\sup_{s > 0} \frac{d_{L^2}(f_{t+s}, f_s)}{s} = |\nabla_{-}F|(f_t)$*, for all t*

iv. $-\frac{d}{dt}F(f_t) = |\nabla_{-}F|^2(f_t)$*, for almost all $t > 0$*

v. $t \mapsto |\nabla_{-}F|(f_t)$ *is right continuous*

vi. $t \mapsto F(f_t)$ *is convex and uniformly Lipschitz continuous on $[t_0, t_1]$ for all $0 < t_0 < t_1 < \infty$*

vii. $|\nabla_{-}F|(f_t)$ *is monotonically non-increasing in t and* $\lim_{t \to \infty} |\nabla_{-}F| = 0$.

Following a lemma in the work of Korevaar and Schoen (see [KS]), we note that if the domain, X, is an admissible simplicial n-complex and Y is a complete length space nonpositively curved in the sense of Alexandrov, the space of L^2 maps between X and Y is itself a complete length space nonpositively curved in the sense of Alexandrov, and the Korevaar-Schoen energy is indeed a convex, lower semicontinuous functional on $L^2(X, Y)$. Hence, there is a flow between X and Y.

One way of demystifying Mayer's definition of the norm of the gradient vector is to apply it to the case when (X, g) and (Y, h) are compact, smooth manifolds and Y additionally has nonnegative sectional curvature. Let $f \in C^2(X, Y)$ and let the metric on $C^2(X, Y)$ be the L^2 distance. Also, let the torsion of f be $\tau(f) := \mathrm{trace}_g \nabla df \in \Gamma(f^{-1}(TY))$ and let E be the Dirichlet energy, a convex, lower semicontinuous functional on $C^2(X, Y)$. We can compute

$$|\nabla_{-}E|(f) = \left(\int_X |\tau(f)|^2 \, dX_g \right)^{\frac{1}{2}}.$$

Although this is the most general result for heat-type flows between metric spaces, it gives almost no information about local phenomena, such as smoothness or whether or not it satisfies some weak definition of a parabolic equation. It may be possible to obtain such local results when $\mathcal{M}$ is a particular space of maps between smoother geometric spaces, but it is not obvious.

We do, however, prove that for an initial C^1 map the Gradient-of-Energy flow defined by Mayer agrees with the harmonic map heat flow that we define in this paper. This is a new result and, as far as the author is aware, the first example showing that another flow coincides with Mayer's Gradient-of-Energy flow. This allows us to obtain certain results for free, mostly revolving around the behavior of energy under the flow, and convergence as time goes to infinity. We show the equivalence in Section 5.3.

It is worth noting that using the idea of "nonlinear" Dirichlet forms, Jost obtained similar results in [J1].

Results of Sturm when the Target is NPC Sturm's work in [St4] on flows between a domain, possibly with boundary, that admits a Markov semigroup (on the set of *functions* on the domain, *not* on the set of *maps* between the domain and target) and a target that admits a barycenter contraction (defined as a contracting map that maps each measure on the space to a single point, the barycenter with respect to the measure). The Markov semigroup on domain is additionally required to satisfy a contraction property in terms of the Wasserstien distance, d^W (see [St4]), given as follows:

$$d^W(h(z,\cdot,t), h(v,\cdot,t)) \leq e^{Ct} d(z,v)$$

where C is bounded, and h is the heat kernel. This condition replaces the assumption in the smooth case of Ricci curvature bounded below. It is uncertain whether or not the domain may be an admissible simplicial n-complex, as it is generally not even an Alexandrov space with curvature bounded below in the sense of Alexandrov. However, as Sturm notes in [St4], a complete, nonpositively curved Alexandrov space always admits a barycenter contraction. In this setting, Sturm proves that a limit map exist (i.e. a harmonic map), and this map is Lipschitz continuous in the interior and Hölder continuous on the boundary.

Results of Chiang and Ratto The methods of Mayer and Sturm (see [Ma] and [St4]) provide weak regularity results for their respective flows because their assumptions do not restrict them to spaces with large, open sets of manifold points. We turn to the results of [C] and [CR], where their results are more "local" in nature.

In [C] and [CR], the authors achieve results similar to [ES]. In the assumptions of [C], the domain, X, is a compact orbifold. In the assumptions of [CR], the domain, X, is a compact metric space containing a finite set of points $\{p_j\}_{i=1}^N$ such that $X \setminus \{p_j\}_{i=1}^N$ is an open, smooth manifold, and $\{p_j\}_{i=1}^N$ are conical singularities. In both [C] and [CR], the target Y is a compact, smooth manifold with nonpositive sectional curvature, and is considered embedded in some higher dimensional Euclidean space. They show the existence of a heat flow similar to Eells and Sampson with many of the same properties. Specifically, they show via heat flow the existence of a harmonic representative in every homotopy class. Also, they show strong regularity for the flow and for harmonic maps in these settings.

X is a metric measure space with singularities the target is $\mathbb{R}$ There are many results for singular domains in this case. They are distinguished by the type of domain. When the domain is a Dirichlet space, we primarily have the results by Sturm (see [St2]). In [St1, St2, St3], Sturm shows methods of defining weak solutions to the heat flow when the domain is a Dirichlet space and satisfies a volume doubling property and have a uniform lower bound for Poincaré constants on balls. When the domain is a metric measure space, often considered with a lower bound on the Alexandrov curvature or a lower bound on a metric definition of Ricci curvature, there are results by Ambrosio, Gigli, and Savaré (see [AGS1] and [AGS2]). In all cases, a gradient flow is defined and its properties expounded. In [AGS1] and [AGS2], the results are presented with attention to applications in probability spaces.

The Results of Pivarski and Saloff-Coste In [PSC], Pivarski and Saloff-Coste use the results of Sturm [St2] to show the existence of a heat semigroup on a suitable space of functions on an admissible n-complex and a corresponding heat kernel. Specifically, in [St2], Sturm shows that a suitable Dirichlet space has a semigroup on a suitable space of functions that satisfies a weak parabolic equation. The Dirichlet space must have a uniform lower bound for the Poincaré constants on geodesic balls of fixed radius and must also satisfy a volume doubling property. Pivarski and Saloff-Coste show that an admissible n-complex satisfies both of these conditions and hence Sturm's results apply. They continue to show properties of the heat semigroup and the heat kernel.

Many of the results of [PSC] are dependent on a paper by Bendikov, Saloff-Coste, Salvatori, and Woess (see [BSCSW]) in which they consider "strip complexes" which, roughly speaking, is an analogue of a polyhedra in which all singularities are manifolds of codimenion 1. Concepts that are briefly covered in [PSC] receive a deeper treatment in [BSCSW].

The Results of Brin and Kifer In [BK], Brin and Kifer focus on Brownian motion on admissible 2-complexes, and show that a heat semigroup can be built out of the probability transition functions and densities (roughly speaking, the probability that a particle starting at a particular point in the domain and then moving randomly is in a specified set at a positive time). They show the existence of a kernel and a pointwise definition of Laplacian that is valid on all points that are not vertices. They show strong regularity properties for the kernel, and that the heat semigroup has a Laplacian-type operator as its infinitesimal generator.

Metric Spaces and Geometric Group Theory Our work in this paper is designed with applications to Geometric Group Theory in mind. We present a fact and an example of the place of heat flows and harmonic maps in Geometric Group Theory.

Theorem. *Let G be a finitely presented group. Then there exists an admissible 2-complex X such that $\pi_1(X, x_0) = G$.*

Hence, let G be a finitely presented group and let Y be a metric measure space with isometry group $\mathrm{Isom}(Y)$. Let $H \subset \mathrm{Isom}(Y)$ be a subgroup, and let $\rho\colon G \to H$ be a group homomorphism. Let $\widetilde{X}$ be the covering space of the admissible 2-complex X with $\pi_0(X, x_0) = G$. A map $f\colon \widetilde{X} \to Y$ is called *ρ-equivariant* if for all $x \in X$ and all $g \in G$,

$$f(g \cdot x) = \rho(g) \cdot f(x),$$

where G acts on $\widetilde{X}$ by deck transformations. If the set of ρ-equivariant maps can be proven to be non-empty, it is possible to extract information about ρ from the set of ρ-equivariant maps (or visa versa). When the domain and target are smooth compact manifolds and the target additionally has non-positive sectional curvature, it is known that every continuous map is homotopic to a harmonic map. Hence, to explore the behavior of ρ, it is often useful to examine the properties of the harmonic ρ-equivariant maps. Although we have presented a simple example of the approach here, it gives a sense of the approach of many papers in the field and we cite in particular [DM1, DM2, DM3, GS, IN, W1, W2].

The difficultly of using the flow methods described above when the domain X is an admissible 2-complex is that simplicial complexes generally are not Alexandrov spaces with curvature bounded below. This is only possible when X is also a topological manifold, which presents a grave restriction on X. Specifically, this topological restriction restricts the fundamental groups that X can have. Many of the methods described above require that the domains have their Alexandrov curvature bounded from below or that they have some variant of a metric definition of Ricci curvature bounded below. We wish to be able to define a flow when X is an arbitrary admissible n-complex. This leaves only a few of the cited the methods as possibilities for the extension of the results of [ES]. Indeed, our approach is essentially to extend the results of [St2]. However, to be able to apply these methods, we must show that an admissible complex is a Dirichlet space that satisfies a volume doubling condition and that, for a fixed radius, has a lower bound on Poincaré constants on balls. This is exactly what is shown in [PSC]. [BK] gives a similar result in the 2-dimensional case. However, in both cases, the target considered is $\mathbb{R}$. There is not much richness to the isometry group of $\mathbb{R}$, so we desire to have a target with more complexity.

Our results in context As noted above, we wish to be able to obtain results about the heat flow when the domain is a compact admissible complex and the target is a smooth compact manifold with non-positive sectional curvature. Also as noted above, we are largely motivated by applications to Geometric Group Theory. To that end, we avoid using methods that restrict the domain excessively. Hence, we avoid [AGS1, AGS2, St4] as they make assumptions about various types of curvature being bounded below, which excludes most admissible complexes, as we have described before. We also do not rely entirely on the methods of [Ma], as it provides no local information about the flow, and hence very weak regularity results; however, it does provide convergence results we shall find useful, and so we show that the Gradient-of-Energy flow of Mayer is equivalent to our heat flow given a suitable initial map. The closest in nature to our results involving flow methods from from [C] and [CR], as they consider domains with large, open sets of manifold point and targets that are smooth Riemannian manifolds with non-positive sectional curvature. Their methods generally follow the approach of [ES], but to obtain information on heat kernels, they use spectral methods that do not appear applicable to the case we consider here. We instead use the results of [St2] and [PSC] to develop methods that give existence of a heat kernel.

A focus of our work involves finding regularity of solutions to Partial Differential Equations on an admissible complex. These do not follow from any of the results of any work cited in the preceding paragraph, but rather come from the work of [BSCSW]. As far as we know, their technique to obtain regularity is novel, and our contribution here is to show that it can be used on other spaces with singularities that are manifolds of codimension 1, such as an admissible complex, and that they can be extend to hold for non-linear and non-homogeneous equations such as the heat flow into a manifold. Our other main regularity result appears to be entirely new. We provide Schauder estimates (of elliptic and parabolic type) on neighborhoods intersecting the $(n-1)$-skeleton of the domain (but away from the $(n-2)$-skeleton) that do not involve bounds on higher order derivatives *on* the $(n-1)$-skeleton. Our approach has been motivated by a construction in [DM3].

2 Preliminaries

2.1 Riemannian Polyhedra

Fundamentally, we define a simplicial complex as a topological space.

Definition 2.1. We define the following:

i. A set of points $s \subset \mathbb{R}^{n+1}$ is an *n-simplex* if there exist $n+1$ points, $\{x_0, \ldots, x_n\}$, in general position such that s is equal to the closed convex hull of $\{x_0, \ldots, x_n\}$.

ii. s is a *face* of simplex t if s is a k-simplex and $s \subset t$.

iii. X is a *simplicial complex* if it contains a set of simplexes with the following relations:

   (a) If $t \in X$ and s is a face of t, then $t \in X$.

   (b) If $s, t \in X$ then $s \cap t \in s, t$.

iv. $X^{[k]}$ denotes the set of all k-simplexes of X.

v. $X^{(k)}$ denotes the *k-skeleton*, which is defined as the union of all closed k-simplexes contained in X (i.e. $X^{(k)} = \cup_{s \in X^{[k]}} \overline{s}$)

vi. If X is a simplicial complex, it is an *n-complex* if every simplex is contained in an n-simplex.

There are additional properties that can be imposed on a simplicial complex so that we may have enough structure to introduce a meaningful sense of harmonic maps and other considerations for later.

Definition 2.2. Let X be a simplicial complex.

i. An n-complex is *dimensionally homogeneous* if every simplex $s \in X$ is contained in an n-simplex.

ii. The *(closed) star of* $s \in X$, denoted $\mathrm{St}(s)$, is the union of all simplexes that contain s. The *open star of* $s \in X$, denoted $\mathrm{st}(s)$, is the union of the interiors of all simplexes that contain s.

iii. The *link of* $s \in X$, denoted $\mathrm{Lk}(s)$, is the set of simplexes that are in $\mathrm{St}(s)$ but not in $\mathrm{st}(s)$.

iv. If X is a dimensionally homogeneous n-complex, X is locally $(n-1)$-*chainable* if for every $p \in X^{(n-2)}$, any two n-simplexes $s_0, s_1 \in \mathrm{St}(p)$ can be *chained*: there exists a finite sequence of n-simplexes $\{A_i\}_{i=1}^{K} \subset \mathrm{St}(p)$ such that $A_1 = s_0$ and $A_K = s_1$ and $\bigcup A_i \setminus \{p\}$ is simply connected.

v. X is an *admissible n-complex* if X is a simplicial complex that dimensionally homogeneous of dimension n and is $(n-1)$-chainable.

It will additionally be useful to put a metric on a simplicial complex so that we may define a length space that, in the interior of each simplex, resembles a manifold.

Definition 2.3. Let X be an admissible n-complex.

i. X is a *Riemannian polyhedron* if there exist a metric g such that for an open k-simplex s, $1 \leq k \leq n$, $g|_s$ is a Riemannian metric in the sense of a manifold.

ii. X is a *smooth Riemannian polyhedron* if

(a) the metric g is smooth on the interior of each simplex;

(b) for any two k-simplexes, s_0, s_1 with a common $(k-1)$-face, F, $g|_F$ is induced by the limits of $g|_{s_0}, g|_{s_1}$;

(c) the sectional curvature is bounded uniformly on the interior of every n-simplex.

It is obvious how to define coordinates on sufficiently small neighborhoods compactly contained in a single n-simplex, but it will often be useful to be able to define coordinates in a compactly neighborhood of an $(n-1)$-simplex, which we shall call *edge coordinates.*

Definition 2.4. Let X be a Riemannian polyhedron, and let $p \in X^{(n-1)} \setminus X^{(n-2)}$. Pick an open neighborhood V such that $V \cap X^{(n-2)} = \emptyset$. Let $\{s_j\}_{j=1}^J$ denote all of the n-simplexes containing p, and let E be their common face (i.e. $E = \cap_{j=1}^J \overline{s_j}$). We define *edge coordinates about* $p \in V$ so that $p = (0, \ldots, 0)$ and for $q \in E$, $q = (x^1, x^2, \ldots, x^{n-1}, 0)$. The n^{th} coordinate denotes the direction normal to E. If we wish to refer to a point in V in a specific n-simplex, s_j, we may denote the n^{th} coordinate, x_j^n.

To make clear what is meant by k^{th}-order derivatives on functions in the case of a Riemannian polyhedron, we provide the following.

Definition 2.5. Let X be an admissible smooth Riemannian polyhedron of dimension 2 or greater. We define the following.

i. Z is a *vector field* on X if it is a vector field when restricted to any n-simplex, considered here as a Riemannian manifold.

ii. Additionally, let X be simplex-wise flat, $\{Z_i\}_{i=1}^n$ is an *orthonormal basis* if its restriction to any n-simplex induces orthonormal coordinates (note that they will not necessarily match on $X^{(n-1)}$).

iii. For $k \in \mathbb{N}$, define

$$I^k := \left\{ (a_1, \dots, a_n) \in \mathbb{N}^n \mid \sum_{i=1}^{n} a_i = k \right\},$$

and for a set of vector fields $Z = \{Z_i\}$, define

$$|D_Z^k f(z)| := \left(\sum_{(a_1,\dots,a_n)\in I^k} |(Z_1)^{a_1} \cdots (Z_n)^{a_n} f(z)|^2 \right)^{\frac{1}{2}}.$$

In the event $k = 1$, we write simply

$$|D_Z f(z)| := \left(\sum_{i=1}^{n} |Z_i f(z)|^2 \right)^{\frac{1}{2}}.$$

Definition 2.6. Let X be an admissible smooth Riemannian polyhedron of dimension n. Let $A \subset X$, we define the following spaces of functions.

i. Let $S^k(A)$ denote the set of functions defined on $A \subset X$ that, when restricted to $A \setminus X^{(n-1)}$, are bounded and continuous with bounded and continuous derivatives up to order k.

ii. Let $C^k(A)$ denote the subset of $S^k(A)$ in which each function is continuous and bounded on A.

iii. Let $\mathsf{BC}_0(A)$ denote the set of $f \in C^1(A)$ such that for $p \in A \cap (X^{(n-1)} \setminus X^{(n-2)})$, and with edge coordinates about p,

$$\sum_{i=1}^{l} \frac{\partial f}{\partial n_i}(p) = 0$$

Where $\frac{\partial}{\partial n_i}$ are the (inward pointing) normals of the edge of each n-simplex, s_i, containing p. This is the so-called *balancing condition*.

iv. Let $C^k_{\text{loc}}(A)$ be the set of functions such that, if $f \in C^k_{\text{loc}}(A)$, then for each compactly contained $A' \subset A$ and each n-simplex, S, $f|_{\overline{A'} \cap \overline{S}} \in C^k(\overline{A'} \cap \overline{S})$.

We note that we may let $k = \infty$ to denote that derivatives of all orders exist, according to the context. We will also append the subscript c to denote compactly supported functions, e.g. $C^k_c(X)$ denotes the subset of functions of $C^k(A)$ that are compactly supported.

For a subset $A \subset X$, and interval $I \subset \mathbb{R}$, we also define analogous spaces on $A \times I$. Indeed, let $C^{k,l}(A \times I)$ denote the set of functions on such that if $f \in C^{k,l}(A \times I)$, then for each $t_0 \in I$, $f(\cdot, t_0) \in C^k(A)$ and $\left(\frac{\partial}{\partial t}\right)^m f$ is continuous on $A \times I$ for $1 \leq m \leq l$. We similarly extend these definitions to apply to $C^{k,l}_{\text{loc}}(A \times I)$.

2.2 Dirichlet Spaces & Energy

Our approach to defining a heat flow from an admissible complex to a nonpositively curved smooth manifold begins by building on the work of [St2], where Sturm provides a setting in which a Laplacian can be defined on spaces more general than manifolds. We begin with the basic definitions.

Definition 2.7. Let X be a separable, measureable space with measure μ. We define the following:

- For $p \in [1, \infty)$, let $L^p(X) := \left\{\text{measurable } f \mid \int_X |f|^p \, d\mu < \infty\right\}$
- For $p = \infty$, let $L^\infty(X) := \{\text{measurable } f \mid \operatorname{ess\,sup}_X |f| < \infty\}$
- If X is an admissible polyhedron with simplex-wise metric tensor g, then the induced measure for an open set A is

$$\mu_g := \sum_{S \in X^{[n]}} \mu_{g|_S}(A \cap S),$$

where $\mu_{g|_S}$ is the induced measure on the Riemannian manifold $A \cap S$ with metric $g|_S$.

Definition 2.8 (See [EF, Chapter 2])**.** Let X be a separable, measurable, locally compact space. A *Dirichlet form E on X* with domain $\mathrm{Dom}(E) \subset L^2(X)$ is a symmetric bilinear map $E\colon \mathrm{Dom}(E) \times \mathrm{Dom}(E) \to \mathbb{R}$ which is nonnegative definite. For $f \in \mathrm{Dom}(E)$, let $E(f) := E(f, f)$. We assume that

i. $\mathrm{Dom}(E)$ is dense in $L^2(X)$.

ii. $\mathrm{Dom}(E)$ is complete in the inner product

$$\langle f, g \rangle_{\mathrm{Dom}(E)} = \langle f, g \rangle_{L^2(X)} + E(f, g), \quad f, g \in \mathrm{Dom}(E)$$

and is hence a Hilbert space.

iii. For any normal contraction T and $f \in \mathrm{Dom}(E)$,

$$E(T \circ f) \leq E(f),$$

where a normal contraction of $\mathbb{R}$ is a map $T\colon \mathbb{R} \to \mathbb{R}$ such that $T(0) = 0$ and $|T(s) - T(t)| \leq |s - t|$ for all $s, t \in \mathbb{R}$.

X is a *Dirichlet space* if there exists a Dirichlet form defined on X. Additionally, for $f, g \in \operatorname{Dom}(E)$ we say that E is *strongly local* if $E(f, g) = 0$ whenever f is constant in some neighborhood of the support of g (or, by symmetry, vice versa).

For our purposes, we wish to define a strongly local Dirichlet form as in [PSC, Definition 1.12], such that for $f \in W^{1,2}(X)$ (defined formally below), $E(f, f)$ coincides with the Korevaar-Schoen-type energy (see [EF, Chapter 9; DM2; KS]) up to a fixed dimensional constant. We show there are equivalent ways of defining the domain of this form. We follow the approach of [PSC].

Definition 2.9. Let X be an admissible Riemannian polyhedron of dimension n with metric g and volume measure μ_g. For two Lipshitz functions f, g, define

$$E_\epsilon(f, g) := \int_X \int_{B(p,\epsilon)\setminus\{p\}} \frac{(f(p) - f(q))\,(g(p) - g(q))}{d_X(p, q)^2} \frac{2n\, d\mu(p) d\mu(q)}{\mu(B(p, \epsilon)) + \mu(B(q, \epsilon))},$$

where μ is the measure induced by the metric g (see above), and $B(p, r)$ is the geodesic ball about p of radius r.

We cite without proof the following.

Lemma 2.10. [PSC, Definition 1.12] *Let X be an admissible Riemannian polyhedron with metric g and volume measure μ_g, and let $E_\epsilon(\cdot, \cdot)$ be as above. Then, as ϵ goes to 0, there exists a limit Dirichlet form in the sense of Dal Maso (see* [PSC, DMa]*) whose closure, denoted $E(\cdot, \cdot)$, is a strongly local Dirichlet form on $L^2(X)$ with a dense subset of compactly supported Lipschitz functions.*

There is an alternate Dirichlet form that one may define. We shall see later that the two forms are essentially the same.

Definition 2.11 (See [PSC, Definition 1.13, Lemma 1.14])**.** Let X be an admissible Riemannian polyhedron, and let $f, g : X \to \mathbb{R}$ be Lipschitz and compactly supported, and let

$$\mathcal{E}_0(f,g) := \sum_{s\in X^{[n]}} \int_s \langle \nabla f, \nabla g\rangle \, d\mu,$$

where $X^{[n]}$ is the set of all n-simplexes of X. Then $\mathcal{E}_0(\cdot,\cdot)$ is a closable form whose domain is the set of compactly supported Lipschitz functions on X. Let its closure be denoted $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$.

Remark 2.12. The above definition makes sense as, for Lipschitz f, $|\nabla f|$ exists almost everywhere.

Definition 2.13. For an admissible Riemannian polyhedron X, and $\mathcal{E}(\cdot,\cdot)$ as in Definition 2.11, and

$$\begin{aligned} W^{1,2}(X) := \ \Big\{ f \in L^2(X) \mid \forall s \in X^{[n]}, f|_s \in W^{1,2}(s), \mathcal{E}(f,f) < \infty, \text{ and} \\ \mathrm{Tr}(\mathrm{f}|_{\mathrm{s}}) = \mathrm{Tr}(\mathrm{f}|_{\mathrm{s'}}) \text{ on } e = s \cap s'; s, s' \in X^{[n]} \Big\}, \end{aligned}$$

where $X^{[n]}$ is the set of all n-simplexes of X and $\mathrm{Tr}\colon W^{1,2}(s) \to L^2(\partial s)$ is the trace map on $s \in X^{[n]}$.

Note that we can naturally extend the definition for a domain $A \subset X$.

Proposition 2.14 (See [PSC, Lemma 1.15, Theorem 1.17])**.** *For E, $\mathcal{E}$, and X as above,*

$$\mathrm{Dom}(E) = \mathrm{Dom}(\mathcal{E}) = W^{1,2}(X).$$

Also, for any $f \in \mathrm{Dom}(E)$,

$$E(f,f) = \mathcal{E}(f,f).$$

From general theory and as a consequence of the Riesz representation theorem, we have the following:

Proposition 2.15. *Let* X *be an admissible Riemannian polyhedron,* E *be a Dirichlet form with domain* $\mathrm{Dom}(E) \subset L^2(X)$*. There is a unique, self-adjoint, negative operator* Δ*, called* the Laplacian*, with domain* $\mathrm{Dom}(\Delta) \subset \mathrm{Dom}(E)$ *defined in the following way.*

$$\mathrm{Dom}(\Delta) := \{f \in \mathrm{Dom}(E) \mid \exists C \ s.t. \ E(f,g) \leq C\|g\|_{L^2}, \forall g \in \mathrm{Dom}(E).\}$$

Remark 2.16. From here forward, we shall fix the Dirichlet, E, to be defined as in Lemma 2.10, and for Δ to be the Laplacian associated to E. We note that by Proposition 2.14, we may equivalently use the Dirichlet form from Definition 2.11.

Referring to the above theorem, we note that by the Riesz representation theorem, this theorem implies the existence of a function $v \in L^2(X)$ such that $E(f,g) = \int_X fv\,d\mu$. Such a function we denote $v = -\Delta g$.

Following the observation that for $\phi \in C^\infty(\mathrm{Int(s)})$, where s is an n-simplex, $\Delta|_\phi$ is the Laplace-Beltrami operator and as a consequence of Green's identity and Proposition 2.14, we have the following proposition.

Proposition 2.17 (See [PSC, Prop. 1.21])**.** *For a function* $f \in C_c^2(X)$*, and Dirichlet form* E *as defined in Lemma 2.10 with associated Laplacian* Δ*,* $f \in \mathrm{Dom}(\Delta)$ *if and only if* f *has the balancing condition.*

2.3 Volume Doubling & Poincaré Inequalities

In the work of Sturm (see [St1, St2, St3]), the principles of volume doubling and local Poincaré inequalities are necessary tools to be able to define heat operators with good properties. We define them here.

Definition 2.18. A metric space X has the *volume doubling property* if there exists a constant N, dependent on X such that for all balls in $B(p, 2r) \subset X$,

$$\mathrm{Vol}(B(p, 2r)) \leq 2^N \mathrm{Vol}(B(p, r)).$$

Note that this definition permits N to change if one inputs a different r, i.e. N is not global for $0 < r < \infty$.

Definition 2.19. A metric space X with Dirichlet form E has a the *uniform lower bound on local Poincaré constants* if there exists a constant C dependent on X such that for all balls in $B(p, r) \subset X$ and for all $u \in \mathrm{Dom}(E)$,

$$\int_{B(p,r)} |u - \bar{u}_{p,r}|^2 \, d\mu \leq Cr^2 \int_{B(p,r)} |\nabla u|^2 \, d\mu,$$

where $|\nabla u|^2$ is the density function with respect to E and

$$\bar{u}_{p,r} = \mathrm{Vol}(B(p, r))^{-1} \int_{B(p,r)} u \, d\mu.$$

For a smooth Riemannian manifold, if compactness is not assumed, the satisfaction of the volume doubling property and the strong local Poincaré inequality are dependent on a lower bound on the Ricci curvature.

Of course, we now ask if an admissible Riemannian polyhedron has the volume doubling property and the strong local Poincaré inequality. The volume doubling property is quite easily satisfied, and we refer to [PSC] for the answer to the other question:

Proposition 2.20. *Let X be an admissible Riemannian polyhedron of dimension n ($n \geq 2$) with the following additional properties:*

i. the metric tensor on X is uniformly elliptic on each n-simplex with constant Λ,

ii. for any k-simplex s_0, $0 \leq k \leq n-1$, the number of n-simplexes containing s_0 is bounded above by a constant M,

iii. the distance between any two vertexes is bounded below by L,

iv. the interior angles of any n-simplex is bounded below by $\alpha > 0$.

For such an X, X satisfies the volume doubling property and the strong local Poincaré inequality.

Proof. The satisfaction of the volume doubling property is immediate. For a proof of the satisfaction of the strong local Poincaré inequality, we refer to [PSC, Corollary 2.10]. □

2.4 Other Spaces of Functions and Maps and their Energy

Our ultimate consideration is flows for maps from smooth Riemannian polyhedra to smooth Riemannian manifolds with nonpositive sectional curvature. Crucial to this program are the balancing condition and regularity. As we will often be referring to the balanced maps and maps that are continuous across the faces, we formally define them when the target is a manifold other than $\mathbb{R}$.

Definition 2.21. Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, and N a finite dimensional smooth Riemannian manifold. Let $A \subset X$ be an open set. We define the following spaces of maps between X and N.

i. Let $S^k(A, N)$ denote the set of maps that, when restricted on the domain to $A \cap \big(X \setminus X^{(n-1)}\big)$, are bounded and continuous and have bounded and continuous derivatives of all orders less than or equal to k.

ii. Let $C^k(A, N)$ denote the subset of $S^k(A, N)$ that is bounded and continuous on all $A \subset X$.

iii. Let $C^k_{\text{loc}}(A, N)$ be the set of maps such that, if $f \in C^k_{\text{loc}}(A, N)$, then for each compactly contained $A' \subset A$ and each n-simplex, S, $f|_{\overline{A'} \cap \overline{S}} \in C^k(\overline{A'} \cap \overline{S}, N)$.

For a subset $A \subset X$, and interval $I \subset \mathbb{R}$, we also define analogous spaces on $A \times I$. Indeed, let $C^{k,l}(A \times I, N)$ denote the set of functions on such that if $f \in C^{k,l}(A \times I, N)$, then for each $t_0 \in I$, $f(\cdot, t_0) \in C^k(A, N)$ and $\left(\frac{\partial}{\partial t}\right)^m f$ is continuous on $A \times I$ for $1 \leq m \leq l$. We similarly extend the definition of $C^{k,l}_{\text{loc}}(A \times I, N)$.

Definition 2.22. Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, and N a finite dimensional smooth Riemannian manifold. Let $\iota \colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $f \colon X \to N$ be continuous with first order derivatives continuous on the interior of the n-simplexes and up to the faces. Let $p \in X^{(n-1)} \setminus X^{(n-2)}$ and let V be a neighborhood of p with edge coordinates. Also, Let $F^\gamma_j = \left(\iota \circ f|_{s_j}\right)^\gamma$, $1 \leq j \leq J$ and $1 \leq \gamma \leq q$. Such map f has the *balancing condition* if for all $p \in X^{(n-1)} \setminus X^{(n-2)}$ and V as above,

$$\sum_{j=1}^{J} \frac{\partial F_j^\gamma}{\partial x^n}(x^1, \ldots, x^{n-1}, 0) = 0$$

for each γ. For open $A \subset X$, we shall denote the subspace of $C^1(A, N)$ that has the balancing condition on A,

$$\mathsf{BC}_0(A, N) := \left\{ f \in C^1(A, N) \mid f \text{ has the balancing condition} \right\}.$$

We also extend the definition of energy to maps between polyhedra and manifolds. We follow the style of [EF, Chapter 9].

Definition 2.23. Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, and N a finite dimensional smooth Riemannian manifold. Let g denote that simplex-wise smooth metric tensor of X and let h denote the metric tensor of N. For a map $f\colon X \to N$ we define the energy density of f at $z \in X \setminus X^{(n-1)}$ relative to coordinates $\{z^i\}$ near z and $\{f^\gamma\}$ near $f(z)$ to be

$$e(f)(z) = g^{ij}(z) \frac{\partial f^\alpha}{\partial z^i} \frac{\partial f^\beta}{\partial z^j}(z) h_{\alpha\beta}(f(z)).$$

If $e(f)$ is locally integrable, we define the (global) energy of f to be

$$E(f) := \int_X e(f)\, dX;$$

otherwise, we define $E(f) := \infty$. We define

$$W^{1,2}(X,N) := \left\{ f \in L^2(X,N) \text{ such that } E(f) < \infty \right\}.$$

We can connect the definition of energy for maps to the Dirichlet form for functions according to the following proposition (see [EF, Lemma 9.3] for proof).

Proposition 2.24. *Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, and N a finite dimensional smooth Riemannian manifold. Let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $f\colon X \to N$ and for each $1 \leq \gamma \leq q$, define F^γ to be the γ^{th} component of $F\colon X \to \iota(N)(\subset \mathbb{R}^q)$, where $F := \iota \circ f$. Then, f is in $W^{1,2}(X,N)$ if and only if F^γ is in $W^{1,2}(X)$ for each $1 \leq \gamma \leq q$.*

Additionally, if f is in $W^{1,2}(X,N)$, then the energy density is given by

$$e(f)(z) = \sum_{\gamma=1}^{q} \left\langle \nabla F^\gamma(z), \nabla F^\gamma(z) \right\rangle,$$

for $z \in X \setminus X^{(n-1)}$.

Corollary 2.25. *Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, and N a finite dimensional smooth Riemannian manifold. Let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $f \in W^{1,2}(X,N)$ and for each $1 \leq \gamma \leq q$, define F^γ to be the γ^{th} component of $F\colon X \to \iota(N)(\subset \mathbb{R}^q)$, where $F := \iota \circ f$. Then,*

$$E(f) = \sum_{\gamma=1}^{q} E(F^\gamma, F^\gamma),$$

where $E(\cdot,\cdot)$ is the Dirichlet form given in either Lemma 2.10 or Definition 2.11.

Proof. This is an immediate consequence of Propositions 2.24 and 2.14. $\square$

We have the usual properties expected from this energy functional, which we again state from [EF].

Proposition 2.26 (Lower-semicontinuity of Energy)**.** *Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, and N a finite dimensional smooth Riemannian manifold, and let $E\colon W^{1,2}(X,N) \to [0,\infty)$ be the energy functional of Definition 2.23. Then E is lower-semicontinous in the following sense: for any sequence of maps $\{f_i\} \subset W^{1,2}(X,N)$ with uniformly bounded energy that converges in L^2 to a map f,*

$$E(f) < \infty \quad \textit{and} \quad E(f) \leq \liminf E(f_i).$$

Proposition 2.27 (Poincaré Inequality)**.** *Let X be an admissible smooth Riemannian polyhedron of dimension n, $(n \geq 2)$, N a finite dimensional smooth Riemannian manifold, and let g denote the simplex-wise smooth metric tensor on X. Let $e(\cdot)$ be the energy density of Definition 2.23. Then for any open, compactly contained subset $X' \subset X$, there is a constant $C > 0$ dependent on X, X', N and the constant of ellipticity on X' with respect to g such that for any $B(p,r) \subset X'$*

$$\int_{B(p,r)} d_N(f(z),\bar{f})^2\, dX(z) \leq Cr^2 \int_{B(p,r)} e(f)\, dX(z), \tag{2.1}$$

where $\bar{f} \in N$ is the barycenter of f on $B(p,r)$. The barycenter is defined as

$$\int_{B(p,r)} d_N(f(z),\bar{f})^2\, dX(z) = \inf_{y_0 \in N} \int_{B(p,r)} d_N(f(z),y_0)^2\, dX(z).$$

Remark 2.28. We note that variations of this statement exist. Sharper constants C can be found if the radius of the ball of the expression of the left-hand side of

Equation (2.1) is shrunk by half. We refer to [EF, Proposition 9.1 & Remark 9.6].

There is an additional precompactness result similar to the one of [KS], which we review later in Proposition 5.21 (see page 102).

3 Partial Differential Equations on Polyhedra

Our ultimate goal is to show the existence of a heat flow between admissible smooth Riemannian polyhedra and smooth Riemannian manifolds with nonpositive sectional curvature. As we will eventually be embedding the manifold isometrically into some higher dimensional Euclidean space, we of course will be curious about parabolic-type differential equations where the target is $\mathbb{R}$. Sturm has treated this subject extensively for Dirichlet spaces with volume doubling conditions and with uniform lower bounds for Poincaré constants on balls (see [St1, St2, St3]). Hence, we can apply this to the case of Riemannian polyhedra.

Assumptions. Unless otherwise specified, we shall assume in this section that X is an admissible smooth Riemannian polyhedron that satisfies the conditions of Proposition 2.20 with Dirichlet form $E(\cdot,\cdot)$ and Laplacian Δ as in Section 2.2, and energy $E(\cdot)$ as in Section 2.4.

3.1 Elliptic & Parabolic Equations on Riemannian Polyhedra

We define some additional function spaces which we shall need to rigorously define differential equations on Dirichlet spaces and to show existence of solutions.

Definition 3.1 (following [St2, Sect. 1.3(A)])**.** For a function $u\colon X \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, we define the following norms and spaces. Let $I = (a, b) \subset \mathbb{R}$.

i. For the Dirichlet form on X, $E(\cdot,\cdot)$, (as defined in Lemma 2.10) with domain $W^{1,2}(X)$, $W^{1,2}(X)$ is a Hilbert space with norm

$$\|f\|_{W^{1,2}(X)} := \left(E(f,f) + \|f\|^2_{L^2(X)}\right)^{\frac{1}{2}}.$$

We note that $W^{1,2}(X) \subset L^2(X) \subset W^{1,2}(X)^*$, where $W^{1,2}(X)^*$ is the dual of $W^{1,2}(X)$.

ii. $C(I \to L^2(X))$ is the set of continuous and bounded functions with respect to t, where $u \in C(I \to L^2(X))$ is of the form $u\colon I \to L^2(X)$, $t \mapsto u(t,\cdot)$ with the (sup) norm

$$\|u\|_{L^\infty(I)} := \sup_{t\in I}\left(\int_X u(t,x)^2\,d\mu(x)\right)^{\frac{1}{2}}$$

iii. $L^2(I \to W^{1,2}(X))$ is the space of functions $u\colon I \to W^{1,2}(X)$ with norm

$$\|u\|_{L^2(I)} := \left(\int_I \|u(t,\cdot)\|_{W^{1,2}(X)}\,dt\right)^{1/2}$$

iv. $H^1(I \to W^{1,2}(X)^*)$ is the space of functions of the form $u\colon I \to W^{1,2}(X)^*$ with distributional time derivative $\frac{\partial}{\partial t}u \in L^2(I \to W^{1,2}(X)^*)$, where $W^{1,2}(X)^*$ is the dual to the space $W^{1,2}(X)$ with the usual L^2 inner product. This space has norm

$$\|u\|_{H^1(I)} := \left(\int_I \|u(t,\cdot)\|^2_{W^{1,2}(X)^*} + \|\tfrac{\partial}{\partial t}u(t,\cdot)\|^2_{W^{1,2}(X)^*}\,dt\right)^{1/2}.$$

v. $\mathcal{F}(I \times X) := L^2(I \to W^{1,2}(X)) \cap H^1(I \to W^{1,2}(X)^*)$ with norm

$$\|u\|_{\mathcal{F}(I\times X)} := \left(\int_I \|u(t,\cdot)\|^2_{W^{1,2}(X)} + \|\tfrac{\partial}{\partial t} u(t,\cdot)\|^2_{W^{1,2}(X)^*}\, dt \right)^{1/2}.$$

It can be shown that

$$\mathcal{F}(I \times X) \subset C(\overline{I} \to L^2(X)).$$

Now that we have defined the function spaces, we have a sense of what a solution to a parabolic equation might be. Specifically, we have the following. We provide existence proofs later.

Definition 3.2 (See [St2, Section 1.4(C)])**.** Let $I = (a, b) \subset \mathbb{R}$.

i. A function u is a *weak solution* of the parabolic equation

$$\frac{\partial}{\partial t} u = \Delta u,$$

on $I \times X$, where I is an interval, if and only if $u \in \mathcal{F}(I \times X)$ and u satisfies

$$\int_I E(u, \phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt = 0,$$

for all $\phi \in \mathcal{F}(I \times X)$. u is a *weak subsolution (super-)* if

$$\int_I E(u, \phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt \le (\ge) 0,$$

for all $\phi \in \mathcal{F}(I \times X)$.

ii. Let $f \in L^2(X)$. u is a *weak solution to the initial value problem*

$$\left.\begin{aligned} \frac{\partial}{\partial t}u = \Delta u \quad & \text{on } I \times X \\ u_a = f \qquad & \text{on } X \end{aligned}\right\}$$

if and only if u is a solution as above, and $\lim_{t \to a^+} u = f$ in $L^2(X)$.

The weak solution being in $\mathcal{F}(I \times X)$ as defined above seems to be a bit cryptic. We give another equivalent condition for u being a weak solution later in Proposition 4.3.

We are also interested in non-homogeneous parabolic-type equations.

Definition 3.3. Let $I = (a, b) \subset \mathbb{R}$, and $f \in L^2\left((a, b) \mapsto W^{1,2}(X)\right)$.

i. A function u is a *weak solution* of the non-homogeneous parabolic equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = f,$$

on $I \times X$, where I is an interval, if and only if $u \in \mathcal{F}(I \times X)$ and u satisfies

$$\int_I E(u, \phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt = \int_I \langle f, \phi \rangle_{L^2(X)}\, dt,$$

for all $\phi \in \mathcal{F}(I \times X)$.

ii. Let $f \in L^2\left((a, b) \mapsto W^{1,2}(X)\right)$ such that $\lim_{t \to a} f(z, t) = g(z)$, $g \in L^2(X)$. u is a *weak solution to the initial value problem*

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) u(z, t) = f(z, t) \quad & \text{on } (z, t) \in X \times (a, b) \\ u(z, a) = g(z) \qquad & \text{on } X \end{aligned}\right\}$$

if and only if u is a solution as above, and $\lim_{t\to a^+} u = g$ in $L^2(X)$.

As time will no longer be a concern, it is far easier to define solutions, weak and otherwise, to elliptic-type equations.

Definition 3.4. Let $f \in L^2(X)$.

i. u is a *weak solution* to the non-homogeneous elliptic-type equation

$$\Delta u = f$$

if $u \in W^{1,2}(X)$ and for all $\phi \in W^{1,2}(X)$,

$$\int_X E(f, \phi) + f\phi \, d\mu = 0.$$

ii. u is a (strong) *solution* to the non-homogeneous elliptic-type equation

$$\Delta u = f$$

if $u \in C^2_{\text{loc}}(X \setminus X^{(n-1)})$ and is balanced and continuous on X, and

$$\Delta_g u(z) = f(z) \quad \text{on } X \setminus X^{(n-1)},$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric tensor g.

We treat existence of these equations (elliptic-/parabolic-type and homogeneous /non-homogeneous) in Section 4.

3.2 The Parabolic Harnack Inequality

As in the case of parabolic equations defined on bounded open regions in Euclidean space, we can show the existence of a parabolic Harnack inequality for weak solutions which can be used to retrieve many of the properties of solutions. Certainly, it may be used to show Hölder continuity of solutions.

Definition 3.5 (See [St3, Property II]). A non-negative weak solution, u, of $\frac{\partial}{\partial t}u = \Delta u$ on $Q = (t - 4r^2, t) \times B(p, 2r)$ satisfies the *parabolic Harnack inequality* if there exists a constant C dependent on X such that for all balls $B(p, 2r) \subset X$ and all $t \in \mathbb{R}$,

$$\sup_{(s,y)\in Q^-} u(s, y) \leq C \cdot \inf_{(s,y)\in Q^+} u(s, y),$$

where $Q^- = (t - 3r^2, t - 2r^2) \times B(p, r)$ and $Q^- = (t - r^2, t) \times B(p, r)$.

Proposition 3.6 (See [St3, Theorem 3.5]). *For a Dirichlet space X, the volume doubling property and the strong local Poincaré inequality hold if and only if the parabolic Harnack inequality holds true for weak solutions to $(\frac{\partial}{\partial t} - \Delta)u = 0$ on $\mathbb{R} \times X$.*

Proposition 3.7 (See [PSC, Cor. 3.4], also [St3, Prop. 3.1]). *Let X be as in the assumptions of this section. For all $R > 0$, there exists C, dependent on X and R, and $\alpha \in (0, 1)$ such that for all $p \in X$ and $T \in R$ and $0 < r < R$,*

$$|u(s, x) - u(t, y)| \leq C \sup_Q |u| \left(\frac{|s - t|^{\frac{1}{2}} + |y - z|^{\frac{1}{2}}}{r} \right),$$

where u is a weak solution of $\frac{\partial}{\partial t}u = \Delta u$ on $Q = (t - 4r^2, t) \times B(p, 2r)$, $s, t \in (T - r^2, T)$ and $y, z \in B(p, r)$.

3.3 Maximum Principles

Again, as in the case of parabolic differential equations defined on bounded open regions in Euclidean space, we can show the existence of a version of the maximum principle for balanced solutions on suitable polyhedra.

Lemma 3.8. *Let $f \in C^{2,1}_{loc}(X \times [0,T))$ and let f be balanced. If for all $p \in X \setminus X^{(n-1)}$ and $t \in [0,T)$,*

$$\left(\frac{\partial}{\partial t} - \Delta\right) f(x,t) \leq 0,$$

then

$$\max_{X \times [0,T)} f(z,t) = \max_{\substack{(X \times \{t=0\}) \\ \cup \left(X^{(n-2)} \times [0,T]\right)}} f(z,t).$$

Proof. We note that we do not have a Laplacian defined pointwise on $X^{(n-1)}$. However, we can use the balancing condition to retrieve the result by the method described in at the beginning of [BSCSW, Section 7(B)]. □

We note that once we have knowledge about the deeper properties of parabolic-type equations, we will have other results similar to maximum principles, notably Proposition 4.4 and Proposition 5.31.

3.4 Higher Regularity of Solutions to Parabolic Equations

Definition 3.9. Let $k \in \mathbb{N}$ be fixed and let $T > 0$ be fixed, too. Let $A \subset X$ be open. For the parabolic equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0,$$

a weak solution $u\colon A \times (0,T) \to \mathbb{R}$ is *time regular to order* k on $X \times (0,T)$ if u is a weak solution to the parabolic equation above as given in Definition 3.2 (see page 38) and for all integers $0 \le m \le k$, $\left(\frac{\partial}{\partial t}\right)^m u$ is also a weak solution on $A \times (0,T)$.

Our main result on the regularity of solutions to parabolic-type equations as above follows. We start here as it is the most difficult case and the elliptic case and non-homogeneous cases are modifications or simplifications of this argument.

Proposition 3.10. *Let $(0,T) \subset (0,\infty)$ and let $A \subset X$ be open such that $d(A, X^{(n-2)}) > 0$. If u is a weak solution to $(\frac{\partial}{\partial t} - \Delta)u = 0$ on $(0,T) \times A$ and is time regular to order k then u satisfies the following:*

i. For each m, $0 \le m \le k$, $\left(\frac{\partial}{\partial t}\right)^m u$ is continuous on $(0,T) \times A$. Specifically, for fixed $0 < t < T$, $\left(\frac{\partial}{\partial t}\right)^m u(\cdot,t) \in C^{k-m+\alpha}(A)$, where $0 < \alpha < 1$.

ii. For any open n-simplex S with metric g,

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) u|_S = 0$$

on $(0,T) \times A \cap S$ pointwise, where Δ_g is the Laplace-Beltrami operator on S. Also, for any $0 < t < T$, $u|_S(\cdot,t) \in C^\infty(A \cap S)$.

iii. For each m, $0 \le m \le k$, $\left(\frac{\partial}{\partial t}\right)^m u$ is balanced on $A \cap X^{(n-1)}$. If $m = k$, then $\left(\frac{\partial}{\partial t}\right)^m u$ is weakly balanced.

We require some lemmas to prove this proposition, and the proof of this proposition (see proof on page 50) follows these lemmas.

Lemma 3.11. *Let F be an $(n-1)$-face and let $U \subset F$ be a compactly contained open subset. Consider the sets*

$$\Omega_+ := (0, L) \times U \subset (0, \infty) \times F,$$

and

$$\Omega_0 := \{0\} \times U.$$

Also let $h_1 \in C^{k+\alpha}(\overline{\Omega_+})$ and $h_2 \in C^{k+\alpha}(\overline{U})$, where k is a nonnegative integer and $\alpha \in (0,1)$. Let f be in $C^\infty(\Omega_+) \cap C^{k+\alpha}(\overline{\Omega_+})$ and let it satisfy

$$\begin{aligned} \left(\left(\tfrac{\partial}{\partial x^n}\right)^2 + \Delta_F \right) f &= h_1 \quad \text{on } \Omega_+ \\ \frac{\partial f}{\partial x^n} &= h_2 \quad \text{on } \Omega_0, \end{aligned}$$

where $\frac{\partial}{\partial x^n}$ is the inward direction normal to Ω_0. If the above holds then, for any set $\Omega'_+ = (0, L') \times U'$ where $L' < L$ and $U' \subset U$ is an open and compactly contained, then f is in $C^{k+1+\alpha}(\overline{\Omega'_+})$.

Proof. This is the proof of [BSCSW, Proposition 5.15]. We begin by showing that this is equivalent to proving the same proposition where f instead solves

$$\left.\begin{aligned} \left(\left(\tfrac{\partial}{\partial x^n}\right)^2 + \Delta_F \right) f &= 0 \quad \text{on } \Omega_+ \\ \frac{\partial f}{\partial x^n} &= h \quad \text{on } \Omega_0, \end{aligned}\right\} \tag{3.1}$$

for some $h \in C^{k+\alpha}(\overline{U})$, where $\frac{\partial}{\partial x^n}$ denotes the inward direction normal to Ω_0. Without loss of generality, we assume there exists a compactly supported function $h \in C^{k+\alpha}(\mathbb{R} \times F)$ such that $h_1 = h|_{\Omega_+}$. This is justified by extension theorems across half spaces as in [S]. So, let $B \subset \mathbb{R} \times F$ be a ball such that $supp(h) \subset B$. Note that we may assume that B is a geodesic ball with respect to the metric on $\mathbb{R} \times F$ given by $\phi\left((\frac{\partial}{\partial x^n})^2 + g_F\right)$ as in Proposition 3.10, where ϕ and the coordinate x^n are smoothly extended to all of $\mathbb{R} \times F$. There is a Green's function $g(z,v)$ on B with respect to the operator $\left(\frac{\partial}{\partial x^n}\right)^2 + \Delta_F$. Let $H(z) := \int_B g(z,v)h(v)\,dv$, and note that $H \in C^{k+2+\alpha}(B)$ and

$$\left(\left(\tfrac{\partial}{\partial x^n}\right)^2 + \Delta_F\right)(f+H) = 0 \quad \text{on } \Omega_+$$
$$\frac{\partial}{\partial x^n}(f+H)|_{\Omega_0} = h_2 + \frac{\partial H|_{\Omega_0}}{\partial x^n} \quad \text{on } \Omega_0.$$

Since $f+H$ is in $C^{k+\alpha}(\overline{\Omega_+})$ and $h_2 + \frac{\partial H|_{\Omega_0}}{\partial x^n}$ is in $C^{k+\alpha}(\overline{\Omega_0})$, we may replace f by $f+H$ and examine instead equation (3.1) above. Pick L' and U' as in the statement of the proposition, so

$$\Omega'_+ = (0,L') \times U' \subset (0,L) \times U \subset (0,\infty) \times F.$$

For some $U'' \subset U$ such that $\overline{U'} \subset U''$, there exists $f_1 \in C_c^{k+\alpha}(\{0\} \times F)$ such that $f_1|_{\{0\}\times U''} = f|_{\{0\}\times U''}$. Then let f_2 be a function on $\Omega''_0 := \{0\} \times U''$ such that

$$f_2 := f_1 - f \quad \text{on } \Omega_0;$$

then, $f_2 = 0$ on Ω''_0. Given f_1 as above on $\{0\} \times F$, let F_1 be a function on $[0,\infty) \times F$ such that

$$\left(\left(\tfrac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) F_1 = 0 \quad \text{on } [0,\infty) \times F$$

and

$$F_1 = f_1 \quad \text{on } \{0\} \times F.$$

As F is an open $(n-1)$-simplex with a smooth metric, F is a manifold, as is $[0, L') \times F$ (which has a boundary). Also, $((\frac{\partial}{\partial x^n})^2 + \Delta_F)$ is an elliptic operator with constant coefficients, so the existence of F_1 is guaranteed. Hence, there is an F_2 such that

$$f = F_1 + F_2,$$

where F_2 satisfies $\left(\left(\frac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) F_2 = 0$ on Ω_+ and $F_2|_{\Omega_0} \equiv 0$. By standard elliptic PDE theory for domains with boundary, F_2 must have continuous, bounded derivatives of all orders on $[0, \infty) \times U''$. Thus, if it can be proven that F_1 is in $C^{k+1+\alpha}(\overline{\Omega'_+})$, then f must be in $C^{k+1+\alpha}(\overline{\Omega'_+})$ and the proof is complete. We concentrate our efforts there. We recall that F is a smooth open manifold with Riemannian metric g and a corresponding Laplace-Beltrami operator Δ_F. Thus, there exists a fractional Laplacian, $\sqrt{-\Delta_F}$ that satisfies for $u \in C^2(F)$ $\left(\sqrt{-\Delta_F}\right)^2 u = -\Delta_F u$ (we refer to [BSCSW, Appendix A] for a tidy summary of the existence and properties of fractional Laplacians on manifolds). We observe that on U'',

$$\begin{aligned} h &= \frac{\partial f}{\partial x^n} \\ &= -\sqrt{-\Delta_F} f_1 + \frac{\partial}{\partial x^n} F_2. \end{aligned}$$

By rearrangment,

$$\left(\mathrm{Id} + \sqrt{-\Delta_F}\right) f_1\Big|_{\Omega''_0} = \left(h + f_1 + \tfrac{\partial}{\partial x^n} F_2\right)\big|_{\Omega''_0}.$$

By assumptions on h and f_1 and F_2, $\left(h + f_1 + \frac{\partial}{\partial x^n} F_2\right)|_{\Omega_0''} \in C^{k+\alpha}_{\text{loc}}(U'')$. So, let $U''' \subset U''$ be an open set such that $\overline{U'} \subset U'''$. Then there exists $f_3 \in C^{k+\alpha}_c(U'')$ such that

$$f_3|_{U'''} = \left(h + f_1 + \tfrac{\partial}{\partial x^n} F_2\right)\Big|_{U'''}.$$

Extend f_3, originally defined on U'', trivially to all of U'' (as $supp(f_3) \subset U''$), and let

$$f_4 = \left(\text{Id} + \sqrt{-\Delta_F}\right)^{-1} f_3.$$

We note that by Appendix A in [BSCSW], $\left(\text{Id} + \sqrt{-\Delta_F}\right)^{-1}$ is a well defined operator on F, with an kernel given by

$$G(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} h^F(x, y, \tfrac{t^2}{4u})\, du\, dt,$$

where h^F is the heat kernel on F. By $f_3 \in C^{k+\alpha}_c(U'')$, f_3 is also in $L^2(F)$ and $f_4 \in C^{k+1+\alpha}_{\text{loc}}(F) \cap L^2(F)$. By definition of f_4, we note

$$\left(\text{Id} + \sqrt{-\Delta_F}\right)(f_1 - f_4) = 0 \quad \text{on } U'''.$$

As F is not compact (its closure is and has a piecewise smooth boundary),

$$\left(\text{Id} + \sqrt{-\Delta_F}\right)(f_1 - f_4)$$

as a function can be extended outside of U to be in $L^2(F)$, as in U it is continuous and compactly supported. Thus, we can apply Theorem A.4 of [BSCSW], and $f_1 - f_4 \in C^\infty_{\text{loc}}(U''')$. Since $f_4 \in C^{k+1+\alpha}_{\text{loc}}(F)$, f_1 must have the same regularity and $f_1 \in$

$C^{k+1+\alpha}_{\text{loc}}(U''')$. As $\overline{U'} \subset U'''$, f_1 is in $C^{k+1+\alpha}(U')$ and F_1 must be in $C^{k+1+\alpha}(U')$, too. As F_2 is has continuous, bounded derivatives of all orders, $f = F_1 + f_2$ must be in $C^{k+1+\alpha}(\overline{\Omega'_+})$, where $\Omega'_+ = (0, L') \times U' \subset (0, L) \times F$, as stated in the proposition. □

Lemma 3.12. *Let F be an $(n-1)$-face, and let $U \subset F$ be a compactly contained open set. Consider the sets*

$$\Omega_+ := (0, L) \times U \subset (0, \infty) \times F,$$

and

$$\Omega_0 := \{0\} \times U.$$

Let J be a fixed positive integer, and for all $1 \le i, j \le J$, let $\delta_j, \tilde{\delta}_j, c_{ij} \in \mathbb{R}$ be constants such that $\delta_j, c_{ij} > 0$. For all $1 \le i, j \le J$, let $w_j, \tilde{w}_j \in C^\infty(\Omega_+)$ and also let them satisfy the following properties.

i. For every j, $1 \le j \le J$, $w_j, \tilde{w}_j \in C^{k+\alpha}(\overline{\Omega_+})$ for some nonnegative integer k and $\alpha \in (0,1)$. Also, for all $1 \le e, j \le J$,

$$w_i|_{\Omega_0} = c_{ij} w_j|_{\Omega_0},$$

and $w_i|_{\Omega_0} \in C^{k+\alpha}(\overline{U})$.

ii. $\left(\left(\frac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) w_j = \tilde{w}_j$ on Ω_+.

iii. On U, for each $1 \le j \le J$, w_j weakly satisfies

$$\sum_{j=1}^{J} \delta_j \frac{\partial w_j}{\partial x^n} = \sum_{j=1}^{J} \tilde{\delta}_j w_j.$$

If all of the above hold, then for all $1 \leq j \leq J$*,* $w_j \in C^{k+1+\alpha}([0, L) \times U)$ *where* $\alpha \in (0, 1)$.

Proof. Define a continuous function, W, on $\overline{\Omega_+}$ as

$$W := \sum_{j=1}^{J} \delta_j w_j.$$

One can verify that W satisfies

$$\left.\begin{aligned} \left(\left(\tfrac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) W &= W_1 \quad \text{on } \Omega_+ \\ \frac{\partial W}{\partial x^n} &= W_2 \quad \text{on } U, \end{aligned}\right\} \tag{3.2}$$

where, by hypothesis,

$$W_1 = \sum_{j=1}^{J} \delta_j \tilde{w}_j$$
$$W_2 = \frac{1}{\delta} \sum_{j=1}^{J} \tilde{\delta}_j w_j,$$

with $\delta := \sum_{j=1}^{J} \delta_j$. We note that $W_1 \in C^{k+\alpha}(\overline{\Omega_+})$ and $W_2 \in C^{k+\alpha}(\overline{U})$. As W satisfies equation 3.2, we note that the conditions of Lemma 3.11 are satisfied and we may apply it to W. Hence, for $\Omega''_+ = (0, L'') \times U''$ where $L'' \in (0, L')$ and $U'' \subset U'$ compactly contained, we have $W \in C^{k+1+\alpha}(\overline{\Omega''_+})$. Since our choice of L'', U'' were arbitrary, we may say $W \in C^{k+1+\alpha}(\overline{\Omega'_+})$ for any $(0, L') \times U'$ relatively compact with respect to the initial set U. We now must prove similar regularity for the functions w_j. We recall that

$$w_i|_{\Omega_0} = c_{ij} w_j|_{\Omega_0},$$

and thus, given the definition of W, each w_j must equal some multiple of W on Ω_0. Hence, as $W|_{\Omega_0} \in C^{k+1+\alpha}(U)$, for all $1 \leq j \leq J$, $w_j|_{\Omega_0} \in C^{k+1+\alpha}(U)$. Now we must obtain regularity on the interior, Ω_+. We note that each w_j satisfies

$$\left.\begin{aligned} \left(\left(\frac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) w_j = \tilde{w}_j \quad \text{on } \Omega_+ \\ w_j = C_j W \quad \text{on } U. \end{aligned}\right\}$$

Hence, we may repeat the arguments of Lemma 3.11 to obtain that for a relatively compact Ω'_+, $w_j \in C^{k+1+\alpha}(\overline{\Omega'_+})$ for each j, $1 \leq j \leq J$. □

Proof of Proposition 3.10 (p. 43). The second part of the proposition follows from standard results in partial differential equations as $A \cap S$ is an open region isometric to a region in some smooth manifold. The third follows easily from the first two parts of the proposition and the observation of Proposition 2.17. Thus, we focus on the first part which gives us regularity across the $(n-1)$-faces of $A \subset X$. As the result is local, we may assume with out loss of generality that our open region $A \subset X$ is a small neighborhood, Ω, around a point p contained in an $(n-1)$-face F such that $p \notin X^{(n-2)}$, $\Omega \cap X^{(n-2)} = \emptyset$ and $\Omega \subset \cup_{j=1}^{J} \overline{S_i}$, where $\{S_i\}_{j=1}^{J} = \mathrm{Star}^{(n)}(F)$ is the set of all n-simplexes whose closure contains the $(n-1)$-face F. Let us pick normal coordinates $(z^1, \ldots, z^n)$ about p such that $(z^1, \ldots, z^{n-1}, 0)$ parameterizes $\Omega \cap F$. Hence, x^n denotes the coordinate normal to F on some closed simplex $\overline{S}$. Define for each $S_j \in \mathrm{Star}^{(n)}(F)$, define $\Omega_j := \Omega \cap S_j$. Also, Let g_j be the metric on S_j,and let g_F be the metric on F (induced by the limit of g_j for each S_j on F). Hence, for each $S_j \in \mathrm{Star}^{(n)}(F)$ there exists smooth positive functions ϕ_j such that

$$g_j = \phi_j \left((dx^n)^2 + g_F\right).$$

Hence, for a weak solution u as above one can compute that it must satisfy

$$\begin{aligned}\Delta_g u &= \left(\left(\tfrac{\partial}{\partial x^n}\right)^2 + \Delta_F + \left(\tfrac{\partial}{\partial x^n}\ln(\phi^{\frac{n-1}{2}})\right)\tfrac{\partial}{\partial x^n}\right)u \\ &= \tfrac{\partial}{\partial t}u,\end{aligned}$$

in each Ω_j where Δ_F is the Laplace-Beltrami operator on F with respect to the metric g_F. Also, it must weakly satisfy

$$\sum_{j=1}^{J}\frac{\partial u_j}{\partial x^n}(x^1,\ldots,x^{n-1},0) = 0.$$

We introduce a change of functions. Let

$$w_i = \phi_j^{\frac{(n-1)}{4}} u|_{S_j}.$$

Upon substitution, our above equations become

$$\left(\left(\frac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) w_j = \frac{\left(\frac{\partial}{\partial x^n}\right)^2 \phi_j^{(n-1)/4}}{\phi_j^{(n-1)/4}} w_j + \phi_j \frac{\partial w_j}{\partial t}, \tag{3.3}$$

in each Ω_j, and on $F \cap \Omega$,

$$\sum_{j=1}^{J}\frac{\partial w_j}{\partial x^n}(x^1,\ldots,x^{n-1},0) = \\ -\sum_{j=1}^{J}\frac{1}{\phi^{(n-1)/2+1/2}}\left(\frac{\partial}{\partial x^n}\phi^{\frac{(n-1)}{4}}\right)\frac{\partial w_j}{\partial x^n}(x^1,\ldots,x^{n-1},0).$$

We can now use Lemma 3.12 and bootstrap a finite number of times to show the regularity of each w_j and hence u. Without loss of generality (as this is a local result), we modify our neighborhood around p, Ω, so that for each n-simplex S_j adjacent to F, $\Omega_j = (0, L) \times U$ in coordinates, where L is small enough that each Ω_j is contained in S_j. Also, let w_j be as above and let

$$\tilde{w}_j = \frac{\left(\frac{\partial}{\partial x^n}\right)^2 \phi_j^{(n-1)/4}}{\phi_j^{(n-1)/4}} w_j + \phi_j \frac{\partial w_j}{\partial t}. \tag{3.4}$$

We note that both w_j and $\frac{\partial}{\partial t} w_j$ both appear on the right hand side. To apply Lemma 3.12 repeatedly, we require some regularity on $\frac{\partial}{\partial t} w_j$. We observe that $\frac{\partial}{\partial t} w_j$ is Hölder continuous as we have assumed that u is a solution that is time regular to order k, so $\frac{\partial}{\partial t} u$ is a solution, too, and we know that it must be Hölder continuous by the parabolic Harnack inequality (see Proposition 3.7 on page 41). Hence, we can apply Lemma 3.12 k times to w. □

Remark 3.13. We note that Proposition 3.10 can also be proven if the assumption of time-regularity were removed and simply replaced by an assumption of smoothness with respect to time; i.e. $\left(\frac{\partial}{\partial t}\right)^m u$ is Hölder continuous. Indeed, the assumption of time-regularity to order k is only used to acquire the knowledge that a solution must have time derivatives of order k that are Hölder continuous.

3.5 Higher Regularity of Solutions to Elliptic Equations and Non-Homogeneous Equations

We provide regularity of solutions to elliptic-type equations.

Proposition 3.14. *Let $A \subset X$ be open such that $d(A, X^{(n-2)}) > 0$. If u is a weak solution to $\Delta u = f$ on A, and $f \in C^{k+\alpha}(A)$ then u satisfies the following:*

i. *$u \in C^{k+1+\alpha}(A)$, where $0 < \alpha < 1$.*

ii. *For any open n-simplex S with metric g,*

$$\Delta_g u|_S = f|_S$$

on A pointwise, where Δ_g is the Laplace-Beltrami operator on S. Also, $u|_S \in C^\infty(A \cap S)$.

iii. *u is balanced on $A \cap X^{(n-1)}$.*

Proof. This is a simplification of Proposition 3.10. We note that equation (3.3) of Proposition 3.10 can be replaced in the case by

$$\left(\left(\frac{\partial}{\partial x^n}\right)^2 + \Delta_F\right) w_j = \frac{\left(\frac{\partial}{\partial x^n}\right)^2 \phi_j^{(n-1)/4}}{\phi_j^{(n-1)/4}} w_j + f,$$

and equation (3.4) can be replaced by

$$\tilde{w}_j = \frac{\left(\frac{\partial}{\partial x^n}\right)^2 \phi_j^{(n-1)/4}}{\phi_j^{(n-1)/4}} w_j + f.$$

and the proof is essentially the same. □

By a similar argument, we have the following.

Proposition 3.15. *Let $(0,T) \subset (0,\infty)$ and let $A \subset X$ be open such that $d(A, X^{(n-2)}) > 0$. If u is a weak solution to $(\frac{\partial}{\partial t} - \Delta)u = f$ on $(0,T) \times A$ for f such that $\left(\frac{\partial}{\partial t}\right)^{m-1} f(\cdot,t) \in C^{k-m+\alpha}(X)$, $1 \le m \le k$, and $\left(\frac{\partial}{\partial t}\right)^m u(\cdot,t)$ is Hölder continuous for $0 \le m \le k$, then u satisfies the following:*

i. $\left(\frac{\partial}{\partial t}\right)^m u(\cdot,t) \in C^{k-m+\alpha}(X)$ *for* $0 \leq m \leq k$.

ii. For any open n*-simplex* S *with metric* g,

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) u|_S = f|_S$$

on $(0,T) \times A \cap S$ *pointwise, where* Δ_g *is the Laplace-Beltrami operator on* S.

iii. For each m, $0 \leq m \leq k$, $\left(\frac{\partial}{\partial t}\right)^m u$ *is balanced on* $A \cap X^{(n-1)}$. *If* $m = k$, *then* $\left(\frac{\partial}{\partial t}\right)^m u$ *is weakly balanced.*

We remark that this regularity result seems a bit weak as it *a priori* assumes that u has a good deal of regularity in time. As we consider the heat flow problem for the linear case, this will not be a problem, as we shall build our flow from the heat kernel and we can show that all time derivatives are Hölder continuous directly. However, the non-linear case requires a bit more care. We give in Proposition 4.28 (see page 80) a set of equations and solutions that possess some higher regularity with respect to time derivatives.

4 The Heat Flow on Polyhedra with Target $\mathbb{R}$

Assumptions. Unless otherwise specified, we shall assume in this section that X is an admissible smooth Riemannian polyhedron that satisfies the conditions of Proposition 2.20 with Dirichlet form $E(\cdot,\cdot)$ and Laplacian Δ as in Section 2.2, and energy $E(\cdot)$ as in Section 2.4.

4.1 Solutions to the Homogeneous Initial Value Problem

There are a number of initial value-type and Dirichet-type problems we can consider. We begin with the initial value problem for homogeneous parabolic-type equations, and then discuss constructive methods for solutions to other problems.

We recall the definition for a weak solution from Definition 3.2:

Definition 4.1. Let $I = (a, b) \subset \mathbb{R}$.

i. A function u is a *weak solution* of the parabolic equation

$$\frac{\partial}{\partial t} u = \Delta u,$$

on $I \times X$, where I is an interval, if and only if $u \in \mathcal{F}(I \times X)$ and u satisfies

$$\int_I E(u, \phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt = 0,$$

for all $\phi \in \mathcal{F}(I \times X)$. u is a *weak subsolution (super-)* if

$$\int_I E(u, \phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt \le (\ge) 0,$$

for all $\phi \in \mathcal{F}(I \times X)$.

ii. Let $f \in L^2(X)$. u is a *weak solution to the initial value problem*

$$\frac{\partial}{\partial t} u = \Delta u, \quad \text{on } I \times X$$
$$u(\cdot, a) = f, \quad \text{on } X$$

if and only if u is a solution as above, and $\lim_{t\to a^+} u = f$ in $L^2(X)$.

Proposition 4.2. *Let X be as in the assumptions for this section. For all $f \in L^2(X)$ there exists a unique weak solution to the initial value problem with initial value f, as in Definition 3.2.*

Proof. This is nearly a direct application of [St2, Prop. 1.2], where existence is guaranteed on a space that has a uniform local Poincaré bound and satisfies the volume doubling properties. The satisfaction is guaranteed by the assumptions on X (see Proposition 2.20 on page 31). □

The assumption on the time derivative can be weakened considerably. Following the observation of Sturm, we cite the following.

Proposition 4.3 (See [St2, Prop. 1.3])**.** *Let $I = (\sigma, \tau)$ be an open interval on $\mathbb{R}$. A function u is a weak solution of the parabolic equation, $\frac{\partial}{\partial t}u = \Delta u$, on $I \times X$ if and only if*

$$u \in L^2(I \to W^{1,2}(X)) \cap C(\overline{I} \times L^2(X)),$$

and

$$\int_\sigma^T E(u,\phi)\,dt - \int_\sigma^T \left\langle u, \tfrac{\partial}{\partial t}\phi\right\rangle_{L^2}\,dt = -\left\langle u(T,\cdot), \phi(T,\cdot)\right\rangle_{L^2} + \left\langle u(\sigma,\cdot), \phi(\sigma,\cdot)\right\rangle_{L^2},$$

for all $T \in (\sigma, \tau)$ and all $\phi \in \mathcal{F}((\sigma, T) \times X)$.

We of course are interested in existence and uniqueness statements of non- homogeneous parabolic-type equations, which are covered in Section 4.5 on page 79. To construct solutions, we require more knowledge about solutions to the homogeneous problem.

4.2 The Heat Semigroup and Heat Kernel

From the existence and uniqueness theorems for weak solutions to the heat equation, we can develop a corresponding semigroup theory.

Proposition 4.4 (See [St2, Section 1.4(C)]). *There exists a uniquely determined, one-parameter set of operators $H_t\colon L^2(X) \to L^2(X)$, such that it has the following properties.*

i. *for every $f \in L^2(X)$, the unique weak solution $u \in \mathcal{F}(I \times X)$ of the initial value problem is given by $u := H_t f$.*

ii. *H_t is a semigroup*

iii. *$t \mapsto H_t$ is strongly continuous*

iv. *H_t has the Markov property. That is, for $f \in L^2(X)$ and all $t \geq 0$,*

$$0 \leq f \leq 1 \Rightarrow 0 \leq H_t f \leq 1.$$

Also, this holds for $f \in L^p(X)$, $p \in [1, \infty]$.

v. *H_t is a contraction operator on $L^p(X)$, $1 \leq p \leq \infty$. That is,*

$$\|H_t f\|_{L^p} \leq \|f\|_{L^p},$$

for $f \in L^p(X)$.

The existence of such a set of operators H_t allows for the definition of an integral kernel as follows. Again, we largely cite the work of Sturm (see [St2]).

Proposition 4.5 (See [St2, Prop. 2.3])**.** *For H_t as above, there exists a measurable function $h\colon X \times X \times \mathbb{R}^+ \to [0,\infty)$ such that*

i. for every $f \in L^1(X) \cup L^\infty(X)$ and $t > 0$,

$$H_t f(x) = \int_X h(z,v,t) f(v)\, d\mu(z),$$

ii. for all $0 < \sigma < \tau$, all $y \in X$, and all $m \in \mathbb{N}$

$$u : (t,x) \mapsto (\tfrac{\partial}{\partial t})^m h(z,v,t),$$

is a weak solution of the equation $\frac{\partial}{\partial t} u = \Delta u$ on $(\sigma,\tau) \times X$,

iii. $\int_{X\times X} h(z,v,t)^2\, d\mu(x) d\mu(y) \leq 1$,

iv. $h(z,v,t+s) = \int_X h(z,w,t) h(w,v,s)\, d\mu(w)$,

v. h is locally bounded on $\{(z,v,t) \mid t > 0\}$

Proof. This is a direct application of [St2, Proposition 2.3], but we give a slightly stronger version here in item *(ii)*. □

Lemma 4.6. *X be compact. There exists an operator $(-\Delta)^{\frac{1}{2}} \colon W^{1,2}(X) \to L^2(X)$ with the following properties.*

i. For $f \in W^{1,2}(X)$, $\|(-\Delta)^{\frac{1}{2}} f\|_{L^2} = \|\nabla f\|_{L^2}$.

ii. For $f \in \mathrm{Dom}(\Delta)$, $\left((-\Delta)^{\frac{1}{2}}\right)^2 f = -\Delta f$.

iii. $(-\Delta)^{\frac{1}{2}}$ *is self-adjoint; i.e. for* $f, g \in W^{1,2}(X)$,

$$\left\langle (-\Delta)^{\frac{1}{2}} f, g \right\rangle_{L^2(X)} = \left\langle f, (-\Delta)^{\frac{1}{2}} g \right\rangle_{L^2(X)}.$$

iv. For the heat kernel associated to Δ, $h(z, v, t)$,

$$(-\Delta_z)^{\frac{1}{2}} h(z, v, t) = (-\Delta_v)^{\frac{1}{2}} h(z, v, t).$$

Proof. Although it is possible to show this without the use of the spectral theorem, it is easier to use it here. We note that [CKP, Proposition 3.20] applies here, as our domain, X, is a compact Dirichlet space with a bound on volume doubling and a local Poincaré inequality. So, we can show that $L^2(X)$ is separable and its basis can be composed of eigenfunctions corresponding to eigenvalues of Δ. The rest follows handily by noting that for $f \in L^2(X)$

$$f(z) = \sum_i \langle f, \phi_i \rangle_{L^2} \phi_i(z);$$

for $f \in W^{1,2}(X)$,

$$(-\Delta)^{\frac{1}{2}} f(z) = \sum_i \lambda_i \langle f, \phi_i \rangle_{L^2} \phi_i(z);$$

and

$$h(z, v, t) = \sum_i e^{-\lambda_i t} \phi_i(z) \phi_i(v),$$

where $\{(\lambda_i, \phi_i)\}$ are pairs of corresponding eigenvalues and eigenfunctions. □

As Δ, $(-\Delta)^{\frac{1}{2}}$, and the heat operator, H_t, are all operators, we also require some basic results for operators.

Definition 4.7. Let A, B be Banach spaces. For an operator $\Lambda\colon A \to B$, we define

$$\|\Lambda\|_{A\to B} := \sup_{f\in A} \frac{\|\Lambda f\|_B}{\|f\|_A},$$

where $\|\cdot\|_A$ is the norm associated to A and $\|\cdot\|_B$ is the norm associated to B. Note that if the operator is unbounded, we set the value to be ∞. Also, in the event that $A = L^p(X)$ and $B = L^q(X)$ for some metric space X and $1 \le p, q \le \infty$, we denote the norm of the operator $\|\Lambda\|_{p\to q}$.

A few lemmas regarding integral operator norms such as the heat operator will be useful.

Lemma 4.8. *For a metric measure space X with volume measure denoted dX, let $\Lambda\colon L^2(X) \to L^\infty(X)$ be an operator with symmetric integral kernel k. That is, for $f \in L^2(X)$,*

$$\Lambda f(z) = \int_X k(z,v) f(v)\, dX(v).$$

Then, for every $v \in X$

$$\sup_{v\in X} \|k(\cdot, v)\|_{L^2} \le \|\Lambda\|_{2\to\infty}.$$

Proof. By definition, we have

$$\begin{aligned}
\|\Lambda\|_{2\to\infty} &= \sup_{f\in L^2(X)} \frac{\|\Lambda f\|_{L^\infty}}{\|f\|_{L^2}} \\
&= \sup_{f\in L^2(X)} \frac{\sup_{z\in X} \left|\int_X k(z,v) f(v)\, dX(v)\right|}{\left(\int_X f(v)^2\, dX\right)^{\frac{1}{2}}}.
\end{aligned}$$

Hence, for arbitrary $f \in L^2(X)$ and $z \in X$, we have

$$\|\Lambda\|_{2\to\infty} \geq \frac{\sup_{z\in X}\int_X k(z,v)f(v)\,dX(v)}{\left(\int_X f(v)^2\,dX\right)^{\frac{1}{2}}}$$

If we let $f(v) = k(z,v)$ we have

$$\begin{aligned}\|\Lambda\|_{2\to\infty} &\geq \sup_{z\in X}\frac{\int_X k(z,v)^2\,dX(v)}{\left(\int_X k(z,v)^2\,dX\right)^{\frac{1}{2}}}\\ &= \sup_{v\in X}\|k(\cdot,v)\|_{L^2}.\end{aligned}$$

□

Lemma 4.9. *Let X be again as it is in the assumptions in the beginning of this section. Let $h(z,v,t)$ and H_t be the corresponding heat kernel and heat operator. Then, for every $v \in X$,*

$$\|H_t\|^2_{2\to\infty} = \sup_{z\in X} h(z,z,2t).$$

Proof. By definition,

$$\begin{aligned}\|H_t\|_{2\to\infty} &= \sup_{f\in L^2(X)}\frac{\|H_tf\|_{L^\infty}}{\|f\|_{L^2}}\\ &= \sup_{f\in L^2(X)}\frac{\sup_{z\in X}\int_X h(z,v,t)f(v)\,dX(v)}{\left(\int_X f(v)^2\,dX(v)\right)^{\frac{1}{2}}}\\ &\leq \sup_{f\in L^2(X)}\frac{\left(\sup_{z\in X}\int_X h(z,v,t)^2\,dX(v)\right)^{\frac{1}{2}}\left(\int_X f(v)^2\,dX(v)\right)^{\frac{1}{2}}}{\left(\int_X f(v)^2\,dX(v)\right)^{\frac{1}{2}}}\\ &= \left(\sup_{z\in X}\int_X h(z,v,t)^2\,dX(v)\right)^{\frac{1}{2}}\\ &= \left(\sup_{z\in X} h(z,z,2t)\right)^{\frac{1}{2}}.\end{aligned}$$

The last step follows by the semigroup property for the heat kernel. We can show the inequality

$$\|H_t\|_{2\to\infty} \geq \left(\sup_{z\in X} h(z,z,2t)\right)^{\frac{1}{2}}$$

by an argument similar to that of Lemma 4.8. □

4.3 Smoothness and L^p Bounds of the Heat Kernel

We recall a previous result with an addition that relates to the heat kernel from [PSC, Cor. 3.4] (see also [St3, Prop. 3.1], and Proposition 3.7).

Proposition 4.10. *For all $R > 0$, there exists C, dependent on X and R, and $\alpha \in (0,1)$ such that for all $p \in X$ and $T \in R$ and $0 < r < R$,*

$$|u(s,x) - u(t,y)| \leq C \sup_Q |u| \left(\frac{|s-t|^{\frac{1}{2}} + |y-z|^{\frac{1}{2}}}{r}\right),$$

where u is a weak solution of $\frac{\partial}{\partial t}u = \Delta u$ on $Q = (t-4r^2,t)\times B(p,2r)$, $s,t \in (T-r^2,T)$ and $y,z \in B(p,r)$. Also, for $t > 0$, $z,v \in X$ and $w \in B(v,\min\{1,\sqrt{t}\})$, we have for $j \in \mathbb{N}$,

$$\left|\left(\tfrac{\partial}{\partial t}\right)^j h(z,v,t) - \left(\tfrac{\partial}{\partial t}\right)^j h(z,w,t)\right| \leq C_j \left(\frac{d(z,v)}{\min\{1,\sqrt{t}\}}\right)^{\alpha} \frac{h(z,v,2t)}{\min\{1,\sqrt{t}\}^j},$$

where $C_j > 0$.

Proposition 4.11. *Let X be as it is in the assumptions of this section. Let $h(z,v,t)$ be the heat kernel on X. For $t > 0$,*

i. $h(z_0,v_0,t)$ is $C^\infty((0,\infty))$ with respect to t for $z_0,v_0 \in X$.

ii. *For any compactly contained open set* $V \subset X$ *bounded away from* $X^{(n-2)}$, $h(z_0, v, t_0) \in C^\infty(\overline{V})$, *for any* $z_0 \in X$.

iii. *For any compactly contained open set* $V \subset X$ *bounded away from* $X^{(n-2)}$, $h(z, v_0, t_0) \in C^\infty(\overline{V})$, *for any* $v_0 \in X$.

iv. $h(z, v, t)$ *is balanced in both* z *and* v.

v. *The above hold for* $\left(\frac{\partial}{\partial t}\right)^m h(z, v, t)$, $m \in \mathbb{N}$.

vi. $u(z, t) := \left(\frac{\partial}{\partial t}\right)^m h(z, v_0, t)$, $m \in \mathbb{N}$, *is a weak solution of* $(\frac{\partial}{\partial t} - \Delta)u = 0$, *for any* $v_0 \in X$.

vii. $\left(\frac{\partial}{\partial t}\right)^m h(z, v, t) = (\Delta)^m h(z, v, t)$ *holds pointwise for* $z, v, t \in X \times X \times \mathbb{R}^+$, $m \in \mathbb{N}$, *where* Δ *may apply to either* z *or* w.

Proof. We begin with (i), which follows easily from the Hölder continuity of time derivatives in Proposition 4.10 for $t > 0$. By the definition of a weak solution and by (i), (vi) follows. For (ii) and (iii), we see that by (vi), $u(z, t) := \left(\frac{\partial}{\partial t}\right)^m h(z, v_0, t)$ must be time regular of order infinity, so we may apply Proposition 3.10, which also gives us (iv). (v) follows from (vi) and Proposition 3.7. (vii) follows by (vi). □

We have one comment about the set on which the equality $\left(\frac{\partial}{\partial t}\right)^j h(z, v, t) = (\Delta)^j h(z, v, t)$ holds. For $j = 1$, clearly this must hold weakly on $(0, \infty)$, but we have stronger results which we shall find useful later.

Lemma 4.12. *For all* $j \in \mathbb{N}$, *all* $v \in X$, *and all* $t \in (0, \infty)$,

$$\left(\frac{\partial}{\partial t}\right)^j h(z, v, t) = (\Delta_z)^j h(z, v, t),$$

for $z \in X \setminus X^{(n-2)}$*, where* Δ_z *denotes the Laplacian applied to the* z*-slot of* h*. Additionally, for each* j *there exists a Hölder continuous version of* $(\Delta_z)^j h(z,v,t)$ *such that it equals* $\left(\frac{\partial}{\partial t}\right)^j h(z,v,t)$ *everywhere.*

Proof. We begin with the case $j = 1$. We recall from Proposition 4.4 that for any $v \in X$, $u(z,t) := h(z,v,t)$ is a weak solution to $\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$. Hence the regularity results of Section 3.4 apply here which is summarized in Proposition 4.11. Hence, $\frac{\partial}{\partial t} h(z,v,t) = \Delta_z h(z,v,t)$ pointwise on $X \setminus X^{(n-1)}$. Again by Proposition 4.11, weak solutions are smooth on $X^{(n-1)} \setminus X^{(n-2)}$, and they must agree pointwise on $X^{(n-1)} \setminus X^{(n-2)}$. Equality on $X^{(n-2)}$ is uncertain, but as $\frac{\partial}{\partial t} h(z,v,t)$ is additionally Hölder continuous by Proposition 4.10 and they agree at all points except $X^{(n-2)}$, then there exists a Hölder continuous version of $\Delta_z h(z,v,t)$ that is equal to $\frac{\partial}{\partial t} h(z,v,t)$ everywhere. For $j \geq 2$, this can be proven similarly by realizing that $\frac{\partial}{\partial t}$, Δ commute at points in $X \setminus X^{(n-1)}$, as both can be considered pointwise-defined differential operators there. □

We are naturally curious about the behavior of constant functions under the heat flow on an admissible Riemannian polyhedron. In the literature on Markov processes, if constant functions are constant under the heat flow they are then the heat operator is *conservative*. If they decrease, the heat operator is called *transitive*.

Proposition 4.13 (See [PSC, Cor. 3.2], see also [St1])**.** *For* $t > 0$ *and all* $z \in X$*, the heat operator is conservative. That is,*

$$\int_X h(z,v,t)\, d\mu(v) = 1.$$

4.4 Gaussian Estimates of the Heat Kernel

There are many Gaussian-type estimates for the heat kernel and it derivatives in both space and time for manifolds. We refer to the work of Saloff-Coste [SC1, SC2] and Davies [D]. We have the following as a direct consequence of Sturm's result for Dirichlet spaces with conditions put on the volume doubling and the existence of a uniform local Poincaré inequality.

Proposition 4.14 (See [PSC, Cor. 3.4], see also [St3, Corollaries 4.2 & 4.10]). *For all $R > 0$, there exist $\{C_j\}_{j\in\mathbb{N}}$, all dependent on X and R, such that for all $z, v \in X$ and $t > 0$,*

$$h(z,v,t) \leq \frac{C_0}{\min\{t,R^2\}^{\frac{n}{2}}} e^{-d^2(z,v)/4t-\lambda_0 t} \left(1+d^2(z,v)/t\right)^{\frac{N}{2}}$$
$$h(z,v,t) \geq \frac{1}{C_0 \mathrm{Vol}\left(B(p,\sqrt{\min\{t,R^2\}})\right)} e^{-Cd^2(z,v)/4t-Ct/R^2}$$
$$\left|\frac{\partial^j}{\partial t^j} h(z,v,t)\right| \leq \frac{C_j(1+\lambda_0 t)^{1+\frac{N}{2}+j}}{t^j\left(\min\{t,R^2\}\right)^{\frac{n}{2}}} e^{-d^2(z,v)/4t-\lambda_0 t}\left(1+d^2(z,v)/t\right)^{\frac{N}{2}+j}$$

where N is dependent on the volume doubling constant, and

$$\lambda_0 = \inf_{f\in W^{1,2}(X)} \frac{E(f,f)}{\|f\|_{L^2(X)}}.$$

We note that for our considerations we have the following corollary.

Corollary 4.15. *Let X additionally be compact. Then, for all $R > 0$, there exist*

$\{C_j\}_{j\in\mathbb{N}}$, *all dependent on* X *and* R, *such that for all* $z, v \in X$ *and* $t > 0$,

$$h(z,v,t) \leq \frac{C_0}{\min\{t, R^2\}^{\frac{n}{2}}} e^{-d^2(z,v)/(4+\epsilon)t}$$
$$\left|\frac{\partial^j}{\partial t^j} h(z,v,t)\right| \leq \frac{C_j}{\min\{t, R^2\}^{\frac{n}{2}}} t^{-j} e^{-d^2(z,v)/(4+\epsilon)t}$$
$$\left|(\Delta_z)^j h(z,v,t)\right| \leq \frac{C_j}{\min\{t, R^2\}^{\frac{n}{2}}} t^{-j} e^{-d^2(z,v)/(4+\epsilon)t}$$

where $C_k = C(k, X, \epsilon, R)$, *and* Δ_z *denotes the Laplacian taken in the* z *slot of* $h(z,v,t)$.

Proof. This follows from Proposition 4.14, by noting that $\lambda_0 = 0$ for a compact admissible Riemannian polyhedron (a constant function suffices to show it must be zero), and that the term $(1 + d^2(z,v)/t)^{\frac{N}{2}+j}$ in the statement of Proposition 4.14 can be absorbed into ϵ by adjusting C. Also, we note that by Lemma 4.12, we have

$$(\Delta_z)^j h(z,v,t) = \left(\frac{\partial}{\partial t}\right)^j h(z,v,t),$$

which shows the equivalence of the last two statements of the corollary. □

We ideally would like similar bounds for spacial derivatives. Indeed, since such Gaussian bounds can be provided for powers of the Laplacian and time derivative, there is hope that similar results can be achieved. Although the strongest result in this regard as far as the author is aware is given in [SC2], a slightly older result offers a suggestion of what one might expect to achieve in our setting:

Proposition 4.16 (See [D]). *Let* M *be smooth* n*-dimensional Riemannian manifold with nonnegative Ricci curvature bounded below. Let* Δ *denote the Laplace-Beltrami* *operator, and H_t and $h(z,v,t)$ the corresponding heat operator and heat kernel. Let ∇_z denote the gradient with respect to z. Then there exist $\{C_j\}$ such that*

$$h(z,v,t) \leq \frac{C_0}{\min\{t,R^2\}^{\frac{n}{2}}} e^{-d^2(z,v)/(4+\epsilon)t}$$
$$\left|\nabla_z \left(\tfrac{\partial}{\partial t}\right)^j h(z,v,t)\right| \leq \frac{C_j}{\min\{t,R^2\}^{\frac{n}{2}}} t^{-j-1/2} e^{-d^2(z,v)/(4+\epsilon)t}$$

where $C_j = C(j,X,\epsilon,R) > 0$.

Remark 4.17. We note this result has been extended to include the possibility of M with Ricci curvature bounded below. See [D].

This result is dependent on a logarithmic parabolic Harnack inequality of P. Li and S.T. Yau in [LY] that assumes either M has no boundary, or that M has a smooth boundary and the existence of a heat kernel that satisfies Neumann boundary conditions. These restrictions present difficulty for the case of a Dirichlet space such as an admissible Riemannian polyhedron considered here, so we attempt to achieve similar results without the use of the work of [LY]. Instead we refer to the approaches of Bendikov and Saloff-Coste [BSC], who address Gaussian bounds for heat kernels in the case of the domain being a (possibly infinite dimensional) group.

Theorem 4.18. *Let X be compact and simplex-wise flat. Let $\{Z_i\}_{i=1}^n$ be an orthonormal basis as in Definition 2.5. Let $k \in \mathbb{N}$. Then, for any $R > 0$, there exists positive* *constants* $B, \{C_j\}, \{C'_j\}$ *such that for all* $z, v \in X$,

$$h(z,v,t) \leq \frac{C_0}{\min\{t,R^2\}^{\frac{n}{2}}} e^{-d^2(z,v)/(4+\epsilon)t}$$
$$\left|\frac{\partial^j}{\partial t^j} h(z,v,t)\right| \leq \frac{C_j}{\min\{t,R^2\}^{\frac{n}{2}}} t^{-j} e^{-d^2(z,v)/(4+\epsilon)t}$$
$$\left|D_Z \frac{\partial^j}{\partial t^j} h(z,v,t)\right| \leq \frac{C'_j}{\min\{t,R\}^{\frac{n}{2}}} t^{-j-\frac{1}{2}} e^{-\frac{d(z,v)^2}{Bt}}$$

where $\{C_j\}, \{C'_j\}$ *are dependent on* j, X, ϵ *and* R*, and* B *is only dependent on* X *and* R

Proof. Clearly, the first two inequalities follow from Corollary 4.15. The last requires considerably more effort and is the conclusion of Proposition 4.23 on page 73 below. □

We require a number of results to prove Theorem 4.18. Some we already have, but the gradient estimate will consume the bulk of our effort.

Proposition 4.19. *Let* X *be compact and simplex-wise flat. Let* $\{Z_i\}_{i=1}^n$ *be an orthonormal basis as in Definition 2.5. Let* $f \in \mathrm{Dom}(\Delta) \cap C^2_{loc}(X \setminus X^{(n-1)})$ *and let* f *and* Δf *be balanced. Then,*

$$\int_X f(-\Delta f)\, dX = \int_X |D_Z f|^2\, dX.$$

Proof. As $f \in C^2(X \setminus X^{(n-1)})$ and $\{Z_i\}_{i=1}^n$ corresponds to orthonormal coordinates when restricted to an n-simplex, say s, we note that for $z \in \mathrm{Int}(s)$,

$$\Delta f(z) = \sum_{i=1}^{n} (Z_i)^2 f(z).$$

Hence,

$$\begin{aligned}
\int_X f\left((-\Delta)f\right)\,dX &= \sum_{s\in X^{[n]}} \int_s f\left((-\Delta)f\right)\,dX \\
&= \sum_{s\in X^{[n]}} \int_s f\left(-\sum_{i=1}^{n}(Z_i)^2 f\right)\,dX \\
&= \sum_{s\in X^{[n]}} \sum_{i=1}^{n} \int_s (Z_i f)(Z_i f)\,dX \\
&= \sum_{s\in X^{[n]}} \sum_{i=1}^{n} \int_s (Z_i f)(Z_i f)\,dX \\
&= \int_X |D_Z f|^2\,dX,
\end{aligned}$$

where $X^{[n]}$ is the set of all n-simplexes. This follows by Green's identity and the balancing condition assumed on f. □

Remark 4.20. We note that the equality of Proposition 4.19,

$$\int_X f(-\Delta f)\,dX = \int_X |D_Z f|^2\,dX$$

implies that this result holds for any choice of orthonormal basis $Z = \{Z_i\}_{i=1}^n$, as we expect in the case of a manifold. In other words, by Proposition 4.19, for any two orthonormal bases, Z, Z' and a balanced function f, we have

$$\int_X |D_Z f|^2\,dX = \int_X |D_{Z'} f|^2\,dX$$

We also require an on-diagonal estimate.

Proposition 4.21. *Let X be compact. For any fixed $z \in X$, and fixed $R > 0$, we* *have for all* $t \in (0, R)$,

$$\frac{1}{Ct^{\frac{n}{2}}} \leq h(z, z, t) \leq \frac{C}{t^{\frac{n}{2}}}$$

for some $C = C(X, R)$.

Proof. This consequence is immediate from Proposition 4.14, although a more direct approach without using Gaussian bounds is possible (see [PSC]). □

Proposition 4.22. *Let* X *be compact and simplex-wise flat. Let* $\{Z_i\}_{i=1}^n$ *be an orthonormal basis as in Definition 2.5. Let* $k \in \mathbb{N}$ *. Then, for any* $R > 0$

$$\|D_Z \Delta^k h(\cdot, v, t)\|_{L^2}^2 \leq \frac{C}{t^{k+\frac{1}{2}} \min\{t^{n/2}, R\}},$$

where $C = C(X, R)$ *and* $v \in X$.

NB: *All spacial derivatives of the heat kernel,* $h(z, v, t)$ *are assumed to apply to the first slot in the* z *variable unless otherwise specified.*

For a proof, we follow the approach of [BSC].

Proof. Without loss of generality, we assume that $R = 1$ and that $0 < t < 1$. We note that $h(z, v, t)$ is balanced and, as $\Delta^k h(z, v, t)$ is a (weak) solution to the heat equation, $\Delta^k h(z, v, t)$ is balanced, too. Hence, by the Proposition 4.19, we have

$$\begin{aligned}
\|D_Z \Delta^k h(\cdot, v, t)\|_{L^2}^2 &= \int_X \left| D_Z \Delta^k h(z, v, t) \right|^2 \, dX(z) \\
&= \int_X \left((-\Delta)^k h(z, v, t) \right) \left((-\Delta)^{k+1} h(z, v, t) \right) \, dX(z) \\
&= \int_X \left((-\Delta)^{k+\frac{1}{2}} h(z, v, t) \right)^2 \, dX(z),
\end{aligned}$$

which finally gives

$$\|D_Z\Delta^k h(\cdot,v,t)\|_{L^2}^2 = \|(-\Delta)^{k+\frac{1}{2}} h(\cdot,v,t)\|_{L^2}^2 \tag{4.1}$$

Let H_t denote the heat operator defined by

$$(H_t f)(z) = \int_X h(z,v,t) f(v)\, dX(v),$$

for $f \in L^p(X)$, $p \in [1,\infty]$. Hence, there is an operator $(-\Delta)^{\frac{k}{2}} H_t$ with kernel $(-\Delta)^{\frac{k}{2}} h(z,v,t)$ that satisfies

$$(-\Delta)^{\frac{k}{2}} H_t f(z) = \int_X (-\Delta)^{\frac{k}{2}} h(z,v,t) f(v) dX(v).$$

We note that for any $f \in L^2(X)$, we have

$$\begin{aligned} 2\|(-\Delta)^{\frac{1}{2}} H_t f\|_{L^2}^2 &= 2\int_X \left(-\Delta H_t f(z)\right)\left(H_t f(z)\right)\, dX(z) \\ &= -2\int_X \left(\frac{\partial}{\partial t} H_t f(z)\right)\left(H_t f(z)\right)\, dX(z) \\ &= -\frac{\partial}{\partial t}\|H_t f\|_{L^2}^2. \end{aligned}$$

We note that, as $H_t f$ is a solution to the heat equation, we can show $-\frac{\partial}{\partial t}\|H_t f\|_{L^2}^2$ is a nonnegative, non-increasing function. Hence,

$$\int_0^t \int_X \left(-\Delta H_\tau f(z)\right)\left(H_\tau f(z)\right)\, dX(z)\, d\tau \geq 2t \int_X \left(-\Delta H_\tau f(z)\right)\left(H_\tau f(z)\right)\, dX(z),$$

which, when combined with the fact that

$$\int_0^t \int_X \left(-\Delta H_\tau f(z)\right) \left(H_\tau f(z)\right)\, dX(z)\, d\tau = \|f\|_{L^2}^2 - \|H_t f\|_{L^2}^2,$$

yields

$$2t\|(-\Delta)^{\frac{1}{2}} H_t f\|_{L^2}^2 \leq \|f\|_{L^2}^2. \tag{4.2}$$

Considering the norm of the operator, we have

$$\|(-\Delta)^{\frac{1}{2}} H_t\|_{2\to 2} \leq \sqrt{\frac{1}{2t}}.$$

We can iterate as follows to get a bound for $\|(-\Delta)^{\frac{k}{2}} H_t\|_{2\to 2}$, for $k \in \mathbb{N}$. Firstly, note that for $f \in L^2(X)$,

$$\begin{aligned}
(-\Delta)^{\frac{k}{2}} H_t f(z) &= (-\Delta)^{\frac{k}{2}} H_{t/2} \left(H_{t/2} f\right)(z) \\
&= (-\Delta_z)^{\frac{k}{2}} \int_X h(z, v, \tfrac{t}{2}) H_{t/2} f(v)\, dX(v) \\
&= \int_X (-\Delta_v)^{\frac{k}{2}-\frac{1}{2}} (-\Delta_z)^{\frac{1}{2}} h(z, v, \tfrac{t}{2}) H_{t/2} f(v)\, dX(v) \\
&= \int_X (-\Delta_z)^{\frac{1}{2}} h(z, v, \tfrac{t}{2}) \left[(-\Delta_v)^{\frac{k}{2}-\frac{1}{2}} H_{t/2} f(v)\right]\, dX(v)
\end{aligned}$$

(by $(-\Delta_z)^{\frac{1}{2}}$ being self-adjoint)

$$\begin{aligned}
&= \int_X (-\Delta_z)^{\frac{1}{2}} h(z, v, \tfrac{t}{2}) \left[(-\Delta_v)^{\frac{k}{2}-\frac{1}{2}} H_{t/2} f(v)\right]\, dX(v) \\
&= (-\Delta)^{\frac{1}{2}} H_{t/2} \left[(-\Delta)^{\frac{k}{2}-\frac{1}{2}} H_{t/2} f\right].
\end{aligned}$$

Hence, by equation (4.2),

$$\begin{aligned} t\|(-\Delta)^{\frac{k}{2}} H_t f\|_{L^2}^2 &= t\|(-\Delta)^{\frac{1}{2}} H_{t/2} \left[(-\Delta)^{\frac{k}{2}-\frac{1}{2}} H_{t/2} f\right] \|_{L^2}^2 \\ &\leq \|(-\Delta)^{\frac{k}{2}-\frac{1}{2}} H_{t/2} f\|_{L^2}^2. \end{aligned}$$

We can repeat this argument and obtain

$$\|(-\Delta)^{\frac{k}{2}} H_t\|_{2\to 2} \leq \left(\frac{k}{2t}\right)^{\frac{k}{2}}.$$

By our results regarding operator norms in Lemmas 4.8 and 4.9, we have

$$\begin{aligned} \|(-\Delta)^{\frac{k}{2}} h(\cdot, v, t)\|_{L^2}^2 &\leq \|(-\Delta)^{\frac{k}{2}} H_t\|_{2\to\infty}^2 \\ &\leq \| \left((-\Delta)^{\frac{k}{2}} H_{t/2}\right) H_{t/2}\|_{2\to\infty}^2 \\ &= \|H_{t/2} \left((-\Delta)^{\frac{k}{2}} H_{t/2}\right) \|_{2\to\infty}^2 \\ &\leq \| \left((-\Delta)^{\frac{k}{2}} H_{t/2}\right) \|_{2\to 2}^2 \|H_{t/2}\|_{2\to\infty}^2 \\ &\leq \left(\frac{k}{t}\right)^k \cdot \left(\sup_{v\in X} h(v, v, t)\right) \\ &\leq \left(\frac{k}{t}\right)^k \cdot \frac{C}{t^{\frac{n}{2}}}. \end{aligned}$$

The last step follows from the on-diagonal estimate of Proposition 4.21. □

Proposition 4.23. *Let X be compact and simplex-wise flat. Let $\{Z_i\}_{i=1}^n$ be an orthonormal basis as in Definition 2.5. Let $k \in \mathbb{N}$ be fixed. Then, for any $R > 0$, there* *exist constants $B, C > 0$ only dependent on X and R such that for all $z, v \in X$,*

$$|D_Z \Delta^k h(z,v,t)| \leq \frac{C}{t^{k+\frac{1}{2}} \min\left\{t^{\frac{n}{2}}, R\right\}} e^{-\frac{d(z,v)^2}{Bt}}.$$

Proof. Lemma 4.24 and Proposition 4.25 below prove the theorem. □

Lemma 4.24. *Presume the conditions of the preceding theorem (Proposition 4.23). If for any $R > 0$, there exist constants $B, C > 0$ such that for all $\alpha > 0$ and all $0 < t < R$,*

$$\|e^{\alpha d(\cdot,v)}|D_Z \Delta^k h(\cdot,v,t)|\|_{L^2} \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{B\alpha^2 t},$$

then

$$|D_Z \Delta^k h(z,v,t)| \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{-\frac{d(z,v)^2}{Bt}}.$$

Proof. Again, without loss of generality, we assume that $R = 1$ and that $0 < t < 1$. We begin by noting that if we can prove there exist constants $B, C > 0$ such that for all $\alpha > 0$ and all $0 < t < 1$,

$$\|e^{\alpha d(\cdot,v)}|D_Z \Delta^k h(\cdot,v,t)|\|_{L^\infty} \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{B\alpha^2 t}, \tag{4.3}$$

then the proof is complete. Our justification is that, assuming equation (4.3) to be true, we have for any $z \in X$,

$$e^{\alpha d(z,v)}|D_Z \Delta^k h(z,v,t)| \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{B\alpha^2 t},$$

which implies

$$|D_Z \Delta^k h(z,v,t)| \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{B\alpha^2 t - \alpha d(z,v)}.$$

Letting, $\alpha = \frac{d(z,v)}{2Bt}$, we have

$$|D_Z \Delta^k h(z,v,t)| \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{-\frac{d(z,v)^2}{4Bt}},$$

which gives Proposition 4.23. So, we aim to prove equation (4.3) by showing that the hypothesis in the statement of this lemma implies equation (4.3). Indeed, suppose our claim is true: there exist constants $B, C > 0$ such that for all $\alpha > 0$ and all $0 < t < 1$,

$$\|e^{\alpha d(\cdot,v)} |D_Z \Delta^k h(\cdot,v,t)|\|_{L^2} \leq \frac{C}{t^{k+n/2+\frac{1}{2}}} e^{B\alpha^2 t}. \tag{4.4}$$

For any $w \in X$, we have by the triangle inequality, $d(z,v) \leq d(z,w) + d(w,v)$, and the semigroup property for $h(z,v,t)$ which gives

$$\begin{aligned} e^{2\alpha d(z,v)} |D_Z \Delta^k h(z,v,t)|^2 &= e^{2\alpha d(z,v)} \left| D_Z \Delta^k \int_X h(z,w,\tfrac{t}{2}) h(w,v,\tfrac{t}{2}) \, dX(w) \right|^2 \\ &\leq \sum_{1 \leq i \leq n} \left| e^{\alpha d(z,w)} \int_X Z_i \Delta^k h(z,w,\tfrac{t}{2}) e^{\alpha d(w,v)} h(w,v,\tfrac{t}{2}) \, dX(w) \right|^2 \\ &\leq \|e^{\alpha d(\cdot,v)} |D_Z \Delta^k h(\cdot,v,\tfrac{t}{2})|\|_{L^2}^2 \|e^{\alpha d(\cdot,v)} h(\cdot,v,\tfrac{t}{2})\|_{L^2}^2. \end{aligned} \tag{4.5}$$

By the Gaussian bound on $h(z,v,\frac{t}{2})$ given in Corollary 4.15,

$$h(z,v,\tfrac{t}{2}) \leq \frac{C}{\left(\frac{t}{2}\right)^{\frac{n}{2}}} e^{-2d^2(z,v)/(4+\epsilon)t}$$

combined with an inequality that follows from the fact that $a^2 + b^2 \geq 2ab$,

$$\alpha d(z,v) - \frac{d(v,z)^2}{ct} \leq \frac{\alpha^2 ct}{4},$$

we have

$$\|e^{\alpha d(\cdot,v)}h(\cdot,v,\tfrac{t}{2})\|_{L^2} \leq \frac{C'}{t^{\frac{n}{2}}}e^{\alpha^2 t/\frac{8}{4+\epsilon}},$$

where $C' = C\sqrt{\mathrm{Vol}(X)}2^n$. Hence, if equation (4.4) holds, then by equation (4.5), equation (4.3) holds for some $B, C > 0$. □

Proposition 4.25. *Presume the conditions of the preceding theorem (Proposition 4.23). For any $R > 0$, there exist constants $B, C > 0$ only dependent on X and R such that for all $\alpha > 0$ and all $t > 0$,*

$$\|e^{\alpha d(\cdot,v)}|D_Z\Delta^k h(\cdot,v,t)|\|_{L^2} \leq \frac{C}{t^{k+\frac{1}{2}}\min\left\{t^{\frac{n}{2}}, R\right\}}e^{B\alpha^2 t}.$$

Proof. We assume without loss of generality that $R = 1$ and that $0 < t < 1$. We begin by noting that

$$\|e^{\alpha d(\cdot,v)}|D_Z\Delta^k h(\cdot,v,t)|\|_{L^2}^2 = \int_X e^{2\alpha d(z,v)}|D_Z\Delta^k h(z,v,t)|^2\,dX(z)$$
$$= \sum_{1\leq i\leq n}\int_X e^{2\alpha d(z,v)}|Z_i\Delta^k h(z,v,t)|^2\,dX(z)$$

by an application of the product rule for derivatives,

$$= -\sum_{1\leq i\leq n}\int_X \left(Z_i e^{2\alpha d(z,v)}\right)\left(\Delta^k h(z,v,t)\right)\left(Z_i\Delta^k h(z,v,t)\right)$$
$$+\left(e^{2\alpha d(z,v)}\right)\left(\Delta^k h(z,v,t)\right)\left((Z_i)^2\Delta^k h(z,v,t)\right)\,dX(z)$$
$$\leq 2\alpha\int_X e^{2\alpha d(z,v)}\left(\sum_{1\leq i\leq n}|Z_i d(z,v)||Z_i\Delta^k h(z,v,t)|\right)\left(\Delta^k h(z,v,t)\right)\,dX(z)$$
$$+\int_X e^{2\alpha d(z,v)}|\Delta^k h(z,v,t)||\Delta^{k+1}h(z,v,t)|\,dX(z)$$
$$= 2\alpha A_1 + A_2. \tag{4.6}$$

We obtain estimates on A_1 and A_2. We note that by [St2,PSC], we have for any $v_0 \in X$ that $e(d(z,v_0)) \leq 1$, where $e(d(z,v_0))$ is the energy density of $d(\cdot,v_0)$ evaluated at the point z. Hence, given the definition of Z, $\sum_{1\leq i\leq n} |Z_i d(z,v)|^2 \leq 1$. We have

$$\begin{aligned}
A_1 &= \int_X e^{2\alpha d(z,v)} \left(\sum_{1\leq i\leq n} |Z_i d(z,v)||Z_i \Delta^k h(z,v,t)| \right) \left(\Delta^k h(z,v,t)\right) \, dX(z) \\
&\leq \int_X e^{2\alpha d(z,v)} \left(\sum_{1\leq i\leq n} |Z_i \Delta^k h(z,v,t)|^2 \right)^{\frac{1}{2}} \left(\Delta^k h(z,v,t)\right) \, dX(z) \\
&\leq \left(\int_X \sum_{1\leq i\leq n} |Z_i \Delta^k h(z,v,t)|^2 \, dX(z) \right)^{\frac{1}{2}} \left(\int_X e^{4\alpha d(z,v)} \left(\Delta^k h(z,v,t)\right)^2 \, dX(z) \right)^{\frac{1}{2}} \\
&= \|D_Z \Delta^k h(\cdot,v,t)\|_{L^2} \|e^{2\alpha d(\cdot,v)} \Delta^k h(\cdot,v,t)\|_{L^2}.
\end{aligned}$$

For A_2, we see immediately that

$$A_2 \leq \|e^{\alpha d(\cdot,v)} |\Delta^k h(\cdot,v,t)|\|_{L^2} \|e^{\alpha d(\cdot,v)} |\Delta^{k+1} h(\cdot,v,t)|\|_{L^2}.$$

We can apply and Proposition 4.22 and Corollary 4.15 to equation (4.6) to obtain

$$\begin{aligned}
\|e^{\alpha d(\cdot,v)} \left|D_Z \Delta^k h(\cdot,v,t)\right| \|_{L^2}^2 \leq\ & 2\alpha \|D_Z \Delta^k h(\cdot,v,t)\|_{L^2} \|e^{2\alpha d(\cdot,v)} \Delta^k h(\cdot,v,t)\|_{L^2} \\
&+ \|e^{\alpha d(\cdot,v)} \Delta^k h(\cdot,v,t)\|_{L^2} \|e^{\alpha d(\cdot,v)} \Delta^{k+1} h(\cdot,v,t)\|_{L^2} \\
&\leq \frac{C\left(1+2\alpha t^{\frac{1}{2}}\right)}{t^{2k+1+n}} e^{B\alpha^2 t} \\
&\leq \frac{C'}{t^{2k+1+n}} e^{B'\alpha^2 t}.
\end{aligned}$$

We justify the last step by noting that the term $\left(1+2\alpha t^{\frac{1}{2}}\right)$ can be absorbed into B' by altering C. By Lemma 4.24, this is sufficient to prove Proposition 4.23. □

There is one consequence of these estimates that will be useful later that we state and prove here.

Proposition 4.26. *Let the top dimension of X be n and let k be a constant such that $2k \in \mathbb{N}$. Let $v \in X$ be fixed and let $R, B > 0$ be fixed. Then there exists a constant $C > 0$ dependent only on X, B and R such that for any $0 \leq t < R$,*

$$\int_X t^{-k} \exp\left(-\frac{d(z,v)^2}{Bt}\right) dX(z) d\tau \leq C t^{\frac{n}{2}-k}.$$

Proof. We see that for a fixed $v \in X$ it is sufficient to show that this result holds in a neighborhood of v, as $t^{-k} \exp\left(-\frac{d(z,v)^2}{Bt}\right)$ is clearly integrable and goes to zero exponentially for any set outside of a neighborhood of v. Without loss of generality, we can assume that X is simplex-wise flat. By the assumption of flatness on the simplexes and the geometry of the polyhedron, we see that it is sufficient to show the following holds,

$$\int_{[0,1]^n} t^{-k} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{Bt}\right) dx_1 \cdots dx_n \leq C t^{\frac{n}{2}-k},$$

where $[0,1]^n \subset \mathbb{R}^n$ is the unit n-dimensional cube. Iterating the integrals we have

$$\int_{[0,1]^n} t^{-k} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{Bt}\right) dx_1 \cdots dx_n$$

$$\begin{aligned}
&= \int_{[0,1]^{n-1}} \left(\int_{[0,1]} t^{-k} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{Bt} \right) dx_1 \right) dx_2 \cdots dx_n \\
&= \int_{[0,1]^{n-1}} \sqrt{\frac{\pi}{4B}} \text{erf}\left((Bt)^{-\frac{1}{2}} \right) t^{-k+\frac{1}{2}} \exp\left(-\frac{x_2^2 + \cdots + x_n^2}{Bt} \right) dx_2 \cdots dx_n \\
&\vdots \\
&= \left(\frac{\pi}{4B} \right)^{\frac{n}{2}} \text{erf}\left((Bt)^{-\frac{1}{2}} \right)^n \frac{C}{\sqrt{t}} \\
&\leq \left(\frac{\pi}{4B} \right)^{\frac{n}{2}} t^{\frac{n}{2}-k} \\
&= Ct^{\frac{n}{2}-k},
\end{aligned}$$

where $\text{erf}\colon \mathbb{R} \to \mathbb{R}^+$ is the error function defined by

$$\text{erf}(a) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{a} e^{-x^2}\, dx.$$

Naturally, erf is monotonically increasing, $\text{erf}(a) \leq 1$, and

$$\lim_{a \to \infty} \text{erf}(a) = 1.$$

□

4.5 Solutions to Non-Homogeneous Parabolic-Type Equations

From the existence of the heat kernel, which provides us a constructive way of finding solutions to homogenous parabolic type equations, we are able to derive existence theorems for solutions to non-homogeneous equations. We recall the following from Definition 3.3.

Definition 4.27. Let $I = (a,b) \subset \mathbb{R}$, and $f \in C\left((a,b) \mapsto L^2(X)\right)$.

i. A function u is a *weak solution* of the non-homogeneous parabolic equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = f,$$

on $I \times X$, if and only if $u \in \mathcal{F}(I \times X)$ and u satisfies

$$\int_I E(u,\phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt = \int_I \langle f, \phi \rangle_{L^2(X)}\, dt,$$

for all $\phi \in \mathcal{F}(I \times X)$.

ii. Additionally, let $g \in L^2(X)$. u is a *weak solution to the initial value problem*

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) u(z,t) &= f(z,t), \quad \text{for } (z,t) \in X \times (a,b) \\ u(\cdot, a) &= g, \qquad\quad \text{on } X \end{aligned}\right\}$$

if and only if u is a solution as above, and $\lim_{t \to a} u = g$ in $L^2(X)$.

Proposition 4.28. *Let X be as in the assumptions of this section, and $I = (a,b)$. Let $f \in C\left((a,b) \mapsto L^2(X)\right)$ be essentially bounded on $X \times (a,b)$ and let f be such that $\lim_{t\to a} f(z,t) = g(z)$, $g \in L^2(X)$. Then there exists a unique weak solution to the initial value problem*

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) u(z,t) &= f(z,t), \quad \textit{for } (z,t) \in X \times (a,b) \\ u(\cdot, a) &= g, \qquad\quad \textit{on } X. \end{aligned}\right\} \tag{4.7}$$

There exist $\alpha, \beta > 0$ such that $u \in C^{1,0}(X \times (a,b)) \cap C^{1+\alpha, 1+\beta}_{loc}(X \setminus X^{(n-2)} \times (a,b))$, *and first-order spacial derivatives of u are in $C^{\alpha,\alpha/2}_{loc}(X \setminus X^{(n-2)} \times (a,b))$. Also, $\|D^2_z u\|, \|\frac{\partial}{\partial t}u\| \in L^\infty_{loc}(X \setminus X^{(n-2)} \times (a,b))$, where D^2_z denotes any second order spacial derivative. The solution is also balanced for all $t > 0$, is smooth on the open set $X \setminus X^{(n-1)}$ and satisfies pointwise Equation (4.7) on $X \setminus X^{(n-1)}$.*

Additionally, if for any open $A \subset X$ such that $d(A, X^{(n-2)}) > 0$, there exists $k \in \mathbb{N}$ such that $\left(\frac{\partial}{\partial t}\right)^m u|_A$ is Hölder continuous for all $0 \leq m \leq k$ and $\left(\frac{\partial}{\partial t}\right)^{m-1} f|_A(\cdot,t) \in C^{k-m+\alpha}(A)$ for each $1 \leq m \leq k$, then $\left(\frac{\partial}{\partial t}\right)^m u|_A(\cdot,t) \in C^{k+1-m+\alpha}(A)$ for each $0 \leq m \leq k$.

Proof. Assume without loss of generality that $(a,b) = (0,T)$. We can actually construct a solution with these properties as follows. For $t \in (0,T)$, define

$$u(z,t) := \int_0^t \int_X h(z,v,t-\tau) f(v,\tau)\, d\mu(v)\, d\tau + \int_X h(z,v,t) g(v)\, d\mu(v). \tag{4.8}$$

We can compute that u solves our problem pointwise, is sufficiently regular, and is balanced. Hence it is certainly a weak solution. We shall draw heavily on the properties of the heat kernel. We require one auxiliary result before diving in. Specifically, we require that for each $z \in X$

$$\int_X h(z,v,t-\tau) f(v,\tau)\, d\mu(v)$$

be continuous with respect to $\tau \in (0,T)$. The condition that $f \in C((0,T) \mapsto L^2(X))$ will be sufficient to show this. We show that for any $\tau \in (0,T)$

$$\lim_{\delta \to 0} \int_X h(z,v,t-\tau-\delta) f(v,\tau+\delta)\, d\mu(v) = \int_X h(z,v,t-\tau) f(v,\tau)\, d\mu(v).$$

Note that using the heat operator

$$H_{t-\tau}f(z,\tau) = \int_X h(z,v,t-\tau)f(v,\tau)\,d\mu(v),$$

which is handier notation. We have

$$\begin{aligned}
&|H_{t-\tau-\delta}f(z,\tau+\delta) - H_{t-\tau}f(z,\tau)| \\
&\qquad = \left|\int_X h(z,v,t-\tau-\delta)f(v,\tau+\delta) - h(z,v,t-\tau)f(v,\tau)\,dv\right| \\
&\qquad = \left|\int_X h(z,v,t-\tau-\delta)f(v,\tau+\delta) - h(z,v,t-\tau-\delta)f(v,\tau)+\right. \\
&\qquad\qquad \left. h(z,v,t-\tau-\delta)f(v,\tau) - h(z,v,t-\tau)f(v,\tau)\,dv\right| \\
&\qquad \le \left|\int_X h(z,v,t-\tau-\delta)\left[f(v,\tau+\delta) - f(v,\tau)\right]\,dv\right| \\
&\qquad\qquad + \left|\int_X \left[h(z,v,t-\tau-\delta) - h(z,v,t-\tau)\right]f(v,\tau)\,dv\right| \\
&\qquad = h(z,z,t-\tau-\delta)\|f(\cdot,\tau+\delta) - f(\cdot,\tau)\|_{L^2(X)} \qquad . \\
&\qquad\qquad + \left|\int_X \left[h(z,v,t-\tau-\delta) - h(z,v,t-\tau)\right]f(v,\tau)\,dv\right|
\end{aligned}$$

Letting δ go to zero, we have the first expression going to zero by $f \in C(I \mapsto L^2(X))$, and the second expression goes to zero by the Hölder continuity of h in time given by Proposition 4.10. Hence, for any fixed z, $\int_X h(z,v,t-\tau)f(v,\tau)\,d\mu(v)$ is continuous with respect to τ on $(0,T)$. We note that

$$\lim_{t\to 0} u(\cdot,t) = g, \quad \text{in } L^2(X)$$

as the first term of (4.8) goes to zero uniformly and, by the property of the heat operator, the second must go to $g(z)$ in $L^2(X)$.

Now on $X \times (0,T)$ we must verify that almost everywhere

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(z,t) = f(z,t).$$

As $\left(\frac{\partial}{\partial t} - \Delta\right) \int_X h(z,v,t)g(v)\,dv = 0$, we must verify that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \int_0^t \int_X h(z,v,t-\tau) f(v,\tau)\,d\mu(v)\,d\tau \overset{ae}{=} f(z,t).$$

We focus on computing the time derivative, which requires some care.

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t \int_X h(z,v,t-\tau) f(v,\tau)\,d\mu(v) d\tau \\
&= \lim_{\delta\to 0} \frac{1}{\delta}\Bigg(\int_0^{t+\delta}\int_X h(z,v,t-\tau+\delta) f(v,\tau)\,d\mu(v) d\tau \\
&\qquad\qquad - \int_0^t \int_X h(z,v,t-\tau) f(v,\tau)\,d\mu(v) d\tau\Bigg) \\
&= \lim_{\delta\to 0}\left(\frac{1}{\delta}\int_t^{t+\delta}\int_X h(z,v,t-\tau+\delta) f(v,\tau)\,d\mu(v) d\tau\right) \\
&\qquad + \int_0^t \int_X \lim_{\delta\to 0}\frac{1}{\delta}\big(h(z,v,t-\tau+\delta) - h(z,v,t-\tau)\big) f(v,\tau)\,d\mu(v) d\tau
\end{aligned}$$

(by the integral mean value theorem and the continuity with respect to τ of $\int_X h(z,v,t-\tau+\delta) f(v,\tau)\,d\mu(v)$)

$$\begin{aligned}
&= \lim_{\epsilon\to 0}\left(\int_X h(z,v,\epsilon) f(v,t)\,d\mu(v)\right) + \int_0^t\int_X \frac{\partial}{\partial t} h(z,v,t-\tau) f(v,\tau)\,d\mu(v) d\tau \\
&\overset{ae}{=} f(z,t) + \int_0^t\int_X \frac{\partial}{\partial t} h(z,v,t-\tau) f(v,\tau)\,d\mu(v) d\tau.
\end{aligned}$$

The last step follows from the fact that $\int_X h(\cdot,v,\epsilon) f(v,t)\,d\mu(v)$ goes to $f(\cdot,t)$ as $\epsilon \to 0$

in $L^2(X)$. Hence, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) \int_a^t \int_X h(z,v,t-\tau) f(v,\tau)\, d\mu(v)\, d\tau \\ &\overset{ae}{=} f(z,t) + \int_0^t \left(\frac{\partial}{\partial t} - \Delta\right) \int_X h(z,v,t-\tau) f(v,\tau)\, d\mu(v)\, d\tau \\ &= f(z,t).\end{aligned}$$

With some regularity on f, this can be shown to be a pointwise equality on certain open sets. Indeed, we can show that $u \in C^0(X \times I)$ following an argument identical to Theorem 2 of [F, Chap. 1, Sect 3.]. We may even show that first order derivatives exist and are bounded on $X \setminus X^{(n-1)}$. By the smoothness of $h(z,v,t)$ given in Proposition 4.11 and by the Gaussian estimates of Theorem 4.18, we have for $z \in X \setminus X^{(n-1)}$ and $t > 0$,

$$\begin{aligned}|\nabla u(z,t)| \leq\ & \int_0^t \int_X |\nabla_z h(z,v,t-\tau)|\, |f(v,\tau)|\ dv\, d\tau \\ & \qquad + \int_X |\nabla_z h(z,v,t)|\, |g(v)|\ dv \\ \leq\ & \left(\int_0^t \int_X \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} e^{-\frac{d(z,v)^2}{Bt}}\ dv\, d\tau\right) \|f\|_{L^\infty(X\times I)} \\ & \qquad + \int_X \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} e^{-\frac{d(z,v)^2}{Bt}}\ |g(v)|\ dv \\ \leq\ & C'\sqrt{t}\|f\|_{L^\infty(X\times I)} + \frac{C''}{t^\alpha}\|g\|_{L^2(X)},\end{aligned}$$

where C, C', C'' are dependent on T and X only and $\alpha \in (0,1)$. Since f is essentially bounded on $X \times I$ and g is in $L^2(X)$, we see the right side is bounded and not dependent on the choice of $z \in X$. Hence, $u \in C^{1,0}(X \times (0,T))$. Additionally, we can see from the construction of u that it must be balanced by the balancing condition on $h(z, v, t)$. Hence, if we pick $\phi \in \mathcal{F}(I \times X)$, u satisfies weakly

$$\int_I E(u, \phi)\, dt + \int_I \left\langle \tfrac{\partial}{\partial t} u, \phi \right\rangle_{L^2(X)} dt = \int_I \langle f, \phi \rangle_{L^2(X)}\, dt,$$

and is hence a weak solution to the initial value problem in the statement of this theorem. To show that $u \in C^{1+\alpha, 1+\beta}_{\text{loc}}(X \setminus X^{(n-2)} \times (a, b))$, we use a technique from [DM3]. Let $p \in X^{(n-1)} \setminus X^{(n-2)}$. Let $\{s_j\}_{j=1}^J$ denote all of the n-simplexes adjacent to p meeting on an $(n-1)$-face F. Define $u_j := u|_{s_j}$. Also let $R > 0$ be such that $d(B(p, R), X^{(n-2)}) > 0$ and $B(p, R) \subset \cup_{j=1}^J \overline{s_j}$, and pick edge coordinates centered at p so that for each u_j, $(x_1, \ldots, x_{n-1}, 0)$ denotes points on F and $(0, \ldots, 0)$ denotes p. For each $1 \le k \le J$, we construct $\overline{u}_k \colon B(0, R) \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ as

$$\overline{u}_k\big((\bar{x}, x_n), t\big) := \begin{cases} u_k\big((\bar{x}, x_n), t\big), & x_n \ge 0 \\ -u_k\big((\bar{x}, -x_n), t\big) + \dfrac{2}{J} \displaystyle\sum_{j=1}^J u_j\big((\bar{x}, -x_n), t\big), & x_n < 0 \end{cases} \tag{4.9}$$

where $\bar{x} = (x_1, \ldots, x_{n-1})$. Also, let $f_k := f|_{s_k}$ and similarly define for each k, $\overline{f}_k \colon B(0, R) \to \mathbb{R}$ as

$$\overline{f}_k\big((\bar{x}, x_n), t\big) := \begin{cases} f_k\big((\bar{x}, x_n), t\big), & x_n \ge 0 \\ -f_k\big((\bar{x}, -x_n), t\big) + \dfrac{2}{J} \displaystyle\sum_{j=1}^J f_j\big((\bar{x}, -x_n), t\big), & x_n < 0 \end{cases} \tag{4.10}$$

By our solution u being balanced and in $C^{1,0}(X \times (a, b))$, we can see that, for each k, $\overline{u}_k$ is a weak solution to a set of equations (indexed by k) given as

$$\left(\Delta - \tfrac{\partial}{\partial t}\right) \overline{u}_k = \overline{f}_k$$

defined on $B(0,R)\times(a,b)\,(\subset \mathbb{R}^n\times\mathbb{R})$. We can use standard results in parabolic differential equations (see in particular [LSU, Theorem 12.1, Chapter III]), to see that for any $0<R'<R$, each $\overline{u}_k|_{B(0,R')}$ is in $C^{1+\alpha,\beta}(B(0,R'))$ and $\frac{\partial \overline{u}_k}{\partial x^i}\in C^{\alpha,\alpha/2}(B(0,R'))$ for any $1\le i\le n$ by the virtue of f being essentially bounded (we can put weaker conditions on f to achieve weaker results, but this will suffice for the settings considered elsewhere in this paper). Additionally, $\|D_z^2\overline{u}_k\|,\|\frac{\partial}{\partial t}\overline{u}_k\|\in L^\infty(B(0,R')\times(a,b))$, where D_z^2 denotes any second-order spacial derivative. To derive the Hölder continuity of $\frac{\partial}{\partial t}\overline{u}_k$, we appeal to a recent result of [TP] which shows the existence of a solution of Equation (4.10) that does possess this regularity. However, it does not suggest uniqueness of such a solution. We find our solution is unique by our argument later in this proof and apply [TP, Theorem 2.1]. This concludes our initial regularity results.

For higher regularity, where it is presumed *a priori* that for some $k\in\mathbb{N}$, $\left(\frac{\partial}{\partial t}\right)^m u$ is Hölder continuous for all $1\le m\le k$, we can appeal to Proposition 3.15 directly.

To show uniqueness, suppose there is a second distinct $\bar{u}(z,t)$ that is a weak solution of (4.7). We create a new function,

$$U(z,t):=u(z,t)-\bar{u}(z,t).$$

We note that $U(z,0)=0$ everywhere (in particular $\|U(\cdot,0)\|_{L^2(X)}=0$) and is a weak solution to the homogeneous equation

$$\left(\frac{\partial}{\partial t}-\Delta\right)U=0.$$

From Proposition 4.4, U is the unique solution to the weak initial value problem with initial data that is identically zero. By the contraction property for the heat semigroup (see Proposition 4.4), for all $t > 0$, $\|U(\cdot, t)\|_{L^2(X)} = 0$. Also by regularity results for the homogeneous problem in Proposition 4.11, and regularity on u noted above, we have that $\bar{u}$ must be continuous, and thus U must be identically zero.

To show convergence in C^0 to the initial map, we may use Theorem 4.18 and Proposition 4.26 to show that the first term of Equation (5.9) goes to 0 uniformly. We can then use [PSC, Theorem 3.10] to show that, as F_0 is continuous and defined a compact domain, $\int_X h(z,v,t)F^\gamma(v)\,d\mu(v)$ goes to F^γ uniformly. □

5 Heat Flows between Polyhedra and Manifolds

5.1 Harmonic Maps and The Harmonic Map Heat Flow Problem

The results of [CR] and [C] include defining flows between compact singular domains (one a manifold with conical singularities and the other an orbifold) and compact, nonpositively curved manifolds. The approach in both cases is a modification of the results of [ES]. In particular, they take care to give energy bounds and heat kernel bounds in these singular cases and then show that the methods of [ES] apply and give the existence of heat flows that are smooth away from the singularities and that converge to harmonic maps with good regularity. In the process, they also show existence of unique harmonic representatives in each homotopy class of smooth maps between the spaces. The primary difficulties that these papers resolve are defining appropriate heat kernels and showing good convergence and regularity properties. However, these methods do not immediately abstract to the case of the domain being an admissible smooth Riemannian polyhedron. The results of [St2] and [PSC] may be modified to allow to the methods of [ES] to apply in the case of maps between compact admissible Riemannian polyhedron and compact Riemannian manifolds with nonpositive sectional curvature. We state some results and outline the argument below.

Assumptions. Unless otherwise specified, we shall assume in this section that X is an admissible smooth Riemannian polyhedron that satisfies the conditions of Proposition 2.20 with Dirichlet form $E(\cdot,\cdot)$ and Laplacian Δ as in Section 2.2, and energy $E(\cdot)$ as in Section 2.4. We also assume that N is a compact smooth Riemannian manifold with nonpositive sectional curvature.

Following [EF], we define the following.

Definition 5.1. Let $f \in W^{1,2}(X, N)$ and let E denote the energy from the assumptions of this section.

i. f is *locally E-minimizing* if, for every open cover $\{U_\alpha\}$ of X such that each U_α is compactly contained in X, $E(f|_{U_\alpha}) \leq E(g|_{U_\alpha})$ for every $g \in W^{1,2}(X, N)$ where $g \overset{ae}{=} f$ on $X \setminus U_\alpha$.

ii. f is *harmonic* if it is continuous and *bi-locally E-minimizing*: for every open cover $\{U_\alpha\}$ of X such that each U_α is compactly contained in X, there is an open set $V_\alpha \subset N$ such that $E(f|_{U_\alpha}) \leq E(g|_{U_\alpha})$ for every continuous map $g \in W^{1,2}(X, N)$ where $g = f$ on $X \setminus U_\alpha$ and $g(U_\alpha) \subset V_\alpha$.

Definition 5.2. $f \in W^{1,2}(X, N)$ is *weakly harmonic* if in any coordinate chart $V \subset$

N, and any open set $U \subset f^{-1}(V)$, it satisfies

$$\int_U \left(\langle \nabla f^\gamma, \nabla \phi \rangle - \Gamma^\gamma_{\alpha\beta}(f) \left\langle \nabla f^\alpha, \nabla f^\beta \right\rangle \phi \right) \, d\mu = 0$$

for all $\phi \in C^\infty(X)$ such that $\operatorname{supp}(\phi)$ is compactly contained in U, and for all γ, $1 \leq \gamma \leq q = \dim(N)$. Here, $\Gamma^\gamma_{\alpha\beta}$ is the Christoff symbol of N in the chart V.

By [EF, Theorem 12.1], we have the following.

Proposition 5.3. *For a continuous map $f \in W^{1,2}(X, N)$, f is harmonic if and only if it is weakly harmonic.*

We note that the definition of a weak harmonic map makes no assumption on the continuity of the map.

For harmonic maps between polyhedra and manifolds, we have the following result by [DM3].

Proposition 5.4 (See [DM3])**.** *Let X be a flat compact Riemannian polyhedron of dimension n ($n \geq 2$), and let N be a complete smooth Riemannian manifold. Let $f \colon X \to N$ be harmonic. Then, $f \in C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}, N)$ and is balanced. Additionally, if $n = 2$, then $f \in C^\infty_{\text{loc}}(X \setminus X^{(n-2)}, N)$.*

Remark 5.5. We note that the result in [DM3] is actually more general than stated here, as they consider harmonic maps in the context of admissible weights, which we do not consider here.

We naturally are interested in flows between polyhedra and manifolds, and so we define the following.

Definition 5.6. Let X be compact. For an interval $(a,b) \subset \mathbb{R}$, f is a *strong solution to the heat flow on* $X \times (a,b)$ if f is continuous on $X \times (a,b)$, $f \in C^{2,1}_{\mathrm{loc}}(X \setminus X^{(n-1)} \times (a,b), N)$, f is balanced for all $t \in (a,b)$, and

$$\frac{\partial}{\partial t} f(z,t) = \tau(f(z,t)), \quad (z,t) \in X \setminus X^{(n-1)} \times (a,b),$$

where $\tau(f) = \mathrm{Trace}_g \nabla df$ is the torsion operator, with simplex-wise metric tensor g.

Let $f_0 \in C^1(X,N)$. Then f is a *strong solution to the heat flow with initial value* f_0 *on* $X \times [a,b)$ if f is continuous on $X \times [a,b)$, $f \in C^{2,1}_{\mathrm{loc}}(X \setminus X^{(n-1)} \times (a,b), N)$, f is balanced for all $t \in (a,b)$, and

$$\left.\begin{aligned} \frac{\partial}{\partial t} f(z,t) = \tau(f(z,t)) \quad & \text{for } (z,t) \in X \setminus X^{(n-1)} \times (a,b), \\ \lim_{t \to a} f_t = f_0 \quad & \text{in } C^0(X,N). \end{aligned}\right\}$$

Naturally, to show existence, one typically begins with a weak solution but, as we shall see later, it is easier to begin with a constructive solution that requires embedding N into Euclidean space. Embedding the target, however, will have many benefits and solving an embedded problem is equivalent to solving the problem above, as we shall see.

5.2 The Embedded Problem

To construct a solution, it is useful to consider isometrically embedding the target into a higher dimensional Euclidean space. Then one must also verify that a solution to a flow problem into an embedded target is equivalent to finding one for a non-embedded target. In [N] and [ES], they provide methods for doing so when the domain is a manifold. We adapt their methods to apply here.

We recall from the case where the domain is a compact smooth Riemannian manifold the following:

Definition 5.7. Let (M, g) and (N, h) be a compact smooth Riemannian manifolds such that N has nonpositive sectional curvature. Let $f_0 \colon M \to N$ be a C^2 map. $f \colon M \times [0, T) \to N$ is a *solution to the heat flow problem with initial value* f_0 *on* $M \times [0, T)$ if $f \in C^0(M \times [0, T), N) \cap C^{2,1}(M \times (0, T), N)$ and

$$\left.\begin{aligned} \frac{\partial f}{\partial t} &= \Delta f + \Gamma(df, df) \quad \text{on } M \times (0, T) \\ \lim_{t \to 0} f &= f_0 \quad \text{in } C^0 \end{aligned}\right\} \tag{5.1}$$

where Δ is the Laplace-Beltrami operator on M and $\Gamma(df, df)$ is defined locally as follows. For a fixed coordinate in the target, y^γ, about a point $f(p) \in N$,

$$\Gamma^\gamma(df, df)(p) := \sum_{\alpha,\beta,i,j} \Gamma^\gamma_{\alpha\beta}(f(p)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} g^{ij}(p),$$

where $\Gamma^\gamma_{\alpha\beta}$ denotes the Christoffel symbol of the Levi-Civita connection on (N, h), and x^i denotes coordinates about $p \in M$.

We must reformulate Equation (5.1) so that it satisfies a differential equation when the target is a submanifold of $\mathbb{R}^q$. We follow the construction of [N].

Let $q \in \mathbb{N}$ be large enough so that there exists $\iota \colon N \to \mathbb{R}^q$ as a smooth isometric embedding. Let $\tilde{N} \subset \mathbb{R}^q$ be an open, tubular neighborhood of $\iota(N)$ so that the nearest-point projection map $\pi \colon \tilde{N} \to \iota(N)$ is well defined. For $y \in N$ with local coordinates $\{y^i\}_{i=1}^{n=\dim(N)}$ we may denote ι locally as

$$\iota : (y^1, \ldots, y^n) \mapsto (\iota^1(y^1, \ldots, y^n), \ldots, \iota^q(y^1, \ldots, y^n))$$

and for $z \in \tilde{N}$ with coordinates inherited from $\mathbb{R}^q$, we denote π as

$$\pi : (z^1, \ldots, z^q) \mapsto (\pi^1(z^1, \ldots, z^q), \ldots, \pi^q(z^1, \ldots, z^q)).$$

By [N], we have the following.

Proposition 5.8. *Let (M, g) and (N, h) be compact smooth Riemannian manifolds and let N have nonpositive sectional curvature. Also, let $\pi, \tilde{N}$ and ι be as above, and let $f_0 \colon M \to N$ be a C^2 map, and let $F_0 := \iota \circ f_0$. If $f \colon M \times [0, T) \to N$ satisfies*

$$\left.\begin{aligned} \frac{\partial f}{\partial t} &= \Delta f + \Gamma(df, df) \quad \text{on } M \times (0, T) \\ \lim_{t \to 0} f &= f_0 \quad \text{in } C^0 \end{aligned}\right\} \tag{5.2}$$

where $\Gamma^\gamma(df, df)(p) = \sum_{\alpha,\beta,i,j} \Gamma^\gamma_{\alpha\beta}(f(p)) \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} g^{ij}(p)$, then $F := \iota \circ f$ satisfies

$$\left.\begin{aligned} \frac{\partial F}{\partial t} &= \Delta F + A(dF, dF) \quad \text{on } M \times (0, T) \\ \lim_{t \to 0} F &= F_0 \quad \text{in } C^0 \end{aligned}\right\} \tag{5.3}$$

where for a fixed coordinate γ, $1 \leq \gamma \leq q$ and in a neighborhood of $p \in M$ with coordinates $\{x^i\}$,

$$A^\gamma(dF, dF)(p) = \sum_{\alpha,\beta,i,j} A^\gamma_{\alpha\beta}(f(p)) \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} g^{ij}(p),$$

and $A^{\gamma}_{\alpha\beta} := \frac{\partial^2 \pi^\gamma}{\partial z^\alpha \partial z^\beta}$ *with* $\{z^i\}$ *denoting the standard coordinates of* $\mathbb{R}^q$*.*

The converse is also true, where, given $F \in C^2(M \times [0,T), \tilde{N})$ *and* $F_0 \in C^2(M, \tilde{N})$*, we define* $f := \iota^{-1} \circ F$ *and* $f_0 := \iota^{-1} \circ F_0$*.*

In light of this, we define the following.

Definition 5.9. Let X be compact and let $\iota \colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding.

i. Let $(a,b) \subset \mathbb{R}$. F is a *weak embedded solution to the heat flow on* $X \times (a,b)$ if it satisfies for each coordinate $1 \leq \gamma \leq q$,

$$\int_{(a,b)} \int_X \Big(\langle \tfrac{\partial}{\partial t} F^\gamma, \phi \rangle + \langle \mathrm{d}F^\gamma, \mathrm{d}\phi \rangle - \langle A^\gamma(F)(\mathrm{d}F, \mathrm{d}F), \phi \rangle \Big) \, d\mu \, dt = 0, \tag{5.4}$$

for every $\phi \in C^\infty_c(X, \mathbb{R}^q)$, where $A^\gamma(\cdot,\cdot)$ is defined as in Proposition 5.8, given in local coordinates by

$$A^\gamma(F)(dF, dF) = A^\gamma_{\alpha\beta}(F) \frac{\partial F^\alpha}{\partial z^i} \frac{\partial F^\beta}{\partial z^j} g^{ij},$$

and $1 \leq \gamma, \alpha, \beta \leq q$ denote coordinates in $\mathbb{R}^q$.

ii. Let $F_0 \in C^1(X, \mathbb{R}^q)$ be such that $\mathrm{Image}(F_0) \subset \iota(N)$. F is a *weak embedded solution to the heat flow with initial value* F_0 *on* $X \times [a,b)$ if it satisfies for each coordinate $1 \leq \gamma \leq q$,

$$\left.\begin{aligned} \int_{(a,b)} \int_X \Big(\langle \tfrac{\partial}{\partial t} F^\gamma, \phi \rangle + \langle \mathrm{d}F^\gamma, \mathrm{d}\phi \rangle - \langle A^\gamma(F)(\mathrm{d}F, \mathrm{d}F), \phi \rangle \Big) \, d\mu \, dt = 0, \\ \lim_{t \to a} F = F_0 \quad \text{in } L^2 \end{aligned}\right\} \tag{5.5}$$

for all $\phi \in C_c^\infty(X, \mathbb{R}^q)$, where $A^\gamma(\cdot,\cdot)$ is defined as in Proposition 5.8.

iii. F is a *strong embedded solution to the heat flow* on $X \times (a,b)$ if F is continuous on $X \times (a,b)$, $F \in C^{2,1}_{\text{loc}}(X \setminus X^{(n-1)} \times (a,b), \mathbb{R}^q)$, F is balanced for all $t \in (a,b)$, and it satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) F^\gamma = A^\gamma(F)(\mathrm{d}F, \mathrm{d}F) \quad \text{on } (X \setminus X^{(n-1)}) \times (a,b),$$

for all γ, $1 \le \gamma \le q$, where Δ_g denotes the Laplace-Beltrami operator on $X \setminus X^{(n-1)}$ with respect to the simplex-wise smooth metric g.

iv. Let $F_0 \in C^1(X, \mathbb{R}^q)$ be balanced and such that $\text{Image}(F_0) \subset \iota(N)$. F is a *strong embedded solution to the heat flow with initial value* F_0 on $X \times [a,b)$ if F is continuous on $X \times [a,b)$, $F \in C^{2,1}_{\text{loc}}(X \setminus X^{(n-1)} \times (a,b), \mathbb{R}^q)$, F is balanced for all $t \in (a,b)$, and it satisfies

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_g\right) F^\gamma = A^\gamma(F)(\mathrm{d}F, \mathrm{d}F) \quad &\text{on } (X \setminus X^{(n-1)}) \times (a,b), \\ \lim_{t \to a} F = F_0 \quad &\text{in } C^0 \end{aligned}\right\} \tag{5.6}$$

for all γ, $1 \le \gamma \le q$.

Remark 5.10. The balancing condition is necessary for a strong solution to the be a weak solution and for a sufficiently smooth weak solution to be a strong solution.

The above does not specifically address whether or not the image of an embedded strong solution stays in $\iota(N) \subset \mathbb{R}^q$ for positive time. It is possible that it "floats" off $\iota(N)$ in time even though at time zero it is, by assumption, contained in $\iota(N)$. As the solution is continuous, we can appeal to a result of [N], which indicates this does not happen.

Proposition 5.11. *Let X be compact and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in $C^1(X)$, and suppose F is a strong embedded solution to the heat flow on $[0,T)$ with initial value F_0. Then $F(X,t) \subset \iota(N)$ for all $0 < t < T$.*

Proof. Suppose that there exists a $t_0 \in (0,T)$ such that $F(X,t_0) \not\subset \iota(N)$. As the flow is continuous and, specifically, $F(\cdot,t)$ is continuous in space for fixed t, if $(F(X,t_0) \not\subset \iota(N)$, then there must exist an open set $A \subset X \setminus X^{(n-1)}$ where $d(A, X^{(n-1)}) > 0$ and $F(A,t_0) \cap \iota(N) = \emptyset$. As A is isometric to a smooth manifold, we may appeal to the proof of [N, Proposition 4.6], which proves this statement pointwise in a smooth manifold. □

Remark 5.12. We should note that the nonpositive sectional curvature of N is crucial to [N, Proposition 4.6]. We have not before indicated the necessity of the curvature assumptions on N, but it appears here.

Now that the embedded problem is known to keep the image of the solution as a subset of $\iota(N)$, we may also ask about the relationship between the embedded problem and the non-embedded problem. Ostensibly, given one, we should be able to retrieve the other. We have again the following as a consequence of [N, Proposition 4.6].

Proposition 5.13. *Let $f_0 \in C^1(X,N)$ and let f be balanced, and let $F_0 := \iota \circ f_0$. For $f\colon X \times [0,T) \to N$, let $F := \iota \circ f$. f is a strong (unembedded) solution to the heat* *flow on $X \times [0, T)$ with initial value f_0 if and only if F is strong embedded solution to the heat flow on $X \times [0, T)$ with initial value F_0.*

We also consider uniqueness. We can use our results in the linear, one-dimensional case to achieve a uniqueness result for strong embedded solutions.

Proposition 5.14. *Let $F_0 \in C^1(X, \mathbb{R}^q)$ be balanced and such that* $\mathrm{Image}(F_0) \subset \iota(N)$*, and let F be a strong embedded solution to the heat flow with initial value F_0 on $X \times [a, b)$. If F' is also a strong embedded solution with initial value F_0, then for each $t \in [a, b)$, $F(\cdot, t) = F'(\cdot, t)$ almost everywhere on X.*

Proof. We show uniqueness by considering each coordinate individually. Let $1 \leq \gamma \leq q$ be fixed. Then both F^γ and F'^γ are balanced for $t \in (a, b)$ and solve the differential equation

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_g\right) F^\gamma = A^\gamma(F)(\mathrm{d}F, \mathrm{d}F), \quad \text{on } (X \setminus X^{(n-1)}) \times (a, b) \\ \lim_{t \to a} F^\gamma = F_0^\gamma \qquad \text{in } C^0 \end{aligned}\right\} \tag{5.7}$$

Consider $G^\gamma := F^\gamma - F'^\gamma$. It must solve

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_g\right) G^\gamma = 0 \quad \text{on } (X \setminus X^{(n-1)}) \times (a, b), \\ \lim_{t \to a} G^\gamma = 0 \quad \text{in } C^0 \end{aligned}\right\} \tag{5.8}$$

Also, G must be balanced for $t \in (a, b)$. Hence, it must be a weak solution to the heat equation with initial value 0, and by the Markov property of Proposition 4.4, for each $t \in (a, b)$, $G = 0$ almost everywhere. □

We have not yet treated existence of solutions. To this end, we also have another proposition that will be our focus subsequently.

Proposition 5.15. *Let* $\iota\colon N \to \mathbb{R}^q$ *be an isometric embedding. Let* $F_0 \in C^1(X, \mathbb{R}^q)$ *be such that* $\mathrm{Image}(F_0) \subset \iota(N)$*. If there exists* W *such that it is continuous on* $X \times [0,T)$, $W \in C^{1,0}(X \times [0,T), \mathbb{R}^q)$*, and it satisfies in each coordinate* $1 \leq \gamma \leq q$*,*

$$W^\gamma(z,t) = \int_0^t \int_X h(z,v,t-\tau) G^\gamma(v,\tau)\, d\mu(v) d\tau + \int_X h(z,v,t) F_0(v)\, d\mu(v), \quad (5.9)$$

where, in local coordinates,

$$G^\gamma(v,\tau) := A^\gamma_{\alpha\beta}(W) \frac{\partial W^\alpha}{\partial x^i} \frac{\partial W^\beta}{\partial x^j} g^{ij},$$

then $W(z,t)$ *is a strong embedded solution with initial value* F_0 *on* $X \times [0,T)$ *and, in particular, solves equation* (5.8) *pointwise on* $X \setminus X^{(n-1)} \times (0,T)$*. Additionally,* $W \in C^{1+\alpha,1+\beta}_{\mathrm{loc}}(X \setminus X^{(n-2)} \times [0,T))$ *for some* $\alpha, \beta > 0$ *and satisfies all of the other conclusions of Proposition 4.28.*

Proof. We shall leave the higher regularity for later, but we shall show that if such a W satisfies equation (5.9) on $[0,T)$ for some $T > 0$, then it must be a strong embedded solution. We verify that if W satisfies (5.9) and $W \in C^{1,0}(X \times [0,T), \mathbb{R}^q)$, then W satisfies weakly

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) W^\gamma = A^\gamma(W)(\mathrm{d}W, \mathrm{d}W),$$

in each coordinate γ, $1 \leq \gamma \leq q$. Let γ be fixed. From the proof of Proposition 4.28

on page 80, we see that such a function W^γ satisfies almost everywhere

$$\frac{\partial}{\partial t} W^\gamma = G^\gamma(z,t) + \int_0^t \int_X \frac{\partial}{\partial t} h(z,v,t-\tau) G^\gamma(v,\tau)\, d\mu(v) d\tau.$$

Hence, we compute

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) W^\gamma(z,t) = \left(\frac{\partial}{\partial t} - \Delta_g\right) \int_0^t \int_X h(z,v,t-\tau) G^\gamma(v,\tau)\, d\mu(v) d\tau,$$

which follows from the fact that $\left(\frac{\partial}{\partial t} - \Delta_g\right) \int_X h(z,v,t) F_0(v)\, d\mu(v) = 0$, and we have almost everywhere,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_g\right) W^\gamma(z,t) &= G^\gamma(z,t) + \int_0^t \left(\frac{\partial}{\partial t} - \Delta_g\right) \int_X h(z,v,t-\tau) G^\gamma(v,\tau)\, d\mu(v) d\tau \\ &= G^\gamma(z,t).\end{aligned}$$

By the regularity assumptions on W, we know that $G^\gamma(\cdot,t) \in C(X)$, and we see that for each γ, W^γ is a weak solution to

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_g\right) W^\gamma &= G^\gamma \\ W^\gamma(\cdot,0) &= F_0^\gamma\end{aligned}\right\}$$

We note that by Theorem 4.18 and by Proposition 4.26, we can bring first order spacial derivatives inside both integrals of the first term of equation (5.9). Thus, we see that by the fact that W satisfies equation (5.9) and by the balancing of $h(z,v,t)$, W must be balanced. Hence, we can apply Proposition 4.28, which gives that $W \in C^{1+\alpha,1+\beta}_{\text{loc}}(X \setminus X^{(n-2)} \times [0,T))$ and that W is smooth in the open manifold

$X \setminus X^{(n-1)}$, which gives pointwise satisfaction of equation (5.8).

To show convergence in C^0 to the initial map, we may use Theorem 4.18 and Proposition 4.26 to show that the first term of Equation (5.9) goes to 0 uniformly. We can then use [PSC, Theorem 3.10] to show that, as F_0 is continuous and defined a compact domain, $\int_X h(z,v,t) F^\gamma(v)\, d\mu(v)$ goes to F^γ uniformly. □

Naturally, we wish to know if solutions exist on some interval $[0, T)$ and, if so, about the maximum T for which this holds. Also, if a solution holds for $T \to \infty$, we ask if W converges to a harmonic map. We break our approach into three parts.

i. (Short-time existence) we show for an initial value f_0, there exists a $T > 0$ such that a solution exists on $[0, T)$.

ii. (Long-time existence) we show that if a solution exists on $[0, T)$, then it must exist on $[0, T]$, which implies the solution exists on $[0, \infty)$.

iii. (Long-time convergence) we show that the solution converges in energy as $t \to \infty$ to a harmonic map.

5.3 Gradient-of-Energy Flow

It will be useful to show that the harmonic map heat flow defined here coincides with the heat flow of Mayer, the so-called Gradient-of-Energy flow. The advantage of Mayer's method here is the very general, long-time convergence results he obtains in [Ma]. It also gives results results on the behavior of energy over time.

Definition 5.16. Let $(\mathcal{M}, d)$ be a complete length space nonpositively curved in the sense of Alexandrov and let $F\colon \mathcal{M} \to \mathbb{R} \cup \infty$ be a lower semi-continuous, convex functional. We define the *norm of the gradient vector* at f_0 as

$$|\nabla_- F|(f_0) := \max\left\{ \limsup_{f \to f_0, f \in \mathcal{M}} \frac{F(f_0) - F(f)}{d(f_0, f)}, 0 \right\}.$$

$f_0 \in \mathcal{M}$ is called *stationary* if

$$|\nabla_- F|(f_0) = 0.$$

Proposition 5.17. *For a complete NPC space $(\mathcal{M}, d)$ and a lower semi-continuous, convex functional $F \colon \mathcal{M} \to \mathbb{R} \cup \infty$, there exists a map*

$$(\cdot)_t \colon \mathcal{M} \times \mathbb{R}_{\geq 0} \to \mathcal{M}$$

that has the following properties:

i. *For $f \in \mathcal{M}, \lim_{t \to 0} f_t = f$*

ii. *For $f \in \mathcal{M}$, $s, t > 0$, $((f)_s)_t = (f)_{s+t}$ (the semi-group property)*

iii. $\lim_{s \to 0} \dfrac{d_{L^2}(f_{t+s}, f_s)}{s} = |\nabla_- F|(f_t)$, *for all t*

iv. $\sup_{s > 0} \dfrac{d_{L^2}(f_{t+s}, f_s)}{s} = |\nabla_- F|(f_t)$, *for all t*

v. $-\frac{d}{dt} F(f_t) = |\nabla_- F|^2(f_t)$, *for almost all $t > 0$*

vi. $t \mapsto |\nabla_- F|(f_t)$ *is right continuous*

vii. $t \mapsto F(f_t)$ *is uniformly Lipschitz continuous on $[t_0, t_1]$ for all $0 < t_0 < t_1 < \infty$*

viii. $|\nabla_- F|(f_t)$ *is monotonically non-increasing in t and* $\lim_{t \to \infty} |\nabla_- F|(f_t) = 0.$

This is the Gradient-of-Energy flow.

There are additional properties of the flow that we will find useful. From [Ma] we have the following.

Proposition 5.18. *Let* $(\mathcal{M}, d)$ *be a complete length space nonpositively curved in the sense of Alexandrov and let* $F\colon \mathcal{M} \to \mathbb{R} \cup \infty$ *be a lower semi-continuous, convex functional. Let* f_t *be the gradient-of-energy flow defined above. Then,*

i. $t \mapsto F(f_t)$ *is convex.*

ii. $\lim_{t\to\infty} F(f_t) = \inf_{u\in\mathcal{M}} F(u)$.

iii. If there exists a convergent subsequence $\{f_{t_i}\}$*, then* $\lim_{t\to\infty}\ f_t$ *exists and is a minimizer of* F.

We shall be particularly interested in convex functionals and their minimizers. We have again from [Ma] the following.

Proposition 5.19. *Let* $(\mathcal{M}, d)$ *be a complete length space nonpositively curved in the sense of Alexandrov and let* $F\colon \mathcal{M} \to \mathbb{R} \cup \infty$ *be a lower semi-continuous, convex functional.* $f_0 \in \mathcal{M}$ *is a stationary point as defined above if and only if* f_0 *minimizes* F.

Now that the norm of the gradient and Gradient-of-Energy flow are defined, we can show that they define a flow for certain maps between polyhedra and manifolds. We must first show that the setting considered elsewhere in this paper is a special case of the setting used in [Ma].

Lemma 5.20. *Let* X *be a compact Riemannian polyhedron and* N *a compact smooth Riemannian manifold with nonpositive sectional curvature. Let* E *be the the energy* *functional on* $L^2(X,N)$ *as in Definition 2.23 (see page 33). Then,* E *is a lower-semicontinous, convex functional bounded below by* 0, *and the space* $L^2(X,N)$ *is a complete, nonpositively curved length space with respect to the metric*

$$d_{L^2}(f,g) := \left(\int_X d_N(f(z),g(z))^2\,dX\right)^{\frac{1}{2}}$$

where $d_N(y_0,y_1)$ *is the geodesic distance between* $y_0,y_1 \in N$*. In particular, for two homotopic maps* $f_1,f_2 \in W^{1,2}(X,N)$*, and* f_t *representing the geodesic homotopy between them, we have*

$$E(f_t) \leq (1-t)E(f_0) + tE(f_1) - C(1-t)t\int_X |\nabla d_N(f_0,f_1)|^2\,dX, \tag{5.10}$$

where C *is a positive dimensional constant dependent on the top dimension of* X*. Additionally,* $W^{1,2}(X,N)$ *is a complete, nonpositively curved length space with respect to* $d_L^2(\cdot,\cdot)$.

Proof. These are standard results and we refer to [EF, Chapters 9 & 11] for proofs and a clear exposition. In particular, for the proof of Equation (5.10), see the proof of [EF, Proposition 11.2]. For the results for $W^{1,2}(X,N)$, we note that convexity is given by Equation (5.10). By an extension of the precompactness result of [KS] described below, we have that $W^{1,2}(X,N)$ is a closed subset of $L^2(X,N)$. □

We have a precompactness theorem similar to the one obtained in [KS]. For the present setting, it is supplied by [EF].

Proposition 5.21. *Let* X *be a compact Riemannian polyhedron and* N *a compact smooth Riemannian manifold with nonpositive sectional curvature, and let* E

be the the energy functional on $L^2(X,N)$ as in Definition 2.23 (see page 33). If $\{f_\alpha\} \subset L^2(X,N)$ is a set of maps bounded uniformly in $W^{1,2}(X,N)$, then there is a subsequence that converges in $L^2(X,N)$ to a map that is in $W^{1,2}(X,N)$.

We now show that the harmonic map heat flow and the gradient-of-energy flow are, in fact, the same. We begin with a definition and a lemma.

Definition 5.22. Let X be a Riemannian polyhedron and N a smooth Riemannian manifold with nonpositive sectional curvature, and let E be the the energy functional on $W^{1,2}(X,N)$ as in Definition 2.23 (see page 33). For $f_0 \in W^{1,2}(X,N)$ such that $|\nabla_- E|(f_0) > 0$, a sequence of maps $\{f_\alpha\} \subset L^2(X,N)$ converging to f_0 is a *(global) maximizing realization of the gradient of energy at f_0* if

$$\begin{aligned}\lim_{\alpha\to\infty} \frac{E(f_0)-E(f_\alpha)}{d_{L^2}(f_0,f_\alpha)} &= \limsup_{f\to f_0, f\in W^{1,2}(X,N)} \frac{E(f_0)-E(f)}{d_{L^2}(f_0,f)} \\ &:= |\nabla_- E|(f_0).\end{aligned}$$

Let $\Omega \subset X$ be open, connected and compactly contained and f_0 be as above. Define

$$\{f^\Omega\} := \left\{ f \in W^{1,2}(X,N) \mid f|_{X\setminus\Omega} \overset{ae}{=} f_0|_{X\setminus\Omega} \right\}$$

A sequence of maps $\{f_\alpha\} \subset W^{1,2}(X,N)$ is a *locally maximizing realization of the gradient of energy at f_0* if for every open, connected, compactly contained set $\Omega \subset X$,

$$\lim_{\alpha\to\infty} \frac{E(f_0)-E(f_\alpha|_\Omega)}{d_{L^2}(f_0,f_\alpha|_\Omega)} = \limsup_{f\to f_0, f\in\{f^\Omega\}} \frac{E(f_0)-E(f)}{d_{L^2}(f_0,f)}.$$

If we make assumptions about the energy and compactness we can easily see these definitions are equivalent, as we state below.

Lemma 5.23. *Let X be a compact Riemannian polyhedron and N a compact smooth Riemannian manifold with nonpositive sectional curvature, and let E be the the energy functional on $L^2(X, N)$ as in Definition 2.23. Let $f_0 \in W^{1,2}(X, N)$ such that $|\nabla_- E|(f_0) > 0$. Then a sequence of maps $\{f_\alpha\} \subset W^{1,2}(X, N)$ is a global maximizing realization of the gradient of energy at f_0 if and only if $\{f_\alpha\}$ is a local maximizing realization of the gradient of energy at f_0.*

We now begin the process of show that the two flows in question are equivalent. We begin by showing that the harmonic map heat flow beginning at a suitable map satisfies the following.

Proposition 5.24. *Let X be a compact Riemannian polyhedron and N a compact smooth Riemannian manifold with nonpositive sectional curvature, and let E be the the energy functional as in Definition 2.23. Let $f_0 \colon X \to N$ be in $C^1(X)$ and let f_0 be an initial map with bounded energy density such that it is not a minimizer of E, and presume that $E(f_t)$ does not achieve a minimum on $(0, T)$. Let $f_t \colon X \times [0, T) \to N$ be a strong solution to the harmonic map heat flow as in Definition 5.9. Then, for any fixed $t_0 \in (0, T)$,*

$$|\nabla_- E|(f_{t_0}) = \lim_{t \to t_0} \frac{E(f_{t_0}) - E(f_t)}{d_{L^2}(f_{t_0}, f_t)} = \|\tau(f_{t_0})\|_{L^2(X)} \tag{5.11}$$

where $\tau(f_{t_0}) = \Delta f_{t_0} + A(f_{t_0})(df_{t_0}, df_{t_0})$ is the torsion field of f_{t_0}.

The same conclusion also holds for the embedded problem.

Proof. We pick $t_0 > 0$ to gain a bit more regularity that will be necessary to make this result hold. We note that for a strong solution to the heat flow as in Definition 5.9, we must necessarily have that for positive t, f_t is in $C^1(X) \cap C^2_{\text{loc}}(X \setminus X^{(n-1)})$ and, if

$\iota\colon N \to \mathbb{R}^q$ is a smooth isometric embedding, $(\iota \circ f_0)^\gamma \in \mathrm{Dom}(\Delta)$ for each coordinate $1 \le \gamma \le q$. In particular, this means that we have that $\tau(f_{t_0}) \in L^2(X)$. As f_{t_0} is not an energy minimizer, Proposition 5.19 gives us that $|\nabla_- E|(f_{t_0}) > 0$. We proceed to prove our result by showing

$$\|\tau(f_{t_0})\|_{L^2(X)} \le |\nabla_- E|(f_{t_0}) \quad \text{and} \quad \|\tau(f_{t_0})\|_{L^2(X)} \ge |\nabla_- E|(f_{t_0}).$$

As an immediate result from the computations, we shall find that

$$\lim_{t \to t_0} \frac{E(f_{t_0}) - E(f_t)}{d_{L^2}(f_{t_0}, f_t)} = \|\tau(f_{t_0})\|_{L^2(X)}.$$

Step 1: $\|\tau(f_{t_0})\|_{L^2(X)} \le |\nabla_- E|(f_{t_0})$.

We begin by computing (5.11) on open, connected sets compactly contained in an n-simplex, and then take an exhaustion of sets to reach our conclusion.

Let $\Omega \subset X$ be an open connected set such that it is compactly contained in an open n-simplex. Let

$$\left\{f^\Omega\right\} := \left\{f \in W^{1,2}(X, N) \mid f|_{X\setminus\Omega} \overset{\text{ae}}{=} f_{t_0}|_{X\setminus\Omega}\right\}.$$

Similarly, for the strong solution to the harmonic map heat flow, f_t (the subscript does *not* denote a time derivative), let

$$f_t^\Omega := \begin{cases} f_t & \text{on } \Omega \\ f_{t_0} & \text{on } X \setminus \Omega. \end{cases}$$

Hence, we note that, if the limits exist,

$$\lim_{t\to 0} \frac{E(f_{t_0}) - E(f_t^\Omega)}{d_{L^2}(f_{t_0}, f_t^\Omega)} = \lim_{t\to 0} \frac{E(f_{t_0}|_\Omega) - E(f_t|_\Omega)}{d_{L^2}(f_{t_0}|_\Omega, f_t|_\Omega)}.$$

We compute the right-hand side. We note that

$$\lim_{t\to t_0} \frac{E(f_{t_0}|_\Omega) - E(f_t|_\Omega)}{d_{L^2}(f_{t_0}|_\Omega, f_t|_\Omega)} = \left(\lim_{t\to t_0} \frac{E(f_{t_0}|_\Omega) - E(f_t|_\Omega)}{t - t_0}\right)\left(\lim_{t\to t_0} \frac{d_{L^2}(f_{t_0}|_\Omega, f_t|_\Omega)}{t - t_0}\right)^{-1}.$$

As $f_t|_\Omega$ is a map from a manifold to another manifold, we can follow standard arguments (see [J2, Section 8.1]), to find that

$$\begin{aligned} \lim_{t\to t_0} \frac{E(f_{t_0}|_\Omega) - E(f_t|_\Omega)}{t - t_0} &= \int_\Omega \left\langle \tau(f_{t_0}), \tfrac{\partial}{\partial t} f \right\rangle \, dX \\ &= \int_\Omega \langle \tau(f_{t_0}), \tau(f_{t_0}) \rangle \, dX \qquad (5.12) \\ &= \|\tau(f_{t_0})|_\Omega\|^2_{L^2(\Omega)}, \end{aligned}$$

as f_t is a pointwise solution of $\frac{\partial}{\partial t} f_t = \tau(f_t)$ on the interior of an n-simplex. Also, we have by similar computations

$$\lim_{t\to t_0} \frac{d_{L^2}(f_{t_0}|_\Omega, f_t|_\Omega)}{t - t_0} = \|\tau(f_{t_0})|_\Omega\|_{L^2(\Omega)},$$

which, by f_{t_0} not being an energy minimizer and Ω being large enough, is greater than zero. Thus, we have

$$\lim_{t\to t_0} \frac{E(f_{t_0}) - E(f_t^\Omega)}{d_{L^2}(f_{t_0}, f_t^\Omega)} = \|\tau(f_{t_0})|_\Omega\|_{L^2(\Omega)}.$$

Let $\{\Omega_\alpha\}_{\alpha\in\mathbb{N}}$ be a sequence of nested, open subsets of $X \setminus X^{(n-1)}$ such that, for each α, Ω_α is bounded away from $X^{(n-2)}$ and $\{\Omega_\alpha\}$ forms an exhaustion of X (i.e. $\overline{\cup_{\alpha=1}^{\infty}\Omega_\alpha} = X$). We presume that Ω_1 is so large that $\|\tau(f_{t_0})|_{\Omega_1}\|_{L^2(\Omega_1)} > 0$. Obviously, $\|\tau(f_{t_0})|_{\Omega_\alpha}\|_{L^2(\Omega_\alpha)}$ is an increasing sequence as $\alpha \to \infty$, bounded above by $\|\tau(f_{t_0})\|_{L^2(X)}$, which is itself bounded. Finally, we can prove

$$\begin{aligned}
\|\tau(f_{t_0})\|_{L^2(X)} &= \lim_{\alpha\to\infty} \|\tau(f_{t_0})|_{\Omega_\alpha}\|_{L^2(\Omega_\alpha)} \\
&\le \lim_{\alpha\to\infty} \limsup_{t\to t_0} \frac{E(f_{t_0}) - E(f_t^{\Omega_\alpha})}{d_{L^2}(f_{t_0}, f_t^{\Omega_\alpha})} \\
&\le \limsup_{f\to f_{t_0}, f\in W^{1,2}(X,N)} \frac{F(f_{t_0}) - F(f)}{d_{L^2}(f_{t_0}, f)} \\
&= |\nabla_- E|(f_{t_0}),
\end{aligned} \tag{5.13}$$

which follows by above work and the definition of $|\nabla_- E|(f_{t_0})$.

Step 2: $\|\tau(f_{t_0})\|_{L^2(X)} \ge |\nabla_- E|(f_{t_0})$. We follow the proof of [IKN, Proposition 5.2]. By definition of $|\nabla_- E|(f_{t_0})$, we have that for every $\epsilon > 0$ there exists a $g \in W^{1,2}(X,N)$ such that

$$\frac{E(f_{t_0}) - E(g)}{d_{L^2}(f_{t_0}, g)} > |\nabla_- E|(f_{t_0}) - \epsilon.$$

We construct a constant speed geodesic homotopy in $W^{1,2}(X,N)$ between f_{t_0} and g by setting a curve $c\colon [0,1] \to W^{1,2}(X,N)$ such that for $z \in X$, $c_0(z) = f_{t_0}(z)$ and $c_1(z) = g(z)$. The existence of such a c is guaranteed in [EF, Chapter 11]. We let ${c'}_t(z)$ denote the directional derivative of c at time t at the point z (i.e. ${c'}_t(z) \in T_{c_t(z)}N$), and by construction we have $|{c'}_t(z)| = d_N(f_{t_0}(z), g(z))$. By the convexity of energy on geodesic homotopies given in Equation (5.10), we have

$$\frac{E(f_{t_0}) - E(c_t)}{t - t_0} \geq E(f_{t_0}) - E(g).$$

We have from [EF, Chapter 11] that a geodesic homotopy is uniformly Lipschitz, so we may compute as before

$$\begin{aligned}
\lim_{t \to t_0} \frac{E(f_{t_0}) - E(c_t)}{t - t_0} &= \int_X \langle c_0'(z), \tau(f_{t_0})(z) \rangle \, dX \\
&= \int_X \left\langle d_N(f_{t_0}(z), g(z)) \frac{c_0'(z)}{|c_0'(z)|}, \tau(f_{t_0})(z) \right\rangle dX \\
&= \int_X d_N(f_{t_0}(z), g(z)) \left\langle \frac{c_0'(z)}{|c_0'(z)|}, \tau(f_{t_0})(z) \right\rangle dX \\
&\leq \int_X d_N(f_{t_0}(z), g(z)) \left\langle \frac{\tau(f_{t_0})(z)}{|\tau(f_{t_0})(z)|}, \tau(f_{t_0})(z) \right\rangle dX \\
&= \int_X d_N(f_{t_0}(z), g(z)) |\tau(f_{t_0})(z)| \, dX \\
&\leq \left(\int_X d_N(f_{t_0}(z), g(z))^2 \, dX \right)^{\frac{1}{2}} \left(\int_X |\tau(f_{t_0})(z)|^2 \, dX \right)^{\frac{1}{2}} \\
&= d_{L^2}(f_{t_0}, g) \|\tau(f_{t_0})\|_{L^2(X)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
d_{L^2}(f_{t_0}, g) \left(|\nabla_- E|(f_{t_0}) - \epsilon \right) &\leq E(f_{t_0}) - E(g) \\
&\leq \lim_{t \to t_0} \frac{E(f_{t_0}) - E(c_t)}{t - t_0} \\
&\leq d_{L^2}(f_{t_0}, g) \|\tau(f_{t_0})\|_{L^2(X)}.
\end{aligned} \tag{5.14}$$

This gives that for all $\epsilon > 0$,

$$|\nabla_- E|(f_{t_0}) - \epsilon \leq \|\tau(f_{t_0})\|_{L^2(X)},$$

and we have by this and step 1

$$|\nabla_- E|(f_{t_0}) = \|\tau(f_{t_0})\|_{L^2(X)}.$$

To show that

$$\lim_{t \to t_0} \frac{E(f_{t_0}) - E(f_t)}{d_{L^2}(f_{t_0}, f_t)} = \|\tau(f_{t_0})\|_{L^2(X)}$$

we refer to the computation of Equation (5.12) (following [J2, Section 8.1]) and we note that the pointwise limit for $z \in X \setminus X^{(n-1)}$ is

$$\lim_{t \to t_0} \frac{1}{t} \left(\langle \nabla f_{t_0}(z), \nabla f_{t_0}(z) \rangle - \langle \nabla f_t(z), \nabla f_t(z) \rangle \right) = \tau(f_{t_0}(z))^2.$$

We recall that $\tau(f_{t_0}) = \Delta f_{t_0} + A(f_{t_0})(df_{t_0}, df_{t_0})$ and that $|\Delta f_{t_0}| \in L^2(X)$ and

$$|A(f_{t_0})(df_{t_0}, df_{t_0})| \in L^\infty(X).$$

Hence, $|\tau(f_{t_0})| \in L^2(X)$ and we have by dominated convergence from Equation (5.13) that

$$\lim_{t \to t_0} \frac{E(f_{t_0}) - E(f_t)}{d_{L^2}(f_{t_0}, f_t)} = \|\tau(f_{t_0})\|_{L^2(X)}.$$

□

We now require one last result regarding the uniqueness of the Gradient-of-Energy flow given in [Ma, Theorem 2.16].

Proposition 5.25. *Let X be a compact Riemannian polyhedron and N a compact smooth Riemannian manifold with nonpositive sectional curvature, and let E be the the energy functional as in Definition 2.23. Let $f_0 \in W^{1,2}(X,N)$ such that $0 < |\nabla_{-}E|(f_0) < \infty$ and let f_t denote the Gradient-of-Energy flow starting at f_0. Also, let $\{g_\alpha\} \subset W^{1,2}(X,N)$ be a sequence of maps tending to f_0 such that*

$$\lim_{\alpha\to\infty} \frac{E(f_0) - E(g_\alpha)}{d_{L^2}(f_0, g_\alpha)} = |\nabla_{-}E|(f_0).$$

Then there exists a sequence $t_\alpha \to 0$ such that

$$\lim_{\alpha\to\infty} \frac{d_{L^2}(f_{t_\alpha}, g_\alpha)}{d_{L^2}(f_0, g_\alpha)} = 0.$$

We can now show that the Gradient-of-Energy flow and the harmonic map heat flow are identical as long as the initial map is sufficiently well behaved.

Proposition 5.26. *Let X be a compact Riemannian polyhedron and N a compact smooth Riemannian manifold with nonpositive sectional curvature, and let E be the the energy functional as in Definition 2.23. For a C^1 initial map $f_0 \colon X \to N$ with bounded energy density, let $\tilde{f}_t \colon X \times [0,T) \to N$ be a strong solution to the harmonic map heat flow as in Definition 5.9, and let $f_t \colon X \times [0,T) \to N$ denote the gradient-of-energy flow beginning at f_0 as above. Then for all $0 < t < T$, and almost all $z \in X$*

$$f_t(z) = \tilde{f}_t(z).$$

Proof. Our approach is to create a function $D(t) := d_{L^2}(f_t, \tilde{f}_t)$, and show that it is continuous with initial value $D(0) = 0$ and $D'(t)$ is defined for all $t > 0$ and $D'(t) = 0$

and, hence, $D(t) = 0$ on all of $[0, T)$ and f_t and $\tilde{f}_t$ must be equal almost everywhere. We presume without loss of generality that $E(f_t)$ and $E(\tilde{f}_t)$ do not achieve minimums on $[0, T)$ for, if they do, each flow must remain constant after such a time and the statement becomes trivial.

We note that for a map $g \in W^{1,2}(X, N)$, $d_{L^2}(f_t, g)$ and $d_{L^2}(\tilde{f}_t, g)$ are both continuous functions of t, so continuity of $D(t)$ is immediate.

We now show that $D'(t)$ is defined for all $0 < t < T$ and is always zero. We use Propositions 5.24 and 5.26. Fix $t_0 \in (0, T)$. By Proposition 5.24, we have that

$$\lim_{t \to t_0} \frac{E(f_{t_0}) - E(\tilde{f}_t)}{d_{L^2}(f_{t_0}, \tilde{f}_t)} = |\nabla_- E|(f_{t_0}),$$

so we may apply Proposition 5.26 and have that we have that for any sequence $t_\alpha \to t_0$ there exists a sequence $t'_\alpha \to t_0$ such that

$$\lim_{\alpha \to \infty} \frac{d_{L^2}(f_{t'_\alpha}, \tilde{f}_{t_\alpha})}{d_{L^2}(f_{t_0}, \tilde{f}_{t_\alpha})} = 0.$$

From the construction of the sequence $\{t'_\alpha\}$ given $\{t_\alpha\}$ in the proof of [Ma, Theorem 2.16], we see that the sequences must be equal. That is, $\{t'_\alpha\} = \{t_\alpha\}$. We note that

$$\frac{d_{L^2}(f_{t_\alpha}, \tilde{f}_{t_\alpha})}{d_{L^2}(f_{t_0}, \tilde{f}_{t_\alpha})} = \left(\frac{d_{L^2}(f_{t_\alpha}, \tilde{f}_{t_\alpha})}{t_\alpha - t_0} \right) \left(\frac{d_{L^2}(f_{t_0}, \tilde{f}_{t_\alpha})}{t_\alpha - t_0} \right)^{-1}.$$

By f_{t_0} not being an energy minimizer, we have from work in Proposition 5.24

$$\lim_{t_\alpha \to t_0} \frac{d_{L^2}(f_{t_0}, \tilde{f}_{t_\alpha})}{t_\alpha - t_0} = \|\tau(f_{t_0})\|_{L^2(X)} > 0.$$

Hence, we have that

$$\lim_{t_\alpha \to t_0} \frac{d_{L^2}(f_{t_\alpha}, \tilde{f}_{t_\alpha})}{t_\alpha - t_0} = 0$$

and, as our choice of sequence was arbitrary, $D'(t_0) = 0$. Although we apply this argument at $t = t_0$, we can easily reconstruct it for any $0 < t_0 < T$, and we have for all $0 < t < T$, $D'(t) = 0$. Also by $D(t)$ continuous and $D(0) = 0$, $D(t)$ must be zero everywhere. □

We now that we have shown the two flows are identical on intervals on which they are defined, we now glean some properties of the harmonic map heat flow quite easily.

Proposition 5.27. *Let X be a compact Riemannian polyhedron and N a compact smooth Riemannian manifold with nonpositive sectional curvature, and let E be the the energy functional as in Definition 2.23. For a C^1 initial map $f_{t_0}\colon X \to N$ with bounded energy density, suppose that $f_t\colon X \times [0,T) \to N$ is a strong solution to the harmonic map heat flow as in Definition 5.9. Then f_t has the following properties.*

i. f_t is a semigroup (i.e. for $s, t > 0$, $(f_t)_s = f_{t+s}$).

ii. $\|\frac{\partial}{\partial t} f_t\|_{L^2} (= \|\tau(f_t)\|_{L^2})$ is monotonically non-increasing in t on $[0,T)$. Also, if $T = \infty$, $\lim_{t\to\infty} \|\frac{\partial}{\partial t} f_t\|_{L^2} = 0$.

iii. $t \mapsto E(f_t)$ is monotonically non-increasing and convex on $[0,T)$.

iv. If $T = \infty$, then $\lim_{t\to\infty} E(f_t) = \inf_{u \in W^{1,2}(X,N)} E(u)$.

v. If $T = \infty$, then $\lim_{t\to\infty} f_t$ exists and is harmonic (i.e. is a minimizer of E).

Proof. We begin by noting that Lemma 5.20 and Proposition 5.26 imply that we can use Propositions 5.17, 5.18 and 5.24. The only statement that does not follow directly is the last one. We can see that f_t as $t \to \infty$ is a minimizing sequence, and so we may apply the precompactness result of Proposition 5.21, which shows the existence of a convergent subsequence. We can then apply the last statement of Proposition 5.18. $\square$

Remark 5.28. We note that the preceding theorem, part of the statement presumes the existence of a harmonic map heat flow. That is an assumption we have not yet justified, but is proven in the following sections.

5.4 Short-Time Existence of a Solution for Initial Data with Bounded Energy Density

Our approach is to show that if one prescribes a C^1 map f, one can embed the target in Euclidean space (say via a smooth embedding ι) and find a solution to the heat flow equation with initial data $\iota \circ f$. This is not as simple as the case where the target is $\mathbb{R}$ (treated in previous sections). Curvature of the target will become an issue. Indeed, we show that we can find a sequence of maps that converge to a strong solution to the heat flow. Following the observation of Proposition 5.13, we wish to prove that for X compact and simplex-wise flat and $f_0 \in C^1(X, N)$, there exists $T > 0$, such that a strong solution with initial value $F_0 := \iota \circ f_0$ exists on $X \times [0, T)$. We develop a sequence of maps that will converge to (5.9). Specifically, fix a map $f_0 \colon X \to N$ such that f_0 is C^1 with bounded energy density. Let $F_0 = \iota \circ f_0$, where $\iota \colon N \hookrightarrow \mathbb{R}^q$ is isometric. Also, let

$$h \colon X \times X \times \mathbb{R}^+ \to \mathbb{R}$$

denote the heat kernel of Proposition 4.4. We develop a sequence of maps, $\{W^l\}_{l=0}^{\infty}$, such that

$$W^l(z,t) = (W^{l,1}(z,t), W^{l,2}(z,t), \ldots, W^{l,q}(z,t)),$$

and are defined as follows:

$$\left.\begin{aligned} W^{0,\gamma}(z,t) &= \int_X h(z,v,t)F_0^\gamma \, dv, \\ W^{l,\gamma}(z,t) &= \int_0^t \int_X h(z,v,t-\tau)G^{l-1,\gamma}(v,\tau)\, dv\, d\tau + W^{0,\gamma}(z,t), \end{aligned}\right\} \tag{5.15}$$

where

$$G^{l,\gamma}(v,\tau) = A^\gamma_{\alpha\beta}(W^l)\left(\frac{\partial W^{l,\alpha}}{\partial v^i}\right)\left(\frac{\partial W^{l,\beta}}{\partial v^j}\right) g^{ij}.$$

We shall show that these maps converge to a solution of (5.9). We follow the arguments of [ES].

Proposition 5.29. *Let X be compact and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in C^1. Let $\{W^l\colon X \to \mathbb{R}^q\}_{l=0}^{\infty}$ be the maps described in equation* (5.15)*. Then, there exists $T \geq 0$, such that*

$$\lim_{l\to\infty} W^l(z,t) = W(z,t)$$

uniformly in C^1 on $X \times [0,T)$, where $W(z,t)$ is a strong embedded solution with initial value F_0 on $X \times [0,T)$. Also $W(z,t) \in C^{1+\alpha,1+\beta}_{\mathrm{loc}}(X \setminus X^{(n-2)} \times [0,T), \iota(N))$, where $\iota(N) \subset \mathbb{R}^q$. Additionally, W satisfies all of the conclusions of Proposition 4.28.

Proof. To show convergence, we follow the arguments of [ES, Section 10] nearly verbatim. Of particular interest here is that the maps $\{W^l(z,t)\}$ converge in energy to a map, $W(z,t)$, which satisfies in each coordinate $1 \le \gamma \le q$

$$W^\gamma(z,t) = \int_0^t \int_X h(z,v,t-\tau) G^\gamma(v,\tau)\, d\mu(v) d\tau + \int_X h(z,v,t) F_0(v)\, d\mu(v), \quad (5.16)$$

where

$$G^\gamma(v,\tau) = A^\gamma_{\alpha\beta} \left(\frac{\partial W^\alpha}{\partial v^i}\right)\left(\frac{\partial W^\beta}{\partial v^j}\right) g^{ij}.$$

We recall our approximating maps are defined as follows.

$$\left.\begin{aligned} W^{0,\gamma}(z,t) &= \int_X h(z,v,t) F_0^\gamma(v)\, dv, \\ W^{l,\gamma}(z,t) &= \int_0^t \int_X h(z,v,t-\tau) G^{l-1,\gamma}(v,\tau)\, dv\, d\tau + W^{0,\gamma}(z,t). \end{aligned}\right\} \quad (5.17)$$

For notational simplicity, in local coordinates, let the subscript i denote a derivative in the i^{th} coordinate and hence let

$$W_i^{l,\alpha} := \frac{\partial W^{l,\alpha}}{\partial v^i}.$$

Also, let

$$e^l(\tau) := \sup_{z\in X} \left(\sum_{\gamma=1}^q W_i^{l,\gamma}(z,\tau) W_j^{l,\gamma}(z,\tau) g^{ij} \right)^{\frac{1}{2}}.$$

From Theorem 4.18 we have bounds on the gradient of the heat kernel and so, following Proposition 4.26, in local coordinates we have

$$W_i^{l,\gamma}(z,t) = \int_0^t \int_X h_i(z,v,t-\tau) G^{l-1,\gamma}(v,\tau)\, dv\, d\tau + W_i^{0,\gamma}(z,t).$$

From the bounds on $A^{\gamma}_{\alpha\beta}$, g, and on $\int_t \int_X |h_i(z,v,\tau)|\, d\mu(v) d\tau$ from Theorem 4.18 and Proposition 4.26, we again compute

$$e^l(t) \leq C \int_0^t (t-\tau)^{-\frac{1}{2}} (e^{l-1}(\tau))^2 \, d\tau + e^0(t),$$

for $t \in [0,\epsilon]$, and C is not dependent on the choice of the initial map. Let

$$\bar{e}^l := \sup_{t\in[0,\epsilon]} e^l(t).$$

From the above, we have

$$\bar{e}^l \leq 2C\sqrt{\epsilon}(\bar{e}^{l-1})^2 + \bar{e}^0.$$

If $2C\sqrt{\epsilon}\bar{e}^{l-1} \leq \frac{1}{2}$ and $2C\sqrt{\epsilon}\bar{e}^0 \leq \frac{1}{4}$, then

$$2C\sqrt{\epsilon}\bar{e}^l \leq \left(2C\sqrt{\epsilon}\bar{e}^{l-1}\right)^2 + 2C\sqrt{\epsilon}\bar{e}^0 \leq \frac{1}{2}.$$

Hence, by induction, we have that $W^l(z,t) \in V$ and $\bar{e}^l \leq C\sqrt{\epsilon}$ for all $t \in [0,\epsilon]$ and all $l \in \mathbb{N}$, as $W^l(z,0) \in \iota(N) \subset \widetilde{N}$ and we have a uniform bound on the energy given that scales as $\sqrt{\epsilon}$ for $t \in [0,\epsilon]$.

We can now show the convergence of the maps $\left\{W^l\right\}$ to W in energy. Let

$$\begin{aligned} K^l(t) = & \sup_{\substack{z\in X\setminus X^{(n-1)} \\ 1\leq\gamma\leq q}} \left|W^{l,\gamma}(z,t) - W^{l-1,\gamma}(z,t)\right| + \\ & \sup_{\substack{z\in X\setminus X^{(n-1)} \\ 1\leq\gamma\leq q}} \left|g^{ij}\left(W^{l,\gamma}_i(z,t) - W^{l-1,\gamma}_i(z,t)\right)\left(W^{l,\gamma}_j(z,t) - W^{l-1,\gamma}_j(z,t)\right)\right|^{\frac{1}{2}} \end{aligned}$$

We see that examining the terms in this expression relies on information about $G^{l,\gamma} -$

$G^{l-1,\gamma}$. We compute

$$G^{l,\gamma} - G^{l-1,\gamma} = A^{\gamma}_{\alpha\beta}(W^{l-1}) \left(W_i^{l,\alpha} W_j^{l,\beta} - W_i^{l-1,\alpha} W_j^{l-1,\beta} \right) g^{ij} + \left(A^{\gamma}_{\alpha\beta}(W^{l}) - A^{\gamma}_{\alpha\beta}(W^{l-1}) \right) W_i^{l,\alpha} W_j^{l,\beta} g^{ij}.$$

By a judicious use of Talyor's theorem, bounds on $A^{\gamma}_{\alpha\beta}$ and its first-order derivatives, and the Cauchy-Schwarz inequality for bilinear forms, we have

$$\left| G^{l,\gamma} - G^{l-1,\gamma} \right| \leq C' \left(\bar{e}^{l} + \bar{e}^{l-1} + (\bar{e}^{l})^2 \right) K^{l}(t).$$

By our bounds on $\bar{e}^{l}$ above, we have

$$\begin{aligned} \left| G^{l,\gamma} - G^{l-1,\gamma} \right| &\leq C' \left(C\sqrt{\epsilon} + C^2 \epsilon \right) K^{l}(t) \\ &= C'' K^{l}(t). \end{aligned}$$

By the conservativeness of $h(z,v,t)$ of Proposition 4.13, the Gaussian bounds of Theorem 4.18 and Proposition 4.26, we can show

$$K^{l+1}(t) \leq C''' \int_0^t (t-\tau)^{-\frac{1}{2}} K^{l}(\tau)\, d\tau.$$

Let

$$\bar{K}^{l}(t) \sup_{\tau \in [0,t]} K^{l}(\tau).$$

Then, for all $l \in \mathbb{N}$, we have

$$\bar{K}^{l}(t) \leq (2C'''\sqrt{t})^{l} \bar{K}^{0}(t),$$

and so, for sufficiently small $t > 0$, which we now let be T, we have convergence of

$$\sum_0^\infty \bar{K}^l(t).$$

Hence, we have our sequence of maps $\{W^l\}_{l=0}^\infty$ converging uniformly in C^1 on $[0,T)$, $T > 0$. Thus, the terms $G^{l,\gamma} = A^\gamma_{\alpha\beta}(W^l)W_i^{l,\alpha}W_j^{l,\beta}g^{ij}$ must converge, too. The limit of $\{W^l\}_0^\infty$ must be W as in equation (5.16) with continuous first-order derivatives, and the limit of $\{G^l\}$ must be G which, for each coordinate γ, satisfies $G^\gamma = A^\gamma_{\alpha\beta}(W)W_i^\alpha W_j^\beta g^{ij}$. By the definition of W in equation (5.16) and by the continuity of $h(z,v,t)$, we have that W must be continuous in space with continuous first-order spacial derivatives on all X. To see this is a strong solution and achieve higher regularity we can apply Proposition 5.15 (see also Proposition 4.28). □

To show long term existence, we will need to show that the energy density stays bounded on compact time intervals. We begin this by examining how scaling the metric on the target affects the embedding, the solution and its energy density. We have some elementary observations. As we are considering N as embedded in $\mathbb{R}^q$, we track how scaling the metric of the compact, smooth Riemannian manifold (N,h) affects the embedding. Indeed, let $\delta \in (0,\infty)$ and define $h_\delta := \delta h$, and, by abuse of notation, let $N_\delta := (N, h_\delta)$. Given the isometric embedding $\iota\colon N \to \mathbb{R}^q$, defined locally near $y \in N$ as

$$\iota(y^1,\dots,y^n) = (\iota^1(y^1,\dots,y^n),\dots,\iota^q(y^1,\dots,y^n)),$$

we can define an isometric embedding $\iota_\delta \colon N_\delta \to \mathbb{R}^q$ as

$$\begin{aligned}\iota_\delta(y^1,\dots,y^n) &:= (\phi(\delta)\iota^1(y^1,\dots,y^n),\dots,\phi(\delta)\iota^q(y^1,\dots,y^n))\\ &:= (\iota_\delta^1(y^1,\dots,y^n),\dots,\iota_\delta^q(y^1,\dots,y^n)),\end{aligned}$$

where $\phi(\delta)=\sqrt{\delta}$. Let $\tilde{N}$ be an open, tubular neighborhood of $\iota(N)$ in $\mathbb{R}^q$ such that the nearest-point projection map $\pi\colon \tilde{N}\to\iota(N)$ is well defined. For N_δ we can define $\tilde{N}_\delta$ and π_δ similarly by scaling the (global) coordinates each by a factor of $\sqrt{\delta}$. Hence, we have

$$\begin{aligned}\pi_\delta(z^1,\dots,z^q) &:= \phi(\delta)\pi(z^1,\dots,z^q)\\ &= (\phi(\delta)\pi^1(z^1,\dots,z^q),\dots,\phi(\delta)\pi^q(z^1,\dots,z^q))\\ &:= (\pi_\delta^1(z^1,\dots,z^q),\dots,\pi_\delta^q(z^1,\dots,z^q)).\end{aligned}$$

Thus, $\tilde{N}_\delta$ is a tubular neighborhood of $\iota_\delta(N_\delta)$ and π_δ is a nearest-point projection map of $\tilde{N}_\delta$ onto $\iota_\delta(N_\delta)$. We have the following.

Lemma 5.30. *Let X be a compact, simplexwise flat Riemannian polyhedron, N a compact Riemannian manifold with Riemannian metric h, $\iota\colon N\hookrightarrow\mathbb{R}^q$ a smooth isometric embedding, $F_0\colon X\to\iota(N)\subset\mathbb{R}^q$ a map in C^1 and $F\colon X\times[0,T)\to\mathbb{R}^q$ a strong embedded solution to the heat flow with initial value F_0. Notably, F satisfies*

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)F^\gamma = \sum_{i,\alpha,\beta}A^\gamma_{\alpha\beta}(F)\frac{\partial F^\alpha}{\partial x_i}\frac{\partial F^\beta}{\partial x_i} \quad \text{on } X\setminus X^{(n-1)}\times(0,T)\\ \lim_{t\to 0}F(\cdot,t)=F_0 \quad \text{in } C^0.\end{aligned}\right\}$$

Additionally, let $\delta \in (0, \infty)$*, and let* $\tilde{N}$, π, $\tilde{N}_\delta$, π_δ*, and* ι_δ *be as above. If* h_δ *is the metric* h *scaled by* δ *(i.e.* $h_\delta = \delta h$*), then* ι_δ *is an isometric embedding of* N_δ *in* $\mathbb{R}^q$*, and for the corresponding initial map* $F_{0,\delta} \colon X \to \iota_\delta(N)(\subset \mathbb{R}^q)$ *given by*

$$F_{0,\delta}(z) = \sqrt{\delta} F_0(z)$$

the strong embedded solution to the heat flow with initial value $F_{0,\delta}$*, denoted* F_δ*, exists on* $X \times [0, T)$ *and satisfies*

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) F_\delta^\gamma = \sum_{i,\alpha,\beta} {}_\delta A^\gamma_{\alpha\beta}(F_\delta) \frac{\partial F_\delta^\alpha}{\partial x_i} \frac{\partial F_\delta^\beta}{\partial x_i} &\quad \text{on } X \setminus X^{(n-1)} \times (0, T) \\ \lim_{t \to 0} F_\delta(\cdot, t) = F_{0,\delta} &\quad \text{in } C^0, \end{aligned}\right\}$$

where

$${}_\delta A^\gamma_{\alpha\beta} = \sqrt{\delta} A^\gamma_{\alpha\beta}.$$

Additionally, for on $X \setminus X^{(n-1)} \times [0, T)$

$$F_\delta(z, t) = \sqrt{\delta} F(z, t)$$

and

$$e(F_\delta(z, t)) = \delta e(F(z, t)).$$

The converse also holds.

Proof. We compare the flows of maps between X and N, and between X and N_δ. We note that the flow from X to $\iota(N)$ beginning at F_0, denoted F, satisfies on $X \setminus X^{(n-1)}$

in each coordinate $1 \leq \gamma \leq q$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) F^\gamma = \sum_{i,\alpha,\beta} A^\gamma_{\alpha\beta}(F) \frac{\partial F^\alpha}{\partial x_i} \frac{\partial F^\beta}{\partial x_i}, \tag{5.18}$$

where $A^\gamma_{\alpha\beta} := \frac{\partial^2 \pi^\gamma}{\partial z^\alpha \partial z^\beta}$ with $\{z^i\}$, denoting the standard coordinates of $\mathbb{R}^q$, and where π is the nearest-point projection of $\tilde{N}$ to $\iota(N)$. Similarly, the flow from X to $\iota_\delta(N_\delta)$ beginning at $F_{0,\delta}$, denoted F_δ, satisfies on $X \setminus X^{(n-1)}$

$$\left(\frac{\partial}{\partial t} - \Delta\right) F^\gamma_\delta = \sum_{i,\alpha,\beta} {}_\delta A^\gamma_{\alpha\beta}(F_\delta) \frac{\partial F^\alpha_\delta}{\partial x_i} \frac{\partial F^\beta_\delta}{\partial x_i}, \tag{5.19}$$

where ${}_\delta A^\gamma_{\alpha\beta} := \frac{\partial^2 \pi^\gamma_\delta}{\partial z^\alpha \partial z^\beta}$, where π_δ is defined above. We note that the only difference between the equations are the coefficients ${}_\delta A^\gamma_{\alpha\beta}$ and $A^\gamma_{\alpha\beta}$. We note the by the definition of each coefficient and the definitions of π and π_δ, they are related by

$$\begin{aligned} {}_\delta A^\gamma_{\alpha\beta} &:= \frac{\partial^2 \pi^\gamma_\delta}{\partial z^\alpha \partial z^\beta} \\ &= \sqrt{\delta} \frac{\partial^2 \pi^\gamma}{\partial z^\alpha \partial z^\beta} \\ &= \sqrt{\delta} A^\gamma_{\alpha\beta}. \end{aligned}$$

From this observation, the rest of statement follows easily. □

Proposition 5.31. *Let X be simplexwise flat, $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding, and $F\colon X \times [0,T) \to \mathbb{R}^q$ be a strong embedded solution to the heat flow with initial value F_0. Also, let F_0 have bounded energy density. Then, the energy* *density is essentially bounded on* $[0,T)$, $0 < T < \infty$*. Specifically, we have*

$$\operatorname*{ess\,sup}_{X\times(0,T)} e(F(z,t)) \leq C$$

where C *is dependent on* F_0, T, X *and* N*.*

Proof. Our approach is to scale the metric of N and use the previous short-time existence argument of Proposition 5.29 (see page 114), which shows there always exists $\epsilon = \epsilon(\sup e(F_0)) > 0$ such that the essential supremum of the energy of the flow on $[0,\epsilon)$ is bounded. As N is a smooth, compact Riemannian manifold, for each $k \in \mathbb{N}$, there exists a constant C dependent on N and k such that $A^{\gamma}_{\alpha\beta}$ corresponding to the embedding of N in $\mathbb{R}^q$ and all of its derivatives of order k or less are bounded by C. Hence, by the preceding Lemma 5.30, for every $B > 0$ and $k \in \mathbb{N}$, there exists $\delta \in (0,1)$ such that if we scale the metric of N by δ, ${}_{\delta}A^{\gamma}_{\alpha\beta}$ (corresponding to the embedding of N_δ in $\mathbb{R}^q$) and all of its derivatives of order k or less are bounded by B.

We now reexamine the short-time existence of Proposition 5.29. We recall that in Proposition 5.29, we show that there exists a time interval (possibly small) such that a sequence of approximating maps converge in C^1 to a solution on this interval. We also recall some of the definitions. The approximating maps to a strong embedded solution with initial map F_0 are given as

$$\left.\begin{aligned} W^{0,\gamma}(z,t) &= \int_X h(z,v,t)F_0^{\gamma}(v)\,dv,\\ W^{l,\gamma}(z,t) &= \int_0^t \int_X h(z,v,t-\tau)G^{l-1,\gamma}(v,\tau)\,dv\,d\tau + W^{0,\gamma}(z,t).\end{aligned}\right\} \tag{5.20}$$

Let the subscript i denote a derivative in the i^{th} coordinate and hence let

$$W_i^{l,\alpha} := \frac{\partial W^{l,\alpha}}{\partial v^i}.$$

Also, recall

$$e^l(\tau) := \sup_{z\in X}\left(\sum_{\gamma=1}^{q} W_i^{l,\gamma}(z,\tau)W_j^{l,\gamma}(z,\tau)g^{ij}\right)^{\frac{1}{2}}.$$

We have from equation (5.17) in local coordinates

$$W_i^{l,\gamma}(z,t) = \int_0^t \int_X h_i(z,v,t-\tau)G^{l-1,\gamma}(v,\tau)\,dv\,d\tau + W_i^{0,\gamma}(z,t). \tag{5.21}$$

By Proposition 5.29, we have that there exists ϵ dependent on the energy density of the initial map F_0 such that a solution exists with bounded energy density on the time interval $[0,\epsilon)$. Let $|A^{\gamma}_{\alpha\beta}| < B$, and let $C > 0$ be such that $\int_X |h_i(z,v,t)|\,d\mu(v) < Ct^{-1/2}$, the existence of which is guaranteed by Theorem 4.18 and Proposition 4.26. Hence we compute from Equation (5.21),

$$e^l(t) \leq BC\int_0^t (t-\tau)^{-\frac{1}{2}}(e^{l-1}(\tau))^2\,d\tau + e^0(t),$$

for $t \in [0,\epsilon]$. We note that B, C are not dependent on the choice of the initial map. Let

$$\bar{e}^l := \sup_{t\in[0,\epsilon_1]} e^l(t).$$

From the above, we have

$$\bar{e}^l \leq 2BC\sqrt{\epsilon}(\bar{e}^{l-1})^2 + \bar{e}^0.$$

If $2BC\sqrt{\epsilon}\bar{e}^{l-1} \leq \frac{1}{2}$ and $2BC\sqrt{\epsilon}\bar{e}^{0} \leq \frac{1}{4}$, then

$$2BC\sqrt{\epsilon}\bar{e}^{l} \leq \left(2BC\sqrt{\epsilon}\bar{e}^{l-1}\right)^2 + 2BC\sqrt{\epsilon}\bar{e}^{0} \leq \frac{1}{2}.$$

By induction, we have that for all $l \in \mathbb{N}$, $\bar{e}^l \leq (4BC\sqrt{\epsilon})^{-1}$ as long as $\sqrt{\epsilon} \leq (8BC\bar{e}^0)^{-1}$. Hence, as W^l converges to a limit in C^1 as $l \to \infty$, $\bar{e}^l$ converges to $\sup_{X\times[0,\epsilon)} e(F(z,t))$. We have that $\sqrt{\epsilon} \leq (8BC\bar{e}^0)^{-1}$ implies that

$$\sup_{X\times[0,\epsilon)} e(F(z,t)) \leq \frac{1}{4BC\sqrt{\epsilon}}.$$

We can show from the Markov property of Proposition 4.4 and Lemma 4.6 that $\bar{e}^0 \leq e^0(0)$ and, as a consequence of Theorem 4.18 and Proposition 4.26, $e^0(0) = \sup_X e(F_0(z))$. Thus,

$$\sqrt{\epsilon} \leq \frac{1}{8BC\sup_X e(F_0(z))} \implies \sup_{X\times[0,\epsilon)} e(F(z,t)) \leq \frac{1}{4BC\sqrt{\epsilon}}. \tag{5.22}$$

We now show that for any $T > 0$, a solution to the embedded heat flow problem exists on $[0,T)$ for an initial map F_0 with bounded energy density, and this solution has bounded energy density on $[0,T)$. Pick any $T > 0$. By our previous work, there exists $\delta \in (0,1)$ such that the B satisfying $|{}_{\delta}A^{\gamma}_{\alpha\beta}| < B$ can be made arbitrarily small. So, pick δ small so that

$$\sqrt{T} \leq \frac{1}{8BC\sup_X e(F_{0,\delta}(z))},$$

where we note that $\sup_X e(F_{0,\delta}(z))$ is monotonically non-increasing as δ goes to zero.

By Equation (5.22), we find that

$$\sqrt{T} \leq \frac{1}{8BC \sup_X e(F_{0,\delta}(z))} \implies \sup_{X\times[0,T)} e(F_\delta(z,t)) \leq \frac{1}{4BC\sqrt{T}}.$$

We note that by scaling the metric of the target,

$$\sup_{X\times[0,T)} e(F_\delta(z,t)) = \delta \sup_{X\times[0,T)} e(F(z,t))$$

Hence,

$$\sup_{X\times[0,T)} e(F(z,t)) \leq \frac{1}{4\delta BC\sqrt{T}}$$

□

5.5 Schauder Estimates of Elliptic- and Parabolic-type Equations

Although we have provided regularity properties in previous sections, we will find Schauder and L^p estimates of elliptic- and parabolic-type equations useful here and interrupt our exposition to explore them before returning to our examination of long-time solutions to the heat flow. We follow and extend a method proposed in [DM3]. For results for elliptic and parabolic differential equations in smooth domains, see [GT] and [LSU], respectively.

We begin by citing a few fundamental results for elliptic differential equations for smooth domains from [GT].

Definition 5.32 (Hölder norms)**.** Let $\Omega \subset \mathbb{R}^n$ be open, and let $0 < \alpha < 1$. $f : \Omega \to \mathbb{R}$

is *uniformly Hölder continuous with exponent* α, if

$$\sup_{p,q\in\Omega, p\neq q} \frac{|f(p)-f(q)|}{|p-q|^\alpha} \leq \infty.$$

We define the following semi-norms and norms for an integer $k \geq 0$.

$$[f]_{C^\alpha(\Omega)} := \sup_{p,q\in\Omega, p\neq q} \frac{|f(p)-f(q)|}{|p-q|^\alpha}.$$
$$|f|_{C^{k+\alpha}(\Omega)} := \sum_{0\leq|\gamma|\leq k} [D^\gamma f]_{C^\alpha(\Omega)},$$

where γ is a multi-index.

Additionally, for $p, q \in \Omega$ define $d_p := \mathrm{dist}(p, \partial\Omega)$ and $d_{p,q} := \min\{d_p, d_q\}$. For $\rho \in \mathbb{R}$, define

$$[f]^{(\rho)}_{C^k(\Omega)} := \sup_{p\in\Omega, |\gamma|=k} d_p^{k+\rho}|D^\gamma f(p)|$$
$$[f]^{(\rho)}_{C^{k+\alpha}(\Omega)} := \sup_{p,q\in\Omega, |\gamma|=k} d_{p,q}^{k+\alpha+\rho}\frac{|D^\gamma f(p) - D^\gamma f(q)|}{|p-q|^\alpha}$$
$$|f|^{(\rho)}_{C^k(\Omega)} := \sum_{i=0}^{k}[f]^{(\rho)}_{C^i(\Omega)}$$
$$|f|^{(\rho)}_{C^{k+\alpha}(\Omega)} := |f|^{(\rho)}_{C^k(\Omega)} + [f]^{(\rho)}_{C^{k+\alpha}(\Omega)}.$$

See [GT, Theorem 6.2] for the following result.

Proposition 5.33 (Interior elliptic Schauder estimate for classical solutions)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open domain. Let* $u \in C^{2+\alpha}(\Omega)$ *be a classical solution to the elliptic* *equation* $Lu = f$ *for some* $f \in C^\alpha(\Omega)$*, where* L *is the elliptic operator defined by*

$$Lu(z) = \sum_{i,j=1}^{n} a_{ij}(z) D_{ij}u(z) + b_i(z)D_iu(z) + c(z)u(z).$$

with coefficients such that $|a_{ij}|^{(0)}_{C^\alpha(\Omega)}, |b_{ij}|^{(1)}_{C^\alpha(\Omega)}, |c|^{(2)}_{C^\alpha(\Omega)} \leq D < \infty$. *Then*

$$|u|^{(0)}_{C^{2+\alpha}(\Omega)} \leq C \left(|u|_{C^0(\Omega)} + |f|^{(2)}_{C^\alpha(\Omega)} \right).$$

See [GT, Theorem 8.32] for the following result.

Proposition 5.34 (Interior elliptic Schauder estimate for $C^{1+\alpha}$ weak solutions)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open domain. Let* $u \in C^{1+\alpha}(\Omega)$ *be a weak solution to the elliptic equation* $Lu = g + \sum_i D_i f_i$*, where* L *is the strictly elliptic operator defined by*

$$Lu(z) = \sum_{i,j=1}^{n} D_i \left(a_{ij}(z) D_j u(z) + b_i(z)u(z) \right) + c_i(z) D_i u(z) + d(z)u(z),$$

and the coefficients of L*,* f_i*, and* g *are locally integrable, and*

$$\max_{i,j} \left\{ |a_{ij}|_{C^\alpha(\Omega)}, |b_i|_{C^\alpha(\Omega)}, |c_i|_{C^0(\Omega)}, |d|_{C^0(\Omega)} \right\} \leq B.$$

Then for any compactly contained $\Omega' \subset \Omega$*, we have*

$$|u|_{C^{1+\alpha}(\Omega')} \leq C \left(|u|_{C^0(\Omega)} + |g|_{C^0(\Omega)} + \sum_i |f_i|_{C^\alpha(\Omega)} \right)$$

where $C = C\left(n, B, \Lambda, \mathrm{dist}(\partial\Omega, \Omega')\right)$ *and* Λ *is the constant of ellipicity.*

Proposition 5.35 (See [GT, Theorem 9.19])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an open domain. Let*

$u \in W^{2,2}_{\text{loc}}(\Omega)$ satisfy the elliptic equation $Lu = f$ almost everywhere, where L is the elliptic operator defined by

$$Lu(z) = \sum_{i,j=1}^{n} a_{ij}(z) D_{ij} u(z) + b_i(z) D_i u(z) + c(z) u(z).$$

If $a_{ij}, b_i, c, f \in C^{k-1+\alpha}(\Omega)$ for some $k(\in \mathbb{N}) \geq 1$, $0 < \alpha < 1$, then $u \in C^{k+1+\alpha}(\Omega)$.

Additionally, if $\Omega \in C^{k+1+\alpha}$, L is strictly elliptic in Ω and $a_{ij}, b_i, c, f \in C^{k-1+\alpha}(\overline{\Omega})$, then $u \in C^{k+1+\alpha}(\overline{\Omega})$.

Proposition 5.36 (Elliptic L^p Estimate, see [GT, Theorem 9.19])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain. Let $u \in W^{2,2}_{\text{loc}}(\Omega)$ satisfy the elliptic equation $Lu = f$ almost everywhere, where L is as above. If all $(k-1)$-order derivatives of a_{ij}, b_i, c are Lipschitz on Ω and $f \in W^{k,2}_{\text{loc}}(\Omega)$ for some $k(\in \mathbb{N}) \geq 1$, then $u \in W^{k+2,2}_{\text{loc}}(\Omega)$.*

We now use a construction of [DM3] to create a solution to a particular differential equation on a ball in $\mathbb{R}^n$ where we can apply these estimates to arrive at a Schauder-type estimate for harmonic maps in neighborhoods away from $X^{(n-2)}$.

Proposition 5.37. *Let $u \colon X \to N(\subset \mathbb{R}^q)$ be a harmonic map. Then for some $0 < \alpha < 1$, $u \in C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}) \cap W^{2,2}_{\text{loc}}(X \setminus X^{(n-2)})$, and we have the following estimate. For every open $\Omega \subset X \setminus X^{(n-2)}$,*

$$|u|_{C^{1+\alpha}(\Omega')} \leq C \left(|u|_{C^0(\Omega)} + |\Delta u|_{C^0(\Omega)} \right) \tag{5.23}$$

where $C = C(n, B, \Lambda, \text{dist}(\partial\Omega, \Omega'))$, Λ is the constant of ellipicity, and $\Omega' \subset \Omega$ is any compactly contained open subset.

Proof. Without loss of generality, we proceed locally and prove the estimate in a ball around a point p. Obviously for any sufficiently small neighborhood of a point $p \in X \setminus X^{(n-1)}$ this follows from established results. Hence, let $p \in X^{(n-1)} \setminus X^{(n-2)}$. Let $\{s_j\}_{j=1}^{J}$ denote all of the n-simplexes adjacent to p meeting on an $(n-1)$-face F. Define $u_j := u|_{s_j}$. Let $1 \leq \gamma \leq q$. Also let $R > 0$ be such that $B(p,R) \cap X^{(n-2)} = \emptyset$ and $B(p,R) \subset \cup_{j=1}^{J} \overline{s_j}$, and pick edge coordinates centered at p so that for each u_j, $(x_1, \ldots, x_{n-1}, 0)$ denotes points on F and $(0, \ldots, 0)$ denotes p. For each $1 \leq \gamma \leq q$ and $1 \leq k \leq J$, we construct $\overline{u}_k^{\gamma} \colon B(0,R) \to \mathbb{R}$ as

$$\overline{u}_k^{\gamma}(\bar{x}, x_n) := \begin{cases} u_k^{\gamma}(\bar{x}, x_n), & x_n \geq 0 \\ -u_k^{\gamma}(\bar{x}, -x_n) + \dfrac{2}{J} \displaystyle\sum_{j=1}^{J} u_j^{\gamma}(\bar{x}, -x_n), & x_n < 0 \end{cases}$$

where $\bar{x} = (x_1, \ldots, x_{n-1})$. Also, let

$$\phi_k^{\gamma}(\bar{x}, x_n) := \sum_{\alpha,\beta,i} A_{\alpha\beta}^{\gamma}\left(u_k(\bar{x}, x_n)\right) \frac{\partial u_k^{\alpha}}{\partial x_i}(\bar{x}, x_n) \frac{\partial u_k^{\beta}}{\partial x_i}(\bar{x}, x_n),$$

and similarly define

$$\overline{A}_k^{\gamma}(\bar{x}, x_n) := \begin{cases} \phi_k^{\gamma}(\bar{x}, x_n), & x_n \geq 0 \\ -\phi_k^{\gamma}(\bar{x}, -x_n) + \dfrac{2}{J} \displaystyle\sum_{j=1}^{J} \phi_j^{\gamma}(\bar{x}, -x_n), & x_n < 0 \end{cases}$$

Clearly, on $B(0,R) \setminus \{x_n = 0\}$, $\Delta \overline{u}_k^{\gamma} = \overline{A}_k^{\gamma}$ and $\overline{u}_k^{\gamma}$ is C^{∞} on this set. We also see that $\overline{A}_k^{\gamma}(\bar{x}, x_n)$ is bounded on $B(0,R)$. By u harmonic, it satisfies the balancing and matching conditions and we can see that each $\overline{u}_k^{\gamma}$ is a weak solution to

$$\Delta \overline{u}_k^{\gamma} = \overline{A}_k^{\gamma}.$$

From general results in elliptic partial differential equations (see in particular Proposition 5.34), we can show that each $\overline{u}_k^\gamma$ is in $W^{2,2}(B(0,r))$ and $C^{1+\alpha}(B(0,r))$ for any $0 < r < R$. Also, as $\overline{A}_k^\gamma \in C^0(B(0,R))$, we may apply Proposition 5.34 to show

$$|\overline{u}_k^\gamma|_{C^{1+\alpha}(B(0,r))} \leq C\left(|\overline{u}_k^\gamma|_{C^0(B(0,R))} + |\overline{A}_k^\gamma|_{C^0(B(0,R))}\right),$$

where $C = C(X, |R-r|)$ for any $1 \leq \gamma \leq q$ and $1 \leq k \leq J$. Hence, we conclude that for $r < R$,

$$|u|_{C^{1+\alpha}(B(p,r))} \leq C\left(|u|_{C^0(B(0,R))} + \left|\sum_{\alpha,\beta,i} A^\gamma_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_i}\right|_{C^0(B(0,R))}\right).$$

□

Proposition 5.38. *Let $u\colon X \times \mathbb{R}_{\geq 0} \to N(\subset \mathbb{R}^q)$ be a strong solution to the embedded heat flow with initial value f_0. Then for some $0 < \alpha < 1$, $u \in C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}) \cap W^{2,2}_{\text{loc}}(X \setminus X^{(n-2)})$, and we have the following estimate. For every open $\Omega \subset X \setminus X^{(n-2)}$ and each $t \geq 0$,*

$$|u(\cdot,t)|_{C^{1+\alpha}(\Omega')} \leq C\Bigg(|u(\cdot,t)|_{C^0(\Omega)} + \left|\frac{\partial u}{\partial t}(\cdot,t)\right|_{C^0(\Omega)} + \left|\sum_{\alpha,\beta,i} A^\gamma_{\alpha\beta}(u(\cdot,t))\frac{\partial u^\alpha}{\partial x_i}(\cdot,t)\frac{\partial u^\beta}{\partial x_i}(\cdot,t)\right|_{C^0(\Omega)}\Bigg), \quad (5.24)$$

where $C = C\left(n, B, \Lambda, \text{dist}(\partial\Omega, \Omega')\right)$, Λ is the constant of ellipicity, and $\Omega' \subset \Omega$ is any compactly contained subset.

Proof. We can follow the preceding argument for the elliptic case almost verbatim. Indeed, we can treat the parabolic case here by noting that for a parabolic solution, it is elliptic in the sense that it satisfies (strongly)

$$\Delta u^\gamma = \frac{\partial u^\gamma}{\partial t} + \sum_{\alpha,\beta,i} A^\gamma_{\alpha\beta}(u)\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_i},$$

where we treat the entirety of the right-hand side as the inhomogeneous part. Specifically, let $p \in X^{(n-1)} \setminus X^{(n-2)}$. Let $\{s_j\}_{j=1}^J$ denote all of the n-simplexes adjacent to p meeting on an $(n-1)$-face F. Define $u_j := u|_{s_j}$. Let $1 \leq \gamma \leq q$. Also let $R > 0$ be such that $B(p,R) \cap X^{(n-2)} = \emptyset$ and $B(p,R) \subset \cup_{j=1}^J \overline{s_j}$, and pick edge coordinates centered at p so that for each u_j, $(x_1, \ldots, x_{n-1}, 0)$ denotes points on F and $(0, \ldots, 0)$ denotes p. For each $1 \leq \gamma \leq q$ and $1 \leq k \leq J$, we construct $\overline{u}^\gamma_k \colon B(0,R) \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ as

$$\overline{u}^\gamma_k\big((\bar{x}, x_n), t\big) := \begin{cases} u^\gamma_k\big((\bar{x}, x_n), t\big), & x_n \geq 0 \\ -u^\gamma_k\big((\bar{x}, -x_n), t\big) + \dfrac{2}{J}\displaystyle\sum_{j=1}^J u^\gamma_j\big((\bar{x}, -x_n), t\big), & x_n < 0 \end{cases} \tag{5.25}$$

where $\bar{x} = (x_1, \ldots, x_{n-1})$. Also, let

$$\phi^\gamma_k\big((\bar{x}, x_n), t\big) := \sum_{\alpha,\beta,i} A^\gamma_{\alpha\beta}\left(u_k\big((\bar{x}, x_n), t\big)\right) \frac{\partial u^\alpha_k}{\partial x_i}\big((\bar{x}, x_n), t\big) \frac{\partial u^\beta_k}{\partial x_i}\big((\bar{x}, x_n), t\big), \tag{5.26}$$

and similarly define

$$\overline{A}^\gamma_k\big((\bar{x}, x_n), t\big) := \begin{cases} \phi^\gamma_k\big((\bar{x}, x_n), t\big), & x_n \geq 0 \\ -\phi^\gamma_k\big((\bar{x}, -x_n), t\big) + \dfrac{2}{J}\displaystyle\sum_{j=1}^J \phi^\gamma_j\big((\bar{x}, -x_n), t\big), & x_n < 0 \end{cases} \tag{5.27}$$

From here, we proceed as in the elliptic case (Proposition 5.37) and apply Proposition 5.34 as before. □

We require a parabolic version of this elliptic result to get Hölder continuity with respect to time. This will be important to show that the limit map for the heat flow as time goes to infinity is balanced. We first introduce relevant norms.

Definition 5.39 (Hölder norms with time)**.** Let $\Omega \subset \mathbb{R}^n$ be open, $T > 0$, and define $Q_T := \Omega \times (0, T)$. We define the following semi-norms and norms for $0 < \alpha < 1$. For continuous $f\colon Q_T \to \mathbb{R}$ define

$$
\begin{aligned}
[f]_{C_x^\alpha(Q_T)} &:= \sup_{(p,t),(q,s)\in Q_T, p\neq q} \frac{|f(p) - f(q)|}{|p-q|^\alpha} \\
[f]_{C_t^\alpha(Q_T)} &:= \sup_{(p,t),(q,s)\in Q_T, t\neq s} \frac{|f(p) - f(q)|}{|t-s|^\alpha} \\
|f|_{C^\alpha(Q_T)} &:= [f]_{C_x^\alpha(Q_T)} + [f]_{C_t^{\alpha/2}(Q_T)}.
\end{aligned}
$$

We additionally define the norms for $j \in \mathbb{N}$

$$
\begin{aligned}
[f]_{C^0(Q_T)} &:= |f|_{C^0(Q_T)} = \sup_{(p,t)\in Q_T} |f(p,t)| \\
[f]_{C^j(Q_T)} &:= \sum_{2k+l=j} \left[\left(\tfrac{\partial}{\partial t}\right)^k \left(\tfrac{\partial}{\partial x}\right)^l \ f \right]_{C^0(Q_T)} \\
|f|_{C^j(Q_T)} &:= \sum_{i=1}^{j} [f]_{C^j(Q_T)},
\end{aligned}
$$

where $\left(\frac{\partial}{\partial x}\right)^l = \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_l}}$, and $\left(\frac{\partial}{\partial t}\right)^k$ denotes differentiation with respect to t applied k times.

Finally, for Hölder norms with exponent $j + \alpha$, where $j \in \mathbb{N}$ and $0 < \alpha < 1$, we have

$$[f]_{C_x^{j+\alpha}(Q_T)} := \sum_{2k+l=j} \left[\left(\tfrac{\partial}{\partial t}\right)^k \left(\tfrac{\partial}{\partial x}\right)^l \ f \right]_{C_x^{\alpha}(Q_T)}$$

$$[f]_{C_t^{j+\alpha}(Q_T)} := \sum_{0<2j+2\alpha-2r-s<2} \left[\left(\tfrac{\partial}{\partial t}\right)^r \left(\tfrac{\partial}{\partial x}\right)^s \ f \right]_{C_t^{\frac{2j+2\alpha-2r-s}{2}}(Q_T)}$$

$$[f]_{C^{j+\alpha}(Q_T)} := [f]_{C_x^{j+\alpha}(Q_T)} + [f]_{C_t^{(j+\alpha)/2}(Q_T)}$$

$$|f|_{C^{j+\alpha}(Q_T)} := [f]_{C^{j+\alpha}(Q_T)} + \sum_{i=0}^{j} [f]_{C^i(Q_T)}.$$

We can now proceed stating a classical result of [LSU].

Proposition 5.40 (See [LSU, Theorem 1.1, Chapter VI]). *Let Ω be an open domain in $\mathbb{R}^n$, $Q_T := \Omega \times (0,T)$ for fixed $T > 0$, and let $u \colon Q_T \to \mathbb{R}$ be a weak solution to the equation*

$$\left(\frac{\partial}{\partial t} - \Delta \right) u = f(x,t,u,Du),$$

such that u is continuous in Q_T, $\frac{\partial}{\partial t} u \in L^2(Q_T)$, and first-order derivatives of u are bounded in Q_T. Then, first-order derivatives of u are Hölder continuous on Q_T and, for any compactly contained $Q' \subset Q_T$, there exists $\alpha \in (0,1)$ and $C > 0$ such that

$$[Du]_{C^\alpha(Q')} \leq C,$$

where α is dependent on $\sup_{Q_T} |Du|$, n and $\sup_{Q_T} |f(x,t,u,Du)|$, and C is additionally dependent on $\operatorname{dist}(Q', \partial Q_T)$.

Additionally, if $|Du(\cdot,0)|_{C^\beta(\Omega)}$ is bounded, then for any compactly contained $\Omega' \subset \Omega$ ($Q'_T := \Omega' \times (0,T)$),

$$[Du]_{C^\alpha(Q'_T)} \leq C,$$

where C is dependent on $\sup_{Q_T}|Du|$, n, $\sup_{Q_T}|f(x,t,u,Du)|$, $|Du(\cdot,0)|_{C^\beta(\Omega)}$, $\operatorname{dist}(\partial\Omega,\Omega')$, *and* $\beta(\geq\alpha)$.

The result cited in [LSU] is more general that what is stated here, but we select only what we need for subsequent results. We note that if $u(\cdot,0)$ has Hölder continuous first-order derivatives and $\sup_{Q_T}|Du|$ and $\sup_{Q_T}|f(x,t,u,Du)|$ are bounded uniformly for all time, the constants C and α do not depend on T. This will be a consideration in Proposition 5.45 later.

From [LSU], we also have the following standard result for solutions with higher regularity.

Proposition 5.41. *Let Ω be an open domain in $\mathbb{R}^n$, $Q_T := \Omega\times(0,T)$ for fixed $T>0$, and let $u\in C^{2,1}(Q_T)$ be a solution to the equation*

$$\left(\frac{\partial}{\partial t}-\Delta\right)u=f,$$

where $f\in C^{\alpha,\alpha/2}(Q_T)$, $0<\alpha$. If $u(\cdot,0)\in C^{2+\beta}(\Omega)$ for some $\beta\geq\alpha$, then for any compactly contained $\Omega'\subset\Omega$ ($Q'_T := \Omega'\times(0,T)$), there exists $C>0$ such that

$$[u]_{C^{2+\alpha}(Q'_T)}\leq C\left([f]_{C^\alpha(Q_T)}+[u]_{C^0(Q_T)}\right),$$

where C is dependent only on $\beta(\geq\alpha)$, n and $\operatorname{dist}(\partial\Omega,\Omega')$.

Again, this result holds in greater generality than what is stated here. In particular, instead of the heat operator, the operator may be any parabolic operator with sufficiently smooth coefficients.

Along the lines of Proposition 5.38, we have by the same construction the following.

Proposition 5.42. *Let $u\colon X \times \mathbb{R}_{\geq 0} \to N(\subset \mathbb{R}^q)$ be a strong solution to the embedded heat flow with initial value f_0 such that $f \in C^1(X, \mathbb{R}^q) \cap C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}, \mathbb{R}^q)$. Then $u \in C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}, \mathbb{R}^q)$, and we have the following estimate. For every open $\Omega \subset X \setminus X^{(n-2)}$ and each $0 < T < \infty$,*

$$|u(\cdot, t)|_{C^{1+\alpha}(Q'_T, \mathbb{R}^q)} \leq C,$$

where $\Omega' \subset \Omega$ is compactly contained, $Q_T := \Omega \times (0, T)$, $Q'_T := \Omega' \times (0, T)$, and C is dependent on $X, N, f, \alpha, \operatorname{dist}(\partial\Omega, \Omega')$, $|u|_{C^0(Q_T)}$, $|Du|_{C^0(Q_T)}$, $|Df_0|_{C^\beta(\Omega)}$, and $\beta(\geq \alpha)$.

Proof. The proof is nearly identical to the proof of Proposition 5.38 which, instead, relies on Proposition 5.40. In particular, we note that the term $f(x, t, u, Du)$ in the statement of Proposition 5.40 will correspond to the terms $\overline{A}^\gamma_k(z, t)$ of the proof of Proposition 5.38. Hence, we note that C in the statement of this proposition is dependent on $|Du|_{C^0(Q_T)}$. By Proposition 5.31, first-order derivatives in space are bounded on bounded time intervals, so $|Du|_{C^0(Q_T)}$ is bounded for finite T, as is $|u(\cdot, t)|_{C^{1+\alpha}(Q'_T, \mathbb{R}^q)}$. □

We note that the differences between Propositions 5.38 and 5.42 are slight, but the latter does not require a bound on the time derivative of the solution and additionally gives a bound on the Hölder continuity of a solution with respect to time. Although the bound of the first result does not depend on time, we note that the second result does.

5.6 Long Term Existence and Convergence to a Harmonic Map

We have show that, given an initial map f_0 that is $C^1(X,N)$ with bounded energy density, there exists some $T > 0$ dependent on $\sup e(f_0)$ such that a solution to the heat flow exists on $[0,T)$ with good regularity properties. Our goal now is to show that the solution can be extended to $[0,T]$. If $f(\cdot,T)$ is continuous with bounded energy density, this gives us that there exists a solution to the initial value problem on $[0,\infty)$, as we can always extend the flow past T.

The simplest place to start is the precompactness theorem of [EF] (see also [KS]): given a sequence of maps in $W^{1,2}(X,N)$ with uniformly bounded energy, there exists a subsequence that converges in $L^2(X,N)$ to a map in $W^{1,2}(X,N)$. Clearly, this applies here, as Proposition 5.27 implies that any sequence of $W(\cdot,t_\alpha)$, $t_\alpha \to T$, has bounded (global) energy. This does not give us a useful limit map, as we want our limit map to have bounded energy density and some regularity. Also, to use this precompactness result, we must appeal to subsequences of $(0,T)$ that converge to T, which is undesirable as we wish to have convergence for *every* subsequence of $(0,T)$ that converges to T. Formally, we show the following for the embedded problem.

Proposition 5.43. *Let X be compact and simplex-wise flat and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in $C^1(X)$ and have bounded energy density. Let W be a strong embedded solution with initial value F_0 on $X\times[0,T)$ for some $T \in (0,\infty)$. Then as $t \to T$, $W(\cdot,t)$ converges uniformly to a continuous map that is in $C^1(X)$ with bounded energy density, and for any open set $A \subset X$ such that $d(A, X^{(n-2)}) > 0$, $W(\cdot,t)|_A$ converges in C^1 to a limit in $C^{1+\alpha}(A)$.*

Proof. Our main tool is the Arzelà-Ascoli theorem. We note that for a smooth compact Riemannian manifold M, possibly with boundary, we can use the mean value theorem and Arzelà-Ascoli theorem to show that if there is a sequence of functions that are uniformly bounded with uniformly bounded derivatives, then there is a subsequence that converges uniformly to a continuous function on M. To apply this principle to maps from X to N, we can either appeal to local coordinates in N and prove convergence locally in each coordinate or we may isometrically embed N in $\mathbb{R}^q$ and show convergence in each coordinate. Here, we prefer the latter approach, as it is set up in the statement of this proposition and, by abuse of notation, use W to denote the γ^{th} coordinate function, W^γ, where $1 \leq \gamma \leq q$.

We note that by Propositions 5.15 and 5.29, we have for any open $A \subset X$ such that $d(A, X^{(n-2)}) > 0$, $W \in C^{1+\alpha,1+\beta}(\overline{A} \times (0,T), N)$, for some $\alpha, \beta > 0$. Hence, by the Arzelà-Ascoli theorem on W and its spacial derivatives, we can say there exists a subsequence $t_i \to T$ such that $W(\cdot, t_i)|_{\overline{A}}$ converges to a limit in C^1 on $\overline{A}$. By the same proposition, on $\overline{A} \times (0,T)$, we note that $\frac{\partial}{\partial t} W(\cdot, t)$ must be bounded. Hence, *any* sequence of maps $W(\cdot, t)$ indexed by t must converge on $\overline{A}$ to a unique limit as $t \to T$.

Now, for an n-simplex, s, consider $W(\cdot, t)|_{\overline{s}}$. As we know that for each $t \in (0,T)$, $W(\cdot, t)|_{\overline{s}} \in C^1(\overline{s})$ and that for each $t \in (0,T)$, $W(\cdot, t)|_{\overline{s}}$ and its first order derivatives are uniformly bounded (this follows from Proposition 5.31), there exists a sequence $\{t_i\}$ whose limit is T such that $W(\cdot, t_i)|_{\overline{s}}$ converges uniformly to a continuous map. We claim that this map is unique. Indeed, assume that for two distinct sequences $\{t_i\}, \{t'_i\} \subset (0,T)$ that converge to T, $W(\cdot, t_i)|_{\overline{s}}$ and $W(\cdot, t'_i)|_{\overline{s}}$ converge to two distinct limits, say $W(\cdot, T)|_{\overline{s}}$ and $W'(\cdot, T)|_{\overline{s}}$ respectively (which may possibly be distinct from the one in the previous paragraph). By our previous argument, they must agree on $X \setminus X^{(n-2)}$. Hence they can only be distinct on $X^{(n-2)}$. Let $z_0 \in X^{(n-2)}$ be such that $W(z_0, T)|_{\overline{s}} \neq W'(z_0, T)|_{\overline{s}}$. Then, as $W(\cdot, T)|_{\overline{s}}$ and $W'(\cdot, T)|_{\overline{s}}$ are continuous, they must be distinct in an open neighborhood of z_0, which is impossible. Hence for any sequence in $(0, T)$ converging to T, there exists a subsequence $\{t_i\}$ such that $W(\cdot, t_i)|_{\overline{s}}$ goes uniformly to a unique continuous limit $W(\cdot, T)|_{\overline{s}}$. We can repeat this process for each n-simplex, s, to build a function from all of X such that for any sequence in $(0, T)$ that converges to T there exists a subsequence $\{t_i\}$ where $W(\cdot, t_i)$ converges uniformly to a unique continuous limit $W(\cdot, T)\colon X \to \mathbb{R}$. We still must show that for *any* sequence $\{t_i\}$ going to T, $W(\cdot, t_i)$ converges uniformly to a unique continuous limit with bounded energy density, $W(\cdot, T)$.

Recall that any bounded sequence of continuous functions on a compact domain that converges pointwise to a continuous function must converge uniformly. As we know that $W(\cdot, T)$ is continuous on X and any sequence of functions $W(\cdot, t_i)$ indexed by $i \to \infty$ is bounded, all we must show is that for any sequence in t going to T, $W(\cdot, t_i)$ converges pointwise. We proceed by contradiction. Suppose there is a sequence $\{t_i\}$ such that $W(\cdot, t_i)$ does *not* converge pointwise to $W(\cdot, T)$, defined as above. Clearly, pointwise convergence must hold on $X \setminus X^{(n-2)}$, so the only possibility is that pointwise convergence fails on $X^{(n-2)}$. Suppose there is a point z_0 such that either

$$\lim_{t_i \to T} W(z_0, t_i) \neq W(z_0, T)$$

or this limit does not exist. Either case produces a contradiction. Given our work above, given a sequence $\{t_i\}$ converging to T, there exists a subsequence $\{t_i'\}$ also converging to T such that $\{W(\cdot, t_i)\}$ converges uniformly to the unique continuous limit map $W(\cdot, T)$. Hence, if there is a sequence $\{t_i\}$ such that $\lim_{t_i \to T} W(z_0, t_i) \neq W(z_0, T)$, there is a subsequence $\{t_i'\} \subset \{t_i\}$ where $\lim_{t_i' \to T} W(z_0, t_i) = W(z_0, T)$, which is absurd. In the second case, we assume that $\lim_{t_i \to T} W(z_0, t_i)$ does not exist. This is also not possible: we claim that $\{W(z_0, t_i)\}$ has a single accumulation point equal to $W(z_0, T)$, which is sufficient. Obviously, the sequence $\{W(z_0, t_i)\}$ is bounded, so it must have a least one accumulation point. By our work above, one accumulation point must be $W(z_0, T)$. Suppose that $\{W(z_0, t_i)\}$ has another accumulation point not equal to $W(z_0, T)$, and let $\{t_i'\} \subset \{t_i\}$ be the subsequence such that $\{W(z_0, t_i')\}$ converges to it. Again, by the work above, there must be a subsequence $\{t_i''\} \subset \{t_i'\}$ such that $\{W(z_0, t_i'')\}$ converges to $W(z_0, T)$, which is absurd.

By Proposition 5.31, we can also show that our limit map $W(\cdot, T)$ must have bounded energy density. Also by $W(\cdot, t)$ being balanced for all $t \in (0, T)$ and by the regularity shown above, it can be shown that $W(\cdot, T)$ is balanced. $\square$

Proposition 5.44. *Let X be compact and simplex-wise flat, and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in $C^1(X)$ and have bounded energy density. Then there exists a strong embedded solution W with initial value F_0 on $X \times [0, \infty)$ with energy density bounded on compact time intervals.*

Proof. By Proposition 5.29, we have for F_0 so defined, the existence of a $T > 0$ such that a strong solution to the embedded heat flow problem with initial map F_0 exists on $X \times [0, T)$. Let T be the largest possible real number for which such a solution W exists on $X \times [0, T)$ By Proposition 5.43, we have that $\lim_{t \to T} W$ exists, is continuous, has bounded energy density and for every open $A \subset X$ such that $d(A, X^{(n-2)}) > 0$, $W(\cdot, T) \in C^{1+\alpha}(\overline{A})$. Hence we can apply Proposition 5.29 again to $W(\cdot, T)$ and extend the flow to $X \times [0, T')$, $T' > T$. Thus, $T = \infty$. We can thus apply Proposition 5.31

to show boundedness of the energy density on compact time intervals. $\square$

Proposition 5.45. *Let X be compact and simplex-wise flat and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in $C^1(X)$ and have bounded energy density. Let W be a strong embedded solution with initial value F_0 on $X \times [0,\infty)$. Then there exists a sequence $\{t_i\}$ tending to infinity such that $W(\cdot, t_i)$ converges uniformly and in $W^{1,2}$ as i goes to infinity to a harmonic map. Additionally, if for any open set $A \subset X \setminus X^{(n-2)}$ with $d(A, X^{(n-2)}) > 0$ $E(W(\cdot,t_i)|_A)$ remains bounded (where the bound may depend on A), then $W(\cdot,t_i)$ converges in $C^1_{loc}(X \setminus X^{(n-2)})$ to a limit in $C^{1+\alpha}_{loc}(X \setminus X^{(n-2)})$. In particular, if the energy density of $W(\cdot,t_i)$ remains bounded, then $W(\cdot,t_i)$ converges in $C^1_{loc}(X \setminus X^{(n-2)})$ to a limit in $C^{1+\alpha}_{loc}(X \setminus X^{(n-2)})$.*

Proof. We break our proof into two cases. In the first case, we presume that the supremum of the energy density remains bounded as time goes to infinity. In the second, we presume that the supremum of the energy density could be unbounded; in this case, we show that the solution remains converges to a limit as time goes to infinity in $C^0(X) \cap W^{1,2}(X)$.

Case 1: $\sup_{X\times[0,\infty)} e(F(z,t)) < \infty$.

We can replicate most of the argument of Proposition 5.43, however we note that in Proposition 5.43, we only had to show convergence on a set with compact closure. This was guaranteed by the finiteness of time (i.e. only for $T < \infty$). Here we must show convergence for some sequence in $[0,\infty)$ going to infinity. For simplicity, we fix a coordinate $1 \le \gamma \le q$ in the target, and we can apply the same analysis of Proposition 5.43 to show that for some sequence of $\{t_i\}$ tending to infinity $W^\gamma(\cdot,t_i)$ converges in $C^0(X)$ to a continuous limit as i goes to infinity.

We wish to improve our regularity of this convergence. We now show for some subsequence of $\{t_i\}$ (which we shall continue to denote $\{t_i\}$) $\{W^\gamma(\cdot, t_i)\}$ converges in $C^1_{\text{loc}}(X \setminus X^{(n-2)})$ to a limit in $C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)})$. Let $\Omega' \subset X \setminus X^{(n-2)}$ be a compactly contained open set. Without loss of generality, we presume that Ω' is entirely contained in the union of the closure of a set of n-simplexes with a common $(n-1)$-face. Hence, we can find an open set Ω such that $\overline{\Omega} \subset X \setminus X^{(n-2)}$ and $\Omega' \subset \Omega$ is compactly contained. We can apply Proposition 5.42, to find that

$$|W^\gamma(\cdot, t)|_{C^{1+\alpha}(Q'_T)} \leq C, \tag{5.28}$$

where $Q_T := \Omega \times (0, T)$, $Q'_T := \Omega' \times (0, T)$, and C is dependent on X, N, α, $\text{dist}(\partial\Omega, \Omega')$, $|W^\gamma|_{C^0(Q_T)}$, $|DW|_{C^0(Q_T),N}$, and $|DF_0|_{C^\beta(\Omega,N)}$. By Proposition 5.31 and the compactness of N, we know that all of the terms on which C is dependent are uniformly bounded for all $t \in [0, \infty)$, so (5.28) holds for $T = \infty$. Hence, by the Arzéla-Ascoli theorem, there is a subsequence of $\{t_i\}$ tending to infinity such that $\{W^\gamma(\cdot, t_i)|_{\Omega'}\}$ converges in C^1 to a limit in $C^{1+\alpha}(\Omega')$. As the energy density is uniformly bounded in for all time as given in Proposition 5.31, we can extend this result to show for all X there is a limit map in $C^1(X)$.

For let us denote the sequence $\{t_i\}$ above as $\mathcal{T}$ and let $W_\mathcal{T}$ denote the limit map in $C^1(X, N) \cap C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}, N)$. We note that this limit has not been proven to be unique, nor have we proven that any sequence in t tending to infinity must correspond to a convergent sequence of maps. We now prove that $W_\mathcal{T}$ must be harmonic. Per Definition 5.1 and Proposition 5.3, we must show that $W_\mathcal{T}$ is balanced and weakly satisfies the harmonic map equation; we already know it is continuous. We begin by showing that at any point in $X \setminus X^{(n-1)}$, $W_\mathcal{T}$ satisfies the embedded harmonic map equation

$$\Delta W_{\mathcal{T}}^{\gamma} + \sum_{\alpha,\beta,i} A_{\alpha\beta}^{\gamma}(W_{\mathcal{T}}) \frac{\partial W_{\mathcal{T}}^{\alpha}}{\partial x^i} \frac{\partial W_{\mathcal{T}}^{\beta}}{\partial x^i} = 0$$

for each coordinate γ. Pick a point $p \in X \setminus X^{(n-1)}$ and let $\Omega \subset X \setminus X^{(n-1)}$ be a connected, compactly contained neighborhood about p. By definition of the flow, we know that

$$\Delta W^{\gamma} + \sum_{\alpha,\beta,i} A_{\alpha\beta}^{\gamma}(W) \frac{\partial W^{\alpha}}{\partial x^i} \frac{\partial W^{\beta}}{\partial x^i} = \frac{\partial W^{\gamma}}{\partial t}$$

pointwise on $X \setminus X^{(n-1)} \times [0, \infty)$. We show that $\frac{\partial W^{\gamma}}{\partial t}$ must go to zero uniformly on Ω' as t goes to infinity, where $\Omega' \subset \Omega$ is any compactly contained subset. Let $Q_T := \Omega \times (0, T)$ $Q'_T := \Omega' \times (0, T)$, $T > 0$. By Proposition 5.41, we have

$$[W]_{C^{2+\alpha}(Q'_T)} \leq C \left(\left[\sum_{\alpha,\beta,i} A_{\alpha\beta}^{\gamma}(W) \tfrac{\partial W^{\alpha}}{\partial x^i} \tfrac{\partial W^{\beta}}{\partial x^i} \right]_{C^{\alpha}(Q_T)} + [u]_{C^0(Q_T)} \right),$$

where C is dependent only on α, X, N, and $\mathrm{dist}(\partial\Omega, \Omega')$. We note that by (5.28), the right-hand terms are bounded uniformly in for all $t \in [0, \infty)$. Hence, for some sequence $\{t_i\}$ tending to infinity, $\frac{\partial}{\partial t} W^{\gamma}$ and all second order spacial derivatives must converge uniformly on Ω' to a continuous limit. We note that by Proposition 5.27, $\frac{\partial}{\partial t} W^{\gamma}$ must go to zero almost everywhere, so we have for any $p \in \Omega'$

$$\begin{aligned} \lim_{t_i \to \infty} \frac{\partial W}{\partial t}(p, t_i) &= \lim_{t_i \to \infty} \Delta W^{\gamma}(p, t_i) + \sum_{\alpha,\beta,i} A_{\alpha\beta}^{\gamma}(W(p, t_i)) \frac{\partial W^{\alpha}}{\partial x^i} \frac{\partial W^{\beta}}{\partial x^i}(p, t_i) \\ &= \frac{\partial W_{\mathcal{T}}}{\partial t}(p) \\ &= 0. \end{aligned}$$

To show that $W_{\mathcal{T}}$ is balanced, we appeal to the method of Proposition 5.38.

Specifically, we refer to equations (5.25), (5.26) and (5.27) of Proposition 5.38. As balancing is a local condition, we examine it at a point $p \in X^{(n-1)} \setminus X^{(n-2)}$. According to Proposition 5.38, given a solution to the heat flow W near p, for some $0 < R < d(p, X^{(n-2)})$ we can develop functions $\overline{u}_j^\gamma \colon B(0,R)(\subset \mathbb{R}^n) \times [0,\infty) \to \mathbb{R}$ for each $1 \leq \gamma \leq q$ and $1 \leq j \leq J$, and for each is in $C^{1+\alpha}(B(0,r) \times [0,\infty))$. By Proposition 5.34 and the bounds on the nonhomogeneous terms, we can show for each function that, as $\{t_i\}$ goes to infinity, there is a limit in $C^{1+\alpha}(B(0,r) \times [0,\infty))$, for some $r < R$. denote the limit of each function restricted to $B(0,r)$, $\overline{u}_{j,\mathcal{T}}^\gamma \colon B(0,r) \to \mathbb{R}$. As $\overline{u}_{j,\mathcal{T}}^\gamma$ is in $C^{1+\alpha}(B(0,r))$, then by definition of $\overline{u}_{j,\mathcal{T}}^\gamma(\bar{x}, x^n)$ from Equation (5.25), we must have

$$\frac{\partial}{\partial x^n}\bigg|_{x^n=0} \left(\frac{2}{J} \sum_{j=1}^{J} u_{j,\mathcal{T}}^\gamma\big((\bar{x}, x^n)\big) \right) = 0,$$

which means that our limit map $W_{\mathcal{T}}$ must be balanced at p and at all points on $X^{(n-1)} \setminus X^{(n-2)}$ near p.

Hence, there exists a sequence $\{t_i\}$ tending to infinity such that $W(\cdot, t_i)$ converges in C^1 to a limit in $C^1(X,N) \cap C^{1+\alpha}_{\text{loc}}(X,N)$ that is balanced and solves the harmonic map equation pointwise at all manifold points. Hence, by Proposition 5.3, the limit is a harmonic map.

Case 2: $\sup_{X \times [0,\infty)} e(F(z,t))$ is unbounded.

In this case, we appeal to the results of [Ma], we note that by Proposition 5.26, the harmonic map heat flow defined here is the same as the Gradient-of-Energy flow defined by Mayer. Thus by Proposition 5.27, as time goes to infinity, $F(\cdot, t)$ converges in $W^{1,2}(X)$ to a harmonic map. All of the local regularity results away from $X^{(n-2)}$ from *Case 1* apply here. To see that is must be uniform, we appeal to the proof of [EF, Theorem 11.1], which shows that the limit of a energy-minimizing sequence of continuous maps in the same homotopy class is a Hölder continuous harmonic map also in the same homotopy class.

We note that we have proven our auxiliary claims during the course of proving *Case 1.* □

Corollary 5.46. *Every map $f\colon X \to N$ in $C^1(X)$ is free-homotopic to a harmonic map.*

Proof. This follows immediately from the preceding proposition. If we isometrically embed N into $\mathbb{R}^q$ by ι and begin the heat flow W with initial data $F_0 := \iota \circ f$, there exists a sequence $\{t_i\}$ tending to infinity such that $\{W(\cdot, t_i)\}$ converges uniformly to a harmonic map. As the heat flow is continuous with respect to t, each $W(\cdot, t_i)$ is homotopic to each other and also to the limit map, which is harmonic. □

We of course wish to address the issue of whether or not *every* sequence in t tending to infinity corresponds to a sequence of maps converging to a harmonic map. Such results exist for maps between manifolds, but the approaches of [ES, N] do not immediately appear to lend themselves to our present situation. As we have show in Proposition 5.26 that for certain initial maps the harmonic map heat flow coincides the the Gradient-of-Energy flow, we instead appeal to [Ma], which gives a very general result that applies here.

Theorem 5.47. *Let X be compact and simplex-wise flat and let $\iota\colon N \hookrightarrow \mathbb{R}^q$ be a smooth isometric embedding. Let $F_0\colon X \to \iota(N) \subset \mathbb{R}^q$ be in $C^1(X)$ and have bounded energy density. Let W be a strong embedded solution with initial value F_0 on $X \times [0, \infty)$. Then $\lim_{t\to\infty} W(\cdot, t)$ exists and it is a harmonic map.*

Proof. This follows directly from Propositions 5.26 and 5.27. □

We have as a consequence of our convergence arguments, a confirmation of a result of [DM3].

Proposition 5.48. *Let X be a flat compact Riemannian polyhedron of dimension n ($n \geq 2$), and let N be a complete smooth Riemannian manifold. Let $f\colon X \to N$ be harmonic. Then, f has the balancing condition and $f \in C^{1+\alpha}_{\text{loc}}(X \setminus X^{(n-2)}, N)$.*

Remark 5.49. We note again that the result in [DM3] is actually more general than stated here, as they consider harmonic maps in the context of admissible weights, which we do not consider here. Additionally, they prove that when the dimension of the domain is 2, then a harmonic maps is in $C^{\infty}_{\text{loc}}(X \setminus X^{(n-2)}, N)$.

5.7 Topology, Energy and Harmonic Maps on 2-Dimensional Domains

The results of Eells and Sampson in [ES] regard heat flows and harmonic maps between compact Riemannian manifolds where the domain is without boundary and the target has nonpositive sectional curvature. In this case, if there exists smooth maps between these spaces, each can be deformed by a heat flow to a harmonic map (i.e. a map with a vanishing torsion field) such that first order derivatives converge. By considering the smoothness of a harmonic map in this setting, the smoothness of the metrics on the domain and target, and the compactness of the domain, one can show that such a harmonic map must have bounded pointwise energy.

However, it is easy to see that these arguments do not apply to harmonic maps between an admissible n-complex, X, and a compact Riemannian manifold, N, with nonpositive sectional curvature. Although we do have a version of parabolic and elliptic maximum principles (see Lemma 3.8 on page 42; note that an elliptic maximum principle follows easily), it simply states that maximums for subsolutions cannot occur on $X \setminus X^{(n-2)}$. Indeed, it appears that energy can concentrate near $X^{(n-2)}$ distributionally. That is, for a harmonic map f, $e(f)$ is in $L^1(X)$, but $e(f)$ need not be bounded, and for any open, bounded region $\Omega \subset X$ such that $d(\Omega, X^{(n-1)}) > 0$, $e(f)$ will be bounded on Ω.

In certain cases, whether or not $e(f)$ is bounded on X is dependent on the topology of X. In [DM3], the case of harmonic maps between an admissible 2-complex with simplex-wise flat curvature and a Riemannian manifold, N, with nonpositive sectional curvature is considered. They give criteria dependent on the topology near vertices to determine for a harmonic maps f when $e(f)$ is bounded, and give some consequence for when it holds. Specifically, they give the following:

Proposition 5.50 (from [DM3, Theorem 4])**.** *Let X be an admissible 2-complex with a simplex-wise flat metric, let N be a manifold with nonpositive sectional curvature and let $f\colon X \to N$ be an energy minimizing map. Let $p_0 \in X$. Also, let $\alpha(p_0) = \mathrm{Ord}^f(p_0)$ denote the order of f at $p_0 \in X$ (see* [DM3, Section 4] *for definition). Then there exists $\sigma > 0$ such that for all $q \in B(p_0, \sigma)$,*

$$e(f(q)) \leq C \cdot d(q, p_0)^{2\alpha(p_0)-2},$$

where $C > 0$ depends on only on $E(f)$ and σ.

Proposition 5.51 (See [DM3, Corollary 14])**.** *Let X be an admissible 2-complex with a simplex-wise flat metric, let N be a Riemannian manifold with nonpositive sectional curvature. Let $f\colon X \to N$ be energy minimizing, and let $\alpha(p) = \mathrm{Ord}^f(p)$ denote the* *order of f at $p \in X$ (see* [DM3, Section 4] *for definition). Also, fix $p_0 \in X^{(0)}$ and let $\lambda(\mathrm{Lk}^{(0)}(p_0), T_{f(p_0)}N)$ denote the discrete eigenvalue as in* [DM3, Prop. 13]. *Then,*

$$\lambda(\mathrm{Lk}^{(0)}(p_0), T_{f(p_0)}N) > (\geq)\frac{1}{2} \Rightarrow \alpha(p_0) > (\geq) 1.$$

For balanced, energy minimizing maps, we can also show by slightly modified arguments of [ES] or [N] that there exists a Bochner-type formula as follows.

Proposition 5.52. *Let X be an admissible n-complex ($n \geq 2$) with a smooth simplex-wise metric g, let N be a Riemannian manifold with nonpositive sectional curvature. Let f be a balanced, energy minimizing map that is in $C^2_{loc}(X \setminus X^{n-1}, N)$. Then,*

$$\begin{aligned}\Delta_g e(f) = |Ddf|^2 + \sum \left\langle df\left(\sum \mathrm{Ric}_g(e_i, e_j)e_j\right), df(e_i)\right\rangle_{f^{-1}(TN)} \\ - \sum_{i,j=1}^{m} \left\langle R^N\left(df(e_i), df(e_j)\right) df(e_j), df(e_i)\right\rangle_{f^{-1}(TN)}.\end{aligned}$$

Remark 5.53. We note that in [DM3], it is proven that energy minimizing maps are necessarily balanced. So the requirement of the previous proposition is redundant.

Given an assumption of flatness on the domain and nonpositive sectional curvature on the target, we have that the energy density is subharmonic on $X \setminus X^{(n-1)}$ by the Bochner formula above. This combined with Proposition 5.50 and an elliptic maximum principle for balanced functions, analogous to the parabolic one in Lemma 3.8, gives us the following result.

Proposition 5.54. *Let X be an admissible 2-complex with a simplex-wise flat metric, let N be a Riemannian manifold with nonpositive sectional curvature. Let $f\colon X \to N$ be energy minimizing. We have the following:*

i. If for all $p \in X^{(0)}$, $\mathrm{Ord}^f(p) > 1$, *then* f *is trivial.*

ii. If for all $p \in X^{(0)}$, $\mathrm{Ord}^f(p) \geq 1$, *then* f *is totally geodesic.*

iii. If for all $p \in X^{(0)}$, $\mathrm{Ord}^f(p) \geq 1$ *and additionally* N *has strictly negative sectional curvature, then* f *maps each* 2*-simplex into a geodesic on* N*.*

By the noted relationship between discrete eigenvalues and the order of energy minimizing maps at points in $X^{(0)}$, we have the following.

Proposition 5.55. *Let* X *be an admissible* 2*-complex with a simplex-wise flat metric, let* N *be a Riemannian manifold with nonpositive sectional curvature. Let* $f\colon X \to N$ *be energy minimizing. We have the following:*

i. If for all $p \in X^{(0)}$, $\lambda(\mathrm{Lk}^{(0)}(p), T_qN) > \frac{1}{2}$ *for all* $q \in N$, *then* f *is trivial.*

ii. If for all $p \in X^{(0)}$, $\lambda(\mathrm{Lk}^{(0)}(p), T_qN) \geq \frac{1}{2}$ *for all* $q \in N$, *then* f *is totally geodesic.*

iii. If for all $p \in X^{(0)}$, $\lambda(\mathrm{Lk}^{(0)}(p), T_qN) \geq \frac{1}{2}$ *for all* $q \in N$ *and, additionally,* N *has strictly negative sectional curvature, then* f *is maps each* 2*-simplex into a geodesic on* N*.*

Remark 5.56. We emphasize consideration of the conditions of the preceding theorems. They all are dependent on a Bochner formula, the simplex-wise metric on the domain being flat and the sectional curvature of the target being nonnegative. If even a single assumption is missing, these results do not generally hold. For our considerations in this paper, we do not always assume that the simplex-wise metric is flat—we only that the metric is always smooth—and so the preceding theorems do not always apply here.

Continuing with our examination of energy on a simplex-wise flat domain, we note with X, N, and f as above that for fixed $p_0 \in X^{(0)}$ with a small bounded open neighborhood Ω, and an energy-minimizing map f with order $\mathrm{Ord}^f(p_0) < 1$, we can provide some information about in which L^p space the energy density might be on Ω. Indeed, we can easily verify the following as a consequence of Proposition 5.50.

Proposition 5.57. *Let X be an admissible 2-complex with a simplex-wise flat metric, let N be a Riemannian manifold with nonpositive sectional curvature. Let $f\colon X \to N$ be energy minimizing. Let α be such that $\mathrm{Ord}^f(p) \leq \alpha < 1$ for all $p \in X^{(0)}$. Then for any $\beta < \frac{1}{2}(1-\alpha)^{-1}$, $e(f) \in L^{\beta}_{\mathrm{loc}}(X)$.*

Corollary 5.58. *Let X be a compact admissible 2-complex with a simplex-wise flat metric, let N be a Riemannian manifold with nonpositive sectional curvature. Let $f \in L^2(X,N)$ be energy minimizing. If there exists $\alpha > \frac{1}{2}$ such that for all $p \in X^{(0)}$, $\mathrm{Ord}^f(p) \geq \alpha$, then $f \in W^{1,2}(X,N)$. Additionally, in the event that X is not compact, then $e(f) \in L^1_{\mathrm{loc}}(X)$.*

Remark 5.59. We note that for a map f in $L^2(X,N)$, its energy density $e(f)$ being in $L^1(X)$ is equivalent to f being in $W^{1,2}$. We note that, in coordinates,

$$e(f(p)) = \|\mathrm{d}f(p)\|^2_{f^{-1}(TN)} = g^{ij}\frac{\partial f^{\alpha}}{\partial x^i}\frac{\partial f^{\beta}}{\partial x^j}h_{\alpha\beta},$$

where g is the simplex-wise metric on X and h is the metric on N. Thus, $e(f) \in L^1(X)$ is equivalent to stating $\|\mathrm{d}f\|_{f^{-1}(TN)} \in L^2(X)$.

Now that we have examined the possibility of energy densities not being smooth and bounded as in the cases considered in [ES], we consider two questions: firstly, do the results of [DM3] regarding discrete eigenvalues and orders of maps abstract to dimensions greater than 2 and, secondly, can the heat kernel and flow methods of [ES] be modified to treat initial maps that are merely in $C^2_{\mathrm{loc}}(X \setminus X^{(n-1)}, N)$ and whose energy density is in $L^1(X)$ but not bounded?

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